Chapter 5
Population Dynamics

This chapter is devoted to the application of Volterra difference equations in population dynamics and epidemics. We begin the chapter by introducing different types of population models including predator-prey models. Most commonly studied version of population models are described by continuous-time dynamics, whereas in real ecosystem the changes in populations of each species due to competitive interaction cannot occur continuously. Hence, discrete-time dynamical systems are often more suitable tool for modeling the dynamics in competing species. Cone theory is introduced and utilized to prove the existence of positive periodic solutions for functional difference equations. We introduce an infinite delay population model which governs the growth of population $N(n)$ of a single species whose members compete among themselves for the limited amount of food that is available to sustain the population, and use the results on cone theory to obtain the existence of a positive periodic solution. Moreover, from a biologist's point of view, the idea of permanence plays a central role in any competing species.

5.1 Background

We begin with a brief history regarding the early work of Vito Volterra on modeling fish population utilizing what we call today: Volterra integral equations and Volterra integro-differential equations. Volterra did not limit himself to academic research. Volterra became interested in mathematical ecology late in 1925. His interest in the field was stimulated by conversations with the young zoologist Umberto D'Ancona, then engaged to marry his daughter Luisa. D'Ancona, studying the records of the fish markets in the upper Adriatic, had noticed a curious phenomenon. He observed that during and after the war, when fishing was severely limited, the proportion of predators among the total catch had increased correspondingly, an effect predicted by Volterra's models. D'Ancona was thus reinforced in his

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belief that the two facts were causally correlated. In other words, the proportion of food fish markedly decreased during the war years. This beginning led Volterra to attack more general problems in ecology. Volterra emphasized consistently that differential equations are, at best, only rough approximations of actual ecological systems. They would apply only to animals without age or memory, which eat all the food they encounter and immediately convert it into offspring. Anything more realistic would yield integro-differential rather than differential equations. The field soon became his major research. More on the next discussion can be found in [158]. To put things into perspective we give some background on the famous Lotka prey-predator model. In 1925 Lotka published his *Elements of Physical Biology*, in which he developed and studied the interaction between two species via the model

$$\frac{dN_1}{dt} = (\varepsilon_1 + \gamma_1 N_2)N_1, \;\; \frac{dN_2}{dt} = (\varepsilon_2 + \gamma_2 N_1)N_2,$$

where ε_1 and ε_2 are the "coefficients of self-increase," while γ_1 and γ_2 account for the interaction between two species, and the N_1, N_2, are population sizes. This model can represent species preying on another, depending on the sign of the constants in the model. In the case of predation, Lotka showed the existence of close periodic orbits. Later on, Lotka considered more advanced models that dealt with multiple species preying on a single specie. For the sake of the next discussion, we write the above Lotka system to suit a predator-prey model. That is, by assuming all the constants are positive we have that

$$\frac{dN_1}{dt} = (\varepsilon_1 - \gamma_1 N_2)N_1, \;\; \frac{dN_2}{dt} = (-\varepsilon_2 + \gamma_2 N_1)N_2, \tag{5.1.1}$$

where N_1 and N_2 represent the populations of the preys and predators at time t, respectively. Note that the predators would die out without the presence of the preys. To better explain this, we multiply the first equation of (5.1.1) by dt and then it is clear that an amount of $\varepsilon_2 N_2$ of them will die in a time interval dt. Suppose that predator tendency to eat the prey when encountered does not depend on age, τ, nor on the state of the association. Assume also that the age distribution of the predators, $\lambda(\tau,t)$, can be considered as independent of time, $\lambda(\tau)$. The individuals of age not younger than τ will be in the proportion

$$f(t-\tau) = \int_{t-\tau}^{\infty} \lambda(\eta)d\eta.$$

Then $f(t-\tau)N_2(t)$ will be the number of predators that is active at time $t-\tau$. Their feeding rate is proportional to $f(t-\tau)N_2(t)N_1(\tau)$ that is $\phi(t-\tau)f(t-\tau)N_2(t)N_1(\tau)$, measuring the effect of feeding through all previous time on the chances of survival and the rate of reproduction at a subsequent time. Setting

$$\phi(t-\tau)f(t-\tau)N_2(t)N_1(\tau)d\tau = F(t-\tau)N_2(t)N_1(\tau)d\tau$$

and integrating over all previous time we obtain, as a positive term for the predators' equation,

$$N_2(t)\int_{t-\tau}^{t} F(t-\tau)N_1(\tau)d\tau.$$

Similar argument can be made for the preys and hence we arrive at the system of integro-differential equations

$$\frac{dN_1}{dt} = N_1(t)\big[\varepsilon_1 - \gamma_1 N_2(t) - \int_{t-\tau}^{t} F_1(t-\tau)N_1(\tau)d\tau\big]$$

$$\frac{dN_2}{dt} = N_2(t)\big[-\varepsilon_2 + \gamma_2 N_1(t) - N_2(t)\int_{t-T}^{t} F_2(t-\tau)N_1(\tau)d\tau\big].$$

Another aspect of importance for application of Volterra difference equations is their usefulness in numerical approximations of Volterra integro-differential equations, see [161] and the reference therein. This notion was briefly discussed in Chapter 1.

5.2 Formulation of Predator-Prey Discrete Models

In this section we obtain the Lotka-Volterra predator-prey model from its continuous counterpart. Researchers have argued that discrete time models governed by difference equations are more appropriate for describing the dynamics relationship among populations than the continuous ones when the populations have nonoverlapping generations. There is no unique way of deriving discrete time version of dynamical systems corresponding to continuous time formulations. One of the ways of deriving difference equations modeling the dynamics of populations with nonoverlapping generations is based on appropriate modifications of models with overlapping generations. In this approach, differential equations with piecewise constant arguments have been useful (see [170]). Thus, we consider the continuous Lotka-Volterra predator-prey model given by (5.1.1) and use differential equations with piecewise constant arguments to obtain a discrete analogue of it. We follow the work given in [70] and [170]. That is, we assume that the average growth rates in system (5.1.1) change at regular intervals of time. We can incorporate this aspect in (5.1.1) and obtain the following modified system

$$\frac{dN_1(t)}{dt}\frac{1}{N(t)} = (\varepsilon_1 - \gamma_1 N_2([t])) \tag{5.2.1}$$

$$\frac{dN_2(t)}{dt}\frac{1}{N_2(t)} = (-\varepsilon_2 + \gamma_2 N_1([t]))$$

where $t \neq 0,1,2,\ldots,[t]$ denotes the integer part of t, $t \in (0,\infty)$. Equations of the form (5.2.1) are known as differential equations with piecewise constant arguments and they occupy a position midway between differential equations and difference equations. By a solution of (5.2.1), we mean a function $N = (N_1,N_2)^T$ which is defined for $t \in (0,\infty)$ and has the properties that,

1. N is continuous on $[0,\infty)$.

2. The derivatives $\frac{dN_1(t)}{dt}$, $\frac{dN_2(t)}{dt}$ exist at each point $t \in (0,\infty)$ with the exception of

the points $t \in \{0,1,2,\cdots,\}$,where left-sided derivatives exist.
3. System (5.2.1) is valid on each interval $[k,k+1]$ with $k = 0,1,2,\cdots$

Next we integrate both sides of (5.2.1) over any interval $[k,k+1)$, $k = 0,1,2,\cdots$ to arrive at for $k \le t \le k+1$ $k = 0,1,2,\cdots$

$$N_1(t) = N_1(k)e^{\{[\varepsilon_1-\gamma_1 N_2(k)](t-k)\}} \tag{5.2.2}$$

$$N_2(t) = N_2(k)e^{\{[-\varepsilon_2+\gamma_2 N_1(k)](t-k)\}}$$

Letting $t \to k+1$, then system takes the form

$$\begin{cases} N_1(k+1) = N_1(k)e^{\{[\varepsilon_1-\gamma_1 N_2(k)]\}} \\ N_2(k+1) = N_2(k)e^{\{[-\varepsilon_2+\gamma_2 N_1(k)]\}} \end{cases} \tag{5.2.3}$$

where $k = 0,1,2,\cdots$.
General forms of discrete-generation host-parasite

$$\begin{cases} P(k+1) = \lambda P(k) f(P(k),H(k)) \\ H(k+1) = c\lambda P(k)(1-f(P(k),H(k)) \end{cases}$$

have been used to model the interaction between host species (a plant, $P(k)$) and a parasite species (a herbivore, $H(k)$). The term $1-f$ represents the probability of being parasitized. Nicholson-Bailey [13] used one of the simplest version of the above general form by considering

$$\begin{cases} P(k+1) = \lambda P(k)e^{-aH(k)} \\ H(k+1) = c\lambda P(k)(1-e^{-aH(k)}) \end{cases}$$

in which $f = e^{-aH(k)}$ is the proportion of hosts escaping parasitism, where a is the mean encounters per host. Hence, $1-e^{-aH(k)}$ is the probability that host will be attacked. In a later study, Beddington et al. [15] considered a generalization of the above model by studying

