The Role of the Jacobi Last Multiplier in Nonholonomic Systems and Locally Conformal Symplectic Structure

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To Mahouton Norbert Hounkonnou, collaborator, colleague and friend, dedicated to his 60th birthday with admiration and gratitude

Abstract In this pedagogic article we study the geometrical structure of nonholonomic system and elucidate the relationship between Jacobi's last multiplier (JLM) and nonholonomic systems endowed with the almost symplectic structure. In particular, we present an algorithmic way to describe how the two form and almost Poisson structure associated to nonholonomic system, studied by L. Bates and his coworkers (Rep Math Phys 42(1–2):231–247, 1998; Rep Math Phys 49(2–3):143–149, 2002; What is a completely integrable nonholonomic dynamical system, in Proceedings of the XXX symposium on mathematical physics, Toruń, 1998; Rep Math Phys 32:99–115, 1993), can be mapped to symplectic form and canonical Poisson structure using JLM. We demonstrate how JLM can be used to map an integrable nonholonomic system to a Liouville integrable system. We map the toral fibration defined by the common level sets of the integrals of a Liouville integrable Hamiltonian system with a toral fibration coming from a completely integrable nonholonomic system.

Keywords Jacobi last multiplier · Nonholonomic system · Conformal Hamiltonian · Liouville integrability · Torus fibration

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1 Introduction

The Jacobi last multiplier (JLM) is a useful tool for deriving an additional first integral for a system of n first-order ODEs when $n-2$ first integrals of the system are known. Besides, the JLM allows us to determine the Lagrangian of a second-order ODE in many cases [15, 25, 31]. In his sixteenth lecture on dynamics Jacobi uses his method of the last multiplier [19, 20] to derive the components of the Laplace–Runge–Lenz vector for the two-dimensional Kepler problem. In recent years a number of articles have dealt with this particular aspect [10, 16, 24–26]. However, when a planar system of ODEs cannot be reduced to a second-order differential equation the question of interest arises whether the JLM can provide a mechanism for finding the Lagrangian of the system.

Let M be an even dimensional differentiable manifold endowed with a non-degenerate 2-form Ω, (M, Ω) is an almost symplectic manifold. An almost symplectic manifold (M, Ω) is called locally conformally symplectic (l.c.s.) manifold by Vaisman [29] if there is a global 1-form η, called the Lee form on M such that

$$d\Omega = \eta \wedge \Omega,$$

where $d\eta = 0$. (M, Ω) is globally conformally symplectic if the Lee form η is exact and when $\eta = 0$, then (M, Ω) is a symplectic manifold. The notion of locally conformally symplectic forms is due to Lee and, in more modern form, to Vaisman. Chinea et al. [8, 9] showed an extension of an observation made by I. Vaisman [29] that locally conformal symplectic manifolds can be seen as a natural geometrical setting for the description of time-independent Hamiltonian systems. In a seminal paper Wojkowski and Liverani [32] studied the Lyapunov spectrum in locally conformal Hamiltonian systems. It was demonstrated that Gaussian isokinetic dynamics, Nośe–Hoovers dynamics and other systems can be studied through locally conformal Hamiltonian systems. It must be noted that the conformal Hamiltonian structure appears in various dissipative dynamics as well as in the activator-inhibitor model connected to Turing pattern formation. It has been shown by Haller and Rybicki [18] that the Poisson algebra of a locally conformally symplectic manifold is integrable by making use of a convenient setting in global analysis. In this paper we explore the role of the Jacobi last multiplier in nonholonomic free particle motion and nonholonomic oscillator. These systems were studied extensively by L. Bates and his coworkers [2–5]. The two forms associated with these nonholonomic systems are not closed, in fact they satisfy l.c.s. condition. We apply JLM to such systems which guarantees that at least locally the symplectic form can be multiplied by a nonzero function to get a symplectic structure. In an interesting paper Bates and Cushman [4] compared the geometry of a toral fibration defined by the common level sets of the integrals of a Liouville integrable Hamiltonian system with a toral fibration coming from a completely integrable nonholonomic system. We apply JLM to study and compare these two toral fibrations. All the examples considered in this paper are taken from Bates et al. papers [2–5]. Relatively very little has been done when the flow is not complete. A quarter of a century ago, Flaschka [14] raised a number of questions concerning a simple class of integrable Hamiltonian systems in $\mathbf{R}^4$ for which the orbits lie on surfaces.

This paper is organized as follows. The first section recalls the definitions of the locally conformal symplectic structure and the Jacobi last multiplier. In Sect. 4 we study nonholonomic dynamics through an example—nonholonomic free particle motion, using constrained Lagrangian dynamics [7] and Bizyaev, Borisov, and Mamaev [6] method. We apply Jacobi last multiplier (JLM) method to transform nonholonomic dynamics into symplectic dynamics, a notion which, to our knowledge, does not appear explicitly in the literature. We study integrability property of the nonholonomic system in Sect. 5. The paper ends with a list of remarks regarding the further applications of JLM in nonholonomic systems. Finally, it is worthwhile to note that the first draft of this paper was circulated as an IHES preprint in 2013.

