Chapter 4
Periodic Solutions

This chapter is devoted to the study of periodic solutions of functional difference systems with finite and infinite delay. We will obtain different results concerning Volterra difference equations with finite and infinite delays, using fixed point theory. Fixed point theory will enable us to obtain results concerning stability, classification of solutions, existence of positive solutions, and the existence of periodic solutions and positive periodic solutions. In the analysis, we make use of Schaefer fixed point theorem, [159], Krasnoselskii's fixed point theorem, [97], and Schauder fixed point theorem. We apply our results to infinite delay Volterra difference equations, by constructing suitable Lyapunov functionals to obtain the a priori bound on all possible solutions. We transition to systems or coupled Volterra infinite delay difference equations and show the existence of a periodic solution and asymptotically periodic solution. For some classes of nonlinear systems with delay, it is shown that the presence of the time delay results in the existence of periodic solutions. We end the chapter by considering functional difference equation that has the characteristic that every constant is a solution. Then by means of fixed point theory we show that the unique solution converges to a pre-determined constant or a periodic solution. In addition we show the solution is stable and that its limit function serves as a global attractor. Most of the results of this chapter can be found in [1, 52, 127, 131, 135], and [137].

4.1 Periodic Solutions in Finite and Infinite Delays Equations

This chapter is entirely devoted to the study of existence of periodic solutions of functional difference equations and in particular Volterra infinite delay difference equations. We begin by discussing some results from the celebrated paper of Elaydi [52], in which the existence of a periodic solution is directly tied up to (UAS). We consider the following systems of difference equations of non-convolution type

Y. N. Raffoul, *Qualitative Theory of Volterra Difference Equations*,
https://doi.org/10.1007/978-3-319-97190-2_4

$$x(n+1) = A(n)x(n) + \sum_{r=0}^{n} B(n,r)x(r) \tag{4.1.1}$$

and its corresponding perturbed system

$$x(n+1) = A(n)x(n) + \sum_{r=0}^{n} B(n,r)x(r) + g(n) \tag{4.1.2}$$

where a, B are $k \times k$ matrix functions on $\mathbb{Z}^+$ and $\mathbb{Z}^+ \times \mathbb{Z}^+$, respectively, and g is a vector function on $\mathbb{Z}^+$. As before, we let $R(n,m)$ be the resolvent matrix of (4.1.1). Our objective is to find a periodic solution for the difference system with infinite delay

$$z(n+1) = A(n)x(n) + \sum_{r=-\infty}^{n} B(n,r)z(r) + g(n), \tag{4.1.3}$$

where

$$A(n+N) = A(n),\ B(n+N, m+N) = B(n,m),\ g(n+N) = g(n). \tag{4.1.4}$$

It can be easily shown, see [52], that

$$R(n+N, m+N) = R(n,m).$$

Hence we have the following theorem.

Theorem 4.1.1 ([52]). *Suppose that the zero solution of Equation* (4.1.1) *is (UAS). Then Equation* (4.1.3) *has the unique N-periodic solution*

$$z(n) = \sum_{m=-\infty}^{n-1} R(n, m+1)g(m).$$

The next theorem provides criteria for the (UAS) of Equation (4.1.1).

Theorem 4.1.2 ([52]). *Let*

$$|x| = \sum_{i=1}^{k} |x_i|,\ \beta_{jn}(n) = \sum_{s=n}^{\infty} b_{ji}(s,n)| < \infty.$$

Assume that

$$\sum_{j=1}^{k} [|a_{ji}(n)| + |\beta_{ji}(n)| \le 1 - c, 1 \le i \le k,\ n \ge n_0, \textit{for some } c \in (0,1).$$

Then the zero solution of Equation (4.1.1) *is (UAS).*

As in the case of Chapter 2, we feel the need for the development of a more general theory for the existence of periodic solutions that will accommodate a wider range of equations. Thus, in this section we consider the functional nonlinear system of difference equations with either finite or infinite delay,

$$\triangle x(n) = F(n, x_n), n \in \mathbb{Z} \tag{4.1.5}$$

where $F : \mathbb{Z} \times BC \to \mathbb{R}^k$ is continuous in x and T-periodic in n. Here BC is the space of bounded sequences $\phi : (-\infty, 0] \to \mathbb{R}^k$ with the maximum norm $||\cdot||$. By x_n we mean that $x_n(s) = x(n+s)$ for $s \le 0$. Let $(P_T, ||\cdot||)$ be the Banach space of T-periodic sequences $\phi : \mathbb{Z} \to \mathbb{R}^k$ with the maximum norm

$$||x|| = \max_{n \in [0, T-1]} |x(n)|.$$

Also, we let

$$P_T^0 = \Big\{\phi \in P_T : \sum_{s=0}^{T-1} \phi(s) = 0\Big\}.$$

Proving the existence of a periodic solution of (4.1.5) rest on the following Schaefer fixed point theorem.

Theorem 4.1.3 ([159]). *Let $(\mathbb{B}, |\cdot|)$ be a normed linear space, H a continuous mapping of $\mathbb{B}$ into $\mathbb{B}$ which is compact on each bounded subset of $\mathbb{B}$. Then either (i) the equation $x = \lambda Hx$ has a solution for $\lambda = 1$, or (ii) the set of all solutions x, for $0 < \lambda < 1$, is unbounded.*

We make the following assumptions.
(a) For every $\phi \in P_T^0$, there exists a constant $d_\phi \in \mathbb{R}$ such that $\sum_{s=0}^{T-1} F(s, \psi_s) = 0$ where

$$\begin{cases} \psi(n) = d_\phi + \sum_{s=0}^{n-1} \phi(s), \text{ for } n \ge 1, \\ \psi(n) = d_\phi + \sum_{s=0}^{j-1} \phi(s), \text{ for } n \le 0,\ n = j \mod T,\ 1 \le j \le T. \end{cases} \tag{4.1.6}$$

(b) Let $E(\phi)(n) = \psi(n)$ be continuous in ϕ with $E : P_T^0 \to P_T$ such that for each $\alpha > 0$, there exists a constant $L_\alpha > 0$ such that $|d_\phi| \le L_\alpha$ whenever $||\phi|| \le \alpha$.
The following proposition assures that ψ is well defined. Its proof is straightforward and therefore omitted.

Proposition 4.1. *Let n and $T \ge 1$ be any given integers. Then there exist unique integers K and j, $1 \le j \le T$, such that $n = KT + j$.*

Theorem 4.1.4 ([135]). *Suppose conditions (i) and (ii) hold. For $0 < \lambda < 1$, define $G_\lambda : P_T^0 \to P_T^0$ by*

$$G_\lambda(\phi)(n) = \lambda F(n, \psi_n).$$

If there is a constant $D > 0$ such that $||\phi|| < D$ whenever ϕ is a fixed point of G_λ, then Equation (4.1.5) has a T-periodic solution .

Proof. First we note that P_T is equivalent to $\mathbb{R}^{Tk}$. Let $n \in \mathbb{Z}$. By the continuity of F and condition (i), we can easily see that

$$\sum_{s=0}^{T-1} G_\lambda(\phi)(s) = \lambda \sum_{s=0}^{T-1} F(s, \psi_s) = 0.$$

Hence, we have that $G_\lambda(\phi) \in P_T^0$. For each $\alpha > 0$, the set $S = \{E(\phi) : \phi \in P_T^0, ||\phi|| \leq \alpha\}$ is closed and bounded by (ii). Let $Q = \{G_\lambda(\phi)(n) : \phi \in S\}$. Then S is a subset of $\mathbb{R}^{Tk}$ which is closed and bounded and thus compact. As G_λ is continuous in ϕ, it maps compact sets into compact sets. Therefore, $Q = G_\lambda(S)$ is compact. The hypothesis $||\phi|| < D$ rules out Condition *(i)* of Theorem 4.1.3 and thus applying Schaefer's theorem to $\phi = G_\lambda(\phi)$ we conclude that G_λ has a fixed point for $\lambda = 1$. That is $\phi = G_1\phi = F(n, \psi_n)$. On the other hand, $\phi(n) = \triangle\psi(n) = F(n, \psi_n)$. Thus, ψ is a T-periodic solution of (4.1.5). This completes the proof.

Corollary 4.1. *Suppose conditions (a) and (b) hold. Assume the functional F maps bounded sets into compact sets. If there exists a positive constant J such that any T-periodic solution $x(n)$ of*

$$\triangle x(n) = \lambda F(n,\, x_n),\ \lambda \in (0,1) \tag{4.1.7}$$

satisfies $||x|| < J$, then (4.1.5) *has a T-periodic solution .*

Proof. Since $G_\lambda(\phi)(n) = \lambda F(n, \psi_n)$, then any fixed solution ϕ of G_λ implies the existence of a T-periodic solution of (4.1.5). As the functional F maps bounded sets into compact subsets, we have, whenever $||\psi|| \leq J$, that $|F(n, \psi_n)| \leq R$, where R depends on the a priori bound J. Let ϕ be a fixed solution of G_λ. Then $\phi(n) = \triangle\psi(n) = \lambda F(n, \psi_n)$. Since all T-periodic solutions of (4.1.7) have a priori bound J, by Theorem 4.1.4, Equation (4.1.5) has a T-periodic solution. This completes the proof.

Corollary 4.2. *Suppose conditions (i) and (ii) hold. If there exist constants $M, r,\ 0 < r < 1$ such that*

$$|F(n, \psi_n)| \leq r||\phi|| + M, \textit{ for all } \phi \in P_T^0$$

where ψ is given by (4.1.6)*, then Equation* (4.1.5) *has a T-periodic solution.*

Proof. The proof is straightforward. To see this, let ϕ be a fixed solution of G_λ. Then for $\phi \in P_T^0$

$$|\phi(n)| = \lambda|F(n, \psi_n)| \leq r||\phi|| + M,$$

from which we arrive at

$$||\phi|| \leq \frac{M}{1-r}.$$

Hence, Equation (4.1.5) has a T-periodic solution by Theorem 4.1.4.

For the next theorem we consider the functional delay equation

$$\triangle x(n) = L(n,\, x_n) + p(n), n \in \mathbb{Z} \tag{4.1.8}$$

where $L : \mathbb{Z} \times BC \to \mathbb{R}^k$ is continuous and linear in x, T-periodic in n and $p \in P_T$.

Theorem 4.1.5 ([135]). *Suppose that for every d in $\mathbb{R}^k$, the $k \times k$ matrix $L(n, \cdot)$ satisfies the relation*

$$L(n,\cdot)d = L(n,d) \textit{ and } \sum_{n=0}^{T-1} L(n,\cdot) \textit{ is invertible.}$$

If there is a priori bound on all possible T-periodic solutions of

$$\triangle x(n) = \lambda \Big[L(n,\ x_n) + p(n)\Big],\ \lambda \in (0,1) \tag{4.1.9}$$

then Equation (4.1.8) *has a T-periodic solution .*

Proof. First we note that

$$\sum_{n=0}^{T-1} L(n,\cdot) \text{ is invertible if and only if the matrix } \Big(\sum_{n=0}^{T-1} L(n,\cdot)\Big)^{-1}$$

exists. In view of Corollary 4.1, we only need to verify *(i)* and *(ii)*. Set $F(n,\psi_n) = L(n,\ \psi_n) + p(n)$ and

$$d_\phi = -\Big(\sum_{n=0}^{T-1} L(n,\cdot)\Big)^{-1}\Big[\sum_{n=0}^{T-1} L\Big(n,(\sum_{s=0}^{n-1}\phi(s))_n\Big) + \sum_{n=0}^{T-1} p(n)\Big]. \tag{4.1.10}$$

For $\phi \in P_T^0$, $d_\phi \in \mathbb{R}^k$ is uniquely determined by (4.1.10). Since $L(n,\cdot)d = L(n,d)$ we have

$$\sum_{n=0}^{T-1} L\Big(n,(d_\phi + (\sum_{s=0}^{n-1}\phi(s))_n\Big) + \sum_{n=0}^{T-1} p(n) = 0.$$

Thus, $\sum_{s=0}^{T-1} F(s,\psi_s) = 0$. Let E be defined as in *(ii)*, then it is readily verified that $E : P_T^0 \to P_T$ and continuous in ϕ. Now, since L is linear and continuous in the second argument, there exists a $\beta > 0$ such that for any $\psi \in BC$, $|L(n,\psi_n)| \le \beta||\psi||$. This yields

$$\Big|L\Big(n,(\sum_{s=0}^{n-1}\phi(s))_n\Big)\Big| \le \beta T||\phi||.$$

Thus, from (4.1.10) one obtains for $||\phi|| \le \alpha$ that

$$d_\phi \le \Big|\Big(\sum_{n=0}^{T-1} L(n,\cdot)\Big)^{-1}\Big|\Big(\beta T\alpha + ||p||\Big)T =: L_\alpha.$$

Thus, by Corollary 4.1 Equation (4.1.8) has a T-periodic solution and the proof is complete.

4.2 Application to Functional Difference Equations

It is obvious that Theorem 4.1.4 is of general nature and hence we will apply it to different types of functional difference equations. Namely, we will obtain existence of periodic solutions of scalar Volterra difference equations with finite or infinite delay.

4.2.1 Finite Delay Difference Equations

We will use Theorem 4.1.4 to prove the existence of a periodic solution for a scalar difference equation with finite delay.

Theorem 4.2.1 ([135]). *Consider the scalar delay difference equation*

$$\triangle x(n) = a(n)x(n) + b(n)x(n-h) + p(n),\ n \in \mathbb{Z}, \tag{4.2.1}$$

where the sequences $a(n), b(n)$, and $p(n)$ are T-periodic sequences, and $h \in \mathbb{Z}$ with $h \geq 0$.
Suppose that either $a(n) > 0$ or $a(n) < 0$ for all $n \in \mathbb{Z}$. Suppose there exists a constant $N > 1$ such that

$$|a(n)| - N|b(n+h)| \geq 0.$$

If
(i) $\rho - ||b|| - \rho T(||a|| + ||b|| > 0$
where $\rho = \min_{n\in[0,T-1]} |a(n)|$, then Equation (4.2.1) has a T-periodic solution.

Proof. First we note that since either $a(n) > 0$ or $a(n) < 0$ for all $n \in \mathbb{Z}$, we have $\sum_{n=0}^{T-1} a(n) \neq 0$. Define L by $L(n, x_n) = a(n)x(n) + b(n)x(n-h)$. Then L is linear and $L(n,\cdot) = a(n) + b(n)$. In view of Theorem 4.1.5, we need to show that $\sum_{n=0}^{T-1} L(n,\cdot) \neq 0$ and all T-periodic solutions of

$$\triangle x(n) = \lambda \Big[a(n)x(n) + b(n)x(n-h) + p(n)\Big],\ \lambda \in (0,1) \tag{4.2.2}$$

have a priori bound. By noting that $b(n+h)$ is also T-periodic, we have

$$\sum_{n=0}^{T-1} |b(n)| = \sum_{s=-h}^{T-h-1} |b(s+h)| = \sum_{s=0}^{T-1} |b(s+h)|.$$

Thus for $a(n) > 0$,

$$\sum_{n=0}^{T-1} (a(n) + b(n)) \geq \sum_{n=0}^{T-1} (|a(n)| - |b(n)|) = \sum_{n=0}^{T-1} (|a(n)| - |b(n+h)|).$$

By making use of $|a(n)| - N|b(n+h)| \geq 0$ in the above inequality, we get

$$\begin{aligned}\sum_{n=0}^{T-1} (a(n) + b(n)) &\geq \sum_{n=0}^{T-1} (|a(n)| - |b(n+h)|)\\ &= \frac{N-1}{N} \sum_{n=0}^{T-1} |a(n)| + \frac{1}{N} \sum_{n=0}^{T-1} (|a(n)| - N|b(n+h)|)\\ &\geq \frac{N-1}{N} \sum_{n=0}^{T-1} a(n) > 0.\end{aligned}$$

Next, suppose $a(n) < 0$ for all $n \in \mathbb{Z}$. Then

$$\begin{aligned}\sum_{n=0}^{T-1}(a(n)+b(n)) &\leq \sum_{n=0}^{T-1}(-|a(n)|+|b(n+h)|)\\ &= \frac{1-N}{N}\sum_{n=0}^{T-1}|a(n)| - \frac{1}{N}\sum_{n=0}^{T-1}(|a(n)|-N|b(n+h)|)\\ &\leq \frac{1-N}{N}\sum_{n=0}^{T-1}|a(n)| < 0.\end{aligned}$$

Hence, we have shown that $\sum_{n=0}^{T-1} L(n,\cdot) \neq 0$ for all $n \in \mathbb{Z}$. Now we turn our attention to finding the a priori bound. Let $x(n)$ be a T-periodic solution of (4.2.2). By summing equation (4.2.2) from n to $n+T-1$ we get

$$0 = x(n+T) - x(n) = \lambda \sum_{s=n}^{n+T-1}\Big[a(s)x(s) + b(s)x(s-h) + p(s)\Big].$$

Thus,

$$\sum_{s=n}^{n+T-1} a(s)x(s) = -\sum_{s=n}^{n+T-1}(b(s)x(s-h)+p(s)).$$

Since there exists an $n^* \in [n, n+T-1]$ such that

$$T|a(n^*)|x(n^*)| \leq \sum_{s=n}^{n+T-1}|a(s)||x(s)|,$$

we arrive from the above relation that

$$\begin{aligned}T|a(n^*)|x(n^*)| &\leq \sum_{s=n}^{n+T-1}(|b(s)||x(s-h)|+|p(s)|)\\ &\leq T||b||\,||x|| + T||p||.\end{aligned}$$

As a consequence, we get

$$|x(n^*)| \leq \frac{||b||}{\rho}||x|| + \frac{||p||}{\rho}.$$

Using Equation (4.2.2) we have

$$\begin{aligned}|\triangle x| &\leq |a(n)||x(n)| + |b(n)||x(n-h)| + |p(n)|\\ &\leq ||a||\,||x|| + ||b||\,||x|| + ||p||\\ &= (||a|| + ||b||)||x| + ||p||.\end{aligned}$$

For all $n \in \mathbb{Z}$, we write $x(n) \in P_T$ as

$$x(n) = x(n^*) + \sum_{s=n^*}^{n+T-1} \triangle x(s). \tag{4.2.3}$$

Using (4.2.3) and then the norms of $x(n^*), \triangle x$ and x we get

$$\begin{aligned}
|x(n)| &\leq |x(n^*)| + \sum_{s=n}^{n+T-1} |\triangle x(s)| \\
&\leq ||x(n^*)|| + T||\triangle x|| \\
&\leq \frac{||b||}{\rho}||x|| + \frac{||p||}{\rho} + T\Big((||a|| + ||b||)||x|| + ||p||\Big).
\end{aligned}$$

The above inequality yields

$$||x|| \leq \frac{T\rho||p||}{\rho - ||b|| - \rho T(||a|| + ||b||)}.$$

This defines a priori bound on all possible T-periodic solutions of Equation (4.2.2). Hence, Equation (4.2.1) has a T-periodic solution by Theorem 4.1.4.

In the next corollary, we relax condition *(i)* of Theorem 4.2.1.

Corollary 4.3 ([135]). *Suppose the hypothesis of Theorem 4.2.1 holds with (i) being replaced by*

$$|a(n) + b(n)|\Big(|d^{-1}|T^2(||a|| + ||b||) + T\Big) = \varsigma < 1,$$

where

$$d^{-1} = \Big[\sum_{n=0}^{T-1}(a(n) + b(n))\Big]^{-1}.$$

Then Equation (4.2.1) *has a T-periodic solution.*

Proof. Take ϕ, ψ, and $L(n,\cdot)$ to be as in Theorem 4.2.1. In view of Corollary 4.2 we only need to show that $|F(n,\psi_n)| \leq r||\phi|| + M$, $M > 0$ is a constant and $0 < r < 1$. By a similar argument as in Theorem 4.1.5, one may easily find that

$$d_\phi = -\Big[\sum_{n=0}^{T-1}(a(n) + b(n))\Big]^{-1}\Big\{\sum_{n=0}^{T-1}(a(n) + b(n))\sum_{n=0}^{n-1}\phi(s) + \sum_{n=0}^{T-1}p(n)\Big\}.$$

Now,

$$\begin{aligned}
|d_\phi| &\leq |d^{-1}|T(|a| + |b|)T|\phi| + T|P| \\
&\leq |d^{-1}|T^2(|a| + |b|)T||\phi|| + T||P||.
\end{aligned}$$

This yields to

$$|F(n,\psi_n)| \le |a(n)+b(n)||d_\phi| + |a(n)+b(n)|T||\phi|| + ||p||$$
$$\le |a(n)+b(n)|\Big(|d^{-1}|T^2(||a||+||b||)+T\Big) = \varsigma < 1.$$

Thus, by Corollary 4.2, Equation (4.2.1) has a T-periodic solution. This completes the proof.

4.2.2 Infinite Delay Volterra Difference Systems

In this section, we apply Corollary 4.2 and Theorem 4.1.5 to show that the Volterra difference system with infinite delay given by

$$\triangle x(n) = A(n)x(n) + \sum_{s=-\infty}^{n} B(n,s)x(s) + g(n), \; -\infty < s \le t < \infty \tag{4.2.4}$$

where A, B are $k \times k$ T-periodic matrices and g is a given $k \times 1$ T-periodic sequence, has a T-periodic solution. We begin with the following theorem.