$$\begin{cases} P(k+1) = \lambda P(k)e^{r(1-P(k)/P_{max})-aH(k)} \\ H(k+1) = c\lambda P(k)(1-e^{-aH(k)}) \end{cases}$$

where P_{max} is the carrying capacity imposed by the environment for the host in the absence of the parasite.
Following in the footsteps of Beddington et al. [15], Elaydi et al. [90] considered the predator-prey model

$$\begin{cases} x(n+1) = x(n)e^{r(1-\frac{x(n)}{K})-by(n))} \\ y(n+1) = ey(n)(1-e^{-ay(n)}) \end{cases} \tag{5.2.4}$$

where $x(n) \ge 0$ and $y(n) \ge 0$ represent population densities of a prey and a predator, respectively, and a,b,e,K, and r are positive. The constant K is the carrying capacity and represents the maximum population size that can be supported by the available limited resources and r is the growth rate. In [90] Elaydi et al. investigated the stability and invariant manifolds and the stability of the coexisting fixed point of model (5.2.4). Motivated by Elaydi et al. [90], in [10], Asheghi revisited model (5.2.4) and analyzed the stability of feasible fixed points and the period-doubling. In addition, the author studied the Neimark-Sacker bifurcation diagrams. In 2014, Li and Xu [176] considered the discrete predator-prey model with infected prey

$$\begin{cases} S(n+1) = S(n)exp\{r_1(n)(1-S(n)-I(n)) - \dfrac{a(n)Z(n)}{1+b(n)(S(n)+I(n))} - \dfrac{\alpha(n)I(n)}{S(n)+I(n)}\} \\ I(n+1) = I(n)exp\{r_2(n)(1-S(n)-I(n)) - \dfrac{a(n)Z(n)}{1+b(n)(S(n)+I(n))} + \dfrac{\alpha(n)I(n)}{S(n)+I(n)} - m_2(n)\} \\ Z(n+1) = Z(n)exp\{\dfrac{a(n)(S(n)+I(n))}{1+b(n)(S(n)+I(n)} - m_3(n)\} \end{cases} \tag{5.2.5}$$

where $S(n)$ and $I(n)$ are the susceptible phytoplankton population and the infected phytoplankton population, respectively, and $Z(n)$ grazes on both the susceptible and infected phytoplankton. The parameter $\alpha > 0$ is the frequency-dependent transmissions rate and the parameter m_2 is the disease-induced mortality of infected prey. The parameters r_1 and r_2 are the intrinsic growth rates of susceptible and infected population, respectively. Rate m_3 represents the natural mortality rate of zooplankton. In addition, a and b are constants. For more on the biological meaning and development of the model in the continuous case, we refer to [81] and [157]. Li and Xu [176] assumed periodicity conditions on the coefficients and used the Continuation theorem due to Gaines and Malvin [71] and showed the existence of a positive periodic solution. Moreover, they effectively used Lyapunov functions and proved the positive periodic solution is indeed globally asymptotically stable.

We remark that all the above models display positive solutions for positive initial data.

5.3 Cone Theory and Positive Periodic Solutions

We begin the chapter by utilizing cone theoretic fixed point theorem to study the existence of positive periodic solutions of the nonlinear nonautonomous system of functional difference equations

$$x(n+1) = A(n)x(n) + f(n,x_n) \tag{5.3.1}$$

where $A(n) = \text{diag}[a_1(n), a_2(n), \ldots, a_k(n)]$, a_j is ω-periodic, $f : \mathbb{Z} \times \mathbb{R}^k \to \mathbb{R}^k$ is continuous in x, and $f(n,x)$ is ω-periodic in n and x, whenever x is ω-periodic.

Most contents can be found in [149] and the references therein. Such results will be applied to the infinite delay scalar Volterra discrete population model

$$N(n+1) = \alpha(n)N(n)\Big[1 - \frac{1}{N_0(n)}\sum_{s=-\infty}^{0} B(s)N(n+s)\Big],\ n\in\mathbb{Z} \qquad (5.3.2)$$

which governs the growth of population $N(n)$ of a single species whose members compete among themselves for the limited amount of food that is available to sustain the population. We emphasize that our conditions can only imply the existence of positive and periodic solutions for model (5.4.1). We note that equation (5.3.2) is a generalization of the known logistic model

$$N(n+1) = \alpha N(n)\Big[1 - \frac{N(n)}{N_0}\Big], \qquad (5.3.3)$$

where α is the intrinsic per capita growth rate and N_0 is the total carrying capacity. For more biological information on equation (5.3.2), we refer the reader to [57]. We remark that in (5.3.2), the term $\sum_{s=-\infty}^{0} B(s)N(n+s)$ is equivalent to $\sum_{u=-\infty}^{n} B(u-s))N(u)$. We chose to write (5.3.2) that way so that it can be put in the form of $x(n+1) = a(n)x(n) + f(n, x_n)$.

Let $\mathscr{X}$ be the set of all real ω-periodic sequences $\phi : \mathbb{Z} \to \mathbb{R}^k$. Endowed with the maximum norm $||\phi|| = \max_{\theta\in\mathbb{Z}} \sum_{j=1}^{k} |\phi_j(\theta)|$ where $\phi = (\phi_1, \phi_2, \ldots, \phi_k)^t$, $\mathscr{X}$ is a Banach space. Here t stands for the transpose. If $x \in \mathscr{X}$, then $x_n \in \mathscr{X}$ for any $n \in \mathbb{Z}$ is defined by $x_n(\theta) = x(n+\theta)$ for $\theta \in \mathbb{Z}$.

The existence of multiple positive periodic solutions of nonlinear functional differential equations has been studied extensively in recent years. Some appropriate references would be [34] and [168]. We are particularly motivated by the work in [88] on functional differential equations and the work of Raffoul in [67, 129], and [151] on boundary value problems involving functional difference equations. When working with boundary value problems whether in differential or difference equations, it is customary to display the desired solution in terms of a suitable Green's function and then apply cone theory (see [8, 45, 67, 78, 79, 80], and [118]). Since our equation (5.3.1) is not the type of boundary value problem, we obtain a variation of parameters formula and then try to find a lower and upper estimates for the kernel inside the summation. Once those estimates are found we use Krasnoselskii's fixed point theorem [97] to show the existence of a positive periodic solution. In [129], Raffoul studied the existence of periodic solutions of an equation similar to equation (5.3.1) using Schauder's Second fixed point theorem. Moreover, In [151], Raffoul considered the scalar difference equation

$$x(n+1) = a(n)x(n) + h(n)f(x(n-\tau(n))) \qquad (5.3.4)$$

where a, h, and τ are ω-periodic for ω is an integer with $\omega \geq 1$. Under the assumptions that $a(n), f(x)$, and $h(n)$ are nonnegative with $0 < a(n) < 1$ for all $n \in [0, \omega-1]$, it was shown that (5.3.4) possesses a positive periodic solution. In this work we extend (5.3.4) to systems with infinite delay and address the existence of positive periodic solutions of (5.3.1) in the case $a(n) > 1$. Let $\mathbb{R}_+ = [0, +\infty)$, for each $x = (x_1, x_2, \dots, x_k)^t \in \mathbb{R}^k$, the norm of x is defined as $|x| = \sum_{j=1}^{k} |x_j|$. $\mathbb{R}_+^k = \{(x_1, x_2, \dots, x_k)^t \in \mathbb{R}^k : x_j \geq 0,\ j = 1, 2, \dots, k\}$. Also, we denote $f = (f_1, f_2, \dots, f_k)^t$, where t stands for transpose. Now we list the following conditions.

(H1) $a(n) \neq 0$ for all $n \in [0, \omega - 1]$ with $\prod_{s=0}^{\omega-1} a_j(s) \neq 1$ for $j = 1, 2, \dots, k$.
(H2) If $0 < a(n) < 1$ for all $n \in [0, \omega - 1]$, then $f_j(n, \phi_n) \geq 0$ for all $n \in \mathbb{Z}$ and $\phi : \mathbb{Z} \to \mathbb{R}_+^n$, $j = 1, 2, \dots, k$ where $\mathbb{R}_+ = [0, +\infty)$
(H3) If $a(n) > 1$ for all $n \in [0, \omega - 1]$, then $f_j(n, \phi_n) \leq 0$ for all $n \in \mathbb{Z}$ and $\phi : \mathbb{Z} \to \mathbb{R}_+^n$, $j = 1, 2, \dots, k$ where $\mathbb{R}_+ = [0, +\infty)$
(H4) For any $L > 0$ and $\varepsilon > 0$, there exists $\delta > 0$ such that $[\phi, \psi \in \mathscr{X}, \|\phi\| \leq L, \|\psi\| \leq L, \|\phi - \psi\| < \delta, 0 \leq s \leq \omega]$ imply

$$|f(s, \phi_s) - f(s, \psi_s)| < \varepsilon. \tag{5.3.5}$$

We begin by stating some preliminaries in the form of definitions and lemmas that are essential to the proofs of our main results. We start with the following definition.