2 Preliminaries

We start with a brief review [17, 18, 29, 30] of the locally conformal symplectic structure. A differentiable manifold M of dimension $2n$ endowed with a non-degenerate 2-form ω and a closed 1-form η is called a locally conformally symplectic (l.c.s.) manifold if

$$d\omega + \omega \wedge \eta = 0. \tag{2.1}$$

The 1-form η is called the Lee form of ω [21]. This allows us to introduce the Lichnerowicz deformed differential operators

$$d_\eta : \Omega^*(M) \longrightarrow \Omega^{*+1}(M),$$

such that $d_\eta\theta = d\theta + \eta \wedge \theta$. Clearly $d_\eta^2 = 0$ and $d_\eta\omega = 0$. It must be worthwhile to note that l.c.s manifold is locally conformally equivalent to a symplectic manifold provided $\eta = df$ and $\omega = e^f \omega_0$, such that $d\omega_0 = 0$.

If (ω, η) is an l.c.s. structure on M and $f \in C^\infty(M, \mathbb{R})$, then $(e^f\omega, \eta - df) = (\omega', \eta')$ is again an l.c.s. structure on M then these two are conformally equivalent, and these two operators and Lee forms are cohomologous: $\eta' = \eta - df$. Hence d_η and $d_{\eta'}$ are gauge equivalent

$$d_{\eta'}(\beta) = (d_\eta - df\wedge)\beta = e^f\, d(e^{-f}\beta).$$

The r.h.s. is connected to Witten's differential. If $f \in C^\infty(M)$ and $t \geqslant 0$, Witten deformation of the usual differential $d_{tf} : \Omega^*(M) \longrightarrow \Omega^{*+1}(M)$ is defined by $d_{tf} = e^{tf} d e^{-tf}$, which means $d_{tf}\beta = d\beta + t\beta \wedge df$. Since d_η and $d_{\eta'}$ are gauge equivalent, the Lichnerowicz cohomology groups $H^*(\Omega^*(M), d_\eta)$ and

$H^*(\Omega^*(M), d_\eta)$ are isomorphic and the isomorphism is given by the conformal transformation $[\beta] \longmapsto [e^f \beta]$.

It is clear from the definition that d_η does not satisfy the Leibniz property:

$$d_\eta(\theta \wedge \psi) = (d + \eta\wedge)(\theta \wedge \psi) = d_\eta\theta \wedge \psi + (-1)^{deg\,\theta}\theta \wedge d\psi$$
$$= d\theta \wedge \psi + (-1)^{deg\,\theta}\theta \wedge d_\eta\psi.$$

For an l.c.s. manifold, we denote by

$$\text{Diff}\,_c^\infty(M, \omega, \eta) := \{f \in \text{Diff}_c^\infty(M) | (f^*\omega, f^*\eta) \simeq (\omega, \eta)\}$$

the group of compactly supported diffeomorphisms preserving the conformal equivalence class of (ω, η). The corresponding Lie algebra of vector fields is

$$\chi_c(M, \omega, \eta) := \{X \in \chi_c(M) \mid \exists c \in \mathbb{R} : L_X^\eta \omega = c\omega\},$$

where $L_X^\eta \beta = L_X\beta + \eta(X)\beta$. The Cartan magic formula for L_X^η is given by

$$L_X^\eta = d_\eta \circ i_X + i_X \circ d_\eta.$$

Here we list some of the important properties of the Lie derivative.

1. $L_X^\eta L_Y^\eta - L_X^\eta L_X^\eta = L_{[X,Y]}^\eta$.
2. $L_X^\eta d_\eta - d_\eta L_X^\eta = 0$
3. $L_X^\eta i_Y - i_Y L_X^\eta = 0$.
4. Let η_1 and η_2 be two Lee forms then $L_X^{\eta_1+\eta_2}(\theta \wedge \psi) = (L_X^{\eta_1}\theta) \wedge \psi + \theta \wedge (L_X^{\eta_2}\psi)$.

Let X and Y be the two conformal vector fields then $[X, Y]$ becomes the symplectic vector field. The proof of this claim is very simple, can easily show that $L_{[X,Y]}^\eta \omega = 0$.

2.1 *Inverse Problem and the Jacobi Last Multiplier*

We start with a brief introduction [10, 15, 24, 25, 31] of the Jacobi last multiplier and inverse problem of calculus of variations [22]. Consider a system of second-order ordinary differential equations

$$y_i'' = f_i(y_j, y_j') \qquad \text{for} \quad 1 \leqslant i, j \leqslant n.$$

Geometrically these are the analytical expression of a second-order equation field Γ living on the first jet bundle $J^1\pi$ of a bundle $\pi : E \to \mathbb{R}$, so

$$\Gamma = y_i' \frac{\partial}{\partial y_i} + f_i(y_j, y_j') \frac{\partial}{\partial y_i'}.$$

The local formulation of the general inverse problem is the question for the existence of a non-singular multiplier matrix $g_{ij}(y, y')$, such that

$$g_{ij}(y_j'' - f_j) \equiv \frac{d}{dt}\Big(\frac{\partial L}{\partial y_i}\Big) - \frac{\partial L}{\partial y_i'},$$

for some Lagrangian L. The most frequently used set of necessary and sufficient conditions for the existence of the g_{ij} are the so-called Helmholtz conditions due to Douglas [13, 27, 28].

