Chapter 3
Everyday Context and Mathematical Learning: On the Role of Spontaneous Mathematical Focusing Tendencies in the Development of Numeracy

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Introduction

In the past 30 years, researchers have increasingly paid attention to not only mathematical learning in the classroom and formal contexts but also to how informal learning outside of the classroom impacts children's mathematical development, especially in developing conceptually rich mathematical knowledge and skills (e.g., Resnick, 1987). Differences in young students' mathematical learning and learning difficulties cannot be explained only by the experiences students have during the deliberately organized teaching and training situations. However, it is only fairly recently that mathematics education and developmental psychology research have also begun to examine the role of children's spontaneous, self-initiated mathematical activities in this development (Hannula & Lehtinen, 2005; McMullen, Hannula-Sormunen, & Lehtinen, 2011). In this chapter, we summarize classical studies on major milestones of numeracy development and furthermore discuss the impact of children's and students' own activities in informal everyday situations on learning trajectories leading to an advanced number sense, which optimally supports their future mathematical learning.

Early Development of Numeracy

Preschool mathematical development forms the necessary basis for later mathematical skills learnt in school (Clements & Sarama, 2014; Fuson, 1988; Gelman & Gallistel, 1978; Mix, Huttenlocher, & Levine, 2002; Nunes & Bryant, 1996).

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Children's mathematical skills and concepts develop highly individually, both in the rate at which children attain essential mathematical skills and in substance within mathematical concepts and skills, as well as in relations among different aspects of numbers (Fuson, 1988; Sophian, 2007).

Early Approximate and Exact Number Recognition

Two separate representational systems allow dealing with small numerosities in infants and toddlers: a fast but relatively imprecise discrimination of numerical magnitudes, which is affected by set size ratio limit, and an exact object tracking system functioning in the small number range (Feigenson & Carey, 2003). These early systems for representing objects and approximate quantities can also be found in other animal species, like in primates and birds (for reviews, see, e.g., Dehaene, 1997).

In addition to these mechanisms forming the basis for magnitude representations, the concept of a "set of individuals" is central for natural number concept including counting and simple arithmetical operations (Spelke, 2003). The toddler learns to connect pre-attentional basic-level perceptual information about the exact values of small numerosities with the socioculturally supported nonverbal and verbal expressions of small cardinal values. First, exact nonverbal number representations allow the child to represent, identify, categorize, and compare sets of objects within a very small range of numbers. These first pre-numeric skills are gradually integrated into cultural enumeration practices with verbal number words (Hannula, Räsänen & Lehtinen, 2007; Mattinen, 2006; Mix et al., 2002; Wynn, 1990). Children know that number words refer to specific, unique numerosities before they know exactly to which numerosity each number word refers (Sarnecka & Gelman, 2004; Wynn, 1992b). Children seem to develop piecemeal in acquiring cardinal meanings of "one, two, and three." After this, understanding of cardinality of set results in the cardinality meanings for all number words within the child's number sequence.

Subitizing and Counting

Two processes are used for recognition of exact numbers of items (Sathian et al., 1999), and they can be distinguished from approximate number recognition (Lemer, Dehaene, Spelke, & Cohen, 2003). These are a highly accurate, very fast, parallel apprehending of items up to around three or four, often called as subitizing, and verbal counting, which is much slower, requires coordination of several attentional sub-processes, and works also for the enumeration of larger numbers (e.g., Jevons, 1871; Trick, Enns, & Brodeur, 1996). Children's subitizing-based enumeration skills develop during childhood (Starkey & Cooper, 1995). Subitizing-based enumeration of children is slower than that of adults, and the subitizing range is smaller (Chi & Klahr, 1975; Trick et al., 1996).

Counting involves reciting the list of number words one by one, synchronizing number words with individuating acts including planning the moves of attention, keeping track of counted targets and inhibition of previously counted targets, and finally activating the cardinal value of the last recited number word as the result of the counting (e.g., Fuson, 1988; Trick & Pylyshyn, 1994). Five how-to-count principles need to be respected when items are counted (Gelman & Gallistel, 1978). These are (a) one-to-one correspondence (all the objects in the target set must be counted and each of them only once), (b) constant order (number words need to be listed in the same order), (c) order irrelevance and (d) abstraction (during counting it does not matter in which order the items are counted or what kind of things are counted), and (e) the cardinality principle (referring to the last number tag used as the cardinal value of the whole set) (Gelman & Gallistel, 1978).

Counting skills develop slowly, which could be explained by several issues: differences in nonverbal and verbal number recognition systems, the demanding integration of different representations and procedures, and the need for lots of practice in acquiring accuracy in counting procedures (e.g., Fuson, 1988; Wynn, 1990). Counting practice with number words provides a child with the basis for constructing the hows and whys of counting, as well as the essential features of correct counting (Briars & Siegler, 1984; Cowan, Dowker, Christakis, & Bailey, 1996). Once the basic skills of counting items in lines are achieved, children move on to learning the marking strategies necessary in counting random arrangements of objects (Fuson, 1988). According to the reciprocal developmental views of Saxe, Guberman, and Gearhart (1987) and Sophian (1998, 2007), children's goal-based numerical activities are related to their conceptual knowledge about numbers and social goals of enumeration change along with the development of skills, and they direct children's attention to different aspects and uses of numbers and counting.

The number sequence production indicates the child's participation in sociocultural numerical activities. Learning the first number words has been described as a serial recall task, in which the cardinal and ordinal aspects, numerical relations as well as the base-ten structure of number words, are only later integrated in the number sequence (Fuson, Richards, & Briars, 1982). Eventually, after different developmental phases, the number sequence becomes a mental construction of the number line, including exact cardinal meanings and ordinal relations between numbers. Later on number words become countable objects themselves (Fuson et al., 1982).

Basic Arithmetic Skills

The basic arithmetic skills that enable verbal adding and subtracting develop together with enumeration, number sequence skills, and separate schemes of quantitative increasing and decreasing. Similar to the early basis for exact number recognition, procedures for numerical operations are also constructed in infancy on object files individuating small exact numbers of items and an analogical magnitude-based estimation for representing numerosities (e.g. McCrink & Wynn, 2004; Wynn, 1992a) and in toddlerhood, on experiences with combining and separating sets of objects. These nonverbal skills form the basis for the development of conventional verbal arithmetic methods (Levine, Jordan, & Huttenlocher, 1992). The physical and social world of young children provides plenty of opportunities for them to develop concepts about amounts of material, their comparisons, and the different effects of actions on these amounts. Resnick and Greeno (1990) propose that children can perceive and reason about aggregations of amounts and objects before they represent them systematically. Their knowledge of arithmetic number facts and their methods of counting to find answers to addition and subtraction tasks are gradually integrated into a unified set of numerical relations, which form the natural number system. Thus the number facts are based on counting methods for arithmetical operations, and the numbers represent members of sets with true cardinal values (see also Fuson, 1988; Sophian, 2007).

