Annemarie Fritz
Vitor Geraldi Haase
Pekka Räsänen *Editors*

International Handbook of Mathematical Learning Difficulties

From the Laboratory to the Classroom

Foreword by
Brian Butterworth

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To our children
Lukas, Ramiro, Olavo, Catarina, Venla,
Anttoni, Aleksi and Viola

Foreword

Mathematical attainment is important for both individuals and societies. Despite widespread understanding that this is the case, systematic reviews of the impact of poor numeracy on life outcomes are relatively scarce and confined to the more advanced economies. In 2008, the UK Government Office for Science published a massive, authoritative and carefully researched report, the *Foresight Mental Capital and Wellbeing Project* subtitled *Making the Most of Ourselves in the 21st Century*. It was led by a very distinguished board of scientists and the government's Chief Scientific Officer, Sir John Beddington. The report summarised the consequences of very low numeracy, dyscalculia, which affects between 4% and 7% of children. Dyscalculia "has a much lower profile than dyslexia but can also have substantial impacts: it can reduce lifetime earnings by £114,000 and reduce the probability of achieving five or more GCSEs (A*-C) by 7–20 percentage points." [GCSEs are the main 16-year-old exam, and a requirement for further or higher education, and most decent jobs.] A large cohort study by the National Research and Development Centre in the UK concluded that men and women with poor numeracy have poorer educational prospects, earn less and are more likely to be unemployed, more likely to be in trouble with the law and more likely to be sick physically and mentally.

The consequences for society are also dramatic. Again for the UK, the accountancy firm, KPMG, estimated the cost to the UK of poor maths in terms of lost direct and indirect taxes, unemployment benefits, justice costs and additional educational costs was £2.4 *billion* per year. In 2011, the OECD's report, *The High Cost of Low Educational Performance*, demonstrated that the standard of maths drives GDP growth: the standard in 1960 was a good predictor of economic growth up to 2000; and the improvement in educational standard from 1975 to 2000 was highly correlated with improvement in economic growth. In particular, the report looked at the potential effects of improving standards in maths.

OECD's PISA (Programme for International Student Assessment) defines poor numeracy as "Level 2" and below, which means that at best children can only manage simple calculations with whole numbers. 11% of UK children fail to reach 400 PISA test points, the minimum level (which is not very high) for a numerate society. So, for example, the economic report found that if the UK improved the performance of those 11% from below minimum level to minimum level, the effect on the gross domestic product (GDP) would be a growth of about 0.44%. Not much you might think, but with an average rate of GDP growth of 1.5%, this would be a massive and cumulative increase of nearly one-third. The average long-run annual improvement in economic growth for all OECD countries (i.e. the richest countries) was 0.68%.

It is clear from international comparisons that there are enormous national differences in average levels of mathematical attainment and also in the proportion of children who are effectively innumerate in a way that affects their life chances and the health of their society more generally. According to the most recent study carried out by the OECD, "23% of students in OECD countries, and 32% of students in all participating countries and economies, did not reach the baseline Level 2 in the PISA mathematics assessment of 15 year olds. At that level, students can only extract relevant information from a single source and can use basic algorithms, formulae, procedures or conventions to solve problems involving whole numbers" (OECD 2016, p4). This failure to reach Level 2 and be effectively innumerate varied from 3.8% in Shanghai China to 74.6% in Peru. In the USA, it is 25.8%, and in the UK it is 21.8%.

So what leads to the debilitating effects of low numeracy? There are, of course, many factors.

How children begin school is known to affect how they will continue. Canadian scientists put it like this, "Children who start school with poor knowledge and skills in … numeracy … are unlikely to catch up to their peers. Individual differences in … numeracy skills are evident at school entry—prior to formal instruction—suggesting that children acquire fundamental skills at home." All this is well-known and has particular relevance to mathematical development. Our study, carried out in Italy, revealed that the number and perhaps type of numerical activities in the home of the pre-schooler seems to be a key factor, and though this is related to parental income and education, especially maternal education, it constitutes a separate driver of early attainment. These factors have been systematically investigated in large-scale international comparisons like PISA.

Valuable and important as these international comparisons are, not to mention their political influence, there are two critical aspects of mathematical cognition and mathematical education that they are not designed to investigate, and perhaps cannot be designed to investigate:

- Explore the interaction of cognitive factors underlying mathematical attainment, especially in specific cultural and educational contexts
- Suggest particular methods of improving educational outcomes

Filling these important gaps makes this volume a vital and necessary complement to TIMSS and PISA.

Typical mathematical development depends on two distinct cognitive sources. One is domain-general capacities that affect almost all aspects of education. These will include reasoning and spatial abilities, long- and short-term memory, attention and motivation. Individual differences in these capacities can make a big difference to mathematical development over and above the cultural and educational context in which the learner is located. However, arithmetical competence is also based on domain-specific innate foundations that we share with other creatures. That is, without formal education, instruction of any kind, a wide range of creatures, from primates to birds to fish and even to insects, have been found to be able to extract information about the number of relevant objects in their environment. Lions can assess the number of invaders to their territory and decide whether there are enough of their own pride to fight them off, or whether they should retreat to fight another day. It is possible to train fish to swim to the larger (or smaller) of two arrays of abstract shapes. Bees count landmarks to estimate the distance between a food source and the hide.

Progress in mathematics and, especially, in numeracy depends on all these cognitive capacities.

Less well studied are the innate bases of geometry, but there is good evidence that human groups without access to geometrical education have good Euclidean intuitions.

This volume is unique in being able to locate the roles of these cognitive capacities, and incapacities, in their specific national contexts. It also offers an unparalleled perspective of how to assess mathematical development and to identify learners with atypical development in a way that respects cultural differences. Finally, it suggests ways in which the slow developers and the maths anxious can be helped. It is to be hoped that practitioners and policy-makers will read these essays carefully and be guided by the wealth of evidence herein provided.

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Reference

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"Expect anything worthwhile to take a long time" (Debbie Millman).

This maxim also applies to our book. Writing and gathering the different contributions took much longer than expected, but the work on this book has proven to be an adventurous journey around the world together with multiple experts in the field of mathematical learning difficulties. The present editors are grateful for having experienced this and would firstly like to thank each of the numerous authors who have enhanced this book with their contributions. Different theoretical concepts, explanatory approaches and research findings allow for a profound insight into the topic of mathematical difficulties and largely encompass different perspectives of various countries.

Some more people have accompanied and supported us on our journey, helping to shape our project. At this point, we would like to thank them cordially.

First and foremost, we would like to express our appreciation to Bruno Fiuza, who was immediately convinced of our book's concept and who made our publication in the Springer publishing house, Brazil, possible. His confidence, as well as his understanding and sympathy in case of modifications and delays, has greatly supported our work. He provided expert advice at all times. Brinda Megasyamalan and Kalaiselvi Ramalingam held responsibility for coordinating and monitoring the progress. They were at all times ready to help and accompanied every step of the realisation competently and with great patience.

Alina Simon, University of Duisburg-Essen, took responsibility for the final editing of the book and prepared the manuscript for the official handover to the publisher. With remarkable competence and patience, she cared about all the aspects that often go unnoticed. We would like to sincerely thank her for this. Also, we would like to thank Moritz Herzog, University of Duisburg-Essen, who has volunteered for numerous responsibilities and coordinated a team of assistants in Essen.

We would like to express our most heartfelt thanks to everybody mentioned above, as cooperating on this book with them has always been so very delightful.

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About the Editors

Annemarie Fritz is full professor for psychology at the University of Duisburg-Essen, Germany, since 1998. She graduated in psychology and habilitated in psychology of special education and rehabilitation. Since 2015, she is distinguished visiting professor at the University of Johannesburg.

At the University of Duisburg-Essen, she runs a research laboratory for children with learning difficulties. In the past 25 years, her research turned to children with mathematical learning difficulties. Here, the focus of her scientific work was the empirical validation of a development model of key numerical concepts and arithmetic skills from ages 4 to 8.

Based on this model, some diagnostic assessments (MARKO-Series) and training programmes for preschool and elementary school-children were developed. In cooperation with the University of Johannesburg, the MARKO-test for preschoolers and first graders was validated and normed in four South African languages. Recently, her interest turned to mathematical assessments for older children and mathematical anxiety.

Vitor Geraldi Haase is full professor of psychology at the Federal University of Minas Gerais (UFMG), Belo Horizonte, Brazil. He graduated in medicine, did his medical residency in paediatric neurology and has an M.A. in applied linguistics and a Ph.D. in medical psychology (Ludwig-Maximilians-Universität zu München). Working at UFMG since 1995, he heads the Laboratory for Developmental Neuropsychology and Número, a clinic for mathematical learning difficulties. He has been doing clinical work and research on numerical cognition applied to mathematical learning difficulties for the last 10 years. The main focus of this research is the characterisation of the molecular genetic variability underlying the cognitive mechanisms associated with mathematical learning difficulties and mathematical anxiety.

Pekka Räsänen is clinical neuropsychologist, principal investigator and executive vice director in the Niilo Mäki Institute, Jyväskylä, Finland. After his graduation, he has worked as a junior researcher in the Academy of Finland and as an interim associate professor of psychology in the University of Lapland, but since 1997 he worked as a researcher in the Niilo Mäki Institute. The Niilo Mäki Institute is a non-profit foundation-based research and development centre on learning and learning disabilities. The Niilo Mäki Institute is the largest and most influential research unit on learning disabilities in Finland offering also clinical services, further education and a publication unit. Pekka has developed most of the standardised tests of mathematical disabilities in Finland, and together with his colleagues, he has developed intervention programmes on language and mathematics for early education and computer-assisted tests and rehabilitation programmes for mathematical learning disabilities and recently also to visuospatial skills.

Chapter 1
Introduction

Annemarie Fritz, Vitor Geraldi Haase, and Pekka Räsänen

Twenty years ago, the main global educational issue was how to arrange schooling for all children in the world (Dakar, 2000; World Conference on Education for all, Jomtien, 1990). Still, in 1997, more than 100 million children did not have access to education (Roser & Ortiz-Ospina, 2017). The recent UNESCO report (2017) states that we have managed to half that figure, but at the same time a new issue has been raised: even though children would go to school, over 600 million (56%) do not reach even the basic level of skills in reading and mathematics. Globally, six out of ten children and adolescents are not able to read or handle mathematics with proficiency by the time they are in the age to complete primary education (UNESCO Institute for Statistics, 2017).

The aim of our book is to offer a global view of mathematical learning difficulties and their different causes, whether they are connected to quality of education or other reasons. In this book, these difficulties are covered from genetic as well as cognitive, neuroscientific and pedagogical perspectives. We also describe the current situation of mathematical learning difficulties in different parts of the world. This trip around the world provides the reader with a unique insight in how difficul-

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ties in mathematics are approached in different cultural settings. The amount of research on mathematical learning difficulties has doubled every decade, but the question of how the lessons learned from research and laboratories can be applied to everyday practice at schools still remains. Many teachers struggle with the question of how to identify those children who face problems in learning, how to support them and how to select the best methods to intervene.

In the current information society, machines can do calculations faster and more accurately than humans. Thus, does it appear necessary for each and every one to learn the basics of mathematics? The answer can be found all around us. Like colour, quantity is in everything. Whatever we do, whatever we see, there always is a number of something. Being numerate means that we are able to communicate about these numbers. Children start this journey to numeracy by comparing "more" and "less". They learn to talk about amounts, changes and differences in quantities when we teach them the quantitative concepts and the number system. It has taken thousands of years for mankind to develop the current efficient system, shared by the whole world. It was developed to describe the exact number of something with words and symbols. These are tools that we can use in almost any kind of context: to inform others about amounts or a change in an amount; to build different kinds of scales and measurement systems like time, length, weight or money; to describe amounts or ratios; and to share, multiply or divide. Acquiring basic mathematical knowledge means to acquire basic competencies for participating in our human culture.

Worldwide the number of children who do not learn these basic competencies during the primary education varies from 15% in North America and Europe to about 85% in sub-Saharan Africa (UNESCO, 2017). These are clearly higher figures than usually described in studies on developmental dyscalculia, a persistent difficulty in learning arithmetic, where the prevalence estimates vary from 2% to 7% (Devine, Soltész, Nobes, Goswami, & Szűcs, 2013; Rapin, 2016). While in dyscalculia the difficulties in learning are considered to stem from more or less specific neurocognitive factors of the child's brain, the high number of low-performing children tells us that the majority of the difficulties are related to an interplay of environmental factors like quality of education and opportunities to learn, not to forget the early development and home learning environment. Children in different parts of the world have very unequal opportunities to learn mathematics.

However, we can also find a strong heterogeneity among the children even within developmental dyscalculia (Fias, Menon, & Szucs, 2013; Rubinsten & Henik, 2009; Rykhlevskaia, Uddin, Kondos, & Menon, 2009), and researchers continue the quest of depicting the key variables behind the individual variation in numerical abilities and difficulties. Large steps in understanding this variation have been taken recently. This book gives an overview of the current state of the art about the mechanisms behind the difficulties, about recognition and diagnostics at school or in clinical practice as well as about the effectiveness of different types of interventions.

The book has been divided into five parts. Each part provides multiple perspectives to the topic area with a summarising discussion chapter for four sections at the end. The first part (*Part I: Development of Number Understanding: Different Theoretical Perspectives*) covers the different theoretical perspectives on the developmental issues. In the last three decades, research on mathematical difficulties and developmental dyscalculia has boomed. Difficulties in mathematical learning have been approached from different theoretical perspectives. Research has been carried out most actively in neuropsychology and cognitive neurosciences. The rapid development in technology and the drop of price in using these technologies have surged the amount of research using brain imaging or genetic analyses. After the initial excitement, the discussion turned to the applicability of neuroscientific findings to educational practice (Ansari & Coch, 2006; De Smedt & Grabner, 2016; Goswami, 2006; Howard-Jones, 2014).

At the same time, developmental psychologists conducted pioneering research with infants and toddlers to understand the development of the complex construct of number.

Learning difficulties cannot be traced back to a monocausal explanatory model. Such models have proven to be generally insufficient for the conception of learning difficulties. Instead, complex interaction models are to be favoured. Among the multitude of influencing factors, socioeconomic factors have to be emphasised. Also important to consider are the complex interactions between individual differences and contextual factors, such as public policies, poverty, culture, school as well as classroom effects and, of course, the quality of the pedagogy in the classroom. Pedagogical models have also seen many revisions. The goal of learning is not anymore considered to be studying specific subjects in order to reach curricular goals, but education is seen via broader concepts of acquiring competencies like "thinking skills" (Marope, Griffin, & Gallagher, 2017).

In the second part (*Part II: Mathematical Learning and Its Difficulties Around the World*), we focus on learning difficulties around the world. The authors draw a picture of the similarities and differences in research, education and public policies. Progress and obstacles in translating basic cognitive research to the classroom are discussed from the perspectives of different countries and areas. The quality of available diagnostic instruments and intervention programmes is evaluated.

Based on the international comparison, the levels of learning even the basic skills vary extensively in different parts of the world. This variation shapes the topics discussed at the local level. When in the welfare societies, learning motivation has risen to be one of the topics of discussion, in developing countries, the questions of social factors causing low performance are urgent problems. The chapters offer us the view of scientific definitions of learning disabilities being universal but at the same time point out how the school systems react, recognise and support those who struggle with learning that varies from one country to another. Likewise, there are large differences in how much evidence-based tools, like assessment materials standardised and normed in the country or field-tested intervention programmes, are available for practice. We would be glad if these chapters would encourage researchers in different countries to engage cross-country collaborations in these efforts.

Part III discusses the cognitive, motivational and emotional underpinnings of mathematical learning difficulties. The development of arithmetic can be approached from different perspectives: the neurobiological, the cognitive and the behavioural level. The development of arithmetic skills is based on the complex interplay of these different levels, which are dependent on each other. In this part, authors present the recent development in research, trying to uncover the relationship between neurocognitive, motivational and emotional variables and learning or learning difficulties in mathematics.

Part III starts with a chapter about genetics, which is one of the most rapidly developing research fields. From the perspective of genetics, developmental dyscalculia is a heterogeneous phenotype. There are multiple methods to evaluate the effects of genetic factors. Different methods show varying levels of genetic impact on learning mathematics, especially about the role of genetics in developmental dyscalculia and comorbid disorders like the common overlap between dyscalculia and dyslexia. The interplay of genetic and environmental factors is an important issue. One way to analyse the role of genetics to dyscalculia has been to focus on specific syndromes where mathematical learning difficulties have been found frequently. The cognitive disorders within these syndromes illustrate the different sources of dyscalculia, most typically difficulties in language, working memory and spatial skills. The following chapters deepen our understanding about these relationships between cognition and mathematical learning. To get a full picture, separate chapters are dedicated to describing how the motivational and emotional factors are tied to learning and learning disabilities.

A considerable body of fMRI studies with healthy adults has increased our knowledge about the brain networks which are relevant for mathematical performance. However, there is less information on how this network develops during learning and on how the findings of the differences between the dyscalculic and typically developing brain could inform us about the interventions needed. Children with dyscalculia show functional as well as structural abnormalities in this network. However, this information only gives us very little knowledge to guide education. As described in the chapter about the comorbid disorders, a detailed analysis at cognitive and behavioural levels is needed for designing the support at individual level.

The multilingual classroom is an understudied topic, considering that the majority of countries in the world are multilingual by nature and that immigration has increased, reaching almost 250 million persons (United Nations, 2016). Migratory movements on a mass scale have brought various new languages to other countries and continents; the Internet has dramatically affected the way in which language and languages are used for communication and indeed for learning (Education in a Multilingual World, 2003). One of the key ideas of this international handbook was to make this diversity in the classrooms visible, whether we are talking about the teaching and learning languages of mathematics in the classroom or of how educational policies take diversity into account.

Part IV turns the discussion to development, learning and teaching (*Part IV: Understanding the Basics: Building Conceptual Knowledge and Characterising Obstacles to the Development of Arithmetic Skills*). High-quality education requires our understanding of the basic steps children take in learning and of how the education should be designed to support them in their progress to more advanced and complex representations. Likewise in this part, we ask for the specific mathematical obstacles which make learning and understanding mathematics so difficult. Implications for teaching and learning will be discussed in all papers.

One of the main problems of students with mathematical difficulties is that most of them have not been able to develop more advanced calculation strategies than counting by one. This is disadvantageous in many ways: counting takes a long time, is error prone and burdens the working memory. But worst of all is that these children do not represent numbers as sets (cardinal aspect) but only as a single number on the mental number line (ordinal aspect). For the understanding of effective addition and subtraction strategies as well as multiplication and division, it is imperative to understand numbers as sets which can be decomposed.

While the very basics of the number system can also be learnt outside of school, more advanced skills require explicit education starting from calculating with multi-digit or decimal numbers and especially grasping the idea of the rational number system. In mathematics, a rational number is any number that can be represented as a fraction. As most of the research has focused on fractions in typically achieving students, the specific challenges that fractions pose to learners with mathematical difficulties are less well understood. Learning to solve problems using calculation skills adds one more demand for pedagogy.

The definitions of dyscalculia usually do not even mention geometry. However, geometrical knowledge has extensive practical implications and creates the basis for understanding more advanced mathematics.

Part V describes different approaches to recognition and intervention, elaborating ways of how to assess mathematical learning difficulties and focusing on different types of interventions and how the research on learning difficulties could guide education.

Research has opened our eyes to multidimensional diversity of skills in the classroom, which asks for improved methods of recognising the individual differences. Likewise, the discussion about how to define and assess the mathematical learning difficulties is vivid. The old diagnostic models of discrepancy between math and other skills have been challenged with new ideas, ideas that try to connect education, remediation and interventions more tightly to assessments. In this last part, we hence want to give an overview about the range of diagnostic procedures as well as intervention approaches and their theoretical bases.

All around the world, countries have plans on how to digitalise education. In a very near future, the access to technologies like the Internet will revolutionise assessment, e.g. national assessments which are carried out more and more often with computer-assisted tests. Likewise, there are increasingly more electronic educational and intervention materials available. Technology will provide teachers and practitioners with new tools for different ways to assess skills and progress of learning.

Research has demonstrated that children from low-resource communities who experience gaps in opportunities for learning may also have lower executive functioning (EF), and this risk is exacerbated for children who are second-language learners. Differences in EF between groups raise important *equity* issues that we must address to meet the needs of all children and thus the entire community of learners in a fair way. As such, children with special needs likely require special interventions. The authors introduce what we know about group-based interventions as well as about how the rapidly increasing technology changes the educational world.

This last chapter also focuses on the title of the book by elaborating on how the findings from the labs turn into effective practice.

With this book, we would like to help in bridging the research on numerical cognition and learning to practical applications in the classrooms, schools and clinics. There are a lot of national and international initiatives and actions underway on improving learning and teaching numeracy globally. We, together with our almost 100 authors, hope that this book will encourage you to walk over that bridge.

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Part I

Development of Number Understanding: Different Theoretical Perspectives

Chapter 2
Neurocognitive Perspective on Numerical Development

Karin Landerl

Introduction

Numbers are an integral part of our everyday life, not only as the basic elements of arithmetic (2 + 4, 247 × 39) but also as lexical numbers (e.g., 7Eleven, 9/11) or ordinal numbers (e.g., 7.2. – the seventh day of the second month of the year). How does our cognitive system process all this valuable information? How do children develop efficient numerical processing skills? And why is it that in a considerable number of individuals, the cognitive system does not tune into processing numbers efficiently during development, inducing the neurodevelopmental disorder of dyscalculia? Numerical processing and the cognitive representation of numbers in different formats have been demonstrated to constitute a central core mechanism underlying arithmetic development in typically developing children and a core deficit strongly associated with developmental dyscalculia (e.g., Butterworth, 2005). The present chapter aims to give a short overview of current research on the development of this core mechanism during early childhood.

The Triple-Code Model of Numerical Processing and the Mental Number Line

Numbers appear in three different formats: (1) as analog magnitude information (4 candles on the birthday cake, about 700 people at the concert, more apples than pears on the fruit stand), which is also referred to as "number sense," (2) in the form of spoken or written number words (three, fourteen), or (3) as Arabic numerals and

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multi-digit numbers (7, 423). The triple-code model (Dehaene, 1992; Dehaene & Cohen, 2007) postulates that in competent adults, these three numerical representations are strongly interconnected, so that, for example, we cannot suppress activation of the analog magnitude representation when we see an Arabic number: In the numerical Stroop paradigm, participants are asked to select the physically larger of two digits. The numerical value of the presented number clearly interferes with the nonnumerical judgment on physical size, inducing faster response times for items in which the numerical value is congruent with the physical size (e.g., 2 8) than in incongruent items (e.g., 2 8, Henik & Tzelgov, 1982). However, our knowledge on how these highly efficient neural networks develop during infancy and childhood is still limited.

In adults, numbers are typically conceived as falling along a mental number line (Dehaene, 1997), which is spatially oriented with smaller numbers on the left and larger numbers on the right. This mental number line is logarithmically condensed thus that the space between pairs of numbers becomes smaller as numerical magnitude increases. Another important question then is when and how children develop such a spatially oriented mental number line.

The Approximate Number System

Processing of analog magnitudes via the so-called approximate number system (ANS) or "number sense" is assumed to be inborn: It has been studied in different kinds of animals (e.g., Agrillo, 2015; Beran, Perdue, & Evans, 2015). Preverbal numerical processing skills have also been demonstrated in newborns and infants (Libertus & Brannon, 2010; Lipton & Spelke, 2004; Schleger et al., 2014), who have the basic ability to differentiate between two numerical set sizes. If the numerical difference is sufficiently clear, infants show significant effects of dishabituation (usually measured as an increase in visual inspection time) when set size changes. Some studies provide evidence that two systems of numerical processing can be differentiated even in infancy (e.g., Feigenson, Dehaene, & Spelke, 2004). Hyde and Spelke (2011) suggest that a "parallel individuation" system underlies infants' ability to enumerate small item sets up to four, while a ratio-dependent numerical magnitude system is involved in processing larger numerical sets. Interestingly, Hyde and Spelke (2011) also found evidence for distinct neural pathways in 6–7.5 months old infants: Processing of small numerical sets evoked a positivity about 400 ms after stimulus presentation (P400) in an ERP analysis, while for larger set sizes the positivity appeared somewhat later (P500) and was ratio-dependent.

Number Words and Verbal Counting

While the non-symbolic number sense is biologically driven, symbolic representations like number words and Arabic numbers are clearly culturally transmitted. As part of their language acquisition, children learn simple number words and acquire the skill of verbal counting. This process entails the one-to-one principle, meaning that each counted object corresponds to one specific number in the ordinal counting sequence. Importantly, it also entails the cardinality principle, that the last number in the ordinal counting sequence represents the set size, prompting children to match distinct set sizes with their corresponding symbolic number representation (Gallistel & Gelman, 2000). However, there is an ongoing debate whether these skills, which require an exact representation of distinct numerical sets, are rooted in the approximate number system or whether the two systems develop independently (Izard & Dehaene, 2008; Karolis & Butterworth, 2016; Leslie, Gelman, & Gallistel, 2008; Mussolin, Nys, Leybaert, & Content, 2016; Odic, Le Corre, & Halberda, 2015).

Interestingly, the acquisition of the verbal representation of number in terms of number words is surprisingly slow. Between 2 and 3 years of age, most children learn to recite number words in (roughly) correct order. However, usually no specific numerical meaning is attached to these number words; they are rather recited like the ABC or a nursery rhyme. The first number word that is usually associated with a numerical meaning is “one”: Children can correctly report that a set contains one item, and they will give correctly one item when asked to do so (Le Corre & Carey, 2007; Le Corre, Van de Walle, Brannon, & Carey, 2006). At this early stage, higher number words are not differentiated at first. So when these children are asked to give a distinct number of items higher than one (e.g., three), they will grab a random set of items, without considering their exact numerosity (Le Corre et al., 2006). Only over a relatively extended period of 6–12 months, children will associate a numerical meaning to the number words “two,” then “three,” and then “four,” before they fully understand the cardinality principle. Note that during this period, children are usually already able to carry out Verbal Counting procedures by correctly applying the one-to-one principle and assigning number words in the correct order to each item that they are serially pointing at (Gelman & Gallistel, 1978; Le Corre & Carey, 2007).

Odic et al. (2015) recently reported that during this intermediate period, the mapping between ANS and verbal number representations may not be bidirectional: While children showed some competence to generate an approximate response to a verbal number (producing finger taps), the reverse ability to respond with a corresponding number word to an analog magnitude (watching a rapid sequence of dots or finger taps) seemed to require full understanding of the cardinality principle. Thus, mapping from precise number words to approximate numerosities may help children to calibrate between the two systems. Furthermore, it is possible that the acquisition of exact number skills may in turn help children to further refine their approximate number system (Mussolin et al., 2016).

Visual-Arabic Code

There is still a lot to learn about the early acquisition of Arabic numerals and on how they are integrated with the two other numerical codes, ANS and number words. These mappings are usually acquired between the ages of 3 and 5 years (Benoit, Lehalle, Molina, Tijus, & Jouen, 2013; Hurst, Anderson, & Cordes, 2016), and most children will be familiar with basic number words when they acquire their first Arabic numerals. So the important question arises whether Arabic numerals are directly mapped with the ANS like number words or whether the mapping is indirect via an association with the corresponding number word which in turn is mapped with the ANS.

Empirical findings on this issue are scarce and as yet contradictory. Benoit et al. (2013) found that four-year-olds were better in matching Arabic numerals with non-symbolic numerosities than in matching them with their corresponding number words, suggesting a direct mapping between numerosities and Arabic numerals. However, their displays of non-symbolic numerosities presented dots as canonical dice patterns, which perhaps provided children with an advantage on enumerating the non-symbolic displays. In contrast to this study, Hurst et al. (2016) reported that three-to-four-year-olds found it clearly harder to map numerosities with Arabic numerals than with number words and that performance on Arabic numeral-number word mapping tasks was better than on Arabic numeral-non-symbolic numerosities mapping. Furthermore, while matching non-symbolic sets to symbolic representations was easier for small compared to larger set sizes, such a set-size effect was lacking for the Arabic numerals-number word mapping. This lack of a set-size effect for matching of the two symbolic codes indicates that it does not depend on the magnitude of the numerosities represented. In summary, these findings rather support the view that children first map Arabic numerals to number words.

While the basic associations between (small) non-symbolic numerosities, number words, and Arabic numerals are established during the preschool years, we do not know much about how long it takes until children have them sufficiently automatized. Such automatization probably entails a neural integration process based on comprehensive experience and practice. That such processes of neural integration take longer than might be expected becomes clear from research on another domain of symbolic learning, development of letter-sound associations. Even 11-year-old children, who were advanced readers and showed highly automatized letter processing on the behavioral level, did not yet show adult-like brain responses in an audiovisual ERP paradigm (Froyen, Van Atteveldt, Bonte, & Blomert, 2008). The corresponding research on the developmental integration of the three numerical codes needs as yet to be done. The lack of this research is even more surprising as current evidence suggests that the quality and precision of the mappings between ANS and symbolic presentations of number are a specific predictor of arithmetic skills even after controlling for age, vocabulary, and precision of the ANS (Libertus, Odic, Feigenson, & Halberda, 2016).

Place Value and Number Syntax

Another challenge of the symbolic numerical codes is that they represent place-value systems. This is particularly tricky in the Arabic notation as the very same numeral has completely different numerical meanings depending on its position in multi-digit numbers (e.g., 503, 35, 305). Transcoding between Arabic numbers and number words as well as deriving the numerical value of multi-digit numbers requires the competent application of a number of rules (Barouillet, Camos, Perruchet, & Seron, 2004), some of which are additive (25 = 20 + 5), while others are multiplicative (700 = 7 × 100), and children are not always aware which one is correct (e.g., writing 7000 as 71,000 by adding the 7 to the numeral for 1000). In order to adequately comprehend the Arabic notation system, children need to understand that the zero does not equal "nothing," but plays an important role as a placeholder. On the other hand, children do not always understand that zeros must be overwritten in multi-digit numbers and combine Arabic numbers like other compound words (e.g., writing 700,050,042 for 7542). The ability to transcode multi-digit numbers is significantly associated with children's arithmetic skills in elementary school (e.g., Moura et al., 2013).

Understanding of the place-value system depends on how transparent place value is represented in the particular language. In some languages, like Japanese, number words provide a one-to-one translation of the corresponding Arabic number (e.g., 7546 corresponds to "seven thousands, five hundreds, four tens, six units"), so all children need to learn is the number words from zero to nine and the words for unit, ten, hundred, and thousand, and once they have worked out the place-value system, they can transcode any particular number. This is of course very different in English and many other languages. Note that European languages typically are particularly intransparent for numbers of the second decade: "Eleven" and "Twelve" do not provide any information that they correspond to 10 + 1 and 10 + 2. The other "teens" in the English number word system ("thirteen" to "nineteen") are composed differently from number words higher than 20, as the unit is named before the ten. In English number words higher than 20 are largely consistent to the corresponding Arabic number, but still, children need to acquire the decade names (twenty, thirty, forty,…).

Many languages like German or Dutch consistently inverse the ten and the unit of their two-digit numbers (e.g., "one-and-twenty"). In children's performance on number reading and number writing, inversion errors (reversing the ten and the unit) constitute a major error source in languages with inversion like German or Dutch, while such errors are (obviously) rather exceptional in languages without inversion like Japanese or French (Imbo, Vanden Bulcke, De Brauwer, & Fias, 2014; Moeller, Zuber, Olsen, Nuerk, & Willmes, 2015). In a longitudinal study in German, the number of inversion errors in Grade 1 predicted children's math grades two years later (Moeller, Pixner, Zuber, Kaufmann, & Nuerk, 2011). In this study, the number of inversion errors was specifically related to children's later performance in carry problems. While arithmetic problems involving carry are always more difficult than problems without carry due to heavier working memory load, the carry effect seems to be larger in languages with inverted verbal mapping of the place-value structure, like German, than in Italian, where number words are consistent with the Arabic notation (Göbel, Moeller, Pixner, Kaufmann, & Nuerk, 2014). Thus, the transparency of the mappings between the different numerical codes appears to directly influence children's arithmetic development.

Experimental Effects of Numerical Processing

A number of numerical processing paradigms have been used to explore, how numbers and numerosities are represented in the cognitive system. As these paradigms are relatively simple and can be used with young children, they have also been highly useful to examine the development of numerical processing skills.

Subitizing vs. Counting in Dot Enumeration

In the dot enumeration paradigm, participants have to count a limited number of dots (usually no more than ten) as quickly as possible. Such enumeration tasks induce a characteristic pattern of performance, indicating two distinct enumeration systems (Vetter, Butterworth, & Bahrami, 2011): Small numerosities up to three or four are typically responded to with high accuracy and speed. This process of rapid identification of small dot numbers is called *subitizing*. For numerosities higher than four, reaction times and error rates rise systematically with increasing numerosity, indicating the execution of a sequential counting procedure. While sequential processing of higher numerosities is likely to involve verbal counting, the subitizing process seems to be more visually based and is sometimes interpreted as a reflection of the preverbal enumeration skills evident in infants (Libertus & Brannon, 2010). In a cross-sectional study, Schleifer and Landerl (2011) found adult-like subitizing performance in 11-year-olds, but not younger children. Full competence in sequential counting of larger dot arrays was only evident in 14-year-olds, while younger age groups performed at less proficient levels. Interestingly, a number of studies (Landerl, 2013; Moeller, Neuburger, Kaufmann, Landerl, & Nuerk, 2009; Reeve, Reynolds, Humberstone, & Butterworth, 2012; Schleifer & Landerl, 2011) found specific subitizing problems in poor math achievers, while in the counting range, responses were generally slower, but the gradients of response time slopes were similar to typically developing control groups. Of particular interest are two longitudinal studies: Landerl (2013) followed a sample of 41 children with dyscalculia from Grade 2 to Grade 4 and observed an atypically large increase in response times in the subitizing range (enumeration of one to three dots) across five assessment points. Reeve et al. (2012) could show that children who were particularly slow at dot enumeration at the age of six also had a significantly reduced subitizing range (two dots at 6 years, three dots at 8.5 years, and four dots at 9 years). These children also had relatively poor arithmetic skills at the ages of 9.5 and 11 years. Thus, although most children with arithmetic deficits also show the typical discontinuity of counting performance between lower and higher numerosities (but see an individual case described by Moeller et al., 2009), their subitizing mechanism seems to be impaired (Butterworth, 1999), and their sequential counting is overall slower.

Ratio Effect in Non-symbolic Number Comparison

Another simple experimental paradigm that is highly informative with respect to the cognitive representation of number is the magnitude comparison task (also called non-symbolic number comparison). Individuals are asked to select the numerically larger of two non-symbolic magnitudes (e.g., dots). The accuracy and speed with which this decision is made depend on the numerical distance between the two numerosities. The larger this distance, the faster and more accurately the decision is made due to a smaller internal overlap between the two magnitude representations. The acuity of analog magnitude processing is usually specified as Weber fraction which is the smallest ratio of two numerosities that a person can reliably judge as larger or smaller (Halberda, Mazzocco, & Feigenson, 2008). The Weber fraction increases during development, allowing children to discriminate similar numerical sets more precisely. Acuity of non-symbolic magnitude processing in kindergarten was found to predict arithmetic competence at age six (Mazzocco, Feigenson, & Halberda, 2011), and interindividual differences in the acuity of magnitude processing were found to be directly related to arithmetic competence (Libertus et al., 2016). However, findings on children with mathematical learning difficulties are inconclusive. While virtually all studies report deficient performance in dyscalculia samples for symbolic comparison tasks presenting Arabic numbers (see 7.3.), results on non-symbolic comparison paradigms are mixed. In their review, De Smedt, Noël, Gilmore, and Ansari (2013) point out that age might be a critical factor: Weaker performance compared to typically developing children was mainly found in samples older than 9 years, while samples aged 6 to 9 years did not show significant deficits. This pattern suggests that poor non-symbolic processing might be a consequence rather than a cause of mathematical development, which helps to refine the acuity of the ANS (Noël & Rousselle, 2011).

Distance Effect in Symbolic Number Comparison

In the symbolic number comparison paradigm, participants are typically asked to select the numerically larger of two Arabic digits. Note that this task requires participants to efficiently access the analog magnitude representations that correspond to the Arabic symbols, which then constitute the basis of the decision. A recent meta-analysis of 45 studies with more than 17,000 participants (Schneider et al., 2016) indicated that mathematical competence (particularly mental arithmetic) is more closely associated with symbolic than with non-symbolic number comparison. Again, a typical and highly robust pattern of performance arises from this task: The decision is more efficient in terms of accuracy and speed for large than for small numerical distances between the two presented numbers (Moyer & Landauer, 1967). This distance effect is interpreted as evidence for the access of the analog magnitude representations corresponding to the numbers. The more overlap between these magnitudes, the harder it is to make the decision.

A distance effect in symbolic comparison tasks has been demonstrated even among preschoolers (Sekuler & Mierkiewicz, 1977). Some studies suggested that the symbolic distance effect might decrease during development (Holloway & Ansari, 2008), indicating a continuing specification of the cognitive representation of number, and that the size of the distance effect might therefore be related to arithmetic skills (De Smedt, Verschaffel, & Ghesquière, 2009). However, other studies found a rather stable influence of numerical distance on symbolic number comparison across different age or achievement groups, accompanied by a general decrease in response times (Girelli, Lucangeli, & Butterworth, 2000; Landerl, 2013; Landerl & Kölle, 2009; Reeve et al., 2012). Thus, it seems that the distance effect may not be directly related to children's math achievement (Gibson & Maurer, 2016). However, significantly increased overall response times in comparison of simple one digit numbers seems to be a persistent and specific deficit in children with dyscalculia (De Smedt et al., 2013; Landerl, 2013; Reeve et al., 2012; Rousselle & Noël, 2007) as well as adults (Rubinsten & Henik, 2006).

Size-Congruity Effect in Symbolic Comparison

The number comparison paradigm has also been applied to investigate the automaticity of numerical processing. When individuals are asked to decide which of two digits is physically larger, numerical value interferes with their physical judgments. Generally, incongruent items (e.g., 4 9) are responded to more slowly than congruent items (e.g., 4 9; Bugden & Ansari, 2011; Girelli et al., 2000; Landerl & Kölle, 2009). This Size-Congruity Effect indicates automatic activation of numbers and requires a certain amount of experience. Some studies show interference between physical and numerical size even in first grade (Landerl, 2013; Rubinsten, Henik, Berger, & Shahar-Shalev, 2002), while in others it was not even found in fourth graders (Landerl, Bevan, & Butterworth, 2004). Interindividual differences in the degree of automatization and differences in task format make it difficult to compare findings across studies. Bugden and Ansari (2011) as well as Landerl (2013) did not find a direct relationship of the size-congruity effect with children's arithmetic skills and concluded that automatic processing of numbers is not related to mathematical competence.