Theorem 4.2.2 ([135]). *Suppose that*

$$D = \sum_{s=0}^{T-1}\Big(A(n) + \sum_{s=-\infty}^{n} B(n,s)\Big) \textit{ is invertible}, \tag{4.2.5}$$

$$\max_{n\in[0,T-1]}\Big[\Big|A(n)+\sum_{s=-\infty}^{n}B(n,s)\Big|MT + (|A(n)| + \sum_{s=-\infty}^{n}|B(n,s)|)T\Big] =: \varsigma < 1, \tag{4.2.6}$$

where

$$M = |D^{-1}|\sum_{u=0}^{T-1}\Big(|A(u)| + \sum_{s=-\infty}^{u}|B(u,s)|\Big).$$

Then Equation (4.2.4) *has a T-periodic solution.*

Proof. Set $F(n,x_n) = A(n)x(n) + \sum_{s=-\infty}^{n} B(n,s)x(s) + p(n)$.
Then (4.2.5) and Theorem 4.1.4 imply that for each $\phi \in P_T^0$, there exists a unique $d_\phi \in \mathbb{R}$ such that $\sum_{n=0}^{T-1} F(n,\psi_n) = 0$, where $\psi(n)$ is defined by (4.1.6). In fact for $\sum_{n=0}^{T-1} F(n,\psi_n) = 0$ gives

$$\sum_{n=0}^{T-1}(A(n)(d+\sum_{s=0}^{n-1}\phi(s)) + \sum_{n=0}^{T-1}\Big[\sum_{s=-\infty}^{n}B(n,s)(d+\sum_{u=0}^{s-1}\phi(u))\Big] + \sum_{n=0}^{T-1}p(n) = 0.$$

This yields

$$d_\phi = -D^{-1}\Big[\sum_{n=0}^{T-1}\Big(A(n)\sum_{s=0}^{n-1}\phi(s) + \sum_{s=-\infty}^{n}B(n,s)\sum_{u=0}^{s-1}\phi(u) + p(n)\Big)\Big].$$

Thus,

$$|d_\phi| \le MT||\phi|| + |D^{-1}|||p||T.$$

On the other hand,

$$\begin{aligned}
|F(n,\psi_n)| &\le \left|A(n)\psi_n + \sum_{s=-\infty}^{n} B(n,s)\psi_s + p(n)\right| \\
&\le \left|A(n)(d_\phi + \sum_{s=0}^{n-1}\phi(s))\right. \\
&\quad \left. + \sum_{s=-\infty}^{n} B(n,s)(d_\phi + \sum_{u=0}^{s-1}\phi(u)) + \sum_{n=0}^{T-1} p(n)\right| \\
&\le \left|A(n) + \sum_{s=-\infty}^{n} B(n,s)\right| |d_\phi| \\
&\quad + \Big(|A(n)| + \sum_{s=-\infty}^{n} |B(n,s)|\Big) T||\phi|| + ||p||.
\end{aligned}$$

Replacing $|d_\phi|$ by its value, we get

$$|F(n,\psi_n)| \le \varsigma||\phi|| + K$$

where $K = max_{n\in[0.T-1]}\left|A(n) + \sum_{s=-\infty}^{n} B(n,s)\right||D^{-1}|\ ||p||T + ||p||$. Thus, Equation (4.2.4) has a T-periodic solution by Corollary 4.2. This completes the proof.

Remark 4.1. Condition (4.2.6) is severe and therefore in the next theorem we avoid it by appealing to Lyapunov functional.

But first, if $A = (a_{ij})$ is a $k \times k$ real matrix, then we define the norm of A by

$$|A| = \max_{1\le i\le k} \sum_{j=1}^{k} |a_{ij}|.$$

Theorem 4.2.3 ([135]). *Consider the 2-dimensional system*

$$\triangle x(n) = \lambda \left[Ax(n) + \sum_{j=-\infty}^{n} C(n-j)x(j) + g(n)\right],\ \lambda \in (0,1) \tag{4.2.7}$$

$A = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$, $\sum_{j=-\infty}^{n-1}\sum_{s=n}^{\infty}|C(s-j)| < \infty$, $g(n) \in P_T$, $\sum_{u=0}^{\infty}|C(u)| = \alpha \le \frac{2}{25}$ *and* $C^T(u) = C(u)$*(transpose). Assume that* $Q = \sum_{n=0}^{T-1}\Big(A(n) + \sum_{s=-\infty}^{n} B(n,s)\Big)$ *is invertible. Then* (4.2.7) *has a solution in* P_T *for* $\lambda = 1$.

Proof. Set $F(n,x_n) = A(n)x(n) + \sum_{s=-\infty}^{n} C(n-s)x(s) + p(n)$. If we let

$$d_\phi = -Q^{-1}\Big[\sum_{n=0}^{T-1}\Big(A(n)\sum_{s=0}^{n-1}\phi(s) + \sum_{s=-\infty}^{n} C(n-s)\sum_{u=0}^{s-1}\phi(u) + p(n)\Big)\Big],$$

then by a similar argument as in Theorem 4.2.2, it is readily verified that

$$\sum_{n=0}^{T-1} F(n,\psi_n) = 0,$$

where $\psi(n)$ is defined by (4.1.6). Also, by a similar argument as in the Theorem 4.2.2, it can be easily shown that there exists a constant $L_\alpha > 0$ such that $|d_\phi| \le L_\alpha$. Next we show that F maps bounded sets into bounded sets. Let J be a given positive constant. Then if ψ is given by (4.1.6), we set $S = \{\psi : \phi \in P_T^0, ||\psi|| \le J\}$ which is closed and bounded. Now

$$\begin{aligned}
|F(n,\psi_n)| &\le \Big|A(n)\psi_n + \sum_{s=-\infty}^{n} C(n-s)\psi_s + p(n)\Big| \\
&\le |A|J + \sum_{u=0}^{\infty} |C(u)|JT + |p| \\
&\le |A|J + JT\frac{2}{25} + ||p|| \le M, M > 0.
\end{aligned}$$

This shows that F maps bounded sets into bounded sets. According to Corollary 4.1, it is left to show that all T-periodic solutions of (4.2.7) have a priori bound. Note that (4.2.7) has an a priori bound on all its T-periodic solutions if and only if

$$x(n+1) = Dx(n) + \lambda \sum_{j=-\infty}^{n} C(n-j)x(j) + \lambda g(n) \tag{4.2.8}$$

does, where $D = \begin{pmatrix} \lambda+1 & 0 \\ 0 & 1-\lambda \end{pmatrix}$. Find $E = E^T$ such that

$$D^T ED - E = -2\lambda I, \text{ as follows.}$$

Let $E = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$. Then $D^T ED - E = -2\lambda I$ implies that

$$\begin{pmatrix} a(\lambda+1)^2 - a & b(1-\lambda^2) - b \\ c(1-\lambda)^2 - c & d(1-\lambda^2) - d \end{pmatrix} = -2\lambda \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

from which it follows

$$E = \begin{pmatrix} -\frac{2}{\lambda+2} & 0 \\ 0 & \frac{2}{2-\lambda} \end{pmatrix}.$$

Thus

$$|E| \leq 2 \text{ for } \lambda \in (0,1).$$

Also,

$$D^T E = \begin{pmatrix} \lambda+1 & 0 \\ 0 & 1-\lambda \end{pmatrix} \times \begin{pmatrix} \frac{-2}{\lambda+2} & 0 \\ 0 & \frac{2}{2-\lambda} \end{pmatrix} = \begin{pmatrix} -2\frac{\lambda+1}{\lambda+2} & 0 \\ 0 & 2\frac{1-\lambda}{2-\lambda} \end{pmatrix}.$$

Thus $|D^T E| \leq 2$ for $\lambda \in (0,1)$. Find $\gamma > 2+2\alpha$ such that $(2+\gamma)\alpha < 2$. This is possible because for $\alpha \in (0, \frac{2}{25}]$, it is elementary to verify that $2+2\alpha < \frac{2}{\alpha} - 2$. Hence we may choose γ such that $2+2\alpha < \gamma < \frac{2}{\alpha} - 2$.
Define a Lyapunov type functional

$$V(n, x(\cdot)) = x^T(n)Ex(n) + \lambda\gamma \sum_{j=-\infty}^{n-1} \sum_{s=n}^{\infty} |C(s-j)| x^2(j).$$

It is of interest to note that V is not positive definite. Then along solutions of (4.2.8) we have

$$\begin{aligned}
\triangle V &= x^T(n+1)Ex(n+1) + \lambda\gamma \sum_{s=n+1}^{\infty} |C(s-n)| x^2(n) \\
&\quad -\lambda\gamma \sum_{j=-\infty}^{n-1} |C(n-j)| x^2(j) - x^T(n)Ex(n) \\
&= \left[x^T(n)D^T + \lambda \sum_{j=-\infty}^{n} x^T(j)C^T(n-j) + \lambda g^T(n) \right] \\
&\quad E \left[Dx(n) + \lambda \sum_{j=-\infty}^{n} C(n-j)x(j) + \lambda g(n) \right] \\
&\quad -x^T(n)Ex(n) + \lambda\gamma \sum_{s=n+1}^{\infty} |C(s-n)| x^2(n) \\
&\quad -\lambda\gamma \sum_{j=-\infty}^{n-1} |C(n-j)| x^2(j) \\
&= x^T(n)D^T EDx(n) + \lambda x^T(n)D^T E \sum_{j=-\infty}^{n} C(n-j)x(j) \\
&\quad +\lambda x^T(n)D^T Eg(n) + \lambda \sum_{j=-\infty}^{n} x^T(j)C^T(n-j)EDx(n) \\
&\quad +\lambda^2 \sum_{j=-\infty}^{n} x^T(j)C^T(n-j)E \sum_{j=-\infty}^{n} C(n-j)x(j) \\
&\quad +\lambda^2 \sum_{j=-\infty}^{n} x^T(j)C^T(n-j)Eg(n) + \lambda g^T(n)EDx(n)
\end{aligned}$$

$$+\lambda^2 g^T(n)E\sum_{j=-\infty}^{n} C(n-j)x(j)+\lambda^2 g^T(n)Eg(n)-x^T(n)Ex(n)$$

$$+\lambda\gamma\sum_{s=n+1}^{\infty}|C(s-n)|x^2(n)-\lambda\gamma\sum_{j=-\infty}^{n-1}|C(n-j)|x^2(j).$$

Hence

$$\begin{aligned}\triangle V &= x^T(n)(D^TED-E)x(n)+2\lambda\sum_{j=-\infty}^{n} x^T(n)D^TEC(n-j)x(j)\\ &\quad +2\lambda x^T(n)D^TEg(n)+2\lambda^2 g^T(n)E\sum_{j=-\infty}^{n} C(n-j)x(j)\\ &\quad +\lambda^2\sum_{j=-\infty}^{n} x^T(j)C^T(n-j)E\sum_{j=-\infty}^{n} C(n-j)x(j)\\ &\quad +\lambda\gamma\sum_{s=n+1}^{\infty}|C(s-n)|x^2(n)\\ &\quad -\lambda\gamma\sum_{j=-\infty}^{n-1}|C(n-j)|x^2(j)+\lambda^2 g^T(n)Eg(n).\end{aligned}$$

Note that

$$\begin{aligned}2\sum_{j=-\infty}^{n} x^T(n)D^TEC(n-j)x(j) &\le 2\sum_{j=-\infty}^{n}|x^T(n)|\,|D^TE|\,|C(n-j)|\,|x(j)|\\ &= |D^TE|\sum_{j=-\infty}^{n}|C(n-j)|\,2\,|x(n)^T|\,|x(j)|\\ &\le |D^TE|\sum_{j=-\infty}^{n}|C(n-j)|\,(x^2(n)+x^2(j))\\ &\le 2\sum_{j=-\infty}^{n}|C(n-j)|\,(x^2(n)+x^2(j))\\ &= 2\alpha x^2(n)+2\sum_{j=-\infty}^{n}|C(n-j)|x^2(j).\end{aligned}$$

In the next two terms we make use of the following fact: for any real numbers a, b, and L with $L\neq 0$, $2ab\le\frac{a^2}{L^2}+L^2b^2$, which can be easily proven by using the fact that $(\frac{a}{L}-Lb)^2\ge 0$. As a consequence, for some $L>0$ we have

$$2x^T(n)D^TEg(n) \le 2|x^T(n)|\,|D^TEg(n)| \le \frac{x^2(n)}{L^2} + L^2|D^TEg(n)|^2$$

and

$$\begin{aligned}
2\lambda g^T(n)E\sum_{j=-\infty}^{n} C(n-j)x(j) &\le 2|\lambda g^T(n)E|\sum_{j=-\infty}^{n}|C(n-j)|\,|x(j)| \\
&\le \sum_{j=-\infty}^{n}|C(n-j)|2|g^T(n)E|\,|x(j)| \\
&\le \sum_{j=-\infty}^{n}|C(n-j)|\frac{x^2(j)}{L^2} + \sum_{j=-\infty}^{n}|C(n-j)|(g^T(n)EL)^2 \\
&= \sum_{j=-\infty}^{n}|C(n-j)|\frac{x^2(j)}{L^2} + \alpha(\lambda g^T(n)EL)^2.
\end{aligned}$$

For $u = s-n$,

$$\begin{aligned}
\gamma\sum_{s=n+1}^{\infty}|C(s-n)|x^2(n) &= \gamma\sum_{u=1}^{\infty}|C(u)|x^2(n) \\
&= \gamma\alpha x^2(n) - \gamma|C(0)|x^2(n).
\end{aligned}$$

Also,

$$\gamma\sum_{j=-\infty}^{n-1}|C(n-j)|x^2(j) = \gamma\sum_{j=-\infty}^{n}|C(n-j)|x^2(j) - \gamma|C(0)|x^2(n).$$

Thus

$$\begin{aligned}
&\gamma\sum_{s=n+1}^{\infty}|C(s-n)|\;x^2(n) - \gamma\sum_{j=-\infty}^{n-1}|C(n-j)|x^2(j) \\
&\quad= \quad \gamma\alpha x^2(n) - \gamma\sum_{j=-\infty}^{n}|C(n-j)|x^2(j).
\end{aligned}$$

Finally,

$$\begin{aligned}
&\lambda\sum_{j=-\infty}^{n}x^T(j)\;C^T(n-j)E\sum_{j=-\infty}^{n}C(n-j)x(j) \\
&\quad\le\quad 2\sum_{j=-\infty}^{n}|x^T(j)|\,|C^T(n-j)|\sum_{j=-\infty}^{n}|C(n-j)|\,|x(j)| \\
&\quad\le\quad \left(\sum_{j=-\infty}^{n}|x^T(j)|\,|C^T(n-j)|\right)^2 + \left(\sum_{j=-\infty}^{n}|C(n-j)|\,|x(j)|\right)^2
\end{aligned}$$

$$
\begin{aligned}
&= 2\Big(\sum_{j=-\infty}^{n} |C(n-j)|\,|x(j)|\Big)^2 \\
&= 2\Big(\sum_{j=-\infty}^{n} |C(n-j)|^{\frac{1}{2}}|C(n-j)|^{\frac{1}{2}}|x(j)|\Big)^2 \\
&\le 2\sum_{j=-\infty}^{n} |C(n-j)| \sum_{j=-\infty}^{n} |C(n-j)|x^2(j) \\
&= 2\alpha \sum_{j=-\infty}^{n} |C(n-j)|x^2(j), \quad \text{by Schwartz inequality for series.}
\end{aligned}
$$

Putting everything together we obtain

$$
\begin{aligned}
\triangle V \le \lambda \Big[& -2x^2(n) + 2\alpha x^2(n) + 2\sum_{j=-\infty}^{n} |C(n-j)|x^2(j) \\
& + \frac{x^2(n)}{L^2} + L^2(D^T E g(n))^2 + \sum_{j=-\infty}^{n} |C(n-j)|\frac{x^2(j)}{L^2} \\
& + \alpha(\lambda g^T(n)EL)^2 + \alpha\gamma x^2(n) - \gamma \sum_{j=-\infty}^{n} |C(n-j)|x^2(j) \\
& + 2\alpha \sum_{j=-\infty}^{n} |C(n-j)|x^2(j) + |\lambda g^T(n)Eg(n)| \Big] \\
= \lambda \Big[& \Big(-2 + 2\alpha + \alpha\gamma + \frac{1}{L^2}\Big)x^2(n) \\
+ & \Big(2 - \gamma + 2\alpha + \frac{1}{L^2}\Big) \sum_{j=-\infty}^{n} |C(n-j)|x^2(j) \\
& + |g^T(n)Eg(n)| + ((D^T E g(n))^2 + (g^T(n)E)^2\alpha)L^2 \Big].
\end{aligned}
$$

Since $-2+2\alpha+\alpha\gamma < 0$ and $2-\gamma+2\alpha < 0$ we may choose L large enough so that $-2+2\alpha+\alpha\gamma+\frac{1}{L^2} < 0$ and $2-\gamma+2\alpha+\frac{1}{L^2} < 0$. Then we have

$$
\begin{aligned}
\triangle V &\le \lambda \Big[(-2+2\alpha+\alpha\gamma+\frac{1}{L^2})x^2(n) + M\Big] \\
&\le \lambda \big[-\mu x^2(n) + M\big]
\end{aligned}
$$

for some positive constants μ and M. Using the fact that $V \in P_T$, we have

$$
0 = V(n+T) - V(n) = \sum_{i=n}^{n+T-1} \triangle V(i) \le \lambda \Big[-\mu \sum_{i=n}^{n+T-1} x^2(i) + TM\Big]
$$

from which it follows

$$\sum_{i=n}^{n+T-1} x^2(i) \leq \frac{TM}{\mu}$$

and

$$\sum_{j=1}^{T} |x(j+n-1)|^2 \leq \frac{TM}{\mu}.$$

Thus $|x(n)|^2$ is bounded over one period, and hence

$$\|x(n)\| \leq K, \quad \text{for some } K > 0.$$

Thus, every possible T-periodic solution $x(n)$ of (4.2.8) for $\lambda \in (0,1]$ is bounded. Therefore, by Corollary 4.1, Equation (4.2.8) has a T-periodic solution for $\lambda = 1$. It is obvious that condition (4.2.6) of the Theorem 4.2.2 cannot be satisfied for Equation (4.2.7) with $\lambda = 1$.

4.3 Periodicity in Scalar Nonlinear Neutral Systems

Next we use Krasnoselskii's fixed point theorem (Theorem 3.5.1) to show that the nonlinear neutral difference equation with functional delay

$$x(t+1) = a(t)x(t) + c(t)\triangle x(t-g(t)) + q\big(t, x(t), x(t-g(t))\big) \tag{4.3.1}$$

has a periodic solution. As usual, in order to apply Krasnoselskii's fixed point theorem, one would need to construct two mappings; one is contraction and the other is compact. Also, by making use of the variation of parameters techniques we are able, using the contraction mapping principle, to show that the periodic solution is unique. Let T be an integer such that $T \geq 1$. We assume the periodicity conditions

$$a(t+T) = a(t), \;\; c(t+T) = c(t), \;\; g(t+T) = g(t), \;\; g(t) \geq g^* > 0 \tag{4.3.2}$$

for some constant g^*. Let BC is the space of bounded sequences $\phi : (-g^*, 0] \to \mathbb{R}^k$ with the maximum norm $||\cdot||$. Materials of this section can be found in [111]. Define $P_T = \{\phi \in BC, \phi(t+T) = \phi(t)\}$. Then P_T is a Banach space when it is endowed with the maximum norm

$$\|x\| = \max_{t \in [0, T-1]} |x(t)|.$$

Also, we assume that

$$\prod_{s=t-T}^{t-1} a(s) \neq 1. \tag{4.3.3}$$

Throughout this section we assume that $a(t) \neq 0$ for all $t \in [0, T-1]$. It is interesting to note that equation (4.3.1) becomes of advanced type when $g(t) < 0$. Since we are searching for periodic solutions, it is natural to ask that $q(t,x,y)$ is periodic in t and Lipschitz continuous in x and y. That is

$$q(t+T,x,y) = q(t,x,y) \tag{4.3.4}$$

and

$$|q(t,x,y) - q(t,z,w)| \le L\|x-z\| + K\|y-w\| \tag{4.3.5}$$

for some positive constants L and E. Note that

$$\begin{aligned} |q(t,x,y)| - |q(t,0,0)| &\le |q(t,x,y) - q(t,0,0)| \le L\|x-0\| + K\|y-0\| \\ &= L\|x\| + K\|y\|. \end{aligned}$$

As a result,

$$|q(t,x,y)| \le L\|x\| + K\|y\| + |q(t,0,0)|. \tag{4.3.6}$$

We have the following lemma.

Lemma 4.1. *Suppose* (4.3.2)–(4.3.4) *hold. If* $x(t) \in P_T$*, then* $x(t)$ *is a solution of equation* (4.3.1) *if and only if*

$$\begin{aligned} x(t) &= c(t-1)x(t-g(t)) \\ &\quad + \frac{1}{1-\prod_{s=t-T}^{t-1} a(s)} \sum_{r=t-T}^{t-1} \Big[x(r-g(r)) \Big(a(r)c(r-1) - c(r) \Big) \\ &\quad + q(r,x(r),x(r-g(r))) \Big] \prod_{s=r+1}^{t-1} a(s). \end{aligned} \tag{4.3.7}$$

Proof. The proof is the same as for (3.5.2) by summing from $t-T$ to $t-1$ and noting that for $x \in P_T$, $x(t) = x(t-T)$.