Theorem 2.1 (Douglas [13]) *There exists a Lagrangian $L : TQ \to \mathbb{R}$ such that the equations are its Euler–Lagrange equations if and only if there exists a non-singular symmetric matrix g with entries g_{ij} satisfying the following three Helmholtz conditions:*

$$g_{ij} = g_{ji}, \qquad \widehat{\Gamma}(g_{ij}) = g_{ik}\Gamma^k_j + g_{jk}\Gamma^k_i,$$

$$g_{ik}\Phi^k_j = g_{jk}\Phi^k_i, \qquad \frac{\partial g_{ij}}{\partial y_k'} = \frac{\partial g_{ik}}{\partial y_j'},$$

$$\Gamma^k_j := -\frac{1}{2}\frac{\partial f_i}{\partial y_j'}, \qquad \Phi^k_i := -\frac{\partial f^k}{\partial x^i} - \Gamma^l_i\Gamma^k_l - \widehat{\Gamma}(\Gamma^k_i),$$

where $\widehat{\Gamma} = \frac{\partial}{\partial t} + y^i\frac{\partial}{\partial x^i} + f^i\frac{\partial}{\partial y^i}$.

When the system is one-dimensional we have $i = j = k = 1$ and then the three set of conditions become trivial and the fourth one reduces to one single P.D.E.

$$\Gamma(g) + g\frac{\partial f}{\partial v} \equiv v\frac{\partial g}{\partial x} + f\frac{\partial g}{\partial v} + g\frac{\partial f}{\partial v} = 0.$$

This is the equation defining the Jacobi multipliers, because $\mathrm{div}\Gamma = \frac{\partial f}{\partial v}$. The main equation can also be expressed as

$$\frac{dg}{dt} + g \cdot div\,\Gamma = 0.$$

Then, the inverse problem reduces to find the function g (often denoted by μ) which is a Jacobi multiplier and L is obtained by integrating the function μ two times with respect to velocities.

An autonomous second-order differential equation $y'' = F(y, y')$ has associated a system of first-order differential equations

$$y' = v, \qquad v' = F(y, v) \tag{2.2}$$

whose solutions are the integral curves of the vector field in $\mathbb{R}^2$

$$\Gamma = v\frac{\partial}{\partial y} + F(y, v)\frac{\partial}{\partial v}. \tag{2.3}$$

A Jacobi multiplier μ for such a system must satisfy divergence free condition

$$\frac{\partial}{\partial y}(\mu v) + \frac{\partial}{\partial v}(\mu F) = 0,$$

which implies μ must be such that

$$v\frac{\partial \mu}{\partial y} + \frac{\partial \mu}{\partial v}F + \mu\frac{\partial F}{\partial v} = 0.$$

which taking into account $\frac{dM}{dx} = v\frac{\partial M}{\partial y} + F\frac{\partial M}{\partial v}$ above equation can be written as

$$\frac{d\log\,\mu}{dx} + \frac{\partial F}{\partial v} = 0. \tag{2.4}$$

The normal form of the differential equation determining the solutions of the Euler–Lagrange equation defined by the Lagrangian function $L(y, v)$ admits as a Jacobi multiplier the function

$$\mu = \frac{\partial^2 L}{\partial v^2}. \tag{2.5}$$

Conversely, if $\mu(y, v)$ is a last multiplier function for a second-order differential equation in normal form, then there exists a Lagrangian L for the system related to μ by the above equation.

Let L be such that condition $M = \frac{\partial^2 L}{\partial v^2}$ be satisfied, then

$$\frac{\partial L}{\partial v} = \int^{v} M(y, \zeta)d\zeta + \phi_1(y)$$

which yields

$$L(y, v) = \int^{v} dv' \int^{v'} M(y, \zeta)d\zeta \; + \phi_1(y)v + \phi_2(y).$$

Geometrical Interpretation of JLM Let M be a smooth, real, n-dimensional orientable manifold with fixed volume form Ω. Let $\dot{x}_i(t) = \gamma_i(x_1(t), \cdots, x_n(t))$, $1 \leqslant i \leqslant n$ generated by the vector field Γ and we consider the $(n-1)$-form $\Omega_\gamma = i_\Gamma \Omega$. The function $\mu \in C^\infty(M)$ is called a JLM of the ODE system generated by Γ, if $\mu\omega$ is closed, i.e.,

$$d(\mu\Omega_\gamma) = d\mu \wedge \Omega_\gamma + \mu d\Omega_\gamma.$$

This is equivalent to $\Gamma(\mu) + \mu.\,\mathrm{div}\,\Gamma = 0$. Characterizations of the JLM can be obtained in terms of the deformed Lichnerowicz operator $d_\mu(\theta) = d\mu \wedge \theta + d\theta$, where the Lee form in terms of the last multiplier, i.e. $\eta = d\mu$. Hence, μ is a multiplier if and only if [11]

$$d(\mu\Omega_\gamma) \equiv d_\mu\Omega_\gamma + (m-1)d\Omega_\gamma = 0. \tag{2.6}$$