Compatibility Effect in Comparison of Two-Digit Numbers

The number comparison paradigm has also been used with two-digit numbers, where it also induces a robust distance effect. More interestingly, though, it is also informative in as to how children learn to integrate the ten and unit position into one numerical value. This is evident from the development of the compatibility effect (Nuerk, Kaufmann, Zoppoth, & Willmes, 2004): Response accuracy is generally higher and response time lower when both tens and units are higher in one number (e.g., 83_62, 8 > 6, and 3 > 2) than when tens and units of the two numbers are incompatible (e.g., 82_63, 8 > 6, but 2 < 3). The compatibility effect indicates that multi-digit numbers are not processed holistically, but require adequate integration of the composite numerals and their place value. The compatibility effect is especially marked in young and unexperienced children (Landerl, 2013; Landerl & Kölle, 2009; Pixner, Moeller, Zuber, & Nuerk, 2009) and predicts later arithmetic skills (Moeller et al., 2011). Although verbal skills are not directly required in the number comparison paradigm, the size of the compatibility effect seems to be influenced by language properties: The difference between compatible and incompatible number pairs was found to be larger in German, a language that inverts the tens and units in number words, than in Italian and Czech (Pixner, Moeller, Hermanova, Nuerk, & Kaufmann, 2011). Quite impressively, Landerl (2013) found that German children with persistent arithmetic deficits performed on chance level for incompatible items at the beginning of second grade, suggesting that these children have not yet worked out the place-value system, although according to the math curriculum, they are already expected to learn their times tables.

SNARC Effect

Convincing evidence for a spatially oriented mental number line in adults comes from the so-called SNARC effect (Dehaene, Bossini, & Giraux, 1993). SNARC is an acronym for “spatial-numerical association of response code,” and the SNARC effect denotes the finding that during nonnumerical decisions like parity judgment, decisions are made systematically faster with the left hand for small numbers and faster with the right hand for large numbers. A SNARC effect was reported from about seven years of age on, while for children younger than seven years, evidence is mixed and seems to strongly depend on the experimental paradigm applied. One limitation is that young children may not always automatically activate the analog magnitude representation when presented with Arabic numbers. Interestingly, Chan and Wong (2016) recently reported evidence that the SNARC effect in Chinese preschoolers might be related to their understanding of ordinality and not (yet) to magnitude processing. Thus, the left-to-right orientation of number representation might be based on children’s experience with counting in that direction. This finding may explain why in children the SNARC effect seems to be unrelated to early numerical abilities (Chan & Wong, 2016) and to mathematical skills (Gibson & Maurer, 2016).

To sum up, a variation of different paradigms has been used to investigate the development of the cognitive representation of number. The general pattern is that the representational system of numbers and numerosities becomes more precise and more efficient during typical development, while this is not the case (at least not to the same extent) in individuals with dyscalculia. Interestingly, very few studies examined the development of numerical processing itself in a longitudinal design. So we do not have sufficient knowledge about the stability of the effects described above across time. Note that the longitudinal study by Landerl (2013) suggests high stability for overall task performance, but stability of the within-task effects described above was quite low until Grade 3. Continuing research efforts focusing on the development of numerical processing is even more important as an increasing number of cross-sectional as well as longitudinal studies clearly show that these skills, particularly the early understanding of symbolic representations of number, are related to children's arithmetic development.

Numbers in the Brain

While numerous studies have investigated brain structures and activation patterns underlying numerical processing in adults, our knowledge of how the developing brain deals with numerical information is still limited. According to the triple-code model (Dehaene & Cohen, 1995; Dehaene, Molko, Cohen, & Wilson, 2004), competent adults process numerical codes in distinct brain areas: While the analog magnitude representation is associated with bilateral parietal brain areas, the verbal code is linked with left perisylvian language regions (and subcortical structures, like basal ganglia and thalamus) and the visual-Arabic code with bihemispheric inferior ventral occipitotemporal regions. In addition, prefrontal areas are activated during more complex arithmetic problem solving. Within the parietal lobe, Dehaene, Piazza, Pinel, and Cohen (2003) specified three distinct neural circuits: the intraparietal sulcus (IPS) is assumed to be involved in analog magnitude processing, while adjacent regions, particularly the posterior superior parietal sulcus (PSPL) and the angular gyrus, are associated with spatial orienting on the mental number line and verbal number fact knowledge. However, it takes many years of experience and interactions with different number formats and number tasks in order to develop these differentiated neuronal networks.

Interestingly, a recent meta-analysis (Kaufmann, Wood, Rubinsten, & Henik, 2011) revealed an age-dependent activation shift from anterior to more posterior brain regions within the parietal cortex for non-symbolic numerical processing. The authors interpreted this shift as stronger reliance on finger-based calculation strategies among children compared to adults. This interpretation is plausible, as observed activations in the anterior IPS among children are relatively close to the finger areas in the sensory homunculus on the postcentral gyrus. However, it may also be associated with the fact that non-symbolic number comparison in young children is sometimes presented via finger patterns (Kaufmann et al., 2011).

For symbolic number processing, findings indicate a decrease in frontal and an increase in parietal brain areas (see Kaufmann et al., 2011 for a review). This frontoparietal shift indicates an increasing functional specialization of (intra)parietal areas. This interpretation is supported by findings on individuals with arithmetic deficits, who have been found to show reduced activation during non-symbolic comparison of numerosities (Mussolin et al., 2009; Price, Holloway, Räsänen, Vesterinen, & Ansari, 2007), symbolic comparison (Mussolin et al., 2009), and arithmetic (Kucian et al., 2006).

Even more impressively, a number of studies also reported structural brain differences in terms of reduced gray matter in individuals with arithmetic deficits in areas known to be involved in basic numerical processing, including the IPS (Isaacs, Edmonds, Lucas, & Gadian, 2001; Rotzer et al., 2008; Rykhlevskaia, Uddin, Kondos, & Menon, 2009).

It should be noted that so far findings are mostly based on cross-sectional comparisons of different age groups, while longitudinal studies are largely lacking. Of particular interest are longitudinal studies that follow specific changes in brain activation, structure, and connectivity during the preschool period, when the basic associations between numerical codes are developed. Admittedly, the current brain imaging methodologies are not always very well equipped to study this young age group.

Implications for Instruction and Intervention

Arithmetic is a complex, multi-componential skill that depends on numerous cognitive mechanisms (Dowker, 2005). Children who are attentive toward the task to be solved and who have available sufficient working memory capacity and executive control show better arithmetic skills than those who have problems in these domains (e.g., Clark, Sheffield, Wiebe, & Espy, 2013; De Smedt et al., 2009). Children also need to have sufficient verbal skills in order to cope with the verbal components of arithmetic (e.g., Donlan, Cowan, Newton, & Lloyd, 2007). Arithmetic development also depends on more general factors like motivation (Moore, Rudig, & Ashcraft, 2015), instruction (Hattie, 2008), parental support (e.g., Benavides-Varela et al., 2016), and socioeconomic background (e.g., Jordan & Levine, 2009). Importantly, comprehensive research of the last decade clearly shows that the neurocognitive representation of number and numerical processing constitute a core mechanism that is pivotal to arithmetic Development. During typical development, a genetically based ability to differentiate numerosities provides the basis for understanding the meaning of culturally transmitted symbolic representations of number in terms of number words and Arabic numbers. The mapping between these numerical codes is complex and takes years in order to develop neuronal networks that enable the highly efficient numerical processing that is typical among adults. Children who are able to develop efficient numerical processing skills early on can use these skills to acquire increasing competence in doing arithmetic, which includes understanding the underlying concepts, developing adequate procedures, and building up comprehensive knowledge of number facts (e.g., Barrouillet & Fayol, 1998). These competences in turn will provide an important basis for mathematical problem solving.

Children with dyscalculia, however, seem to have a hard time to develop their numerical processing skills and to map the different forms of numerical representations on each other. Thus, at the start of formal math instruction, these children lack the foundational ability to understand number (Butterworth, 2005), and their mental number line is perhaps still rather imprecise and underspecified (Landerl, 2013). Without sufficiently efficient numerical processing skills, their acquisition of arithmetic concepts and procedures will be slow and compensatory, and the buildup of number fact knowledge will be seriously impaired. Indeed, deficient number fact knowledge is one of the main symptoms of dyscalculia (e.g., Jordan, Hanich, & Kaplan, 2003).

For typically developing children, experience with number words, counting games, and Arabic numerals is usually sufficient to trigger associations with children's analog magnitude representations. In case of dyscalculia, however, the analog magnitude representations are either themselves underspecified or they are not readily accessible in order to form efficient connections. Thus, teaching numbers and calculation skills is not sufficient for these children; instead, they need support in establishing and specifying their cognitive representation of number as well as efficient interconnections between the numerical codes. An increasing number of evidence-based intervention programs, many of them computerized, provide excellent and engaging means to train children in these basic skills (see Chodura, Kuhn, & Holling, 2015, for a recent meta-analysis). Importantly, intervention programs should focus on symbolic number processing skills and arithmetic, as currently there is no conclusive evidence that training the non-symbolic ANS alone transfers to symbolic arithmetic (Szűcs & Myers, 2017).

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Chapter 3
Everyday Context and Mathematical Learning: On the Role of Spontaneous Mathematical Focusing Tendencies in the Development of Numeracy

Minna M. Hannula-Sormunen, Jake McMullen, and Erno Lehtinen

Introduction

In the past 30 years, researchers have increasingly paid attention to not only mathematical learning in the classroom and formal contexts but also to how informal learning outside of the classroom impacts children's mathematical development, especially in developing conceptually rich mathematical knowledge and skills (e.g., Resnick, 1987). Differences in young students' mathematical learning and learning difficulties cannot be explained only by the experiences students have during the deliberately organized teaching and training situations. However, it is only fairly recently that mathematics education and developmental psychology research have also begun to examine the role of children's spontaneous, self-initiated mathematical activities in this development (Hannula & Lehtinen, 2005; McMullen, Hannula-Sormunen, & Lehtinen, 2011). In this chapter, we summarize classical studies on major milestones of numeracy development and furthermore discuss the impact of children's and students' own activities in informal everyday situations on learning trajectories leading to an advanced number sense, which optimally supports their future mathematical learning.

Early Development of Numeracy

Preschool mathematical development forms the necessary basis for later mathematical skills learnt in school (Clements & Sarama, 2014; Fuson, 1988; Gelman & Gallistel, 1978; Mix, Huttenlocher, & Levine, 2002; Nunes & Bryant, 1996).

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Children's mathematical skills and concepts develop highly individually, both in the rate at which children attain essential mathematical skills and in substance within mathematical concepts and skills, as well as in relations among different aspects of numbers (Fuson, 1988; Sophian, 2007).

Early Approximate and Exact Number Recognition

Two separate representational systems allow dealing with small numerosities in infants and toddlers: a fast but relatively imprecise discrimination of numerical magnitudes, which is affected by set size ratio limit, and an exact object tracking system functioning in the small number range (Feigenson & Carey, 2003). These early systems for representing objects and approximate quantities can also be found in other animal species, like in primates and birds (for reviews, see, e.g., Dehaene, 1997).

In addition to these mechanisms forming the basis for magnitude representations, the concept of a "set of individuals" is central for natural number concept including counting and simple arithmetical operations (Spelke, 2003). The toddler learns to connect pre-attentional basic-level perceptual information about the exact values of small numerosities with the socioculturally supported nonverbal and verbal expressions of small cardinal values. First, exact nonverbal number representations allow the child to represent, identify, categorize, and compare sets of objects within a very small range of numbers. These first pre-numeric skills are gradually integrated into cultural enumeration practices with verbal number words (Hannula, Räsänen & Lehtinen, 2007; Mattinen, 2006; Mix et al., 2002; Wynn, 1990). Children know that number words refer to specific, unique numerosities before they know exactly to which numerosity each number word refers (Sarnecka & Gelman, 2004; Wynn, 1992b). Children seem to develop piecemeal in acquiring cardinal meanings of "one, two, and three." After this, understanding of cardinality of set results in the cardinality meanings for all number words within the child's number sequence.

Subitizing and Counting

Two processes are used for recognition of exact numbers of items (Sathian et al., 1999), and they can be distinguished from approximate number recognition (Lemer, Dehaene, Spelke, & Cohen, 2003). These are a highly accurate, very fast, parallel apprehending of items up to around three or four, often called as subitizing, and verbal counting, which is much slower, requires coordination of several attentional sub-processes, and works also for the enumeration of larger numbers (e.g., Jevons, 1871; Trick, Enns, & Brodeur, 1996). Children's subitizing-based enumeration skills develop during childhood (Starkey & Cooper, 1995). Subitizing-based enumeration of children is slower than that of adults, and the subitizing range is smaller (Chi & Klahr, 1975; Trick et al., 1996).

Counting involves reciting the list of number words one by one, synchronizing number words with individuating acts including planning the moves of attention, keeping track of counted targets and inhibition of previously counted targets, and finally activating the cardinal value of the last recited number word as the result of the counting (e.g., Fuson, 1988; Trick & Pylyshyn, 1994). Five how-to-count principles need to be respected when items are counted (Gelman & Gallistel, 1978). These are (a) one-to-one correspondence (all the objects in the target set must be counted and each of them only once), (b) constant order (number words need to be listed in the same order), (c) order irrelevance and (d) abstraction (during counting it does not matter in which order the items are counted or what kind of things are counted), and (e) the cardinality principle (referring to the last number tag used as the cardinal value of the whole set) (Gelman & Gallistel, 1978).

Counting skills develop slowly, which could be explained by several issues: differences in nonverbal and verbal number recognition systems, the demanding integration of different representations and procedures, and the need for lots of practice in acquiring accuracy in counting procedures (e.g., Fuson, 1988; Wynn, 1990). Counting practice with number words provides a child with the basis for constructing the hows and whys of counting, as well as the essential features of correct counting (Briars & Siegler, 1984; Cowan, Dowker, Christakis, & Bailey, 1996). Once the basic skills of counting items in lines are achieved, children move on to learning the marking strategies necessary in counting random arrangements of objects (Fuson, 1988). According to the reciprocal developmental views of Saxe, Guberman, and Gearhart (1987) and Sophian (1998, 2007), children's goal-based numerical activities are related to their conceptual knowledge about numbers and social goals of enumeration change along with the development of skills, and they direct children's attention to different aspects and uses of numbers and counting.

The number sequence production indicates the child's participation in sociocultural numerical activities. Learning the first number words has been described as a serial recall task, in which the cardinal and ordinal aspects, numerical relations as well as the base-ten structure of number words, are only later integrated in the number sequence (Fuson, Richards, & Briars, 1982). Eventually, after different developmental phases, the number sequence becomes a mental construction of the number line, including exact cardinal meanings and ordinal relations between numbers. Later on number words become countable objects themselves (Fuson et al., 1982).

Basic Arithmetic Skills

The basic arithmetic skills that enable verbal adding and subtracting develop together with enumeration, number sequence skills, and separate schemes of quantitative increasing and decreasing. Similar to the early basis for exact number recognition, procedures for numerical operations are also constructed in infancy on object files individuating small exact numbers of items and an analogical magnitude-based estimation for representing numerosities (e.g. McCrink & Wynn, 2004; Wynn, 1992a) and in toddlerhood, on experiences with combining and separating sets of objects. These nonverbal skills form the basis for the development of conventional verbal arithmetic methods (Levine, Jordan, & Huttenlocher, 1992). The physical and social world of young children provides plenty of opportunities for them to develop concepts about amounts of material, their comparisons, and the different effects of actions on these amounts. Resnick and Greeno (1990) propose that children can perceive and reason about aggregations of amounts and objects before they represent them systematically. Their knowledge of arithmetic number facts and their methods of counting to find answers to addition and subtraction tasks are gradually integrated into a unified set of numerical relations, which form the natural number system. Thus the number facts are based on counting methods for arithmetical operations, and the numbers represent members of sets with true cardinal values (see also Fuson, 1988; Sophian, 2007).

The development of natural and later rational number concept and arithmetic skills is a gradual and long-lasting process, which is supported and constrained by different experiences during childhood and adolescence. In the second part of this chapter we deal with some of the experiences and present a novel approach for understanding the role of children's own activity in this development.

Children's Mathematical Activities in School and Home

Individual differences in young children's early mathematical skills have been explained by the amount of deliberate mathematically related activities in homes (e.g., Skwarchuk, Sowinski, & LeFevre, 2014). However, Lefevre, Clarke, and Stringer, (2002) focused on parents' direct teaching of early number skills and showed that the frequency of this kind of home teaching was positively related to children's school-based mathematical achievement. However in many other studies parent's self-reported engaging in numeracy activities was not related to children's number skills development (e.g., Blevins-Knabe, Austin, Musun, Eddy, & Jones, 2000; Missall, Hojnoski, Caskie, & Repasky, 2015).

The nature of mathematical learning environments at home has been analyzed in many studies, but still little is known about the specific types of home numeracy activities in which children are engaged with their parents (Cahoon, Cassidy, & Simms, 2017). Ginsburg and his colleagues (Ginsburg, Duch, Ertle, & Noble, 2012) concluded that still parents do relatively little to encourage their children's numeracy learning and instead focus on teaching literacy (Ginsburg et al., 2012). Skwarchuk et al. (2014) made a distinction between formal and informal mathematical activities in parent-child interaction. LeFevre et al. (2009) found that children's mathematical skills were related to the indirect numeracy activities in which learning was incidental and embedded in regular family life.

These results suggest that there are potentially more subtle connections between numerical activities at home and success in mathematics. While examinations of the mathematical home environment provide some hints as the potential causes of individual differences in mathematical development, it is also possible that less explicit mathematical behaviors and activities play a role in mathematical success.

Role of Children's Own Practice in Numeracy Development

Our previous studies have focused on the development of exact number recognition skills and children's early mathematical development (e.g., Hannula & Lehtinen, 2005; a review Hannula-Sormunen, 2014). This work suggests that young children's early development and especially their developmental individual differences in exact number recognition and utilization cannot be adequately described in terms of earlier theories and methods capturing only the processes and skills which are used after a child has already focused attention on the numerical aspect of the task. This work shows that young 3–7-year-old children have substantial individual differences in their self-initiated, spontaneous focusing on numerosity (SFON) in tasks in which their possible failure to regard exact numerosity is not entirely explained by their inability to deal with the cognitive requirements of the tasks. It seems that individual differences in this self-initiated numerosity focusing explain some of the individual differences in children's numerical development, i.e., why some children develop better than others in numeracy during their childhood years. Exact number recognition and utilization are not totally automatic processes; instead, they need to be triggered in natural surroundings. When a child's tendency to spontaneously focus his or her attention on the aspect of number is very strong, this produces lots of practice in number recognition and utilization and thus enhances the child's understanding of numerical aspects as affordances of sets (Hannula & Lehtinen, 2005). By using the term "spontaneous," we do not refer to the innate origins of the tendency, but the self-initiated nature of focusing in a particular situation. Focusing on the aspect of exact numerosity requires determination of the set being perceived on some basis (e.g., shall I count the blue, red, big, small, or all flowers?), and this is needed in exact cardinality determination based on both subitizing and counting in a natural environment. Not all possible subitizable or countable numbers of items in a natural setting can be brought to the conceptual, conscious levels of processing. Mechanisms of object individuation are mid-level processes (Trick & Pylyshyn, 1994). They produce only pre-numeric individuation information on the objects. Thus, Hannula and Lehtinen (2005) proposed that an attentional process of focusing attention on the aspect of exact number in the set of items or incidents is needed for recognition of number on a conceptual level. It triggers exact number recognition processes and utilization of the recognized exact number in action.

Focusing on numerical changes while sets of objects are manipulated could be a crucial part of understanding the meaning for numerical operations, which could explain significant predictive relations between SFON and arithmetical skills (Hannula, Lepola, & Lehtinen, 2010; Nanu, McMullen, Munck, Pipari Study Group, & Hannula-Sormunen, 2018).

How to Measure SFON?

In their early studies, Hannula and Lehtinen (2005) found that there were inter-individual differences in young children’s tendency to focus spontaneously on the number of objects or events. They organized play-like situations where it was possible to observe if, without explicit guidance to do so, children focused on the number of objects or events and used this number in their actions. The tasks and activities were created in a way that would remind children of the games and other daily activities they do at home, in preschool, and day care. In the tasks there were many features on which it was possible to focus attention and children were not told that the activities in the tasks were related to numbers. For example, in the feeding games (e.g., imitation task), there was a plate of glass berries and a toy parrot with a big mouth into which it was possible to put berries. The researcher started the game by explaining that the idea is to feed the parrot. They then introduced the materials and said: “Watch carefully what I do, and then you do just like I did.” After that the researcher put two berries, one at a time, into the parrot’s mouth, and they disappeared with a bumping sound into the parrot’s stomach. The child was then told: “Now you do exactly like I did.” These activities were repeated with different numbers of berries. A parallel game-like task was, for example, putting envelopes into a postbox. For older children, both imitation tasks were used with two sets of different colored items. Overall, there now exist more than 20 different SFON task versions suitable for children and adults (for a review, see Hannula-Sormunen, 2014). These measures are based on activities which are close to children’s familiar play situations but which, at the same time, make it possible to measure the strength of children’s tendencies to spontaneously focus on numerosity by using well-defined standard procedures. Even in using these procedures, measuring spontaneous focusing tendencies is challenging.

Studies using the original SFON measures have shown that it is possible to measure the strength of children’s SFON tendency in a rather reliable way (Hannula & Lehtinen, 2005; Hannula-Sormunen, Lehtinen, & Räsänen, 2015; Nanu et al., 2018). Recently, several other measures have been developed by various researchers, which highlight different aspects of spontaneous focusing (see Rathé, Torbeyns, Hannula-Sormunen, De Smedt, & Verschaffel, 2016 for an extensive review). The design principles for SFON assessments include the following aspects: (1) mathematically unspecified settings, (2) multiple (mathematical and non-mathematical) interpretations possible, (3) fully engaging for all, and (4) within competence range (Hannula, 2005). Nothing in the task situation gives any hints to the participants that the SFON tasks would be in any way numerical in nature. The experimenter gives no feedback. The child’s attention and interest are carefully captured in the beginning of the task by the experimenter. It is important to carefully make sure that tasks involve only numbers so small that every child is capable of enumerating them. Similarly, all other cognitive requirements of the SFON tasks need to be at manageable level for all participants, so that participants’ insufficient motor skills, inhibition, verbal production, and working memory do not explain individual differences in the SFON tasks (Hannula, 2005; Nanu et al., 2018).

Guided focusing on numerosity (GFON) task versions have demonstrated that the children who had zero SFON responses were able to deal with the cognitive task requirements after their focus was explicitly guided toward the numerical aspects of the SFON task (Hannula et al., 2010; Hannula & Lehtinen, 2005). Children's performance on the guided tasks supports the hypotheses that SFON is a dissociable part of utilizing exact number recognition in action in (mathematically) non-guided settings. The analyses of video-recorded performance in the SFON tasks allow all quantification acts or indications of child's understanding of the quantitative goal of the task to be acknowledged as SFON. It is notable that by using a number range beyond participating children's capabilities, children's failure in producing equal sets could be caused by their inability to enumerate the sets, their lack of focus on numerosity, or even both of these reasons.

The use of the above described methods made it possible for Hannula and Lehtinen (2005) to separate attentional process SFON which is defined as a process of spontaneously focusing attention on the exact number of a set of items or incidents. This attentional process triggers exact number recognition and using the recognized exact number in action, particularly in natural situations where the numerical magnitudes are not artificially made evident, which is typical for most educational materials. It appears that even though there are task-individual interaction effects that may cause differences in SFON tendency across, for example, action and verbally based tasks, a confirmatory factor analysis revealed a second-order latent variable referring to underlying general SFON tendency (Hannula-Sormunen et al., in preparation).

Findings of SFON Studies

Since the initial study of Hannula and Lehtinen (2005), cross-sectional and longitudinal studies on SFON have been conducted by many research groups in several countries (Hannula-Sormunen, 2014; Rathé et al., 2016). SFON tendency is positively related to the development of cardinality recognition, subitizing-based enumeration, object counting, and number sequence skills before school age (Batchelor, Inglis, & Gilmore, 2015; Bojorque, Torbeyns, Hannula-Sormunen, Van Nijlen, & Verschaffel, 2016; Edens & Potter, 2013; Hannula, 2005; Hannula & Lehtinen, 2001, 2005; Hannula, Räsänen, & Lehtinen, 2007; Potter, 2009). SFON tendency can be enhanced through guided focusing activities in preschool at the age of 3 years (Hannula, Mattinen, & Lehtinen, 2005; Mattinen, 2006). Path models of the development of SFON and counting skills from 3 to 6 years of age indicate a reciprocal relationship between SFON and counting skills before school age (Hannula & Lehtinen, 2005). SFON tendency in kindergarten is a significant, domain-specific predictor of arithmetical, but not reading, skills assessed at the end of second grade (Hannula et al., 2010). At primary school age, dyscalculic students have a lower SFON tendency than their normally developing peers (Kucian et al., 2012). Spontaneous focusing on mathematically meaningful aspects seems to be one of the specific forms of mathematical behavior in children whose mathematical skills develop optimally during childhood years, and the lack of which is associated with mathematical learning difficulties. Individual differences in children's SFON are not explained by children's lack of enumeration skills or other cognitive skills needed for SFON tasks (Hannula & Lehtinen, 2005), and focusing on other aspects, such as spatial locations, is a separate process which does not explain away the correlation between SFON and counting skills (Hannula et al., 2010).

The theoretical explanation for the strong predictive role of SFON tendency is based on the hypothesis that SFON is an indicator of the amount of self-initiated practice in using exact enumeration that a child gets in her or his natural surroundings (Hannula & Lehtinen, 2005). High SFON tendency would result in much higher amounts of practice with enumeration than what those children get who only deal with numbers when they are guided by adults (Hannula et al., 2010; Lehtinen, Hannula-Sormunen, McMullen, & Gruber, 2017). So far this theoretical hypothesis has been supported by a few studies. First, as part of a SFON enhancement study of 3-year-olds (Hannula, Mattinen, & Lehtinen (2005)), the analyses showed a positive correlation ($r = 0.55$) between children's scores in SFON tasks and their SFON tendency, observed by the personnel in all day care settings. Similar results were presented by Batchelor (2014) who found a positive association between children's task-based SFON measures and their spontaneous focusing on numbers in play-based behavior as observed during parent-child play interactions.

There have been a few intervention studies in which SFON has been trained. In the first intervention study by Hannula et al. (2005), SFON training was conducted by using only small exact numbers up to three. Early educators guided children's attention to small exact numbers both in everyday situations and also in structured numerical games. Guided focusing on numerosities was done by talking about, showing, and manipulating small numbers of, e.g., toys, snacks, socks, or other things during everyday interaction. Structural games involved variations with the numbers of objects of a set (Marton & Booth, 1997). The number of the fishes was first changed and observed together with the children in a cartoon aquarium, and then after adults kept secretly changing the numbers of fishes, which made the aquarium an exciting numerical focusing target. The aim of guiding children to focus on numerosities within daily routines and games was to enhance children's spontaneous focusing on numerosities. Children who participated in the SFON enhancement program outperformed the control group in SFON tendency on a delayed posttest, which was conducted half a year after the pretest (Hannula et al., 2005; Mattinen, 2006).

Beyond Mere Numerosity: The Development of Relational Reasoning as the Foundation for Rational Number Knowledge

Exact numerical and whole number reasoning is only part of numerical development that is relevant for mathematics. Already young infants have been shown to recognize the halfway point in objects (McCrink & Wynn, 2007). The system of approximate number found in infants and even nonhuman animals has features of relational reasoning (e.g., Dehaene, Izard, Spelke, & Pica, 2008). Beyond these innate capacities, young children can also solve tasks using mathematical relations (e.g., Boyer, Levine, & Huttenlocher, 2008). Early primary school-age children are able to match proportional quantities, especially those represented by continuous quantities (Boyer et al., 2008; Spinillo & Bryant, 1999). Four-year-olds have also been found to be able to reason with proportional quantities (Sophian, Harley, & Martin, 1995). Mix, Levine, and Huttenlocher (1999) found that 4- and 5-year-olds were able to calculate simple fraction arithmetic problems with pieces of foam. Finally, Frydman and Bryant (1988) found that 5-year-old children could reason about fair sharing even with different sizes of candy.

Resnick (1992) described the development of mathematical reasoning by focusing on the nature of the objectified mathematical reasoning. Figure 3.1 applies this model to the development of relational reasoning in mathematics in the development of rational number knowledge. At the most basic level in Resnick's model is the mathematics of protoquantities, which have no explicit quantitative value. The nonverbal notion of half may be used in reasoning at this level, as has been found in infant habituation studies (e.g., McCrink & Wynn, 2007), although the explicit identification of these mathematical features by children is not possible (cf. Spinillo & Bryant, 1999). The next level of reasoning in Resnick's model is the mathematics of quantities, which involves reasoning about physical material with explicit quantities. In the case of relational development, this level may describe young children's reasoning about proportional relations (e.g., Boyer & Levine, 2012).

The level of the mathematics of numbers is where numbers begin to act as "nouns" or "conceptual entities that can be manipulated and acted upon" (Resnick, 1992, p. 414). This level would include the first skills and processes with symbolic

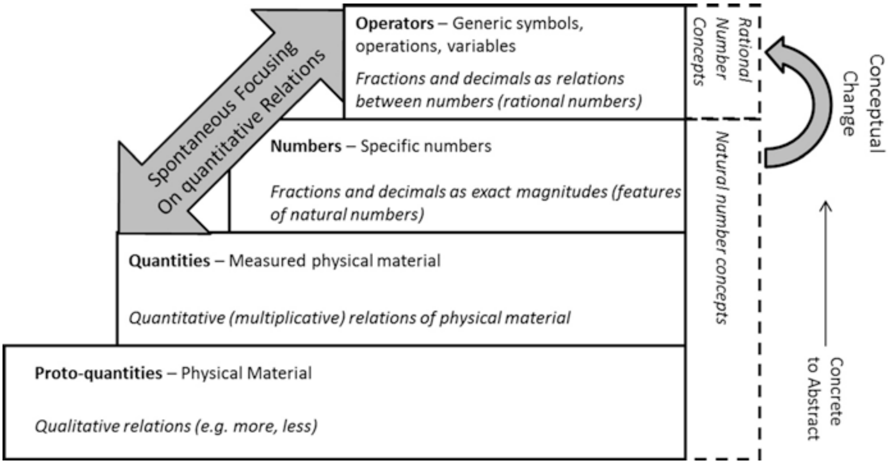

Fig. 3.1 The development of quantitative relations and rational number knowledge. (Based on model from Resnick (1992): From protoquantities to operators: Building mathematical competence on a foundation of everyday knowledge (McMullen, 2014))

fractions and decimals, where fractions and decimals are symbolic entities that can be acted upon and reasoned about independent of physical material. At this level, ½ is not the relationship of 1 part to 2, but may merely represent the magnitude of one-half of 1 (halfway between 0 and 1 on the number line), a notion supported by the continuity in the development of magnitude estimation skills from natural to rational numbers (e.g., Siegler, 2016). While many features of natural numbers can be attached to fractions and decimals, often in a supportive manner (Nunes & Bryant, 2008), it is also at this level that the natural number bias would cause problems with reasoning about fractions and decimals (Ni & Zhou, 2005).

It is only in moving into the mathematics of operations level that mathematically correct concepts of rational numbers appear. At this level, it is possible to reason about rational numbers as a concept, independent of specific numbers (Resnick, 1992, p. 414) representing the mathematical relations inherent in fractions (the relation between numerator and denominator) and decimals (the relation between place value and terms). Thus, at this level, rational numbers become mathematical objects that have specific features that are partially distinct from natural numbers (e.g., Vamvakoussi & Vosniadou, 2004). However, reaching this level is not a simple progression as described by Resnick, but instead may require radical change in the conception of the nature of number (Merenluoto & Lehtinen, 2004; Vamvakoussi & Vosniadou, 2004).

Spontaneous Focusing on Quantitative Relations

The quantitative relations that children experience in everyday situations are often approximate and dynamically changing. For example: "… a 7-year-old child traveling with her mother to visit their grandparents in the countryside. During the boring car drive the child starts spontaneously to think about the trip in terms of quantitative relations, asking 'Are we halfway there yet?'" (McMullen, Hannula-Sormunen, Laakkonen, & Lehtinen, 2016). In these situations the distances are often approximated, and, as well, the car can be approaching halfway, and after that it can be considered approaching halfway of the remaining distance to be traveled (i.e., 3/4) (McMullen et al., 2016). If children are involved in thinking about these kinds of "messy" quantitative relations of everyday contexts, it could have important effects on the way they think about the nature of numbers and how they can reason in novel situations with complex mathematical concepts. Based on a series of studies, McMullen and colleagues (McMullen, 2014; McMullen, Hannula-Sormunen, & Lehtinen, 2013, 2014) proposed that there is a tendency similar to SFON which indicates that children and school pupils can also focus spontaneously on the relation between two or more quantities in non-explicitly mathematical settings. Crucially, analogous to SFON tendency, individual differences in the tendency of spontaneous focusing on quantitative relations (SFOR) have been found to predict mathematical development in late primary school and lower secondary school (e.g., McMullen et al., 2016).

Studies with younger children conducted in Finland and the USA show that children begin to focus spontaneously on quantitative relations at the age from 6 to 7, but there are substantial inter-individual differences in the strength of the tendency still during the early school years (McMullen et al., 2013, 2014). Likewise, even after controlling for the ability to solve the tasks when explicitly guided to do so, there remain substantial inter-individual differences, even within grade levels, in SFOR tendency in studies of late primary school in Finland (McMullen et al., 2016) and Belgium (Van Hoof et al., 2016) and lower secondary school in the USA (McMullen, Hannula-Sormunen, Lehtinen, & Siegler, submitted).

A 4-year follow-up from the age of 7 to fourth grade reveals that individual differences in SFOR tendency may be related to later fraction knowledge (McMullen et al., 2014). As well, SFOR tendency was found to be a unique predictor of rational number conceptual development in late primary school students in Finland (McMullen et al., 2016) and Belgium (Van Hoof et al., 2016). In these studies the SFOR tendency was particularly related to the development of conceptual understanding of rational numbers, which has been difficult to support by traditional mathematics teaching and practice. A later study (McMullen, Hannula-Sormunen, & Lehtinen, 2017) showed that SFOR tendency is in a reciprocal relationship with rational number knowledge, similar to that which has been found between SFON and natural number knowledge (see Hannula & Lehtinen, 2005). The most recent findings indicate that besides rational number conceptual knowledge, SFOR tendency also predicts lower secondary school students' algebra knowledge (McMullen, et al., in press) and their flexible and adaptive use of rational number knowledge in novel tasks (McMullen et al., submitted).

There has yet to be established a causal relation between SFOR tendency and mathematical development; however, this relation has not been completely explained by a myriad of potential mediators, including nonverbal intelligence, arithmetic fluency, grade level, or prior knowledge (McMullen et al., 2016, submitted; Van Hoof et al., 2016) nor mathematical achievement (McMullen et al., submitted; Van Hoof et al., 2016), spatial reasoning, or mathematical motivation (McMullen et al., submitted). While SFOR tendency was relatively consistent, in terms of rank-order stability, over a 1-year period in late primary school, this does not suggest that it is impossible to enhance students' SFOR tendency. In fact, preliminary evidence suggests that by using innovative out-of-classroom activities, it is possible to increase students' tendency to focus on quantitative relations in SFOR tasks (McMullen et al., 2017). Thus, SFOR tendency is more likely a product of environmental factors rather than a static personal trait and may be malleable to explicit intervention.

Self-Initiated Practice and Number Sense

Not only is it expected that children would gain more practice with their numerical or mathematical skills and knowledge, but given the nature of the mathematical aspects embedded in everyday life, it may be that they also gain qualitatively better practice with mathematics than their peers that mostly encounter mathematics only in formal classroom situations. When children focus on number or quantitative relations in their natural environment, they are confronted with a higher variation of cognitive challenges than in dealing with deliberately planned school tasks. From a mathematical point of view, natural situations are "messy" in many ways. For example, the objects to be enumerated are seldom clear-cut objects clearly delineated from a monochromatic background, and it is often not completely clear what to include in the set that is to be enumerated. The number of objects and the observed quantitative relations are often changing dynamically in regular or irregular ways. Dealing with these situations requires flexible thinking about numbers and relations (Baroody, 2003). Dealing with numbers and relations in informal contexts makes it more difficult to rely on written procedures or algorithms but instead mental and oral processes (Schliemann & Nunes, 1990) that are often approximate and constrained by external factors (Lave, 1988). Self-initiated practice is often based on the inherent need to use the developing skills in novel situation (Piaget, 1955), which may result in strong personal involvement. In an everyday context, the feedback and practical consequences of enumeration or mathematical operations require a different control of correctness than in formal educational contexts. For example, Carraher, Schliemann, and Brizuela (2001) presented that in a formal school contexts, students "would sometimes claim that the amount of change to be returned to a customer after a purchase would be greater than the amount of money originally handed to the seller." In fact, even formal instruction often does not lead to coherent conclusions in formal mathematical settings, as a similar finding has been shown with students' estimations of positive fraction addition problems often being less than both addends (Braithwaite, Tian, & Siegler, 2017).

Discussion

Studies on spontaneous mathematical focusing tendencies (SFON and SFOR) suggest that theories of early development and later extensions of the number concept should also take into account the role of attentional processes and children's self-initiated practice. These studies highlight the role that mathematical practice in informal everyday contexts may play in mathematical development. Research on spontaneous mathematical focusing tendencies makes an important contribution by providing a novel explanation for the learning trajectories leading to differences in students' mathematical development, including their number sense. Mathematics curricula in many countries highlight the need to develop a more advanced number sense that can support the flexible and adaptive use of mathematical strategies (Mullis, Martin, Goh, & Cotter, 2016).

Studies on everyday mathematics are often presented as evidence for the need for a fundamentally different alternative approach (situated cognition) which differs from the traditional mathematics learning in schools (decontextualized cognition) (e.g., Lave, 1988). The interpretation of spontaneous focusing tendencies and self-initiated practice in informal activity contexts is different from the situated cognition approach. As studies show, spontaneous focusing tendencies are in reciprocal development with the formal mathematics learning in formal context. Focusing on mathematically relevant aspects of informal context and learning mathematics in formal education are fundamentally different experiences. However, students' own mathematical activities in informal situations do not lead to different mathematical knowledge, but instead provide opportunities to strengthen and enrich their mathematical development (see the criticism of Lave by Greiffenhagen & Sharrock, 2008).