Definition 5.3.1. Let X be a Banach space and K be a closed, nonempty subset of X. The set K is a cone if

(i) $\alpha u + \beta v \in K$ for all $u, v \in K$ and all $\alpha, \beta \geq 0$
(ii) $u, -u \in K$ imply $u = 0$.

We now state the Krasnoselskii's fixed point theorem [97].

Theorem 5.3.1 (Krasnoselskii [97]). *Let $\mathscr{B}$ be a Banach space, and let $\mathscr{P}$ be a cone in $\mathscr{B}$. Suppose Ω_1 and Ω_2 are open subsets of $\mathscr{B}$ such that $0 \in \Omega_1 \subset \overline{\Omega}_1 \subset \Omega_2$ and suppose that*

$$T : \mathscr{P} \cap (\overline{\Omega}_2 \backslash \Omega_1) \to \mathscr{P}$$

is a completely continuous operator such that

(i) $\|Tu\| \leq \|u\|$, $u \in \mathscr{P} \cap \partial\Omega_1$, and $\|Tu\| \geq \|u\|$, $u \in \mathscr{P} \cap \partial\Omega_2$; or
(ii) $\|Tu\| \geq \|u\|$, $u \in \mathscr{P} \cap \partial\Omega_1$, and $\|Tu\| \leq \|u\|$, $u \in \mathscr{P} \cap \partial\Omega_2$.
Then T has a fixed point in $\mathscr{P} \cap (\overline{\Omega}_2 \backslash \Omega_1)$.

For the next lemma we consider

$$x_j(n+1) = a_j x_j(n) + f_j(n, x_n),\ j = 1, 2, \dots, k. \tag{5.3.6}$$

The proof of the next Lemma can be easily deduced from [129] and hence we omit it.

Lemma 5.1 ([149]). *Suppose (H1) hold. Then $x_j(n) \in \mathscr{X}$ is a solution of equation (5.3.6) if and only if*

$$x_j(n) = \sum_{u=n}^{n+\omega-1} G_j(n, u) f_j(u, x_u),\ j = 1, 2, \dots, k \tag{5.3.7}$$

where

$$G_j(n,u) = \frac{\prod_{s=u+1}^{n+\omega-1} a_j(s)}{1-\prod_{s=n}^{n+\omega-1} a_j(s)}, \quad u \in [n, n+\omega-1],\ j = 1,2,\dots,k. \tag{5.3.8}$$

Set

$$G(n,u) = \mathrm{diag}[G_1(n,u), G_2(n,u), \dots, G_k(n,u)].$$

It is clear that $G(n,u) = G(n+\omega, u+\omega)$ for all $(n,u) \in \mathbb{Z}^2$. Also, if either (H2) or (H3) holds, then (5.3.8) implies that

$$G_j(n,u) f_j(u, \phi_u) \geq 0$$

for $(n,u) \in \mathbb{Z}^2$ and $u \in \mathbb{Z}$, $\phi : \mathbb{Z} \to \mathbb{R}_+^k$. In defining the desired cone we observe that if (H2) holds, then

$$\frac{\prod_{s=0}^{\omega-1} a_j(s)}{1-\prod_{s=n}^{n+\omega-1} a_j(s)} \leq |G_j(n,u)| \leq \frac{\prod_{s=0}^{\omega-1} a_j^{-1}(s)}{1-\prod_{s=n}^{n+\omega-1} a_j(s)} \tag{5.3.9}$$

for all $u \in [n, n+\omega-1]$. Also, if (H3) holds, then

$$\frac{\prod_{s=0}^{\omega-1} a_j^{-1}(s)}{\left|1-\prod_{s=n}^{n+\omega-1} a_j(s)\right|} \leq |G_j(n,u)| \leq \frac{\prod_{s=0}^{\omega-1} a_j(s)}{\left|1-\prod_{s=n}^{n+\omega-1} a_j(s)\right|} \tag{5.3.10}$$

for all $u \in [n, n+\omega-1]$. For all $(n,s) \in \mathbb{Z}^2$, $j = 1,2,\dots,k$, we define

$$\sigma_2 := \min\left\{ \Big(\prod_{s=0}^{\omega-1} a_j(s)\Big)^2, j = 1,2,\dots,n \right\}$$

and

$$\sigma_3 := \min\left\{ \Big(\prod_{s=0}^{\omega-1} a_j^{-1}(s)\Big)^2, j = 1,2,\dots,n \right\}.$$

We note that if $0 < a(n) < 1$ for all $n \in [0, \omega-1]$, then $\sigma_2 \in (0,1)$. Also, if $a(n) > 1$ for all $n \in [0, \omega-1]$, then $\sigma_3 \in (0,1)$. Conditions (H2) and (H3) will have to be handled separately. That is, we define two cones; namely, $\mathscr{P}2$ and $\mathscr{P}3$. Thus, for each $y \in \mathscr{X}$ set

$$\mathscr{P}2 = \{y \in \mathscr{X} : y(n) \geq 0, n \in \mathbb{Z}, \text{and } y(n) \geq \sigma_2 \|y\|\}$$

and

$$\mathscr{P}3 = \{y \in \mathscr{X} : y(n) \geq 0, n \in \mathbb{Z}, \text{ and } y(n) \geq \sigma_3 \|y\|\}.$$

Define a mapping $T : \mathscr{X} \to \mathscr{X}$ by

$$(Tx)(n) = \sum_{u=n}^{n+\omega-1} G(n,u) f(u, x_u) \tag{5.3.11}$$

where $G(n,u)$ is defined following (5.3.8). We denote

$$(Tx) = \Big(T_1x, T_2x, \ldots, T_kx\Big)^t.$$

It is clear that $(Tx)(n+\omega) = (Tx)(n)$.

Lemma 5.2 ([149]). *If (H1) and (H2) hold, then the operator $T\mathscr{P}2 \subset \mathscr{P}2$. If (H1) and (H3) hold, then $T\mathscr{P}3 \subset \mathscr{P}3$.*

Proof. Suppose (H1) and (H2) hold. Then for any $x \in \mathscr{P}2$ we have

$$(T_jx(n)) \geq 0, \ j = 1,2,\ldots k.$$

Also, for $x \in \mathscr{P}2$ by using (5.3.8)–(5.3.11) we have that

$$(T_jx)(n) \leq \frac{\prod_{s=0}^{\omega-1} a_j^{-1}(s)}{1-\prod_{s=n}^{n+\omega-1} a_j(s)} \sum_{u=n}^{n+\omega-1} |f_j(u,x_u)|$$

and

$$\|T_jx\| = \max_{n\in[0,\omega-1]} |T_jx(n)| \leq \frac{\prod_{s=0}^{\omega-1} a_j^{-1}(s)}{1-\prod_{s=n}^{n+\omega-1} a_j(s)} \sum_{u=n}^{n+\omega-1} |f_j(u,x_u)|.$$

Therefore,

$$\begin{aligned}(T_jx)(n) &= \sum_{u=n}^{n+\omega-1} G_j(n,u)f_j(u,x_u)\\ &\geq \frac{\prod_{s=0}^{\omega-1} a_j(s)}{1-\prod_{s=n}^{n+\omega-1} a_j(s)} \sum_{u=n}^{n+\omega-1} |f_j(u,x_u)|\\ &\geq (\prod_{s=0}^{\omega-1} a_j(s))^2 \|T_jx\| \geq \sigma_2\|T_jx\|.\end{aligned}$$

That is, $T\mathscr{P}2$ is contained in $\mathscr{P}2$. The proof of the other part follows in the same manner by simply using (5.3.10), and hence we omit it. This completes the proof.

To simplify notation we denote,

$$A_2 = \min_{1\leq j\leq k} \frac{\prod_{s=0}^{\omega-1} a_j(s)}{1-\prod_{s=n}^{n+\omega-1} a_j(s)}, \tag{5.3.12}$$

$$B_2 = \max_{1\leq j\leq k} \frac{\prod_{s=0}^{\omega-1} a_j^{-1}(s)}{1-\prod_{s=n}^{n+\omega-1} a_j(s)}, \tag{5.3.13}$$

$$A_3 = \min_{1\leq j\leq k} \frac{\prod_{s=0}^{\omega-1} a_j^{-1}(s)}{\left|1-\prod_{s=n}^{n+\omega-1} a_j(s)\right|}, \tag{5.3.14}$$

and

$$B_3 = \max_{1\le j\le k} \frac{\prod_{s=0}^{\omega-1} a_j(s)}{\left|1-\prod_{s=n}^{n+\omega-1} a_j(s)\right|}. \tag{5.3.15}$$

Lemma 5.3 ([149]). *If (H1), (H2), and (H4) hold, then the operator $T : \mathscr{P}2 \to \mathscr{P}2$ is completely continuous. Similarly, if (H1), (H3), and (H4) hold, then the operator $T : \mathscr{P}3 \to \mathscr{P}3$ is completely continuous.*