3 Nonholonomic Free Particle, Conformal Structure, and Jacobi Last Multiplier

Let us start with the discussion of Hamiltonian formulation of nonholonomic systems [6, 7]. Consider a mechanical system in 3D space with coordinates x, y, z. Let the coordinate z be cyclic. The motion takes place in the presence of a nonholonomic constraint which is given by

$$f = \dot{z} - y\dot{x} = 0. \tag{3.1}$$

We express the equation of motion in the form of Euler–Lagrange equations with undetermined multiplier λ

$$\frac{d}{dt}\Big(\frac{\partial L}{\partial \dot{x}_i}\Big) - \frac{\partial L}{\partial x_i} = \lambda\frac{\partial f}{\partial \dot{x}_i}, \qquad i = 1, 2, 3. \tag{3.2}$$

It is clear from the cyclic condition and definition of f that λ satisfies $\lambda = \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{z}}\right)$.

We consider the motion of a free particle with unit mass subjected to a constraint (3.1) and the Lagrangian is $L = \frac{1}{2}(\dot{x}^2 + \dot{y}^2 + \dot{z}^2)$, although the results presented in this paper are quite general. We use (3.2) to obtain the equations of motion[1]

$$\dot{x} = p_x, \quad \dot{y} = p_y, \quad \dot{z} = p_z, \quad \dot{p_x} = -\lambda y, \quad \dot{p_y} = 0 \quad \dot{p_z} = \lambda. \tag{3.3}$$

Using the constraint equation $\dot{z} = y\dot{x}$ we can find

$$\lambda = p_x p_y - \lambda y^2, \qquad \text{or} \qquad \lambda = \frac{p_x p_y}{1 + y^2}$$

and this is equivalent to $\lambda = (\frac{\partial L}{\partial \dot{z}})$. Hence eliminating the multiplier λ we obtain

$$\dot{x} = p_x, \quad \dot{y} = p_y, \quad \dot{p_x} = -y\frac{p_x p_y}{(1 + y^2)}, \quad \dot{p_y} = 0. \tag{3.4}$$

[1]The physicist way of looking the constrained dynamics is different from our presentation, it is described by $L = \frac{1}{2}\big(\dot{x}^2 + \dot{y}^2 + \dot{z}^2\big) + \lambda\big(\dot{z} - y\dot{x}\big)$, where momenta are given by $p_x = \dot{x} - \lambda y, \quad p_y = \dot{y}, \quad p_z = \dot{z} + \lambda, \quad p_\lambda = 0$, The usual Dirac analysis of constraints then identifies the following two constraints, $\phi_1 = p_\lambda = 0 \quad \phi_2 = p_z - yp_x - \lambda(1 + y^2)$ primary and secondary, respectively, which are second class, $\{\phi_1, \phi_2\} = (1 + y^2)$. It would be interesting to bridge the gap between these two methods.

3.1 Reduction, Constrained Hamiltonian and Nonholonomic Systems

Let $L_c(\mathbf{x}, \dot{\mathbf{x}})$ be the Lagrangian of the system after substituting the expression of $\dot{z}$ or $\dot{x_3}$. Thus we obtain a close system of equations for the variables $(\mathbf{x}, \dot{\mathbf{x}})$ and constraint $f = \dot{z} - y\dot{x} = 0$, given by

$$\frac{d}{dt}\Big(\frac{\partial L_c}{\partial \dot{x}}\Big) - \frac{\partial L_c}{\partial x} = \Big(\frac{\partial L}{\partial \dot{z}}\Big)^* \dot{y}, \qquad \frac{d}{dt}\Big(\frac{\partial L_c}{\partial \dot{y}}\Big) - \frac{\partial L_c}{\partial y} = -\Big(\frac{\partial L}{\partial \dot{z}}\Big)^* \dot{x}, \tag{3.5}$$

where $(\frac{\partial L}{\partial \dot{z}})^*$ means that the substitution $\dot{z}$ is made after the differentiation. This reduces to study the system with two degrees of freedom and preserves the energy integral

$$E = \frac{\partial L_c}{\partial \dot{x}}\dot{x} + \frac{\partial L_c}{\partial \dot{y}}\dot{y} - L_c.$$