In this way, it is important that formal mathematics education in school takes into account children's mathematically relevant activities in out-of-school situations and tries to better connect learning at school and learning in informal situations (Carraher et al., 2001; Resnick, 1987; Wager, 2012). In traditional mathematics teaching in all levels from early education to high school, mathematical content is often learned in an isolated, piecemeal way, and teaching rarely aims to deliberately trigger students to use mathematical knowledge as a way to see the world. In contrast, we have developed specific pedagogical tools aimed at improving students' and young children's awareness of opportunities to pay attention to mathematical aspects of their daily lives. For example, encouraging young preschool-age children to search for instances of particular numerosities in their everyday lives, in concert with practicing enumeration skills in a tablet-based game (Fingu; Holgersson et al., 2016), was able to improve students arithmetic skills in comparison with a control group (Hannula-Sormunen, Alanen, McMullen, & Lehtinen, 2016). As well, providing students with a framework and opportunities to explore their everyday surroundings for examples of quantitative relations, which are then analyzed for their mathematical features, has been shown to be effective in getting students to spontaneously focus on quantitative relations in task contexts (McMullen et al., 2017). In general, providing students with more opportunities to practice everyday problem solving in which mathematics are embedded may be valuable for promoting their deeper understanding of mathematical content (Pongsakdi, Laine, Veermans, Hannula-Sormunen, & Lehtinen, 2016). In the end, all of these approaches share the same aim – to break out of conventional preschool and school mathematics instruction and provide children and students with opportunities to explore mathematical phenomena in connection with everyday experiences.

Within this framework, also mathematical learning difficulties could be partly explained in novel ways. In some cases, these difficulties can be consequences of lacking self-initiated practice, whereas in other cases the mathematical difficulties appearing in school can mean that the child does not apply the mathematical thinking developed in out-of-school situations in formal tasks in classroom.

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Chapter 4
Competence Models as a Basis for Defining, Understanding, and Diagnosing Students' Mathematical Competences

Kristina Reiss and Andreas Obersteiner

Competence Models as Normative Definitions of Educational Goals

What students should learn in the mathematics classroom and, in particular, what they should understand and be able to do has been discussed intensively for many years. While in former years curricula focused mainly on the mathematical contents as input of instruction, the attention shifted to its outcome more recently. In consequence, standards for school mathematics were implemented in numerous countries in the last years (e.g., Kultusministerkonferenz, 2003, 2004, 2012, in Germany; Common Core State Standards Initiative, 2012, in the USA). Standards are normative tools in education. They describe the aims of schooling and illustrate what students are supposed to understand and to achieve. Moreover, they define the mathematical problems students should be able to solve.

Educational standards typically address students' competences. The concept of competences encompasses content-related knowledge as well as ways and means to apply this knowledge within a subject or in a general context. In this sense, competences have been defined by Weinert (2001, p. 27 f.; original citation in German, translation see Klieme et al., 2004, p. 16) as "cognitive abilities and skills possessed by or able to be learned by individuals that enable them to solve particular problems, as well as the motivational, volitional, and social readiness and capacity to utilize the solutions successfully and responsibly in variable situations."

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Competences are according to this definition neither personal traits nor general characteristics. They may be regarded as domain-specific requirements in order to solve a problem and may be acquired by an individual via learning. Standards thus also reflect mathematical literacy as defined in PISA, the Programme for International Student Assessment that emphasizes “an individual’s capacity to formulate, employ, and interpret mathematics in a variety of contexts” (OECD, 2013, p. 25). In PISA, as in most other contexts where applicable knowledge is addressed, motivation, volition, and social readiness do not play a prominent role. In the following we will therefore concentrate on the cognitive aspects of competences.

To define what competence means in a particular domain, standards rely on competence models. In the case of mathematical competence, these models include descriptions of mathematical contents like numbers and operations, algebraic thinking, functions, geometry, statistics, and probability as well as mathematical activities like problem-solving, reasoning and argumentation, modeling, use of tools, communication, and identification of mathematical structures. For example, the German standards for school mathematics at the primary level (Kultusministerkonferenz, 2004) state that with respect to numbers, students should acquire a variety of abilities: An important aspect is to understand place value and numbers up to 1,000,000 and their properties. Students should also be able to add, subtract, multiply, and divide whole numbers both mentally and in written form and recognize the relations between these basic arithmetic operations. They should develop different solutions to arithmetic problems, identify errors, control results, and use arithmetic rules. In addition, students are supposed to apply their knowledge in different contexts. They should be able to solve real-world problems using exact or approximate calculation and verify the results.

With such definitions, educational standards can provide guidance concerning the goals of learning. However, intending to define mathematical competence from a normative perspective, standards often seem to presuppose that teaching and learning take place under good or even optimal conditions, for example, in well-appointed rooms, with well-educated teachers, and in front of attentive students (e.g., National Council of Teachers of Mathematics, 2000; cf. Reiss, 2009). Thus, standards do not provide information about results of less successful learning and especially not about the knowledge acquisition of students with learning difficulties. Furthermore, educational standards specify the goals of classroom instruction but usually do not give recommendations or show ways how teachers should actually reach the goals in the classroom (Klieme et al., 2004). Accordingly, standards lack information about concrete steps leading to students’ competences. For doing so, more fine-grained models would be necessary that describe mathematical competence on various levels and provide information on possible learning gains and learning gaps. Such models may also be used for empirical evaluations of students’ competences and should be apt for understanding successful as well as ineffective learning processes.

Competence Models to Understand and Evaluate Students' Learning

In order to evaluate to what extent students meet the goals described in educational standards, the introduction of standards has often been accompanied by the implementation of testing procedures. Assessment instruments were based on models of mathematical competence that describe this competence in a hierarchical manner. Competence models that have been used for international comparison studies such as TIMSS (the Trends in International Mathematics and Science Study) for primary school students (e.g., Mullis, Martin, Foy, & Hooper, 2016; described as "international benchmarks") or PISA for secondary students (e.g., OECD, 2016; described as "levels of proficiency"; see also Reiss, Sälzer, Schiepe-Tiska, Klieme, & Köller, 2016, p. 226, for a more detailed report on competence levels in PISA) characterize mathematical competence based on empirical data. Accordingly, they do not aim at describing *desirable* knowledge as educational standards do but *realistic* and mostly empirically confirmed knowledge. In particular they reflect the important differences in students' performance and allow the appreciation of high-achieving students as well as the assessment of performance at a lower level and of students with learning difficulties.

These descriptions of proficiency or competence levels in the large-scale studies mentioned above were presented first in the late 1990s. They were accompanied by sample tasks and turned out to be useful for getting an idea of what students' performance at a certain level really meant. However, the levels of proficiency within these models were not sufficiently "fine-grained" but lacked details of mathematical processes and their products. The models mentioned above provided only rough information and, in particular, could not be used to explain how students would proceed from one level of proficiency to the next.

As a consequence, Reiss and Winkelmann (2009) presented a model of competency for the primary mathematics classroom (grade 4), which took into account more details of the students' actual problem-solving behavior. The model was based on data of a representative sample of students and of test items and was particularly used in the course of further test development. Moreover, it was extended and refined by Reiss, Roppelt, Haag, Pant, and Köller (2012) based on a larger number of test items and of participating students, thus using more representative data. The model includes descriptions of levels of competence with respect to different mathematical topics, such as numbers and operations or geometry or probability. The different levels of competence within the model were defined in a way that each level covered the same range of test points. In the following, the levels are described for competences concerning numbers and arithmetical operations.

Level I (Lowest Level): Basic Technical Knowledge (Routine Procedures Based on Elementary Conceptual Knowledge)

At this level, students know the basic structure of the decimal system such as the classification of numbers into ones, tens, hundreds, etc. Students are familiar with basic single-digit multiplication and addition problems. Subtraction and addition of lower numbers can be completed in partly written form. While doing this, students are able to check for the accuracy of their solutions. Written addition can be utilized correctly if two summands are involved. Written subtraction can be utilized if the carry is less than ten. In simple problems, students make use of the relationship between addition and subtraction. Strategies that students have learned during their first years at school – such as doubling a number – are applied to larger numbers. One-digit numbers or numbers below 1000 with last digits 0 or 00 can be placed on a number line with appropriate scale. Such numbers can be compared according to their size.

Level II: Basic Use of Elementary Knowledge (Routine Procedures Within a Clearly Defined Context)

Students use the structure of the decimal system when dealing with various representations of numbers. They recognize ordering principles and utilize these principles when continuing number patterns or during structural counting. Simple problems related to basic types of calculation are conducted mentally but also in a partly written or fully written form; occasionally, students find the solutions through systematic trial and error. During such trials, students make rough estimations and use them to determine the value range of their solutions. They correctly utilize fundamental mathematical terms (such as "sum") as well as basic mathematical procedures to solve simple word problems.

Level III: Recognition and Utilization of Relationships Within a Familiar Context (Both Mathematical and Factual)

The numbers that were taught as part of the curriculum are securely read and written in various representations (such as in a place value panel). Also, the number zero can be assigned correctly. Students are proficient in every type of a partly written or of a fully written calculation procedure that is part of the curriculum, but division is limited to single-digit divisors. They can use basic procedures of mental arithmetic even in unfamiliar contexts. They can transfer the multiplication table to a larger range of numbers, perform rough estimations, and round the results meaningfully, even with large numbers. Students recognize the relationship between addition and subtraction, as well as between multiplication and division. They can recognize and communicate simple structural aspects (e.g., in relation to sequences of the multiplication table) if the contents were practiced before. In addition, they model simple object matters and find solutions – as long as the numbers used are within the number range covered by the curriculum.

Level IV: Secure and Flexible Utilization of Conceptual Knowledge and Procedures Within the Curricular Scope

Students solve problems securely using all types and variations of the calculations taught as part of the curriculum. In particular, this includes written division. During calculations, students systematically utilize the attributes of the decimal system and relations between operations. They also apply this knowledge when investigating number sequences, for example, when finding incorrect numbers in a sequence or when explaining the underlying procedures for the sequence. Different calculation procedures are combined flexibly, and solutions are estimated or rounded appropriately. Students use solution strategies such as systematic trial and error even for more complex problems. Students are familiar with rules for calculation, and they can apply these rules meaningfully. Students are able to adequately model, and correctly work on, complex situations, and to present their solutions appropriately. Students' conceptual knowledge also includes special technical terms they can use and communicate appropriately.

Level V: Modeling Complex Problems and Independent Development of Adequate Strategies

Difficult mathematical problems can be solved correctly using various strategies. Relations between numbers are recognized according to the situation. Mathematical rules, such as the factorability of natural numbers, are utilized in problem-solving processes. Based on basic mathematical principles, even difficult solutions can be worked on and are solved utilizing procedures such as systematic trial and error. Special aspects such as calculations with fractions or numbers in decimal notation do not pose any problems. Moreover, students are able to comprehend and describe different solution approaches.

The model covers the key topics of numbers and operations and includes computation, estimation and number sense, word problems, and the structure of the whole-number system, which may be regarded as important aspects of this knowledge domain (Verschaffel, Greer, & DeCorte, 2007). Moreover, it takes into account that regarding products and processes, respectively, conceptual and procedural aspects of knowledge interact in problem-solving processes and complement each other (Hiebert, 1986).

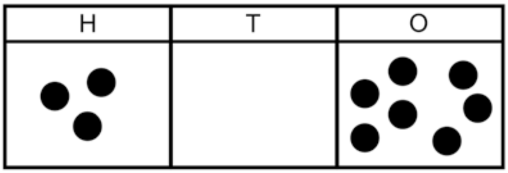
According to Pant, Böhme, and Köller (2012), students performing at level II are regarded to master a minimum standard in mathematics at the end of grade 4. These students should be able to successfully participate in further instructions in the next grade. Students performing at level III and IV perform on average or slightly above; students at level V show outstanding mathematical competence. Thus, the model may help to identify the individual level of performance and may be suitable to describe gaps of knowledge and competence.

Competence Models to Better Understand the Difficulty of Mathematical Problems: Examples

Teachers' diagnostic proficiency encompasses knowledge about the competences students need to have in order to solve specific mathematical problems. The competence model presented above can be used to describe these competences. This way, the model may help teachers to classify the requirements of a particular task and the proficiency of their students in solving this task. The following examples will illustrate how these aspects complement one another. The items shown below were used in a nationwide mathematics test for German primary schools. This test was completed by nearly all students and administered by teachers. The data presented below come from a pilot study administered by professional test personnel. The study yielded data on item characteristics like difficulty and solution rates as well as written solutions of students.

The first item presented here addressed the place value of whole numbers (Fig. 4.1; see also Obersteiner, Moll, Reiss, & Pant, 2015). For a correct solution, students were supposed to argue why the place value table did not represent the number 370. The item asked for a basic understanding of the place value system and was thus regarded at competence level II from a theoretical point of view. The empirical data substantiated the classification in level II as 56% of the children gave a correct solution. The information that theoretical and empirical difficulty were identical does not only verify the model but may also help teachers in understanding what low performance means with respect to students' knowledge.

Moreover, the artifacts as part of the empirical data provided information on students' errors or erroneous strategies. Obviously, a dichotomous coding may cause



Paul wants to show the number 370 in a place value table. However, he makes a mistake. Explain his mistake.

Fig. 4.1 Sample problem "Place Value" (translated from the German original) a loss of important details about individual solution processes. It is accordingly more apt for evaluating the performance of groups like schools or classrooms and less apt for understanding the individual need for support (see also Klieme et al., 2004) whereas looking at the solutions in detail provides relevant information. In this specific task, as mentioned above, 56% of the students gave the correct answer, but 23% of students did not answer at all. The remaining 21% of (wrong) solutions could be analyzed in depth. They showed that most of the students who did not succeed but tried a solution had at least rough ideas about the place value system but were not able to give a coherent argumentation. The problem lied in formulating and presenting the mathematical claim and not so much in understanding place values as such. From the mathematics education point of view, this information is helpful in particular for teachers. As level II is regarded the minimum standard, students' wrong or missing solutions are particularly important to know. They provide evidence why students fail in answering correctly and thus precisely identify their learning problems.

Another item aimed at the knowledge of number patterns (Fig. 4.2). In order to solve this item correctly, students needed to understand that all pairs but one added up to 100. From a theoretical point of view, the item was assigned to competence level III, namely, "recognize and explain the principles in number patterns if numbers are used that are part of the curriculum." However, 34% correct solutions showed that the empirically verified difficulty was higher and placed the item at level IV. Children who were not able to give a correct solution often referred to irrelevant aspects and stated, for example, that the number 5 was missing in the pattern or that the number pair given in the question was part of the set of pairs. Some tried to apply operations other than addition to the number pairs (e.g., multiplication: "93 is not a multiple of six."). None of these solutions provided a consistent pattern and could not be rated correct from a mathematical point of view. Presumably, the correct solution did not only presuppose an understanding of patterns but asked for a specific kind of number sense (Dehaene, 1997), which was an obstacle for many students. As mentioned above, all ideas – whether correct or incorrect – are valuable information for classroom work and might particularly lead to an explicit understanding of deficits and errors.

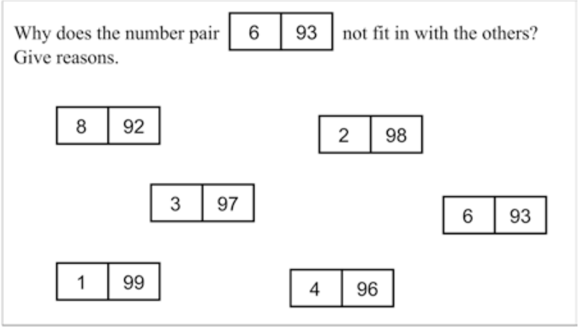


Fig. 4.2 Sample problem "Number Patterns" (translated from the German original)

Identifying students' learning progress in detail relies on knowing when they succeed as well as when and where they fail. These aspects are part of a teacher's knowledge on the diagnosis of learning. It is important that teachers are able to correctly interpret the test results in order to take advantage of them. This is a challenging task as educational standards in mathematics and mathematical competence are closely related concepts, but knowing one does not necessarily imply knowing the other. Research suggests that a profound knowledge of mathematics is the basis for teaching, but this content knowledge is not sufficient for being a successful teacher and should be accompanied by pedagogical content knowledge (Kunter et al., 2007; Shulman, 1987). Accordingly, teachers should not only learn whether a student's answer is right or wrong but they should also be assisted in understanding these answers in more detail. In particular, it is not only the product that counts in the classroom but also – and probably much more – the process leading to a correct or erroneous product.

Competence Models as Tools to Support Teachers' Diagnostic Processes

Understanding students' mastery of mathematical topics and evaluating their difficulties with mathematical problems are most challenging for teachers (Baumert et al., 1997). However, diagnosing students' learning processes is a task that teachers face in their everyday classroom. It is important that they fulfill this task according to high standards as it is the basis for adaptive teaching and thus affects the overall instructional quality (Helmke & Schrader, 1987). Diagnosing presupposes to systematically collect useful information in order to plan and initiate appropriate interventions (Hoge & Coladarci, 1989). Accordingly, diagnosing is based on data and the proper reflection of these data (Helmke, 2010; Herppich et al., 2017).

As part of their diagnostic activities, teachers should be able to evaluate students' learning processes and the requirements of specific contents of teaching (Helmke, Hosenfeld, & Schrader, 2004; Hill, Rowan, & Ball, 2005; Lorenz, 2011; Schrader, 2009). Diagnosing requires diverse professional competences of teachers and asks for content knowledge, pedagogical content knowledge, as well as for pedagogical knowledge (Shulman, 1987). All these components are regarded to be important in order to understand a students' behavior in the mathematics classroom (Helmke, 2010; Weinert, Schrader, & Helmke, 1990). However, some authors emphasize the role of pedagogical content knowledge (Brunner, Anders, Hachfeld, & Krauss, 2011) because a sound diagnosis will often be based on students' solutions to mathematical problems. Accordingly, teachers need to choose adequate tasks, to assess their difficulty, to identify errors, and to judge possible reasons for faulty solutions.

Obviously, competence models provide rather general ideas about students' knowledge and skills and describe outcomes but do not include ways how to acquire a specific type of knowledge or how to solve a certain mathematical problem. Still, competence models can support teachers in diagnosing their students' competences in at least three ways.

First, competence models may support teachers in understanding the structure and composition of their students' mathematical knowledge. Classroom instruction usually follows a domain-specific arrangement taking into account the organization and logic of a specific subject. In mathematics, for example, addition and subtraction of whole numbers or fractions are taught in parallel as they are regarded to be complementing operations: subtraction is the inverse operation of addition. Multiplication is taught at a later point in time as the definition of multiplication asks for the definition of addition: multiplication is regarded to be repeated addition (e.g., Common Core State Standards Initiative, 2012). The way in which these arithmetic operations are seen from the mathematics point of view and accordingly instructed in school does not necessarily reflect the views of children on the subject. Many children perceive addition and subtraction as different operations with different degrees of difficulty or miss the linking of addition and multiplication. This means that their knowledge structure does not necessarily correspond to the structure of the curriculum or of the subject as a scientific discipline. Moreover, children might have prior knowledge on a specific topic from everyday experiences, making a seemingly more difficult topic easier for them to understand. For example, children might encounter fractions much earlier than fractions are introduced at school (in German classrooms, for example, fractions are mostly part of the grade 6 curriculum). These views are reflected in competence models for the early grades (cf. Reiss, Heinze, & Pekrun, 2007). They provide evidence that the structure and composition of mathematics cannot be easily transformed into the structure and composition of students' mathematical knowledge. Teachers' understanding of their children's views may be enhanced by a comparative analysis of competence levels.

A second way in which competence models may support teachers' diagnostic processes is through their functioning as tools for classifying, evaluating, and interpreting empirical results. Understanding children's mathematical competences is not only important with respect to an individual but also with respect to schools, school districts, or even countries. Teachers as well as the general public are therefore confronted with empirical studies describing the results of tests and give evaluations and interpretations. Competence models can help interpret results from empirical studies. For example, the German national assessment of mathematical competence in third grades (VERA; https://www.iqb.hu-berlin.de/vera) is based on the competence model suggested by Reiss et al. (2012). The results of this assessment are reported back to the teachers. These test results are not sufficiently elaborated for diagnosing individual students, but they will give an overview on the level of classrooms.

Third, competence models can support teachers' diagnostic processes by providing detailed information about students' competences based on theoretical considerations and empirical data. Competence models can thus serve as a reference point to which a specific students' performance can be compared. The reference point provides more information than teachers usually receive when taking the average performance of their classroom as benchmark for individual achievement. This way, models may help to initiate more accurate and theoretically as well as empirically substantiated judgments of students' competence and will thus support a general comparison of students with their peers. The absence of reference points has often been an issue in research on teachers' judgments of their students' competences (Südkamp, Kaiser, & Möller, 2012).

Diagnosing is a process that encompasses a number of steps with varying demands (Fischer et al., 2014). When teachers diagnose their students' mathematical competence based on students' written work to a specific problem, competence models can be beneficial at several steps of the diagnostic process. At first, the teacher has to understand the affordances of the particular problem. This includes knowledge of the mathematical content but also knowledge about whether and why a problem is generally difficult for students. As described above, competence models can provide guidance for this judgment. As a second step, the teacher needs to identify possible mistakes in the students' work. Doing so is a more challenging task than it may seem at first sight. As we will discuss in more detail in the next section, whether a student's solution to a problem should be considered correct or incorrect is not only a matter of the content itself. Rather, this judgment depends on many other factors, particularly at the primary school level. Once the teacher has identified faulty solutions, he or she needs to be able to understand the nature of the mistakes and hypothesize about potential reasons. In particular, it is of interest whether student errors are of a systematic nature. To examine whether a student consistently shows a specific error pattern, the teacher should ask the student to solve another problem. The proper selection of this problem is critical in order to be able to actually capture a hypothesized error pattern. At this stage, a competence model can be useful because it helps selecting a problem at a competence level that is just suitable for the particular student. Eventually, this iterative process may end when the teacher is convinced of the student's error pattern that may correspond to a certain level of competence according to the model. Suitable interventions should follow this process with the aim of helping the student reach the next level of competence.

This detailed description of a diagnostic process has revealed where competence models can be useful. However, our description also points to limitations of current competence models: They describe what students know at certain levels, but they do not describe what students do not yet know or what typical mistakes at a certain level might look like. Integrating this information may, however, improve teachers' understanding of their students' learning.

Advancing Mathematical Competence Models: The Role of Student Errors

As already mentioned, diagnostic processes require an understanding of what students know but also what they do not know at a specific level of competence. This information is relevant to identify error patterns that students might have with regard to a certain problem. More fundamentally, as errors are an essential part of learning, understanding student errors and misconceptions is required in order to describe their learning progress and development.

Errors are sometimes regarded as interference of a learning process that should be avoided if possible. However, constructivist theories of learning suggest that errors should be regarded as fruitful learning opportunities (Bodemer & Ruggeri, 2015; Oser, Hascher, & Spychiger, 1999). This view is particularly important because it is not always feasible to precisely define what an error is. Even in mathematics and above all in primary school mathematics, it may depend on the context whether a student's answer is rated as correct or incorrect (Beitlich, Moll, Nagel, & Reiss, 2015). In general, the answer to an arithmetic problem will be true or false, but if a solution requires reasoning, it is not always self-evident which argument is acceptable at a certain stage of learning and which is not. Likewise, depending on the specific problem and its requirements, the mathematical language in general may be rated as correct or incorrect. For example, if a problem primarily asks for a numerical result, faulty argumentation or an inaccurate use of the terminology may play a minor role. As a consequence, regarding an error as divergence from a given norm (Oser et al., 1999) is a useful approach also in mathematics.

When teachers diagnose student's mathematical competences, they need to be able to identify whether a student's response deviates from the norm given by academic mathematics. However, they also need to consider whether it fits into norms developed in the classroom, and these norms are difficult to define and to evaluate. Accordingly, if competence models would include information about which sort of mistakes are to be expected on the various competence levels, it would be easier for teachers to define the norm.

There is a further facet of knowledge, which has been introduced by the group of Oser (e.g., Oser et al., 1999). They defined the concept of negative knowledge: in order to solve a mathematical problem correctly, students need specific knowledge, such as the rules of mental and written calculation or properties of the decimal system. However, in many situations, it is also useful to know which methods or contents will not help solving the problem. An example is knowing that specific operations like ignoring brackets or mixing up addition and multiplication will generally lead to a wrong result. This knowledge may come from experiences when application led to a wrong solution or no solution at all. As acquiring negative knowledge can support conceptual learning (Heemsoth & Heinze, 2016), teachers should not only know about the facets of (positive) knowledge that constitute mathematical competence but also about the negative knowledge that may support students' development. Although it is probably a challenge to integrate negative knowledge into competence models, an explicit knowledge of what does not work should be helpful for students. There are typical errors in mathematics that could be part of competence models. Moreover, a better understanding of this view on knowledge might be enriching for teachers' diagnostic competence.

Desiderata

Mathematical competence is a complex construct, and diagnosing students' mathematical competence is a complex task of teachers. Models of mathematical competence that are based on theories and empirical evidence can provide guidance because they help in understanding what mathematical competence means and how it develops. Moreover, models that include fine-grained descriptions of competence levels can be used as reference points and thus support teachers in diagnosing students. Empirical research is needed to evaluate the effectiveness of using competence models during diagnostic processes.

Research into teachers' diagnostic processes should also assess the role of different types of knowledge that are most relevant to support these processes. Although research has identified gaps in teachers' diagnostic competences (Heinrichs, 2014; Ostermann, Leuders, & Nückles, 2015), it is to date unclear which knowledge components teachers actually rely on and should rely on when diagnosing students. Knowledge about students' errors and misconceptions might be just one facet that has as of yet received little attention.

Research suggests that errors play an important role for successful learning. Accordingly, models of competence should be accompanied by information about typical errors and misconceptions students might have. Such information may help teachers in getting a clearer picture of their students' potentials and limitations. This information might also help in recognizing developmental steps and in defining supporting steps. In particular, models that describe the development of competences (e.g., Fritz, Ehlert, & Balzer, 2013; Reiss et al., 2007) might benefit from such a broader perspective.

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Chapter 5
Mathematical Performance among the Poor: Comparative Performance across Developing Countries

Janeli Kotzé and Servaas van der Berg

Introduction

Education is widely regarded as an important policy lever for providing poor children with an escape route out of poverty. However, the strong positive relationship between economic status and educational outcomes, often referred to as the social gradient, has become one of the great regularities of our time, with the inequalities in educational outcomes mirroring inequalities in social status. The steeper a country's social gradient, the larger the achievement gap between the affluent and the poor. If a steep social gradient is combined with convex returns to education, as found in many middle-income countries, and with low economic growth, social mobility will inevitably stagnate (Van der Berg, 2015).

Steep social gradients are common in most developing countries (Cruces, Domenech, & Gasparini, 2014; Gregorio & Lee, 2002; Rolleston, James, & Aurino, 2013). Comparing social gradients across countries is therefore useful for providing a framework for reviewing countries' experiences. Accurate intercountry comparisons require a comparable measure of both socio-economic status and educational outcomes. With the burgeoning of international educational assessments, comparative measures of educational outcomes have become readily available. However, very little work has been done on developing a comparative measure of economic status across different contexts and datasets (Chudgar, Luschei, Fagiolo, & Lee, 2012).

Asset indices have become the generally accepted measure of socio-economic status (SES) in the absence of household income or expenditure data (Filmer & Pritchett, 2001). Asset indices have proven very useful in determining a student's socio-economic conditions, as it is more infomative to ask children what items their

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household possess rather than what the monthly household income or expenditure is. Such information has commonly been used to derive socio-economic gradients and to compare these gradients across countries. Although the large body of literature on asset indices has contributed to more accurate measurement of SES, some potential biases still exist due to differences in asset prices and the utility of assets in different contexts. Does ownership of a bicycle convey the same information on underlying wealth of a household in Mozambique as it does in the United States? Or a radio?

Given the comparability constraints, this chapter makes use of a new, more comparable measure of SES as put forward by Kotzé & Van der Berg (in press). This method uses consumption per capita, denoted in international dollars, as a common yardstick for comparing economic status across countries and datasets. This allows the comparison of educational outcomes in education systems in sub-Saharan African countries and Latin-American countries that participated in two separate international educational evaluations, by looking at the quality of educational outcomes for children living under the $3.10-a-day poverty line, for instance. In this report, this yardstick of $3.10 in per capita income per month will be used to distinguish children in poverty from other children.

Background

The strong positive relationship between poverty and education documented by the Coleman Report (1966) has become a stylised fact in both the fields of economics and education. There is evidence of this relationship both within countries and between countries. There are still large achievement gaps between poorer and richer individuals even within rich countries such as the United States and England, despite half a century of research and also policy that has often been dedicated to reducing such gaps (Coleman, 1966; OECD, 2001; Reardon, 2011). The achievement differentials between higher income and lower income countries further confirm that this trend also applies at a global level, as the latest TIMSS 2015 results confirm. Figure 5.1 illustrates the relationship between the average Grade 4 TIMSS mathematics score for each country and the country's GDP per capita converted to US$ using purchasing power parity.

Within countries, and especially developing countries, the relationship between economic status and educational outcomes is often positive and convex, with the affluent gaining much higher returns from education than the poor. This relationship is portrayed through a social gradient that potentially provides a framework with which to compare the inequality of education within various counties. For such a comparison, two standard measures are required: (1) a comparable measure of learning outcomes and (2) a comparable measure of learner socio-economic status. The comparable measure of learning outcomes has become relatively easily accessible with the availability of large-scale international assessments such as TIMSS, PIRLS and PISA, although it is more difficult to compare across different assessments. However, very little has been done with regard to creating a comparable measure of learner SES.

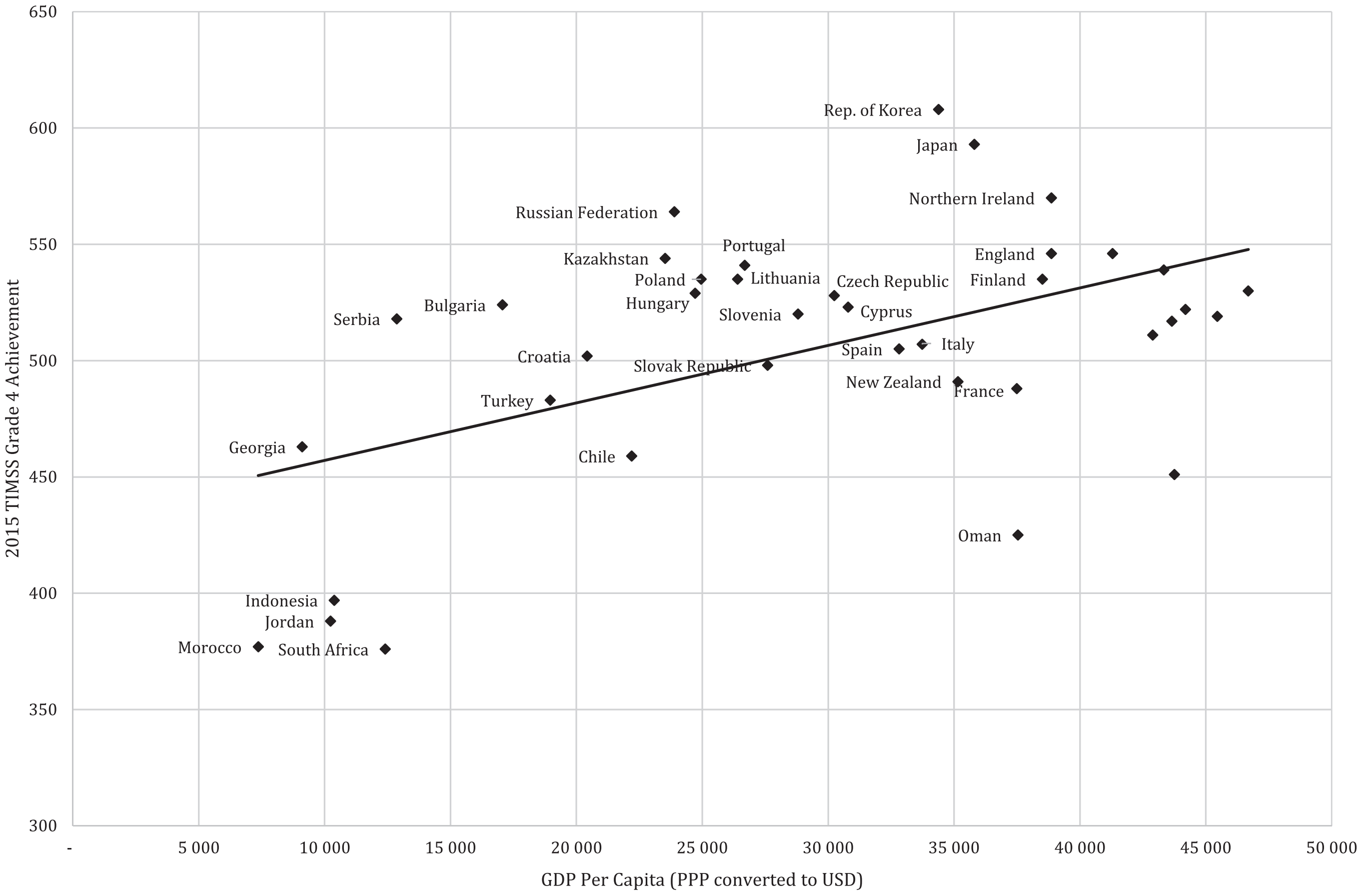

Fig. 5.1 Association between GDP per capita and a country's performance in Grade 4 mathematics. (Source: 2015 TIMSS Grade 4 Achievement; World Bank Indicators. Notes: The graph is restricted to only include countries with a GDP per capita below $40,000 as the trend becomes less defined among the high-income countries)

Since the landmark paper by Filmer and Pritchett (2001), asset indices have become a popular proxy for income or expenditure measures in the economic, demographic and sociological literature. Conceptually, the appropriateness of an asset index as a proxy for socio-economic status or wealth is based on two assumptions. The first assumption is that the latent trait underlying the possession of a range of assets, along with a set of housing characteristics and household education levels, is a good approximation of a household's economic status. The second assumption is that the ranking of households on this index is relatively strongly correlated to the ranking of households when using the household-size-adjusted expenditures, in this way making it an accurate proxy for SES. Since the advent of the asset index approach, a large body of literature has developed around the appropriateness of this proxy, the construction of such an index, and the application of the index in a vast array of empirical studies.

Although there are important issues that still need greater attention when measuring educational outcomes across international assessments, this issue will not be dealt with in this chapter. Rather, the comparison that was made by Gustafsson (2012) will be used for this purpose, as explained later.

In the decade and a half that they have been in use, asset indices as a proxy for SES have been applied in a wide variety of fields. Filmer and Pritchett (2001) originally applied an asset index to inequality in schooling outcomes, but since then the tool has been applied to health outcomes (Bollen, Glanville, & Stecklov, 2002; Chuma & Molyneux, 2009; Filmer, 2005; Gwatkon et al., 2000; Lindelow, 2006; Njau et al., 2006; Schellenberg et al., 2003), child health outcomes (Fay, Leipziger, Wodon, & Yepes, 2005; Montgomery, Gragnolati, Burke, & Paredes, 2000; Sahn & Stifel, 2003; Sastry, 2004; Tarozzi & Mahajan, 2005; Wagstaff & Watanabe, 2003), early childhood development (Ghuman, Behrman, Borja, Gultiano, & King, 2005; Paxson & Schady, 2005), and further studies of educational inequalities (Caro & Cortes, 2012; Case, Paxson, & Ableidinger, 2004; Das, Habyarimana, & Krishnan, 2004; Taylor & Yu, 2009).

Methodology and Data

Most internationally standardised assessments include the administration of a learner background questionnaire to capture information on learners' home background. These questionnaires often include questions on the assets available at the learners' house, parental education and, in developing country contexts, the infrastructure available at a learners' house. Using this information, an asset index is constructed as a unidimensional composite indicator of a set of assets that reflects the underlying wealth of a household. To gain the most accurate asset index using the available information, each item included in the index is assigned a weight on the basis of the variance and covariance of the items included, using methods such as factor analysis (FA), principal component analysis (PCA) or multiple correspondence analysis (MCA). Regardless of the method used, the construction of an asset index involves attributing unique weights to each of the various possessions on the basis of the amount of common information the asset contributes in relation to the latent variable (in this case wealth).

When constructing an asset index across various countries, the same process is followed, given the common information across all these countries participating in the same assessment and administering the same household background questionnaire. By construction, these weights will only vary by country if an asset index is constructed for each country individually. If an asset index is constructed for a combined sample of countries, the implicit assumption is that the same possessions will carry the same weights in different countries, regardless of the different contexts. While this assumption may be plausible for countries at roughly similar economic development levels, it may not be as accurate for countries with greatly varying economic structures (Filmer & Pritchett, 2001; Harttgen & Vollmer, 2011). For instance, ownership of a radio in Malawi is associated with a completely different percentile in the expenditure distribution than ownership of a radio in Finland but may also convey very different information about the underlying wealth of the household concerned.

In order to circumvent this problem and have an asset index which is both an accurate reflection of within country socio-economic status and a comparable measure between countries, this chapter applies a method proposed by Kotzé and Van der Berg (in press). The method uses asset indices that have been constructed using country-specific weights, and to address the problem of comparability, the method links the asset index distribution to the national consumption distribution in order to simulate household consumption for each wealth percentile. Consumption per capita, denoted in international dollars (converted at purchasing power parity (PPP) rates), then serves as a common yardstick with which to compare country-specific asset indices across different countries.

The resultant measure is a single, internationally comparable measure of SES and can be applied to every international evaluation for which an asset index can be derived and for which a household survey containing per capita consumption is available. Moreover, this new indicator of socio-economic status will enable the comparison of equally poor students under different education systems. For example, the level of numeracy of a child in a household that earns less than $3.10 per capita per day in Malawi can be compared with the level of numeracy of a child who is equally poor in Peru.