The development of natural and later rational number concept and arithmetic skills is a gradual and long-lasting process, which is supported and constrained by different experiences during childhood and adolescence. In the second part of this chapter we deal with some of the experiences and present a novel approach for understanding the role of children's own activity in this development.

Children's Mathematical Activities in School and Home

Individual differences in young children's early mathematical skills have been explained by the amount of deliberate mathematically related activities in homes (e.g., Skwarchuk, Sowinski, & LeFevre, 2014). However, Lefevre, Clarke, and Stringer, (2002) focused on parents' direct teaching of early number skills and showed that the frequency of this kind of home teaching was positively related to children's school-based mathematical achievement. However in many other studies parent's self-reported engaging in numeracy activities was not related to children's number skills development (e.g., Blevins-Knabe, Austin, Musun, Eddy, & Jones, 2000; Missall, Hojnoski, Caskie, & Repasky, 2015).

The nature of mathematical learning environments at home has been analyzed in many studies, but still little is known about the specific types of home numeracy activities in which children are engaged with their parents (Cahoon, Cassidy, & Simms, 2017). Ginsburg and his colleagues (Ginsburg, Duch, Ertle, & Noble, 2012) concluded that still parents do relatively little to encourage their children's numeracy learning and instead focus on teaching literacy (Ginsburg et al., 2012). Skwarchuk et al. (2014) made a distinction between formal and informal mathematical activities in parent-child interaction. LeFevre et al. (2009) found that children's mathematical skills were related to the indirect numeracy activities in which learning was incidental and embedded in regular family life.

These results suggest that there are potentially more subtle connections between numerical activities at home and success in mathematics. While examinations of the mathematical home environment provide some hints as the potential causes of individual differences in mathematical development, it is also possible that less explicit mathematical behaviors and activities play a role in mathematical success.

Role of Children's Own Practice in Numeracy Development

Our previous studies have focused on the development of exact number recognition skills and children's early mathematical development (e.g., Hannula & Lehtinen, 2005; a review Hannula-Sormunen, 2014). This work suggests that young children's early development and especially their developmental individual differences in exact number recognition and utilization cannot be adequately described in terms of earlier theories and methods capturing only the processes and skills which are used after a child has already focused attention on the numerical aspect of the task. This work shows that young 3–7-year-old children have substantial individual differences in their self-initiated, spontaneous focusing on numerosity (SFON) in tasks in which their possible failure to regard exact numerosity is not entirely explained by their inability to deal with the cognitive requirements of the tasks. It seems that individual differences in this self-initiated numerosity focusing explain some of the individual differences in children's numerical development, i.e., why some children develop better than others in numeracy during their childhood years. Exact number recognition and utilization are not totally automatic processes; instead, they need to be triggered in natural surroundings. When a child's tendency to spontaneously focus his or her attention on the aspect of number is very strong, this produces lots of practice in number recognition and utilization and thus enhances the child's understanding of numerical aspects as affordances of sets (Hannula & Lehtinen, 2005). By using the term "spontaneous," we do not refer to the innate origins of the tendency, but the self-initiated nature of focusing in a particular situation. Focusing on the aspect of exact numerosity requires determination of the set being perceived on some basis (e.g., shall I count the blue, red, big, small, or all flowers?), and this is needed in exact cardinality determination based on both subitizing and counting in a natural environment. Not all possible subitizable or countable numbers of items in a natural setting can be brought to the conceptual, conscious levels of processing. Mechanisms of object individuation are mid-level processes (Trick & Pylyshyn, 1994). They produce only pre-numeric individuation information on the objects. Thus, Hannula and Lehtinen (2005) proposed that an attentional process of focusing attention on the aspect of exact number in the set of items or incidents is needed for recognition of number on a conceptual level. It triggers exact number recognition processes and utilization of the recognized exact number in action.

Focusing on numerical changes while sets of objects are manipulated could be a crucial part of understanding the meaning for numerical operations, which could explain significant predictive relations between SFON and arithmetical skills (Hannula, Lepola, & Lehtinen, 2010; Nanu, McMullen, Munck, Pipari Study Group, & Hannula-Sormunen, 2018).

How to Measure SFON?

In their early studies, Hannula and Lehtinen (2005) found that there were inter-individual differences in young children's tendency to focus spontaneously on the number of objects or events. They organized play-like situations where it was possible to observe if, without explicit guidance to do so, children focused on the number of objects or events and used this number in their actions. The tasks and activities were created in a way that would remind children of the games and other daily activities they do at home, in preschool, and day care. In the tasks there were many features on which it was possible to focus attention and children were not told that the activities in the tasks were related to numbers. For example, in the feeding games (e.g., imitation task), there was a plate of glass berries and a toy parrot with a big mouth into which it was possible to put berries. The researcher started the game by explaining that the idea is to feed the parrot. They then introduced the materials and said: "Watch carefully what I do, and then you do just like I did." After that the researcher put two berries, one at a time, into the parrot's mouth, and they disappeared with a bumping sound into the parrot's stomach. The child was then told: "Now you do exactly like I did." These activities were repeated with different numbers of berries. A parallel game-like task was, for example, putting envelopes into a postbox. For older children, both imitation tasks were used with two sets of different colored items. Overall, there now exist more than 20 different SFON task versions suitable for children and adults (for a review, see Hannula-Sormunen, 2014). These measures are based on activities which are close to children's familiar play situations but which, at the same time, make it possible to measure the strength of children's tendencies to spontaneously focus on numerosity by using well-defined standard procedures. Even in using these procedures, measuring spontaneous focusing tendencies is challenging.