Remark One can obtain the equations of motion (3.4) using the constrained Lagrangian. We now define the *constrained* Lagrangian by substituting the constraint equation $\dot{z} = y\dot{x}$ into Lagrangian:

$$L_c = \frac{1}{2}\big((1 + y^2)\dot{x}^2 + \dot{y}^2\big). \tag{3.6}$$

The equations of motion can be obtained from the constrained Lagrangian $L_c(y, \dot{x}, \dot{y}) = L(\dot{x}, \dot{y}, y\dot{x})$ using chain rule. This is a special case of nonholonomic treatment given in Tony Bloch's book [7]. The general equations of motion for a nonholonomic system with the constraint equation $\dot{w} = -A^a_\alpha \dot{r}^\alpha$ in terms of constrained Lagrangian $L_c(r^\alpha, w^a, \dot{r}^\alpha) = L[(r^\alpha, w^a, \dot{r}^\alpha, -A^a_\alpha(r, w)\dot{r}^\alpha]$ are given as

$$\frac{d}{dt}\frac{\partial L_c}{\partial \dot{r}^\alpha} - \frac{\partial L_c}{\partial r^\alpha} + A^a_\alpha(r, w)\frac{\partial L_c}{\partial w^a} = -\frac{\partial L_c}{\partial \dot{w}^a}B^b_{\alpha\beta}r^\beta, \tag{3.7}$$

where

$$B^b_{\alpha\beta} = \left(\frac{\partial A^b_\alpha}{\partial r^\beta} - \frac{\partial A^b_\beta}{\partial r^\alpha} + A^a_\alpha\frac{\partial A^b_\beta}{\partial w^a} - A^a_\beta\frac{\partial A^b_\alpha}{\partial w^a}\right).$$

Note that the system is holonomic if and only if the coefficients $B^b_{\alpha\beta}$ vanish.

The Lagrangian of the reduced system is $L_c = 1/2\big((1 + y^2)\dot{x}^2 + \dot{y}^2\big)$. Let S be the configuration space and $Leg_c : TS \to T^*S$ be the Legendre transformation of the reduced system. Using Legendre transformation

$$m_i = \frac{\partial \tilde{L}}{\partial \dot{x}}, \qquad H = \sum_{i=1}^{2} m_i\dot{x_i} - L_c, \qquad i = 1, 2 \tag{3.8}$$

we obtain the following system of equations

$$\dot{x}_i = \frac{\partial H}{\partial m_i}, \qquad \dot{m}_1 = -\frac{\partial H}{\partial x_1} + \frac{\partial H}{\partial m_2}\mathcal{S}, \qquad \dot{m}_2 = -\frac{\partial H}{\partial x_2} - \frac{\partial H}{\partial m_2}\mathcal{S}, \tag{3.9}$$

where $\mathcal{S} = (\frac{\partial L}{\partial \dot{z}})^*$ and $i = (1, 2)$. Then the momenta corresponding to the reduced equations are given by

$$m_x = \frac{\partial L_c}{\partial \dot{x}} = (1 + y^2)\dot{x}, \qquad m_y = \frac{\partial L_c}{\partial \dot{y}}$$

and the corresponding Hamiltonian of the reduced system is given by

$$H_c = \frac{1}{2}\Big(\frac{m_x^2}{1 + y^2} + m_y^2\Big). \tag{3.10}$$

It is easy to find Hamiltonian equations from (3.9) as $\dot{x} = \frac{\partial H}{\partial m_x}$, $\dot{y} = \frac{\partial H}{\partial m_y}$, $\dot{m_x} = -\frac{\partial H}{\partial x} + \frac{\partial H}{\partial m_y}\mathcal{S} = -0 + y m_y m_x/(1 + y^2)$, $\dot{m_y} = -\frac{\partial H}{\partial y} - \frac{\partial H}{\partial m_x}\mathcal{S} = y m_x^2/(1 + y^2)^2 - y m_x^2/(1 + y^2)^2 = 0$. Here we tacitly use $S = \dot{z} = y m_x/(1 + y^2)$.

The new set of equations is given by

$$\dot{x} = \frac{m_x}{1 + y^2}, \quad \dot{y} = m_y, \quad \dot{m_x} = \frac{y m_x m_y}{1 + y^2}, \quad \dot{m_y} = 0. \tag{3.11}$$

The vector field

$$\Gamma = \frac{m_x}{1 + y^2}\partial_x + m_y \partial_y + \frac{y m_x m_y}{1 + y^2}\partial_{m_x} \tag{3.12}$$

satisfies

$$i_\Gamma \omega_{nh} = -dH_c,$$

where the two form is given by

$$\omega_{nh} = dm_x \wedge dx + dm_y \wedge dy - \frac{m_x y}{1 + y^2} dy \wedge dx. \tag{3.13}$$

Here ω_{nh} is the nondegenerate two form on phase space P, however it is not closed, i.e.,

$$d\omega_{nh} = \frac{y dy}{1 + y^2} \wedge dm_x \wedge dx = d\Big(\frac{1}{2}\ln(1 + y^2)\Big) \wedge \omega_{nh}. \tag{3.14}$$