To increase the accuracy of a comparable SES measure, the social gradients are also adjusted to account for children who are out of school. Although access to schooling has increased significantly, it is evident from Fig. 5.2 that a 100% attendance rate is not yet a reality for the poorest households in most sub-Saharan African countries. Figure 5.2 also shows that there is very little difference between countries with regard to attendance rates in primary schooling. However, gradients probably will be much steeper and differentiated for higher school grades. Furthermore, to make the consumption distributions comparable across the countries, household consumption is shown in purchasing power parity terms (PPP $) using the World

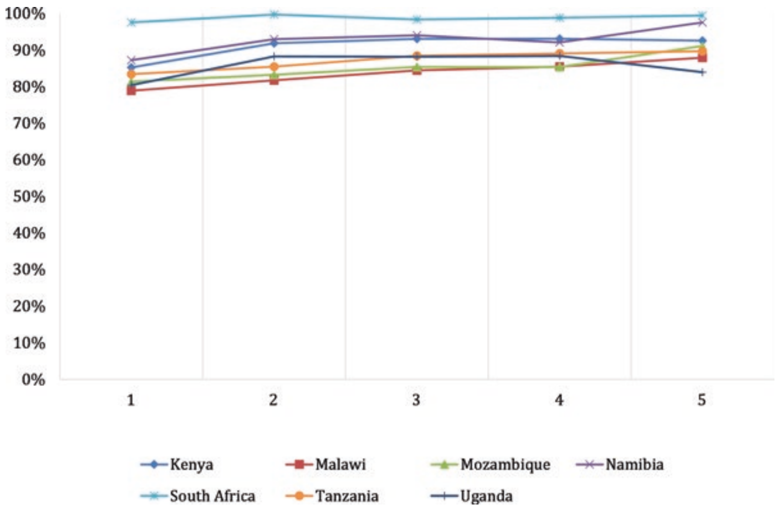

Fig. 5.2 Percentage of 11- to 15-year-olds currently in school by consumption per capita quintile

Bank indicators gross domestic product deflator (GDP deflator) and 2007 purchasing power parity (PPP) values.

This method will be used to compare the SACMEQ III (sub-Saharan Africa) and SERCE (Latin America) datasets, as both evaluated sixth grade students in mathematics and were conducted quite soon after each other. SACMEQ is a consortium of education ministries, policymakers and researchers that aims to improve educational planning in participating countries in Southern and Eastern Africa by measuring educational quality (Moloi & Strauss, 2005:12). SACMEQ III was administered in 2007 and collected data from about 61,000 learners, 8000 teachers and 2800 school principals (SACMEQ, 2014).

SERCE (Segundo Estudio Regional Comparativo y Explicativo) was conducted by LLECE (the Laboratorio Latinoamericano de Evaluacion de la Calidad de la Education) among 16 Latin-American countries. Both third grade and sixth grade students were assessed in mathematics, reading, writing and natural sciences. SERCE was conducted in 2006, and 95,288 sixth grade students were assessed (UNESCO, 2008). These surveys collect extensive background information on the schooling and home environments of students and, in addition, test students and teachers in both numeracy and literacy (Hungi et al., 2010; Ross et al., 2005).

International student assessments are generally constructed to discriminate between performances around an international mean. The performance of sub-Saharan and some Latin-American countries on these achievement tests, however, is so far below the mean performance of OECD countries that their scores cease to be meaningful if they are tested on some of the other international test programmes. For this reason, tests such as SACMEQ and SERCE are very useful, as they are constructed to match the context and standards of the region. Furthermore, both assessments assessed Grade 6 learners in mathematics and language at around the same time period (2006/2007), which makes them highly comparable. Both these datasets also contain information on home possessions, thus allowing country-specific asset indices to be constructed.

There is still an issue of comparing achievement scores across tests. In 2001, Barro and Lee compared the achievement scores of TIMSS and IALS in their groundbreaking paper. They did not, however, specifically adjust these scores for differences between the scores. Since this pioneering paper, various researchers have adapted this method. Currently there are at least four different methods used for compiling a global dataset of educational quality that calibrate achievement scores across international student achievement tests (Angrist, Patrinos, & Schlotter, 2013; Barro & Lee, 2001; Gustafsson, 2012; Hanushek & Woessman, 2012). Hanushek and Woessman (2009) put forward a method for transforming country test scores to a single comparable scale using the standardised National Assessment of Educational Progress (NEAP) in the United States as the anchor assessments that joins the various different assessments. Their method, however, does not allow the inclusion of test programmes in which the United States did not participate. The approach taken by Gustafsson (2012) is similar to theirs but also includes achievement scores from two major regional testing programmes in Africa and Latin America, SACMEQ and SERCE. This is done by using at least two bridging countries between any two assessment programmes to link the assessments across the programmes and to transform countries' average achievement scores in international testing programmes to a single normalised scale. Using this dataset and linking it to available household surveys make it possible to analyse socio-economic gradients for cognitive outcomes for 7 African countries and 14 Latin-American countries.

Comparing Social Gradients Across Contexts

Figure 5.3 illustrates the socio-economic gradients of seven of the sub-Saharan countries that participated in SACMEQ III as well as an indication of the $1.90 and $3.10 international poverty lines, which reveal the failure of these education systems for many of the world's poorest. Interestingly, Kenya and Tanzania outperform the other countries at all levels of SES. At the poverty lines, Kenya and Tanzania perform equally well, but wealthier Kenyan learners outperform wealthier Tanzanian learners. Malawi, on the other hand, consistently performs the worst and shows little difference in test scores between the poorest and wealthiest students in the country. Both South Africa and Namibia have steep socio-economic gradients, with a relatively well-performing upper class. Poor students in Mozambique and Uganda that fall under the $3.10-a-day poverty line far outperformed equally poor children in South Africa and Namibia, who performed at about the same very low level as children from Malawi. This is quite striking since South Africa has a GDP per capita that is twenty-five times that of Malawi, sixteen times that of Mozambique and one-and-a-half times that of Namibia.

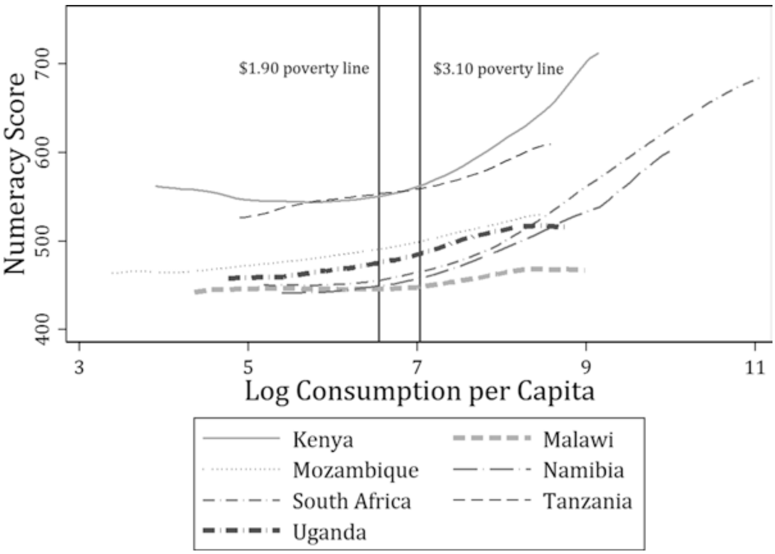

Fig. 5.3 Mathematics performance and SES in SACMEQ III

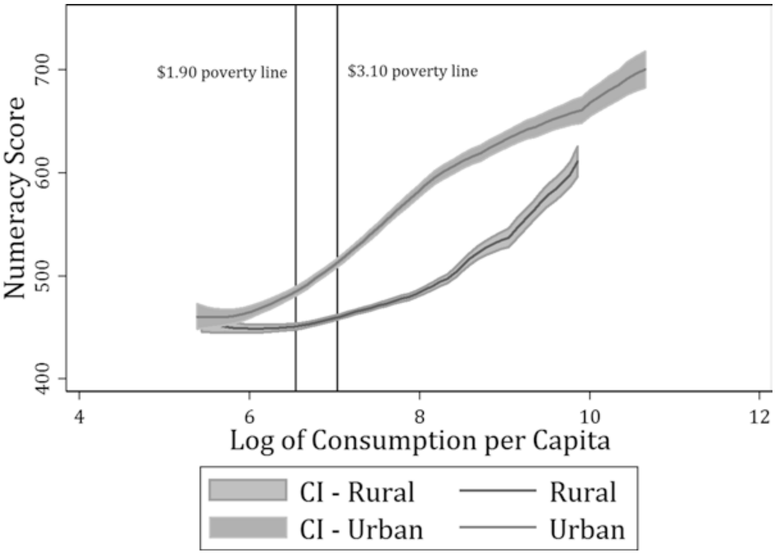

Fig. 5.4 Comparable measure of SES across urban and rural subsamples within South Africa

Socio-economic gradients can also be derived for different subsamples within a country. Asset prices and asset availability often differ among urban and rural settings, which implies that different assets should carry different weights in an asset index. Figure 5.4 illustrates the separate socio-economic gradients for education

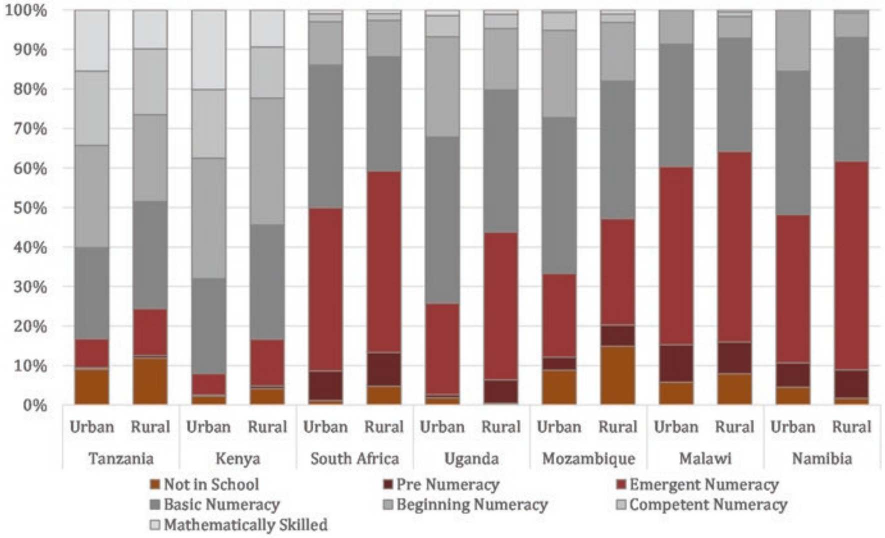

Fig. 5.5 Proportion of poor students (living under $3.10 per day) per competency level, by urban/rural location in different SACMEQ countries

outcomes in South Africa's urban and rural areas when separate urban and rural weights are used. As expected, urban students have, on average, better educational outcomes than their equally poor rural counterparts. Interestingly though, the poorest rural children do not perform much worse than their urban peers, but a significant difference emerges as per capita consumption increases.

It is possible to gain a deeper understanding of the heterogeneity in the quality of schooling in different contexts by categorising student achievement scores into skill levels. SACMEQ distinguishes eight competency levels on the basis of the difficulty of the questions and the skills required to give correct responses. These competency levels provide a more concrete understanding of student capabilities. Figure 5.5 shows what proportion of students in SACMEQ countries in both urban and rural settings are in households above the $3.10-a-day poverty line (the white parts of the bars) and then classifies the performance of those in poverty by competency level. The stark difference in performance between Kenya and the rest of the SACMEQ countries is once again apparent. Even though Kenya is a low-income country and has a poverty headcount rate of one-and-a-half times that of a middle-income country such as South Africa, it still manages to provide a much larger proportion of these students with sufficient mathematical skills. This means that the most marginalised students in Kenya are better prepared for participation in a modern labour market than marginalised students in South Africa. Furthermore, the difference in the quality of educational outcomes between Mozambique and Malawi is quite remarkable. Mozambique has one-and-a-half times the poverty headcount rate of Malawi, but half of Mozambique's poor students are proficient in numeracy, whereas only a small number of students in Malawi function at the same level,

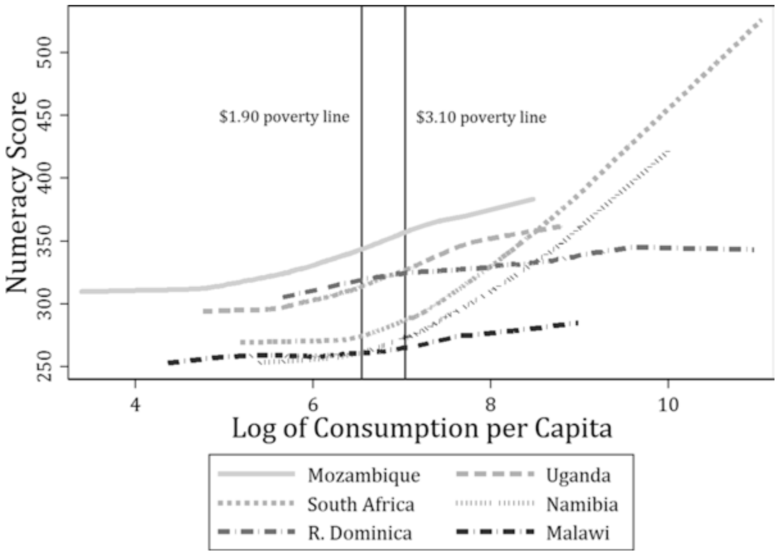

Fig. 5.6 Socio-economic gradient for the six poorest-performing countries across Latin America and sub-Saharan Africa

despite the fact that the Malawian government has supplied free primary education for the past 20 years.

Using Gustafsson's (2012) recalibrated 2007 SACMEQ and 2006 SERCE achievement scores and the comparable measure of SES, Fig. 5.6 portrays the relationship between the transformed numeracy scores and SES for the six weakest performing countries in SACMEQ and SERCE combined. The Latin-American countries in this group outperformed the sub-Saharan countries at given levels of poverty, and as they contain more middle-income countries, there are far fewer children that live under the $3.10-a-day poverty line in the Latin-American region. The Dominican Republic was the only Latin-American country among the poorest-performing countries. It is striking that at equal income levels Mozambique and Uganda seem to be much more effective at producing educational outcomes than middle-income countries such as South Africa and the Dominican Republic. Children in households that live in poverty receive an education of a much higher quality in Mozambique and Uganda than in South Africa.

Figure 5.7 shows the relationship between the numeracy scores and SES for the six best-performing countries in the Latin-American and sub-Saharan regions. Kenya is the only sub-Saharan country that managed to make it to the top six countries and, remarkably, outperforms much wealthier countries such as Uruguay and Costa Rica for given levels of SES. This graph, however, does not take into account the total number of students in these countries.

Figure 5.8 shows the proportion of students living under the $3.10-a-day poverty line, per competency level for eight Latin-American and sub-Saharan countries.

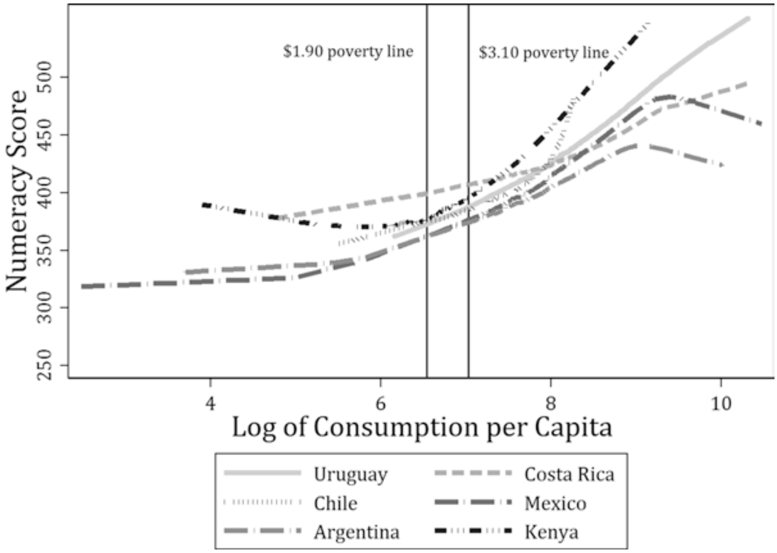

Fig. 5.7 Socio-economic gradient for the six best-performing countries across Latin America and sub-Saharan Africa

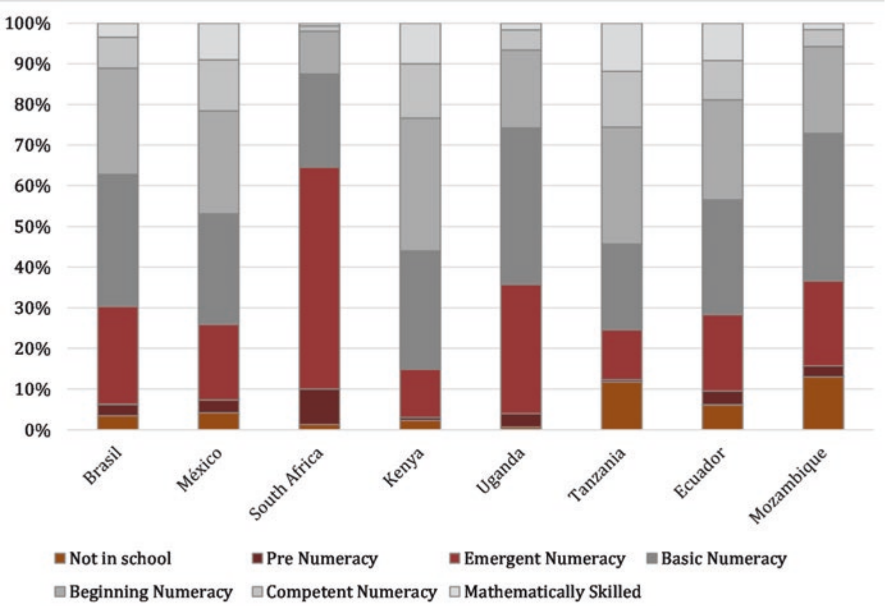

Fig. 5.8 Proportion of students living under $3.10 per day, per competency level

Among these countries, Brazil, Mexico and South Africa have the smallest proportion of learners living under the poverty line. South Africa has the highest proportion of students who are functioning at below-acceptable levels of numeracy (pre-numeracy and emergent numeracy). Kenya and Tanzania both have a much higher proportion of students living in poverty but also have a much higher proportion of these students performing at acceptable (basic numeracy) to above-average levels (beginning numeracy, competent numeracy and mathematically skilled). In Kenya, specifically, 30% of poor students (below the $3.10-a-day poverty line) perform at acceptable levels, and 56% perform at above-average levels of numerical skills. This is quite remarkable when taking into account that Brazil has a GDP per capita that is ten times that of Kenya and South Africa has a GDP per capita that is four times larger than Kenya.

Conclusion

The interplay between learner SES and learner outcomes is an important indicator of the ability of a country's education system to provide the most marginal learners with a level of education similar to that of wealthier learners. This relationship is often portrayed through a social gradient, but since no credible method has previously existed for comparing social gradients across different contexts, comparisons made were often inaccurate as researchers have had to choose between either the accuracy of a measure within countries or the comparability of the measures across countries.

Using a new method, a comparable measure of SES was applied to the 2007 SACMEQ and 2006 SERCE datasets and enabled the comparison of educational outcomes for 7 sub-Saharan and 14 Latin-American countries. This measure has allowed a comparison of the direction and strength of the association between SES and educational outcomes in different countries and settings. From these comparisons it has become clear that, in certain wealthier but more unequal countries, such as South Africa and Brazil, the poorest children are much worse off in terms of the quality of education as reflected in cognitive scores on international tests than the poorest children in some much poorer countries. This signifies that some countries are managing to provide their poor learners with a much higher quality of education than other wealthier countries. The differences in the mathematics performance most likely stem from various institutional and contextual differences. Teacher content knowledge and pedagogical skill as well as teacher motivation undoubtedly are strong predictors of learner performance. Finally, administrative institutions and accountability mechanisms are often thought to have the ability to influence overall learner performance. However, little research has been done in this regard in developing country contexts.

Appendix

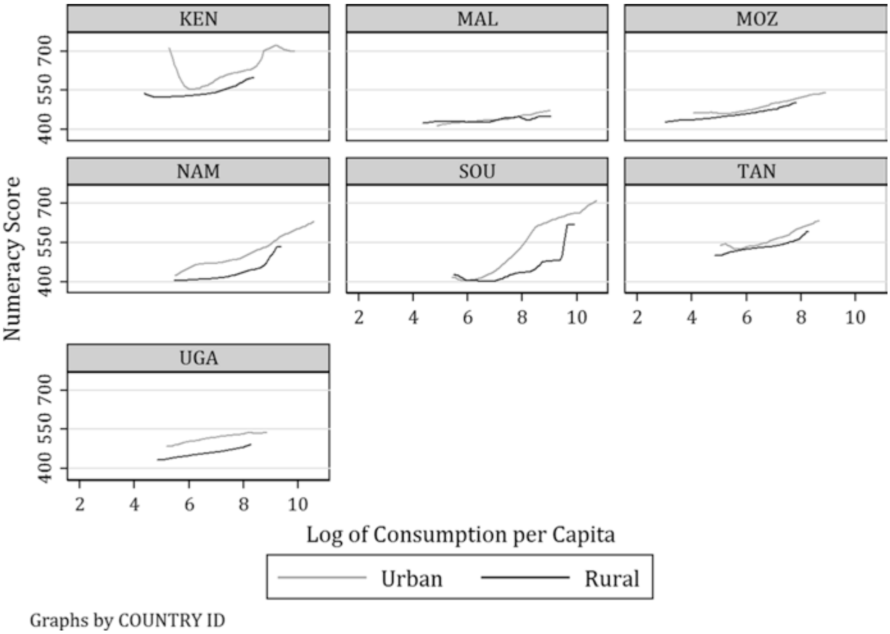

Fig. 5.9 Socio-economic gradients for urban and rural samples

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Chapter 6
Didactics as a Source and Remedy of Mathematical Learning Difficulties

Michael Gaidoschik

A Lack of Certain Arithmetical Abilities or a Certain Way of Doing Arithmetic?

Persisting deficits in the field of basic addition and subtraction are widely held as a main characteristic of mathematics learning difficulties (MLD) by scholars in the field of mathematics education (e.g., Baroody, Bajwa, & Eiland, 2009; Schipper, 2009), as well as in the field of (neuro)psychology (e.g., Boets & De Smedt, 2010; Geary, 2004). However, there are quite different ways to review what is characteristic for many children. Whereas some researchers stress the *lack of automaticity* (e.g., Landerl & Kaufmann, 2013), others emphasize that children with MLD tend to *rely heavily on counting* for solving addition and subtraction tasks even in higher classes (e.g., Cumming & Elkins, 1999; Dowker, 2009; Gray, 2005; Moser Opitz, 2013; Ostad, 1998).

This is *not* only talking about two sides of the same coin. Of course, children who have not yet automated a basic fact have to find another way to solve it. Yet, it makes a difference whether research sets out to explain an *absence* (that of fact retrieval) or a *presence* (that of counting as the predominant computing strategy).

When you set out to explain why an ability is *missing,* a consistent next step is to look at *prerequisites* that might underlie that ability. Thus, there is a body of research on how basic fact fluency statistically relates to deficits with regard to various cognitive measures. Such are, inter alia, working memory (cf. Alloway & Passolunghi, 2011), speed of processing (Willcutt et al., 2013), and the ability to quickly compare numerosities (cf. Vanbinst, Ceulemans, Ghesquière, & De Smedt, 2015).

From a mathematics education viewpoint, the results of this kind of research are unsatisfactory on both the practical and theoretical levels.

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To start with the latter: Doing mathematics is not simply putting together the *prerequisites* of doing mathematics. In order to understand what learning mathematics is about, as well as what hinders a child from learning successfully, we have to explore what children do and think while actually performing mathematics. We do not characterize this performance adequately by saying what children do *not* do (retrieve facts); it is what they *do* that we have to understand. As is outlined in the following sections, educational research can show that the strategies children exhibit when solving basic tasks relate to what they *know and think* about numbers and arithmetical operations. This, in turn, relates to their foregoing and ongoing arithmetic instruction.

As to the practical level, the better we understand underachieving children's mathematical thinking and doing, the more we are able to design and further develop preventive as well as remedial measures.

This chapter will try to both elucidate why so many children stick to counting strategies even in higher grades and outline what arithmetic instruction can and should do to counteract this.

Computing by Counting: What Else Could a Child Do to Solve a Basic Task?

Numerous studies have shown that children who grow up in developed countries usually are already able to solve verbally posed addition and subtraction tasks at least in the number range up to ten before they enter formal schooling. As a rule, they do so by using some sort of counting strategy to find the solution (Verschaffel, Greer, & De Corte, 2007).

In autonomously executing a counting strategy to solve a given problem, children seem to demonstrate at least basic understanding of the underlying arithmetic *operation*: addition as "putting together" or "establishing how much it is all together"; subtraction as "taking away" or, when using counting-up, perhaps even as "establishing the difference" (cf. Baroody & Tiilikainen, 2003; Fuson, 1992). However, we have to be aware of the possibility that a child might be quite proficient in solving basic tasks *without* having or activating such an understanding. Instead, he or she might reduce addition to "going further" within the learned sequence of number words, and subtraction to "going back" (Gaidoschik & Beier, 2017; Schultz, Jakob, & Gerster, 2017). These interpretations are insufficient insofar as they remain on a procedural level and (as discussed below) tend to impede the recognition of quantitative relations (Gray, 1991; Gray & Tall, 1994).

There is a similar and interconnected ambiguity regarding the way children understand *numbers* when they compute by counting. A child who solves an addition task by "counting all" using his or her fingers or some external counting material to represent both the summands and the sum (Verschaffel et al., 2007), by doing so autonomously, certainly demonstrates that he or she has learned that counting is a way to establish quantities ("cardinal principle", Gelman & Gallistel, 1978). Compared to this rather cumbersome strategy, we may regard "counting-on" and especially "counting-on from larger" or "min-counting" (e.g., solving 3 + 4 by counting "five, six, seven") as progress (Fuson, 1992; Verschaffel et al., 2007). However, there are children who use counting-on as their main strategy and disregard cardinality. Thus, they might perform 3 + 4 as "four, five, six," stating that "six" is the solution. The error results from taking the larger number "four" as their starting point for counting-on three number words, including "four." From a quantitative view, it is clear that "four" stands for the quantity of the bigger summand and you have to add *another* three. From the procedural view that is characteristic of some children, this might be seen as a mere rule that has to be memorized (Gaidoschik, 2003; Schipper, 2009).

So children who compute by using a counting strategy might have quite different concepts of numbers and operations. What concepts would they need to develop noncounting ways of computing? To answer this question, some didactically oriented content analysis might be useful.

Direct Fact Retrieval

There are essentially just three ways to solve an addition or subtraction problem without resorting to counting. The first one is, quite trivially, fact retrieval. You do not have to use counting for solving (e.g., 3 + 4) if you "just know it." With regard to the conceptual basis of this knowledge, Gray (1991, p. 554) stated appropriately: "Such a strategy can be used without any evidence of meaning." Of course, fact retrieval *may* go with understanding, and there is widespread agreement in the current literature on mathematics education that children *should* acquire this very combination of understanding and knowing by heart for all basic facts in early grades (cf. NCTM, n.d.; Schipper, 2009). However, this is a goal but not a starting point and, in any case, a child who recalls the solution of a basic task from long-term memory does not thereby demonstrate deeper understanding than a child who relies on a counting strategy.

Deriving Unknown Facts from Known Facts

Things are different if the child avoids counting by using a second alternative referred to in the literature as *derived facts strategies* (DFS) (Dowker, 2009, 2014; Steinberg, 1985) or *reasoning strategies* (Baroody, Purpura, Eiland, Reid, & Paliwal, 2016). Such strategies build upon *relations* between arithmetic facts. To stay with the example of 3 + 4, using a DFS, a child who has memorized 3 + 3, but not yet 3 + 4, could derive the solution of the latter by "adding one more" to the solution of the former.

DFS rest on arithmetic laws, such as:

- The commutative property of addition: a + b = b + a; therefore, e.g., 1 + 8 can be derived from 8 + 1.
- The associative property of addition: a + (b + c) = (a + b) + c; therefore, e.g., 3 + 4 can be derived from 3 + 3 [3 + 4 = 3 + (3 + 1) = (3 + 3) + 1 = 6 + 1].
- The inverse relation between addition and subtraction: a + b = c ↔ c − b = a ↔ c − a = b; therefore, e.g., 7 − 4 = 3 can be derived from 3 + 4 = 7.

Gaidoschik (2010, 2012) found that some 20% of a random sample of 139 Austrian first graders employed such strategies even before any formal arithmetic instruction had started.

Certainly, to use a certain DFS correctly and autonomously, a child needs to see some kind of relation between the derived fact and the fact chosen as a basis for deriving it. This is not to say that we can take any application of a DFS as proof of the child's thorough understanding of the underlying arithmetic laws. For instance, when expanding the "number-after rule" (Baroody & Tiilikainen, 2003) from 8 + 1 to 1 + 8, a child may use a "commutativity permission" in terms of being confident that this "shortcut" is "allowed." However, the same child may not be aware of "true commutativity" as may be assessed by tasks that ask them to explain and justify that procedure or to transfer it to larger numbers (Baroody, Wilkins, & Tiilikainen, 2003).

Likewise, a child who solves, e.g., 8 − 4 by inverting 4 + 4 = 8 may not be ready to transfer the same strategy to, e.g., 12 − 6, and may solve 12 − 6 by finger counting, although he or she easily retrieves 6 + 6 = 12 from memory (Baroody, 1999; Gaidoschik, 2010, 2012).

Therefore, the utilization of a DFS may be based on a "weak schema—a generalization narrow in scope (tied to a particular context) and, perhaps, lacking logical coherence" (Baroody et al., 2003, p. 145). Baroody and colleagues (2003) hypothesize that such limitations are typical of an initial stage of DFS use. However, it is important to see that further development toward a "strong schema" is not just a matter of time but bears on changes in the ways a child is thinking about numbers and operations (Baroody et al., 2003; Gaidoschik, 2010).

Numbers as Compositions of Other Numbers

This leads to the third and last in our list of alternatives to computing by counting: *utilization* of the concept of "numbers as compositions of other numbers" (Resnick, 1983, p. 114), also referred to as the (numerical) "part–whole schema" (Resnick, 1983, p. 115). Following this concept, a child in thinking and dealing with a given number is aware of other numbers forming parts of it. Children thus may conceive, e.g., the number eight as a whole that is composed of the numbers five and three as its parts. Drawing on this concept, they might solve at least two addition tasks (5 + 3, 3 + 5) as well as two subtraction tasks (8 − 5, 8 − 3) without resorting to a counting strategy, simply by putting the parts together into the whole or separating

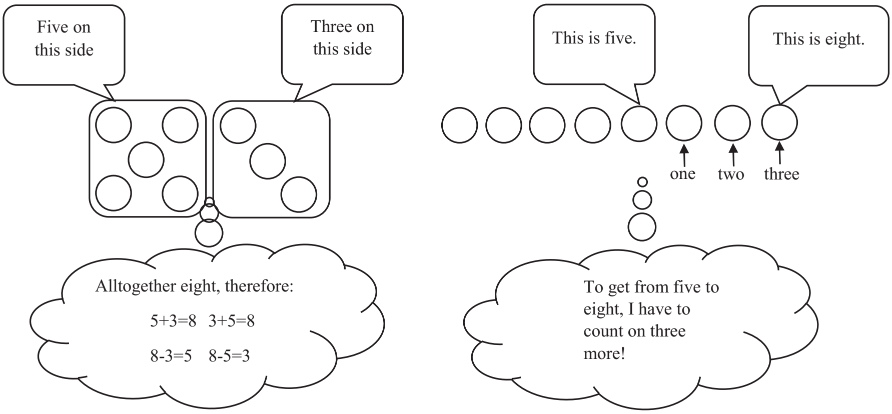

Fig. 6.1 Numbers conceived as compositions of other numbers, or as positions within the number sequence

one part of the whole (Fig. 6.1). Employing another view on the same number triple, they could also detect that each part forms the difference between the whole and the other part. This would enable them to determine this difference without any counting.

It is important to see that numerical part–whole thinking is more than cardinality. Understanding cardinality comprises awareness of quantity but not necessarily of *structure* within that quantity. A numerical part–whole conception, however, implies awareness of structure. It is this awareness that "permits forms of mathematical problem solving and interpretation that are not available to younger children" (Resnick, 1983, p. 114). More precisely, it permits strategies not available to children of all ages who are restricted to the concept of numbers as positions within the number word sequence (an ordinal interpretation; see Fig. 6.1) or to an unstructured cardinal interpretation (Gaidoschik, 2010; Schultz et al., 2017).

Of course, when a child derives sums and differences from the interiorized part–whole structure of a number, we could label this as just another type of DFS. From a didactical point of view, though, it is important to state that there are different levels of understanding in the use of a DFS. In this context, Baroody (2006) distinguishes strategies that are "highly salient" (such as using commutativity), and may be learned as a procedure, from strategies that children are unlikely to use if they have not yet developed a numerical part–whole concept (cf. Gaidoschik, 2010). To give an example of such a less salient strategy, children may solve 6 + 7 by using the "power of five" (Flexer, 1986) in decomposing 6 into 5 + 1 and 7 into 5 + 2, then recomposing the total as (5 + 5) + (2 + 1) = 10 + 3 = 13. It is hardly conceivable that a child adopts such a strategy as a mere procedure without the conceptual basis of numbers being composed of other numbers.

Therefore, a numerical part–whole concept not only enables direct derivation of addition and subtraction tasks from an interiorized number triple (e.g., 8 − 5 = 3 from thinking of eight as composed of five and three); it also seems to form a prerequisite for conceiving arithmetic properties and relations profoundly enough to also adopt a less salient DFS (Baroody, 1999; Baroody et al., 2016). Accordingly, Canobi (2004) found significant relations between conceptual part–whole knowledge of numbers and the use of fact retrieval and DFS in a sample of 90 6- to 8-year-old British children.

Note that numerical part–whole thinking is not an "all-or-nothing phenomenon" (Baroody, 1999, p. 168). Schultz and colleagues (2017) distinguish "flexible" and "inflexible" part–whole relations and, based on case studies, report on children who utilize part–whole structures for some tasks but not for others, or restrict them to certain number triples. Similarly, Dowker found great differences in the ability of young British children (*N* = 44, mean age = 6.8 years) to utilize a given task for deriving another, depending on what kind of arithmetic principle connects the two tasks. Derivations based on the inverse relationship between addition and subtraction proved most difficult (Dowker, 2014).

Evidence on the Impact of Instructional Efforts Focused on Noncounting Strategies

Empirical evidence that early arithmetic instruction, by focusing on part–whole thinking of numbers and elaborating and practicing a DFS on a solid conceptual basis, does indeed support children to overcome computing by counting comes from longitudinal, cross-sectional, and intervention studies, as well as from international comparative studies.

International Comparisons

Geary, Bow-Thomas, Fan, and Siegler (1996) studied Chinese and US children of different age groups (*N* = 209, age 5.9–8.8 years). They found that among the Chinese sample, at the end of first grade, fact retrieval was by far the most predominant strategy for solving addition and subtraction tasks up to 20, with a share of about 91%. By contrast, their US peers used retrieval in only about 28% of their trials. Even at the end of third grade, the US students used retrieval in only 56% of their tasks. Chinese kindergarteners, as well as Chinese first graders when interviewed early in the year, applied a DFS quite often (in 44% of the tasks in kindergarten and in 36% of tasks in the fourth month of first grade). By contrast, US children of all age groups hardly ever used a DFS (Table 6.1).

Without denying the influence of other factors such as language characteristics and sociocultural conditions, Geary and colleagues (1996) assume a substantial impact of instruction. Even today, in US classrooms it is common for children to be encouraged to solve basic tasks by counting, at least as long as till the end of first

Table 6.1 Frequency of use of different strategies for basic tasks in samples of different age groups from China and the USA

	End of kindergarten (%)		Fourth month of first grade (%)		End of first grade (%)		End of third grade (%)	
	China	USA	China	USA	China	USA	China	USA
Fact retrieval	21	17	43	20	91	28	100	56
Derived facts	44	2	36	1	6	4	0	4
Counting	35	81	18	78	3	68	0	39

cf. Geary et al. (1996)

grade (Henry & Brown, 2008; Van de Walle, 2004). The current US Common Core State Standards for Mathematics (CCSSM) are ambiguous in that respect. On the one hand, they clearly advocate that children in first grade should “understand and apply properties of operations and the relationship between addition and subtraction” and learn a range of DFS (NCTM, n.d., p. 15). On the other hand, the standards consider counting-on as an equally adequate strategy for first graders. As for kindergarten, recommended activities clearly point at reaching cardinality but not explicitly at a numerical part–whole concept. In fact, *all* kindergarten activities the CCSSM advocate under the heading “operations and algebraic thinking” invite children to solve arithmetic problems by counting fingers or (drawn) objects (NCTM, n.d., p. 11). It is important to see that such activities are *not* sufficient to guide children to detect, reflect, and utilize numerical part–whole structures (Gaidoschik, 2010; Schultz et al., 2017).

By contrast, in China (Zhou & Peverly, 2005), as well as in other East Asian countries (Fuson & Kwon, 1992), there is a long and widespread tradition of instructing first graders and even kindergarteners *not* to use counting but a DFS to solve basic tasks. These strategies draw on part–whole structures with numbers structured by ten (Fuson & Kwon, 1992) and five (Hatano, 1992). Geary and colleagues hypothesize that the high share of fact retrieval among the Chinese children in their study is, to a substantial extent, due to this teaching practice. By repeatedly using a DFS during their early formal instruction, Chinese children would develop powerful mental associations between tasks and solutions, as well as between related tasks. This would help them memorize more and more basic facts within a comparatively short period (Geary et al., 1996).

Longitudinal and Cross-Sectional Data and Related Theories

The longitudinal study by Gaidoschik (2010, 2012) points in the same direction. The author interviewed a random sample of Austrian first graders three times: at the beginning of the school year, midyear and, finally, at the end of the school year ($N = 139$; average age at first interview 6.5 years). Note that the arithmetic instruction of this sample had *not* contained targeted measures to convey DFS, as was established through teacher interviews and qualitative content analyses of textbooks and learning materials. During the second and third interviews, the children solved inter alia the same 14 addition and subtraction tasks up to ten. The interviewer presented these tasks one by one verbally and, simultaneously, as written on a flash card. The children had to solve each task mentally in the way they usually would, and state the result verbally. Immediately thereafter, they should explain or demonstrate how they had arrived at the solution (Gaidoschik, 2010, 2012).