We use the following notion of compact mapping.
Let $\mathscr{S}$ be a subset of a Banach space $\mathscr{B}$ and $f : \mathscr{S} \to \mathscr{B}$. If f is continuous and $f(\mathscr{S})$ is contained in a compact subset of $\mathscr{B}$, then f is a compact mapping.
We express equation (4.3.7) as

$$(H\varphi)(t) = (B\varphi)(t) + (A\varphi)(t) \tag{4.3.8}$$

where $A, B : P_T \to P_T$ are given by

$$(B\varphi)(t) = c(t-1)\varphi(t-g(t)) \tag{4.3.9}$$

and

$$(A\varphi)(t) = \Big(1 - \prod_{s=t-T}^{t-1} a(s)\Big)^{-1} \sum_{r=t-T}^{t-1} \Big[\varphi(r-g(r))[a(r)c(r-1) - c(r)]$$
$$+q(r, \varphi(r), \varphi(r-g(r)))\Big] \prod_{s=r+1}^{t-1} a(s). \tag{4.3.10}$$

Lemma 4.2. *Suppose* (4.3.2)–(4.3.5) *hold. If A is defined by* (4.3.10), *then* $A : P_T \to P_T$ *and is compact.*

Proof. First we want to show that $(A\varphi)(t+T) = (A\varphi)(t)$.
Let $\varphi \in P_T$. Then using (4.3.10) we arrive at

$$(A\varphi)(t+T) = \Big[1 - \prod_{s=t}^{t+T-1} a(s)\Big]^{-1} \sum_{r=t}^{t+T-1} \Big[\varphi(r-g(r))[a(r)c(r-1) - c(r)]$$
$$+q(r, \varphi(r), \varphi(r-g(r)))\Big] \prod_{s=r+1}^{t+T-1} a(s).$$

Let $j = r - T$, then

$$(A\varphi)(t+T) =$$
$$\Big[1 - \prod_{s=t}^{t+T-1} a(s)\Big]^{-1} \sum_{j=t-T}^{t-1} \Big[\varphi(j+T-g(j+T))[a(j+T)c(j+T-1) - c(j+T)]$$
$$+q(j+T, \varphi(j+T), \varphi(j+T-g(j+T)))\Big] \prod_{s=j+T+1}^{t+T-1} a(s)$$
$$= \Big[1 - \prod_{s=t}^{t+T-1} a(s)\Big]^{-1} \sum_{j=t-T}^{t-1} \Big[\varphi(j-g(j))[a(j)c(j-1) - c(j)]$$
$$+q(j, \varphi(j), \varphi(j-g(j)))\Big] \prod_{s=j+T+1}^{t+T-1} a(s).$$

Now let $k = s - T$, then

$$(A\varphi)(t+T) = \Big[1 - \prod_{k=t-T}^{t-1} a(k)\Big]^{-1} \sum_{j=t-T}^{t-1} \Big[\varphi(j-g(j))[a(j)c(j-1) - c(j)]$$
$$+q(j, \varphi(j), \varphi(j-g(j)))\Big] \prod_{k=j+1}^{t-1} a(s)$$
$$= (A\varphi)(t).$$

To see that A is continuous, we let $\varphi, \psi \in P_T$ with $\|\varphi\| \leq C$ and $\|\psi\| \leq C$. Let

$$\eta = \max_{t\in[0,T-1]} \left|\frac{1}{(1-\prod_{s=t-T}^{t-1} a(s))}\right|, \quad \beta = \max_{r\in[t-T,t]} |a(r)c(r-1) - c(r)|,$$
$$\gamma = \max_{t\in[0,T-1]} \prod_{s=t-T}^{t-1} a(s). \tag{4.3.11}$$

Given $\varepsilon > 0$, take $\delta = \varepsilon/M$ such that $\|\varphi - \psi\| < \delta$, where $M = T\gamma\eta[\beta + L + K]$. By making use of (4.3.5) into (4.3.10) we obtain

$$\begin{aligned}
&\left\|\Big(A\varphi(t)\Big) - \Big(A\psi(t)\Big)\right\| \\
&= \left\|\frac{1}{1-\prod_{s=t-T}^{t-1} a(s)} \sum_{r=t-T}^{t-1} \Big[\Big(\varphi(r-g(r)) - \psi(r-g(r))\Big)\Big(c(r-1)a(r) - c(r)\Big)\right. \\
&+ \left.\Big(q(r,\varphi(r),\varphi(r-g(r))) - q(r,\psi(r),\psi(r-g(r)))\Big)\Big] \prod_{s=r+1}^{t-1} a(s)\right\| \\
&\leq \eta \sum_{r=t-T}^{t-1} \Big[\|\varphi-\psi\|\beta + L\|\varphi-\psi\| + K\|\varphi-\psi\|\Big]\gamma \\
&\leq \gamma\eta \sum_{r=t-T}^{t-1} (\beta + L + K)\|\varphi-\psi\| = \eta\gamma T(\beta + L + K)\|\varphi-\psi\| \\
&= M\|\varphi-\psi\| = M\delta < \varepsilon
\end{aligned}$$

where L and K are given by (4.3.5). This proves A is continuous.
Next, we show that A maps bounded subsets into compact sets. Let J be given, $S = \{\varphi \in P_T : \| \varphi \| \leq J\}$ and $Q = \{(A\varphi)(t) : \varphi \in S\}$, then S is a subset of R^T which is closed and bounded thus compact. As A is continuous in φ it maps compact sets into compact sets. Therefore $Q = A(S)$ is compact.

It is trivial to show that the map B is a contraction provided we assume that

$$\Big\|c(t-1)\Big\| \leq \zeta < 1. \tag{4.3.12}$$

Theorem 4.3.1 ([111]). *Let $\alpha = ||q(t,0,0)||$. Let η, β and γ be given by (4.3.11). Suppose (4.3.2)–(4.3.5) and (4.3.12) hold. Suppose there is a positive constant G such that all solutions $x(t)$ of (4.3.1), $x(t) \in P_T$ satisfy $|x(t)| \leq G$, the inequality*

$$\Big\{\zeta + \eta\gamma T(\beta + L + K)\Big\}G + \eta\gamma T\alpha \leq G \tag{4.3.13}$$

holds. Then equation (4.3.1) has a T-periodic solution.

Proof. Define $\mathbb{M} = \{\varphi \in P_T : ||\varphi|| \leq G\}$. Then Lemma 4.2 implies $A : \mathbb{M} \to P_T$ and A is compact and continuous. Also the mapping B is a contraction and it is clear that $B:\mathbb{M}\to P_T$. Next, we show that if $\varphi,\psi\in\mathbb{M}$, we have $||A\phi+B\psi||\le G$. Let $\varphi,\psi\in\mathbb{M}$ with $||\varphi||,||\psi||\le G$. Then from (4.3.8)–(4.3.12) and the fact that $|q(t,x,y)|\le L\|x\|+K\|y\|+\alpha$, we have

$$\begin{aligned}
\Big\|\Big(A\varphi(t)\Big)+\Big(B\psi(t)\Big)\Big\| &= \Big\|\frac{1}{1-\prod_{s=t-T}^{t-1}}\sum_{r=t-T}^{t-1}\Big[\varphi(r-g(r))\Big(c(r-1)a(r)-c(r)\Big) \\
&\quad + q(r,\varphi(r),\varphi(r-g(r)))\Big]\prod_{s=r+1}^{t-1}a(s)+c(t-1)\psi(t-g(t))\Big\| \\
&\le \eta\gamma\sum_{r=t-T}^{t-1}\Big[L\|\varphi\|+K\|\varphi\|+\beta\|\varphi\|+\alpha\Big]+\zeta\|\psi\| \\
&\le \eta\gamma[(\beta+L+K)\|\varphi\|+\alpha]T+\zeta\|\psi\| \\
&\le \eta\gamma T(\beta+L+K)G+\eta\gamma T\alpha+G\zeta \\
&= \Big\{\zeta+\eta\gamma T(\beta+L+K)\Big\}G+\eta\gamma T\alpha \\
&\le G.
\end{aligned}$$

We see that all the conditions of Krasnoselskii's theorem are satisfied on the set $\mathbb{M}$. Thus there exists a fixed point z in $\mathbb{M}$ such that $z=Az+Bz$. By Lemma 4.1 this fixed point is a solution of (4.3.1). Hence (4.3.1) has a T-periodic solution.

Remark 4.2. The constant G of Theorem 4.3.1 serves as a priori bound on all possible T-periodic solutions of equation (4.3.1) as we shall see in the Example 4.1.

Next we use the contraction mapping principle to show the periodic solution is unique.

Theorem 4.3.2 ([111]). *Suppose* (4.3.2)–(4.3.5) *and* (4.3.12) *hold. Let* η,β*, and* γ *be given by* (4.3.11). *If*

$$\zeta+T\gamma\eta\Big(\beta+L+K\Big)\le\nu<1,$$

then equation (4.3.1) *has a unique T-periodic solution.*

Proof. Let the mapping H be given by (4.3.8). For $\varphi,\psi\in P_T$, in view of (4.3.8), we have

$$\begin{aligned}
\Big\|\Big(H\varphi(t)\Big)-\Big(H\psi(t)\Big)\Big\| &= \Big\|\Big(B\varphi(t)\Big)+\Big(A\varphi(t)\Big)-\Big(B\psi(t)\Big)-\Big(A\psi(t)\Big)\Big\| \\
&= \Big\|\Big(\Big(B\varphi(t)\Big)-\Big(B\psi(t)\Big)\Big)+\Big(\Big(A\varphi(t)\Big)-\Big(A\psi(t)\Big)\Big)\Big\| \\
&\le \Big\|\Big(B\varphi(t)\Big)-\Big(B\psi(t)\Big)\Big\|+\Big\|\Big(A\varphi(t)\Big)-\Big(A\psi(t)\Big)\Big\|
\end{aligned}$$

$$\leq \zeta \|\varphi - \psi\|| + \gamma\eta \sum_{r=t-T}^{t-1} \Big[L\|\varphi - \psi\| + K\|\varphi - \psi\| + \beta\|\varphi - \psi\|\Big]$$
$$\leq \Big[\zeta + T\gamma\eta\Big(\beta + L + K\Big)\Big]\|\varphi - \psi\|$$
$$< \nu\|\varphi - \psi\|.$$

By the contraction mapping principle, (4.3.1) has a unique T-periodic solution.

We have the following example.

Example 4.1 ([111]). Consider equation (4.3.1) along with conditions (4.3.2)–(4.3.5). Suppose that $a(t) \neq 1$ for all $t \in [0, T-1]$. Set

$$\rho = \min_{t\in[0,T-1]} |a(t) - 1| \ , \quad \delta = \max_{t\in[0,T-1]} k(t),$$

where $k(t) = c(t) - c(t-1)$. Suppose $1 - ||c|| > 0$. If

$$\rho(1 - ||c||) > (1 - ||c||)(\delta + L + K) + T\rho(||a-1|| + L + K)$$

holds, and G is defined by

$$G = \frac{\alpha(1 - ||c|| + T\rho}{\rho(1-||c||) - (1-||c||)(\delta + L + K) - T\rho(||a-1|| + L + K)}$$

satisfies inequality (4.3.13), then (4.3.1) has a T-periodic solution.

Proof. We rewrite (4.3.1) as

$$\triangle x(t) = \Big(a(t) - 1\Big)x(t) + c(t)\triangle x(t - g(t)) + q\big(t, x(t), x(t - g(t))\big). \tag{4.3.14}$$

Let the mappings A and B be defined by (4.3.10) and (4.3.9), respectively.
Let $x(t) \in P_T$. A summation of equation (4.3.14) from 0 to $T-1$ gives

$$\sum_{s=0}^{T-1} \triangle x(s) = \sum_{s=0}^{T-1} \Big[(a(s)-1)x(s) + c(s)\triangle x(s - g(s)) + q\big(s, x(s), x(s-g(s))\big)\Big].$$

Or,

$$x(T) - x(0) = \sum_{s=0}^{T-1} \Big[(a(s)-1)x(s) + c(s)\triangle x(s - g(s)) + q\big(s, x(s), x(s-g(s))\big)\Big].$$

Since $x(t) \in P_T$, $x(T) = x(0)$. Therefore

$$0 = \sum_{s=0}^{T-1} \Big[(a(s)-1)x(s) + c(s)\triangle x(s - g(s)) + q\big(s, x(s), x(s-g(s))\big)\Big]. \tag{4.3.15}$$

Rewrite and then sum by parts, using the summation by parts formula

$$\sum Ey\triangle z = yz - \sum z\triangle y$$

with $Ey(s) = c(s)$ and $z = x(s-g(s))$. As a consequence, we have

$$\begin{aligned}\sum_{s=0}^{T-1} c(s)\triangle x(s-g(s)) &= c(s-1)x(s-g(s))\Big|_{s=0}^{T} - \sum_{s=0}^{T-1} x(s-g(s))\triangle c(s-1)\\ &= c(T-1)x(T-g(T)) - c(-1)x(0-g(0))\\ &\quad - \sum_{s=0}^{T-1} x(s-g(s))[c(s)-c(s-1)]\\ &= -\sum_{s=0}^{T-1} x(s-g(s))[c(s)-c(s-1)].\end{aligned}$$

As a result (4.3.15) becomes

$$\begin{aligned}&\sum_{s=0}^{T-1}[a(s)-1]x(s)\\ &= \sum_{s=0}^{T-1} x(s-g(s))[c(s)-c(s-1)] - q\big(s, x(s), x(s-g(s))\big).\end{aligned} \tag{4.3.16}$$

Let $S = \sum_{s=0}^{T-1}\Big|a(s)-1\Big|\Big|x(s)\Big|$. We claim that there exists a $t^* \in [0, T-1]$ such that

$$T\Big|a(t^*)-1\Big|\Big|x(t^*)\Big| \le \sum_{s=0}^{T-1}\Big|a(s)-1\Big|\Big|x(s)\Big|.$$

Suppose such t^* does not exist. Then

$$T\Big|a(t^*)-1\Big|\Big|x(t^*)\Big| > S,$$

which implies that

$$T\Big|a(t^*)-1\Big|\Big|x(t^*)\Big| > S+\varepsilon.$$

Or

$$\sum_{t^*=0}^{T-1}\Big|a(t^*)-1\Big|\Big|x(t^*)\Big| > \sum_{t^*=0}^{T-1}\frac{S+\varepsilon}{T}.$$

Hence, $S > S+\varepsilon$, which is a contradiction. Therefore, such t^* exists.
From (4.3.16), it implies that there exists a $t^* \in (0, T-1)$ such that

$$T\Big|a(t^*)-1\Big|\Big|x(t^*)\Big| \le \sum_{s=0}^{T-1}\Big|k(t)\Big|\Big|x(s-g(s))\Big| + \Big|q\big(s, x(s), x(s-g(s))\big)\Big|.$$

By taking the maximum over $t \in [0, T-1]$, we obtain from the above inequality

$$
\begin{aligned}
T\rho||x(t^*)|| &\leq \sum_{s=0}^{T-1}\Big(\delta||x|| + L||x|| + E||x|| + \alpha\Big)\\
&= \sum_{s=0}^{T-1}\Big((\delta + L + E)||x|| + \alpha\Big)\\
&= T\Big((\delta + L + E)||x|| + \alpha\Big),
\end{aligned}
$$

which gives us

$$
||x(t^*)|| \leq \frac{1}{\rho}(\delta + L + K)||x|| + \frac{\alpha}{\rho}. \tag{4.3.17}
$$

Since for all $t \in [0, T-1]$

$$
x(t) = x(t^*) + \sum_{s=t^*}^{t-1} \triangle x(s),
$$

taking maximum over $t \in [0, T-1]$ and using

$$
||x(t)|| \leq ||x(t^*)|| + \sum_{s=0}^{T-1} |\triangle x(s)|
$$

yields

$$
||x(t)|| \leq \frac{1}{\rho}(\delta + L + E)||x|| + \frac{\alpha}{\rho} + T||\triangle x||. \tag{4.3.18}
$$

Taking the norm in (4.3.1) yields

$$
||\triangle x(t)|| \leq ||a-1||\;||x|| + ||c||\;||\triangle x|| + K||x|| + L||x|| + \alpha.
$$

Or

$$
\Big(1 - ||c||\Big)||\triangle x(t)|| \leq \Big(||a-1|| + E + L\Big)||x|| + \alpha.
$$

Thus

$$
||\triangle x(t)|| \leq \frac{\Big(||a-1|| + E + L\Big)||x|| + \alpha}{1 - ||c||}. \tag{4.3.19}
$$

A substitution of (4.3.19) into (4.3.18) yields

$$||x(t)|| \leq \frac{1}{\rho}(\delta + L + K)||x|| + \frac{\alpha}{\rho} + T\frac{\Big(||a-1|| + K + L\Big)||x|| + \alpha}{1 - ||c||}.$$

Hence

$$||x(t)|| \leq \frac{\alpha(1 - ||c|| + T\rho)}{\rho(1 - ||c||) - (1 - ||c||)(\delta + L + E) - T\rho(||a-1|| + L + E)} = G.$$

Thus, for all $x(t) \in P_T$ we have shown that

$$||x(t)|| \leq G.$$

Define $\mathbb{M} = \{\varphi \in P_T : ||\varphi|| \leq G\}$. Then by Theorem 4.3.1, Equation (4.3.1) has a T-periodic solution. This completes the proof.

4.4 Periodicity in Vector Neutral Nonlinear Functional Difference Equations

Motivated by the work of Hale on functional differential equations [74], in this section we consider the nonlinear neutral difference equation

$$\triangle x(t) = A(t)x(t) + \triangle Q(t, x(t - g(t))) + G(t, x(t), x(t - g(t))) \qquad (4.4.1)$$

where A is an $n \times n$ matrix function, $g : \mathbb{Z} \to \mathbb{Z}^+$ is scalar and the functions $Q : \mathbb{Z} \times \mathbb{R}^n \to \mathbb{R}^n$ and $G : \mathbb{Z} \times \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n$ are continuous in x. The purpose of this work is to make use of the notion of the fundamental matrix and invert (4.4.1) so that fixed point theory can be used. Krasnoselskii's fixed point theorem is one of the tools that we use in this research in order to show the existence of a periodic solution. The obtained mapping is the sum of two mappings; one is a contraction and the other is compact. The need to use Krasnoselkii's fixed point theorem may be necessary if one of the mappings is not compact nor satisfies a Lipschitz condition. Inverting equation (4.4.1) to a fixed point problem enables us to show the uniqueness of the periodic solution by appealing to the contraction mapping principle.
For an integer $T > 1$ let P_T be the set of all n-vector functions $x(t)$, periodic in t of period T. Then $(P_T, ||\cdot||)$ is a Banach space when it is endowed with the maximum norm

$$||x|| = \max_{t\in\mathbb{Z}} |x(t)| = \max_{t\in[0,T-1]} |x(t|.$$

Note that P_T is equivalent to the Euclidean space $\mathbb{R}^{nT}$. If A is an $n \times n$ real matrix, then we define the norm of A by $|A| = \max_{1\leq i\leq n} \sum_{j=1}^{n} |a_{ij}|$. First we make the following definition.

Definition 4.4.1. If the matrix $B(t)$ is periodic of period T, then the linear system

$$y(t+1) = B(t)y(t) \tag{4.4.2}$$

is said to be *noncritical with respect to T*, if it has no periodic solution of period T except the trivial solution $y = 0$.

Since we are searching for the existence of periodic solution for system (4.4.1), it is natural to assume that

$$A(t+T) = A(t), \ \ g(t+T) = g(t), \ \ g(t) \geq g^* > 0 \tag{4.4.3}$$

with $g : \mathbb{Z} \to \mathbb{Z}^+$ being scalar and $Q(t,x)$ and $G(t,x,y)$ are continuous functions and periodic in t of period T. That is

$$Q(t+T,x) = Q(t,x), \ \ G(t+T,x,y) = G(t,x,y). \tag{4.4.4}$$

Throughout this section it is assumed that the matrix $B(t) = I + A(t)$ is nonsingular and system (4.4.2) is noncritical, where I is the $n \times n$ identity matrix. Also, if $x(t)$ is a sequence, then the forward operator E is defined as $Ex(t) = x(t+1)$. Next we state some known results about system (4.4.2). Let $K(t)$ represent the fundamental matrix of (4.4.2) with $K(0) = I$. Then
(i) $detK(t) \neq 0$.
(ii) $K(t+1) = B(t)K(t)$ and $K^{-1}(t+1) = K^{-1}(t)B^{-1}(t)$.
(iii) System (4.4.2) is noncritical if and only if $det(I - K(T)) \neq 0$.
(iv) There exists a nonsingular matric L such that $K(t+T) = K(t)L^T$ and $K^{-1}(t+T) = L^{-T}K^{-1}(t)$.
With the above-mentioned $K(t)$ in mind we have the following lemma.