Studies using the original SFON measures have shown that it is possible to measure the strength of children's SFON tendency in a rather reliable way (Hannula & Lehtinen, 2005; Hannula-Sormunen, Lehtinen, & Räsänen, 2015; Nanu et al., 2018). Recently, several other measures have been developed by various researchers, which highlight different aspects of spontaneous focusing (see Rathé, Torbeyns, Hannula-Sormunen, De Smedt, & Verschaffel, 2016 for an extensive review). The design principles for SFON assessments include the following aspects: (1) mathematically unspecified settings, (2) multiple (mathematical and non-mathematical) interpretations possible, (3) fully engaging for all, and (4) within competence range (Hannula, 2005). Nothing in the task situation gives any hints to the participants that the SFON tasks would be in any way numerical in nature. The experimenter gives no feedback. The child's attention and interest are carefully captured in the beginning of the task by the experimenter. It is important to carefully make sure that tasks involve only numbers so small that every child is capable of enumerating them. Similarly, all other cognitive requirements of the SFON tasks need to be at manageable level for all participants, so that participants' insufficient motor skills, inhibition, verbal production, and working memory do not explain individual differences in the SFON tasks (Hannula, 2005; Nanu et al., 2018).

Guided focusing on numerosity (GFON) task versions have demonstrated that the children who had zero SFON responses were able to deal with the cognitive task requirements after their focus was explicitly guided toward the numerical aspects of the SFON task (Hannula et al., 2010; Hannula & Lehtinen, 2005). Children's performance on the guided tasks supports the hypotheses that SFON is a dissociable part of utilizing exact number recognition in action in (mathematically) non-guided settings. The analyses of video-recorded performance in the SFON tasks allow all quantification acts or indications of child's understanding of the quantitative goal of the task to be acknowledged as SFON. It is notable that by using a number range beyond participating children's capabilities, children's failure in producing equal sets could be caused by their inability to enumerate the sets, their lack of focus on numerosity, or even both of these reasons.

The use of the above described methods made it possible for Hannula and Lehtinen (2005) to separate attentional process SFON which is defined as a process of spontaneously focusing attention on the exact number of a set of items or incidents. This attentional process triggers exact number recognition and using the recognized exact number in action, particularly in natural situations where the numerical magnitudes are not artificially made evident, which is typical for most educational materials. It appears that even though there are task-individual interaction effects that may cause differences in SFON tendency across, for example, action and verbally based tasks, a confirmatory factor analysis revealed a second-order latent variable referring to underlying general SFON tendency (Hannula-Sormunen et al., in preparation).

Findings of SFON Studies

Since the initial study of Hannula and Lehtinen (2005), cross-sectional and longitudinal studies on SFON have been conducted by many research groups in several countries (Hannula-Sormunen, 2014; Rathé et al., 2016). SFON tendency is positively related to the development of cardinality recognition, subitizing-based enumeration, object counting, and number sequence skills before school age (Batchelor, Inglis, & Gilmore, 2015; Bojorque, Torbeyns, Hannula-Sormunen, Van Nijlen, & Verschaffel, 2016; Edens & Potter, 2013; Hannula, 2005; Hannula & Lehtinen, 2001, 2005; Hannula, Räsänen, & Lehtinen, 2007; Potter, 2009). SFON tendency can be enhanced through guided focusing activities in preschool at the age of 3 years (Hannula, Mattinen, & Lehtinen, 2005; Mattinen, 2006). Path models of the development of SFON and counting skills from 3 to 6 years of age indicate a reciprocal relationship between SFON and counting skills before school age (Hannula & Lehtinen, 2005). SFON tendency in kindergarten is a significant, domain-specific predictor of arithmetical, but not reading, skills assessed at the end of second grade (Hannula et al., 2010). At primary school age, dyscalculic students have a lower SFON tendency than their normally developing peers (Kucian et al., 2012). Spontaneous focusing on mathematically meaningful aspects seems to be one of the specific forms of mathematical behavior in children whose mathematical skills develop optimally during childhood years, and the lack of which is associated with mathematical learning difficulties. Individual differences in children's SFON are not explained by children's lack of enumeration skills or other cognitive skills needed for SFON tasks (Hannula & Lehtinen, 2005), and focusing on other aspects, such as spatial locations, is a separate process which does not explain away the correlation between SFON and counting skills (Hannula et al., 2010).

The theoretical explanation for the strong predictive role of SFON tendency is based on the hypothesis that SFON is an indicator of the amount of self-initiated practice in using exact enumeration that a child gets in her or his natural surroundings (Hannula & Lehtinen, 2005). High SFON tendency would result in much higher amounts of practice with enumeration than what those children get who only deal with numbers when they are guided by adults (Hannula et al., 2010; Lehtinen, Hannula-Sormunen, McMullen, & Gruber, 2017). So far this theoretical hypothesis has been supported by a few studies. First, as part of a SFON enhancement study of 3-year-olds (Hannula, Mattinen, & Lehtinen (2005)), the analyses showed a positive correlation (r = 0.55) between children's scores in SFON tasks and their SFON tendency, observed by the personnel in all day care settings. Similar results were presented by Batchelor (2014) who found a positive association between children's task-based SFON measures and their spontaneous focusing on numbers in play-based behavior as observed during parent-child play interactions.

There have been a few intervention studies in which SFON has been trained. In the first intervention study by Hannula et al. (2005), SFON training was conducted by using only small exact numbers up to three. Early educators guided children's attention to small exact numbers both in everyday situations and also in structured numerical games. Guided focusing on numerosities was done by talking about, showing, and manipulating small numbers of, e.g., toys, snacks, socks, or other things during everyday interaction. Structural games involved variations with the numbers of objects of a set (Marton & Booth, 1997). The number of the fishes was first changed and observed together with the children in a cartoon aquarium, and then after adults kept secretly changing the numbers of fishes, which made the aquarium an exciting numerical focusing target. The aim of guiding children to focus on numerosities within daily routines and games was to enhance children's spontaneous focusing on numerosities. Children who participated in the SFON enhancement program outperformed the control group in SFON tendency on a delayed posttest, which was conducted half a year after the pretest (Hannula et al., 2005; Mattinen, 2006).

Beyond Mere Numerosity: The Development of Relational Reasoning as the Foundation for Rational Number Knowledge

Exact numerical and whole number reasoning is only part of numerical development that is relevant for mathematics. Already young infants have been shown to recognize the halfway point in objects (McCrink & Wynn, 2007). The system of approximate number found in infants and even nonhuman animals has features of relational reasoning (e.g., Dehaene, Izard, Spelke, & Pica, 2008). Beyond these innate capacities, young children can also solve tasks using mathematical relations (e.g., Boyer, Levine, & Huttenlocher, 2008). Early primary school-age children are able to match proportional quantities, especially those represented by continuous quantities (Boyer et al., 2008; Spinillo & Bryant, 1999). Four-year-olds have also been found to be able to reason with proportional quantities (Sophian, Harley, & Martin, 1995). Mix, Levine, and Huttenlocher (1999) found that 4- and 5-year-olds were able to calculate simple fraction arithmetic problems with pieces of foam. Finally, Frydman and Bryant (1988) found that 5-year-old children could reason about fair sharing even with different sizes of candy.