At the end of their first school year, only about one third of the sample used either fact retrieval or a DFS for at least 10 of the 14 tasks. This very subgroup had used a DFS already in the second or even in the first interview of the study. Conversely, the approximately 27% of children who at the end of first grade used counting for at least 10 of the 14 tasks did not use any DFS in any of the three interview sessions (Gaidoschik, 2010, 2012).

The author compared the frequency of strategy change for ten identical tasks between the second and third interviews. In the middle of the year, children in that sample had used a DFS for these tasks in 83 cases. In 58 of them (69.9%), the respective child solved the same task by fact retrieval at the end of the year (if not, he or she would apply a DFS again). Conversely, there had been 179 instances of task solving by counting-on or counting-back at midyear. At the end of the year, in 61 cases the children had moved on to fact retrieval (34.1%), but in the majority of cases, a child who had used a counting strategy for a certain task at midyear would again use counting for that same task at the end of the year. This difference in the frequency with which a change toward fact retrieval occurred was highly significant (Gaidoschik, 2010, 2012), strengthening the assumption that the repeated use of a DFS subsequently contributes to the automation of basic facts (Baroody, 1999; Van de Walle, 2004).

This also coincides with a central finding of Gray's (1991) cross-sectional study with 72 British 7- to 12-year-old students. Gray interviewed and observed the computing strategies of children who, according to their teachers, performed in mathematics above average, average, and below average. The below-average performers in all age groups, besides using more counting and less fact retrieval on basic facts up to 10 than average and above-average performers, characteristically did not use DFS at all. In contrast, the 7- and especially the 8-year-old children rated as above average regularly used a DFS to solve tasks they had not yet automated, whereas the older above-average performers solved all the basic facts up to 10 by retrieval. Based upon his findings, Gray suggested that "the use of derived facts for the younger children is an indispensable stage in developing knowledge of the number bonds" (Gray 1991, p. 571).

Findings like these cast doubt on development models according to which children would abandon computing by counting basically by using counting-on successfully over a sufficient time period, as such a routine would strengthen "bonds of association" between tasks and solutions in long-term memory (Siegler, 1996). Baroody and Tiilikainen (2003, p. 82) point out that this expectation rests on "oversimplified and untested assumptions," taking learning as a "mechanical and passive process." According to them, computational practice is important as it provides "an opportunity to see and reflect on patterns and relations," but it is the very *reflection* that makes the difference (Baroody & Tiilikainen, 2003, p. 83).

If computational practice consists of repeated problem solving by counting, this might even *prevent* children from reflecting part–whole relations within numbers, as well as the relations between tasks that allow for DFS. Gray observed that the lower-achieving children in his sample, in solving a basic task on their own, typically focused their attention on the counting procedure. Subsequently, the operands were "marginalized" to such a degree that many younger children in this group, after having stated the solution, did not "remember the problem that had triggered the solution" (Gray, 1991, p. 569). Steinberg (1985) points out another issue that might be important in this context: in her intervention study (see below), she observed only five out of the 23 students in her experimental second-grade class who did not respond satisfyingly to 8-week instruction on DFS. Four of these children had been "very good counters" when the intervention started. The author hypothesizes that "these children may have become so proficient in their counting that they did not see the need, and were unwilling to invest the effort, to learn new strategies that might be slower and less accurate when first used" (Steinberg, 1985, p. 351).

In line with this is what Hopkins and Russo (2017) report on case studies of six Australian students who participated in what the authors label "problem-based practice." The students either had been rated "almost proficient" in basic addition for having solved through fact retrieval about two thirds of tasks in an initial assessment, or belonged to the "accurate min-counting" cluster who had used counting-on for more than 50% of the tasks. The subsequent practice consisted of 15 training sessions distributed over consecutive school days. In each session, the students had to solve the same 36 basic tasks, using whatever strategy they themselves would choose. Whereas two of the three "almost proficient" students increased the number of facts solved by retrieval, the students in the accurate min-counting group did not at all. Of course, Brownell (1929, p. 105) already assumed that if "a child habitually counts in using the number combinations, drill, far from breaking this habit or giving him a better one, merely affords him opportunity to increase his proficiency in counting."

On the other hand, though, there is a caveat also against hoping that early instruction in DFS *in itself* would guarantee basic fact fluency in higher grades. Cumming and Elkins (1999) report on a high share of Australian students who heavily rely on counting strategies even in sixth grade, though belonging to a sample of students ($N = 109$) whose arithmetic instruction in first and second grade had stressed "thinking strategies." Only some 30% of the fifth and sixth graders had automated the basic addition facts. About the same proportion had a DFS as their main strategy (Cumming & Elkins, 1999). Of course, it might be of importance that these children's teachers had treated counting-on as an explicitly welcome "thinking strategy." As elucidated above, this might have been counterproductive for at least some children (cf. the warning from Van de Walle, 2004, p. 164). What is more, their instruction obviously did not include what Van de Walle (2004, p. 159) calls "drill of strategies"—that is, targeted effort to elicit children's "repeated use" of a DFS. Empirical evidence (Clarke & Holmes, 2011; Gaidoschik, 2017; Woodward, 2006), as well as theoretical considerations (Gaidoschik, 2010), indicate that, at least for some children, it might indeed be crucial to combine DFS instruction with some form of repetitive practice in order to develop basic fact fluency (cf. Gaidoschik, 2017).

Intervention and Field Studies

As for intervention studies, Rechtsteiner-Merz (2013) compared 12 German first graders (aged 6–7 years) in five experimental classes with eight students in control classes. All these children had shown learning difficulties in mathematics during the first months of first grade. Only the teachers of the experimental classes focused on part–whole relations within numbers and on operational relations between tasks throughout the year. At the beginning of second grade, six of the eight children in the control classes still clung to counting strategies, whereas all but one student in the experimental classes used either fact retrieval or a DFS as their predominant strategy up to 20 (Rechtsteiner-Merz, 2013).

This is in line with a range of small-scale, mainly qualitatively oriented studies, which suggest that children as a rule respond positively to an instruction that puts DFS center stage in the first school year (e.g., Buchholz, 2004; Thornton, 1978, 1990) and in the second school year (Steinberg, 1985). More specifically, Baroody and colleagues (2016) recently evaluated the efficacy of computer-based fostering of two less salient DFS with 81 students (5.7–10.1 years of age) randomly assigned to intervention and control groups. The intervention groups outperformed the control groups significantly in progress toward basic fact fluency also with unpracticed items, indicating that the students were able to transfer the acquired strategies to new tasks (Baroody et al., 2016).

Evidence of the beneficial influence of targeted instruction on children's replacement of counting with noncounting strategies, but also of the great challenge such an approach poses to teachers, comes from a field study by Gaidoschik, Fellmann, Guggenbichler and Thomas (2017). The authors interviewed children in eight Austrian classes at the end of their first school year ($N = 117$). Their teachers, attendees of an in-service teacher development program, had strived to work out a solid understanding of numbers as compositions of other numbers in the first months of schooling. On that basis, they took a guided discovery-learning approach, complemented with direct instruction when needed by individual children, to convey DFS as a convenient way to solve basic tasks (Gaidoschik et al., 2017).

Four of the eight teachers had received support from expert teachers who paid a visit to them once a week throughout the year, joining them for one lesson in the classroom for either team teaching, observing, or giving extra support to individual children. Additionally, the expert teacher would spend another hour per week with the class teacher and give feedback about what she had observed in the classroom. Moreover, the two would talk about problems that had occurred during the previous week, and discuss plans for the one to follow (Gaidoschik et al., 2017).

Interviews with all eight teachers clearly indicate that the four of them who got this kind of tutoring had indeed been paying considerably more attention especially to the *consolidation* of single DFS once they had introduced them in the classroom. On the other side, student interviews based upon the same tasks and method already used for the random sample by Gaidoschik (2010) (see above) revealed significant differences between the two groups within the eight classes regarding the frequency of computing by counting and, conversely, fact retrieval and DFS (Table 6.2).

Table 6.2 Percentage of use of counting strategies in samples with different forms of arithmetic instruction

	2010 sample (no DFS instruction) (%)	2017 sample (DFS instruction based on elaboration of the numerical part–whole concept) (%)							
		Support from expert teachers				No support from expert teachers			
		Class A	Class B	Class C	Class D	Class E	Class F	Class G	Class H
Sums up to 10	39	1	0	5	7	17	16	9	14
Sums up to 20	52	0	0	8	11	21	16	12	22

cf. Gaidoschik (2010, 2012) and Gaidoschik et al. (2017)

In the four classes whose teachers had been getting support (A–D), students resorted to counting significantly less often than those in classes E–H, whose teachers had to get along without that support. In two of the supported classes, counting strategies virtually did not occur at all during the interviews. However, in all eight classes, computing by counting was significantly less frequent than it had been in the random sample studied by Gaidoschik (2010), which had not received targeted instruction in using DFS (cf. Table 6.2) (Gaidoschik et al., 2017).

Overcoming Computing by Counting as a Didactic Challenge

Against the backdrop of the evidence and theoretical considerations outlined above, we may formulate some summarizing statements on the relation between instruction and the widespread phenomenon of children who compute by counting even in higher grades.

First of all, there is empirical evidence that it is *not* just a matter of time and/or individual disposition whether or not—and if so, at what age—children give up counting as their main computing strategy. On the one hand, international comparative studies (e.g., Geary et al., 1996), as well as longitudinal studies (e.g., Gaidoschik, 2010, 2012) suggest that in classes in which instruction is *not* clearly focused on the elaboration of DFS, a large share of children will cling to counting as their main computation strategy at least till the end of first grade. On the other hand, comparative as well as intervention studies (e.g., Rechtsteiner-Merz, 2013) and field studies (e.g., Gaidoschik et al., 2017) allow the assumption that already by the end of first grade the vast majority of children will have moved on to a combination of fact retrieval and DFS *if* DFS have been worked out and consequently fostered in the classroom.

In a 5-year longitudinal study, Geary (2011) found that first-grade use of fact retrieval and DFS predicted mathematics achievement through fifth grade ($N = 117$). From a didactic point of view, this is a consequence of the “hierarchy of learning” that is constitutive for primary grade mathematics (Wittmann, 2015). Within this hierarchy, a child who relies on counting to solve single-digit tasks will hardly adopt noncounting strategies for multidigit tasks, not only because of missing number knowledge, but also as computing by counting tends to become a habit, giving the child a certain sense of security (Gray, 2005).

More fundamentally, children who permanently compute by counting are at high risk of missing what is at the heart of mathematics: structures and patterns (Devlin, 1994). This starts from the pivotal structure of numbers being composed of other numbers, creating some kind of a vicious circle: using counting to solve basic tasks tends to impede realizing the numerical part–whole structures that underlie these tasks, the awareness of which would render counting superfluous (Gaidoschik, 2010). Consequently, computing by counting may hamper the perception of other important relations and structures such as the relation between doubles and halves, multiples, proportionality, and even the decimal structure of two-digit numbers (Schipper & Wartha, 2017).

It is therefore essential for a child's further mathematical development to learn how to add and subtract without counting in the *early* primary grades, preferably in first grade (Schipper, 2009). Of course, this is not a matter of some weeks or months more or less; children learn at different paces. Nor are we talking about children who occasionally refer to counting as a backup strategy to play it safe. Most adults will do that now and then, including those who do not have any problems with elementary mathematics. What we are talking about is counting as a child's *main strategy* to solve addition and subtraction problems. It is *this* form of computing that Lorenz and Radatz (1993, p. 117) rate as a "dead end," stating that children would hardly have a chance to get out of it once they have reached second or third grade.

The latter statement might be too pessimistic, but reliable evidence in the form of longitudinal studies on this issue is missing. We have plenty of evidence, though, from a range of countries worldwide, that a substantial proportion of children and adolescents use counting as their predominant if not sole computing strategy way beyond their third school year (e.g., Moser Opitz, 2013; Ostad, 1998). Hopkins and Bayliss (2017) found that about 35% of 200 randomly selected Western Australian seventh graders still heavily relied on—typically accurate—counting-on with single-digit additions (share of tasks solved by counting-on in this group: 47% on average). Unsurprisingly, these students showed significant lower achievement in a standardized test of general mathematical achievement than their peers who used retrieval-based strategies.

Considering this high incidence, it is obvious that common definitions of MLD do not apply to *all* children, adolescents, and adults who compute by counting beyond first grade. Nevertheless, all these individuals have good reason to find learning and performing mathematics difficult, as it *is* difficult without having basic fact fluency. Furthermore, given the hierarchy of learning outlined above, it is not at all surprising that those who are by definition acknowledged as "having an MLD" are to a large extent (but not always: cf. Dowker, 2005), persons who heavily rely on counting for adding and subtracting.

From a didactic viewpoint, the consequences seem rather clear. They are subject of the final section of this chapter.

Learning Difficulties, Teaching Difficulties, and the Role of Education Policies

In his overview of research on early arithmetic learning, Cowan (2003, p. 68) draws the sobering conclusion: "Mathematics education continues to be more a matter of faith than of fact." While this may have been bettered in the last 15 years, it is quite clear that mathematics education, in any case, is a matter of culture and national traditions. There are substantial differences in the ways teachers teach mathematics in different nations, comprising curricular decisions on what content children should learn at what age, and the choice of methods and learning materials (Li & Lappan, 2015).

As has been outlined, in some countries there is a tradition of purposefully fostering noncounting computation from an early age, whereas in many others it is common to encourage children to use counting for adding and subtracting at least until the end of first grade.

However, given the empirical evidence and theoretical considerations outlined in the preceding sections, we can assume that arithmetic instruction, for the benefit of children's further development, should strive to enable children to add and subtract without counting *as soon as possible*. To reach that goal, it seems helpful to first concentrate on elaborating and consolidating a viable part–whole concept of numbers and then draw on that concept to convey conceptual understanding of DFS and procedural fluency in using them.

There are numerous sound and quite detailed proposals on how to implement these principal considerations in daily classroom work (e.g., Gaidoschik, 2007; Van de Walle, Karp, & Bay-Williams, 2015). We definitely need more evaluation and design research studies to refine these concepts and adapt them to different classroom conditions. However, the by far greater challenge for the time being is to bring into classrooms what mathematics education already knows about how to overcome computing by counting. It takes adequately educated teachers to put that into action, and studies like those by Gaidoschik and colleagues (2017) and by Pfister, Moser Opitz and Pauli (2015) show how difficult this is even for those teachers who willingly participate in in-service development programs. As long as we have not met that challenge, it seems like a sound hypothesis that the large number of children who currently do not reach basic fact fluency is due to adults' *teaching difficulties*.

To avoid misunderstandings: This is not to say that whatever we may eventually achieve in raising the effectiveness of classroom instruction across the nations, we should expect to hereby eliminate *differences* between children regarding how easily, fast, deeply, and comprehensively they acquire arithmetic abilities. Presumably, there will always be children who struggle to reach a higher level. For the time being, though, a large number of children receive arithmetic instruction over the years and still leave school *without even the most basic* of these abilities, fluency with basic facts being one of them. Research of the kind reported in this chapter gives reason to hope that we could reduce this number drastically through changes in the ways we teach. This is the one essential message of this chapter.

The other is that theoretical considerations (e.g., Van de Walle, 2004; Wittmann, 2015), as well as empirical evidence (e.g., Gaidoschik, Deweis, & Guggenbichler, 2018; Moser Opitz, 2001) strongly suggest that those children who presumably need more than state-of-the-art classroom instruction to acquire basic mathematical competence indeed need *more*, but not something substantially *different* from what is recommended for *all* children.

To specify this hypothesis for the subject of this article: Some children seem to need *more* thought-provoking *impulses* to acquire a part–whole interpretation of numbers and use it for problem solving (Gaidoschik & Beier, 2017). To do so, they need *more support* in how to interpret and use material (including their own fingers) and visual representations so they may detect the parts within the whole and reflect the part–whole relation comprehensively (Gaidoschik, 2010; Schultz et al., 2017). To autonomously apply any single DFS, they may need *more guided instruction*, always geared at understanding (Gaidoschik, 2007). In many cases, they will simply need *more time* to do so and *more feedback* on what they do (Lorenz, 2003)—more than an individual teacher is able to offer in a regular classroom.

If so, a *good school system* would supply, in addition to highly qualified class teachers, equally highly qualified supporting teachers, who work as team teachers or with small groups of children or foster individual children on a one-to-one basis. This would substantially boost the possibilities to adapt the didactical measures outlined in this chapter to the individual needs of these children. As has been stated, we need more research to refine existing concepts on how to foster basic fact fluency. Yet, there is no indication that one special group of children would need completely different *concepts*. A strong argument against such an assumption stems from the mathematical content, which is the same for all children, and predefines which steps have to be taken to make further steps possible (Wittmann, 2015). To develop instructional designs that make this content feasible for all children is the job of mathematics education. To install a school system that is fit to implement these designs—including, above all, a sufficient number of adequately educated teachers—is not the job of mathematics education or any other science, but of politics.

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Chapter 7
Development of Number Understanding: Different Theoretical Perspectives

Daniel Ansari

Introduction

In this section, five different perspectives are presented on the development of numerical understanding and mathematical disabilities. Different groups of experts present neurocognitive and developmental perspectives on mathematics as a competence, as well as low achievement of mathematics from sociological and didactic perspectives. These different perspectives and opinions about learning and not learning mathematics raise several questions. Using these questions as a starting point, I try to put these chapters into a broader perspective about the current state in research and practice and what challenges and unresolved issues we have in this field of research.

What Kind of Perspectives on Learning Mathematics Have Developed Most During the Last Decade?

Research on the development of numerical and mathematical cognition continues to be heavily influenced by a theoretical framework that posits that humans are born with a sense of number and that this "number sense" is the basis upon which symbolic numerical and mathematical abilities are built (Dehaene, 1997). I would say that a large proportion of research that approaches the development of numerical and mathematical competencies from psychological and neuroscientific

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perspectives has focused on better characterizing "number sense" and studying its developmental trajectory and influence on the acquisition of higher-level, symbolic numerical and mathematical abilities (Feigenson, Dehaene, & Spelke, 2004; Libertus & Brannon, 2009). I think there has been a tremendous amount of development in this field. We have learned much more about children's numerical competency development; we have also gained tremendous insights into how nonhuman animals process number and what similarities there exist between humans and nonhuman species. We know much more about the neural correlates of number processing from large-scale brain systems that seem to be involved when we process numbers and engage in calculations to the sensitivity of single neurons to numerical quantity (Nieder & Dehaene, 2009). The behavioral and brain imaging data from infants, children, adults, and nonhuman primates has been taken to suggest that there are dedicated systems for the representation and processing of numerical quantity which exhibit both phylogenetic and ontogenetic continuities.

Notwithstanding, there have also been challenges to the position that humans are born with a sense of number which provides the ontogenetic foundation for symbolic numerical and mathematical abilities (Gebuis, Cohen Kadosh, & Gevers, 2016; Leibovich & Ansari, 2016; Reynvoet & Sasanguie, 2016). There currently exists a vibrant debate in the field around this question. Researchers have questioned the degree to which the purported innate sense of number scaffolds human symbolic numerical abilities. Moreover, research on how numerical information is extracted from arrays of objects has highlighted the roles played by non-numerical variables such as density and area in the processing of numerical information (for an extensive review with commentaries see: Leibovich, Katzin, Harel, & Henik, 2016).

Have Some Views About MLD Dominated the Discussion?

The view that we are born with an innate sense of number has dominated the discussion. This is, of course, not surprising because the search for what the origins of our numerical abilities are lies at the heart of research into numerical and mathematical cognition. Any developmental account of numerical and mathematical competencies must specify the ontogenetic starting point of such abilities. It is therefore important that current debates around the origins of numerical abilities continue. I think it is fair to say that we have yet to have any consensus on fundamental questions such as "how do children learn the meaning of number words"? We are, in my view, still far away from resolving some of the most fundamental questions. At the same time, there are numerous other questions in the field of numerical and mathematical abilities that are in sore need of attention. These topics have been somewhat neglected because of the preoccupation with the origins of numerical abilities.

Have Some Perspectives Got Too Little Attention in General Discussion?

There are a number of topics that I feel have gotten too little attention. What follows below is not meant to be an exhaustive list, but perspective and approaches that have, in my view, not received sufficient attention despite repeated calls for doing so:

1. First of all, I think there needs to be more research on older children with mathematical difficulties. When I talk to educators about research on early number development and predictors of individual differences, I invariably get the question: How should we help older children who still lack foundational skills and concepts? I do not have the answer to this critical question. So much research in our field is focused on young children and how to scaffold mathematical learning from a young age, but less effort has been devoted to understanding individual differences in higher-level numerical and mathematical learning and how cognitive science, psychology, neuroscience, and educational research might help us to address the issues older children and younger adults clearly face.
2. Secondly, we need to continue to move beyond models of mathematical learning difficulties, such as dyscalculia, that posit that such difficulties are caused by very specific impairments to domain-specific representational systems for number. Instead, we need to embrace the growing data that suggest that mathematical learning difficulties are highly heterogeneous and, importantly, very rarely occur in isolation of other difficulties, such as developmental dyslexia or ADHD (Fias, Menon, & Szucs, 2013; Kaufmann et al., 2013). Researchers, including myself, have viewed comorbidity of mathematical learning difficulties with learning challenges in other domains as a confound. This view was driven by the bias to find highly specific causal factors, such as specific genes, specific brain regions, and behavioral impairments on extremely basic (thus presumably biologically primary) behavioral tasks of numerical processing. Therefore, children who did not present with "clean" deficits were often excluded from empirical investigation since the study of these children would not inform a better understanding of domain-specific causal factors of mathematical learning difficulties. It is now clear from research in genetics (e.g., Kovas & Plomin, 2007) and neuroimaging, that there are many factors that are shared between different learning disorders. Thus, researchers need to embrace the overlap between disorders, better try to understand its causes and derive models for the understanding of developmental learning difficulties that go beyond very circumscribed, domain-specific causal factors.
3. Researchers such as myself have not sufficiently included what is known about the cultural history and cross-cultural variability in the systems used for number notation, spoken number words, and the kind of artifacts used to represent number (Beller & Bender, 2008; Bender & Beller, 2013; Núñez, 2017). Given that numerical symbols, such as Arabic numerals, have had a relatively short history,

it is, in my view, important to also include anthropological data and perspective in the study of the cognitive and brain processes that underlie numerical and mathematical cognition and the associated developmental trajectory. By not only focusing on the possible evolutionary antecedents and through the additional integration of historical and anthropological data, we will gain a richer understanding of the diversity of human representational systems for symbolic number. I believe understanding the cultural and historical factors that have shaped our numerical and mathematical abilities will provide us with greater insight into developmental processes and the roles played by different cultural practices and language.

4. Over the past decade or so, there has been a growing interest in the role that parents play in their children's numerical and mathematical development. In this vein, factors such as parental number talk (Gunderson & Levine, 2011), their ability to enhance children's mathematical skills through and the role played by parental mathematical anxiety have been studied (Maloney, Ramirez, Gunderson, Levine, & Beilock, 2015). While this line of work is undoubtedly valuable, the majority of the extant studies are likely reporting inflated effect sizes with respect to the effect of parental factors on children's numerical and mathematical skills. This is because children and their parents share not only an environment but also genes. Therefore, without, at least partially, accounting for the biological factors that might explain relationships between parental and child factors, the environment is overestimated. One way to better understand the roles played by parents is to control for the numerical and mathematical abilities of parents (van Bergen, van Zuijen, Bishop, & de Jong, 2017). This would allow for an indirect control of biological factors and thus yield more accurate estimates of the effects of particular parental behaviors on their children.
5. Finally, and to my mind of the greatest importance for the progress of the field, it is necessary that influential effects in the fields of numerical and mathematical cognition are replicated. It is well established that many influential effects in the social sciences have been found to have a surprisingly poor replication rate (Open Science Collaboration, 2015). This implies that many theories may be built on shaky foundations and are in dire need of revision. In order to build a cumulative science of numerical and mathematical cognition, where every advance is built on foundations that are represented by replicable effects and findings, it is necessary to come together as a field to conduct multi-lab, registered replication studies of key findings in the field (Munafò et al., 2017). For example, theories that postulate that we are born with a sense of approximate number are built, at least in part, on the evidence that infants and newborns can discriminate between dot arrays and numbers of sounds. While I do not mean to challenge, a priori, the existence of such effects, I believe that our confidence in key theoretical accounts concerning the ontogenetic foundations of numerical competencies would be significantly strengthened by adequately powered, registered replication studies of the infant number processing studies. Similar efforts are underway in other domains of infant cognition (see http://babieslearninglanguage.blogspot.ca/2015/12/the-manybabies-project.html). This need applies equally to many other studies in the field of

numerical cognition. For example, we reported in 2007 that children with developmental dyscalculia exhibit reduced activation of their right intraparietal sulcus (a brain region thought to be critical for numerical and mathematical processing) during a dot discrimination task (Price, Holloway, Räsänen, Vesterinen, & Ansari, 2007). However, the data reported involved a comparison of eight children with developmental dyscalculia and eight age-matched typically developing controls. It is clear that the results we reported in 2007 were woefully underpowered (Button et al., 2013). There is a need for replicating findings such as these and many more. Researchers in the field of numerical and mathematical cognition, including myself, must be prepared for accepted wisdom to be shattered and for a process of revision to be initiated.

Can We Compare the Results from Studies on Dyscalculia from Different Countries to Each Other?

It is difficult to compare the performance of dyscalculics across countries because of the way in which dyscalculia is typically defined. Most children with dyscalculia are diagnosed on the basis of their performance on standardized tests. Therefore, their performance is always measured against the mean performance of a particular population. So, for instance, one might decide that children will be labeled as having dyscalculia, more important to study if their performance on a set of standardized tests falls below two standard deviations of the mean of the population of which they are member of (country A). Now consider children in a different country (country B) where the mean is two standard deviations higher. In other words, the population means of the two countries differ by two standard deviations. It follows mathematically that children who would be labeled as having dyscalculia in country B would be considered to fall within the average range of mathematical ability in country A. Therefore comparing “dyscalculics” from countries A and B would result in the comparison of children with very different mathematical abilities. This does not necessarily imply that the factors that contribute to the low mathematical performance of children in both countries may not be similar. It may be that the causal factors are very comparable but that the educational system in country B is more effective at shifting the mean of the countries math performance distribution than country A.

A more meaningful comparison of children with mathematical difficulties across countries would be to move away from the present way of selecting children on the basis of arbitrary cutoff points (e.g., two standard deviations below the mean) toward studying the predictors of individual differences in children’s mathematical skills across countries. If, say, number comparison predicts individual differences across countries, this would imply that children with mathematical difficulties across countries all have weaknesses in number comparison. Put differently, if we accept that there will be mean differences in the level of performance of children defined as presenting with dyscalculia, then we can begin to investigate more interesting questions concerning both the common mechanisms (despite mean differences) that underpin mathematical learning difficulties across countries. We can also learn from high-performing countries about the ways in which they succeeded in raising the mean level of performance of students in their systems (and therefore also the level of performance of children with dyscalculia).

Finally, it is important to mention that there are, of course, other approaches of determining whether a child has a mathematical learning difficulty that go beyond using cutoffs on standardized measures. In particular, approaches such as response to intervention/instructions may be very fruitful in the context of international comparisons of dyscalculia. Here students are identified as having difficulties when they persistently fail to respond to instruction (even when that instruction is varied and uses different approaches to teach the same problems).

How Far Are We in Understanding the Mathematical Brain?

We have made tremendous progress in understanding the brain circuitry that underpins our numerical and mathematical abilities (Ansari, 2008; Cantlon, 2012; Dehaene, Piazza, Pinel, & Cohen, 2003). Not only have noninvasive neuroimaging methods helped us to better understand what brain regions, at a large scale, are involved in numerical and mathematical processing, but we know more about how single neurons might process numerical quantity (Nieder & Dehaene, 2009). Furthermore, we are beginning to make progress in understanding how the mathematical brain develops (Peters & De Smedt, 2017) and we are starting to understand how mathematical processing interacts with other domains such as language and visuospatial cognition in the brain.

What Are the Key Questions to Focus on Next to Improve the Understanding of the Mathematical Brain?

While I strongly believe that the developments in the study of the neural mechanisms have progressed significantly, I can also see outstanding questions (at least a selection):

1. How replicable are the data on the neuroscience of numerical and mathematical cognition? As is the case for most other domains of cognitive neuroscience, what we know about the neural correlates of numerical and mathematical processing is built on a body of neuroimaging studies with small samples. This necessarily means that the body of knowledge that we have garnered to date is underpowered. Therefore, we need studies into the neural basis of numerical and mathematical cognition that are more adequately powered and thus have larger sample sizes. This will require researchers in our fields to collaborate more intensively on collecting larger datasets.

2. Some may argue that we have made progress in understanding the neural correlates of mathematical learning problems, such as developmental dyscalculia. My impression is that we have not. The research on the neural mechanisms underpinning mathematical learning difficulties is conflicting and not conclusive. As is the case for the study of mathematical learning difficulties more generally, the study of their neural correlates is confounded by small sample sizes, vast differences in the classification of what constitutes a mathematical learning disorder (Szűcs, 2016), and publication bias (studies that report differences in functional and structural neuroanatomical correlates between children with and without mathematical difficulties are far more likely to be published than studies showing the contrary). Furthermore, as I have argued above, I believe that we need to move beyond trying to find the "unique" signatures of mathematical learning difficulties to better understand how mathematical difficulties intersect with other domains of learning and dysfunctions therein.
3. Research on the neural architecture that supports numerical and mathematical cognition, including my own research, is still primarily correlational in nature. We tend to try and understand the neural correlates of a particular task that we are interested in. There is nothing wrong with that, of course, and such lines of research will and should, in my opinion, continue. However, we need to have more investigations of how experimental manipulations change the neural networks engaged in numerical and mathematical cognition (Zamarian, Ischebeck, & Delazer, 2009). Training studies can help us to better understand causal mechanisms and to inform our understanding of the direct link between neural mechanisms and behavioral outcomes. For instance, there is much interest in whether training nonsymbolic representations of magnitude can enhance symbolic number processing (e.g., whether the numerical magnitude processing mechanisms apparently available to infants and nonhuman animals can be trained and, importantly, whether their training-induced enhancements will lead to improvements in symbolic number processing (e.g., symbolic calculation)). This hypothesis should be tested at the level of the brain: Can the neural mechanisms underlying nonsymbolic number processing be changed and does that change affect the brain representation of number symbols? Beyond resolving similar theoretical questions, training studies provide the opportunity to inform education questions. For example, do the particular materials used to train a particular mathematical concept differentially influence the brain mechanisms engaged, and, in turn, do those changes have a differential impact on behavioral outcomes?

Are There Some Breakthroughs in Science that You Think Would Change Our Picture in the Near Future?

The breakthroughs in science that will change our field are already beginning. Scientist are realizing that in order to make real progress, the traditional model of each lab running studies by themselves should come to an end. To truly make progress and to be able to study phenomena of interest in a serious way, we need to collaborate at a larger scale. We need to collect data using the same methods across laboratories and combine our data to be able to make the most powerful inferences. In this way, it will also be necessary to engage in more adversarial collaborations. That is researchers who hold opposing theoretical positions should be collaborating with one another to resolve their disagreements by agreeing on experimental protocols that would help to arbitrate their positions and then to run those preregistered empirical studies in their labs and agree a priori to publish their results no matter which (if any) of their theoretical positions is supported by the data.

What Is the Role of Spontaneous Focusing on Numerosity (SFON) in MLD?

That is a rather difficult question. I suspect that the neurocognitive mechanisms that underpin SFON are, at least in part, the same ones that underpin visual attention. I think it is highly unlikely that SFON is underpinned by very specific neurocognitive processes that do not apply in other domains of cognition, such a single brain area or even a dedicated network of regions. Instead, it is, in my estimation, more likely that SFON is linked to processes of visual attention. For instance, one might hypothesize that processes of so-called bottom-up or stimulus-driven attention play a role (Corbetta & Shulman, 2002). These attentional processes must, at some level, interact with representations of number and the cardinality of sets. In general, any precise account of the mechanisms underpinning SFON must tease out the relative, interactive processes of the process of “focusing” and pre-existing conceptual number knowledge. In order to SFON, children must be capable of representing the cardinality of the sets they are focusing on.

I think it is not fruitful to look for a core cause of developmental dyscalculia and related mathematical difficulties. Instead, future research should embrace the heterogeneity of mathematical difficulties and the associated developmental trajectories. Having said that, it needs to be acknowledged that, as discussed by Lethinen et al., individual differences in SFON have been found to correlate with variability in students’ concurrent and future performance on measures of mathematical competence. Therefore, it is highly likely that children with mathematical learning difficulties also show lower levels of SFON. However, that does not mean that SFON is the core cause of mathematical learning difficulties. Given that, to date, our understanding of the relationship between SFON and mathematical competence comes from correlational studies, it is also a possibility that lower frequency of SFON behavior among children with mathematical learning difficulties is driven by their poor conceptual understanding of number. In other words, if children have impoverished representations of the cardinality of sets, they may not be able to “spontaneously” deploy this knowledge to engage in behaviors that can be characterized as SFON. Taken together, I believe it is useful to study SFON in the context of children with mathematical learning difficulties, but I believe the causal relationship between SFON and such difficulties may be far from straightforward.

Can a Child Be at Different Levels in Different Math Contents in the Way Described by Reiss or Is the Development More Based on Some General Factors?

I strongly believe that a detailed characterization of the difficulties that children with MLD have is critical in order to inform educational strategies. The levels proposed here can be easily identified by educators and can help them to design strategies to scaffold the skills and understanding that a given student is lacking. In this way, what Reiss proposes is likely to be incredibly valuable for educators working with children who have mathematical learning difficulties.

As for the question of whether more general factors are involved, I think there is ample evidence that competencies such as working memory, inhibitory control, and attention are associated with mathematical learning difficulties (Cragg & Gilmore, 2014). However, I am not convinced that measuring such variables in the context of assessing the specific difficulties that individual students have (and the level of their difficulties) is as essential as characterizing the specific difficulties within the domain of learning (in this case mathematics). Consider the following two scenarios:

1. A teacher receives a report that states that a student who they know has mathematical learning difficulties scored low on a test of working memory.
2. A teacher uses the diagnostic scheme put forward by Reiss and discovers that the student who presents with mathematical learning difficulties has real difficulties understanding the place value structure of the decimal system. For example, if the student is asked to write down “thirty-six,” the student writes 306.

Now consider which of the two scenarios is more informative from a pedagogical point of view. I think it is likely that most educators would be able to have a clearer action plan based on the second compared to the first scenario. Of course, being aware of both scenarios may lead to the richest pedagogical interventions. However, without a clear characterization of the mathematical difficulties that students experience, educators will be less well equipped to help their students.

What Are the Roles of Informal and Formal Learning in Mathematics?

The distinction between informal and formal learning is a useful one in mathematics education. Currently most (but by no means all) educational systems are structured in such a way that young children, before they enter elementary school, are in environments that emphasize play, unstructured time, and exploration. When children enter grade 1, they often find themselves in a strikingly different environment in which they have much more structured time, homework, etc. Therefore, the difference between informal and formal education is imposed by the structures of education. In contrast, I am not convinced that the differentiation between informal and formal learning is one that is empirically useful. I am not aware of any data that suggest that there is a qualitative difference between the neurocognitive systems that operate in so-called informal and formal contexts of learning. In other words, for me, the difference between informal and formal learning is a structural, contextual one, not one that reflects the way in which children learn.

What is the Role of Socioeconomic Status in the Development of Math Skills

It is widely agreed that SES is a very broad index of the environments in which children develop. It can be reflective of the educational opportunities they are afforded, the amount of time that they spend with their parents, and the amount of input they consequently receive from their parents and other family members. I think the mechanisms by which SES exerts its effects on individual differences in children's math skills are complex. Furthermore, the mechanisms by which SES affects individual differences are poorly understood. As much as I believe that it is important to study the mechanisms by which SES exerts its effects, I think it is equally, if not more important to study how the environment can be structured to counteract the effects of SES on academic outcomes. I will go into more detail on this in my answer to the next question.

Kotzé (see Chap. 5) gives an example that children in different countries with the same absolute SES perform at different levels of skills. The SES in that chapter is defined based on only buying power. One of the questions for research is what other aspects of SES should be included into the equation? For example, cross-national comparisons, such as PISA (programme in International student assessment), have consistently revealed that the effect of SES on educational outcomes varies between countries. The relationship between SES and educational attainment is sometimes referred to as the "SES gradient." In educational systems that strive for a high degree of equity (affording the same educational opportunity and quality to all learners regardless of neighborhood, etc.), the gradient is flatter. Therefore, it is entirely possible that in one country, low SES might be associated with negative outcomes, while in another the association may not be as severe. I think the lesson from these findings is that educational policy has the potential to work against the association between SES and educational attainment. This is why I think it is critical to examine closely how one can structure the educational system in such a way that it does not exacerbate the negative effects of SES. Of course, there might also be other policies that could lessen SES differences within any countries, such as universal income or greater tax breaks to low income earners. However, in the presence of SES differences, the educational system can be structured in a way that lessens the gradient.

What Is the Interplay Between Different Perspectives of Numerical Development? Do They Talk to Each Other?

Like in any field of science, there should, of course, be more cross talk between different perspectives. But that is easier said than done. How many conferences have you been to where the conclusion is something along the lines of “We need more interdisciplinary research” or “There should be more collaboration,” and how many times have you gone away and actually changed the way you do your research and with whom you collaborate? I think we all know the answers to these questions. This is not to say that it is impossible, and it definitely does happen, but to a lesser degree than we might hope. This is not because researchers are lazy or unmotivated to collaborate with others. However, researchers do operate within different circles of a disciplines’ Venn diagram, and therefore their views and approaches are only ever partially overlapping with that of others. Despite the growing access to information from across different fields, all scientists are biased toward theories and methodological approaches that are close to their own expertise, theoretical perspective, and methodological toolkit.

How Could We Improve the Discussion Between Different Views?

The most likely answer, given our current models of doing science, would be to hold a conference and publish an edited volume that brings together individual researchers with different perspectives. But I am not convinced that this model has been successful in providing the fertile grounds for greater interdisciplinary, serious consideration of differing perspectives, and the growth of new knowledge that builds on different views. In my view, real breakthroughs can only be achieved by incentivizing interdisciplinary discussions and research through funding and institutional support structures. To put it bluntly, researchers, like most individuals, follow the money. That is not to deny that we all have good intentions, but we also need to keep our labs and research programs funded. Therefore, unless there are real changes in the way that research is funded and the way that institutions measure our success, the journey of truly integrating different viewpoints, research methodologies, and theories will continue to be difficult. Again, this is just my perspective. I am truly optimistic that change is on the horizon. However, change is not just up to the individual investigator and their good will and idealism, but change needs to be systemic and provide the means and infrastructure to make it happen. We need different grant funding systems and different metrics for hiring and promotion of researchers.