Proof. Suppose (H1), (H2), and (H4) hold. First show that T is continuous. By (H4), for any $L > 0$ and $\varepsilon > 0$, there exists a $\delta > 0$ such that $[\phi,\ \psi \in \mathscr{X},\ \|\phi\| \le L,\ \|\psi\| \le L,\ \|\phi-\psi\| < \delta]$ imply

$$\max_{0\le s\le\omega-1} |f(s,\phi_s) - f(s,\psi_s)| < \frac{\varepsilon}{B_2\omega}$$

where B_2 is given by (5.3.13). If $x,\ y \in \mathscr{P}2$ with $\|x\| \le L$, $\|y\| \le L$, and $\|x-y\| < \delta$, then

$$\begin{aligned}|(Tx)(n)-(Ty)(n)| &\le \sum_{u=n}^{n+\omega-1} |G(n,u)||f(u,x_u)-f(u,y_u)| \\ &\le B_2 \sum_{u=0}^{\omega-1} |f(u,x_u)-f(u,y_u)| < \varepsilon\end{aligned}$$

for all $n \in [0,\omega-1]$, where $|G(n,u)| = \max_{1\le j\le n} |G_j(n,u)|,\ j = 1,2,\ldots,k$. This yields $\|(Tx)-(Ty)\| < \varepsilon$. Thus, T is continuous. Next we show that T maps bounded subsets into compact subsets. Let $\varepsilon = 1$. By (H4), for any $\mu > 0$ there exists $\delta > 0$ such that $[x,\ y \in \mathscr{X},\ \|x\| \le \mu,\ \|y\| \le \mu,\ \|x-y\| < \delta]$ imply

$$|f(s,x_s)-f(s,y_s)| < 1.$$

We choose a positive integer N so that $\delta > \frac{\mu}{N}$. For $x \in \mathscr{X}$, define $x^i(n) = \frac{ix(n)}{N}$, for $i = 0,1,2,....,N$. For $||x|| \le \mu$,

$$\begin{aligned}||x^i - x^{i-1}|| &= \max_{n\in\mathbb{Z}} \left|\frac{ix(n)}{N} - \frac{(i-1)x(n)}{N}\right| \\ &\le \frac{||x||}{N} \le \frac{\mu}{N} < \delta.\end{aligned}$$

Thus, $|f(s,x^i)-f(s,x^{i-1})| < 1$. As a consequence, we have

$$f(s,x_s)-f(s,0) = \sum_{i=1}^{N} \Big(f(s,x^i)-f(s,x^{i-1})\Big),$$

which implies that

$$
\begin{aligned}
|f(s,x_s)| &\le \sum_{i=1}^{N} |f(s,x_s^i) - f(s,x_s^{i-1})| + |f(s,0)| \\
&< N + |f(s,0)|.
\end{aligned}
$$

Thus, f maps bounded sets into bounded sets. It follows from the above inequality and (5.3.11) that

$$
\begin{aligned}
||(Tx)(n)|| &\le B_2 \sum_{j=1}^{k} \Big(\sum_{u=n}^{n+T-1} |f_j(u,x_u)| \Big) \\
&\le B_2 \omega (N + |f(s,0)|).
\end{aligned}
$$

If we define $S = \{x \in \mathscr{X} : ||x|| \le \mu\}$ and $Q = \{(Tx)(n) : x \in S\}$, then S is a subset of $\mathbb{R}^{\omega k}$ which is closed and bounded and thus compact. As T is continuous in x, it maps compact sets into compact sets. Therefore, $Q = T(S)$ is compact. The proof for the other case is similar by simply invoking (5.3.15). This completes the proof.

Next, we state two theorems and two corollaries. Our theorems and corollaries are stated in a way that unify both cases; $0 < a(n) < 1$ and $a(n) > 1$ for all $n \in [0, \omega - 1]$.

Theorem 5.3.2 ([149]). *Assume (H1).*
(a) Suppose (H2) and (H4) hold and that there exist two positive numbers R_1 and R_2 with $R_1 < R_2$ such that

$$
\sup_{\|\phi\|=R_1,\, \phi \in \mathscr{P}2} |f(s,x_s)| \le \frac{R_1}{\omega B_2}, \tag{5.3.16}
$$

and

$$
\inf_{\|\phi\|=R_2,\, \phi \in \mathscr{P}2} |f(s,x_s)| \ge \frac{R_2}{\omega A_2}, \tag{5.3.17}
$$

where A_2 and B_2 are given by (5.3.12) and (5.3.13), respectively. Then, there exists $\bar{x} \in \mathscr{P}2$ which is a fixed point of T and satisfies $R_1 \le \|\bar{x}\| \le R_2$.
(b) Suppose (H3) and (H4) hold and that there exist two positive numbers R_1 and R_2 with $R_1 < R_2$ such that

$$
\sup_{\|\phi\|=R_1,\, \phi \in \mathscr{P}3} |f(s,x_s)| \le \frac{R_1}{\omega B_3}, \tag{5.3.18}
$$

and

$$
\inf_{\|\phi\|=R_2,\, \phi \in \mathscr{P}3} |f(s,x_s)| \ge \frac{R_2}{\omega A_3}, \tag{5.3.19}
$$

where A_3 and B_3 are given by (5.3.14) and (5.3.14), respectively. Then, there exists $\bar{x} \in \mathscr{P}3$ which is a fixed point of T and satisfies $R_1 \le \|\bar{x}\| \le R_2$.

Proof. Suppose (H1), (H2), and (H4) hold. Let $\Omega_\xi = \{x \in \mathscr{P}2 | \|x\| < \xi\}$. Let $x \in \mathscr{P}2$ which satisfies $\|x\| = R_1$. in view of (5.3.16), we have

$$\begin{aligned}|(Tx)(n))| &\le \sum_{u=n}^{n+\omega-1} |G(n,u)||f(u,x_u)| \\ &\le B_2\omega\frac{R_1}{\omega B_2} = R_1.\end{aligned}$$

That is, $\|Tx\| \le \|x\|$ for $x \in \mathscr{P}2 \cap \partial\Omega_{R_1}$. let $x \in \mathscr{P}2$ which satisfies $\|x\| = R_2$ we have, in view of (5.3.17),

$$|(Tx)(n)| \ge A_2 \sum_{u=n}^{n+\omega-1} |f(u,x_u)| \ge A_2\omega\frac{R_2}{\omega A_2} = R_2.$$

That is, $\|Tx\| \ge \|x\|$ for $x \in \mathscr{P}2 \cap \partial\Omega_{R_2}$. In view of Theorem 5.3.1, T has a fixed point in $\mathscr{P}2 \cap (\bar{\Omega}_2 \setminus \Omega_1)$. It follows from Lemma 5.2 that (5.3.1) has an ω-periodic solution $\bar{x}$ with $R_1 \le \|\bar{x}\| \le R_2$. The proof of (b) follows in a similar manner by simply invoking conditions (5.3.18) and (5.3.19).

As a consequence of Theorem 5.3.2, we state a corollary which its proof we omit.

Corollary 5.1 ([149]). *Assume that (H1) holds.*
(a) Suppose (H2) and (H4) hold and

$$\lim_{\phi\in\mathscr{P}2,\, \|\phi\|\to 0} \frac{|f(s,\phi_s)|}{\|\phi\|} = 0, \tag{5.3.20}$$

$$\lim_{\phi\in\mathscr{P}2,\, \|\phi\|\to \infty} \frac{|f(s,\phi_s)|}{\|\phi\|} = \infty. \tag{5.3.21}$$

Then (5.3.1) *has a positive periodic solution.*
(b) Suppose (H3) and (H4) hold and

$$\lim_{\phi\in\mathscr{P}3,\, \|\phi\|\to 0} \frac{|f(s,\phi_s)|}{\|\phi\|} = 0, \tag{5.3.22}$$

$$\lim_{\phi\in\mathscr{P}3,\, \|\phi\|\to \infty} \frac{|f(s,\phi_s)|}{\|\phi\|} = \infty. \tag{5.3.23}$$

Then (5.3.1) *has a positive periodic solution.*

Theorem 5.3.3 ([149]). *Suppose that (H1) holds.*
(a) Suppose (H2) and (H4) hold and that there exist two positive numbers R_1 and R_2 with $R_1 < R_2$ such that

$$\inf_{\|\phi\|=R_1,\, \phi\in\mathscr{P}2} |f(s,x_s)| \ge \frac{R_1}{\omega B_2}, \tag{5.3.24}$$

and

$$\sup_{\|\phi\|=R_2,\,\phi\in\mathscr{P}2} |f(s,x_s)| \le \frac{R_2}{\omega A_2}, \tag{5.3.25}$$

where A_2 and B_2 are given by (5.3.12) *and* (5.3.13)*, respectively. Then, there exists $\bar{x} \in \mathscr{P}2$ which is a fixed point of T and satisfies $R_1 \le \|\bar{x}\| \le R_2$.*
(b) Suppose (H3) and (H4) hold and that there exist two positive numbers R_1 and R_2 with $R_1 < R_2$ such that

$$\inf_{\|\phi\|=R_1,\,\phi\in\mathscr{P}3} |f(s,x_s)| \ge \frac{R_1}{\omega B_3}, \tag{5.3.26}$$

and

$$\sup_{\|\phi\|=R_2,\,\phi\in\mathscr{P}3} |f(s,x_s)| \le \frac{R_2}{\omega A_3}, \tag{5.3.27}$$

where A_3 and B_3 are given by (5.3.14) *and* (5.3.15)*, respectively. Then, there exists $\bar{x} \in \mathscr{P}3$ which is a fixed point of T and satisfies $R_1 \le \|\bar{x}\| \le R_2$.*

The proof is similar to the proof of Theorem 5.3.2 and hence we omit it. As a consequence of Theorem 5.3.3, we have the following corollary.