Resnick (1992) described the development of mathematical reasoning by focusing on the nature of the objectified mathematical reasoning. Figure 3.1 applies this model to the development of relational reasoning in mathematics in the development of rational number knowledge. At the most basic level in Resnick's model is the mathematics of protoquantities, which have no explicit quantitative value. The nonverbal notion of half may be used in reasoning at this level, as has been found in infant habituation studies (e.g., McCrink & Wynn, 2007), although the explicit identification of these mathematical features by children is not possible (cf. Spinillo & Bryant, 1999). The next level of reasoning in Resnick's model is the mathematics of quantities, which involves reasoning about physical material with explicit quantities. In the case of relational development, this level may describe young children's reasoning about proportional relations (e.g., Boyer & Levine, 2012).

The level of the mathematics of numbers is where numbers begin to act as "nouns" or "conceptual entities that can be manipulated and acted upon" (Resnick, 1992, p. 414). This level would include the first skills and processes with symbolic

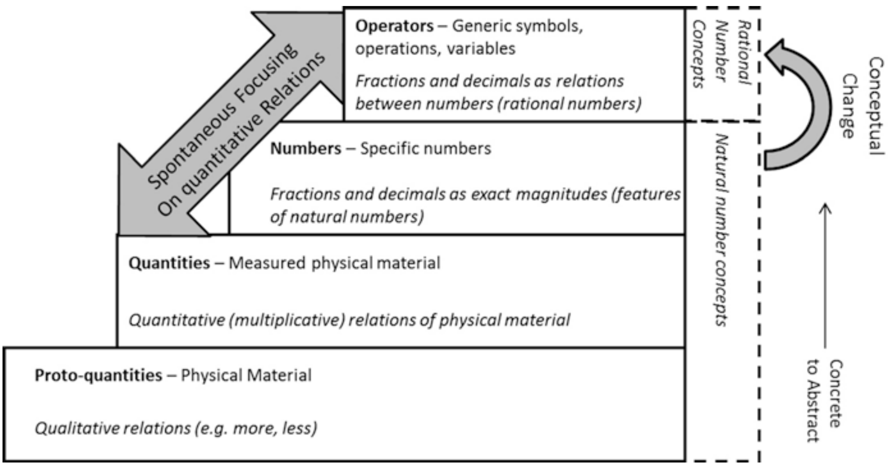

Fig. 3.1 The development of quantitative relations and rational number knowledge. (Based on model from Resnick (1992): From protoquantities to operators: Building mathematical competence on a foundation of everyday knowledge (McMullen, 2014))

fractions and decimals, where fractions and decimals are symbolic entities that can be acted upon and reasoned about independent of physical material. At this level, ½ is not the relationship of 1 part to 2, but may merely represent the magnitude of one-half of 1 (halfway between 0 and 1 on the number line), a notion supported by the continuity in the development of magnitude estimation skills from natural to rational numbers (e.g., Siegler, 2016). While many features of natural numbers can be attached to fractions and decimals, often in a supportive manner (Nunes & Bryant, 2008), it is also at this level that the natural number bias would cause problems with reasoning about fractions and decimals (Ni & Zhou, 2005).

It is only in moving into the mathematics of operations level that mathematically correct concepts of rational numbers appear. At this level, it is possible to reason about rational numbers as a concept, independent of specific numbers (Resnick, 1992, p. 414) representing the mathematical relations inherent in fractions (the relation between numerator and denominator) and decimals (the relation between place value and terms). Thus, at this level, rational numbers become mathematical objects that have specific features that are partially distinct from natural numbers (e.g., Vamvakoussi & Vosniadou, 2004). However, reaching this level is not a simple progression as described by Resnick, but instead may require radical change in the conception of the nature of number (Merenluoto & Lehtinen, 2004; Vamvakoussi & Vosniadou, 2004).

Spontaneous Focusing on Quantitative Relations

The quantitative relations that children experience in everyday situations are often approximate and dynamically changing. For example: "… a 7-year-old child traveling with her mother to visit their grandparents in the countryside. During the boring car drive the child starts spontaneously to think about the trip in terms of quantitative relations, asking 'Are we halfway there yet?'" (McMullen, Hannula-Sormunen, Laakkonen, & Lehtinen, 2016). In these situations the distances are often approximated, and, as well, the car can be approaching halfway, and after that it can be considered approaching halfway of the remaining distance to be traveled (i.e., 3/4) (McMullen et al., 2016). If children are involved in thinking about these kinds of "messy" quantitative relations of everyday contexts, it could have important effects on the way they think about the nature of numbers and how they can reason in novel situations with complex mathematical concepts. Based on a series of studies, McMullen and colleagues (McMullen, 2014; McMullen, Hannula-Sormunen, & Lehtinen, 2013, 2014) proposed that there is a tendency similar to SFON which indicates that children and school pupils can also focus spontaneously on the relation between two or more quantities in non-explicitly mathematical settings. Crucially, analogous to SFON tendency, individual differences in the tendency of spontaneous focusing on quantitative relations (SFOR) have been found to predict mathematical development in late primary school and lower secondary school (e.g., McMullen et al., 2016).

Studies with younger children conducted in Finland and the USA show that children begin to focus spontaneously on quantitative relations at the age from 6 to 7, but there are substantial inter-individual differences in the strength of the tendency still during the early school years (McMullen et al., 2013, 2014). Likewise, even after controlling for the ability to solve the tasks when explicitly guided to do so, there remain substantial inter-individual differences, even within grade levels, in SFOR tendency in studies of late primary school in Finland (McMullen et al., 2016) and Belgium (Van Hoof et al., 2016) and lower secondary school in the USA (McMullen, Hannula-Sormunen, Lehtinen, & Siegler, submitted).