Will Science Change Math Education in the Near Future?

I do not believe that science can directly change math education, as in prescribing how educators should teach, nor does it need to. In this way the difference between evidence-based and evidence-informed is really useful. I do not think that any single empirical research study, or even a body of research findings, will directly change the way that educators work on a daily basis. It might inform how they structure their classrooms, the materials they use, the way they space activities, etc. But I don't think that science will fundamentally change math education. Education is a dynamic process between adults who are educators and children who are part of a social system that demands them to be in classrooms. I view the role of scientific research on learning and development as one that is informative rather than prescriptive. By stating this, I do not wish to diminish the role that empirical research has to play in education, rather I wish to acknowledge the complexities of the educational landscape that are informed by the sociocultural context, the priorities of a particular educational system, and the training and attitudes of the educators within that system. That is far from saying that research is irrelevant – no it is critical, but not in a prescriptive way but in an informative one. Researchers have, in my view, a social responsibility to make their evidence available to educators and educational policy makers. But they cannot expect their insights and results to lead to immediate uptake. By the same token, researchers also have the responsibility of pointing out educational policy and practices that run counter to what the current received scientific wisdom is. There needs to be a conversation and alignment between education practice and empirical research. Importantly, such as process of alignment needs to be bidirectional, research can inform math education, and, equally, math education can inform research.

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Part II
Mathematical Learning and Its Difficulties Around the World

Chapter 8
Mathematical Learning and Its Difficulties: The Case of Nordic Countries

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The Nordic countries are located in the North corner of Europe and consist of Denmark, Finland, Iceland, Norway, and Sweden. They form a culturally and politically isomorphic group with tight relationships. These welfare societies share the ideology of a strong responsibility of the state on the well-being of the members of the society. The strong economies (World Bank, 2013) and high levels of taxation (see KPMG International, 2013) have been the guarantees for that the states have had the assets to organize the welfare including health, social services, and education. During the current millennium, the Nordic countries have consistently been at the top

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of different international comparisons on welfare, health, quality of living, economic competitiveness, and even happiness of the citizens (Helliwell, Layard, & Sachs, 2013). Likewise, these countries are similar in a high expenditure on education, relatively small class sizes in schools, and long academic teacher education (see OECD, 2012). Investing in education has been one of the core features of the success of these countries. Despite many similarities, there are differences how the educational systems work and how education is conceptualized. For example, Finland and Denmark have *compulsory learning*, while Iceland, Norway, and Sweden have *compulsory schooling*. Compulsory schooling means that a pupil is obliged to attend school, while compulsory learning means that the educational authorities are obliged to ensure that pupils acquire the knowledge laid down in the curriculum (Tomas, 2009).

One of the marked differences between the Nordic countries has been the results of the OECD PISA studies, where Finland since the first study in 2000 has been among the top performers in mathematics and Denmark significantly above OECD average, while the other Nordic countries have been close to the OECD average (OECD, 2013a). In PISA 2015, Finland's and Denmark's students performed equally high and significantly higher than students in the other three countries. However, in Denmark, Finland, Sweden, and Iceland, the trend of performance level since 2003 has been declining (OECD, 2016). The percentage of low performers (defined as below Level 2) was as low as 6 in Finland in 2006 but has raised to 14% in the 2015 assessment. In other Nordic countries, the percentage of low performers in mathematics has varied from 14% in Denmark in 2006 and 2015 up to 27% in Sweden 2012. The latest TIMMS studies for fourth- and eighth-grade students have shown similar trends (Mullis, Martin, Foy, & Arora, 2012) (Fig. 8.1).

Despite these differences at school age, the Nordic countries reach the world's highest levels of numeracy in adulthood. In the recent study on the numeracy proficiency in adulthood (16–65 years of age), all Nordic countries topped the list together with Japan and the Netherlands (Iceland did not participate) (OECD, 2013b).

Likewise, the participation rates in adult lifelong learning and training have been the highest in the world in the Nordic countries (Eyridice, 2012). The high levels of basic skills in adulthood may be connected to the dynamic nature of the work life.

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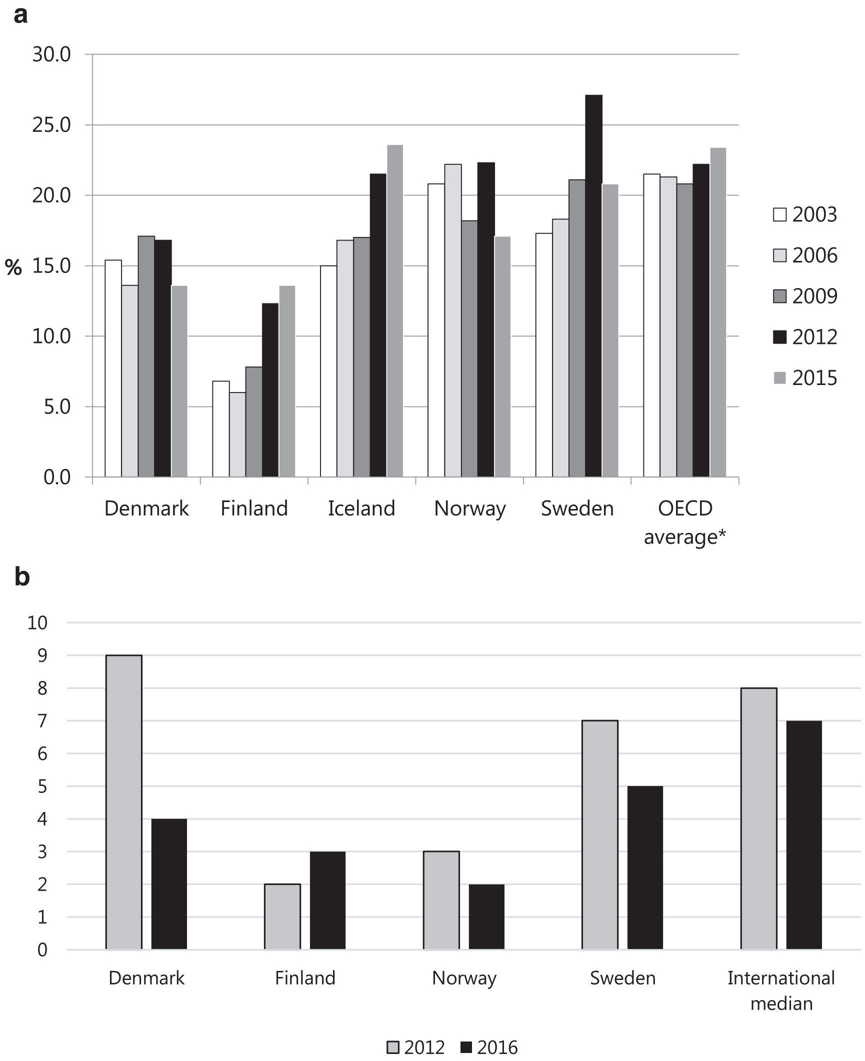

Fig. 8.1 (**a**) Percentage of low performers (below Level 2) in Nordic countries in PISA studies 2003–2015, (**b**) percentage of low performers in fourth grade in TIMSS 2012 and 2016

The Nordic countries had the highest percentage of workers who reported changes that affected their work environment (substantial restructuring or reorganization and an introduction of new processes or technologies) in their current workplace during the previous 3 years (OECD, 2013b). Continual changes in work life require continuous training of employees to be successful. At the same time, the workers need to have strong basic skills to be able to assimilate new skills and ways to work and to adjust to the changes what digitalization and automatization bring. Those with low achievement

Table 8.1 Comparison of the countries in different features of education and education policy

	DK	SE	FI	NO	IS
Compulsory education (age)					
	6–16	7–16	7–16	6–16	6–16
Public expenditure on education, % of GDP (2007, source: Eurostat, UOE) (EU average 4,98%)					
	7,83	6,69	5,91	6,76	7,36
Ratio of pupils to teachers in ISCED 1 (2008, source: Eurostat, UOE)(EU average 16)					
	10,1	12,2	14,4	10,8	10,0
Decision-making authorities involved in developing and approving the principal steering documents for mathematics teaching					
Curriculum	(a)	Central	Central	Central	Central
Guidelines for teachers	Central	Central	Regional/ school	All levels	–
School plans	Schools	Schools	Schools	Regional/ schools	Schools
Evaluation of the effectiveness of curriculum implementation					
External	Yes	Yes	No	No	No
School self-evaluation	No	Yes	Yes	Yes	(b)
Assessment criteria prescribed					
Learning objectives/outcomes	Yes/yes	Yes/yes	Yes/no	Yes/no	Yes/yes
Recommended minimum taught time compared to total time					
Primary	15,3	13,5	17,5	17,2	15,1
Compulsory secondary	12,9	13,5	11,8	11,0	13,5
Total time primary, estimation for (9 years)	1200	900 (from 2016 1125	912	1092	1200
Textbooks					
Autonomy	Yes	Yes	Yes	Yes	No
Monitoring of consistency	No	No	No	No	No
Central level guidelines for teaching methods					
Prescribed or recommended	Yes	No	Yes	Yes	No
Types of grouping	Yes	No	No	No	No
Low achievement					
Surveys or report on low achievement	Yes	Yes	No	Yes	No
Central level support	Yes (c)	No	Yes (d)	Yes (e)	No
Differentiation of curriculum content according to ability	No (f)	No	Yes	Yes	No (f)
Support for low achievers					
Standardized tests	Yes	No	Yes	Yes	No
Intervention of a specialized teacher			No	Yes	
Small group tuition			Yes	No	
Compulsory diagnostic tests at grades	Third, sixth, ninth	Third, sixth, ninth		Second	

(continued)

Table 8.1 (continued)

	DK	SE	FI	NO	IS
National surveys on motivation	Yes	No	Yes	Yes	No
Strategy to increase motivation	No	Yes	Yes	Yes	No
Lack of qualified teachers in upper primary education (% reported by principles)	2	2,6	2,9	17,8	7,6
Teacher training (advocated by central authorities)					
Differentiating teaching for pupils with different abilities and motivation levels	No	No	Yes	Yes	No
Detecting and tackling pupils' difficulties in mathematics	No	No	Yes	No	Yes

Source: Eurydice (2011)

(a) Denmark: National authorities develop and publish a document entitled Fælles Mål which includes central guidelines and objectives for mathematics teaching, but this is not defined as a curriculum in national regulations

(b) Iceland: School self-evaluation is obligatory, but schools do not have to focus on the curriculum

(c) In Denmark, the Ministry of Education has produced a special document that contains several recommendations on how to address learning difficulties in mathematics. It recommends that mathematics teachers carefully observe low achievers, engage in a dialogue with them, and focus on what they can do, rather than on what they cannot do. Beyond assigning such students easier tasks, teachers should also guide them toward new strategies to cope with their difficulties

(d) In Finland, the core curriculum contains guidelines on general support for students. The most common approach is early detection and support. The Ministry of Education organizes targeted in-service teacher training and maintains a website (10) with information on the most common learning problems in mathematics in the early school years. The site provides access to computer-assisted instruction methods for mathematics (Number Race, Ekapeli-Matikka, and Neure). In addition, specific tests for the diagnosis of learning problems are available for purchase from private companies

(e) In Norway, the main elements of the national policy to reduce low achievement are based on early intervention, national tests and mapping (diagnostic) tests, and the integration of basic mathematics skills in all subject curricula. The national strategy, Science for the future: Strategy for strengthening mathematics, science and technology (MST) 2010–2014, and the National Centre for Mathematics Education (see Annex) are important agents in promoting mathematics education

(f) Same content but at different levels of difficulty

in mathematics are vulnerable in work life and often among the first to suffer from economic turbulences, or as Parsons and Bynner (2005, p. 7) state it, "Poor numeracy skills makes it difficult to function effectively in all areas of modern life."

There are large differences how the educational systems in the Nordic countries (see Table 8.1) have responded to low achievement in mathematics at school age and how the educational system provides support both to the children with low achievement and to the teachers in the schools to work with these children. Therefore, it is reasonable to look at the similarities and differences how low achievement in mathematics is treated in these countries. To enlighten the similarities and differences in the educational support systems on learning disorders in mathematics between the Nordic countries, we presented five questions.

1. How are special needs in mathematics education (mathematical learning disabilities, MLD) defined?
2. What kind of support do children get at school for severe MLD?
3. Who gives the support, and what qualifications they have for this work?
4. Are the evidence-based assessment tools and intervention methods available?
5. What are the key issues and current trends in MLD at the moment?

Sweden

In Sweden, the legislative text does not use the terms MLD, "dyscalculia," "math disabilities," or even special needs in math education. Instead, the legislative text says that all children that are in risk of not attaining the national knowledge goals in a school subject have the right to special support, that is, some form of special education. The legislative text says nothing about what kind of or how much support the children have the right to receive. It only says that the schools must have a competence for special education. It is the responsibility of a school to provide the child individually adapted and adequate support.

In practice, there is large variability how support is organized at the school level. For example, some municipalities/schools require that the child has an ICD-10-based (KSH, 2011; WHO, 2005) medical diagnosis for mathematical disabilities to receive any special support, whereas other municipalities/schools focus on the functional level, which is in line with the legislation. Unfortunately, not all children receive the support that they need and have the right for it according to the law, because the schools do not have the required financial resources or the special education competence. In principle, there are two types of support: individual support with special needs teacher (one-to-one teaching) and level-groupings in small groups (2–5 children) with special needs teacher or regular class teacher.

An additional problem in Sweden is that not all schools acknowledge the concept or term mathematical disabilities (dyscalculia). Accordingly, it is very difficult to estimate the prevalence of children with mathematical disabilities. About 15% of the students usually do not get a pass on the national test in mathematics for the final grade (Skolverket, 2013). Immigrant students or students whose parents have low levels of education are overrepresented among those who do not reach the goals. There is great variation between schools and municipalities in performance levels. There are suburban schools in metropolitan areas with large numbers of immigrant students where the majority of students do not get a pass on the national test.

In Sweden, there is a considerable lack of special needs teachers on mathematical difficulties, because the university-based special needs teacher program started as late as 2008. This program is a 1.5-year-long training program for teachers. Therefore, most of the teachers responsible for helping children with MLD are not specialized to these pedagogical questions. The legislative text does not specify that the schools should have special needs teachers and/or special education teachers.

It only says that the schools must have special educational competence in some form. Likewise, there are no organized systems for continuing education or further training for special needs teachers or special education teachers.

The schools do not use any evidence-based assessment tools or intervention methods because there are none available. Furthermore, there are very few experts in Sweden who do assessments on MLD. However, at Danderyds Hospital in Stockholm, which is one of the few places where this kind of assessments is done, they use the British Dyscalculia Screener (Butterworth, 2003), and recently they have started to use the Panamath test (Halberda, Mazzocco, & Feigenson, 2008).

The new Swedish Education Act from 2010 stipulates that the education and instructions used in Swedish schools must be founded on scientific evidence and established experience. Thus, in the future, the Swedish school authorities will probably put more emphasis on matters regarding evidence-based teaching methods and evidence-based assessment tools. There is, however, some skepticism about the "evidence movement" developed in Anglo-Saxon countries.

Norway

The Norwegian educational policy is founded on the principles of inclusion and adapted education. However, to develop educational practices that achieve these overarching principles is a continuous challenge (Haug, 2010; Mathisen & Vedøy, 2012).

Laws and regulations in Norway do not define or apply the terms dyscalculia and mathematical disabilities. The term learning difficulties is used. According to the Educational Act, the focus is on pupils who do not benefit satisfactorily from ordinary teaching and thereby have the right to be assessed for being in some kind of special needs (See section "A Lack of Certain Arithmetical Abilities or a Certain Way of Doing Arithmetic?" in Chap. 6). Pupils should be referred to educational and psychological counseling service (EPS) for an expert assessment. The expert assessment shall consider and determine the following:

- The pupil's learning outcome from the ordinary educational provisions
- Learning difficulties the pupil has and other special conditions of importance to education
- Realistic educational objectives for the pupil
- Whether it is possible to provide help for the pupil's difficulties within the ordinary educational provisions
- What kind of instruction is appropriate to provide (See section "Evidence on the Impact of Instructional Efforts Focused on Non-counting Strategies" in Chap. 6)

In 2013 an amendment became in force that describes more details about administrative procedures in connection with decisions concerning special education. "Before an expert assessment is undertaken, the school must have considered and tested out, if relevant, measures within the ordinary education facilities that might make the pupil benefit satisfactorily" (See section "Overcoming Computing by Counting as a Didactic Challenge" in Chap. 6).

This can be interpreted as pointing toward a more systematic problem-solving approach in line with recent response to intervention models (Glover & Vaughn, 2010). Further descriptions or guidelines regarding how to assess satisfactory learning outcome and/or the substance of the local schools' investigations are not provided. However, obligatory standardized national test (grades 2, 5, 8, and 9) is a part of the assessment of the children's mathematics at school. The tests aim to be a tool for the teachers to adapt the teaching to each child.

An emerging use of the term dyscalculia is taking place in Norway, and related diagnostic practices evolve. There is, however, no unified and agreed upon definition overall related to mathematics difficulties. On this grounding, it is not straightforward to find the extent of pupils with MLD. If difficulties are defined as getting a low grade in mathematics in school (low achievers), the results from the exam of Norwegian 15-year-old pupils show that 35–40% got 1 or 2 in mathematics (the grading system 6–1, with 6 as the highest). In 2012–2013 the percentage of pupils with individual decisions about special needs education was 8.6% in total (The Ministry of Education, 2013). How many of them with special needs in mathematics is not known. In research, e.g., Ostad (1997) used the term mathematical disable for the lowest performing 10% of children in Norwegian schools and found this level of low performance to be stable through all school years.

The support provided by schools varies. Lessons can be given in smaller groups or individually, outside or inside the regular classes and classrooms. The quality of the support also varies in line with the helper's background, from adequate support from a teacher in special needs with competence in mathematics to an assistant without teacher training at all. The use of assistants in special education increased from 2001 to 2008 (Bonesrønning, Iversen, & Pettersen, 2010).

Due to a lack of research-based knowledge about what goes on in segregated and inclusive special education in Norway, a joint research project was carried out from 2012 to 2015. The project had as main research questions: "What special education is about, and what is its function?" (http://www.hivolda.no/speed).

One of the main points from the research is to build education for all on a professional ground, to understand the complexity of the challenges, and to make institutions responsible, not only individuals (Haug, 2016).

Laws and regulations in education put emphasis on identifying pedagogical needs and developing supportive actions. Categorizing students or groups by diagnostic labeling is subordinate. However, this question of diagnosis and labeling causes a tension in the public and is a constant topic of the educational debate.

New practices of assessment in contexts (Nielsen, 2013) are being developed and tried out by Statped and EPS (Daland & Dalvang, 2009, 2016). It adopts a stance toward curiosity on how mathematical learning situations can be understood and further developed. This assessment approach seeks to investigate and analyze relations between three main dimensions: developing as a person, learning mathematics, and participating in learning communities.

Iceland

Like in other Nordic countries, laws and regulations in Iceland do not define difficulties with mathematics or dyscalculia at any school level. Schools set their own targets of competency in mathematics in coherence with the national curriculum guidelines, and pupils are offered support based on them, as well as on outcomes from standardized testing in mathematics in grades 4, 7, and 10. On those tests, between 17% and 24% of pupils score 0–22 points on a normal scale and fall into the category of poor performance (Sverrisson & Skúlason, 2012).

Support in schools for pupils with difficulties in mathematics is either in the hands of special education needs (SEN) teachers or mathematics (or other) teachers. In a survey from 2010 (Óskarsdóttir, 2011), different approaches to grouping and teaching were evident. In some schools, the tradition is that the SEN teachers work with pupils that need support in small groups of two to four pupils two to four lessons a week usually in a separate room. In other schools, pupils in the same year group are tracked into groups in mathematical classes depending on their level of performance, and the low-performing pupils work in small groups often with a mathematics teacher (or other experienced teachers) up to six lessons a week. In a minority of schools, SEN teachers or mathematics teachers go into classrooms and assist pupils that need support.

SEN teachers, according to the survey, map pupils' abilities before they begin working with them and tend to work with tailor-made assignments. They use manipulatives and physical models in their teaching and do not necessarily follow the textbook that is used for mainstream mathematics teaching. The focus in their teaching is on how to learn algorithms as a means of solving problems and to establish ways of working with mathematics. Mathematics teachers on the other hand use the textbooks and other teaching materials used by the year group and tend to follow the curriculum guidelines. The emphasis in teaching is placed on basic algorithms, teaching pupils how to calculate but less on how to use manipulatives other than computers and calculators.

In Iceland teachers and SEN teachers have a university degree. There is one course aimed at preparing SEN teachers to teach mathematics, and it is called "Mathematics for all." The focus of this course is on mathematics learning and how children develop mathematical thinking. The participants of the course also work to develop their own understanding of mathematics and discuss their different ways of approaching mathematical problems. The aim is to be able to understand children's diverse ways of developing mathematical thinking. The main goal of the course is to prepare teachers to map pupils' abilities and to learn how to support children to overcome their difficulties in learning. Also, there is a discussion about diverse pupils' difficulties and how SEN teachers need to collaborate with mathematics teachers in assisting pupils. The course is based on research on how children learn mathematics as well as on research on learning difficulties in mathematics and teacher development in teaching mathematics in inclusive settings (Guðjónsdóttir, Kristinsdóttir, & Óskarsdóttir, 2007, 2009, 2010).

One standard-based assessment tool is available to SEN teachers as well as mathematics teachers. This test, Talnalykill (Guðmundsson & Arnkelsson, 1998), is standardized and criterion-referenced in Iceland. Those who want to use it must be licensed. The test is made up of two main test components, written group tests, and individual oral testing. Some schools in Reykjavik and other places have used the written part of the test to screen third grades for mathematics difficulties. The test has been criticized for focusing mainly on children's fluency in using traditional algorithms and not screening for other mathematical competencies such as the ability to deal with mathematical language and tools. Many teachers in schools also find it too time-consuming, both regarding assessing the pupil and the time it takes to calculate the results. School psychologists also assess pupils for difficulties with mathematics using tests such as WISC-IV (Guðmundsson, Skúlason, & Salvarsdóttir, 2006), which has been standardized and localized for the Icelandic context.

In the new national curriculum (Ministry of Education, Science and Culture, 2011), the emphasis is on equal opportunities for all pupils regardless of their abilities or circumstances. At the compulsory school level, all pupils have the right to compulsory education in their inclusive neighborhood school. The focus in the mathematics chapter is on the right of all children to develop their mathematical thinking and get the support they need to develop mathematical competencies (Mennta-og menningarmálaráðuneytið, 2013).

Finland

The Finnish educational system is state governed and funded but municipally organized. The private school sector is practically non-existing. The leading principle of the educational policy has been to offer free, high-quality education to all in local schools. There are no standardized or national assessments in primary education, but every school and teacher have a freedom to decide how they monitor the development and learning of their pupils. Typically, teachers use a lot of formal and informal exams to follow the progress of their students.

The number of pupils in special education increased rapidly in Finland during the last two decades from less than 3% up to 8.5% in 2010. At the same time, the number of children receiving part-time special education peaked at 23.3% (The Finnish Centre for Statistics, 2013) resulting in about one-third of children at the early-grade education to receive some individualized support. Even though supporting reading skills was clearly the largest subject, special needs education in mathematics showed the largest growth (Räsänen & Koponen, 2011). In 2010 about one-fourth of part-time special education was devoted to mathematics. All these figures were world records at their time.

This unexpected growth in special education caused the Finnish special education system to be reformed. It started to be a too expensive solution for treating individual differences in learning. The changes in the Basic Education Act were passed in 2010. The new strategy emphasized inclusion over segregation and stressed the importance of a pedagogical approach over medical and psychological approaches. The aim was to change the old diagnostic terminology to a more pedagogically oriented language. According to the "new educational talk," medical or psychological terms like mathematical learning disorders or dyscalculia were not recommended. Instead, the focus should be given to identifying pedagogical needs and taking supportive measures (Thuneberg et al., 2013).

The new support system is divided into three levels of intensity. *General support*, targeted to all children, is for temporary needs in learning. The second level, concerning about 20% of children with needs for more regular support, is referred to a pedagogical assessment and to an *intensified support* with a time limit. Main tools are part-time special needs education, individual guidance counseling, and use of flexible teaching groups, as well as home-school cooperation. The third level, targeted to about 5% of the children, *special support*, is provided for those who cannot adequately achieve their growth, development, or learning objectives through other support measures. The most serious cases, defined in the previous system, as having severe mathematical learning disorders, go through a broad pedagogical evaluation and if needed may study according to an individual learning plan (ILP). The pedagogical evaluation is coordinated by the school teachers and typically contains a consultation of a child psychologist who has many options for standardized tests of mathematical achievement to be used as part of the assessment.

Even though the system reminds the descriptive conventions of the response to intervention (RtI) model (Fuchs & Fuchs, 2007), it was not the foundation for the new model in Finland. The key differences between the RtI and the Finnish models are the absence of standardized assessments and structured evidence-based interventions in the Finnish model. In the Finnish model, the teachers are at the helm, and they are given freedom and responsibility to tailor the needed processes to support each child. This requires a well-organized system at the school level and continuous further training for teachers. In larger cities, there are "Mathlands," which are support and learning centers for teachers. Likewise, there is a government-funded web service (lukimat.fi) run by Niilo Mäki Institute, a research center on learning disorders. The service offers information and free tools for early interventions and assessment of reading and mathematical difficulties in early primary education.

In Finland, practically every school has qualified special education teachers with a university degree. The majority of them give part-time special education in collaboration with the classroom and subject teachers. Likewise, every school has a student welfare group for multi-professional collaboration. However, even though the school system offers a lot of individualized support, there are still a lot of challenges to meet. According to the two recent analyses from the national assessments on mathematical achievement from sixth (Räsänen, Närhi, & Aunio, 2010) and ninth grades (Räsänen & Närhi, 2013), close to half of the children identified having a low achievement in mathematics (about 5–6% in total) get only a little attention from the school or teachers.

These results were a surprise because the criterion for low achievement in these evaluations was a combination of assessment and teacher's identification. Therefore, the reason for ignoring these children with low achievement from support was not due to non-identification. The biggest challenge in Finland is not whether the pupils will be identified having mathematical learning disabilities but how to guarantee that they all are offered the support and care they need.

Denmark

The Danish educational system is free and publicly funded. Even private schools get public funding for as much as 73% of the amount given to public schools, while the rest is paid by the parents (per private school student around 130 Euro per month). Private schools are getting more popular. While in 2000 the percentage was 12% in private schools, in 2016 the percentage was 18%.

All public schools prepare the students for national exam at the end of grade 9. From 2017, national compulsory assessments also include 14 digital, adaptive tests from grades 2 to 8, including 3 mathematics tests in grades 3, 6, and 8. Most private schools offer these national tests, too. For teachers, the aims of the national testing program are to provide a tool for teachers' own formative assessment of their students' progress and a tool for monitoring their own teaching. Nevertheless, many teachers find it difficult to use the national tests according to these aims. Other assessment tools are provided by publishers or the teachers themselves, and every school has the freedom to decide how to act upon test results. Besides, some schools and adult learning centers use the British Dyscalculia Screener (Butterworth, 2003).

In the present national curriculum guidelines for mathematics (Common Goals, 2016), no student characteristics (i.e., special needs students) are described. But for some specific skills and knowledge, eight "attention points" are described: they refer to the level of basic skills that are a prerequisite in order to acquire sufficient skills later on.

The political agreement in the Parliament June 2013 on improving Danish school children's performance in school subjects included initiatives for "students with dyscalculia." On this background, a test for dyscalculia for grade 4 in Danish schools, guidelines for test takers, and ideas for follow-up assistance are being developed in 2015–2018. A proposed definition of dyscalculia serves as a starting point: "Dyscalculia is an impairment that may influence education and work. Weak calculation skills are not matched by corresponding weak skills in other fields" (SFI, 2013). Expected percentage of students to be identified by this future dyscalculia test is as low as 1%. Many more students than 1% are facing mathematics difficulties and in need for focused support, either just in mathematics or also in other subjects, drawing on social, psychological, physical, and didactical perspectives. Support in mathematics is needed for students in segregated special schools and classes as well as in regular school and classes.

Since the Salamanca Declaration (UNESCO, 1994), professionals and politicians have argued for increasing efforts for inclusion. The number of students in special education and the costs of special education have been steadily growing in Denmark. Data from public schools showed that in 2008–2009 support organized in special classes and special schools was provided for 5,6% of the students, while special education in ordinary classes was provided for another 8,7%. When these and other data were brought up and analyzed (Finansministeriet et al., 2010), political efforts were intensified to include more students in ordinary classes and schools and to replace special education with another instrument. Economistic arguments were put forward but also humanistic arguments for better learning and well-being when "special students" would be more along with "regular students."

As several special schools have been closed the last years, also some students with diagnoses as autism, Tourette syndrome, conduct disorders, or general learning difficulties are now being included in regular classes and schools. However, it has in many cases proven to be problematic, as several teachers have not been trained, are not knowledgeable, or are not given sufficient resources to create the needed inclusive learning environment.

After the law "No 379 – 28 April 2012," less than 9 specialized lessons of 60 min (equivalent to 12 lessons of 45 min) per week are not seen or regulated as a special education program. Support less than nine lessons is given as part of mainstream education. In instruction, can be used, inter alia, differentiated teaching, tracking for shorter periods, two teachers in class, teaching assistants who can both help each student and the class as a whole, or supplementary teaching and other kinds of support (www.uvm.dk 2015). Some programs for supplementary teaching are developed and used as an early intervention in mathematics; see, for instance, Lindenskov and Weng (2014).

Available data on mathematics in special education in special and regular schools is extremely sparse (Lindenskov, 2012). Nevertheless, the interest in special needs in mathematics has been growing since 2000 among school teachers in public and private schools, adult educators, high school teachers, school psychologists, special education teachers, consultants, teacher educators, and researchers. To increase the overall quality of school mathematics, 10 years ago a diploma program was set up for mathematics teachers in service to become "math tutors." The 1-year program includes six modules, and one module focuses on students in mathematics difficulties. Several seeds have been sowed for continuous interest and for development projects and initiatives at school and municipality levels. The educated math tutors have organized a national network covering about 1000 tutors spread over 800 out of 1400 schools.

In 2010, the association DanSMa (Danish Special Mathematics) was founded as a common meeting place for these professionals to discuss typical issues concerning people with special needs in mathematics in order to improve offers for children, adolescents, and adults. DanSMa initiates public debates, disseminates the latest research on the character and background of mathematics difficulties, as well as on identification and interventions, and arranges seminars with invited speakers (dansma.dk).

Summing Up

We presented five questions to analyze the similarities and differences between the Nordic countries how children with MLD are recognized and how their learning is supported. To summarize our findings, we go through the replies question by question.

The first question concerned how special needs in mathematics education and mathematical learning disabilities (MLD) are recognized and defined in each country. In all countries, the legislations recognize low achievement as a special question, but none of those take any stance on ICD or other clinical diagnostic systems. There are no commonly accepted criteria for diagnosing MLD. The assessment procedures used in Iceland, Denmark, and Norway are rather close to those defined in ICD, namely, combining standardized achievement tests and cognitive assessment. In Finland, standardized tests and a psychological assessment are a common practice in a case with persistent learning disabilities, but giving a diagnostic label for MLD is exceptional. The educational reform in 2010 pushes Finland closer to the Swedish approach where there is a strong aim to avoid assessments and diagnostic labeling and to concentrate on methods of inclusive education.

The Finnish and Icelandic schools have been extremely sensitive to define a child as having special needs in education (SNE). In Iceland, about 24% of children are defined as having special needs, while in Finland about 8% of children are defined as pupils with SNE, and an additional 20% receive a part-time special education. Denmark and Norway are in the middle, but a striking contrast is Sweden, where only 1.5% of children are defined having SNE (see Table 8.1, NESSE, 2012).

We can also contrast the Nordic models against the response-to-instruction models of special education. In the RtI models the extremes of a continuum could be called as “a standard protocol” at one and “a problem-solving approach” at the other end (Fuchs, Fuchs, & Stecker, 2010). In the standard protocol, assessment means an evidence-based intervention with standardized measures of improvement before and after the intervention to be able to define those with MLD and needs for more intensified and individualized special educational intervention. The problem-solving approach sees the assessments as a tool for a non-categorical evaluation of skills mastered and yet to be mastered and is used primarily to inform classroom instruction, rather than to guide decision-making on a diagnosis or for a more individualized intervention. In other words, while the first stresses the importance of special education as a separate process, the latter sees that the special education should be blurred inside the regular instruction (for more about this discussion, see, e.g., Fuchs, Fuchs, & Compton, 2012).

If we try to put the Nordic models into this discussion and continuum, none of the countries follow the standards approach. The success of the RtI model in the USA has not attracted the policy-makers in Nordic countries to formalize the support systems or increase the usage of standardized tests. The general discussion has been more about how to develop inclusive models and lessen the needs for separate special needs education (e.g., Statped model in Norway). Finland is the only country where SNE has been formally structured to levels of support with defined procedures how the evaluation should be done when moving between the levels. This mimics vaguely the standards approach with pedagogical evaluations, but without specifications of assessment procedures. At the same time, there is an aim to push forward the inclusive problem-solving RTI model. Sweden has been an extreme on its reluctance toward assessments and diagnostics with a strong inclusive ideology and aims to apply the problem-solving approach.

One of the largest differences between the Nordic countries lies in the details how children's progress in learning is monitored. In Norway, Denmark, and Iceland, there are standardized assessments at specific grade levels, which are absent from the Swedish and Finnish systems. In Finland, the evaluation is totally in the hands of the teachers, who typically use a lot of formal and informal examinations to monitor the children's development in their own classroom. The specific feature in the Swedish discussion on education has been the reservations against assessments, especially standardized assessments and the evidence-based, "quantitative" approaches.

The second and third questions were: what kind of support do children with MLD get at school, and what are the qualifications for the support personnel? In all countries, the importance of inclusive education is stressed, but still, a common way of dealing with MLD is still taking the child out of a classroom to individualized or to a selected small group receiving special education. In none of the countries, there are officially recommended or recognized intervention programs to be used. In Finland, there are research centers on learning disorders, which have developed widely used programs on learning disorders. According to a recent analysis (Sabel, Saxenian, Miettinen, Kristensen, & Hautamäki, 2011), these research centers have had a large role in shaping the Finnish special education. In Norway, a state-funded Statped is developing models for special education. However, their aim is not to produce evidence-based intervention programs but to guide teachers in professional development (cf. problem-solving approach in RtI). In Denmark, the development work is concentrated around the large network of diploma-trained teachers.

In Sweden, there has been a lack of specialized teachers, and the university training of special educators started as late as in 2008, while in Finland it started in 1959, and nowadays the majority of the Finnish universities have units of special education offering studies up to the Ph.D. level. Therefore, it is not a surprise that from Nordic countries, what kind of, and who gives extra support to children with MLD, varies the most in Sweden. The Swedish educational office (Skolverket, 2009) has also raised concerns over the influence of increasing segregation in the Swedish school system after it transformed itself from one of the most centralized school systems into one of the most decentralized (Tomas, 2009). Even though the variance between schools in mathematics has increased in Sweden, the Nordic countries still have the smallest between schools variance in mathematics achievement in the world (Gaber, Cankar, Umek, & Tasner, 2012).

Our fourth question concerned the role of research and evidence-based approaches in interventions on MLD. Following the international trends, research interest toward MLD has been raising in all Nordic countries. There is a biannual Nordic Congress on special needs education in mathematics (NORSMA, The Nordic Research Network on Special Needs Education in Mathematics) where experiences on different types of assessment and interventions and on the effectiveness of special education are shared. However, none of the educational systems require that special educational approaches should be evidence-based. Therefore, research-based tools, even though welcomed at schools, are not a standard, and it depends on teacher's own activity, if they apply any of the models or instruments.

In all Nordic countries, an increasing number of researchers are pushing toward more research-based assessment and intervention procedures. The increasing understanding of the dyscalculic brain and changes in the diagnostic definitions encourage the researchers. At the same time, new questions emerge for the interplay between research and educational practice. The new competency-based curriculums redefine the learning aims and bring new colors to the practices at school and new challenges and research questions for studies on learning disabilities. It seems that the gap between everyday activities and aims in classrooms and the neuroscientific research is not getting narrower in the near future.

Our last question was about the future challenges. We can see a perennial battle between different views on the role of individualized special needs education and inclusive education. The puzzle how to teach the whole classroom effectively but at the same time individualize education within and outside of the classroom is an open question asking for scientific efforts. Neuroscientific research on learning and learning disorders gets the headlines (e.g., Coughlan, 2014) but still gives a little to the actual educational practices in classrooms. A lot of different views are presented, and the only thing where all parties agree is the lack of scientific evidence for any of the opinions.

According to the latest TIMMS study (Mullis et al., 2012), low motivation toward mathematics learning is more apparent and concerning feature of current Nordic students than low achievement. However, in the international assessments, there has been interesting feature: Within each participating country, there is a positive correlation between students' learning motivation and achievement; but when aggregating the data at a country level, the correlation between motivation and achievement becomes negative (He & Van de Vijver, 2016). High-performing countries show lower averages in motivation than lower-performing countries. From the Nordic countries particularly Finland, together with the many Asian top performing countries, they show this strong achievement paradox of high achievement and low motivation. Despite high general well-being of youth in Nordic countries, enjoyment of learning mathematics, especially in the upper primary education, has not been a part of it. The equation how to combine efficient learning, self-efficacy, and motivation in mathematics education is a big challenge for both research and practice to solve.

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Chapter 9
Mathematical Learning and Its Difficulties in the Middle European Countries

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The Big Picture

The acquisition of good mathematical skills is important for academic success throughout the life span (Duncan & Magnuson, 2009). Moreover, difficulties with math learning were found to affect people's ability to gain full-time employment and often restricted employment options to manual and often low-paying jobs (Dowker, 2005). In this chapter we aim to gain insight into the state of the art regarding math learning and its difficulties in the Middle European countries.

The PISA 2015 study of mathematics and literacy of 15-year-olds revealed that Singapore had the highest scores in mathematics, closely followed by Hong Kong, Taipei, Macau and Japan. The Netherland was in 11th place, Belgium in 15th place, Germany in 16th place and the UK in 27th place.