Lemma 4.3. *Suppose* (4.4.3)–(4.4.4) *hold. If* $x(t) \in P_T$, *then* $x(t)$ *is a solution of equation* (4.4.1) *if and only if*

$$\begin{aligned} x(t) &= Q(t,x(t-g(t))) \\ &\quad + K(t)\Big(K^{-1}(T) - I\Big)^{-1} \sum_{u=t}^{t+T} K^{-1}(u)\Big(I - A(u)B^{-1}(u)\Big)\Big[A(u)Q(u,x(u-g(u))) \\ &\quad + G(u,x(u),x(u-g(u)))\Big]. \end{aligned} \tag{4.4.5}$$

Proof. Let $x(t) \in P_T$ be a solution of (4.4.1) and $K(t)$ be a fundamental matrix of solutions of (4.4.2). First we write (4.4.1) as

$$\begin{aligned} \triangle\{x(t) - Q(t,x(t-g(t)))\} &= A(t)\{x(t) - Q(t,x(t-g(t)))\} \\ &\quad + A(t)Q(t,x(t-g(t))) + G(t,x(t),x(t-g(t))). \end{aligned}$$

Since $K(t)K^{-1}(t)=I$, it follows that

$$\begin{aligned}0 &= \triangle\Big(K(t)K^{-1}(t)\Big)=K(t)\triangle(K^{-1}(t))+\triangle(K(t))EK(t)\\ &= K(t)\triangle(K^{-1}(t))+A(t)K(t)K^{-1}(t)B^{-1}(t)\\ &= K(t)\triangle(K^{-1}(t))+A(t)B^{-1}(t).\end{aligned}$$

Or,

$$\triangle(K^{-1}(t))=-K^{-1}(t)A(t)B^{-1}(t). \tag{4.4.6}$$

If $x(t)$ is a solution of (4.4.1) with $x(0)=x_0$, then

$$\begin{aligned}&\triangle\Big\{K^{-1}(t)\Big(x(t)-Q(t,x(t-g(t)))\Big)\Big\}\\ &= K^{-1}(t)\triangle\Big(x(t)-Q(t,x(t-g(t)))\Big)+\triangle(K^{-1}(t))E\Big(x(t)-Q(t,x(t-g(t)))\Big)\\ &= K^{-1}(t)\Big[A(t)\Big(x(t)-Q(t,x(t-g(t)))\Big)+A(t)Q(t,x(t-g(t)))+G(t,x(t),x(t-g(t)))\Big]\\ &\quad -K^{-1}(t)A(t)B^{-1}(t)\Big[B(t)\Big(x(t)-Q(t,x(t-g(t)))\Big)\\ &\quad +A(t)Q(t,x(t-g(t)))+G(t,x(t),x(t-g(t)))\Big], \text{ by (4.4.6)}\\ &= K^{-1}(t)\Big(I-A(t)B^{-1}(t)\Big)\Big(A(t)Q(t,x(t-g(t)))+G(t,x(t),x(t-g(t)))\Big).\end{aligned}$$

Summing the above equation from 0 to $t-1$ yields

$$\begin{aligned}x(t) &= Q(t,x(t-g(t)))+K(t)\Big(x_0-Q(0,x(-g(0)))\Big)\\ &+ K(t)\sum_{u=0}^{t-1}K^{-1}(u)\Big(I-A(u)B^{-1}(u)\Big)\Big[A(u)Q(u,x(u-g(u)))\\ &+ G(u,x(u),x(u-g(u)))\Big].\end{aligned} \tag{4.4.7}$$

For the sake of simplicity, we let

$$D(u)=\Big(I-A(u)B^{-1}(u)\Big)\Big[A(u)Q(u,x(u-g(u)))+G(u,x(u),x(u-g(u)))\Big].$$

Since $x(T)=x_0=x(0)$, using (4.4.7) we get

$$x_0-Q(0,x(-g(0))) = \Big(I-K(T)\Big)^{-1}\sum_{u=0}^{T-1}K(T)K^{-1}(u)D(u). \tag{4.4.8}$$

A substitution of (4.4.8) into (4.4.7) yields

$$x(t) = Q(t,x(t-g(t))) + K(t)\Big(I-K(T)\Big)^{-1}\sum_{u=0}^{T-1} K(T)K^{-1}(u)D(u) + \sum_{u=0}^{t-1} K(t)K^{-1}(u)D(u). \tag{4.4.9}$$

It remains to show that expression (4.4.9) is equivalent to (4.4.5). Since

$$(I-K(T)\Big)^{-1} = \Big(K(T)(K^{-1}(T)-I)\Big)^{-1} = \Big(K^{-1}(T)-I\Big)^{-1}K^{-1}(T),$$

(4.4.9) becomes

$$\begin{aligned}
x(t) &= Q(t,x(t-g(t)))+K(t)\Big(K^{-1}(T)-I\Big)^{-1}\sum_{u=0}^{T-1} K^{-1}(u)D(u) \\
&\quad + \sum_{u=0}^{t-1} K(t)K^{-1}(u)D(u) \\
&= Q(t,x(t-g(t)))+K(t)\Big(K^{-1}(T)-I\Big)^{-1}\Big\{\sum_{u=0}^{T-1} K^{-1}(u)D(u) \\
&\quad + \sum_{u=0}^{t-1} K^{-1}(T)K^{-1}(u)D(u) - \sum_{u=0}^{t-1} K^{-1}(u)D(u)\Big\} \\
&= Q(t,x(t-g(t)))+K(t)\Big(K^{-1}(T)-I\Big)^{-1}\Big\{-\sum_{u=T}^{t-1} K^{-1}(u)D(u) \\
&\quad + \sum_{u=0}^{t-1} K^{-1}(T)K^{-1}(u)D(u)\Big\}.
\end{aligned}$$

By letting $u = s-T$ in the third term on the right side of the above expression, we end up with

$$\begin{aligned}
x(t) &= Q(t,x(t-g(t)))+K(t)\Big(K^{-1}(T)-I\Big)^{-1}\Big\{-\sum_{u=T}^{t-1} K^{-1}(u)D(u) \\
&\quad + \sum_{s=T}^{T+t-1} K^{-1}(T)K^{-1}(s-T)D(s-T)\Big\}.
\end{aligned} \tag{4.4.10}$$

By *(iv)* we have $K(t-T) = K(t)L^{-T}$ and $K(T) = L^T$, where $L^{-T} = (L^T)^{-1}$. Hence, $K^{-1}(T)K^{-1}(s-T) = K^{-1}(s)$. Moreover, since $D(s-T) = D(s)$ expression (4.4.10) becomes

$$
\begin{aligned}
x(t) &= Q(t,x(t-g(t))) + K(t)\Big(K^{-1}(T)-I\Big)^{-1}\Big\{-\sum_{u=T}^{t-1}K^{-1}(u)D(u) \\
&\quad + \sum_{u=T}^{t+T-1}K^{-1}(u)D(u)\Big\} \\
&= Q(t,x(t-g(t))) + K(t)\Big(K^{-1}(T)-I\Big)^{-1}\sum_{u=t}^{t+T-1}K^{-1}(u)D(u).
\end{aligned}
$$

This completes the proof.

Now we are in a position to define a suitable mapping that satisfies all the requirements of Theorem 3.5.1. Define a mapping H by

$$
\begin{aligned}
(H\varphi)(t) &= Q(t,\varphi(t-g(t))) \\
&\quad + K(t)\Big(K^{-1}(T)-I\Big)^{-1}\sum_{u=t}^{t+T-1}K^{-1}(u)\Big(I-A(u)B^{-1}(u)\Big) \\
&\quad \times \Big[A(u)Q(u,\varphi(u-g(u))) + G(u,\varphi(u),\varphi(u-g(u)))\Big]. \qquad (4.4.11)
\end{aligned}
$$

It is clear that $H : P_T \to P_T$ by the way it was constructed in Lemma 4.3.
We note that to apply the above theorem we need to construct two mappings; one is a contraction and the other is compact. Therefore, we express equation (4.4.11) as

$$
(H\varphi)(t) = (B\varphi)(t) + (C\varphi)(t)
$$

where $C, B : P_T \to P_T$ are given by

$$
(B\varphi)(t) = Q(t,\varphi(t-g(t))) \qquad (4.4.12)
$$

and

$$
\begin{aligned}
(C\varphi)(t) &= K(t)\Big(K^{-1}(T)-I\Big)^{-1}\sum_{u=t}^{t+T-1}K^{-1}(u)\Big(I-A(u)B^{-1}(u)\Big) \\
&\quad \times \Big[A(u)Q(u,\varphi(u-g(u))) + G(u,\varphi(u),\varphi(u-g(u)))\Big]. \qquad (4.4.13)
\end{aligned}
$$

We assume the functions Q and G are Lipschitz continuous in x and in x and y, respectively. That is, there are positive constants E_1, E_2, and E_3 such that

$$
|Q(t,x) - Q(t,y)| \le E_1\|x-y\| \text{ and} \qquad (4.4.14)
$$

$$
|G(t,x,y) - G(t,z,w))| \le E_2\|x-z\| + E_3\|y-w\|. \qquad (4.4.15)
$$

Observe that in view of (4.4.14) and (4.4.15) we have

$$\begin{aligned}|Q(t,x)| &= |Q(t,x) - Q(t,0) + Q(t,0)| \\ &\le |Q(t,x) - Q(t,0)| + |Q(t,0)| \\ &\le E_1\|x\| + \alpha.\end{aligned}$$

Similarly,

$$\begin{aligned}|G(t,x,y)| &= |G(t,x,y) - G(t,0,0) + G(t,0,0)| \\ &\le |G(t,x,y) - G(t,0,0)| + |G(t,0,0)| \\ &\le E_2\|x\| + E_3\|y\| + \beta\end{aligned}$$

where $\alpha = \max_{t\in\mathbb{Z}} |Q(t,0)|$ and $\beta = \max_{t\in\mathbb{Z}} |G(t,0,0)|$. The next lemma plays an important role in showing C is compact.

Lemma 4.4. *Suppose the hypothesis of Lemma 4.3 holds. If C is defined by* (4.4.13)*, then*
(I)

$$||C\varphi|| \le r \sum_{u=0}^{T-1} \left\| A(u)Q(u,\varphi(u-g(u))) + G(u,\varphi(u),\varphi(u-g(u))) \right\|,$$

where

$$r = \max_{t\in[0,T-1]} \left(\max_{t\le u\le t+T-1} \left| \left[K(u)(K^{-1}(T)-I)K^{-1}(t)\right]^{-1} \left(I - A(u)B^{-1}(u)\right) \right| \right) \tag{4.4.16}$$

is a constant which is independent of Q and G and depends only upon $T, A(t), B(t)$, and $K(t)$ where $1 \le t \le T$.
(II) C is continuous and compact.

Proof. Let C be defined by (4.4.13) which is equivalent to

$$\begin{aligned}(C\varphi)(t) = \sum_{u=t}^{t+T-1} &\left[K(u)(K^{-1}(T)-I)K^{-1}(t)\right]^{-1} \\ &\left(I - A(u)B^{-1}(u)\right)\left[A(u)Q(u,\varphi(u-g(u))) \right. \\ &\left. + \; G(u,\varphi(u),\varphi(u-g(u)))\right].\end{aligned}$$

As $(C\varphi)(t) \in P_T$, we have

$$\|(C\varphi)(t)\| = \max_{t\in[0,T-1]} \Big| \sum_{u=t}^{t+T-1} \Big[K(u)(K^{-1}(T)-I)K^{-1}(t)\Big]^{-1} \Big(I - A(u)B^{-1}(u)\Big)$$
$$\times \Big[A(u)Q(u,\varphi(u-g(u))) + G(u,\varphi(u),\varphi(u-g(u)))\Big]\Big|$$
$$\leq \max_{t\in[0,T-1]} \Big(\max_{t\leq u\leq t+T-1} \Big| \Big[K(u)(K^{-1}(T)-I)K^{-1}(t)\Big]^{-1} \Big(I - A(u)B^{-1}(u)\Big)\Big| \Big)$$
$$\times \sum_{u=0}^{T-1} \Big\|A(u)Q(u,\varphi(u-g(u))) + G(u,\varphi(u),\varphi(u-g(u)))\Big\|.$$

This completes the proof of *(I)*. To see that C is continuous, we let $\varphi, \psi \in P_T$ with $\|\varphi\| \leq D$ and $\|\psi\| \leq D$. Given $\varepsilon > 0$, take $\delta = \varepsilon/N$ such that $\|\varphi - \psi\| < \delta$. By making use of (4.4.14) and (4.4.15) into (4.4.13) we get

$$\Big\|C\varphi - C\psi\Big\| \leq rT\Big[|A|E_1\|\varphi-\psi\| + (E_2+E_3)\|\varphi-\psi\|\Big]$$
$$\leq N\|\varphi-\psi\| < \varepsilon$$

where E_1, E_2, and E_3 are given by (4.4.14) and (4.4.15) and $N = rT(|A|E_1 + E_2 + E_3)$. This proves C is continuous. Next, we show that C maps bounded subsets into compact sets. Let J be given and let $S = \{\varphi \in P_T : \| \varphi \| \leq J\}$ and $Q = \{C\varphi : \varphi \in S\}$, then S is a subset of $\mathbb{R}^{nT}$ which is closed and bounded thus compact. As C is continuous in φ it maps compact sets into compact sets. Therefore $Q = C(S)$ is compact.

Lemma 4.5. *If B is given by* (4.4.12) *and $E_1 \leq \zeta < 1$, where E_1 is given by* (4.4.14) *then B is a contraction.*

Proof. Let B be defined by (4.4.12). Then for $\varphi, \psi \in P_T$ we have

$$\|B\varphi - B\psi\| = \max_{t\in[0,T-1]} |B\varphi - B\psi|$$
$$\leq E_1 \max_{t\in[0,T-1]} |\varphi(t-g(t)) - \psi(t-g(t))|$$
$$\leq \zeta\|\varphi-\psi\||.$$

Hence B defines a contraction mapping with contraction constant ζ.

Theorem 4.4.1. *Let $\alpha = \max_{t\in\mathbb{Z}} |Q(t,0)|$ and $\beta = \max_{t\in\mathbb{Z}} |G(t,0,0)|$. Let r be given by* (4.4.16). *Suppose* (4.4.3), (4.4.4), (4.4.14), *and* (4.4.15) *hold. Let J be a positive constant satisfying the inequality*

$$\alpha + E_1 J + rT\Big[|A|(E_1+\alpha) + E_2 + E_3\Big]J + rT\beta \leq J. \qquad (4.4.17)$$

Let $\mathbb{M} = \{\varphi \in P_T : ||\varphi|| \leq J\}$. Then equation (4.4.1) *has a solution in M.*

Proof. Define $\mathbb{M} = \{\varphi \in P_T : ||\varphi|| \leq J\}$. Then Lemma 4.4 implies $C : P_T \to P_T$ and C is compact on M and continuous. Also, from Lemma 4.5, the mapping B is a contraction and it is clear that $B : P_T \to P_T$. Next, we show that if $\varphi, \psi \in \mathbb{M}$, we have $||C\phi + B\psi|| \leq J$. Let $\varphi, \psi \in \mathbb{M}$ with $||\varphi||, ||\psi|| \leq J$. Then

$$\begin{aligned}\Big\|C\varphi + B\psi\Big\| &\leq E_1||\psi|| + \alpha + r\sum_{u=0}^{T-1}[|A|(\alpha + E_1||\varphi||) + E_2||\varphi|| + E_3||\varphi|| + \beta] \\ &\leq \alpha + E_1 J + rT\Big[|A|(E_1+\alpha) + E_2 + E_3\Big]J + rT\beta \leq J.\end{aligned}$$

We see that all the conditions of Krasnoselskii's theorem (Theorem 3.5.1) are satisfied on the set $\mathbb{M}$. Thus there exists a fixed point z in $\mathbb{M}$ such that $z = Az + Bz$. By Lemma 4.3, this fixed point is a solution of (4.4.1). Hence (4.4.1) has a T-periodic solution.

Corollary 4.4. *Suppose* (4.4.3)*,* (4.4.4)*,* (4.4.14)*, and* (4.4.15) *hold and* $Q(t, x(t - g(t)))$ *and* $G(t, x(t), x(t - g(t)))$ *are uniformly bounded. Let* M *be defined as in Theorem 4.4.1 such that for* $\varphi \in M$,

$$||Q(, \varphi(t - g(t)))|| \leq J_1,$$

and

$$\begin{aligned}&\Big\|\sum_{u=t}^{t+T-1}\Big[K(u)(K^{-1}(T) - I)K^{-1}(t)\Big]^{-1} \\ &\Big[A(u)Q(u, \varphi(u - g(u))) + G(u, \varphi(u), \varphi(u - g(u)))\Big]\Big\| \leq J_2\end{aligned}$$

for positive constants J_1 *and* J_2*. If*

$$J_1 + J_2 \leq J,$$

then (4.4.1) *has a* T*-periodic solution.*

Proof. Define B and C by (4.4.12) and (4.4.13), respectively and imitate the proof of Theorem 4.4.1.

In the next theorem we use the contraction mapping principle to show that the periodic solution is unique.

Theorem 4.4.2. *Suppose* (4.4.3)*,* (4.4.4)*,* (4.4.14)*, and* (4.4.15) *hold. Then equation* (4.4.1) *has a unique* T*-periodic solution.*

Proof. Due to condition (4.4.21) we have that

$$E_1 + rT(|A|E_1 + E_2 + E_3) < 1.$$

Let the mapping H be given by (4.4.11). For $\varphi, \psi \in P_T$, in view of (4.4.11), we have

$$\Big\|H\varphi - H\psi\Big\| \le \Big(E_1 + rT(|A|E_1 + E_2 + E_3)\Big)\|\varphi - \psi\|.$$

This completes the proof.

It is worth noting that Theorem 4.4.1 and Theorem 4.4.2 are not applicable to functions G of the form

$$G(t,\varphi(t),\varphi(t-g(t))) = f_1(t)\varphi^2(t) + f_2(t)\varphi^2(t-g(t))),$$

where $f_1(t), f_2(t)$, and $g(t) > 0$ are periodic sequences. To accommodate such functions, we state the following corollary, which requires the functions Q and G to be locally Lipschitz.

Corollary 4.5. *Suppose* (4.4.3)–(4.4.4) *hold and let* α *and* β *be the constants defined in Theorem 4.4.1. Let* J *be a positive constant and define* $\mathbb{M} = \{\varphi \in P_T : ||\varphi|| \le J\}$. *Suppose there are positive constants* E_1^*, E_2^*, *and* E_3^* *so that for* x, y, z, *and* $w \in \mathbb{M}$ *we have*

$$|Q(t,x) - Q(t,y)| \le E_1^*\|x-y\|,$$

$$|G(t,x,y) - G(t,z,w))| \le E_2^*\|x-z\| + E_3^*\|y-w\|,$$

and

$$\alpha + E_1^*J + rT\Big[|A|(E_1^* + \alpha) + E_2^* + E_3^*\Big]J + rT\beta \le J. \tag{4.4.18}$$

Then equation (4.4.1) *has a unique solution in* $\mathbb{M}$.

Proof. Let $\mathbb{M} = \{\varphi \in P_T : ||\varphi|| \le J\}$. Let the mapping H be given by (4.4.11). Then the results follow immediately from Theorem 4.4.1 and Theorem 4.4.2, since

$$E_1^* + rT(|A|E_1^* + E_2 + E_3^*) < 1.$$

This completes the proof.

Now we display an example as an application.

Example 4.2. For small positive ε_1 and ε_2, we consider the perturbed discrete Van Der Pol equation

$$\triangle^2 x + (\varepsilon_2 x^2 - 1)\triangle x - x - \varepsilon_1 \triangle\Big(cos(t\pi)x^2(t-g(t))\Big) - \varepsilon_2 cos(t\pi) = 0, \tag{4.4.19}$$

where $g : \mathbb{Z} \to \mathbb{Z}^+$ is scalar and 2-periodic. By letting $\triangle x_1 = x_2$ we can transform (4.4.19) to

$$\triangle\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} + \triangle\begin{pmatrix} 0 \\ \varepsilon_1 cos(\pi t)x_1^2(t-g(t)) \end{pmatrix} + \begin{pmatrix} 0 \\ \varepsilon_2 cos(\pi t) - \varepsilon_2 x_2 x_1^2 \end{pmatrix},$$

where

$$A = \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix}, \quad Q(t,x(t-g(t))) = \begin{pmatrix} 0 \\ \varepsilon_1 cos(\pi t) x_1^2(t-g(t)) \end{pmatrix}$$

and

$$G(t,x(t),x(t-g(t))) = \begin{pmatrix} 0 \\ \varepsilon_2 cos(\pi t) - \varepsilon_2 x_2 x_1^2 \end{pmatrix}.$$

Since the matrix $B = I + A$ has real eigenvalues, the system $x(t+1) = Bx(t)$ is noncritical. Let $\varphi(t) = (\varphi_1(t), \varphi_2(t)),\ \psi(t) = (\psi_1(t), \psi_2(t)) \in \mathbb{M} = \{\phi \in P_2 : ||\varphi|| \leq J\}$. Then,

$$\Big\| G(t,\varphi(t),\varphi(t-g(t))) - G(t,\psi(t),\psi(t-g(t))) \Big\|$$

$$\leq \varepsilon_2 \max_{t\in[0,1]} \Big| (\varphi_2(t)(\varphi_1(t)+\psi_1(t)), \psi_1^2(t)) \begin{pmatrix} \varphi_1(t)-\psi_1(t) \\ \varphi_2(t)-\psi_2(t) \end{pmatrix} \Big|$$

$$\leq 2\varepsilon_2 J^2 \|\varphi - \psi\|.$$

Hence, we see that $\beta = \varepsilon_2, E_2 = 2\varepsilon_2 J^2$, and $E_3 = 0$. In a similar fashion, we obtain $\alpha = 0$ and $E_1 = 2\varepsilon_1 J^2$. Thus, inequality (4.4.21)

$$2\varepsilon_1 J^2 + 2r\Big[2\varepsilon_1 J|A| + 2\varepsilon_2 J^2\Big] J + 2r\varepsilon_2 \leq J$$

is satisfied for small ε_1 and ε_2. Hence, equation (4.4.19) has a 2-periodic solution, by Theorem 4.4.1. On the other hand, the above inequality automatically implies that

$$2\varepsilon_1 J + 2r\Big[2\varepsilon_1 J|A| + 2\varepsilon_2 J^2\Big] < 1$$

for small ε_1 and ε_2, and hence equation (4.4.19) has a unique 2-periodic solution, by Corollary 4.5.

Next we make use of Schauder's fixed point theorem, Theorem 4.7.1, to show that Equation (4.4.1) has a T-periodic solution. This scenario could be encountered when one of the mappings is neither contraction nor compact. Thus, we assume that the function Q is uniformly continuous and bounded. That is there exists a positive constant W such that

$$|Q(t,x)| \leq W, \text{ for all } t \geq 0. \tag{4.4.20}$$

Theorem 4.4.3. *Let* $\beta = \max_{t\in\mathbb{Z}} |G(t,0,0)|$. *Let* r *be given by* (4.4.16). *Suppose* (4.4.3), (4.4.4), (4.4.15), *and* (4.4.20) *hold. Let* J *be a positive constant satisfying the inequality*

$$W + rT\Big[|A|W + E_2 + E_3\Big] J + rT\beta \leq J. \tag{4.4.21}$$

Then for $\mathbb{M} = \{\varphi \in P_T : ||\varphi|| \leq J\}$, *equation* (4.4.1) *has a solution in* M.

Proof. Define $\mathbb{M} = \{\varphi \in P_T : ||\varphi|| \leq J\}$. Let the map H be defined by (4.4.11). Then by similar argument, one can easily show that

$$H : \mathbb{M} \to \mathbb{M}.$$

In addition, using the Lebesgue dominated convergence theorem, one can easily show the map H is compact. For the complete argument we refer to Section 4.7.1. Thus, by Theorem 4.7.1, Equation (4.4.1) has a T-periodic solution.