A 4-year follow-up from the age of 7 to fourth grade reveals that individual differences in SFOR tendency may be related to later fraction knowledge (McMullen et al., 2014). As well, SFOR tendency was found to be a unique predictor of rational number conceptual development in late primary school students in Finland (McMullen et al., 2016) and Belgium (Van Hoof et al., 2016). In these studies the SFOR tendency was particularly related to the development of conceptual understanding of rational numbers, which has been difficult to support by traditional mathematics teaching and practice. A later study (McMullen, Hannula-Sormunen, & Lehtinen, 2017) showed that SFOR tendency is in a reciprocal relationship with rational number knowledge, similar to that which has been found between SFON and natural number knowledge (see Hannula & Lehtinen, 2005). The most recent findings indicate that besides rational number conceptual knowledge, SFOR tendency also predicts lower secondary school students' algebra knowledge (McMullen, et al., in press) and their flexible and adaptive use of rational number knowledge in novel tasks (McMullen et al., submitted).

There has yet to be established a causal relation between SFOR tendency and mathematical development; however, this relation has not been completely explained by a myriad of potential mediators, including nonverbal intelligence, arithmetic fluency, grade level, or prior knowledge (McMullen et al., 2016, submitted; Van Hoof et al., 2016) nor mathematical achievement (McMullen et al., submitted; Van Hoof et al., 2016), spatial reasoning, or mathematical motivation (McMullen et al., submitted). While SFOR tendency was relatively consistent, in terms of rank-order stability, over a 1-year period in late primary school, this does not suggest that it is impossible to enhance students' SFOR tendency. In fact, preliminary evidence suggests that by using innovative out-of-classroom activities, it is possible to increase students' tendency to focus on quantitative relations in SFOR tasks (McMullen et al., 2017). Thus, SFOR tendency is more likely a product of environmental factors rather than a static personal trait and may be malleable to explicit intervention.

Self-Initiated Practice and Number Sense

Not only is it expected that children would gain more practice with their numerical or mathematical skills and knowledge, but given the nature of the mathematical aspects embedded in everyday life, it may be that they also gain qualitatively better practice with mathematics than their peers that mostly encounter mathematics only in formal classroom situations. When children focus on number or quantitative relations in their natural environment, they are confronted with a higher variation of cognitive challenges than in dealing with deliberately planned school tasks. From a mathematical point of view, natural situations are "messy" in many ways. For example, the objects to be enumerated are seldom clear-cut objects clearly delineated from a monochromatic background, and it is often not completely clear what to include in the set that is to be enumerated. The number of objects and the observed quantitative relations are often changing dynamically in regular or irregular ways. Dealing with these situations requires flexible thinking about numbers and relations (Baroody, 2003). Dealing with numbers and relations in informal contexts makes it more difficult to rely on written procedures or algorithms but instead mental and oral processes (Schliemann & Nunes, 1990) that are often approximate and constrained by external factors (Lave, 1988). Self-initiated practice is often based on the inherent need to use the developing skills in novel situation (Piaget, 1955), which may result in strong personal involvement. In an everyday context, the feedback and practical consequences of enumeration or mathematical operations require a different control of correctness than in formal educational contexts. For example, Carraher, Schliemann, and Brizuela (2001) presented that in a formal school contexts, students "would sometimes claim that the amount of change to be returned to a customer after a purchase would be greater than the amount of money originally handed to the seller." In fact, even formal instruction often does not lead to coherent conclusions in formal mathematical settings, as a similar finding has been shown with students' estimations of positive fraction addition problems often being less than both addends (Braithwaite, Tian, & Siegler, 2017).

Discussion

Studies on spontaneous mathematical focusing tendencies (SFON and SFOR) suggest that theories of early development and later extensions of the number concept should also take into account the role of attentional processes and children's self-initiated practice. These studies highlight the role that mathematical practice in informal everyday contexts may play in mathematical development. Research on spontaneous mathematical focusing tendencies makes an important contribution by providing a novel explanation for the learning trajectories leading to differences in students' mathematical development, including their number sense. Mathematics curricula in many countries highlight the need to develop a more advanced number sense that can support the flexible and adaptive use of mathematical strategies (Mullis, Martin, Goh, & Cotter, 2016).

Studies on everyday mathematics are often presented as evidence for the need for a fundamentally different alternative approach (situated cognition) which differs from the traditional mathematics learning in schools (decontextualized cognition) (e.g., Lave, 1988). The interpretation of spontaneous focusing tendencies and self-initiated practice in informal activity contexts is different from the situated cognition approach. As studies show, spontaneous focusing tendencies are in reciprocal development with the formal mathematics learning in formal context. Focusing on mathematically relevant aspects of informal context and learning mathematics in formal education are fundamentally different experiences. However, students' own mathematical activities in informal situations do not lead to different mathematical knowledge, but instead provide opportunities to strengthen and enrich their mathematical development (see the criticism of Lave by Greiffenhagen & Sharrock, 2008).

In this way, it is important that formal mathematics education in school takes into account children's mathematically relevant activities in out-of-school situations and tries to better connect learning at school and learning in informal situations (Carraher et al., 2001; Resnick, 1987; Wager, 2012). In traditional mathematics teaching in all levels from early education to high school, mathematical content is often learned in an isolated, piecemeal way, and teaching rarely aims to deliberately trigger students to use mathematical knowledge as a way to see the world. In contrast, we have developed specific pedagogical tools aimed at improving students' and young children's awareness of opportunities to pay attention to mathematical aspects of their daily lives. For example, encouraging young preschool-age children to search for instances of particular numerosities in their everyday lives, in concert with practicing enumeration skills in a tablet-based game (Fingu; Holgersson et al., 2016), was able to improve students arithmetic skills in comparison with a control group (Hannula-Sormunen, Alanen, McMullen, & Lehtinen, 2016). As well, providing students with a framework and opportunities to explore their everyday surroundings for examples of quantitative relations, which are then analyzed for their mathematical features, has been shown to be effective in getting students to spontaneously focus on quantitative relations in task contexts (McMullen et al., 2017). In general, providing students with more opportunities to practice everyday problem solving in which mathematics are embedded may be valuable for promoting their deeper understanding of mathematical content (Pongsakdi, Laine, Veermans, Hannula-Sormunen, & Lehtinen, 2016). In the end, all of these approaches share the same aim – to break out of conventional preschool and school mathematics instruction and provide children and students with opportunities to explore mathematical phenomena in connection with everyday experiences.

Within this framework, also mathematical learning difficulties could be partly explained in novel ways. In some cases, these difficulties can be consequences of lacking self-initiated practice, whereas in other cases the mathematical difficulties appearing in school can mean that the child does not apply the mathematical thinking developed in out-of-school situations in formal tasks in classroom.