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Overall average scores may not fully tell us the extent of the problem of low numeracy in a country. The same PISA study indicates that 7.6% of the Singapore sample were classified as low achievers in mathematics (at or below PISA's Level 1), as were 16.7% in Belgium, 17.2% in Germany and 20.1% in both the UK and the Netherlands.

It is important to remember that this is a survey by a specific organization using specific measures and that other measures might lead to different results. In fact the other main international survey, TIMSS (2015), gives fairly similar results overall but puts England in a better position (10th) than would be expected by the UK's position in the PISA study. Over the years, England has generally done significantly better in TIMSS than PISA surveys (Sturman, 2015): something which has never been fully explained.

Within-country studies have shown that low numeracy is a common problem in the UK (Butterworth, Sashank, & Laurillard, 2011). Cohort studies of people born in particular weeks in 1958 and 1970 have indicated that about 22% of the population in the UK have severe numeracy difficulties that have a serious impact on their occupational and social chances (Bynner & Parsons, 1997; Parsons & Bynner, 2005), whereas only about 5% have similar levels of difficulty in literacy. Less severe, but still significant, mathematical difficulties are still more common. A recent survey indicates that nearly half of working-age adults in the UK are on the level of numeracy that one would expect at the end of primary school (BIS, 2011). Although 17.2% of the 15-year-old students in Germany did not reach the defined minimum Level 2 in the PISA estimates of mathematical competencies, public awareness of mathematical difficulties as compared to written language difficulties is comparatively small.

A number of children have severe and persistent difficulties with mathematics which are resistant to instruction. In this case they are labelled as individuals with mathematical learning disabilities (MLD). The prevalence of MLD in the Middle European countries has been estimated as approximately 6% (Barbaresi, Katusic, Colligan, Weaver, & Jacobsen, 2005; Desoete & Baten, 2017; Desoete, Roeyers, & De Clercq, 2004; Dowker, 2016) depending on the criteria used to define MLD. The prevalence in siblings of children with MLD is even higher, ranging from 40% to 64% (Desoete, Praet, Titeca, & Ceulemans, 2013). However, the prevalence of MLD varies by age with a maximum in about Grade 3 students (Hasselhorn & Schuchardt, 2006). A recent epidemiological study on the prevalence of MLD in Germany controlling for learning disabilities in reading and spelling reported a percentage of 9.2% of children with MLD at the beginning of Grade 3 with about half of these children even fulfilling the criteria of the WHO definition of dyscalculia, i.e. being identified to have a discrepancy between their poor mathematical skills and their intact general intelligence of at least 1.2 SD (Fischbach et al., 2013). The discrepancy criterion is however not used in all Middle European countries. In Belgium, for example, a child can have below average intelligence and MLD if the low IQ does not explain all of his or her behaviour. In this case this is defined as comorbidity.

Educational Policies on MLD

Educational policies on MLD differ between countries in several regards. We give three examples.

In the UK, children with specific learning difficulties may be given additional support within the school system. The situation is currently in a state of flux, with some significant changes taking place in response to the Children and Families Act 2014; and also there are some differences between the systems in operation in the different countries within the UK. In the system currently being implemented in England, children assessed as having special educational needs may be given within-school support, which may include physical adaptations, extra individual attention often from a teaching assistant or participation in any appropriate intervention program available. A minority with more complex needs may be given a wider-ranging education, health and care plan, which may include recommendations for medical and social support outside as well as within school. Various professionals may be involved in special needs assessments, but specific learning difficulties are most commonly diagnosed by educational psychologists. In addition, it is rare for children to have a main diagnosis of mathematical learning difficulties/dyscalculia in the UK. Dyscalculia was only formally recognized as a specific learning difficulty in the UK in 2001 and is still far less often diagnosed than dyslexia or ADHD, even though the actual incidence may be similar. It is, however, recognized that many pupils with dyslexia have comorbid mathematical difficulties: about 40% of dyslexic people also meet criteria for dyscalculia (Wilson et al., 2015), and even those who do not, may have difficulties with specific areas of arithmetic such as memorizing multiplication tables. Thus, mathematical difficulties may sometimes be diagnosed and addressed in connection with a main diagnosis of another specific learning disability, especially dyslexia. Moreover, not all children, who may benefit from educational interventions, are appropriately diagnosable with specific learning difficulties. The problem of low attainment in arithmetic is very common, and there are many factors that may contribute to it. Poverty and social disadvantage are important contributory factors (Bynner & Parsons, 1997; Parsons & Bynner, 2005). Currently, the government provides a 'pupil premium', an additional funding for publicly funded schools in England, according to their number of pupils from disadvantaged backgrounds (usually defined as those eligible for free school meals) with the purpose of raising their attainment and closing the gap between these children and those from more advantaged backgrounds. The money is sometimes used to fund intervention projects in both literacy and numeracy.

In Germany, the 16 federal states are autonomous with regard to most issues surrounding education and school. However, there is a standing conference of the ministers of education from the 16 federal states (called KMK), who decided to fix a lot of organizational details, contents and strategic issues of schooling in order to achieve a high level of comparability of schooling among the German Federal

States. With regard to most issues surrounding MLD, there are not much common accords. In 4 of the 16 German Federal States, the school law does not even mention the existence of MLD. In most of those federal states having an explicit decree on MLD, almost all issues surrounding the diagnosis are determined. Some of them in addition have rules for the compensation of resulting disadvantages and the provision of individual aids. With regard to the diagnosis in most cases, the individual teacher or the head of the school has to initiate the testing of the children concerned. However, only in one federal state there is an explicit diagnostic service as part of the local education authority. As a consequence, in many regions in Germany, only very few concerned children are really diagnosed by MLD experts. In states in which rules exist for the compensation of resulting disadvantages, mainly the school grades in math are paused for children with diagnosed MLD during the elementary school years to protect them from becoming a repeater. However, the individual aids offered to those children to overcome their MLD are mostly restricted to the expertise of the classroom teachers.

Most recently, a further training program called MARKO-T for poor math learners was developed (Gerlach, Fritz, & Leutner, 2013) and successfully evaluated (e.g. Ehlert & Fritz, 2013). Despite some differences in detail, the theoretical ideas behind the program are similar to those of the MZZ program described above.

In Belgium there are different approaches to education in the Dutch, French and German community with Brussels depending on the French and Dutch education structures. The power for education lies with the communities. Within the Flemish government, the Minister of Education is responsible for almost all aspects of educational policy, from nursery to university education. Nevertheless, the federal authorities are competent for some educational issues, namely, the start and end of compulsory education, establishing minimum conditions for obtaining a diploma and determining education staff pensions, and schools are entirely free in choosing teaching/education methods, curriculums and time tables. The complexity of the Belgian political structure does not facilitate the educational policy on MLD. In Flanders there is the M-decree, making schools less segregated and more inclusive for children with disabilities (such as MLD). Due to the constitutional freedom of education, schools are entirely free in choosing teaching/education methods (mathematics), curriculums and time tables and education views. However, if they want government recognition or funding, they must meet the attainment targets or developmental objectives in the curriculum, teaching materials must be available and the school buildings have to comply with safety provisions and hygiene standards. The support of all pupils (and also of pupils with MLD) is organized by the pupil guidance centre (or CLB), a service financed and evaluated by the government. The CLB is responsible for the diagnosis of MLD (based on their PRODIA protocol) and for the support within a continuum of care. Children remain in regular education. Only if their needs are out of proportion, they can go to special schools for a short period. Before September 2015, there was segregated education for children with MLD with special education type 8 especially aiming on children with learning disabilities. In Flanders the support of teachers of children with (and without) MLD is organized by the pedagogical support centre (PBD), giving educational and methodological advise. The educational inspectorate of the Flemish Ministry of Education and Training acts as external supervisor by assessing the implementation of the 'attainment targets' and 'developmental objectives'.

Theories and Educational Practice

The knowledge about learning disabilities and the need to understand and empower individuals with MLD and their environment increase in most countries. The opportunity propensity model (e.g. Byrnes & Wasik, 2009) states that children are more likely to attain high levels of math achievement if they are given genuine opportunities to enhance their skills (opportunity condition) and if they are willing and able to benefit from these opportunities (propensity condition). This model claims that some low performers are presented with fewer opportunities, while other low performers are presented with opportunities but unable to benefit from them (e.g. due to lack of preparation), and a last group of low performers are presented with opportunities but unwilling to engage fully and benefit from them. Especially the last group is a big challenge when trying to develop interventions and guide practice.

There were relatively few numeracy interventions available in the UK until recently. Individualized programs used in Britain by the early twenty-first century included the Mathematics Recovery program first developed in Australia by Bob Wright and his colleagues (Wright, Martland, & Stafford, 2006); some computerized interventions, such as RM Maths (Earl, 2003); and Dowker's (2001) Numeracy Recovery program, since expanded and developed as Catch Up Numeracy (Dowker & Sigley, 2010; Holmes & Dowker, 2013).

The British government, in the first decade of the twenty-first century, became more active in developing interventions for children with numeracy difficulties (Dowker, 2004; Gross, 2007). They developed the Springboard program for small groups and 'Wave 3' materials for individual use.

Focus on interventions increased further, with the Williams Review of primary mathematics education (2008). This review recommended early intervention for primary school children with difficulties in mathematics. Children with serious difficulties in mathematics should receive intensive one-to-one intervention from a qualified teacher, though paired or small-group work may be appropriate in some cases. Children with somewhat less severe difficulties might receive less intensive individualized or small-group intervention, and teaching assistants could provide some of this, with appropriate training. Williams (2008) proposed that mathematics interventions should be given in the early years of primary school, preferably in Grade 2 (6–7 years). This proposal resulted in schools and local authorities being given some funding to set up such interventions. A variety of such interventions were set up, but most came into one of three categories: (1) those that involved the use of the existing Wave 3 materials, usually with some modifications; (2) those that were based primarily on detailed diagnostic assessments of individual strengths and weaknesses, with activities targeted to these; and (3) those that primarily involved the use of multisensory apparatus such as Numicon. A review (Dowker, 2009) indicated that the interventions were viewed positively by teachers and local authorities. They were described as improving performance and attitudes in children, and it was often stated that teachers and pupils enjoyed them. It was sometimes reported that the schools improved their overall performance in national curriculum tests. Only rarely, however, was it possible to carry out systematically controlled studies on their effectiveness. In order to set up an intensive intervention for children with serious numeracy difficulties and to test its effectiveness, Every Child Counts was set up as a partnership initiative between the Every Child a Chance charity (a coalition of business partners and charitable trusts) and government. The aim was to enable the lowest-attaining children to make greater progress towards expected levels of attainment in mathematics, catching up with their peers and performing at least at average levels on school assessment tests, wherever possible, by the end of the second year of primary school. The original intention was to provide intensive support in mathematics to 30,000 Grade 2 children annually, though this has been significantly reduced due to the financial crisis of 2008 and subsequent government spending cuts.

Dunn, Matthews, and Dowrick (2010) developed the Numbers Count program, which draws on aspects of three existing interventions: Multi-sensory Mathematics (developed in Leeds using Numicon materials), Numeracy Recovery (developed in Hackney) and Mathematics Recovery (Wright et al., 2006). This program involved careful assessment of individual children's strengths and weaknesses, followed by intervention targeted to addressing specific weaknesses, and emphasizes the development of number concepts through multisensory teaching. It included a wide variety of components of arithmetic but places particular emphasis on methods of counting and number representation. Children received half an hour of individualized or sometimes very small-group (two or three children to a teacher) intervention per day. It was delivered by teachers who have received masters-level training. In the initial stages of the project, 2621 Grade 2 children, across 27 English local authorities, took part in Numbers Count. They received an average of 40 half-hour individualized Numbers Count lessons in a term, delivered by teachers who had received masters-level training. The participating children were given the Sandwell test, a standardized arithmetic test, before and after entering the program, and were retested 3 months and 6 months later.

Torgerson et al. (2011) carried out an independent evaluation of the program. About 12 children of 44 schools were randomly allocated to either an intervention group or a waiting-list control group. Children in the intervention group received an average of 40 half-hour individualized Numbers Count lessons in a term, delivered by teachers who had received masters-level training. The participating children were given the Sandwell test, a standardized arithmetic test as a pretest, and were posttested on the Sandwell test after 3 months and 6 months and also the Progress in Maths 6 (PIM 6) test after 3 months. Findings showed that the intervention group performed significantly better than the controls on the PIM 6 test (effect size 0.33).

The changes in the Sandwell scores were greater. Before entering the program, the children's Number Age was on average 11 months below their chronological age. On average, they gained 14 months in Number Age in one term, a ratio gain of over 4 (months gained in mathematical age divided by mean duration of intervention in months), and were scoring at chronological age level by the time they exited the program. However, it must be noted that while the PiM scores were marked by people blind to the children's group assignment, the Sandwell scores were not, so that there could have been unconscious bias with regard to the latter. As always, the question arises of whether the gains will be maintained over the long term. A long-term evaluation is currently being carried out to investigate whether the effects of the intervention persists to the end of primary school and into secondary school.

However, interventions of this level of intensiveness are unlikely to be possible from a practical or economic point of view for the majority of children who experience milder mathematical difficulties and, even if they could be implemented, might take up too much time from such children's other educational activities. Yet, there are many children, who, though they are not among the lowest attainers, still struggle with mathematics and are at risk of persistent numeracy difficulties in adulthood. Lighter-touch interventions are needed for such children. Thus, a somewhat less intensive intervention developed by Edge Hill University in association with Numbers Count is first Class @ Number. In this program, a specially trained teaching assistant delivered 30 half-hour sessions to a group of up to 4 children, for 12–15 weeks. The lessons focussed on number and calculation, developing children's mathematical understanding, communication and reasoning skills. It was used for both primary and secondary pupils. So far it has been used with over 45,000 children. Evaluations so far by Edge Hill University have shown an average Number Age gain of over three times the expected amount of progress. It is now receiving an independent evaluation organized by the Education Endowment Foundation. Another less intensive intervention, in terms of time spent, is Catch UpTM Numeracy (Dowker & Sigley, 2010; Holmes & Dowker, 2013). The main target pupils for the Catch UpTM Numeracy intervention were pupils in Years 2–6 who have numeracy difficulties. Up till now, over 45,000 children in England and Wales took part in this intervention since its development in 2007. The intervention begins by assessing the children on ten components of early numeracy, ranging from counting to word problem-solving to estimation to memory for arithmetical facts. Each child is assessed individually by a trained teacher/teaching assistant using 'Catch UpTM Numeracy formative assessments' which the member of staff then uses to complete the 'Catch Up Numeracy learner profile'. This personalized profile is used to determine the entry level for each of the ten Catch Up Numeracy components and the appropriate focus for numeracy teaching (based on the profile and the individual learner's needs). Children are provided with mathematical games and activities targeted to their specific levels in specific activities. Teachers and teaching assistants receive 3 days of formal training from the Catch Up TM organization in delivering the program. The children receive weekly intervention (two 15-min sessions per week) for approximately one school term, focusing on components with which they have difficulty. Each 15-min teaching session includes (a) a review and introduction to remind the child of what was achieved in the previous session and to outline the focus of the current session, (b) a numeracy activity and (c) linked recording activity where the child records the results of the activity, in oral, written and/or concrete fashion, and where the child receives focused teaching related to his or her performance in the activity and any observed errors. Evaluations so far indicate that Catch Up Numeracy results in children making over twice the improvement in Number Age on the Number Screening Test (Gillham & Hesse, 2001) than would be expected by the passage of time alone and leads to significantly more improvement than a business-as-usual control group (Dowker, 2016; Holmes & Dowker, 2013). It appears also to give more improvement than a matched-time control group (Holmes & Dowker, 2013), but a randomized controlled trial study of this is still in progress.

There are many other interventions in use. While so far, we have focussed on individualized or very small-group interventions, schools are found to use a variety of interventions of varying degrees of intensiveness. Interventions in literacy and numeracy have sometimes been classified into three categories of varying degrees of intensiveness, described as 'waves' in the UK and 'tiers' in the USA. Wave 1 involves whole-class teaching designed to be suitable for children of a variety of attainment levels; Wave 2 involves interventions in small groups with children who are experiencing mild or moderate difficulties in the subject, while Wave 3 involves more intensive, usually individualized interventions for children with more significant problems. Wave 1 or 2 interventions may involve allowing for independent individualized or small-group work within a class. Individualized work within a class usually involves progressing through a textbook at one's own pace, the use of individualized worksheets and/or the individualized use of educational computer software. Small-group approaches may take a similar form or may involve group projects where several pupils work together on the solution or solutions to a problem. On the whole, whole-class approaches have not been regarded as interventions, but in the context of curriculum development. However, as the UK has become increasingly concerned with improving numeracy standards, there is an increasing interest in investigating the possible role of certain new whole-class programs in reducing the incidence of numeracy difficulties. One program which is attracting current interest from this point of view is Mathematics Mastery, a program inspired by some aspects of Singapore mathematics education. Compared to traditional curricula, fewer topics are covered in more depth, and greater emphasis is placed on problem-solving and on encouraging mathematical thinking. A current evaluation by the Education Endowment Fund has so far indicated that the use of the program in Grade 1 results in an average increased gain in mathematics age of 2 months in the first year and the use of the program in Grade 7 results in an average increased gain in mathematics age of 1 month. Further investigation is desirable to see whether these gains are maintained or extended over time.

In Germany there is a widespread scepticism regarding standardized training programs as tools to aid children with MLD accompanied with the dominant belief that the best location for individual math acquisition support is school. Most math teachers agree that adequate interventions for concerned children should follow a guideline fitting at least the following three principles: First, the training activities should focus exclusively on mathematical contents, since the practice and enhancement of domain-general learning skills would not help to overcome the deficits associated with MLD; second, it is necessary to build up the mathematical competencies systematically by starting with basic numerical skills including skills usually acquired by children before school entry; and third, the systematic usage of manipulatives (as a kind of didactic tools) is held to be crucial to help children to understand the relevant mathematical concepts as well as the symbols that are used to represent these concepts. Among the most common manipulatives in German schools are the counting frame (abacus), multisystem blocks (DIENES material) and the number ray. With regard to the usage of the child's ten fingers to count, however, there are controversial debates whether this is a helpful mean of number representation or not (see Moeller, Martignon, Wessolowski, & Nuerk, 2011). Whereas experts in mathematics education recommend fostering mentally based numerical representation and to induce children at least from the second year of math instruction to abandon finger counting, neurocognitive researchers argue that elaborate finger-based numerical representations are beneficial for later numerical development at least for children with problems in the acquisition of arithmetic skills.

Despite the reservations of many teachers with regard to standardized numeracy interventions during the last couple of years, a number of those programs were developed and some of them successfully evaluated. Especially, there are some promising evaluation results of programs focussing at rather young children before formal schooling to prevent them to develop MLD after school entry. The currently most disseminated program in Germany is the Würzburg training program ('quantities, counting, numbers') aiming to promote early awareness of quantities, numbers and relations between numbers (*MZZ*; Krajewski, Nieding, & Schneider, 2007). It systematically builds up conceptual knowledge from practising basic skills to instruction in numerical structures. It draws on the means of representation that incorporate basic mathematical ideas particularly well, offering a clear structure of the numerical space and allowing for effective problem-solving strategies. MZZ tries to teach children the meaning of numbers in a playful manner. The abstract structure of numbers and of the numerical space is thus rendered 'tangible' and 'visible'. During the playful practice, children use materials that represent and clarify the structure of numbers. The children hence do not need to mentally deduct the abstract meaning of numbers themselves, but they gain fundamental insights into basic mathematical knowledge by dealing with the objects as the means of representation, which they can grasp and compare.

The program comprises three focal areas of promotion. The first area focusses on training and linking basic numerical skills of the children (quantities, counting, numbers). Within due course, the children are expected to manage counting and numbers up to ten and develop an awareness of the quantities underlying numbers (awareness of linkage of quantities to number-words). The second priority targets the understanding of numbers in terms of their rising sequence (precise quantity number-words linkage). At this stage, the children are supposed to learn that numbers can be ordered sequentially according to their power and compared. The third priority teaches the children that numbers are related in various ways. They shall thus learn that numbers can be decomposed and recomposed to build different numbers and that the difference between two numbers is a number itself (linking quantity relations with number-words). These insights are promoted by the support of visual representations and adequate verbal descriptions. The children are thus guided to grasp numerical rules not only in a visual sense (e.g. greater numbers take up more space) but to become aware of such rules linguistically and reason verbally (e.g. '4 is larger than 3, because more things belong to 4 than to 3'). A number of evaluation studies did provide evidence for the training's effectivity with preschool children (for an overview, see Krajewski & Simanowski, 2016). The program also was successfully applied to foster numerical skills at school starting in children at risk for mathematical achievement problems (Hasselhorn & Linke-Hasselhorn, 2013) and among first graders with poor basic numerical skills (Ennemoser, Sinner, & Krajewski, 2015).

In Belgium there is also the belief that the best location for individual math acquisition support is school with training activities focusing on mathematical contents, focusing on prenumerical and numerical skills and using manipulatives to represent concepts. Among the most common manipulatives in Flanders are the Multi-Arithmetic Blocks (MAB material) and number ray.

Working memory (De Weerdt, Desoete, & Roeyers, 2013), small number discrimination (Ceulemans et al., 2014), familiarity with math language (Praet, Titeca, Ceulemans, & Desoete, 2013), symbolic numerical processing (Vanbinst, Ghesquière, & De Smedt, 2015), seriation, classification (Stock, Desoete, & Roeyers, 2010) and intelligence (Dix & van der Meer, 2015) are examples of the propensity factors that have been frequently related to later math achievement.

Although all these elements can help explaining a great amount of the variance in mathematical performance, a lot of the variability keeps unexplained, making the development of guidelines for practice difficult. In addition the most compelling finding within the O-P model is why some children getting enough opportunities are unwilling to engage fully and benefit from them. Therefore it might be important to not only focus on cognitive propensity factors but also to investigate the role of metacognition (Baten, Praet, & Desoete, 2017) and noncognitive propensities and of the match with opportunity constructs (Ceulemans et al., 2017). In a recent study, the motivational/interactional profiles of 54 students with low math performances and a high risk for dropout and 40 teachers of a school for vocational secondary education in Flanders were examined. The dataset revealed that the communication, motivation and transactional profiles of the teachers and students differed. Of the students 58% were playful-resister, whereas 47% were enthusiastic and empathic. Of the teachers 50% were empathic and 35% were responsible workaholic. The most compelling finding was that there was a match of motivational needs especially for empathic individuals (47% of the students and 50% of the teachers). However, 30% of the students were motivated by action and excitement, whereas none of their teachers were motivated by such an interaction. Moreover 55% of the teachers were motivated by structure, planning and recognition of their work, whereas 49% students were not motivated by this behaviour. In addition the need for accuracy and precision was very small in the 51% of the students, whereas 23% of the teachers had this as one of their most important motivational needs (Baten & Desoete, 2016; Baten, Desoete, Van de Velde, & Hantson, 2016). Thus, there might be a mismatch of motivational needs leading to miscommunication between teachers and students due to a difference in motivational and transactional preferences, especially with children motivated by playfulness (learning has to be fun) or action (learning has to be exciting and challenging), resulting in reduced engagement and high dropouts. If we want more children to engage and be motivated, we should also provide humour and action opportunities in our math lessons.

In addition in Flanders (given the option of its government for inclusive education), more and more teachers are confronted with and have the knowledge about MLD. To apply the M-decree, more knowledge is needed to coach teachers in their capacity to identify, promote, monitor, pilot, analyse and mainstream dealing with diversity such as children and adolescents with MLD ('cooperative collaboration'). On several places trainings took place to build capacities such as the creating 'universal designs for learning (UDL)' (e.g. Meirsschaut, Monsecour, & Wilssens, 2015) in teachers and 'action-based' assessment and working (HGD, HGW) in school guidance centres. There is also an active organization of parents with children with learning disabilities organizing workshops and paper sessions to inform other parents. In school guidance centres, the action-based approach and the model of continuous care are used. However in practice there is a huge difference between schools and their knowledge and preparedness to deal with individuals with MLD.

What Is the Role of Research Guiding the Practice?

There is an increasing influence of research on educational practice.

In England, reviews of research have been used by governments to influence practice with regard to educational interventions, including those in numeracy (Dowker, 2004, 2009; especially Williams, 2008). There is an increasing interest in gathering evidence on the effectiveness of interventions and other educational policies. The Education Endowment Foundation was set up in 2011, with the aim of gathering evidence as to what works effectively in reducing low attainment, especially in disadvantaged groups, and in particular funding trials of promising but untested programs and approaches and supporting schools and other educational institutions in using the evidence in practice. The Institute of Effective Education in York conducts evaluations of the effectiveness education programs and practices throughout the UK and internationally. It publishes 'Best Evidence in Brief', a free fortnightly online newsletter of education research news, and 'Better: Evidence-based Education', a magazine for teachers about findings with regard to effective practice, and Evidence4Impact, an online database of education programs with nearly 15,000 subscribers worldwide.

In Belgium speech therapists, occupational therapists, special educators and psychologists are trained to assess and treat individuals with MLD. In their training materials, the discrepancy criterion no longer is present as criterion for MLD. This criterion is not included in the PRODIA protocol of school guidance centres. There is a direction to a more evidence-based approach of all professionals assessing and helping individuals with MLD and their environment.

Overall there are more studies on reading disabilities compared to the studies on math disabilities. Nevertheless the main question is how can we define, predict, understand and help individuals with MLD?

Although the criteria for MLD seem clear, there are some disagreements on f.ex, the criteria used to define the ‘substantially below’ performances (Stock et al., 2010). In addition, there is some disagreement as to whether MLD represents a specific and definable impairment or the lower end of the continuum of arithmetical ability. Mazzocco, Devlin, and McKenney (2008) found that children with MLD (and a severe form of disability) showed qualitatively different profiles in fact retrieval performances when compared to typically achieving children, whereas the differences between children at the lower end of the continuum (low achievers, LA, with a mild form of disability) and typically achieving children were of a quantitative turn. Geary, Hoard, Byrd-Craven, Nugent, and Numtee (2007) revealed that children with MLD (a severe disability) had a severe math cognition deficit and underlying deficit in working memory and speed of processing. The LA groups (with a mild disability) had more subtle deficits in few math domains. Finally, although the criterion of nonresponsiveness to intervention (Fuchs et al., 2007; Fuchs, Fuchs, & Prentice, 2004) is an interesting one, some studies suggest that even quite significant arithmetical difficulties are often responsive to interventions targeted at their specific strengths and weaknesses (Dowker, 2016; Dowker & Sigley, 2010).

Another question is whether mathematics should be considered as one component or not. Some authors propose at least a procedural and a semantic memory subtype within MLD (e.g. Geary, 2004; Pieters, Roeyers, Rosseel, Van Waelvelde, & Desoete, 2015). The procedural subtype would be due to executive dysfunction and characterized by a developmental delay in the acquisition of counting and counting procedures used to solve simple arithmetic problems. The semantic memory subtype would be due to verbal memory dysfunction and characterized by errors in the retrieval of arithmetic facts (Wilson, Revkin, Cohen, Cohen, & Dehaene, 2006). However, not all studies have found different profiles for these groups (Landerl, Bevan, & Butterworth, 2004; Rousselle & Noël, 2007).

Nevertheless, the research finding that mathematical difficulties are not unitary is based on studies of both atypical and typical mathematical development (Cowan et al., 2011; Desoete et al., 2004; Dowker, 2005; Gifford & Rockliffe, 2012; Jordan, Mulhern, & Wylie, 2009; Russell & Ginsburg, 1984). Studies of adults with acquired dyscalculia (Cappelletti, Butterworth, & Kopelman, 2012; Delazer, 2003; Warrington, 1982) and behavioural and brain imaging studies of adult mathematical cognition (Stanescu-Cosson et al., 2000; Van Eimeren et al., 2010) have provided increasing and converging evidence that arithmetical cognition is made up of multiple components and that it is quite possible for children and adults to show strong discrepancies, in either direction, between the components. In other words, although the components often correlate with one another, people can show weaknesses in virtually any component. There is still room for debate, however, as to whether these varied difficulties have similar or distinct causes. In this context, there is a lot of debate about the extent to which mathematical difficulties are mainly due to domain-specific causes (e.g. difficulties in subitizing; difficulties in magnitude comparison), to domain-general causes (e.g. verbal difficulties; spatial difficulties; working memory difficulties) or to an interaction between the two. Perhaps the answer may be different for different types or severity levels of mathematical difficulties.

In addition there is still some disagreement about hypotheses to explain research data, leading to confusing suggestions for clinical practice. We give some examples. According to the triple-code model (Dehaene & Cohen, 1995; Noel, 2001; Schmithorst & Brown, 2004), there are three types of representations for numbers. Two of them are symbolic and format-dependent, a visual Arabic number form (e.g. '5') and a verbal word frame with number-words (e.g. 'five'), and one is nonsymbolic and format-independent: the analogue magnitude representation (e.g. five dots). There is an agreement on the importance of these representations (Desoete, Ceulemans, De Weerdt, & Pieters, 2012) and on the problems due to imprecise representations in MLD (e.g. Mussolin, Mejias, & Noël, 2010; Piazza et al., 2010). However there is no agreement on how to explain this finding. Butterworth and his collaborators (Landerl et al., 2004) explained this with their *defective number module* hypothesis, assuming that MLD occur when the basic ability to process numerosity fails to develop normally, resulting in difficulties to understand number concepts and, consequently, in learning numerical information. However Rousselle and Noël (2007) evaluated an alternative explanation with the *access deficit hypothesis* stating that there was no deficit in number sense per se, since when investigating numerosity processing with no symbolic processing requirement, MLD children in second grade were only impaired when comparing Arabic numerals (i.e. symbolic number magnitude) but not when comparing collections of sticks (i.e. nonsymbolic number magnitude). The Walloon authors suggested that children with MLD had difficulty in accessing number magnitude from symbols rather than in processing numerosity per se.

These debates remain important question areas of the basic research on MLD at the moment. The future of MLD education in Western European countries remains unclear. In most countries there is a tendency towards inclusive education empowering children with and without MLD and looking for good practices where everyone benefits (as the Universal Design for Learning, UDL suggests).

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Chapter 10
Mathematical Learning and Its Difficulties in Eastern European Countries

Csaba Csíkos, Szilárd András, Attila Rausch, and Anna Shvarts

Eastern European Mathematics Education as Defined by Geographical, Historical, and Political Factors

In this book, the term "Eastern Europe" is used in accordance with how other chapters in the section titled "Lessons from International System-Level Surveys" have considered their territory and field of interest. Thus, the group of countries to which we refer in this chapter is defined not strictly geographically, but we have taken them as a group of countries that previously belonged to the immediate sphere of interest of the former Soviet Union. According to a current multilingual thesaurus (Eurovoc), published by the Publications Office of the European Union, Eastern Europe consists of 21 states. Other descriptions available in the geographical or political literature may add the Baltic states (Estonia, Latvia, and Lithuania) or even Finland to this group of countries. Furthermore, the Visegrad Group (the Czech Republic, Hungary, Poland, and Slovakia) and Slovenia are often labeled as Central

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European countries, together with Austria, Germany, and Switzerland. Having acknowledged that any such categorizations may be offensive to some or unusual to others, in this chapter we refer to Eastern Europe as a group of 24 countries: those 21 listed in the footnote and the three Baltic states.[1]

Constraints and Promises of Recent Decades in Eastern European Mathematics Education

The Eastern European countries belonged for some decades to the immediate political, economic, and/or military sphere of the former Soviet Union. In this block of countries the leading role of one (or more, but not many) Marxist political parties defined several aspects of the school system. Centralized curricula and textbooks aimed to provide the same pathway for all children. Equality has always been a central issue in the Central and Eastern European socialist states (Bankov, Mikova, & Smith, 2006); however, the Programme for International Student Assessment (PISA) studies revealed that this has not been accomplished.

Adler (1980) praised the intensive study of educational psychology in the Soviet Union and the links between school practice and the newly emerged psychological findings on how children learn. The influence of Soviet educational psychology had its effect in the region, according to Szalontai (2000). Even nowadays, the classical seminal works by Talysina, Stolyar, Davidov, Vygotsky, Leontiev, and others play important roles in Russian math educators' training. According to Goldin (2003), any kinds of ideologically set mathematics education necessarily dismiss the integrity of mathematical knowledge. Nonetheless, Eastern European mathematics and science education were seen with a kind of fear from the other side of the Atlantic Ocean from the time of the Sputnik shock (1957) until the very end of the Cold War. Stefanich and Dedrick (1985) emphasized that in Eastern Europe, 42% of Bachelor of Arts (BA) degrees were awarded in the field of engineering (while only 6% were in the USA). As Valero et al. (2015, p. 290) state, "The narrative that connects progress, economic superiority, and development to citizen's mathematical competence is made intelligible in the 20th century". Emphasizing the importance of mathematics education in the Western world was a reaction in order to maintain the supremacy of the capitalist Western world.

Was there a special kind of mathematics education that might be labeled as socialist mathematics education? In his book, Swetz (see Howson, 1980) compared seven rather different countries (all labeled as socialist countries, including Tanzania). The country profiles were provided by excellent scholars; however, some of them did not live or work in the countries they were writing about. In spite of his critical book review, Howson agrees that "mathematics education in any country

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[1] Albania, Armenia, Azerbaijan, Belarus, Bosnia and Herzegovina, Bulgaria, Croatia, Czech Republic, Former Yugoslav Republic of Macedonia, Georgia, Hungary, Kosovo, Moldova, Montenegro, Poland, Romania, Russia, Serbia, Slovakia, Slovenia, Ukraine.

cannot be divorced from politics" (p. 285). In these seven "socialist" countries (among which the Soviet Union, East Germany, Yugoslavia, and Hungary were included, as part of Eastern Europe) there were three strong common features: (1) a central curriculum (not only in mathematics); (2) textbook word problems stressing industrial and societal phenomena; and (3) a strong emphasis on talent development and competitions. This third feature in itself may be the reason why the "mathematics for all" movement has not spread widely in Eastern Europe (Karp & Furinghetti, 2016). Eastern European mathematics education could rather well fulfill the role of "the gate keeper," i.e., dividing the society into two parts: those who are able to do mathematics and those who cannot (Skovsmose, 1998).

In case the reader finds a country description written by one or more (no matter how excellent) scholars incidental or nonrepresentative, a little larger sample was involved in Hawighorst's (2005) investigation; 15 parents from three different cultures were interviewed (five of them were German parents resettling from the former Soviet Union). The study focuses on some significant differences the "resettlers" expressed in their view. They seem to be particularly critical of school mathematics lessons. They expect a high level of content knowledge in mathematics; therefore, they plan to send their children to the most elitist type of secondary school—the Gymnasium.

If a kind of socialist mathematics education did exist in the bloc of Eastern European countries, it was certainly not uniform. Karp and Furinghetti (2016) analyzed how consecutive eras in the Soviet Union differed from each other with respect to aims and methods. Mathematics as a school subject was once considered a robust system to train the new bureaucracy of a new political era, while later some parts of the mathematics curricula were considered useless. What is more, the methods of instruction and evaluation changed according to what the leaders thought about the development of the so-called collective spirit.

Stoilescu (2014) summarized what many Eastern European citizens felt under the socialist regimes: people's capability to make decisions and take responsibility were weakened by encouraging centralized, uniform thought through uniform and ideologically infiltrated textbooks. The freedom of research was constrained by ideological factors. A direct (and negative) intervention from governmental officers is described by Varga (1988): his teaching experiment, which was interrupted by withdrawal of authorization from the ministry of education, was able to restart later but only with modified experimental factors.

The countries in Eastern Europe have further common features beyond those originating in the decades of the Soviet sphere of interest. Eastern European states had long historical relations with the European centers (Bertomeu-Sánchez, García-Belmar, Lundgren, & Patiniotis, 2006) with respect to the circulation of scientific knowledge. From the sixteenth century it was quite common that encyclopedias written in one of the European languages were translated and used as textbooks in both central and peripheral countries in Europe. The word "textbook" itself came from English in the eighteenth century and had the meaning of a collection of texts that might be used for educative or reference purposes. However, encyclopedias were used as textbooks long before that term was coined. An example is the *Encyclopaedia* written by the Hungarian Apáczai in 1655 (Palló, 2006), which was unique not only at that time but indeed also until the end of the nineteenth century, since textbooks translated from German or French were mainly in use in Eastern Europe.

Historically, educational thoughts in the heart of Europe were formed in such a way that this tradition is now called "Didaktik," as opposed to the Anglo-Saxon curriculum tradition (Westbury, 2000). In a Berlin–Hong Kong comparative study, Lui and Leung (2013) described several aspects of mathematics teaching common in Berlin and in Hong Kong, where the Confucianist tradition is standard. Both traditions emphasize exercises and practice (for more than one third of the time allocated to math lessons).

Even nowadays, Eastern European countries have some common characteristics that might be considered as part of their historical and cultural backgrounds. The PISA study (Mikk, Krips, Säälik, & Kalk, 2016) revealed that students' scores in mathematics and science were, unusually, correlated with their judgment of their relations with their teachers. The five items in the background questionnaire asked them to score, on a five-point Likert scale, how much they trusted their teachers, to what extent their teachers treated them fairly, etc. From Eastern Europe, the Czech Republic, Hungary, Poland, Romania, Russia, and Slovakia were considered, and the −0.63 correlation (albeit not significant in such a small country sample) was at least alarming. No similar tendency was revealed in other country groups in the world.

Lessons from International System-Level Surveys

International educational surveys based on nationwide representative samples started in the twentieth century. The International Association for the Evaluation of Educational Achievement (IEA) focused on mathematics from the outset. The First and Second International Mathematical Studies (FIMS and SIMS) involved far fewer countries than the later TIMSS series. (Note: initially the acronym "TIMSS" was defined as "Third International Mathematics and Science Study" but later the definition was changed to "Trends in International Mathematics and Science Study".) From Eastern Europe, only Hungary took part in SIMS, while Bulgaria, the Czech Republic, Hungary, Latvia, Lithuania, Romania, Russia, Slovakia, and Slovenia participated in TIMSS in 1995. Since 2003, TIMSS has been conducted every 4 years, making it possible to outline developmental trends in countries' profiles. With the advent of the PISA studies, much more attention has been paid to each country's overall educational achievement, especially in comparison with the overall state of the country's economy, health, and culture. Each year the United Nations publishes Human Development Index (HDI) scores, which are composite scores indicating each country's general state of development. From the Eastern European region, Slovenia, Estonia, Slovakia, and Poland had the four highest scores in 2015. There is a fairly strong connection between a country's HDI score and its PISA score. For the 27 countries where both PISA scores and HDI scores were available, the correlation coefficients are shown in Table 10.1.