Corollary 5.2 ([149]). *Assume that (H1) hold.*
(a) Suppose (H2) and (H4) hold and

$$\lim_{\phi\in\mathscr{P}2,\,\|\phi\|\to 0} \frac{|f(s,\phi_s)|}{\|\phi\|} = \infty, \tag{5.3.28}$$

$$\lim_{\phi\in\mathscr{P}2,\,\|\phi\|\to \infty} \frac{|f(s,\phi_s)|}{\|\phi\|} = 0. \tag{5.3.29}$$

Then (5.3.1) *has a positive periodic solution.*

(b) Suppose (H3) and (H4) hold and

$$\lim_{\phi\in\mathscr{P}3,\,\|\phi\|\to 0} \frac{|f(s,\phi_s)|}{\|\phi\|} = \infty, \tag{5.3.30}$$

$$\lim_{\phi\in\mathscr{P}3,\,\|\phi\|\to \infty} \frac{|f(s,\phi_s)|}{\|\phi\|} = 0. \tag{5.3.31}$$

Then (5.3.1) *has a positive periodic solution.*

5.3.1 Applications to Infinite Delay Population Models

We apply the results from the previous section to the model (5.3.2) and show that it admits the existence of a positive periodic solution. Thus, we consider the scalar discrete model that governs the growth of population $N(n)$ of a single species whose members compete among themselves for the limited amount of food that is available to sustain the population. Thus, we consider the infinite delay Volterra scalar model

$$N(n+1) = \alpha(n)N(n)\Big[1 - \frac{1}{N_0(n)}\sum_{s=-\infty}^{0} B(s)N(n+s)\Big],\ n \in \mathbb{Z} \tag{5.3.32}$$

as described in the Introduction. We chose to write (5.3.32) that way so that it can be put in the form of $x(n+1) = a(n)x(n) + f(n, x_n)$.
Before we state our results in the form of a theorem, we assume that

(P1) $\alpha(n) > 1$, $N_0(n) > 0$ for all $n \in \mathbb{Z}$ with $\alpha(n)$, $N_0(n)$ are ω-periodic and
(P2) $B(n)$ is nonnegative on $(-\infty, 0] \cap \mathbb{Z}$ with $\sum_{n=-\infty}^{0} B(n) < \infty$.

Theorem 5.3.4 ([149]). *Under assumptions (P1) and (P2), equation* (5.3.32) *has at least one positive ω-periodic solution.*

Proof. Let $a(n) = \alpha(n)N(n)$ and

$$f(n, x_n) = -\frac{x(n)a(n)}{N_0(n)}\sum_{s=-\infty}^{0} B(s)x(n+s).$$

It is clear that $f(n, x_n)$ is ω-periodic whenever x is ω-periodic and (H1) and (H3) hold since $f(n, \phi_n) \le 0$ for all $(n, \phi) \in \mathbb{Z} \times (\mathbb{Z}, \mathbb{R}_+)$. To verify (H4), we let $x, y : \mathbb{Z} \to \mathbb{R}_+$ with $\|x\| \le L$, $\|y\| \le L$ for some $L > 0$. Then

$$\begin{aligned}
&|f(n,x_n) - f(n,y_n)| \\
&= \Big|\frac{x(n)a(n)}{N_0(n)}\sum_{s=-\infty}^{0} B(s)x(n+s) - \frac{y(n)a(n)}{N_0(n)}\sum_{s=-\infty}^{0} B(s)y(n+s)\Big| \\
&\le \Big|\frac{x(n)a(n)}{N_0(n)}\Big|\sum_{s=-\infty}^{0} B(s)|x(n+s) - y(n+s)| \\
&\quad + \Big|\frac{(x(n)-y(n))a(n)}{N_0(n)}\Big|\sum_{s=-\infty}^{0} B(s)|y(n+s)| \\
&\le \frac{L\|a\|}{N_{0*}}\max_{s\in\mathbb{Z}_-}|x(n+s) - y(n+s)| + \frac{|x(n)-y(n)|\|a\|L}{N_{0*}},
\end{aligned}$$

where $N_{0*} = \min\{N_0(s) : 0 \le s \le \omega - 1\}$. For any $\varepsilon > 0$, choose $\delta = \varepsilon N_{0*}/(2L\|a\|)$. If $\|x - y\| < \delta$, then

$$|f(n,x_n) - f(n,y_n)| < L\|a\|\delta/N_{0*} + \delta\|a\|L/N_{0*} = 2L\|a\|\delta/N_{0*} = \varepsilon.$$

This implies that (H4) holds. We now show that (5.3.22) and (5.3.23) hold. For $\phi \in \mathscr{P}3$, we have $\phi(n) \geq \sigma_3\|\phi\|$ for all $n \in [0, \omega-1]$. This yields

$$\frac{|f(n,\phi)|}{\|\phi\|} \leq \max_{\tau\in[0,\omega-1]} \frac{a(\tau)}{N_0(\tau)} \sum_{s=-\infty}^{0} B(s)\|\phi\| \to 0$$

as $\|\phi\| \to 0$ and

$$\frac{|f(n,\phi)|}{\|\phi\|} \geq \min_{\tau\in[0,\omega-1]} \frac{a(\tau)}{N_0(\tau)} \sum_{s=-\infty}^{0} B(s){\sigma_3}^2\|\phi\| \to +\infty$$

as $\|\phi\| \to \infty$. Thus, (5.3.22) and (5.3.23) are satisfied. By (b) of Corollary 5.1, equation (5.3.32) has a positive ω-periodic solution. This completes the proof.

Next we consider the infinite delay Volterra discrete model

$$x_i(n+1)) = x_i(n)\Big[a_i(n) - \sum_{j=1}^{k} b_{ij}(n)x_j(n) - \sum_{j=1}^{k}\sum_{s=-\infty}^{n} C_{ij}(n,s)g_{ij}(x_j(s))\Big] \quad (5.3.33)$$

where $x_i(n)$ is the population of the ith species, a_i, $b_{ij} : \mathbb{Z} \to \mathbb{R}$ are ω-periodic, and $C_{ij} : \mathbb{Z}\times\mathbb{Z} \to \mathbb{R}$ is ω-periodic. For more on such derivation we refer to [44].

Theorem 5.3.5 ([149]). *Suppose that the following conditions hold for $i, j = 1, 2, \ldots, k$.*

(i) $a_i(n) > 1$, for all $n \in [0, \omega-1]$, and $a_i(n)$ is ω-periodic,
(ii) $b_{ij}(n) \geq 0,\ C_{ij}(n,s) \geq 0$ for all $(n,s) \in \mathbb{Z}^2$,
(iii) $g_{ij} : \mathbb{R}^+ \to \mathbb{R}^+$ is continuous in x and increasing with $g_{ij}(0) = 0$,
(iv) $b_{ii}(s) \neq 0$, for $s \in [0, \omega-1]$,
(v) $C_{ij}(n+\omega, s+\omega) = C_{ij}(n,s)$ for all $(n,s) \in \mathbb{Z}^2$ with $\max_{n\in\mathbb{Z}} \sum_{s=-\infty}^{n} |C_{ij}(n,s)| < +\infty$.

Then equation (5.3.33) has a positive ω-periodic solution.