4.5 Periodicity in Nonlinear Systems with Infinite Delay

As we have seen in the previous section that using Schaefer's fixed-point theorem (Theorem 4.1.3) enabled us to show that if there is an a priori bound on all possible T-periodic solutions of a related auxiliary Volterra difference equation, then there is a T-periodic solution. In this section we apply our results to scalar Volterra difference equations in which the a priori bound is established by means of nonnegative definite Lyapunov functionals. Thus, we consider

$$x(n+1) = Dx(n) + f(x(n)) + \sum_{j=-\infty}^{n} K(n,j)g(x(j)) + p(n), \tag{4.5.1}$$

with the existence of positive constant Q such that

$$sup_{n\in\mathbb{Z}} \sum_{j=-\infty}^{n} |K(n,j)| \leq Q,$$

where D is a $k \times k$ matrix and p is a given $k \times 1$ vector with $p(n+T) = p(n)$ for integer T. The kernel $K(n,j)$ satisfies $K(n+T, j+T) = K(n,j)$ for all $-\infty < j \leq n < \infty$, where $(n,j) \in \mathbb{Z}^2$ and $K(n,j) = 0$ for $j > n$. The period T is taken to be the least positive integer for which these hold. The functions f and g are continuous. Results of this section can be partially found in [137]. In [131] the author studied the existence of periodic solutions of the Volterra difference system with

$$\triangle x(n) = Dx(n) + \sum_{j=-\infty}^{n} C(n-j)x(j) + g(n), n \in \mathbb{Z} \text{ with } \sum_{u=0}^{\infty} |C(u)| < \infty \tag{4.5.2}$$

where D and C are $k \times k$ matrices and g is a given $k \times 1$ vector with $g(n+T) = g(n)$ for integer T, by using Schaefer's fixed point theorem. In [131] the mapping was constructed by taking direct sum in (4.5.2). On the other hand, Elaydi [52] considered (4.5.2) and utilized the notion of the resolvent of an equation associated with (4.5.2) and concluded the existence of a periodic solution of (4.5.2). In arriving at his results, Elaydi had to show that the zero solution of an homogenous equation associated with (4.5.2) is uniformly asymptotically stable . Thus, it was assumed that $|D| < 1$ where $|\cdot|$ is a suitable matrix norm. Later on, for the purpose of relax-

ing $|D| < 1$, Elaydi and Zhang [53] used the notion of degree theory, due to Grannas, and obtained the existence of a periodic solution of (4.5.2).
Once our results are established, we apply them to nonlinear Volterra discrete equations of the form

$$x(n+1) = ax(n) + f(x(n)) + \sum_{j=-\infty}^{n} K(n,j)g(x(j)) + p(n). \tag{4.5.3}$$

In [130] the author considered (4.5.3) with the assumptions that the two functions f and g are uniformly bounded and the coefficient a satisfies the stringent condition $-1 \le a \le 1$. Our objective is to relax those conditions. We achieve our objective by displaying nonnegative definite Lyapunov functionals, which in turn give the a priori bound. Thus, the results of this section will advance the theory of existence of periodic solutions in the most general form of nonlinear Volterra difference equations. For (4.5.1) a homotopy will have to be constructed which we obtain in the following manner.
Let m be a real number such that either $m > 1$ or $m < -1$. For $0 \le \lambda \le 1$, we rewrite (4.5.1) as

$$\begin{aligned} x(n+1) &= \lambda(-m^{-1}I + D)x(n) + m^{-1}x(n) + \lambda f(x(n)) \\ &\quad + \lambda \sum_{j=-\infty}^{n} K(n,j)g(x(j)) + \lambda p(n). \end{aligned} \tag{4.5.4}$$

One may easily verify that

$$\begin{aligned} x(n) &= \lambda \sum_{j=-\infty}^{n-1} m^{-(n-j-1)} \Big[(-m^{-1}I + D)x(j) + f(x(j))\Big] \\ &\quad + \lambda \sum_{s=-\infty}^{n-1} m^{-(n-s-1)} \sum_{j=-\infty}^{s} K(s,j)g(x(j)) \\ &\quad + \lambda \sum_{j=-\infty}^{n-1} p(j)m^{-(n-j-1)} \end{aligned} \tag{4.5.5}$$

is a solution of (4.5.4) and hence of (4.5.1). Define the space P_T by

$$P_T = \Big\{x(n) : x(n+T) = x(n), \text{ for all } n \in \mathbb{Z}\Big\}$$

where T is the least positive integer so that $x(n+T) = x(n)$. Then $\Big(P_T, |\cdot|\Big)$ defines a Banach space of T-periodic $k \times 1$ real vector sequences $x(n)$ with the maximum norm

$$|x| = \max_{i=1,\cdot,\cdot,\cdot,k} \Big\{ \max_{n \in [0,T-1]} |x_i(n)| \Big\}.$$

For $x(n) \in P_T$, using (4.5.5) we define the mapping $H : P_T \to P_T$ by

$$\begin{aligned}(Hx)(n) &= \lambda \sum_{j=-\infty}^{n-1} m^{-(n-j-1)}\Big[(-m^{-1}I+D)x(j)+f(x(j))\Big] \\ &+ \lambda \sum_{s=-\infty}^{n-1} m^{-(n-s-1)} \sum_{j=-\infty}^{s} K(s,j)g(x(j)) \\ &+ \lambda \sum_{j=-\infty}^{n-1} p(j)m^{-(n-j-1)}. \end{aligned} \tag{4.5.6}$$

Thus,

$$x = \lambda Hx$$

is equivalent to (4.5.5). Next we prove two Lemmas that are essential for the application of Schaefer's theorem (Theorem 4.1.3).

Lemma 4.6 ([137]). *If H is defined by* (4.5.6)*, then H is continuous and $H: P_T \to P_T$.*

Proof. For the continuity of H we let $\phi_1, \phi_2 \in P_T$ and use (4.5.6) to obtain,

$$\begin{aligned}\Big|(H\phi_1)-(H\phi_2)\Big| \le & \sum_{j=-\infty}^{n-1} |m^{-(n-j-1)}|\Big|(-m^{-1}I+D\Big||\phi_1-\phi_2| \\ &+ \sum_{j=-\infty}^{n-1} |m^{-(n-j-1)}||f(\phi_1)-f(\phi_2)| \\ &+ Q\sum_{s=-\infty}^{n-1} |m^{-(n-s-1)}||g(\phi_1)-g(\phi_2)|. \end{aligned}$$

By invoking the continuity of f and g and the fact that the infinite series $\sum_{j=-\infty}^{n-1}|m^{-(n-j-1)}|$ is convergent, we deduce that H is continuous. Left to show that $H: P_T \to P_T$. Let $\varphi(n) \in P_T$ and use the substitution $v = j-T$ followed by the substitution $r = s-T$ to obtain $(H\varphi)(n+T) = (H\varphi)(n)$. This concludes the proof of the lemma.

Lemma 4.7 ([137]). *If H is defined by* (4.5.6)*, then H maps bounded subsets into compact subsets.*

Proof. Let $J > 0$ be given and define the two sets $S = \big\{x(n) \in P_T : |x| \le J\big\}$ and $W = \big\{(Hx)(n) : x(n) \in P_T\big\}$. Then W is a subset of $\mathbb{R}^{Tk}$, which is closed and bounded and thus compact. As H is continuous in x it maps compact sets into compact sets. We deduce that $W = H(S)$ is compact. This concludes the proof of the Lemma.

Now we are in a position to state and prove our main theorem that yields the existence of a periodic solution of (4.5.1).

Theorem 4.5.1. *If there exists an $L > 0$ such that for any T-periodic solution of* (4.5.4)*, $0 < \lambda < 1$ satisfies $|x| \le L$, then* (4.5.1) *has a solution in P_T.*

Proof. Let H be defined by (4.5.6). Then, by Lemmas 4.6 and 4.7, H is continuous, compact, and T-periodic. The hypothesis $|x| \leq L$ rules out part *(ii)* of Theorem 4.1.3 and thus $x = \lambda Hx$ has a solution for $\lambda = 1$, which solves (4.5.1). This concludes the proof.

Remark 4.3. When it comes to application, the reader shall see that we may have to require $m \in (-1,0) \cup (0,1)$. Thus, to take care of such situation we note that Equation (4.5.1) is equivalent for $\lambda = 1$ to

$$x(n+1) = \lambda(-mI+D)x(n) + mx(n) + \lambda f(x(n)) + \lambda \sum_{j=-\infty}^{n} K(n,j)g(x(j)) + \lambda p(n). \tag{4.5.7}$$

Then it follows readily that x is a bounded solution of (4.5.7) if and only if

$$\begin{aligned} x(n) = {} & \lambda \sum_{j=-\infty}^{n-1} m^{-(j-n+1)} \Big[(-mI+D)x(j) + f(x(j))\Big] \\ & + \lambda \sum_{s=-\infty}^{n-1} m^{-(s-n+1)} \sum_{j=-\infty}^{s} K(s,j)g(x(j)) \\ & + \lambda \sum_{s=-\infty}^{n-1} p(j)m^{-(j-n+1)}. \end{aligned} \tag{4.5.8}$$

Then one may easily prove a theorem similar to Theorem 4.5.1 for the case $m \in (-1,0) \cup (0,1)$.

4.5.1 Application to Infinite Delay Volterra Equations

Now we apply the results of the previous section to scalar nonlinear Volterra difference equations with of the form

$$x(n+1) = ax(n) + f(x(n)) + \sum_{j=-\infty}^{n} K(n,j)g(x(j)) + p(n), \tag{4.5.9}$$

where the terms f, g, K, and p obey the same conditions as before. The highlight of this work is to prove the existence of periodic solution of Equation (4.5.9) where the magnitude of a could be $|a| > 1$. In most of the literature, it is required that $|a| < 1$. To relax this condition we resort to nonnegative definite Lyapunov functional to obtain the a priori bound on all possible T-periodic solutions of Equation (4.5.9) and then conclude the existence of a periodic solution by invoking Theorem 4.5.1. We shall assume in addition to those assumptions made in the previous section that there exists $F : \mathbb{Z}^+ \to \mathbb{R}$ and $R > 0$ such that

$$|K(n,u+n)| \le F(u), \text{ with } \sum_{u=0}^{\infty} |F(u)| \le R, \tag{4.5.10}$$

and

$$\max_{n\in\mathbb{Z}} \sum_{j=-\infty}^{n-1} \sum_{s=n}^{\infty} |K(s,j)| < \infty. \tag{4.5.11}$$

We note that assumption (4.5.10) implies that

$$\max_{n\in\mathbb{Z}} \sum_{s=n}^{\infty} |K(s,j)| \le R.$$

Now we state two theorems; one will show the existence of a periodic solution of (4.5.9) when $|a| < 1$, and the other when $|a| > 1$. The proof of the first theorem will be established in three different cases on the coefficient a.

Theorem 4.5.2 ([137]). *Assume* (4.5.10) *and* (4.5.11). *Also, we assume that there exists an* $\alpha > 0$ *such that*

$$|f(x)| + R|g(x)| \le \alpha |x|,$$

and

$$|\mu| + \alpha - 1 \le -\beta, \textit{ for some positive constant } \beta, \tag{4.5.12}$$

where μ *is to be defined in the body of the proof and* R *is given by* (4.5.10). *Then, Equation* (4.5.9) *has a* T*-periodic solution.*

Proof. **Case 1.** $0 < a < 1$
Set $m = a$. Then $0 < m < 1$. We shall apply Theorem 4.5.1 with $m \in (0,1)$ to the corresponding family of equations

$$\begin{aligned} x(n+1) &= \lambda(-m+a)x(n) + mx(n) + \lambda f(x(n)) \\ &\quad + \lambda \sum_{j=-\infty}^{n} K(n,j)g(x(j)) + \lambda p(n). \end{aligned} \tag{4.5.13}$$

Our aim is to show that there is a priori bound, say L such that all solutions $x(n)$ of

$$\begin{aligned} x(n) &= \lambda \sum_{j=-\infty}^{n-1} m^{-(j-n+1)} \Big[(-m+a)x(j) + f(x(j))\Big] \\ &\quad + \lambda \sum_{s=-\infty}^{n-1} m^{-(s-n+1)} \sum_{j=-\infty}^{s} K(s,j)g(x(j)) + \lambda \sum_{s=-\infty}^{n-1} p(j)m^{-(j-n+1)} \end{aligned}$$

for $0 < \lambda < 1$ satisfies $|x| \le L$. Once this is accomplished then we can rule out *(ii)* of Schaefer's theorem (Theorem 4.1.3), and then conclude the above equation has a solution for $\lambda = 1$.

We begin by rewriting (4.5.13) in the form

$$x(n+1) = \mu x(n) + \lambda f(x(n)) + \lambda \sum_{j=-\infty}^{n} K(n,j)g(x(j)) + \lambda p(n), \tag{4.5.14}$$

where $\mu = m + \lambda(-m+a)$. Define the Lyapunov functional V by

$$V(n,x(\cdot)) = |x(n)| + \lambda \sum_{j=-\infty}^{n-1} \sum_{s=n}^{\infty} |K(s,j)||g(x(j))|. \tag{4.5.15}$$

It is clear that for $x(n) \in P_T$, $V(n+T,x) = V(n,x)$ and hence V is periodic. Along the solutions of (4.5.14) we have

$$\begin{aligned}
\triangle V(n,x(\cdot)) &= |x(n+1)| - |x(n)| + \lambda \sum_{s=n+1}^{\infty} |K(s,n)||g(x(n))| \\
&\quad - \lambda \sum_{j=-\infty}^{n-1} |K(n,j)||g(x(j))| \\
&\le \Big(|\mu| - 1\Big)|x(n)| + \lambda |f(x)| + \lambda \sum_{s=n}^{\infty} |K(s,n)||g(x(n))| + |p| \\
&\le \Big(|\mu| - 1\Big)|x(n)| + |f(x)| + R|g(x)| + |p| \\
&\le \Big(|\mu| + \alpha - 1\Big)|x(n)| + |p| \\
&\le -\beta\, |x(n)| + |p|.
\end{aligned}$$

Since V is periodic for $x \in P_T$, we have by summing the above inequality over one period that

$$\begin{aligned}
0 = V(n+T,x(\cdot)) - V(n,x(\cdot)) &= \sum_{s=n}^{n+T-1} \triangle V(s,x(\cdot)) \\
&\le -\beta \sum_{s=n}^{n+T-1} |x(s)| + T\,|p|.
\end{aligned}$$

This implies that

$$\sum_{s=n}^{n+T-1} |x(s)| \le \frac{T\,|p|}{\beta}.$$

Thus, $|x(n)|$ is bounded over one period, and hence for any T-periodic solution of (4.5.13) there is an $E > 0$ such that $|x(n)| \le E$, which serves as the a priori bound on every possible T-periodic solution of (4.5.13). Therefore, by Theorem 4.5.1 Equation (4.5.9) has a T-periodic solution for $0 < a < 1$. This concludes the proof of Case 1.

Note that since $0 < \lambda < 1$ condition (4.5.12) reduces to $|a| + \alpha - 1 \le -\beta$.

Case 2. $-1 < a < 0$
Set $m = a$. Then $-1 < m < 0$ and we apply Theorem 4.5.1 with $m \in (-1,0)$ to the corresponding family of equations (4.5.13) with $\mu = m + \lambda(-m+a) = a$. Define the Lyapunov functional V by (4.5.15) and proceed with the proof as in Case 1. Note that since $0 < \lambda < 1$, and $\mu = a$, condition (4.5.12) reduces to $|a| + \alpha - 1 \leq -\beta$.
Case 3. $a = 0$
Let m be any fixed number strictly between 0 and 1. Then, $\mu = m - \lambda m < m$. Choose m small enough so that (4.5.12) is satisfied. Then apply Theorem 4.5.1 with $m \in (0,1)$ to the corresponding family of equations (4.5.13). Define the Lyapunov functional V by (4.5.15) and proceed with the proof as in Case 1.

The next theorem handles the case $|a| > 1$.

Theorem 4.5.3 ([137]). *Assume* (4.5.10) *and* (4.5.11)*. Also, we assume that there exists an* $\alpha > 0$ *such that*

$$|f(x)| + R|g(x)| \leq \alpha |x|,$$

and

$$|\mu| - \alpha - 1 \geq \beta, \textit{ for some positive constant } \beta,$$

where μ *is to be defined in the body of the proof. Then, Equation* (4.5.9) *has a* T*-periodic solution.*

Proof. **Case 1.** $a > 1$
Set $m = a$. We shall apply Theorem 4.5.1 with $m > 1$ to the corresponding family of equations (4.5.13). Then, $\mu = m + \lambda(-m+a) = a$.
Define the Lyapunov functional V by

$$V(n,x(\cdot)) = |x(n) - \lambda \sum_{j=-\infty}^{n-1} \sum_{s=n}^{\infty} |K(s,j)||g(x(j))|. \tag{4.5.16}$$

It is clear that for $x(n) \in P_T$, then $V(n+T,x) = V(n,x)$ and hence V is periodic. Along the solutions of (4.5.14) we have

$$\begin{aligned} \triangle V(n,x(\cdot)) = {} & |x(n+1)| - |x(n)| - \lambda \sum_{s=n}^{\infty} |K(s,n)||g(x(n))| \\ & + \lambda \sum_{j=-\infty}^{n-1} |K(n,j)||g(x(j))| \\ & \geq \Big(|\mu| - 1\Big)|x(n)| - (|f(x)| + R|g(x)|) - |p| \\ & \geq \Big(|\mu| + \alpha - 1\Big)|x(n)| - |p| \\ & \geq \beta\, |x(n)| - |p|. \end{aligned}$$

Since V is periodic for $x \in P_T$, we have by summing the above inequality over one period that

$$0 = V(n+T, x(\cdot)) - V(n, x(\cdot)) = \sum_{s=n}^{n+T-1} \triangle V(s, x(\cdot))$$
$$\geq \beta \sum_{s=n}^{n+T-1} |x(s)| - Tp.$$

This implies that

$$\sum_{s=n}^{n+T-1} |x(s)| \leq \frac{T\,p}{\beta}.$$

Thus, $|x(n)|$ is bounded over one period, and hence for any T-periodic solution of (4.5.13) there is an $E > 0$ such that $|x(n)| \leq E$, which serves as the a priori bound on every possible T-periodic solution of (4.5.13). Therefore, by Theorem 4.5.1 Equation (4.5.9) has a T-periodic solution for $a > 1$. This concludes the proof of Case 1.

Again, we remark that the condition $|\mu| - \alpha - 1 \geq \beta$, for some positive constant β, reduces to $|a| - \alpha - 1 \geq \beta$.

Case 2. $a < -1$

Set $m = a$. Then $m < -1$ and we apply Theorem 4.5.1 to the corresponding family of equations (4.5.13) with $\mu = m + \lambda(-m + a) = a$. Thus, $|\mu| = |a|$. Define the Lyapunov functional V by (4.5.16) and then the proof is the same as in Case 2. This concludes the proof of the theorem.

Remark 4.4. 1) By relaxing the condition $|a| < 1$, we point out that Theorem 4.5.3 significantly improves the literature that is related to the existence of periodic solutions in Volterra difference equations.

2) In [130], for $|a| = 1$, the author was able to show the existence of a periodic solution under the stringent condition that the functions f and g are uniform bounded by certain positive constants. However, we could not do the same here under the condition

$$|f(x)| + R|g(x)| \leq \alpha |x|.$$

4.6 Functional Equations with Constant or Periodically Constant Solutions

Consider the difference equation

$$\triangle x(t) = x(t) - x(t-L), \tag{4.6.1}$$

then any constant is a solution of (4.6.1). In this case we ask ourselves if the constant solution is pre-determined. Therefore, it is convenient to generalize the concept and look at variant forms of the general functional difference equation

$$\triangle x(t) = g(x(t)) - g(x(t-L)), \tag{4.6.2}$$

where $g : \mathbb{R} \to \mathbb{R}$ and is continuous in x. Eqn.(4.6.2) can be easily generalized to functional equations of the form

$$\triangle x(t) = g(x(t-L_1)) - g(x(t-L_1-L_2)), \tag{4.6.3}$$

$$\triangle x(t) = g(x(t)) - \sum_{s=t-L}^{t-1} p(s-t)g(x(s)). \tag{4.6.4}$$

$$\triangle x(t) = \sum_{s=t-L}^{t-1} p(s-t)g(x(s)) - \sum_{s=-\infty}^{t-1} q(s-t)g(x(s)). \tag{4.6.5}$$

Results of this section are partially published in [127] and [139]. In [139] Raffoul, studied systems (4.6.2) and (4.6.3) along with

$$\triangle x(t) = g(t,x(t)) - g(t,x(t-L)), \;\; g(t+L,x) = g(t,x). \tag{4.6.6}$$

The first term on the right takes into account the ideas of (4.6.4) while the second term takes into account the deaths distributed over all past times. Note that if $x = c$ where c is constant, then $\triangle x(t) = 0$ in (4.6.2)–(4.6.5) provided that

$$\sum_{s=-L}^{-1} p(s) = 1, \text{ and } \sum_{s=-\infty}^{-1} q(s) = 1.$$

4.6.1 The Finite Delay System

By means of fixed point theory we show that the unique solution of (4.6.4) converges to a pre-determined constant or a periodic solution. Then, we show the solution is stable and that its limit function serves as a global attractor. The same theory will be extended to two more models. We will use the contraction mapping principle to determine that constant. First, we state what it means for $x(t)$ to be a solution of (4.6.4). Note that since (4.6.4) is autonomous, we lose nothing by starting the solution at 0.
Let $\psi(t) : [-L,0] \to \mathbb{R}$ be a given bounded initial function. We say $x(t,0,\psi)$ is a solution of (4.6.4) if $x(t,0,\psi) = \psi(t)$ on $[-L,0]$ and $x(t,0,\psi)$ satisfies (4.6.4) for $t \geq 0$.
It is of importance to us to know such constants since all of our models have constant solutions. First we rewrite (4.6.4) as

$$\triangle x(t) = \triangle_t \sum_{s=-L}^{-1} p(s) \sum_{u=t+s}^{t-1} g(x(u)), \tag{4.6.7}$$

where $p(s)$ satisfies the condition

$$\sum_{s=-L}^{-1} p(s) = 1. \tag{4.6.8}$$

Also, we assume that the function g is globally Lipschitz. That is, there exists a constant $k > 0$ such that

$$|g(x) - g(y)| \leq k|x - y|. \tag{4.6.9}$$

On the other hand, in order to obtain contraction, we assume there is a positive constant $\xi < 1$ so that

$$k \sum_{s=-L}^{-1} |p(s)|(-s) \leq \xi. \tag{4.6.10}$$

We note that if $p(t) = \dfrac{1}{L}$, then (4.6.8) is satisfied. Moreover, in this case condition (4.6.10) becomes

$$k \sum_{s=-L}^{-1} |p(s)|(-s) = k \sum_{s=-L}^{-1} \frac{1}{L}(-s) = \frac{k(L+1)}{2}.$$

Thus, condition (4.6.10) is satisfied for

$$\frac{k(L+1)}{2} \leq \xi.$$

To construct a suitable mapping, we let $\psi : [-L, 0] \to \mathbb{R}$ be a given initial function. By summing (4.6.7) from $s = 0$ to $s = t-1$ we arrive at the expression

$$x(t) = \psi(0) - \sum_{s=-L}^{-1} p(s) \sum_{u=s}^{-1} g(\psi(u)) + \sum_{s=-L}^{-1} p(s) \sum_{u=t+s}^{t-1} g(x(u)). \tag{4.6.11}$$

If $x(t)$ is given by (4.6.11), then it solves (4.6.4). In the next theorem we show that, given an initial function $\psi : [-L, 0] \to \mathbb{R}$, the unique solution of (4.6.4) converges to a unique determined constant.