The corresponding Poisson structure is given by

$$\{x, m_x\} = 1, \qquad \{y, m_y\} = 1, \qquad \{m_x, m_y\} = \frac{m_x y}{1+y^2}, \tag{3.15}$$

which does not satisfy Jacobi identity, it is known as almost Poisson structure. The (nonholonomic) Poisson bracket between two functions $f_i = f_i(x, y, m_x, m_y)$, $(i = 1, 2)$ is

$$\{f_1, f_2\}_{nh} = \frac{\partial f_1}{\partial x}\frac{\partial f_2}{\partial m_x} - \frac{\partial f_1}{\partial m_x}\frac{\partial f_2}{\partial x} + \frac{\partial f_1}{\partial y}\frac{\partial f_2}{\partial m_y} - \frac{\partial f_1}{\partial m_y}\frac{\partial f_2}{\partial y} + \frac{m_x y}{1+y^2}\Big(\frac{\partial f_1}{\partial m_x}\frac{\partial f_2}{\partial m_y} - \frac{\partial f_1}{\partial m_y}\frac{\partial f_2}{\partial m_x}\Big).$$

The equations of motion may be given in terms of nonholonomic Poisson bracket

$$\dot{f} = \{f, H\}_{nh}, \qquad \forall f : M \to \mathbb{R}. \tag{3.16}$$

A function $f : M \to \mathbb{R}$ is an integral of motion of the nonholonomic system if and only if it satisfies $\{f, H\}_{nh} = 0$.

Using these almost Poisson structures we can still do Hamiltonian dynamics as long as we are willing to give up the existence of canonical coordinates and the Jacobi identities for the Poisson brackets. We will subsequently see that the Jacobi last multiplier plays a crucial role to obtain the canonical coordinates and Poisson structures.

3.2 *Hamiltonization and Reduction Using Jacobi Multiplier*

Let us compute the JLM of the set of Eq. (3.4) from

$$\frac{d}{dt}\log\mu + \big(-\frac{y\dot{y}}{1+y^2}\big) = 0,$$

thus we obtain

$$\mu = (1+y^2)^{1/2}. \tag{3.17}$$

It is worthwhile to note that if we compute the "JLM" of the set of Eq. (3.11) from $\frac{d}{dt}\log\mu + \big(\frac{y\dot{y}}{1+y^2}\big) = 0$, we obtain the inverse multiplier $\mu^{-1} = (1+y^2)^{-1/2}$. It is obvious because we compute it on the *dual space*.

Using the Jacobi last multiplier (JLM) one can show that system (3.10) has an invariant measure that can be represented in the form $\mu(y)d\mathbf{x}d\mathbf{m}$. JLM is a smooth and positive function on the entire phase space, so it acts like a density of the invariant measure and satisfies the Liouville equation

$$\operatorname{div}(\mu\Gamma) = 0,$$

where Γ stands for the vector field determined by system (3.10).

Proposition 3.1 *The function $K = m_x/\sqrt{1+y^2} = \mu^{-1}m_x$ is the integral of motion of the nonholonomic system, thus nonholonomic Poisson bracket with H vanishes, $\{H, K\}_{nh} = 0$.*

It follows directly from the set of Eq. (3.11).

3.3 Conformally Hamiltonian Formulation of Nonholonomic Systems

Let M be a symplectic manifold with symplectic form ω, when it is exact we write $\omega = d\theta$. For a function $H \in C^\infty(M)$ we denote its Hamiltonian vector field by X_H.

Definition 3.2 The diffeomorphism ϕ^a is conformal if $(\phi^a)^*\omega = \omega$ and corresponding to this flow the vector field Γ^a is said to be conformal with parameter $a \in \mathbb{R}$ if $L_{\Gamma^a}\omega = a\omega$.

It is clear

$$\frac{d}{dt}\phi_t^*\omega = \phi_t^* L_{\Gamma^a}\omega = a\phi_t^*\omega$$

which has a unique solution $\phi_t^*\omega = e^{at}\omega$.

The next proposition was given by McLachlan and Perlmutter [23].

Proposition 3.3 *Let M be a symplectic manifold with symplectic form ω. It admits a conformal vector field $a \neq 0$ if and only if $\omega = -d\theta$.*

(a) Given a Hamiltonian $H \in C^\infty(M)$, the conformal Hamiltonian vector field X_H^a satisfies

$$i_{X_H^a}\omega = dH - a\theta. \tag{3.18}$$

(b) If $H^1(M) = 0$, then the set of conformal vector fields on M is given by $\{X_H + cZ\} : H \in C^\infty(M)\}$, where Z is defined by $i_Z\omega = -\theta$ and it is known as the Liouville vector field.

If $H_1(M) = 0$, we know that every conformal vector field can be written as $X_H + cZ$ for some Hamiltonian and a unique $c \in \mathbb{R}$.