Proof. For $x = (x_1, x_2, \ldots, x_k)^T$, define

$$f_i(n, x_n) = -x_i(t)\sum_{j=1}^{k} b_{ij}(n)x_j(n) - \sum_{j=1}^{k}\sum_{s=-\infty}^{n} C_{ij}(n,s)g_{ij}(x_j(s))$$

for $i = 1, 2, \ldots, k$ and set $f = (f_1, f_2, \ldots, f_k)^t$. Then by some manipulation of conditions (i)–(v), the conditions (H1) and (H2) are satisfied. Also, it is clear that f satisfies (H4). Define

$$b^* = \max\{\|b_{ij}\| : i, j = 1, 2, \cdot, \cdot, \cdot, k\},$$
$$C^* = \max\Big\{\sup_{n\in\mathbb{Z}} \sum_{j=1}^{n}\sum_{s=-\infty}^{n} |C_{ij}(n,s)| : i = 1, 2, \cdot, \cdot, \cdot, k\Big\}$$

and
$$g^*(u) = \max\{g_{ij}(u) : i, j = 1, 2, \cdot, \cdot, \cdot, k\}$$

Let $x \in \mathscr{P}3$. Since g is increasing in x, we arrive at

$$|f_i(n, x_n)| \le |x_i(n)| \Big[b^* \|x\| + \sum_{j=1}^{n} \sum_{s=-\infty}^{n} |C_{ij}(n,s)| g_{ij}(\|x_j\|) \Big].$$

Thus
$$|f(n, x_n)| \le \|x\| [b^* \|x\| + C^* g^*(\|x\|)],$$

which implies
$$\frac{|f(n, x_s)|}{\|x\|} \le [b^* \|x\| + C^* g^*(\|x\|)] \to 0$$

as $\|x\| \to 0$. For $x \in \mathscr{P}3$, $x_i(n) \ge \sigma_3 \|x_i\|$ for all $n \in \mathbb{Z}$. Also, from (ii), $b_{ij}(n), C_{ij}(n,s)$ have the same sign. Thus, using condition (iii) we have

$$\begin{aligned}
&|f_i(n, x_n)| \\
&= \sum_{j=1}^{n} x_i(n) |b_{ij}(n)| x_j(n) + \sum_{j=1}^{k} \sum_{s=-\infty}^{n} |C_{ij}(n,s)| g_{ij}(x_j(s)) \\
&\ge |b_{ii}(n)| x_i(n)|^2 \ge {\sigma_3}^2 \|x_i\|^2 |b_{ii}(n)|
\end{aligned}$$

and
$$|f(n, x_s)| \ge {\sigma_3}^2 \sum_{i=1}^{k} \|x_i\|^2 \min_{1 \le i \le k} |b_{ii}(n)| \ge \frac{{\sigma_3}^2}{k} \|x\|^2 \min_{1 \le i \le k} |b_{ii}(n)|.$$

Here we have applied the inequality $\left(\sum_{i=1}^{k} \|x_i\| \right)^2 \le k \sum_{i=1}^{k} \|x_i\|^2$. Thus,

$$\frac{|f(n, x_s)|}{\|x\|} \to +\infty \ \text{ as } \ \|x\| \to +\infty.$$

By (b) of Corollary 5.1, equation (5.3.33) has a positive ω-periodic solution. This completes the proof.

Theorem 5.3.6 ([149]). *Suppose that the following conditions hold for* $i, j = 1, 2, \ldots, k$.

(i) $0 < a_i(n) < 1$, *for all* $n \in [0, \omega - 1]$, *and* $a_i(n)$ *is* ω*-periodic,*
(ii) $b_{ij}(n) \le 0$, $C_{ij}(n,s) \le 0$ *for all* $(n,s) \in \mathbb{Z}^2$,
(iii) $g_{ij} : \mathbb{R}^+ \to \mathbb{R}^+$ *is continuous in x and increasing with* $g_{ij}(0) = 0$,
(iv) $b_{ii}(s) \ne 0$, *for* $s \in [0, \omega - 1]$,
(v) $C_{ij}(n + \omega, s + \omega) = C_{ij}(n,s)$ *for all* $(n,s) \in \mathbb{Z}^2$ *with* $\max_{n \in \mathbb{Z}} \sum_{s=-\infty}^{n} |C_{ij}(n,s)| < +\infty$.

Then equation (5.3.33) *has a positive* ω*-periodic solution.*

Proof. The proof follows from part (a) of Corollary 5.1.

Remark 5.1. In the statements of Theorem 5.3.5 and Theorem 5.3.6 condition *(iv)* can be replaced by

(iv*) $\sum_{j=1}^{k}\sum_{s=-\infty}^{n}|C_{ij}(n,s)|\neq 0$ and $g_{ii}(x)\to+\infty$ as $x\to+\infty$.

5.4 Permanence of Multi-Species Competition Predation

The literature on nonautonomous continuous population models described by differential equations is vast, see [165, 166, 167, 168, 169] and the references cited therein. For example, Wen [169] considered the global attractivity of positive periodic solution of multi-species ecological competition-predation system. Yang and Xu [33] studied the global attractivity and existence of the periodic n-prey and m-predator Lotka-Volterra system of differential equations. It is biologically and mathematically crucial to study the existence and stability of periodic solution. However, a more basic and important biological question to ask is whether or not those involved populations will be alive and well in the long run. In [174], Chen discussed the permanence and global stability of nonautonomous Lotka-Volterra system with multi-species predator-prey and deviating arguments by using comparison theorem and constructing suitable Lyapunov functional. On the other hand, the problems of permanence of time-delay systems have received considerable attention in theoretical ecology due to the fact that more realistic models should include some of the past states of these systems. The dynamic behaviors of population models governed by difference equation have also been studied by many authors (see [31, 50, 95, 126, 162, 164, 165, 174, 175], and [178] and the references therein). This is a relatively new topic and the author believes this study should increase research activities on the subject. In [126], Muroya studied the persistence and global stability of delay discrete system for k-species,

$$x_i(n+1)=x_i(n)\exp\{c_i-a_ix_i(n)-\textstyle\sum_{k=1}^{l}a_{ik}(n)x_k(n-\tau_{ik})\}.$$

Results of this section can be partially found in [152] and [165]. The Jacobian method of Section 5.2 is not suitable for our study here. The aim of this study is to investigate the permanent behavior of the following discrete $(l+m)$-species Lotka-Volterra competition-predation system with several delays

$$\begin{aligned}
x_i(n+1)&=x_i(n)\exp\{r_i(n)-a_i(n)x_i(n)-\textstyle\sum_{k=1}^{l}a_{ik}(n)x_k(n-\tau_{ik})\\
&\qquad -\textstyle\sum_{k=1}^{m}e_{ik}(n)y_k(n-\eta_{ik})\},\\
y_j(n+1)&=y_j(n)\exp\{-b_j(n)-c_j(n)y_j(n)+\textstyle\sum_{k=1}^{l}d_{jk}(n)x_k(n-\delta_{jk})\\
&\qquad -\textstyle\sum_{k=1}^{m}c_{jk}(n)y_k(n-\xi_{jk})\},\\
x_i(\theta)&=\phi_i(\theta)\geq 0,\ y_j(\theta)=\psi_j(\theta)\geq 0,\ \ \theta\in\mathbb{N}[-\tau,0]:=\{-\tau,-\tau+1,...,-1,0\},
\end{aligned}\tag{5.4.1}$$

where $i=1,2,...,l;\ j=1,2,...,m;\ \tau_{ik},\ \eta_{ik},\ \delta_{jk}$ and ξ_{jk} are nonnegative integers; $\phi_i(0)>0,\ \psi_j(0)>0;$

$$\tau=\max\{\max_{1\le i,k\le l}\tau_{ik},\ \max_{1\le i\le l;\,1\le k\le m}\eta_{ik},\ \max_{1\le k\le l;\,1\le j\le m}\delta_{jk},\ \max_{1\le j,k\le m}\xi_{jk}\}>0;$$

$x_i(n)$ is the density of species X_i at nth generation; $y_j(n)$ is the density of species Y_j at nth generation; $r_i(n)$ represents the intrinsic growth rate of the prey species X_i at the nth generation; $b_j(n)$ represents the death rate of the predator species Y_j at the nth generation; $a_{ik}(n)$ and $c_{jk}(n)$ measure the intensity of intraspecific competition or interspecific action of prey species and predator species, respectively; $e_{ik}(n)$ and $d_{jk}(n)$ represent the influence of the $(n-\eta_{ik})$th and $(n-\delta_{jk})$th generation of the predator and prey on the prey and predator population, respectively. For more background of system (5.4.1), one could refer to [169] and [174]. It is clear that model (5.4.1) has positive solutions for positive initial data. We note that the model (5.4.1) generalizes the models in [31, 50], and [126].

Definition 5.4.1. System (5.4.1) is said to be permanent if there are positive constants M_k and L_k, $k=1,2$, such that for each positive solution

$$\{x_1(n),...,x_l(n),y_1(n),...,y_m(n)\}$$

of system (5.4.1) satisfies

$$L_1\le\lim_{n\to\infty}\inf x_i(n)\le\lim_{n\to\infty}\sup x_i(n)\le M_1,$$

$$L_2\le\lim_{n\to\infty}\inf y_j(n)\le\lim_{n\to\infty}\sup y_j(n)\le M_2,$$

for all $i=1,2,...,l;\ j=1,2,...,m.$

Throughout, we always assume $\{r_i(n)\}$, $\{b_j(n)\}$, $\{a_{ik}(n)\}$, $\{e_{ik}(n)\}$, $\{d_{jk}(n)\}$, $\{c_{jk}(n)\}$, $\{a_i(n)\}$ and $\{c_j(n)\}$ are bounded nonnegative sequences, and use the following notations for any bounded sequence $\{u(n)\}$

$$\overline{u}=\sup_{n\in\mathbb{N}}u(n),\ \ \underline{u}=\inf_{n\in\mathbb{N}}u(n).$$

In order to present our main result, we need some preliminaries. Let $\mathbb{R}_+^{l+m}=\{(x_1(n),...,x_l(n),y_1(n),...,y_m(n))|\ x_i(n)\ge 0, y_j(n)\ge 0,\ i=1,2,...,l; j=1,...,m\}$, and let $x(n)=(x_1(n),...,x_l(n),y_1(n),...,y_m(n))\in\mathbb{R}_+^{l+m}$, the notation $x(n)>0$ denotes $x(n)\in \mathrm{Int}\mathbb{R}_+^{l+m}$. For ecological reasons, we consider system (5.4.1) only in $\mathrm{Int}\mathbb{R}_+^{l+m}$. It is easy to obtain the following result.