Theorem 4.6.1 ([127]). *Assume* (4.6.8)–(4.6.10) *and let* $\psi : [-L, 0] \to \mathbb{R}$ *be a given initial function. Then, the unique solution* $x(t, 0, \psi)$ *of* (4.6.4) *satisfies* $x(t, 0, \psi) \to r$, *where* r *is unique and given by*

$$r = \psi(0) + g(r) \sum_{s=-L}^{-1} p(s)(-s) - \sum_{s=-L}^{-1} p(s) \sum_{u=s}^{-1} g(\psi(u)). \tag{4.6.12}$$

Proof. For $|\cdot|$ denoting the absolute value, the metric space $(\mathbb{R}, |\cdot|)$ is complete. Define a mapping $\mathscr{H} : \mathbb{R} \to \mathbb{R}$, by

$$\mathscr{H} r = \psi(0) + g(r) \sum_{s=-L}^{-1} p(s)(-s) - \sum_{s=-L}^{-1} p(s) \sum_{u=s}^{-1} g(\psi(u)).$$

For $a, b \in \mathbb{R}$, we have

$$\left|\mathcal{H}a-\mathcal{H}b\right| \le \sum_{s=-L}^{-1}|p(s)|(-s)|g(a)-g(b)| \le k\sum_{s=-L}^{-1}|p(s)|(-s)|a-b| \le \xi|a-b|.$$

This shows that $\mathcal{H}$ is a contraction on the complete metric space $(\mathbb{R}, |\cdot|)$, and hence $\mathcal{H}$ has a unique fixed point r, which implies that (4.6.12) has a unique solution. It remains to show that (4.6.4) has a unique solution and that it converges to the constant r.

Let $||\cdot||$ denote the maximum norm and let $\mathbb{M}$ be the set bounded functions $\phi : [-L,\infty) \to \mathbb{R}$ with $\phi(t) = \psi(t)$ on $[-L,0]$, $\phi(t) \to r$ as $t \to \infty$. Then $(\mathbb{M}, ||\cdot||)$ defines a complete metric space. For $\phi \in \mathbb{M}$, define $\mathcal{P} : \mathbb{M} \to \mathbb{M}$ by

$$(\mathcal{P}\phi)(t) = \psi(t), \text{ for } -L \le t \le 0,$$

and

$$(\mathcal{P}\phi)(t) = \psi(0) - \sum_{s=-L}^{-1}p(s)\sum_{u=s}^{-1}g(\psi(u)) + \sum_{s=-L}^{-1}p(s)\sum_{u=t+s}^{t-1}g(\phi(u)), \text{ for } t \ge 0. \tag{4.6.13}$$

For $\phi \in \mathbb{M}$ with $\phi(t) \to r$, we have

$$\sum_{s=-L}^{-1}p(s)\sum_{u=t+s}^{t-1}g(\phi(u)) \to g(r)\sum_{s=-L}^{-1}p(s)(-s), \text{ as } t \to \infty.$$

Then, using (4.6.12) and (4.6.13), we see that

$$(\mathcal{P}\phi)(t) \to \psi(0) + g(r)\sum_{s=-L}^{-1}p(s)(-s) - \sum_{s=-L}^{-1}p(s)\sum_{u=s}^{-1}g(\psi(u)) = r.$$

Thus, $\mathcal{P} : \mathbb{M} \to \mathbb{M}$. It remains to show that $\mathcal{P}$ is a contraction.
For $a, b \in \mathbb{M}$, we have

$$\begin{aligned}\left|(\mathcal{P}a)(t)-(\mathcal{P}b)(t)\right| &\le \sum_{s=-L}^{-1}|p(s)|(-s)|g(a(s))-g(b(s))| \\ &\le k\sum_{s=-L}^{-1}|p(s)|(-s)\,|a-b| \le \xi||a-b||.\end{aligned}$$

Thus, $\mathcal{P}$ is a contraction and has a unique fixed point $\phi \in \mathbb{M}$. Based on how the mapping $\mathcal{P}$ was constructed, we conclude the unique fixed point ϕ satisfies (4.6.4).

Remark 4.5. For any given initial function, Theorem 4.6.1 explicitly gives the limit to which the solution converges to. That limit is the unique solution r of (4.6.12).

Remark 4.6. For arbitrary initial function, say $\eta : [-L,0] \to \mathbb{R}$, Theorem 4.6.1 shows that $x(t,0,\eta) \to r$. Thus, we may think of r as being "global attractor."

Remark 4.7. We may think of Theorem 4.6.1 as of stability results. In general, we know that solutions depend on initial functions. That is, solutions which start close remain close on finite intervals. Under conditions Theorem 4.6.1 such solutions remain close forever, and their asymptotic respective constants remain close too.

The next theorem is a verification of our claim in Remark 4.7.

Theorem 4.6.2 ([127]). *Assume the hypothesis of Theorem 4.6.1. Then every initial function is stable. Moreover, if ψ_1 and ψ_2 are two initial functions with $x(t,0,\psi_1) \to r_1$, and $x(t,0,\psi_2) \to r_2$, then $|r_1 - r_2| < \varepsilon$ for positive ε.*

Proof. Let $||\psi||_{[-L,0]}$ denote the supremum norm of ψ on the interval $[-L,0]$. Fix an initial function ψ_1 and let ψ_2 be any other initial function. Let $\mathscr{P}_i, i = 1,2$ be the mapping defined by (4.6.13). Then by Theorem 4.6.1 there are unique functions θ_1, θ_2 and unique constants r_1 and r_2 such that

$$\mathscr{P}_1\theta_1 \to \theta_1,\ \ \mathscr{P}_2\theta_2 \to \theta_2,\ \ \theta_1(t) \to r_1,\ \ \theta_2(t) \to r_2.$$

Let $\varepsilon > 0$ be any given positive number and set $\delta = \dfrac{\varepsilon\big(1 - k\sum_{s=-L}^{-1}|p(s)|(-s)\big)}{1 + k\sum_{s=-L}^{-1}|p(s)|(-s)}$.

Then

$$\begin{aligned}
|\theta_1(t) - \theta_2(t)| &= |(\mathscr{P}_1\theta_1)(t) - (\mathscr{P}_2\theta_2)(t)| \\
&\le |\psi_1(0) - \psi_2(0)| + \sum_{s=-L}^{-1} p(s) \sum_{u=s}^{-1} |g\big(\psi_1(s)\big) - g\big(\psi_2(s)\big)| \\
&+ \sum_{s=-L}^{-1} p(s) \sum_{u=t+s}^{t-1} |g\big(\theta_1(s)\big) - g\big(\theta_2(s)\big)| \\
&\le |\psi_1(0) - \psi_2(0)| + k \sum_{s=-L}^{-1} |p(s)|(-s)||\psi_1 - \psi_2||_{[-L,0]} \\
&+ k \sum_{s=-L}^{-1} |p(s)|(-s)|||\theta_1 - \theta_2||.
\end{aligned}$$

This yields

$$||\theta_1 - \theta_2|| < \frac{1 + k\sum_{s=-L}^{-1}|p(s)|(-s)}{1 - k\sum_{s=-L}^{-1}|p(s)|(-s)}||\psi_1 - \psi_2||_{[-L,0]} < \varepsilon,$$

provided that

$$||\psi_1 - \psi_2||_{[-L,0]} < \frac{\varepsilon\big(1 - k\sum_{s=-L}^{-1}|p(s)|(-s)\big)}{1 + k\sum_{s=-L}^{-1}|p(s)|(-s)} := \delta.$$

This shows that

$$|x(t,0,\psi_1) - x(t,0,\psi_2)| < \varepsilon, \text{ whenever } ||\psi_1 - \psi_2||_{[-L,0]} < \delta.$$

For the rest of the proof we note that $|\theta_i(t) - k_i| \to 0$, as $t \to \infty$ implies that

$$\begin{aligned}|r_1 - r_2| &= |r_1 - \theta_1(t) + \theta_1(t) - \theta_2(t) + \theta_2(t) - r_2| \\ &\le |r_1 - \theta_1(t)| + ||\theta_1 - \theta_2|| + |\theta_2(t) - r_2| \to ||\theta_1 - \theta_2||, \text{ (as } t \to \infty) \\ &< \varepsilon.\end{aligned}$$

4.6.2 The Infinite Delay System

In this section, we consider the infinite delay system . For completeness we restate the infinite delay system

$$\triangle x(t) = \sum_{s=t-L}^{t-1} p(s-t)g(x(s)) - \sum_{s=-\infty}^{t-1} q(s-t)g(x(s)) \tag{4.6.14}$$

and rewrite it as

$$\triangle x(t) = -\triangle_t \sum_{s=-L}^{-1} p(s) \sum_{u=t+s}^{t-1} g(x(u)) + \triangle_t \sum_{s=-\infty}^{t-1} \sum_{u=-\infty}^{s-t} q(u)g(x(s)), \tag{4.6.15}$$

where we have assumed (4.6.8) and

$$\sum_{s=-\infty}^{-1} q(s) = 1. \tag{4.6.16}$$

Let $\psi : (-\infty, 0] \to \mathbb{R}$ be an initial bounded sequence. Then

$$x(t) = -\sum_{s=-L}^{-1} p(s) \sum_{u=t+s}^{t-1} g(x(u)) + \sum_{s=-\infty}^{t-1} \sum_{u=-\infty}^{s-t} q(u)g(x(s)) + c, \tag{4.6.17}$$

where

$$c = \psi(0) + \sum_{s=-L}^{-1} p(s) \sum_{u=s}^{-1} g(x(u)) - \sum_{s=-\infty}^{-1} \sum_{u=-\infty}^{s} q(u)g(\psi(s)) \tag{4.6.18}$$

is a solution of (4.6.14). We have the following theorem.

Theorem 4.6.3 ([127]). *Assume* (4.6.8)*,* (4.6.9)*, and* (4.6.16) *and there exists a constant* α *so that for* $0 < \alpha < 1$*, we have*

$$k\Big(\sum_{s=-L}^{-1}|p(s)(-s)|+\sum_{s=-\infty}^{-1}\sum_{u=-\infty}^{s}|q(u)|\Big)\le\alpha. \tag{4.6.19}$$

Then, the unique solution $x(t,0,\psi)$ of (4.6.14) satisfies $x(t,0,\psi)\to r$, where r is unique and given by

$$r=c-g(r)\sum_{s=-L}^{-1}p(s)(-s)+g(r)\sum_{s=-\infty}^{-1}\sum_{u=-\infty}^{s}q(u), \tag{4.6.20}$$

and c is given by (4.6.18).

Proof. For $|\cdot|$ denoting the absolute value, the metric space $(\mathbb{R},|\cdot|)$ is complete. Define a mapping $\mathscr{H}:\mathbb{R}\to\mathbb{R}$, by

$$\mathscr{H}r=c-g(r)\sum_{s=-L}^{-1}p(s)(-s)+g(r)\sum_{s=-\infty}^{-1}\sum_{u=-\infty}^{s}q(u).$$

For $a,b\in\mathbb{R}$, we have

$$\begin{aligned}\big|\mathscr{H}a-\mathscr{H}b\big| &\le \sum_{s=-L}^{-1}|p(s)(-s)||g(a)-g(b)|+|g(a)-g(b)|\sum_{s=-\infty}^{-1}\sum_{u=-\infty}^{s}|q(u)|\\ &\le k\Big(\sum_{s=-L}^{-1}|p(s)(-s)|+\sum_{s=-\infty}^{-1}\sum_{u=-\infty}^{s}|q(u)|\Big)|a-b|\\ &\le \alpha|a-b|.\end{aligned}$$

This shows that $\mathscr{H}$ is a contraction on the complete metric space $(\mathbb{R},|\cdot|)$, and hence $\mathscr{H}$ has a unique fixed point r, which implies that (4.6.20) has a unique solution. It remains to show that (4.6.14) has a unique solution and that it converges to the constant r.

Let $||\cdot||$ denote the maximum norm and let $\mathbb{M}$ be the set bounded functions $\phi:[-\infty,\infty)\to\mathbb{R}$ with $\phi(t)=\psi(t)$ on $[-\infty,0],\phi(t)\to r$ as $t\to\infty$. Then $(\mathbb{M},||\cdot||)$ defines a complete metric space. For $\phi\in\mathbb{M}$, define $\mathscr{P}:\mathbb{M}\to\mathbb{M}$ by

$$(\mathscr{P}\phi)(t)=\psi(t),\ \text{for}\ t\in(-\infty,0],$$

and

$$(\mathscr{P}\phi)(t)=c-\sum_{s=-L}^{-1}p(s)\sum_{u=t+s}^{t-1}g(\phi(u))+\sum_{s=-\infty}^{t-1}\sum_{u=-\infty}^{s-t}q(u)g(\phi(s)),\ \text{for}\ t\ge 0 \tag{4.6.21}$$

where c is given by (4.6.18). Due to the continuity of g we have that for $\phi\in\mathbb{M}$ with $\phi(t)\to r$,

$$\sum_{s=-L}^{-1}p(s)\sum_{u=t+s}^{t-1}g(\phi(u))\to g(r)\sum_{s=-L}^{-1}p(s)(-s),\ \text{as}\ t\to\infty.$$

Next we show that

$$\sum_{s=-\infty}^{t-1}\sum_{u=-\infty}^{s-t} q(u)g(\phi(s)) \to g(r)\sum_{s=-\infty}^{t-1}\sum_{u=-\infty}^{s-t} q(u), \text{ as } t\to\infty. \tag{4.6.22}$$

Again, due to the continuity of G, for $\phi \in \mathbb{M}$ with $\phi(t) \to r$, one might find positive numbers Q and T such that for any $\varepsilon > 0$ we have

$$|g(\phi(t)) - g(r)| \le Q \text{ for all t and } |\phi(t) - r| < \varepsilon \text{ if } T \le t < \infty.$$

With this in mind, we have

$$\begin{aligned}
\Big|\sum_{s=-\infty}^{t-1}\sum_{u=-\infty}^{s-t} q(u)\big(g(\phi(s)) - g(r)\big)\Big| &\le \sum_{s=-\infty}^{T-1}\sum_{u=-\infty}^{s-t} |q(u)|\Big|\big(g(\phi(s)) - g(r)\big)\Big| \\
&+ \sum_{s=T}^{t-1}\sum_{u=-\infty}^{s-t} |q(u)|\Big|\big(g(\phi(s)) - g(r)\big)\Big| \\
&\le Q\sum_{s=-\infty}^{T-1}\sum_{u=-\infty}^{s-t} |q(u)| + \sum_{s=T}^{t-1}\sum_{u=-\infty}^{s-t} |q(u)||\phi(s) - r| \\
&\le Q\sum_{s=-\infty}^{T-1}\sum_{u=-\infty}^{s-t} |q(u)| + k\varepsilon\sum_{s=T}^{t-1}\sum_{u=-\infty}^{s-t} |q(u)| \\
&\le Q\sum_{s=-\infty}^{T-t-1}\sum_{u=-\infty}^{s} |q(u)| + k\varepsilon\sum_{s=-\infty}^{t-1}\sum_{u=-\infty}^{s-t} |q(u)|.
\end{aligned}$$

Due to the convergence that was assumed in (4.6.19), we have $\sum_{s=-\infty}^{T-t-1}\sum_{u=-\infty}^{s} |q(u)| \to 0$, as $t \to \infty$. Moreover, for $T \le t < \infty$, condition (4.6.19) implies that $k\varepsilon\sum_{s=-\infty}^{t-1}\sum_{u=-\infty}^{s-t} |q(u)| \le \varepsilon\alpha$. Hence (4.6.22) is proved. It remains to show that $\mathscr{P}$ is a contraction. For $a, b \in \mathbb{M}$, we have

$$\begin{aligned}
\big|(\mathscr{P}a)(t) - (\mathscr{P}b)(t)\big| &\le \sum_{s=-L}^{-1} |p(s)|(-s)|g(a(s)) - g(b(s))| \\
&+ \sum_{s=-\infty}^{t-1}\sum_{u=-\infty}^{s-t} |q(u)||g(a(s)) - g(b(s))| \\
&\le k\Big(\sum_{s=-L}^{-1} |p(s)(-s)| + \sum_{s=-\infty}^{t-1}\sum_{u=-\infty}^{s-t} |q(u)|\Big)||a-b|| \\
&\le \alpha||a-b||.
\end{aligned}$$

Parallel remarks to Remarks 4.5–4.7 can be made regarding the infinite delay model given by (4.6.14).

4.6.3 The Finite Delay System Revisited

We revisit the finite delay system given by (4.6.4) with slight adjustment, namely

$$\triangle x(t) = g(t,x(t)) - \sum_{s=t-L}^{t-1} p(s-t)g(s,x(s)), \tag{4.6.23}$$

where

$$g(t+L,x) = g(t,x) \tag{4.6.24}$$

and investigate the existence of periodic solutions. As before, we assume there exists a positive constant k such that for all $x,y \in \mathbb{R}$ we have

$$|g(t,x) - g(t,y)| \leq k|x-y|. \tag{4.6.25}$$

If (4.6.8) holds, then we may rewrite (4.6.23) as

$$\triangle x(t) = \triangle_t \sum_{s=-L}^{-1} p(s) \sum_{u=t+s}^{t-1} g(u,x(u)). \tag{4.6.26}$$

As before, to construct a suitable mapping, we let $\psi : [-L,0] \to \mathbb{R}$ be a given initial function. By summing (4.6.26) from $s=0$ to $s=t-1$ we arrive at the expression

$$x(t) = \sum_{s=-L}^{-1} p(s) \sum_{u=t+s}^{t-1} g(u,x(u)) + c, \tag{4.6.27}$$

where c is given by

$$c = \psi(0) - \sum_{s=-L}^{-1} p(s) \sum_{u=s}^{-1} g(u,\psi(u)). \tag{4.6.28}$$

Theorem 4.6.4 ([127]). *Assume* (4.6.8)–(4.6.10)*,* (4.6.24)*, and* (4.6.25) *and let* $\psi : [-L,0] \to \mathbb{R}$ *be a given initial function. Then, the unique solution* $x(t,0,\psi)$ *of* (4.6.23) *satisfies* $x(t,0,\psi) \to \rho$*, as* $t \to \infty$ *where* ρ *is a unique L-periodic solution of* (4.6.23)*.*

Proof. Let $||\cdot||$ denote the maximum norm and let $\mathbb{M}$ be the set of L-periodic sequences $\phi : \mathbb{Z} \to \mathbb{Z}$. Then $(\mathbb{M}, ||\cdot||)$ defines a Banach space of L-periodic sequences. For $\phi \in \mathbb{M}$, define $\mathscr{P} : \mathbb{M} \to \mathbb{M}$ by

$$(\mathscr{P}\phi)(t) = c + \sum_{s=-L}^{-1} p(s) \sum_{u=t+s}^{t-1} g(u,\phi(u)) \tag{4.6.29}$$

Next we show that

$$(\mathscr{P}\phi)(t+L) = (\mathscr{P}\phi)(t).$$

To see, for $\phi \in \mathbb{M}$, we have

$$\begin{aligned}(\mathscr{P}\phi)(t+L) &= c + \sum_{s=-L}^{-1} p(s) \sum_{u=t+s+L}^{t+L-1} g(u,\phi(u)) \\ &= c + \sum_{s=-L}^{-1} p(s) \sum_{l=t+s}^{t-1} g(l+L,\phi(l+L)),\ (l = u-L) \\ &= c + \sum_{s=-L}^{-1} p(s) \sum_{l=t+s}^{t-1} g(l,\phi(l)) = (\mathscr{P}\phi)(t).\end{aligned}$$

Hence, $\mathscr{P}$ maps $\mathbb{M}$ into $\mathbb{M}$. Also, by similar argument as in the previous theorems, one can easily show that $\mathscr{P}$ is a contraction. Hence, (4.6.29) has a unique fixed point ρ in $\mathbb{M}$, which solves (4.6.23). It remains to show that $(\mathscr{P}\phi)(t) \to \rho(t)$.
Let $||\cdot||$ denote the maximum norm and let $\mathbb{M}$ be the set of bounded functions $\phi : [-L,\infty) \to \mathbb{R}$ with $\phi(t) = \psi(t)$ on $[-L,0]$, $\phi(t) \to \rho(t)$ as $t \to \infty$. Then $(\mathbb{M}, ||\cdot||)$ defines a complete metric space. For $\phi \in \mathbb{M}$, define $\mathscr{P} : \mathbb{M} \to \mathbb{M}$ by

$$(\mathscr{P}\phi)(t) = \psi(t), \text{ for } -L \le t \le 0,$$

and

$$(\mathscr{P}\phi)(t) = c + \sum_{s=-L}^{-1} p(s) \sum_{u=t+s}^{t-1} g(u,\phi(u)), \text{ for } t \ge 0.$$

$$\begin{aligned}\big|(\mathscr{P}\phi)(t) - \rho(t)\big| &= \Big| \sum_{s=-L}^{-1} p(s) \sum_{u=t+s}^{t-1} g(u,\phi(u)) - \sum_{s=-L}^{-1} p(s) \sum_{u=t+s}^{t-1} g(u,\rho(u))\Big| \\ &\le \sum_{s=-L}^{-1} |p(s)| \sum_{u=t+s}^{t-1} k|\phi(u) - \rho(u)| \\ &\le \sum_{s=-L}^{-1} |p(s)| \sum_{u=t-L}^{t-1} k|\phi(u) - \rho(u)| \to 0, \text{ as } t \to \infty,\end{aligned}$$

since $|\phi(u) - \rho(u)| \to 0$, as $t \to \infty$. The proof for showing $\mathscr{P}$ is a contraction is similar to before and hence we omit. Thus we have shown that $\mathscr{P}$ has a unique fixed point in $\mathbb{M}$, which converges to ρ.