Let $\omega = dm_x \wedge dx + dm_y \wedge dy$ be the symplectic form. Then by contraction with respect to the Hamiltonian vector field we obtain

$$i_{X_H}\omega = -dH + \lambda\big(\frac{\partial H}{\partial m_2}dx_1 - \frac{\partial H}{\partial m_1}dx_2\big) \equiv -dH + \lambda\theta.$$

The vector field Z is tangent to the fibers is given by

$$Z = \frac{\partial H}{\partial m_2}\frac{\partial}{\partial m_1} - \frac{\partial H}{\partial m_1}\frac{\partial}{\partial m_2}, \qquad i_Z\omega = \theta.$$

Given the Hamiltonian $H_c \in C^\infty(M)$, the Hamiltonian vector field X_{H_c} corresponding to Hamiltonian H_c satisfies

$$i_{X_{H_c}}\omega^{nh} = dH_c - \theta, \quad \theta = \frac{m_x y}{1+y^2}dx.$$

This yields a conformal vector field. Let $\omega = dm_x \wedge dx + dm_y \wedge dy$ be the symplectic form when the manifold equipped with coordinates (x, y, m_x, m_y). The conformal vector field is given by $X_{H_c} + Z$, where Z is defined by

$$i_Z\omega = -\theta, \qquad \text{where} \qquad Z = \frac{m_x y}{1+y^2}\frac{\partial}{\partial m_x}. \tag{3.19}$$

4 Integrability of Nonholonomic Dynamics and Locally Conformally Symplectic Structure

In this section we unveil the connection between the Jacobi last multiplier, l.c.s. structure and integrability properties of nonholonomic dynamics.

Proposition 4.1 *The nonholonomic two form ω_{nh} and $\tilde{\omega}_{nh}$ satisfy locally conformal symplectic structure and the Lee form is $\eta = d\big(\log(1+y^2))^{1/2}\big) = d(\log\mu)$, where μ is the Jacobi's last multiplier.*

Proof It is straightforward to check

$$d\omega_{nh} = -\big(\frac{ydy}{1+y^2}\big) \wedge dm_x \wedge dx$$

$$= d\big(\log\frac{1}{2}(1+y^2)\big) \wedge \big(dm_x \wedge dx + dm_y \wedge dy - \frac{m_x y}{1+y^2}dy \wedge dx\big) = \eta \wedge \omega_{nh}$$

and similarly for the other case. □

The inverse multiplier plays an important role for changing locally conformal symplectic form ω_{nh} to symplectic form. In this process we find new momemta which satisfy canonical Poisson structure.

Proposition 4.2 *Let μ^{-1} be the inverse multiplier, then $\omega = \mu^{-1}\omega_{nh}$ is a symplectic form, given by*

$$\tilde{\omega} = d\tilde{m}_x \wedge dx + d\tilde{m}_y \wedge dy, \tag{4.1}$$

where the new momenta are

$$\tilde{m}_x = \mu^{-1} m_x = \frac{m_x}{\sqrt{1+y^2}} \qquad \tilde{m}_y = \frac{m_x}{\sqrt{1+y^2}}. \tag{4.2}$$

Proof By direct computation one obtains

$$\mu^{-1}\omega_{nh} = \frac{1}{\sqrt{1+y^2}}\Big(dm_x \wedge dx + dm_y \wedge dy - \frac{m_x y}{1+y^2} dy \wedge dx\Big)$$

$$= \frac{dm_x}{\sqrt{1+y^2}} \wedge dx + \frac{dm_y}{\sqrt{1+y^2}} \wedge dy - \frac{m_x y}{(1+y^2)^3/2} dy \wedge dx \equiv d\tilde{m}_x \wedge dx + d\tilde{m}_y \wedge dy.$$

□

It is clear $d\tilde{\omega} = 0$ and the new momenta satisfy the canonical Poisson structure

$$\{x, \tilde{m}_x\} = 1, \qquad \{y, \tilde{m}_y\} = 1. \tag{4.3}$$

4.1 Role of Jacobi's Multiplier and Integrability of Nonholonomic Dynamical Systems

We now address the question of integrability of the nonholonomic systems as posed by Bates and Cushman [4, 12]. In their papers, they explored to what extent nonholonomic systems behave like an integrable system. The fundamental Liouville theorem states that it suffices to have n $\{f_1 = H, f_2, \cdots, f_n\}$ independent Poisson commuting functions to explicitly (i.e., by quadratures) integrate the equations of motion for generic initial conditions. Let $M_c = \{f_1 = c_1, \cdots, f_n = c_n\}$ be a common invariant level set, which is regular (i.e., $df_1, \cdots df_n$ are independent), compact and connected, then it is diffeomorphic to n-dimensional tori $\mathbb{T}^n = \mathbb{R}^n/\Lambda$, where Λ is a lattice in $\mathbb{R}^n$. These tori are known as the Liouville tori [1, 12], In the neighborhood of M_c there exist canonical variables I, $\phi \bmod 2\pi$, called action-angle variables which satisfy $\{\phi_i, I_j\} = \delta_{ij}$, $\{\phi_i, \phi_j\} = \{I_i, I_j\} = 0$, $i, j = 1, \cdots n$, such that the level sets of the actions $_I, \cdots, I_n$ are invariant tori and $H = H(I_1, \ldots, I_n)$.