References

Baroody, A. J. (2003). The development of adaptive expertise and flexibility: The integration of conceptual and procedural knowledge. In A. J. Baroody & A. Dowker (Eds.), *The development of arithmetic concepts and skills: Constructing adaptive expertise* (pp. 1–33). London: Lawrence Erlbaum.

Batchelor, S. (2014). *Dispositional factors affecting children's early numerical development*. Loughborough University, Leicestershire, United Kingdom. Retrieved from https://dspace.lboro.ac.uk/2134/17474

Batchelor, S., Inglis, M., & Gilmore, C. (2015). Spontaneous focusing on numerosity and the arithmetic advantage. *Learning and Instruction, 40*(2–3), 116–135.

Blevins-Knabe, B., Austin, A. B., Musun, L., Eddy, A., & Jones, R. M. (2000). Family home care providers' and parents' beliefs and practices concerning mathematics with young children. *Early Child Development and Care, 165*(1), 41–58.

Bojorque, G., Torbeyns, J., Hannula-Sormunen, M., Van Nijlen, D., & Verschaffel, L. (2016). Development of SFON in Ecuadorian kindergartners. *European Journal of Psychology of Education*. 32, 449–462.

Boyer, T. W., & Levine, S. C. (2012). Child proportional scaling: Is 1/3=2/6=3/9=4/12? *Journal of Experimental Child Psychology, 111*(3), 516–533.

Boyer, T. W., Levine, S. C., & Huttenlocher, J. (2008). Development of proportional reasoning: Where young children go wrong. *Developmental Psychology, 44*(5), 1478–1490.

Braithwaite, D. W., Tian, J., & Siegler, R. S. (2017). Do children understand fraction addition? *Developmental Science*, e12601. 21, 4.

Briars, D., & Siegler, R. S. (1984). A featural analysis of preschoolers' counting knowledge. *Developmental Psychology, 20*, 607–618.

Cahoon, A., Cassidy, T., & Simms, V. (2017). Parents' views and experiences of the informal and formal home numeracy environment. *Learning, Culture and Social Interaction, 15*(August), 69–79.

Carraher, D., Schliemann, A. D., & Brizuela, B. M. (2001). Can young students operate on unknowns. In *Proceedings of the XXV Conference of the International Group for the Psychology of Mathematics Education* (pp. 130–140).

Chi, M. T. H., & Klahr, D. (1975). Span and rate of apprehension in children and adults. *Journal of Experimental Child Psychology, 3*, 93–102.

Clements, D. H., & Sarama, J. (2014). *Learning and teaching early math. The learning trajectories approach* (2nd ed.). New York: Routledge.

Cowan, R., Dowker, A., Christakis, A., & Bailey, S. (1996). Even more precisely assessing children's understanding of order-irrelevance principle. *Journal of Experimental Child Psychology, 62*, 84–101.

Dehaene, S. (1997). *The number sense: How the mind creates mathematics*. New York: Oxford University Press.

Dehaene, S., Izard, V., Spelke, E., & Pica, P. (2008). Log or linear? Distinct intuitions of the number scale in Western and Amazonian indigene cultures. *Science, 320*(5880), 1217–1220.

Edens, K. M., & Potter, E. F. (2013). An exploratory look at the relationships among Math skills, motivational factors and activity choice. *Early Childhood Education Journal, 41*(3), 235–243.

Feigenson, L., & Carey, S. (2003). Tracking individuals via object-files: Evidence from infants' manual search. *Developmental Science, 6*, 568–584.

Frydman, O., & Bryant, P. (1988). Sharing and the understanding of number equivalence by young children. *Cognitive Development, 3*(4), 323–339.

Fuson, K. (1988). *Children's counting and concepts of number*. New York: Springer Verlag.

Fuson, K., Richards, J., & Briars, D. J. (1982). The acquisition and elaboration of the number word sequence. In C. J. Brainerd (Ed.), *Children's logical and mathematical cognition. Progress in cognitive development research* (pp. 33–92). New York: Springer-Verlag.

Gelman, R., & Gallistel, C. R. (1978). *The child's understanding of number*. Cambridge, MA: Harvard University Press.

Ginsburg, H. P., Duch, H., Ertle, B., & Noble, K. G. (2012). How can parents help their children learn math? In B. H. Wasik (Ed.), *Handbook of family literacy* (pp. 51–65). New York: Routledge/Taylor & Francis Group.

Greiffenhagen, C. G., & Sharrock, W. (2008). School mathematics and its everyday other? Revisiting Lave's 'Cognition in Practice'. *Educational Studies in Mathematic, 69*, 1–21 Hannula-Sormunen, M. M. (2014). Spontaneous focusing on numerosity and its relation to counting and arithmetic. In A. Dowker & R. Cohen Kadosh (Eds.), *Oxford handbook of mathematical cognition* (pp. 275–290). Oxford: Oxford University Press.

Hannula, M. M. (2005). *Spontaneous focusing on numerosity in the development of early mathematical skills (Annales Un)*. Turku, Finland: Painosalama.

Hannula, M. M., & Lehtinen, E. (2001). Spontaneous tendency to focus on numerosities in the development of cardinality. In M. Panhuizen-Van Heuvel (Ed.), *Proceedings of the 25th conference of the International Group for the Psychology of Mathematics Education* (Vol. 3, pp. 113–120). Amersfoort, The Netherlands: Drukkerij Wilco.

Hannula, M. M., & Lehtinen, E. (2005). Spontaneous focusing on numerosity and mathematical skills of young children. *Learning and Instruction, 15*(3), 237–256.

Hannula, M. M., Lepola, J., & Lehtinen, E. (2010). Spontaneous focusing on numerosity as a domain-specific predictor of arithmetical skills. *Journal of Experimental Child Psychology, 107*(4), 394–406.

Hannula, M. M., Mattinen, A., & Lehtinen, E. (2005). Does social interaction influence 3-year-old children's tendency to focus on numerosity? A quasi-experimental study in day care. In L. Verschaffel, E. De Corte, G. Kanselaar, & M. Valcke (Eds.), *Powerful environments for promoting deep conceptual and strategic learning* (pp. 63–80).

Hannula, M. M., Räsänen, P., & Lehtinen, E. (2007). Development of counting skills: Role of spontaneous focusing on numerosity and subitizing-based enumeration. *Mathematical Thinking and Learning, 9*, 51–57.