We note that Remarks 4.5–4.7 and hence Theorem 4.6.2 hold for equations (4.6.14) and (4.6.23). We end with the following corollary.

Corollary 4.6 ([127]). *Assume the hypothesis of Theorem 4.6.4. If there exists an $r \in \mathbb{R}$, such that*

$$g(t,r) = \sum_{s=-L}^{-1} p(s) g(t+s,r), \tag{4.6.30}$$

then ρ of Theorem 4.6.4 is constant.

Proof. Suppose (4.6.23) has a constant solution r. Then from (4.6.26) we have

$$\begin{aligned}0 = \triangle r &= \triangle_t \sum_{s=-L}^{-1} p(s) \sum_{u=t+s}^{t-1} g(u,r)\\&= \sum_{s=-L}^{-1} p(s)\big(g(t,r) - g(t+s,r)\big)\\&= g(t,r) \sum_{s=-L}^{-1} p(s) - \sum_{s=-L}^{-1} p(s)g(t+s,r)\\&= g(t,r) - \sum_{s=-L}^{-1} p(s)g(t+s,r), \text{ due to (4.6.8).}\end{aligned}$$

Or,

$$g(t,r) = \sum_{s=-L}^{-1} p(s)g(t+s,r).$$

This completes the proof.

4.7 Periodic and Asymptotically Periodic Solutions in Coupled Systems

Now we turn our attention to the existence of periodic and asymptotically periodic solutions of a coupled system of nonlinear Volterra difference equations with infinite delay. By means of fixed point theory, namely Schauder's fixed point theorem, we furnish conditions that guarantee the existence of such periodic solutions. Consider the coupled system of nonlinear Volterra difference equations with infinite delay

$$\begin{cases}\triangle x_n = h_n x_n + \sum\limits_{i=-\infty}^{n} a_{n,i} f(y_i)\\ \triangle y_n = p_n y_n + \sum\limits_{i=-\infty}^{n} b_{n,i} g(x_i)\end{cases} \tag{4.7.1}$$

where f and g are real valued and continuous functions, and $\{a_{n,i}\}$, $\{b_{n,i}\}$, $\{h_n\}$, and $\{p_n\}$ are real sequences. In this study, we use *Schauder's fixed point theorem* to provide sufficient conditions guaranteeing the existence of periodic and asymptotically periodic solutions of system (4.7.1). Since we are seeking the existence of periodic solutions it is natural to ask that there exists a least positive integer T such that

$$h_{n+T} = h_n,\ p_{n+T} = p_n, \tag{4.7.2}$$

$$a_{n+T,i+T} = a_{n,i}, \tag{4.7.3}$$

and

$$b_{n+T,i+T} = b_{n,i} \tag{4.7.4}$$

hold for all $n \in \mathbb{N}$, where $\mathbb{N}$ indicates the set of all natural numbers.
There is a vast literature on this subject in the continuous and discrete cases. For instance, in [179] the authors considered the two-dimensional system of nonlinear Volterra difference equations

$$\begin{cases} \triangle x_n = h_n x_n + \sum\limits_{i=1}^{n} a_{n,i} f(y_i) \\ \triangle y_n = p_n y_n + \sum\limits_{i=1}^{n} b_{n,i} g(x_i) \end{cases}, \quad n = 1, 2, \dots$$

and classified the limiting behavior and the existence of its positive solutions with the help of fixed point theory. Also, the authors of [102] analyzed the asymptotic behavior of positive solutions of second order nonlinear difference systems, while the authors of [107] studied the classification and the existence of positive solutions of the system of Volterra nonlinear difference equations. Periodicity of the solutions of difference equations has been handled by [6, 46, 47, 48, 49, 50, 51, 52, 53, 54, 55, 56, 57, 58, 59, 60, 61, 62, 63, 64, 65, 66, 67, 68, 69, 70, 71, 72, 73, 74, 75]. In [48] and [49], the authors focused on a system of Volterra difference equations of the form

$$x_s(n) = a_s(n) + b_s(n) x_s(n) + \sum_{p=1}^{r} \sum_{i=0}^{n} K_{sp}(n,i) x_p(i), \;\; n \in \mathbb{N},$$

where a_s, b_s, $x_s : \mathbb{N} \to \mathbb{R}$ and $K_{sp} : \mathbb{N} \times \mathbb{N} \to \mathbb{R}$, $s = 1, 2, \dots, r$, and $\mathbb{R}$ denotes the set of all real numbers and obtained sufficient conditions for the existence of asymptotically periodic solutions. They had to construct a mapping on an appropriate space and then obtain a fixed point. Furthermore, in [86] the authors investigated the existence of periodic and positive periodic solutions of system of nonlinear Volterra integro-differential equations. The paper [55] of Elaydi was one of the first to address the existence of periodic solutions and the stability analysis of Volterra difference equations. Since then, the study of Volterra difference equations has been vastly increasing. For instance, we mention the papers [93, 113], and the references therein. In addition to periodicity we refer to [96] and [117] for results regarding boundedness.
The main purpose of this study is to extend the results of the above-mentioned literature by investigating the possibility of existence of periodic and the asymptotic periodic solutions for systems of nonlinear Volterra difference equations with infinite delay.

By a solution of the system (4.7.1) we mean a pair of sequences $\{(x_n, y_n)\}_{n \in \mathbb{Z}}$ of real numbers which satisfies (4.7.1) for all $n \in \mathbb{N}$. Let $\mathbb{Z}^-$ denote the set of all negative integers. The initial sequence space for the solutions of the system (4.7.1) can be constructed as follows. Let S denote the nonempty set of pairs of all sequences $(\eta, \zeta) = \{(\eta_n, \zeta_n)\}_{n \in \mathbb{Z}^-}$ of real numbers such that

$$\max\left\{ \sup_{n \in \mathbb{Z}^-} |\eta_n|, \sup_{n \in \mathbb{Z}^-} |\zeta_n| \right\} < \infty$$

and for each $n \in \mathbb{N}$, the series

$$\sum_{i=-\infty}^{0} a_{n,i} f(\eta_i) \text{ and } \sum_{i=-\infty}^{0} b_{n,i} g(\zeta_i)$$

converge. It is clear that for any given pair of initial sequences $\{(\eta_n, \zeta_n)\}_{n\in\mathbb{Z}^-}$ in S there exists a unique solution $\{(x_n, y_n)\}_{n\in\mathbb{Z}}$ of the system (4.7.1) which satisfies the initial condition

$$\begin{pmatrix} x_n \\ y_n \end{pmatrix} = \begin{pmatrix} \eta_n \\ \zeta_n \end{pmatrix} \text{ for } n \in \mathbb{Z}^-. \tag{4.7.5}$$

Such solution $\{(x_n, y_n)\}_{n\in\mathbb{Z}}$ is said to be the solution of the initial problem (4.7.1-4.7.5). For any pair $(\eta, \zeta) \in S$, one can specify a solution of (4.7.1–4.7.5) by denoting it by $\left(x_\eta, y_\zeta\right) := \{(x_n(\eta), y_n(\zeta))\}_{n\in\mathbb{Z}}$, where

$$(x_n(\eta), y_n(\zeta)) = \begin{cases} (\eta_n, \zeta_n) & \text{for } n \in \mathbb{Z}^- \\ (x_n, y_n) & \text{for } n \in \mathbb{N} \end{cases}$$

In our analysis, we apply a fixed point theorem to general operators over a Banach space of bounded sequences defined on the whole set of integers. Unlike the above-mentioned literature that dealt with stability of delayed difference systems, in the construction of our existence type theorems we neglect the consideration of phase space, for simplicity. For similar approach we refer to [28].

Theorem 4.7.1. *[Schauder's Fixed Point Theorem] Let X be a Banach space. Assume that K is a closed, bounded, and convex subset of X. If $T : K \to K$ is a compact operator, then it has a fixed point in K.*

4.7.1 Periodicity

In this section, we use Schauder's fixed point theorem to show that system (4.7.1) has a periodic solution. First, we start by defining periodic sequences on $\mathbb{Z}$.

Definition 4.7.1. Let T be a positive integer. A sequence $x = \{x_n\}_{n\in\mathbb{Z}}$ is called T-periodic if $x_{n+T} = x_n$ for all $n \in \mathbb{Z}$. The smallest positive integer T such that $x_{n+T} = x_n$ holds for all $n \in \mathbb{Z}$ is called the period of the sequence $x = \{x_n\}_{n\in\mathbb{Z}}$.

Let P_T be the set of all T-periodic sequences on $\mathbb{Z}$. Then P_T is a Banach space when it is endowed with the maximum norm

$$\|(x,y)\| := \max\left\{ \max_{n\in[1,T]_{\mathbb{Z}}} |x_n|, \max_{n\in[1,T]_{\mathbb{Z}}} |y_n| \right\}.$$

Let us define the subset $\Omega(W)$ of P_T by

$$\Omega(W) := \{(x,y) \in P_T : \|(x,y)\| \le W\},$$

where $W>0$ is a constant. Then $\Omega(W)$ is bounded, closed, and convex subset of P_T. For any pair $(x,y)=\{(x_n,y_n)\}_{n\in\mathbb{Z}}\in\Omega(W)$, we define the mapping $E:\Omega\to P_T$ by

$$E(x,y):=\{E(x,y)_n\}_{n\in\mathbb{Z}}:=\left\{\begin{pmatrix}E_1(x,y)_n\\E_2(x,y)_n\end{pmatrix}\right\}_{n\in\mathbb{Z}},$$

where

$$E_1(x,y)_n:=\begin{cases}x_n & \text{for } n\in\mathbb{Z}^-\\ \alpha_h\sum\limits_{i=n}^{n+T-1}\left(\prod\limits_{l=i+1}^{n+T-1}(1+h_l)\right)\sum\limits_{m=-\infty}^{i}a_{i,m}f(y_m) & \text{for } n\in\mathbb{N}\end{cases}, \tag{4.7.6}$$

$$E_2(x,y)_n:=\begin{cases}y_n & \text{for } n\in\mathbb{Z}^-\\ \alpha_p\sum\limits_{i=n}^{n+T-1}\left(\prod\limits_{l=i+1}^{n+T-1}(1+p_l)\right)\sum\limits_{m=-\infty}^{i}b_{i,m}g(x_m) & \text{for } n\in\mathbb{N}\end{cases}, \tag{4.7.7}$$

and

$$\alpha_h:=\left[1-\prod_{l=0}^{T-1}(1+h_l)\right]^{-1},$$
$$\alpha_p:=\left[1-\prod_{l=0}^{T-1}(1+p_l)\right]^{-1}.$$

We shall use the following result on several occasions in our further analysis.

Lemma 4.8. *Assume that (4.7.2–4.7.4) hold. Suppose that $1+h_n\neq 0$, $1+p_n\neq 0$ for all $n\in[1,T]_{\mathbb{Z}}:=[1,T]\cap\mathbb{Z}$, and that*

$$\prod_{l=0}^{T-1}(1+h_l)\neq 1 \text{ and } \prod_{l=0}^{T-1}(1+p_l)\neq 1. \tag{4.7.8}$$

The pair $(x,y)=\{(x_n,y_n)\}_{n\in Z}$ satisfies

$$E(x,y)=(x,y)$$

if and only if it is a T-periodic solution of (4.7.1).

Proof. One may easily verify that the pair $(x,y)=\{(x_n,y_n)\}_{n\in\mathbb{Z}}\in\Omega(W)$ satisfying $(x,y)=E(x,y)$ also satisfies the system (4.7.1) for all $n\in\mathbb{N}$. Conversely, suppose that the pair $(x,y)=\{(x_n,y_n)\}_{n\in\mathbb{Z}}$ is a T-periodic sequence satisfying (4.7.1) for all $n\in\mathbb{N}$. Multiplying both sides of the first equation in (4.7.1) with $\left(\prod\limits_{l=0}^{n}(1+h_l)\right)^{-1}$ and taking the summation from n to $n+T-1$, we obtain

$$\sum_{i=n}^{n+T-1}\triangle\left[x_i\left(\prod_{l=0}^{i-1}(1+h_l)\right)^{-1}\right]=\sum_{i=n}^{n+T-1}\left(\prod_{l=0}^{i}(1+h_l)\right)^{-1}\sum_{m=-\infty}^{i}a_{i,m}f(y_m).$$

This implies that

$$x_{n+T}\left(\prod_{l=0}^{n+T-1}(1+h_l)\right)^{-1} - x_n\left(\prod_{l=0}^{n-1}(1+h_l)\right)^{-1}$$
$$= \sum_{i=n}^{n+T-1}\left(\prod_{l=0}^{i}(1+h_l)\right)^{-1}\sum_{m=-\infty}^{i} a_{i,m} f(y_m).$$

Using the equalities $x_{n+T} = x_n$ and $\prod_{l=n}^{n+T-1}(1+h_l) = \prod_{l=0}^{T-1}(1+h_l)$, we have $E_1(x,y)_n = (x_n, y_n)$ for $n \in \mathbb{N}$. The equality $E_2(x,y)_n = (x_n, y_n)$ for $n \in \mathbb{N}$ can be obtained by using a similar procedure. The proof is complete.

In preparation for the next result we assume that there exist positive constants W_1, W_2, K_1, and K_2 such that

$$|f(x)| \le W_1 \tag{4.7.9}$$
$$|g(y)| \le W_2, \tag{4.7.10}$$

$$|\alpha_h| \sum_{i=n}^{n+T-1}\left|\prod_{l=i+1}^{n+T-1}(1+h_l)\right| \sum_{m=-\infty}^{i} |a_{i,m}| \le K_1, \tag{4.7.11}$$

and

$$\left|\alpha_p\right| \sum_{i=n}^{n+T-1}\left|\prod_{l=i+1}^{n+T-1}(1+p_l)\right| \sum_{m=-\infty}^{i} |b_{i,m}| \le K_2 \tag{4.7.12}$$

for all $n \in \mathbb{Z}$ and all $(x,y) \in \Omega(W)$.

Theorem 4.7.2. *In addition to the assumptions of Lemma 4.8 suppose that (4.7.9–4.7.12) hold. Then (4.7.1) has a T-periodic solution.*

Proof. From Lemma 4.8, we can deduce that $E(x,y)_{n+T} = E(x,y)_n$ for any $(x,y) \in \Omega(W)$. Moreover, if $(x,y) \in \Omega(W)$, then

$$|E_1(x,y)_n| \le |\alpha_h| \sum_{i=n}^{n+T-1}\left|\prod_{l=i+1}^{n+T-1}(1+h_l)\right| \sum_{m=-\infty}^{i} |a_{i,m}|\,|f(y_m)| \le W_1K_1, \tag{4.7.13}$$

and

$$|E_2(x,y)_n| \le \left|\alpha_p\right| \sum_{i=n}^{n+T-1}\left|\alpha_p \prod_{l=i+1}^{n+T-1}(1+p_l)\right| \sum_{m=-\infty}^{i} |b_{i,m}|\,|g(x_m)| \le W_2K_2 \tag{4.7.14}$$

for all $n \in \mathbb{N}$. If we set $W = \max\{W_1K_1, W_2K_2\}$, then E maps $\Omega(W)$ into itself. Now we show that E is continuous. Let $\{(x^l, y^l)\}$, $l \in \mathbb{N} = \{0,1,2,\ldots\}$, be a sequence in $\Omega(W)$ such that

$$\lim_{l\to\infty}\left\|(x^l,y^l)-(x,y)\right\| = \lim_{l\to\infty}\left(\max_{n\in[1,T]_{\mathbb{Z}}}\left\{\left|x_n^l-x_n\right|,\left|y_n^l-y_n\right|\right\}\right)$$
$$= 0.$$

Since $\Omega(W)$ is closed, we must have $(x,y)\in\Omega(W)$. Then by definition of E we have

$$\left\|E(x^l,y^l)-E(x,y)\right\| = \max\{\max_{n\in[1,T]_{\mathbb{Z}}}\left|E_1(x^l,y^l)_n-E_1(x,y)_n\right|,$$
$$\max_{n\in\mathbb{Z}}\left|E_2(x^l,y^l)_n-E_2(x,y)_n\right|\},$$

in which

$$\left|E_1(x^l,y^l)_n-E_1(x,y)_n\right| = |\alpha_h|\left|\sum_{i=n}^{n+T-1}\left(\prod_{l=i+1}^{n+T-1}(1+h_l)\right)\sum_{m=-\infty}^{i}a_{i,m}f(y_m^l)-\right.$$
$$\left.\sum_{i=n}^{n+T-1}\left(\prod_{l=i+1}^{n+T-1}(1+h_l)\right)\sum_{m=-\infty}^{i}a_{i,m}f(y_m)\right|$$
$$\le |\alpha_h|\sum_{i=n}^{n+T-1}\left|\prod_{l=i+1}^{n+T-1}(1+h_l)\right|\sum_{m=-\infty}^{i}|a_{i,m}|\left|f(y_m^l)-f(y_m)\right|.$$

Similarly,

$$\left|E_2(x^l,y^l)_n-E_2(x,y)_n\right| \le |\alpha_p|\sum_{i=n}^{n+T-1}\left|\prod_{l=i+1}^{n+T-1}(1+p_l)\right|\sum_{m=-\infty}^{i}|b_{i,m}|\left|g(x_m^l)-g(x_m)\right|.$$

The continuity of f and g along with the Lebesgue dominated convergence theorem imply that

$$\lim_{l\to\infty}\left\|E(x^l,y^l)-E(x,y)\right\| = 0.$$

This shows that E is continuous. Finally, we have to show that $E\Omega(W)$ is precompact. Let $\{(x^l,y^l)\}_{l\in\mathbb{N}}$ be a sequence in $\Omega(W)$. For each fixed $l\in\mathbb{N}$, $\{(x_n^l,y_n^l)\}_{n\in\mathbb{Z}}$ is a bounded sequence of real pairs. Then by *Bolzano-Weierstrass Theorem,* $\{(x_n^l,y_n^l)\}_{n\in\mathbb{Z}}$ has a convergent subsequence $\{(x_{n_k}^l,y_{n_k}^l)\}$. By repeating the diagonalization process for each $l\in\mathbb{N}$, we can construct a convergent subsequence $\{(x^{l_k},y^{l_k})\}_{l_k\in\mathbb{N}}$ of $\{(x^l,y^l)\}_{l\in\mathbb{N}}$ in $\Omega(W)$. Since E is continuous, we deduce that $\{E(x^l,y^l)\}_{l\in\mathbb{N}}$ has a convergent subsequence in $E\Omega(W)$. This means, $E\Omega(W)$ is precompact. By Schauder's fixed point theorem we conclude that there exists a pair $(x,y)\in\Omega(W)$ such that $E(x,y)=(x,y)$.

Theorem 4.7.3. *In addition to the assumptions of Lemma 4.8, we assume that (4.7.9), (4.7.11), and (4.7.12) hold. If g is a nondecreasing function satisfying*

$$|g(x)|\le g(|x|), \tag{4.7.15}$$

then (4.7.1) has a T-periodic solution.

Proof. By (4.7.11) and (4.7.13) we already have

$$|E_1(x,y)| \leq W_1 K_1 \text{ for all } (x,y) \in \Omega(W).$$

This along with (4.7.15) imply

$$\begin{aligned}|E_2(x,y)_n| &\leq \sum_{i=n}^{n+T-1}\left|\alpha_p \prod_{l=i+1}^{n+T-1}(1+p_l)\right| \sum_{m=-\infty}^{i}|b_{i,m}|\,|g(x_m)| \\ &\leq \sum_{i=n}^{n+T-1}\left|\alpha_p \prod_{l=i+1}^{n+T-1}(1+p_l)\right| \sum_{m=-\infty}^{i}|b_{i,m}|\,g(|E_1(x,y)|) \\ &\leq K_2 g(W_1K_1).\end{aligned}$$

If we set $W = \max\{W_1K_1, K_2g(W_1K_1)\}$, then the rest of the proof is similar to the proof of Theorem 4.7.2 and hence we omit it.

Similarly, we can give the following result.

Theorem 4.7.4. *In addition to the assumptions of Lemma 4.8, we assume (4.7.10), (4.7.11), and (4.7.12) hold. If f is a nondecreasing function satisfying*

$$|f(y)| \leq f(|y|),$$

then (4.7.1) has a T-periodic solution.