Lemma 5.4 ([165]). *$\mathrm{Int}\mathbb{R}_+^{l+m}$ is positively invariant set of system* (5.4.1).

Lemma 5.5 ([165]). *Assume that $\{x(n)\}$ satisfies $x(n)>0$ and*

$$x(n+1)\le x(n)\exp\{r(n)(1-ax(n))\}$$

for $n \in [n_1, \infty)$*, where* a *is a positive constant. Then*

$$\limsup_{n\to\infty} x(n) \le \frac{1}{a\bar{r}} \exp(\bar{r} - 1).$$

Lemma 5.6 ([165]). *Assume that* $\{x(n)\}$ *satisfies*

$$x(n+1) \ge x(n)\exp\{r(n)(1 - ax(n))\}, n \ge N_0,$$

$\limsup_{n\to\infty} x(n) \le K$ *and* $x(N_0) > 0$*, where* a *is a constant such that* $aK > 1$ *and* $N_0 \in \mathbb{N}$*. Then*

$$\liminf_{n\to\infty} x(n) \ge \frac{1}{a} \exp\{\bar{r}(1 - aK)\}.$$

The main result will follow directly from the following two propositions.

Proposition 5.1 ([152]). *For every solution* $\{x_1(n), ..., x_l(n), y_1(n), ..., y_m(n)\}$ *of system* (5.4.1)*, we have*

$$\limsup_{n\to\infty} x_i(n) \le M_i \quad (i = 1, 2, ..., l), \qquad \limsup_{n\to\infty} y_j(n) \le W_j \quad (j = 1, 2, ..., m),$$

where

$$M_i = \frac{\exp(\bar{r}_i - 1)}{\underline{a}_i + \underline{a}_{ii}\exp(-\bar{r}_i\tau_{ii})}, \qquad W_j = \frac{\exp(\sum_{k=1}^{l} \bar{d}_{jk}M_k - \underline{b}_j - 1)}{\underline{c}_j + \underline{c}_{jj}\exp((\underline{b}_j - \sum_{k=1}^{l} \bar{d}_{kk}M_k)\xi_{jj})}.$$

Proof. First, we prove $\limsup_{n\to\infty} x_i(n) \le M_i$. From the first equation of (5.4.1), we have

$$x_i(n+1) \le x_i(n)\exp\{r_i(n)\}.$$

It follows that

$$\prod_{s=n-\tau_{ik}}^{n-1} x_i(s+1) \le \prod_{s=n-\tau_{ik}}^{n-1} x_i(s)\exp\{r_i(s)\},$$

that is

$$x_i(n) \le x_i(n - \tau_{ik})\exp\{\sum_{s=n-\tau_{ik}}^{n-1} r_i(s)\}.$$

In other words,

$$x_i(n - \tau_{ik}) \ge x_i(n)\exp\{-\sum_{s=n-\tau_{ik}}^{n-1} r_i(s)\},$$

and hence

$$\begin{aligned} x_i(n+1) &\le x_i(n)\exp\{r_i(n) - a_i(n)x_i(n) - \sum_{k=1}^{l} a_{ik}(n)x_k(n)\exp\{-\sum_{s=n-\tau_{ik}}^{n-1} r_i(s)\} \\ &\le x_i(n)\exp\{r_i(n) - (a_i(n) + a_{ii}(n))\exp\{-\sum_{s=n-\tau_{ii}}^{n-1} r_i(s)\}x_i(n)\} \\ &\le x_i(n)\exp\{\bar{r}_i - (\underline{a}_i + \underline{a}_{ii})\exp(-\bar{r}_i\tau_{ii})x_i(n)\}. \end{aligned}$$

It follows from Lemma 5.5 that

$$\limsup_{n\to\infty} x_i(n) \le M_i.$$

Next, we prove that $\limsup_{n\to\infty} y_j(n) \le W_j$. For sufficiently small $\varepsilon > 0$, there exists sufficiently large n_0 such that $x_i(n) \le M_i + \varepsilon$ for all $n > n_0$. From the second equation of (5.4.1), we have

$$\begin{aligned} y_j(n+1) &\le y_j(n)\exp\{-b_j(n) + \sum_{k=1}^{l} d_{jk}(n)x_k(n-\delta_{jk})\} \\ &\le y_j(n)\exp\{-b_j(n) + \sum_{k=1}^{l} d_{jk}(n)(M_k+\varepsilon)\}. \end{aligned}$$

By a similar argument, we can verify that

$$y_j(n-\xi_{jk}) \le y_j(n)\exp\{\sum_{s=n-\xi_{jk}}^{n-1}(b_j(s) - \sum_{k=1}^{l} d_{jk}(s)(M_k+\varepsilon))\},$$

and hence

$$\begin{aligned} y_j(n+1) \le{}& y_j(n)\exp\{-b_j(n) + \sum_{k=1}^{l} d_{jk}(n)(M_k+\varepsilon)) - c_j(n)y_j(n) \\ &- \sum_{k=1}^{m} c_{jk}(n)y_k(n)\exp\{\sum_{s=n-\xi_{jk}}^{n-1}(b_k(s) - \sum_{k=1}^{l} d_{kk}(s)(M_k+\varepsilon))\}\} \\ &\le y_j(n)\exp\{(-\underline{b}_j + \sum_{k=1}^{l} \overline{d}_{jk}(M_k+\varepsilon)) - (\underline{c}_j + \underline{c}_{jj})\exp\{(\underline{b}_j \\ &- \sum_{k=1}^{l} \overline{d}_{kk}(M_k+\varepsilon))\xi_{jj}\}y_j(n)\}. \end{aligned}$$

Therefore, by Lemma 5.5, we obtain

$$\limsup_{n\to\infty} y_j(n) \le W_j.$$

The proof is complete.

Proposition 5.2 ([152]). *Let* $\{x_1(n),\ldots,x_l(n),y_1(n),\ldots,y_m(n)\}$ *denote any positive solution of system* (5.4.1). *Assume*

$$(H)\ \min_{1\le i\le l;\,1\le j\le m}\left\{\frac{(\overline{a}_i+\overline{a}_{ii})M_i}{\underline{r}_i - \sum_{k=1,k\ne i}^{l}\overline{a}_{ik}M_k - \sum_{k=1}^{m}\overline{e}_{ik}W_k},\ \frac{(\overline{c}_j+\overline{c}_{jj})W_j}{\sum_{k=1}^{l}\underline{d}_{jk}m_k - \overline{b}_j - \sum_{k=1,k\ne j}^{m}\overline{c}_{jk}W_k}\right\} > 1.$$

Then there exist positive constants m_i and w_j such that

$$\liminf_{n\to\infty} x_i(n) \geq m_i, \quad \liminf_{n\to\infty} y_j(n) \geq w_j \quad (i = 1,2,...,l;\ j = 1,2,...,m),$$

where

$$\begin{aligned} m_i &= \frac{\underline{r}_1 - \sum_{k=1}^{l} \overline{a}_{ik} M_k - \sum_{k=1}^{m} \overline{e}_{ik} W_k}{\overline{a}_i + \overline{a}_{ii}} \exp\{(\overline{r}_i - \sum_{k=1}^{l} \underline{a}_{ik} M_k) \\ &\quad \times (1 - \frac{(\overline{a}_i + \overline{a}_{ii}) M_i}{\underline{r}_i - \sum_{k=1}^{l} \overline{a}_{ik} M_k - \sum_{k=1}^{m} \overline{e}_{ik} W_k})\}, \\ w_j &= \frac{\sum_{k=1}^{l} \underline{d}_{jk} m_k - \overline{b}_j - \sum_{k=1}^{m} \overline{c}_{jk} W_k}{\overline{c}_j + \overline{c}_{jj}} \exp\{(-\underline{b}_j + \sum_{k=1}^{l} \overline{d}_{jk} m_k - \sum_{k=1}^{m} \underline{c}_{jk} W_k) \\ &\quad \times (1 - \frac{(\overline{c}_j + \overline{c}_{jj}) W_j}{\sum_{k=1}^{l} \underline{d}_{jk} m_k - \overline{b}_j - \sum_{k=1}^{m} \overline{c}_{jk} W_k})\}. \end{aligned}$$