Hannula-Sormunen, M. M. (2014). Spontaneous focusing on numerosity and its relation to counting and arithmetic. In A. Dowker & R. Cohen Kadosh (Eds.), *Oxford handbook of mathematical cognition* (pp. 275–290). Croydon, UK: Oxford University Press.

Hannula-Sormunen, M. M., Alanen, A., McMullen, J., & Lehtinen, E. (2016) *Integrating SFON enhancement with computerized arithmetical training – A pilot study*. Poster presented at the Workshop "Domain-General and Domain-Specific Foundation of Numerical and Arithmetic Processing", 28–30 September, Tuebingen.

Hannula-Sormunen, M. M., Lehtinen, E., & Räsänen, P. (2015). Preschool children's spontaneous focusing on numerosity, subitizing, and counting skills as predictors of their mathematical performance seven years later at school. *Mathematical Thinking and Learning, 17*(2–3), 155–177.

Hannula-Sormunen, M. M., Nanu, C., Laakkonen, E., Batchelor, S., Simms, V., De Smedt, B., et al. (in preparation). Spontaneous focusing on numerosity as second-order factor. Manuscript in preparation, 2018.

Holgersson, I., Barendregt, W., Emanuelsson, J., Ottosson, T., Rietz, E., & Lindstrom, B. (2016). Fingu-A game to support children's development of arithmetic competence: Theory, design and empirical research. In *International perspectives on teaching and learning mathematics with virtual manipulatives* (Vol. 7, pp. 123–145). Cham, Switzerland: Springer.

Jevons, S. (1871). The power of numerical discrimination. *Nature, 3*, 281–282.

Kucian, K., Kohn, J., Hannula-Sormunen, M. M., Richtmann, V., Grond, U., Käser, T., et al. (2012). Kinder mit Dyskalkulie fokussieren spontan weniger auf Anzahligkeit [Children with Developmental Dyscalculia Focus Spontaneously Less on Numerosities]. *Lernen Und Lernstörungen, 1*(4), 241–253.

Lave, J. (1988). *Cognition in practice: Mind, mathematics, and culture in everyday life*. Cambridge, UK: Cambridge University Press.

Lefevre, J. A., Clarke, T., & Stringer, A. P. (2002). Influences of language and parental involvement on the development of counting skills: Comparisons of French- and English-speaking Canadian children. *Early Child Development and Care, 172*(3), 283–300.

LeFevre, J.-A., Skwarchuk, S.-L., Smith-Chant, B. L., Fast, L., Kamawar, D., & Bisanz, J. (2009). Home numeracy experiences and children's math performance in the early school years. *Canadian Journal of Behavioural Science/Revue canadienne des sciences du comportement, 41*(2), 55–66.

Lehtinen, E., Hannula-Sormunen, M., McMullen, J., & Gruber, H. (2017). Cultivating mathematical skills: From drill-and-practice to deliberate practice. *ZDM – Mathematics Education, 49*(4), 625–636.

Lemer, C., Dehaene, S., Spelke, E., & Cohen, L. (2003). Approximate quantities and exact number words: Dissociable systems. *Neuropsychologia, 41*, 1942–1958.

Levine, S. C., Jordan, N. C., & Huttenlocher, J. (1992). Development of calculation abilities in young children. *Journal of Experimental Child Psychology, 53*, 72–103.

Marton, F., & Booth, S. (1997). *Learning and awareness*. Mahwah, NJ: Erlbaum.

Mattinen, A. (2006). *Huomio lukumääriin: Tutkimus 3-vuotiaiden lasten matemaattisten taitojen tukemisesta päiväkodissa [Focus on numerosities: A study on supporting 3 year-old children's mathematical development in day care]*. Turku, Finland: Painosalama.

McCrink, K., & Wynn, K. (2004). Large-number addition and subtraction by 9-month-old infants. *Psychological Science, 15*, 776–781.

McCrink, K., & Wynn, K. (2007). Ratio abstraction by 6-month- old infants. *Psychological Science, 18*, 740–745.

McMullen, J. (2014). *Spontaneous focusing on quantitative relations and the development of rational number conceptual knowledge*. Turku, Finland: Painosalama.

McMullen, J., Hannula-Sormunen, M., & Lehtinen, E. (2017). Spontaneous focusing on quantitative relations as a predictor of rational number and algebra knowledge. *Contemporary Educational Psychology, 51*, 356–365.

McMullen, J., Hannula-Sormunen, M. M., Kainulainen, M., Kiili, K., & Lehtinen, E. (in press). Moving mathematics out of the classroom: Using mobile technology to enhance spontaneous focusing on quantitative relations. *British Journal of Educational Technology*.

McMullen, J., Hannula-Sormunen, M. M., Laakkonen, E., & Lehtinen, E. (2016). Spontaneous focusing on quantitative relations as a predictor of the development of rational number conceptual knowledge. *Journal of Educational Psychology, 108*(6), 857–868.

McMullen, J., Hannula-Sormunen, M. M., & Lehtinen, E. (2011). Young children's spontaneous focusing on quantitative aspects and their verbalizations of their quantitative reasoning. In B. Ubuz (Ed.), *Proceedings of the 35th conference of the International Group for the Psychology of Mathematics Education* (pp. 217–224). Ankara, Turkey: PME.

McMullen, J., Hannula-Sormunen, M. M., & Lehtinen, E. (2013). Young children's recognition of quantitative relations in mathematically unspecified settings. *Journal of Mathematical Behavior, 32*(3), 450–460.

McMullen, J., Hannula-Sormunen, M. M., & Lehtinen, E. (2014). Spontaneous focusing on quantitative relations in the development of children's fraction knowledge. *Cognition and Instruction, 32*(2), 198–218.

McMullen, J., Hannula-Sormunen, M. M., Lehtinen, E., & Siegler, R. S. (submitted). Adaptive rational number knowledge and spontaneous focusing on multiplicative relations.

Merenluoto, K., & Lehtinen, E. (2004). Number concept and conceptual change: Towards a systemic model of the processes of change. *Learning and Instruction, 14*(5), 519–534.

Missall, K., Hojnoski, R. L., Caskie, G. I. L., & Repasky, P. (2015). Home numeracy environments of preschoolers: Examining relations among mathematical activities, parent mathematical beliefs, and early mathematical skills. *Early Education and Development, 26*(3), 356–376.

Mix, K. S., Huttenlocher, J., & Levine, S. C. (2002). *Math without words: Quantitative development in infancy and early childhood*. New York: Oxford University Press.