Example 4.3. Let

$$\begin{aligned}h_n &= 1+\cos n\pi, \\ p_n &= 1-\cos n\pi, \\ a_{n,i} = b_{n,i} &= e^{i-n},\end{aligned}$$

and

$$f(x) = \sin x \text{ and } g(x) = \sin 2x.$$

Then (4.7.1) turns into the following system

$$\begin{cases}\triangle x_n = (1+\cos n\pi)x_n + \sum_{i=-\infty}^{n} e^{i-n}\sin(y_i), \\ \triangle y_n = (1-\cos n\pi)y_n + \sum_{i=-\infty}^{n} e^{i-n}\sin(2x_i)\end{cases}.$$

It can be easily verified that conditions (4.7.2–4.7.8) and (4.7.9–4.7.12) hold. By Theorem 4.7.2, there exists a 2-periodic solution $(x,y) = \{(x_n,y_n)\}_{n\in\mathbb{Z}}$ of system (4.7.1) satisfying

$$x_n = -\frac{1}{2}\sum_{i=n}^{n+1}\prod_{l=i+1}^{n+1}(2+\cos(l\pi))\sum_{m=-\infty}^{i}e^{m-i}\sin(y_m),$$
$$y_n = -\frac{1}{2}\sum_{i=n}^{n+1}\prod_{l=i+1}^{n+1}(2-\cos(l\pi))\sum_{m=-\infty}^{i}e^{m-i}\sin(2x_m),$$

for all $n \in \mathbb{N}$.

4.7.2 Asymptotic Periodicity

In this section, we study the existence of an asymptotically T-periodic solution of system (4.7.1) by using Schauder's fixed point theorem. First we state the following definition.

Definition 4.7.2. A sequence $\{x_n\}_{n\in\mathbb{Z}}$ is called asymptotically T-periodic if there exist two sequences u_n and v_n such that u_n is T-periodic, $\lim_{n\to\infty} v_n = 0$, and $x_n = u_n + v_n$ for all $n \in \mathbb{Z}$.

First, we suppose that

$$\prod_{j=0}^{T-1}(1+h_j) = 1 \text{ and } \prod_{j=0}^{T-1}(1+p_j) = 1. \tag{4.7.16}$$

Then we define the sequences $\varphi := \{\varphi_n\}_{n\in\mathbb{N}}$ and $\psi := \{\psi_n\}_{n\in\mathbb{N}}$ as follows

$$\varphi_n := \prod_{j=0}^{n-1}\frac{1}{1+h_j} \text{ and } \psi_n := \prod_{j=0}^{n-1}\frac{1}{1+p_j}. \tag{4.7.17}$$

Furthermore, we define the constants m_k, M_k, $k = 1, 2$, by

$$m_1 := \min_{i\in[1,T]_{\mathbb{Z}}}|\varphi_i|, \; M_1 := \max_{i\in[1,T]_{\mathbb{Z}}}|\varphi_i|, \; m_2 := \min_{i\in[1,T]_{\mathbb{Z}}}|\psi_i|, \; M_2 := \max_{i\in[1,T]_{\mathbb{Z}}}|\psi_i|.$$

We note that in this section, we do not assume (4.7.3–4.7.4) but instead we ask that the series

$$\sum_{i=0}^{\infty}\sum_{m=-\infty}^{i}|a_{i,m}| < \infty \text{ and } \sum_{i=0}^{\infty}\sum_{m=-\infty}^{i}|b_{i,m}| < \infty \tag{4.7.18}$$

converge to a and b, respectively. Observe that (4.7.18) implies

$$\lim_{n\to\infty}\sum_{i=n}^{\infty}\sum_{m=-\infty}^{i}|a_{i,m}| = \lim_{n\to\infty}\sum_{i=n}^{\infty}\sum_{m=-\infty}^{i}|b_{i,m}| = 0. \tag{4.7.19}$$

Theorem 4.7.5. *Suppose that (4.7.9–4.7.10), (4.7.16), and (4.7.18–4.7.19) hold. Then system (4.7.1) has an asymptotically T-periodic solution $(x,y) = \{(x_n, y_n)\}_{n\in\mathbb{Z}}$ satisfying*

$$x_n := u_n^{(1)} + v_n^{(1)}$$

$$y_n := u_n^{(2)} + v_n^{(2)}$$

for $n \in \mathbb{N}$, where

$$u_n^{(1)} = c_1 \prod_{j=0}^{n-1} (1 + h_j), \quad u_n^{(2)} = c_2 \prod_{j=0}^{n-1} (1 + p_j),\ n \in \mathbb{Z}^+$$

c_1 and c_2 are positive constants, and

$$\lim_{n\to\infty} v_n^{(1)} = \lim_{n\to\infty} v_n^{(2)} = 0.$$

Proof. Due to the T-periodicity of the sequences $\{h_n\}_{n\in\mathbb{Z}}$ and $\{p_n\}_{n\in\mathbb{Z}}$ and by (4.7.16-4.7.17) we have

$$\varphi_n \in \{\varphi_1, \varphi_2, ..., \varphi_T\} \text{ and } \psi_n \in \{\psi_1, \psi_2, ..., \psi_T\}$$

for all $n \in \mathbb{N}$. This means

$$m_1 \le |\varphi_n| \le M_1 \tag{4.7.20}$$

$$m_2 \le |\psi_n| \le M_2 \tag{4.7.21}$$

for all $n \in \mathbb{Z}$. Define
$\mathbb{B} = \{(\boldsymbol{\Phi}, \boldsymbol{\Psi}) : \boldsymbol{\Phi} = \boldsymbol{\Phi}_1 + \boldsymbol{\Phi}_2, \boldsymbol{\Psi} = \boldsymbol{\Psi}_1 + \boldsymbol{\Psi}_2, (\boldsymbol{\Phi}_1, \boldsymbol{\Psi}_1)_{n+T} = (\boldsymbol{\Phi}_1, \boldsymbol{\Psi}_1)_n$, and $(\boldsymbol{\Phi}_2, \boldsymbol{\Psi}_2)_n \to (0,0)$ as $n \to \infty\}$. Then $\mathbb{B}$ is a Banach space when endowed with the maximum norm

$$\|(x,y)\| = \max\{\sup_{n\in\mathbb{Z}} |x_n|, \sup_{n\in\mathbb{Z}} |y_n|\}.$$

For a positive constant W^* we define

$$\boldsymbol{\Omega}^*(W^*) := \{(x,y) \in \mathbb{B} : \|(x,y)\| \le W^*\}.$$

Then, $\boldsymbol{\Omega}^*(W^*)$ is a nonempty bounded convex, and closed subset of $\mathbb{B}$. Define the mapping $E^* : \boldsymbol{\Omega}^*(W^*) \to \mathbb{B}$ by

$$E^*(x,y) = \{E^*(x,y)_n\}_{n\in\mathbb{Z}} = \left\{ \begin{pmatrix} E_1^*(x,y)_n \\ E_2^*(x,y)_n \end{pmatrix} \right\}_{n\in\mathbb{Z}},$$

where

$$E_1^*(x,y)_n := \begin{cases} x_n & \text{for } n \in \mathbb{Z}^- \\ c_1 \frac{1}{\varphi_n} - \sum_{i=n}^{\infty} \sum_{m=-\infty}^{i} \frac{\varphi_{i+1}}{\varphi_n} a_{i,m} f(y_m) & \text{for } n \in \mathbb{N} \end{cases}, \tag{4.7.22}$$

and

$$E_2^*(x,y)_n := \begin{cases} y_n & \text{for } n \in \mathbb{Z}^- \\ c_2 \frac{1}{\psi_n} - \sum\limits_{i=n}^{\infty} \sum\limits_{m=-\infty}^{i} \frac{\psi_{i+1}}{\psi_n} b_{i,m} g(x_m) & \text{for } n \in \mathbb{N} \end{cases}. \tag{4.7.23}$$

We will show that the mapping E^* has a fixed point in $\mathbb{B}$. First, we demonstrate that $E^*\Omega^*(W^*) \subset \Omega^*(W^*)$. If $\{(x,y)\} \in \Omega^*(W^*)$, then

$$\left|E_1^*(x,y)_n - c_1 \frac{1}{\varphi_n}\right| \le M_1 m_1^{-1} W_1 \sum_{i=n}^{\infty} \sum_{m=-\infty}^{i} |a_{i,m}| \tag{4.7.24}$$

$$\le M_1 m_1^{-1} W_1 \sum_{i=0}^{\infty} \sum_{m=-\infty}^{i} |a_{i,m}|$$

$$= M_1 m_1^{-1} W_1 a, \tag{4.7.25}$$

and

$$\left|E_2^*(x,y)_n - c_2 \frac{1}{\psi_n}\right| \le M_2 m_2^{-1} W_2 \sum_{i=n}^{\infty} \sum_{m=-\infty}^{i} |b_{i,m}| \tag{4.7.26}$$

$$\le M_2 m_2^{-1} W_2 \sum_{i=0}^{\infty} \sum_{m=-\infty}^{i} |b_{i,m}|$$

$$= M_2 m_2^{-1} W_2 b. \tag{4.7.27}$$

This implies that

$$|E_1^*(x_n,y_n)| \le M_1 m_1^{-1} W_1 a + \frac{c_1}{m_1},$$

and

$$|E_2^*(x_n,y_n)| \le M_2 m_2^{-1} W_2 b + \frac{c_2}{m_2}.$$

If we set

$$W^* = \max\{M_1 m_1^{-1} W_1 a + \frac{c_1}{m_1}, M_2 m_2^{-1} W_2 b + \frac{c_2}{m_2}\},$$

then we have $E^*\Omega^*(W^*) \subset \Omega^*(W^*)$ as desired.
Next, we show that E^* is continuous. Let $\{(x^q,y^q)\}_{q\in\mathbb{N}}$ be a sequence in $\Omega^*(W^*)$ such that
$\lim_{q\to\infty} \|(x^q,y^q) - (x,y)\| = 0$, where $(x,y) = \{(x_n,y_n)\}_{n\in\mathbb{Z}}$. Since $\Omega^*(W^*)$ is closed, we must have $(x,y) \in \Omega^*(W^*)$. From (4.7.22) and (4.7.23), we have

$$|E_1^*(x^q,y^q)_n - E_1^*(x,y)_n| \le \sum_{i=n}^{\infty} \sum_{m=-\infty}^{i} \left|\frac{\varphi_{i+1}}{\varphi_n}\right| |a_{i,m}| \, |f(y_m^q) - f(y_m)|$$

and

$$|E_2^*(x^q,y^q)_n - E_2^*(x,y)_n| \le \sum_{i=n}^{\infty} \sum_{m=-\infty}^{i} \left|\frac{\psi_{i+1}}{\psi_n}\right| |b_{i,m}| \, |g(x_m^q) - g(x_m)|.$$

Since f and g are continuous, we have by the Lebesgue dominated convergence theorem that

$$\lim_{q\to\infty} \|E^*(x^q,y^q) - E^*(x,y)\| = 0.$$

As we did in the proof of Theorem 4.7.2 we can show that E^* has a fixed point in $\Omega^*(W^*)$. On the other hand, using a similar procedure that we have employed in the proof of Lemma 4.8, we can deduce that any solution $(x,y) = \{(x_n,y_n)\}_{n\in\mathbb{Z}}$ of the system (4.7.1) is a fixed point for the operator E^*. This means $E^*(x,y) = (x,y)$ or equivalently,

$$x_n = c_1 \frac{1}{\varphi_n} - \sum_{i=n}^{\infty} \sum_{m=-\infty}^{i} \frac{\varphi_{i+1}}{\varphi_n} a_{i,m} f(y_m) \tag{4.7.28}$$

and

$$y_n = c_2 \frac{1}{\psi_n} - \sum_{i=n}^{\infty} \sum_{m=-\infty}^{i} \frac{\psi_{i+1}}{\psi_n} b_{i,m} g(x_m). \tag{4.7.29}$$

Conversely, any pair $(x,y) = \{(x_n,y_n)\}_{n\in\mathbb{Z}}$ satisfying (4.7.28) and (4.7.29) will also satisfy

$$\begin{aligned} x_{n+1} - x_n(1+h_n) = {} & c_1\Big(\prod_{j=0}^{n}(1+h_j) - (1+h_n)\prod_{j=0}^{n-1}(1+h_j)\Big) \\ & + (1+h_n)\sum_{i=n}^{\infty}\sum_{m=-\infty}^{i} \frac{\varphi_{i+1}}{\varphi_n} a_{i,m} f(y_m) \\ & - \sum_{i=n+1}^{\infty}\sum_{m=-\infty}^{i} \frac{\varphi_{i+1}}{\varphi_{n+1}} a_{i,m} f(y_m), \end{aligned}$$

and hence

$$\begin{aligned} x_{n+1} - x_n(1+h_n) = {} & \sum_{i=n}^{\infty}\sum_{m=-\infty}^{i} \frac{(1+h_n)\prod_{j=0}^{n-1}(1+h_j)}{\prod_{j=0}^{i}(1+h_j)} a_{i,m} f(y_m) \\ & - \sum_{i=n+1}^{\infty}\sum_{m=-\infty}^{i} \frac{\prod_{j=0}^{n}(1+h_j)}{\prod_{j=0}^{i}(1+h_j)} a_{i,m} f(y_m) \\ = {} & \sum_{m=-\infty}^{n} a_{n,m} f(y_m). \end{aligned}$$

That is, any fixed point $(x,y) = \{(x_n,y_n)\}_{n\in\mathbb{Z}}$ of the operator E^* satisfies the first equation in (4.7.1). Similarly, one may show that the second equation holds.
For an arbitrary fixed point $(x,y) \in \Omega^*(W^*)$ of E^*, we have

$$\lim_{n\to\infty}\left|x_n - c_1\frac{1}{\varphi_n}\right| = \lim_{n\to\infty}\left|E_1^*(x,y)_n - c_1\frac{1}{\varphi_n}\right| = 0, \tag{4.7.30}$$

and

$$\lim_{n\to\infty}\left|y_n - c_2\frac{1}{\psi_n}\right| = \lim_{n\to\infty}\left|E_2(x,y)_n - c_2\frac{1}{\psi_n}\right| = 0. \tag{4.7.31}$$

Choosing

$$u_n^{(1)} = c_1\frac{1}{\varphi_n}, \quad v_n^{(1)} = -\sum_{i=n}^{\infty}\sum_{m=-\infty}^{i}\frac{\varphi_{i+1}}{\varphi_n}a_{i,m}f(y_m) \tag{4.7.32}$$

and

$$u_n^{(2)} = c_2\frac{1}{\psi_n}, \quad v_n^{(2)} = -\sum_{i=n}^{\infty}\sum_{m=-\infty}^{i}\frac{\psi_{i+1}}{\psi_n}b_{i,m}g(x_m), \tag{4.7.33}$$

we have $x_n = u_n^{(1)} + v_n^{(1)}$ and $y_n = u_n^{(2)} + v_n^{(2)}$. By (4.7.30) and (4.7.31), $v_n^{(1)}$ and $v_n^{(2)}$ tend to 0 when $n\to\infty$. Left to show that $u_n^{(1)}$ and $u_n^{(2)}$ are T-periodic.

$$\begin{aligned}
u_{n+T}^{(1)} &= c_1\prod_{j=0}^{n+T-1}(1+h_j) = c_1\prod_{j=0}^{n-1}(1+h_j)\prod_{j=n}^{n+T-1}(1+h_j)\\
&= c_1\prod_{j=0}^{n-1}(1+h_j)\prod_{j=0}^{T-1}(1+h_j)\\
&= c_1\prod_{j=0}^{n-1}(1+h_j), \text{ by (4.7.16).}
\end{aligned}$$

The proof for $u_n^{(2)}$ is identical and hence we omit.

Example 4.4. Consider the system (4.7.1) with the following entries

$$\begin{aligned}
h_n = p_n &= \begin{cases} 1, & \text{if } n = 2k+1 \text{ for } k\in\mathbb{Z} \\ -\frac{1}{2}, & \text{if } n = 2k \text{ for } k\in\mathbb{Z} \end{cases},\\
a_{n,i} &= e^{i-2n}, \text{ for } n,i\in\mathbb{Z}\\
b_{n,i} &= e^{2i-3n}, \text{ for } n,i\in\mathbb{Z}\\
f(x) &= \cos x \text{ and } g(x) = \cos 2x.
\end{aligned}$$

Then (4.7.1) turns into the following system:

$$\begin{cases} \triangle x_n = h_n x_n + \sum\limits_{i=-\infty}^{n} e^{i-2n}\cos(y_i), \\ \triangle y_n = p_n y_n + \sum\limits_{i=-\infty}^{n} e^{2i-3n}\cos(2x_i) \end{cases}.$$

Obviously, the sequences $\{h_n\}_{n\in\mathbb{Z}}$ and $\{p_n\}_{n\in\mathbb{Z}}$ are 2-periodic and all conditions of Theorem 4.7.5 are satisfied. Hence, we conclude by Theorem 4.7.5 the existence of an asymptotically 2-periodic solution $(x,y) = \{(x_n,y_n)\}_{n\in\mathbb{Z}}$ satisfying

$$x_n = c_1 \frac{1}{\varphi_n} - \sum_{i=n}^{\infty} \sum_{m=-\infty}^{i} \frac{\varphi_{i+1}}{\varphi_n} e^{m-2i} \cos(y_m)$$

$$y_n = c_2 \frac{1}{\psi_n} - \sum_{i=n}^{\infty} \sum_{m=-\infty}^{i} \frac{\psi_{i+1}}{\psi_n} e^{2m-3i} \cos(2x_m),$$

for all $n \in \mathbb{N}$, where c_1 and c_2 are positive constants, $\varphi := \{\varphi_n\}_{n\in\mathbb{N}}$ and $\psi := \{\psi_n\}_{n\in\mathbb{N}}$ are as in (4.7.17).

4.8 Open Problems

In this section we propose seven open problems regarding existence of periodic solutions of Volterra difference equations and functional equations. We begin by considering the scalar Volterra difference equation

$$x(n+1) = c(n) - \sum_{s=-\infty}^{n} D(n,s)g(x(s)), \tag{4.8.1}$$

where g is continuous.

Open Problem 1.
Use the method of Section 4.5 to show (4.8.1) has a periodic solution under suitable conditions. Then prove parallel theorems to Theorems 4.5.2 and 4.5.3.
This will be different due to the absence of a linear term in x in Equation (4.8.1). Actually, it will be very challenging to find a suitable Lyapunov functional that does the trick.

Open Problem 2.
In light of our work in Section 4.7, what can be said about (4.8.1) with respect to periodicity and asymptotic periodicity? Again, the absence of a linear term in x makes (4.8.1) impossible to invert in order to obtain the possible mapping.

Open Problem 3.
Coupled integro-differential equations have many applications in science and engineering. In computational neuroscience, the Wilson–Cowan model describes the dynamics of interactions between populations of very simple excitatory and inhibitory model neurons. It was developed by H.R. Wilson and Jack D. Cowan [171, 172] and extensions of the model have been widely used in modeling neuronal populations [89, 108, 153, 173]. Here we propose a parallel coupled Volterra difference equations model

$$\begin{cases} \triangle x(n) = h_1(n)x(n) + h_2(n)y(n) + \sum_{-\infty}^{n} a(n,s)f(x(s),y(s)), \\ \triangle y(n) = p_1(n)y(n) + p_2(n)x(n) + \sum_{-\infty}^{n} b(n,s)g(x(s),y(s)), \end{cases} \tag{4.8.2}$$

where the functions f and g are assumed to be continuous. It would be of great interest to study the existence of periodic and asymptotically periodic solutions of (4.8.2).

Open Problem 4.
Consider Equation (4.5.9) and let P_T be the space of all periodic sequences of period T. Let $x \in P_T$ and use Theorem 1.1.1 to invert (4.5.9) and then use the Contraction mapping principle and the Schauder second fixed point theorem (see [156], p. 25) to show the existence of a unique periodic solution and a periodic solution. Compare both results and to the results of this chapter.

Open Problem 5 (Our Preferred System)
After careful examination of the three systems that we considered in the Section 4.6.1, we are lead to suggest that the system

$$\triangle x(t) = \sum_{s=t-L}^{t-1} p(s-t)g(x(s)) - \sum_{s=-\infty}^{t-1} q(s-t)g(x(s)) \tag{4.8.3}$$

which incorporates the most realistic properties from each of the systems, is our favorite system to be considered. The first term on the right takes into account the ideas from (4.6.14) in a more general form. Here we assume that

$$\sum_{s=-L}^{-1} q(s) = 1 \text{ and } \sum_{s=-\infty}^{-1} q(s) = 1. \tag{4.8.4}$$

Next, one would need to rewrite (4.8.3) as we did in (4.6.15) and then proceed to prove theorems that are parallel to Theorems 4.6.3 and 4.6.4.

Open Problem 6 (Neutral Systems)
There has been a tremendous effort in extending difference equations to neutral difference equations. Neutral difference equations have not been developed like its counterpart, differential equations. Suppose you are observing an organism that is displaying a normal growth or sub-ordinary growth. Suddenly growth accelerates and results in more accelerated growth. This is typical of neutral growth. Present growth rate depends not only on the past state, but also on the past growth rate. Typical models in the spirit of the previous section would be

$$\triangle\big(x(t) - h(x(t-L_1))\big) = g(x(t)) - g(x(t-L_2)). \tag{4.8.5}$$

It is clear that any constant function is a solution of (4.6.25). Now suppose both functions h and g are Lipschitz continuous. Let $L = \max\{L_1, L_2\}$, define an initial function and then prove a parallel Theorem to Theorem 4.6.3. Another possible neutral model to consider is

$$\triangle\big(x(t) - h(x(t-L_1))\big) = \sum_{s=t-L_2}^{t-1} p(s-t)g(x(s)) - \sum_{s=-\infty}^{t-1} q(s-t)g(x(s)) \tag{4.8.6}$$

If we assume (4.6.24) then any constant function is a solution of (4.8.6).

Open Problem 7 (Minorsky Model)
The second order differential equation

$$x''(t) + cx'(t) + g(x(t-h)) = 0 \tag{4.8.7}$$

is called Minorsky equation which he developed as an automatic steering device controller for the large ship the New Mexico. It was pointed out later on that the model given by (4.8.7) was not that accurate and since then a correction term was added and hence the new model

$$x''(t) + cx'(t) + g(x(t-h)) - g(x(t-h-L)) = 0. \tag{4.8.8}$$

Staying in the spirit of our study, one might consider analyzing the second order difference equation

$$\triangle^2 x(t) + c\triangle x(t) + g(x(t-h)) - g(x(t-h-L)) = 0. \tag{4.8.9}$$

Clearly, any constant is a solution of (4.8.9).