Proof. We first prove that $\liminf_{n\to\infty} x_i(n) \geq m_i$. For any $\varepsilon > 0$, according to Proposition 5.1, there exists a $n_1 \in \mathbb{N}$ such that $x_i(n-\tau) \leq M_i + \varepsilon$, $y_j(n-\tau) \leq W_j + \varepsilon$ for all $n \geq n_1$. Thus, it follows from the first equation of system (5.4.1) that

$$\begin{aligned} x_i(n+1) &\geq x_i(n) \exp\{(r_i(n) - \sum_{k=1,k\neq i}^{l} a_{ik}(n)(M_k + \varepsilon) \\ &\quad - \sum_{k=1}^{m} e_{ik}(n)(W_k + \varepsilon)) - (a_i(n) + a_{ii}(n)) x_i(n)\} \\ &= x_i(n) \exp\{(r_i(n) - \sum_{k=1,k\neq i}^{l} a_{ik}(n)(M_k + \varepsilon) - \sum_{k=1}^{m} e_{ik}(n)(W_k + \varepsilon)) \\ &\quad \times (1 - \frac{a_i(n) + a_{ii}(n)}{r_i(n) - \sum_{k=1,k\neq i}^{l} a_{ik}(n)(M_k + \varepsilon) - \sum_{k=1}^{m} e_{ik}(n)(W_k + \varepsilon)} x_i(n))\} \\ &\geq x_i(n) \exp\{(r_i(n) - \sum_{k=1,k\neq i}^{l} a_{ik}(n)(M_k + \varepsilon) - \sum_{k=1}^{m} e_{ik}(n)(W_k + \varepsilon)) \\ &\quad \times (1 - \frac{a_i(n) + a_{ii}(n)}{\underline{r}_i - \sum_{k=1,k\neq i}^{l} \overline{a}_{ik}(M_k + \varepsilon) - \sum_{k=1}^{m} \overline{e}_{ik}(W_k + \varepsilon)} x_i(n))\}. \end{aligned}$$

By lemma 5.6 and condition (H), we obtain

$$\liminf_{n\to\infty} x_i(n) \geq m_i.$$

From the second equation of (5.4.1), we have

$$
\begin{aligned}
y_j(n+1) \geq\ & y_j(n)\exp\{-b_j(n)+\sum_{k=1}^{l} d_{jk}(n)m_k \\
& -\sum_{k=1,k\neq j}^{m} c_{jk}(n)(W_k+\varepsilon)-(c_j(n)+c_{jj}(n))y_j(n)\} \\
&= y_j(n)\exp\{(-b_j(n)+\sum_{k=1}^{l} d_{jk}(n)m_k-\sum_{k=1,k\neq j}^{m} c_{jk}(n)(W_k+\varepsilon)) \\
&\quad \times(1-\frac{c_j(n)+c_{jj}(n)}{\sum_{k=1}^{l} d_{jk}(n)m_k-b_j(n)-\sum_{k=1,k\neq j}^{m} c_{jk}(n)(W_k+\varepsilon)}y_j(n)\} \\
&\geq y_j(n)\exp\{(-b_j(n)+\sum_{k=1}^{l} d_{jk}(n)m_k-\sum_{k=1,k\neq j}^{m} c_{jk}(n)(W_k+\varepsilon)) \\
&\quad \times(1-\frac{c_j(n)+c_{jj}(n)}{\sum_{k=1}^{l} \underline{d}_{jk}m_k-\overline{b}_j-\sum_{k=1,k\neq j}^{m} \overline{c}_{jk}(W_k+\varepsilon)}y_j(n)\}.
\end{aligned}
$$

By Lemma 5.6 and condition (H), we obtain $\liminf_{n\to\infty} y_j(n) \geq w_j$. This completes the proof.

Now, we state our main results of this section, which its proof is a direct consequence of Propositions 5.1 and 5.2.

Theorem 5.4.1 ([152]). *Assume* (H) *holds. Then system* (5.4.1) *is permanent.*

Now, let us consider the special case of system (5.4.1), i.e., $a_i(n) \equiv c_j(n) \equiv 0$, $\tau_{ik} \equiv 0$, $\eta_{is} \equiv 0$, $\delta_{jk} \equiv 0$ and $\xi_{js} \equiv 0$ $(i,k=1,2,\ldots,l;\ j,s=1,2,\ldots,m)$, in this case, system (5.4.1) can be written as

$$
\begin{aligned}
x_i(n+1) &= x_i(n)\exp\{r_i(n)-\textstyle\sum_{k=1}^{l} a_{ik}(n)x_k(n)-\sum_{k=1}^{m} e_{ik}(n)y_k(n)\}, \\
y_j(n+1) &= y_j(n)\exp\{-b_j(n)+\textstyle\sum_{k=1}^{l} d_{jk}(n)x_k(n)-\sum_{k=1}^{m} c_{jk}(n)y_k(n)\}.
\end{aligned}
\tag{5.4.2}
$$

As a corollary of Theorem 5.4.1, we have

Corollary 5.3 ([152]). *Let* $\{x_1(n),\ldots,x_l(n),y_1(n),\ldots,y_m(n)\}$ *denote any positive solution of system* (5.4.2). *Assume*

$$
\min_{1\leq i\leq l;\,1\leq j\leq m}\left\{\frac{\overline{a}_{ii}M_i'}{\underline{r}_i-\sum_{k=1,k\neq i}^{l}\overline{a}_{ik}M_k'-\sum_{k=1}^{m}\overline{e}_{ik}W_k'},\ \frac{\overline{c}_{jj}W_j'}{\sum_{k=1}^{l}\underline{d}_{jk}m_k'-\overline{b}_j-\sum_{k=1,k\neq j}^{m}\overline{c}_{jk}W_k'}\right\}>1
$$

holds. Then there exist positive constants M_i', W_j', m_i' and w_j' such that

$$m_i' \le \liminf_{n\to\infty} x_i(n) \le \limsup_{n\to\infty} x_i(n) \le M_i' \quad (i = 1,2,...,l),$$

$$w_j' \le \liminf_{n\to\infty} y_j(n) \le \limsup_{n\to\infty} y_j(n) \le W_j' \quad (j = 1,2,...,m),$$

where

$$M_i' = \frac{\exp(\overline{r}_i - 1)}{\underline{a}_{ii}}, \qquad W_j' = \frac{\exp(\sum_{k=1}^{l} \overline{d}_{jk} M_k' - \underline{b}_j - 1)}{\underline{c}_{jj}},$$

$$m_i' = \frac{\underline{r}_1 - \sum_{k=1}^{l} \overline{a}_{ik} M_k' - \sum_{k=1}^{m} \overline{e}_{ik} W_k'}{\overline{a}_{ii}} \exp\{(\overline{r}_i - \sum_{k=1}^{l} \underline{a}_{ik} M_k')$$
$$\times (1 - \frac{\overline{a}_{ii} M_i'}{\underline{r}_i - \sum_{k=1}^{l} \overline{a}_{ik} M_k' - \sum_{k=1}^{m} \overline{e}_{ik} W_k'})\},$$

$$w_j' = \frac{\sum_{k=1}^{l} \underline{d}_{jk} m_k' - \overline{b}_j - \sum_{k=1}^{m} \overline{c}_{jk} W_k'}{\overline{c}_{jj}} \exp\{(-\underline{b}_j + \sum_{k=1}^{l} \overline{d}_{jk} m_k' - \sum_{k=1}^{m} \underline{c}_{jk} W_k')$$
$$\times (1 - \frac{\overline{c}_{jj} W_j'}{\sum_{k=1}^{l} \underline{d}_{jk} m_k' - \overline{b}_j - \sum_{k=1}^{m} \overline{c}_{jk} W_k'})\}.$$

Finally, we give a suitable example to illustrate the feasibility of Theorem 5.4.1.

Example 5.1. We consider the following system:

$$x(n+1) = x(n)\exp\{1 - x(n-1) - \tfrac{1}{60}(3+\sin n)y_1(n)\},$$
$$y_1(n+1) = y_1(n)\exp\{-1 + \tfrac{2}{9}(8+\cos n)x(n) - y_1(n-1) - \tfrac{1}{80}(3+\sin n)y_2(n)\},$$
$$y_2(n+1) = y_2(n)\exp\{-1 - \cos n + \tfrac{2}{9}(8+\cos n)x(n) - y_2(n-1)\}.$$

It is easy to verify that the system satisfies the condition (H). Therefore, by Theorem 5.4.1 the system is permanent.

5.5 Open Problems

Open Problem 1.
In this section we consider the Lotka-Volterra predator-prey model given by Elaydi et al. [90] and extend it to predator-prey model with ratio dependence. Let x and y represent population densities of a prey and a predator, respectively. In [70], Fan and Wang discretized a continuous model with ratio dependence and obtained the Lotka-Volterra discrete predator-prey model with ratio dependence

$$\begin{cases} x(n+1) = x(n)exp\{a(n) - b(n)x(n) - \dfrac{c(n)y(n)}{m(n)y(n)+x(n)}\} \\ \\ y(n+1) = y(n)exp\{-d(n) + \dfrac{f(n)x(n)}{m(n)y(n)+x(n)}\} \end{cases} \tag{5.5.1}$$

We refer to [70] for the specific interpretation of the coefficients. In [70] the authors proved the existence of positive periodic solution of (5.5.1) by using Coincidence Theory or Degree Theory.

We propose the reader uses the idea of [176] and show the positive periodic solution is actually globally asymptotically stable by constructing a suitable Lyapunov function.