Mix, K. S., Levine, S. C., & Huttenlocher, J. (1999). Early fraction calculation ability. *Developmental Psychology, 35*(5), 164–174.

Mullis, I. V. S., Martin, M. O., Goh, S., & Cotter, K. (2016). *TIMSS 2015 Encyclopedia: Education policy and curriculum in mathematics and science*. Retrieved from http://timssandpirls.bc.edu/timss2015/encyclopedia/

Nanu, E. C., McMullen, J., Munck, P., Pipary Study Group, & Hannula-Sormunen, M. M. (2018). Spontaneous focusing on numerosity in preschool as a predictor of mathematical skills and knowledge in the fifth grade. *Journal of Experimental Child Psychology, 169*, 42–58.

Ni, Y., & Zhou, Y.-D. (2005). Teaching and learning fraction and rational numbers: The origins and implications of whole number bias. *Educational Psychologist, 40*(1), 27–52.

Nunes, T., & Bryant, P. (1996). *Children doing mathematics*. Cornwall, UK: T.J. Press.

Nunes, T., & Bryant, P. (2008). Rational numbers and intensive quantities: Challenges and insights to pupils' implicit knowledge. *Anales De Psicologia, 24*, 262–270.

Piaget, J. (1955). The construction of reality in the child. *Psychological Bulletin, 52*(1955), 526–528.

Pongsakdi, N., Laine, T., Veermans, K., Hannula-Sormunen, M. M., & Lehtinen, E. (2016). Improving word problem performance in elementary school students by enriching word problems used in mathematics teaching. *NOMAD, Nordic Studies in Mathematics Education, 21*(2). Retrieved from https://www.mendeley.com/profiles/margareta-engvall/

Potter, E. F. (2009). Spontaneous focusing on numerosity: Motivational and skill correlates in young children in a public preschool and kindergarten program. In S. L. Swars, D. W. Stinson, & S. Lemons-Smith (Eds.), *Proceedings of the 31st annual meeting of the North American chapter of the International Group for the Psychology of Mathematics Education* (pp. 152–155). Atlanta, GA: Georgia State University Press.

Rathé, S., Torbeyns, J., Hannula-Sormunen, M. M., De Smedt, B., & Verschaffel, L. (2016). Spontaneous focusing on numerosity: A review of recent research. *Mediterrean Journal for Research on Mathematics Education, 15*, 1–25.

Resnick, L. B. (1987). The 1987 presidential address: Learning in school and out. *Educational Researcher, 16*(9), 13–20; 54.

Resnick, L. B. (1992). From protoquantities to operators: Building mathematical competence on a foundation of everyday knowledge. In G. Leinhardt, R. T. Putnam, & R. A. Hattrup (Eds.), Analysis of arithmetic for mathematics teaching (pp. 373–429). Hillsdale, NJ, US: Lawrence Erlbaum Associates, Inc.

Resnick, L. B., & Greeno, J. G. (1990). Conceptual growth of number and quantity. Unpublished manuscript.

Sarnecka, B. W., & Gelman, S. A. (2004). Six does not just mean a lot: Preschoolers see number words as specific. *Cognition, 92*, 329–352.

Sathian, K., Simon, T. J., Peterson, S., Patel, G. A., Hoffman, J. M., & Grafton, S. T. (1999). Neural evidence linking visual object enumeration and attention. *Journal of Cognitive Neuroscience, 11*, 36–51.

Saxe, G. B., Guberman, S. R., & Gearhart, M. (1987). Social processes in early number development. *Monographs of the Society for Research in Child Development, 52*, Serial No. 216.

Schliemann, A. D., & Nunes, T. (1990). A situated schema of proportionality. *British Journal of Developmental Psychology, 8*(3), 259–268.

Siegler, R. S. (2016). Continuity and change in the field of cognitive development and in the perspectives of one cognitive developmentalist. *Child Development Perspectives, 10*, 128–133.

Skwarchuk, S. L., Sowinski, C., & LeFevre, J. A. (2014). Formal and informal home learning activities in relation to children's early numeracy and literacy skills: The development of a home numeracy model. *Journal of Experimental Child Psychology, 121*(1), 63–84.

Sophian, C. (1998). A developmental perspective on children's counting. In C. Donlan (Ed.), *The development of mathematical skills* (pp. 27–41). Hove, UK: Taylor & Francis.

Sophian, C. (2007). *The origins of mathematical knowledge in childhood*. New York: Lawrence Erlbaum.

Sophian, C., Harley, H., & Martin, C. S. M. (1995). Relational and representational aspects of early number development. *Cognition and Instruction, 13*(2), 253–268.

Spelke, E. (2003). What makes us smart? Core knowledge and natural language. In D. Gentner & S. Goldin-Meadow (Eds.), *Language in mind*. Cambridge, MA: MIT Press.

Spinillo, A. G., & Bryant, P. E. (1999). Proportional reasoning in young children: Part – Part comparisons about continuous and discontinous quantity. *Mathematical Cognition, 5*(2), 181–197.

Starkey, P., & Cooper, R. G. (1995). The development of subitizing in young children. *British Journal of Developmental Psychology, 13*, 399–420.

Trick, L., Enns, J. T., & Brodeur, D. A. (1996). Life span changes in visual enumeration: The number discrimination task. *Developmental Psychology, 32*, 925–932.

Trick, L. M., & Pylyshyn, Z. W. (1994). Why are small and large numbers enumerated differently? A limited-capacity preattentive stage in vision. *Psychological Review, 101*(1), 80–102.

Vamvakoussi, X., & Vosniadou, S. (2004). Understanding the structure of the set of rational numbers: A conceptual change approach. *Learning and Instruction, 14*(5), 453–467.

Van Hoof, J., Degrande, T., McMullen, J., Hannula-Sormunen, M. M., Lehtinen, E., Verschaffel, L., et al. (2016). The relation between learners' spontaneous focusing on quantitative relations and their rational number knowledge. *Studia Psychologica, 58*(2), 156–170.

Wager, A. A. (2012). Incorporating out-of-school mathematics: From cultural context to embedded practice. *Journal of Math Teacher Education, 15*, 9–23.

Wynn, K. (1990). Children's understanding of counting. *Cognition, 36*, 155–193.

Wynn, K. (1992a). Addition and subtraction by human infants. *Nature, 358*, 749–750.

Wynn, K. (1992b). Children's acquisition of the number words and the counting system. *Cognitive Psychology, 24*, 220–251.