The vector fields $X_{f_1}, \cdots, X_{f_n}$ corresponding to the n integrals of motion $f_1, \cdots f_n$ are independent (it follows from the independency of differentials) and span the tangent spaces of $T_q M_c$ for all $q \in M_c$, since M_c is compact, hence X_{f_i}s are complete. The Poisson commutativity implies the commutativity of vector fields. In other words, the so-called invariant manifolds, which are the (generic) submanifolds traced out by the n commuting vector fields X_{f_i} are Liouville tori, the flow of each of the vector fields X_{f_i} is linear, so that the solutions of Hamilton's equations are quasi-periodic. A proof in the case of a Liouville integrable system on a symplectic manifold was given by Arnold [1].

We will soon figure out that the (reduced) nonholonomic problem which we are considered in this paper has two constants of motion H (Hamiltonian) and K, these are Poisson commuting. However, because the nonholonomic system does not satisfy the Jacobi identity, the associated vector fields X_H and X_K do not commute, i.e. $[X_H, X_K] \neq 0$, on the torus. So Bates and Cushman [4] asked if such system is integrable in some sense or how can it be converted to integrable systems.

4.2 JLM and Commuting of Vector Fields

It has been observed the reduced Hamiltonian equation of motion lies on the invariant manifold given by

$$K = \frac{m_x}{\sqrt{1+y^2}}, \tag{4.4}$$

where K satisfies $\frac{dK}{dt} = 0$. The Hamiltonian vector field

$$X_K = \frac{1}{\sqrt{1+y^2}} \frac{\partial}{\partial x} \tag{4.5}$$

satisfies $X_k \lrcorner \omega_{nh} = -dK$.

The Hamiltonian vector field X_H satisfies

$$L_{X_H} K = X_H(K) = 0, \tag{4.6}$$

which implies

$$\omega_{nh}(X_H, X_K) = X_K \lrcorner X_H \lrcorner \omega_{nh} = X_K \lrcorner \Big(\frac{m_x}{1+y^2} dm_x + m_y dm_y - \frac{mx^2 y}{(1+y^2)^2} dy \Big) = 0.$$

Next observe that the Lie bracket between vector fields X_H and X_K

$$[X_H, X_k] = -\frac{y m_x}{1+y^2} X_K. \tag{4.7}$$

This has been demonstrated by Bates and Cushman the vector fields X_H and X_K do not commute on the torus, because the two form ω_{nh} is not closed. They try to seek an integrating factor g such that $[gX_K, X_H] = 0$. The next proposition addresses the value of g.

Proposition 4.3 *Let μ be the Jacobi last multiplier, then the modified vector field $\mu^{-1}X_K$ commutes with the Hamiltonian vector field X_H, i.e.,*

$$[\mu^{-1}X_K, X_h] = 0. \tag{4.8}$$

Proof We know that the JLM $\mu = \sqrt{1+y^2}$, so that $\mu^{-1}X_K = \partial_x$. Hence we obtain $[\mu^{-1}X_K, X_h] = 0$. □

5 Final Comments and Outlook

Our formalism can be easily extended to nonholonomic oscillator. In this case, Lagrangian is given by $L = \frac{1}{2}(\dot{x}^2 + \dot{y}^2 + \dot{z}^2) + \frac{1}{2}y^2$, subject to the nonholonomic constraint $\dot{z} = y\dot{x}$. The reduced system of equations are given by

$$\dot{x} = p_x, \quad \dot{y} = p_y, \quad \dot{p_x} = -\frac{y}{1+y^2}p_x p_y, \quad \dot{p_y} = -y.$$

One can easily check that the last multiplier is $\mu = (1+y^2)^{1/2}$. The two form associated to the reduced nonholonomic oscillator equation

$$\omega_{as} = (1+y^2)dp_x \wedge dx + dp_y \wedge dy + yp_x dy \wedge dx$$

satisfies locally conformal symplectic structure, $d\omega_{as} + \eta \wedge \omega_{as} = 0$, where the Lee form $\eta = d\big(\log(1+y^2)^{1/2}\big)$. Hence the inverse Jacobi's last multiplier transforms ω_{as} into a symplectic form

$$\mu^{-1}\omega_{as} \equiv \tilde{\omega} = d\tilde{p}_x \wedge dx + d\tilde{p}_y \wedge dy,$$

where the modified momenta are given by $\tilde{p}_x = \sqrt{1+y^2}p_x$ and $p_y = \frac{p_y}{\sqrt{1+y^2}}$. Thus everything can be repeated here.

The application of the Jacobi Last Multiplier (JLM) for finding Lagrangians of any second-order differential equation has been extensively studied. It is known that the ratio of any two multipliers is a first integral of the system, in fact, it plays a role similar to the integrating factor for system of first-order differential equations. But so far, it has not been applied to nonholonomic systems. In this paper we have studied nonholonomic system endowed with a two form, which is closely related to locally conformal symplectic structure. We have applied JLM to map it to symplectic frame work. Also, we have shown how a toral fibration defined by the common level sets of integrable nonholonomic system, studied by Bates and Cushman, can be mapped to toral fibration defined of the integrals of a Liouville integrable Hamiltonian system.

There are some open problems popped up from this article. Firstly, it would be nice to study the time-dependent nonholonomic systems using JLM. Secondly, we have considered examples from the integrable domain, hence it would be great to apply JLM in nonintegrable domain.

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