On one hand, Table 10.1 clearly indicates a strong connection between a country's general developmental level and its PISA score. In this respect, mathematics is

Table 10.1 Correlation coefficients between Human Development Index (HDI) scores and Programme for International Student Assessment (PISA) 2000 scores ($N = 27$ countries)

	Correlation coefficients		
	HDI score	PISA reading score	PISA mathematics score
Reading	0.606		
Mathematics	0.599	0.898	
Science	0.433	0.924	0.919

Source: Modified from Csíkos (2006, p. 184)
All correlation coefficients are significant ($p < 0.05$)

not exceptional at all, implying that fostering students' performance in mathematics should by all means be embedded in the general development of the quality of education. Nonetheless, at the individual and classroom levels, mathematics does have some unique characteristics. On the other hand, Table 10.1 also demonstrates the very strong connection between PISA scores in the three fields. This further strengthens the idea that achievement will develop as a consequence and in accordance with an increase in the general quality of the educational system.

Strengths and Weaknesses as Measured by International Surveys

The Eastern European countries achieved fairly different average scores in the latest TIMSS and PISA surveys. Table 10.2 illustrates how different their positions in the two lists are.

There is a tendency for several countries to refrain from participating in the eighth graders' TIMSS survey since that age group has a large overlap with the PISA target population. Since the sample of countries participating in each survey varies and a score of 500 on the TIMSS scale refers to the actual average mean (while in PISA a score of 500 means the OECD country average in 2000), it is hard to compare the achievement results in the two studies. Of course, the large differences in the ranking lists may attract lay people's attention, but there is a large overall tendency for TIMSS and PISA to both measure the quality of education and, of course, the quality of mathematics education.

In which fields of mathematical thinking do Eastern European countries have an advantageous or disadvantageous position? In "mathematics years," i.e., when mathematics is the central field to be measured in PISA, detailed scores are available regarding three thinking processes in mathematics (Table 10.3) and four content domains within mathematics (Table 10.4).

In each row of Table 10.3, the first aspect of our analysis is whether there is any strikingly high or low score or whether the students' results are balanced in the fields of the three thinking processes. For a detailed analysis of what these processes mean, the reader should consult the PISA 2012 framework (OECD, 2013). Roughly speaking, the *formulating* aspect of mathematical thinking refers to the process of

Table 10.2 Eastern European countries' Trends in International Mathematics and Science Study (TIMSS) and Programme for International Student Assessment (PISA) scores, and their connections with Human Development Index (HDI) scores and gross national income (GNI) per capita

HDI ranking	Country	HDI score	2015 TIMSS score (fourth grade)	2015 TIMSS score (eighth grade)	2015 PISA score	GNI per capita (2011 PPP $)
25	Slovenia	0.880	520	516	510	27,852
28	Czech Republic	0.870	528		492	26,660
30	Estonia	0.861			520	25,214
35	Slovakia	0.844	498		475	25,845
36	Poland	0.843	535		504	23,177
37	Lithuania	0.839	535	511	478	24,500
44	Hungary	0.828	529	514	477	22,916
46	Latvia	0.819			482	22,281
47	Croatia	0.818	502		464	19,409
49	Montenegro	0.802			418	14,558
50	Belarus	0.798				16,676
50	Russian Federation	0.798	564	538	494	22,352
52	Romania	0.793			444	18,108
59	Bulgaria	0.782	524		441	15,596
66	Serbia	0.771	518			12,190
76	Georgia	0.754	463	453	404	7,164
78	Azerbaijan	0.751				16,428
81	Macedonia	0.747			371	11,780
81	Ukraine	0.747				8,178
85	Albania	0.733			413	9,943
85	Armenia	0.733				8,124
85	Bosnia and Herzegovina	0.733				9,638
107	Moldova	0.693			420	5,223
	Correlations with 2015 PISA	*0.84*	*0.72*	*0.92*		*0.88*

Even in this small sample of countries, the correlation coefficients are significant
PPP $ international dollars after conversion using purchasing power parity rates

formulating mathematical models in a given situation, usually described in a word problem. *Employing* (which has a 50% weight, while the two others each have a 25% weight) refers to mathematical concepts, facts, and procedures. *Interpreting* involves evaluation of mathematical outcomes. In several Eastern European countries, students' achievement is well balanced among the three processes. However, when there is a larger difference between columns, it might define a type of mathematical process that is relatively underdeveloped in that country. Quite often, the processes of formulating have a fairly weaker average (Slovenia, Hungary, Croatia), which may point also to the relative strength of routine processes belonging to the employing category. According to Table 10.3, Russia has a relatively weak average

Table 10.3 Eastern European countries' average scores for three different mathematical thinking processes

Country	2012 PISA score	Formulating score	Employing score	Interpreting score
Estonia	521	517	524	513
Poland	518	516	519	515
Slovenia	501	492	505	498
Czech Republic	499	495	504	494
Latvia	491	488	495	486
Slovakia	482	480	485	473
Russian Federation	482	481	487	471
Lithuania	479	477	482	471
Hungary	477	469	481	477
Croatia	471	453	478	477
Serbia	449	447	451	445
Romania	445	445	446	438
Bulgaria	439	437	439	441
Montenegro	410	404	409	413
Albania	394	398	397	379

PISA Programme for International Student Assessment

Table 10.4 Eastern European countries' average scores in four different mathematical content domains

Country	2012 PISA score	Change and relationships score	Space and shape score	Quantity score	Uncertainty and data score
Estonia	521	530	513	525	510
Poland	518	509	524	519	517
Slovenia	501	499	503	504	496
Czech Republic	499	499	499	505	488
Latvia	491	496	497	487	478
Slovakia	482	474	490	486	472
Russian Federation	482	491	496	478	463
Lithuania	479	479	472	483	474
Hungary	477	481	474	476	476
Croatia	471	468	460	480	468
Serbia	449	442	446	456	448
Romania	445	446	447	443	437
Bulgaria	439	432	442	443	432
Montenegro	410	399	412	409	415
Albania	394	388	418	386	386

PISA Programme for International Student Assessment in the interpreting cluster of mathematical thinking as compared to the other mathematical processes.

Table 10.4 indicates whether in a given country there is any field of mathematics that is relatively highly developed or underdeveloped. Often, and understandably, the country profiles are rather well balanced (e.g., Slovenia, Hungary, Romania, and Bulgaria). However, in several cases, countries have a prioritized field (at least, one may infer that the reason is massive curricular coverage or a larger body of learning material in that country). For instance, Albanian students have far better results in geometry than in other fields. Geometry, in general, is thought to have a relatively momentous role within mathematics in Eastern Europe (Aubrey & Godfrey, 2003). On the other hand, Russia (Kolmogorov's country) seems to have a weakness in uncertainty and data, which may be due to a focus on formal mathematics since Kolmogorov's reforms in 1970. These results have been seriously taken into account, and these themes are included in curricula and the national maturation exam. The aforementioned relative strengths and weaknesses usually reflect long-term curricular and instructional methodological traditions in a given country.

Socioeconomic Background and Mathematics Achievement

At the time when Hungary was the only participant from the Eastern European region in the IEA studies, and Japan and Hungary competed for the highest country achievements, the Second International Science Study (SISS) created a measure of inequalities, called the Ratio of Homogeneity (ROH) index. Although in the early 1980s there was a central curriculum in Hungary, with only one textbook, the differences between schools proved to be larger than those in other top-performing countries (see Postlethwaite & Wiley, 1992). These within-country differences reflected both parents' efforts in finding the "best available" school for their children, and traditional geographical and socioeconomic differences in the country.

The PISA studies put a special emphasis on the role of family-related background variables. Over the course of the six PISA cycles, a more and more refined measure of students' socioeconomic status (SES) has been developed. Schleicher (2014) made computations from the PISA 2012 database, and one striking illustration of how SES is related to mathematics achievement was based on comparing groups that belong to different SES deciles. For each country, ten such SES groups can be compared, and while the top decile groups usually do not differ from each other, the lowest or the lower two deciles often lag far behind. The three eye-catching examples in this respect are Slovakia, Hungary, and the Czech Republic. It is very peculiar that the average mathematics scores of the students in the lowest SES decile in these three countries are lower than those of their Mexican peers. However, at the country level, Mexico has an average score of 413.

The PISA studies developed an index to measure students' economic, social, and cultural status (ESCS). In general, in top-performing countries, ESCS tends to have a relatively weak correlation with students' performance. Conceptually, ESCS can be connected to the idea of social inclusion, and the within-school and between-school differences in ESCS in a given country indicate the level of social inclusion. Although social justice and equity have long been catchphrases in Eastern Europe, according to the aforementioned ROH index, these countries may still suffer from lack of inclusion and lack of equity in their mathematics classrooms. Two interconnected phenomena should be investigated here: the question of low performers and whether the school system provides a chance for them to succeed (Table 10.5).

There is a clear connection between a country's overall average achievement and the percentage of low-performing students in that country. Estonia's 10.5% is the lowest value in Europe, a little lower than those of Finland (12.3%) and Switzerland (12.4%). In the majority of Eastern European countries, it is not only the high percentage of low performers that hinders their future development, but also the relatively low level of inclusion, as measured by means of ESCS. In general, ESCS explains around 15% of the PISA score variance in the overall country pool, but in certain countries, a higher percentage (i.e., a more expressed role of) explained variance appears. The OECD (2016) provided a statistical analysis (https://doi.org/10.1787/9789264250246-en) to compare the percentages of low-performing students in the top and bottom quartiles of ESCS. Ranking the countries in ascending order of the difference between the two rates of low-performing students, some Eastern European countries are at the far end of this; Bulgaria, Romania, Hungary, and Slovakia have more than a 40% difference in the rate of low performers in the two groups. It means that in these school systems the socioeconomic differences are deepened or at least not decreased. Conversely, in Estonia there is only a 12.6% difference in the rate of low-performing students in the most advantaged and most disadvantaged ESCS quartiles.

Table 10.5 Eastern European countries' percentages of low-performing students in mathematics in the 2012 Programme for International Student Assessment (PISA)

Country	Low-performing students (%)
Estonia	10.5
Poland	14.4
Slovenia	20.1
Czech Republic	21.0
Latvia	19.9
Slovakia	27.5
Russian Federation	24.0
Lithuania	26.0
Hungary	28.1
Croatia	29.9
Serbia	28.9
Romania	40.8
Bulgaria	43.8
Montenegro	56.6
Albania	60.7

The Organization for Economic Co-operation and Development (OECD) overall average rate is 23.0%

Member states of the European Union should decrease their percentage of low performers (not only in mathematics but also in the other key fields) below 15% by 2020. This aim seems to be unattainable, and in order to approach a 15% or at least 20% rate, school reforms in some Eastern European countries can be exemplary.

In Poland and Estonia, educational reforms have been aimed at decreasing the within-country differences by means of letting prospective vocational school students stay for one more year in the general schooling system (World Bank, 2010). As a philosophical basis for this reform movement, it is worth highlighting that Poland has a relatively fortunate situation within Eastern Europe. According to Turnau (1993), Polish scholars had the freedom to build international relations. Nevertheless, he is critical of the level of scientific achievement (which should have been much stronger in this state of research freedom). As an explanation, the still-existing complicating factor of language difficulties has been mentioned, along with the lack of strong theoretical embeddedness in math educators' scientific works.

The Estonian reforms (Lees, 2016) have contained elements that decreased inequality: individual psychological support, consultancy offered in the case of learning difficulties, free lunch, etc. Some of these elements were started even before the country's independence was regained. These seem to have little to do with mathematics achievement, but—as revealed from system-level data—it is the overall quality of education that will increase mathematical performance.

Russian reforms in mathematics education, according to a "Conception of Mathematics Education Development in the Russian Federation" government document, deal with individualization, where each student is supposed to receive education in accordance with his or her abilities, including talent recognition and support. At the same time, much more attention is paid to the development of gifted children and improving scientific achievements than to the support of children with learning difficulties. The need to establish a system of additional "leisure-time groups in mathematics" and to popularize mathematics is stressed throughout the conception.

There is a long list of educational reforms in Romania with a lot of positive effects in general (see UNESCO, 2015) but, as Nicu (2016) stated, there is a lack of consistency in introducing elements of reform and pursuing their effects in the Romanian education system. One of the reasons could be the fact that there have been too many and too rapid changes at the policy level. In the last 17 years there have been 12 different prime ministers and 17 different persons as minister of education, while the Romanian educational system is still highly centralized with almost no professional autonomy for teachers or teacher organizations.

Some Current Features and Tendencies in Eastern European Mathematics Education

Looking into Classrooms: Methodological Challenges

Blömeke, Suhl, and Döhrmann (2013) conducted an international comparative study on teachers' knowledge. An important aspect of their research is that they conducted an item-level analysis of different aspects of teachers' knowledge: strengths and weaknesses in mathematical pedagogical content knowledge and (pure) mathematical content knowledge. Here, Russia and Poland represented the Eastern European region, and the results pointed to a shared culture of mathematics teacher education in these two countries. Furthermore, Polish and Russian prospective teachers' advantages have been revealed (as compared to other countries in the sample: Taiwan, Hong Kong, Norway, the USA, and Germany) when solving difficult mathematical tasks.

In another study, Kaiser and Blömeke (2013) provided an example of how Eastern–Western dichotomies can be handled in large sample investigations. In this analysis, Poland and Russia, of course, belong to the Western culture countries, but when comparing future mathematics teachers' mathematical content knowledge and mathematical pedagogical content knowledge, these two countries proved to be similar to some traditional Eastern culture countries. In Poland, Russia, Taiwan, Thailand, Germany, Switzerland, and Georgia, students had greater mathematical content knowledge. Conversely, in the USA, Norway, the Philippines, Malaysia, Chile, Spain, and Botswana, prospective teachers' mathematical pedagogical content knowledge proved to be greater. Consequently, in the two representative Eastern European countries, preservice mathematics teachers are relatively well trained in mathematics and relatively poorly trained in pedagogy.

As an example, in Romania a regular teacher training program consists of 180 credits in the scientific field (bachelor's level) and 30 credits in the pedagogical module (offered by the teacher training institute, which has a fixed national curriculum that is 80% the same for all specializations). For those completing a master's program (2 years, 120 credits) there is a second pedagogical module with 30 additional credits. During these studies there is only one course of subject didactics and one discipline (during one semester) of practical training in schools. Thus, the main knowledge of how to be a mathematics teacher is not sourced in the worldwide recognized knowledge base (books and research papers) but in mathematical problem books. This viewpoint and structure (scientific specialization and pedagogical module) is historically rooted in the Romanian educational system; it was used from 1918 onward (when the modern and unified Romanian state was proclaimed) and at the beginning it was determined by the necessity for a large number of teachers in a relatively short time period (at the outset for unifying the four different educational systems that existed in the different regions before the unification, and subsequently for elimination of illiteracy, till 1959). This system practically allows the possibility for each student from higher education to become a teacher with minimal practical training, without any preliminary selection. In this context it is not surprising that according to the TIMSS 2007 teacher reports, rote-learning strategies were used in more than half of the lessons for at least 60% of students in the eighth grade in Bulgaria, Cyprus, Lithuania, Romania, and Turkey.

In Russia, there is also a traditional emphasis on mathematical content knowledge, with a 50–70% rate of preservice math teachers' university courses being dedicated to pure mathematics. According to the aforementioned "Conception of Mathematics Education Development in the Russian Federation" document, the curriculum for teachers needs to be changed in order to add extra tasks in elementary mathematics, including creative tasks and tasks at an advanced level, which teachers need to be able to solve by themselves. The conception also stresses the role of practices at schools, which would motivate teachers to acquire deeper pedagogical and psychological knowledge, but it does not point toward reformation of pedagogical or psychological courses that could be based on contemporary findings in mathematics education research.

There are many anecdotal cases where mathematics teachers from different cultures have observed each other's lessons. Such an experience is described by Woodrow (1997): Hungarian colleagues in a British school observed that instead of forcing students to achieve well, the British colleagues considered it more important not to hurt their students' self-image.

A current trend in Europe is the widespread dissemination of inquiry-based learning (IBL; often called problem-based learning in mathematics education). According to Maaß and Dorier's (2010) analysis, three Eastern European countries participating in the Promoting Inquiry in Mathematics and Science Education across Europe (PRIMAS) project—Hungary, Romania, and Slovakia—can be characterized by late introduction of IBL into their curricula. In this way, these countries proved to be similar to Malta, Spain, and Cyprus. Remarkably, problem-based learning was introduced in Russia back in 1832 by P. S. Gur'yev in his "Arithmetic leaflets" ["Arifmeticheskie listki"] (Polyakova, 2011) and then was spread in curricula during the 1940s and 1950s (Karp, 2011); the current programs still honor this tradition.

Moreover, the analysis of the Mathematics and Science Across Europe (MASCIL) project (see Maaß and Engeln (2016)) revealed that teachers have a positive attitude about IBL and about connecting IBL with the "World of Work" (WoW) in some Eastern European countries (Romania and Bulgaria), but neither IBL nor the WoW context is frequently used in daily teaching practice. Teachers from Romania feel less supported than teachers from other European countries in implementing IBL or using WoW contexts. Romania puts an emphasis on active participatory methods and active learning using cooperative strategies (in pairs or in groups). In other words, it recommends a shift from teaching from the front to cooperative teaching and learning in order to improve motivation and engagement in mathematics.

Fostering Students' Mathematics Learning Talent Development, Remedial Education, School Readiness, and Attitudes

Hungary is said to be first country where a nationwide high school mathematical competition was organized, at the end of the nineteenth century (Kontorovich, 2011). Frank (2012) cited George Pólya's reason as to why mathematics was so important and highly developed in the first decades of the twentieth century in Hungary: it is the least expensive science. The "competition cult" that was so greatly expressed in the Eastern European countries during the Soviet regime had strong antecedents in Hungarian mathematics and physics competitions. Also, the existence of specialized high schools aimed at developing mathematical and science talent originates in Budapest, e.g., the Lutheran High School, where several Nobel Prize winners studied high-level mathematics (see Marx, 1996).

Currently, Russia maintains a strong tradition of specialized mathematical education in schools; in many schools, children are divided into mathematics and humanities classes after the eight to ninth grades. Educational standards and the approximate curriculum for schools (which is officially provided by the ministry of education) assume two levels of competence and corresponding programs: ordinary and advanced. For example, the advanced level for primary school (first to fourth grades) includes the ability to solve logical tasks; to read simple pie charts; to plan, conduct, and analyze simple empirical investigations; and other tasks. There is also a number of special mathematics schools for the most talented children. These schools have a unique system to teach mathematics, which is called the "system of leaflets" (e.g., Shen, 2000). In this two-level system, all curricula for advanced mathematics are presented as sequences of problems, and a student needs to solve them on his own and then explain his solutions to a teacher assistant. This system develops the skill to think, to achieve a new mathematical result on one's own, and to experience a mathematical discovery together with the team of teacher assistants and schoolmates, as they not only go through mathematics problems but also experience out-of-school activities together (Yurkevich & Davidovich, 2008). As a result, each student is able to enter the mathematical departments of the best universities; many students successfully participate in all-Russian and other mathematical competitions, although successful participation in competitions is mostly considered an incidental result of education.

In Eastern European mathematics education, much less attention has been paid to remedial education. Currently the system is undergoing serious reconstruction in Russia: all children are supposed to be taught in inclusive classes, thus the system of specialized schools is going to be renewed and new educational standards are being elaborated in order to adjust school curricula to specify what needs to be taught at each of three disability levels of each disorder (Malofeev, Nikol'skaya, Kukushkina, & Goncharova, 2009).

Talent Development and Participation in the International Mathematics Olympiad

Since the beginning of mathematics competitions, the main aim has been not to win prizes or praise good students, but to find future creative mathematicians (Kontorovich, 2011). The International Mathematics Olympiad (IMO) was initiated in the socialist countries with the aim of promoting excellence in mathematics (Adler, 1980).

The first IMO was held in Romania (and still Romania has hosted the most IMOs—five times). The Romanian team has participated in all 57 contests over the years. Several times, Romania has been first in the unofficial country rankings (the last time was in 1996), and there is a very strong tradition in Romania for preparing children for mathematical contests. If we focus only on the countries of the European Union, we can see that the results for the Romanian team are in the forefront of the rankings: in first position in 2011 and 2012, and in second position in 2014 and 2015. Obviously, in the worldwide ranking, the results of the Romanian team show a declining trend (see Table 10.6).

Of course, the results of the Romanian IMO team can be analyzed from several viewpoints. If we see the numbers of medals won over the years (75 gold, 138 silver, and 98 bronze), they are impressive. But if we relate this number to the number of Romanian students participating in the IMO competition (380), we see that the efficiency is around 82%, which is less than the efficiency obtained by the Hungarian teams (91%) or by the Russian teams (100%).

The way in which talented students are selected in Eastern European countries has a long tradition. In Romania, there is a multistep testing procedure for high-achieving students (they are tested at the local level in schools, at the county level, and nationally), a huge collection of background materials (the *Gazeta Matematica* journal and publications from several specialized publishing houses like GIL), and an excellent study program for those included in the enlarged national teams. It is also worth mentioning that from a professional point of view the educational system is controlled by the ministry of education (through local inspectorates), while the contests are supervised more or less by members of the Romanian Mathematical Society. This duality seems to be persistent in the Romanian educational system, and there are no concrete signs that there is a (common or political) will to change it at the national level. Despite the good Romanian results in international competitions, the talent recognition, talent development, and talent support programs are

Table 10.6 Ranking of the Romanian International Mathematics Olympiad (IMO) team in the unofficial country rankings

	2010	2011	2012	2013	2014	2015	2016
Romania's overall ranking	16	8	10	22	11	13	22
Romania's ranking among European Union countries	6	1	1	5	2	2	6

not visible at a systemic level in schools. Extra classes for remedial education are allowed, but for talent development these are transferred to centers of excellence (mostly created in cities), which are unavailable for most children. There are several civil initiatives (sponsored by foundations), but most of them are not embedded in the regular (not private) school system.

Our second example here is Russia. As the cessionary of the former Soviet Union's several first rankings, Russia traditionally has had quite good results in IMOs. It achieved second to fourth places in the worldwide ranking during the last 20 years, though in the last 2 years the results got worse; Russia won no gold medals and took eighth place in 2015, but it came seventh and won four gold medals in 2016. The mass media stressed that the results in 2016 were noticeably better than those in 2015, but they need to be improved further since Russia is expected by politicians and mathematicians to come first or second every year, as it did many times during the Soviet years. The 11th place it achieved in 2017 signified a failure of the current efforts. As has been mentioned, education for gifted students receives special attention; for example, a special center for gifted children was opened in Sochi under the personal control of the president. The preparations of the Russian team for IMOs, together with other conferences and summer and winter camps, are conducted at this center.

There are a few levels in the selection process of the IMO participants. Results from two all-Russian mathematical competitions, an open Chinese competition, and a "Romanian Masters" competition are taken into account. Around 50 selected participants are invited for 2 weeks of preparations a few times during the last 2 years of school. At the end the final team is formed, and a lot of attention is paid to individual preparation during the last 2 weeks of preparation before the IMO: each pupil solves the tasks that are chosen for him or her in accordance with his or her difficulties during the previous competitions and preparations. Tasks in international and all-Russian competitions are different: tasks in all-Russian competitions need more creative thinking, while tasks in the IMO are more technical, and it is exactly creative thinking in which Russian participants are so strong, while they lose in comparison with their Asian colleagues in technics and stability of calculation skills. This is what needs to be approached during the preparations.

School Readiness in Mathematics

The importance and topicality of school readiness investigations in the Eastern European region is illustrated here by the cases of Hungary and Poland. The first kindergarten in Central and Eastern Europe was established in 1828 in Buda by Terézia Brunszvik. Since that time, kindergarten education has been a central topic in Hungary. The current national kindergarten curriculum considers mathematics "a tool for observing and learning in the world through activities" (Government decree 363/2012). In Poland the development of mathematics skills is an important part of the core curriculum in kindergarten. Children learn about counting, numeracy, classification, addition, and subtraction through playful activities (Smoczyńska et al., 2014).

Children start school at age 6 in both countries. School readiness assessments are not compulsory in Hungary. The decision as to whether a child is ready for primary school is generally made by kindergarten teachers based on mostly social and physical characteristics. Cognitive development is also an important indicator, but language and vocabulary are usually more relevant than early numerical skills. The test battery most commonly used to assess the key cognitive and social skills for school readiness in Hungary is called the Diagnostic System for Assessing Development for Four- to Eight-Year-Old Children (DIFER), which includes a basic counting and numeracy test (see Csapó, Molnár, & Nagy, 2014). However, kindergarten teachers barely use these tools, considering that face-to-face measures are time consuming for them. To overcome these problems there are new research projects to extend technology-based assessments to early childhood as well (Rausch & Pásztor, 2017). A newly developed online test is used at school entry to assess early numerical skills from age 5–7, including five subtests: basic counting, the number word sequence, numeral recognition, magnitudes, and numerals and relations. The results of the first nationwide measurements are promising (Rausch, 2016).

In Poland, kindergarten teachers are requested to make school readiness assessments, which is called preschool diagnosis, based on the instructions of the core curriculum (Smoczyńska et al., 2014). Assessing basic counting skills is usually part of these measurements. Integrating information and communication technologies (ICT) into early mathematics education and assessments is a rapidly developing research area in Poland as well. The Test of Abilities at the Start of School (TUNSS), done using tablet computers, is used to assess school achievements in mathematics, reading, and writing from preschool up to second grade students at primary school. The mathematics subtest has items related to numbers, measurements, space and shape, relations, and dependencies (Szram, 2016).

The worldwide growing importance of mathematics education and the unquestioned importance of the early years of schooling define a research field that has brought some important findings from Eastern European colleagues. A cross-cultural investigation into early arithmetic by Rodic et al. (2015) found similar knowledge structures in 5- to 7-year-old students in the participating countries: the UK, Russia, China, and Kyrgyzstan.

Conclusion

Eastern European countries have a rather famous (sometimes labeled as infamous) heritage of school mathematics education. Having built on both the European didactical tradition and the Soviet ideas of psychology, Eastern Europe's mathematics education has produced impressive results in talent recognition and talent development, as indicated by the outstanding participation at International Mathematics Olympiads. From the 1980s, however, the average results of students in mathematics have tended to decrease, as measured by large-scale international surveys. At the root of the problems is an increasing difference between students with advantaged and those with disadvantaged socioeconomic status, and (not independently of that) the increasing proportion of low-performing students may lead to the conclusion that many countries in the region may and should follow some elements of the Polish and Estonian school reforms.

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Chapter 11
Mathematical Learning and Its Difficulties in Southern European Countries

Maria Gracia-Bafalluy and Miguel Puyuelo-San Clemente

Introduction

In this chapter, we consider the countries of Southern Mediterranean Europe: Portugal, Spain, France, Italy, and Greece. Historically rich and complex, these countries present deep differences both between and within them, with deep territorial North–South divides and gender discrepancies.

They also feature decentralized education policies (except for Greece, which retains mainly centralized education). Aside from European and national policies, regions and school centers have some autonomy in the way central guidelines are applied and in the professionals, materials, and resources they choose.

Some common guidelines are set by the European Union. Concepts such as competencies, special educational needs, etc., are shared among the member countries through entities such as the Eurydice network—part of the Education, Audiovisual and Culture Executive Agency (EACEA)—which supports European cooperation in education systems, and other international entities such as the Organization for Economic Co-operation and Development (OECD).

However, there is a substantial gap between current scientific knowledge and the legislative bodies responsible for ensuring that individuals receive the support they need in mathematics. Bringing about this necessary improvement will require more

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effort in promoting and facilitating communication between researchers, lawmakers, therapists, and other educators.

Also, this geographical area is currently undergoing a deep, decade-long crisis and social transformation. In some cases, that has caused economic resources devoted to education to be blocked or increased by less than the needs of the sector in these countries. By 2014, some countries had had reduced education budgets for three or four consecutive years (Portugal and Spain, respectively), while others had done so for even longer (Greece and Italy).

Educational Policies in Southern Europe

In 2006 the European Union issued a guide to the competencies necessary for active participation in society, including mathematics (Eurydice, 2016). This forced Southern European countries to update their curricula in order to follow the guidelines. These changes were also driven by new benchmarks, such as the shift from traditional subject-based to learning-outcome-based curricula, the inclusion of cross-curricular links between subjects, and the introduction of specific targets as criteria for learning. Moreover, these changes can be understood as a response to the performances obtained by students in external examinations such as the Program for International Student Assessment (PISA) test of 15-year-olds, organized by the OECD (Eurydice, 2011) (see Table 11.1).

However, the OECD itself has issued alerts about increasing inequalities between and within countries over the last 30 years (OECD, 2014), as can be seen in Figs. 11.1 and 11.2.

Table 11.1 Scores of Southern European countries in the last Organization for Economic Co-operation and Development (OECD) Programme for International Student Assessment (PISA) evaluation (2015) and 2012 evaluation scores (the last edition in which mathematics was assessed as a main domain); extracted from OECD (2016)

Country	2015 score (standard error) [OECD average: 490]	2015 standard deviation (standard error)	2012 score (standard error) [OECD average-30: 496]	% score under level 2 in math in 2015 [OECD average-30: 22.9%]
Spain	486[a] (2.2)	85 (1.3)	484[a] (1.9)	22.2
France	493 (2.1)	95 (1.5)	495 (2.5)	23.5
Italy	490 (2.8)	94 (1.7)	485[a] (2.0)	23.3
Greece	454[a] (3.8)	89 (1.8)	453[a] (2.5)	35.8
Portugal	492 (2.5)	96 (1.3)	487 (3.8)	23.8

At level 2, students can interpret and recognize situations in contexts that require no more than direct inference. They can extract relevant information from a single source and make use of a single representational mode. Students at this level can employ basic algorithms, formulas, procedures, and conventions to solve problems involving whole numbers. They are capable of making literal interpretations of the results. This is the baseline level of proficiency required to participate fully in modern society (www.oecd.org)

[a] Mean scores significantly below the OECD average. Significance data are not available for percentage scores under level 2 in mathematics in 2015

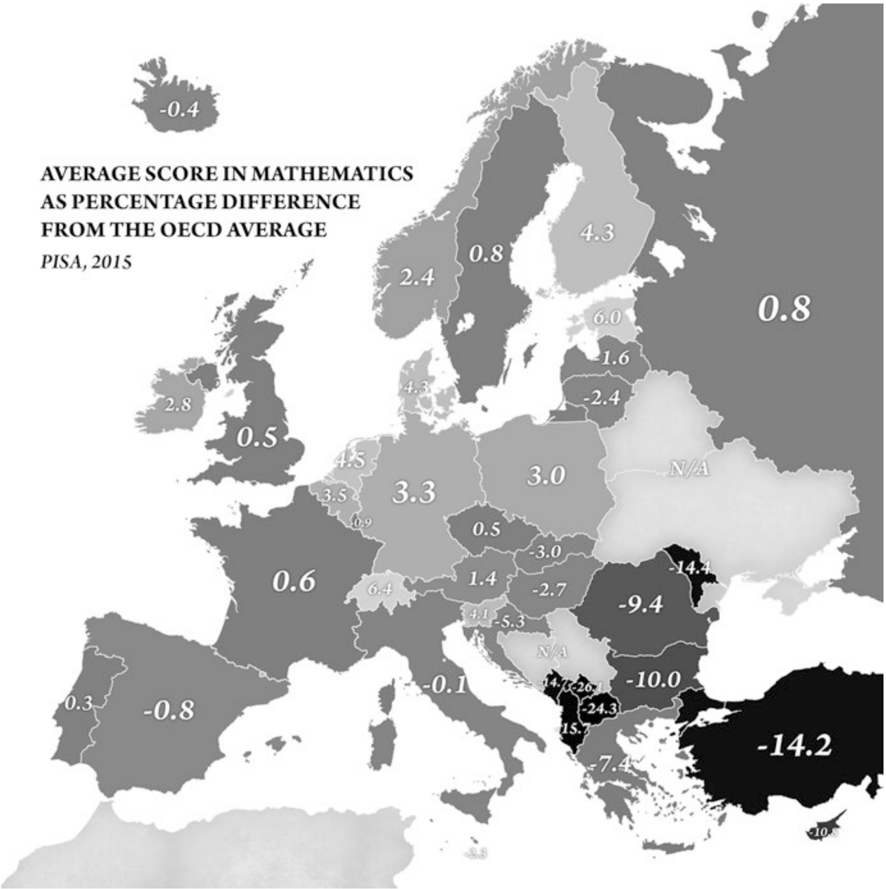

Fig. 11.1 Organization for Economic Co-operation and Development (OECD) Programme for International Student Assessment (PISA) 2015 results for European countries. Average scores are compared with the OECD average. (Courtesy of Jakub Marian, https://jakubmarian.com/)

Due to the common European guidelines, all the countries included in this chapter have similar policies on how and when to assess children for mathematics learning difficulties (MLD). In all of them, teachers are encouraged to detect cases of learning difficulties and assess them or refer them to the relevant pedagogical teams.

For instance, in Italy, a 2010 law (Legge 170/2010 Gazzetta Ufficiale, 2010) set out conditions and supports for including cases of learning difficulties in normal schooling. A more recent guideline (Direttiva Ministeriale 27.12.2012 (Ministero dell'Istruzione, 2012) included dyscalculia as a specific learning difficulty and indicated that these cases should be detected by teachers.

In Portugal, when a case of special education needs is detected, the child must be included in "quality education" (Decreto-Lei n.° 3/2008), even if that requires additional support (see Fernandes, Miranda, & Cruz-Santos, 2014, p. 15). However, the assessment criteria to determine such cases are not prescriptions but recommendations (Eurydice, 2011, p. 36).

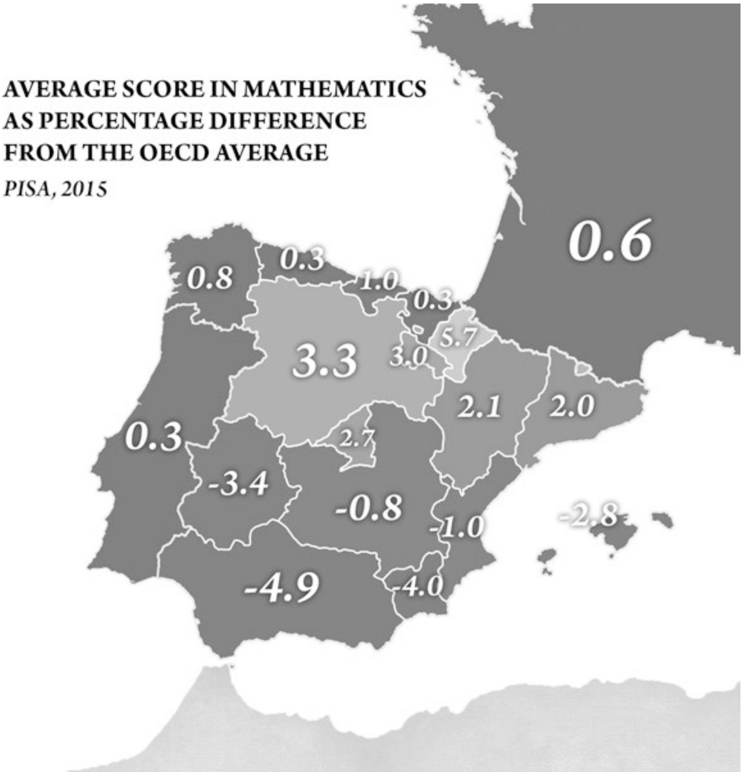

Fig. 11.2 Organization for Economic Co-operation and Development (OECD) Programme for International Student Assessment (PISA) 2015 results for autonomous communities in Spain, the only country with detailed scores. Average scores are compared with the OECD average. (Courtesy of Jakub Marian, https://jakubmarian.com/)

In Greece, MLD are mentioned in Ministerial Decision 24136/Γ7/21-2-2013 regarding lower secondary (compulsory) education (UNESCO, 2015). According to Ministry of National Education Law 2817 (2000, (Syriopoulou-Delli 2010)), the category "people with special educational needs" includes "difficulties in mathematics." These cases must be identified by the Center of Differential Diagnosis and Support of Special Educational Needs (KEDDY; Law 3699/2008), which includes a psychologist, a teacher of special education, and a social worker, and must provide counseling support to teachers and parents (Syriopoulou-Delli 2010).

In France, a decree (décret n° 2005-1014 of 24 August 2005 (Bulletin Officiel, 2005)) sets out that during elementary school, families of children who do not attain the basic competencies will be contacted by the school principal to set up a personalized support plan together (Programme de Réussite Éducative (PPRE)), to be applied during or after school hours from cours élémentaire 1 (CE1) (around 7 years of age). The professionals in charge are psychologists, pediatricians, or specialized teachers, all with specific training.

In Spain, a 1995 law (Real Decreto 696/1995; 28 April, 1995) regulated work with special educational needs (in children who could not attain basic objectives, without further definition) and specified that these cases might be temporary or permanent. Thus, the law referred to a rather heterogeneous population, and specific learning difficulties were not mentioned. It was not until 2006 that a law (Ley Orgánica de Educación (LOE); 3 May, 2006) encompassed "specific learning difficulties" as particular needs requiring educational support, although the specific kinds of difficulties remained unspecified.

More recently, a new organic law for educative quality improvement (Ley Orgánica para la Mejora de la Calidad Educativa (LOMCE), published as Ley Orgánica 8/2013 December 9) was passed as a response to the poor results obtained by Spanish students in PISA and other international evaluations. However, this law does not repeal the previous LOE, and its modifications do not affect concepts such as mathematics learning or special needs in mathematics.

To sum up then, for these countries, when a tutor and/or mathematics teacher detects a child who has serious trouble completing their mathematical assignments, they contact their corresponding educational orientation and psychopedagogical team (in Spain or France), or select tests in order to further assess the child's general and specific abilities. The psychopedagogical teams are made up of psychologists, therapeutic pedagogues, and speech therapists. Their roles include psychopedagogical evaluation, proposing adaptations to the curriculum that best respond to the educational needs of the student, and orienting teachers. Only in the case of Spain can the teams or individual professionals also develop a specific curricular proposal or individual curricular adaptation (not significant or significant) for these students; similarly, in Greece the content difficulty can be adapted to the performance level of the pupil (but with no modifications of the curriculum objectives) (Eurydice, 2011).

Definition of Mathematics Learning Difficulties, and Assessment and Diagnostic Criteria

Countries providing a clinical definition of specific learning difficulties make reference to the criteria in the International Classification of Diseases 10 (ICD-10) (World Health Organization, 1992) or the *Diagnostic and Statistical Manual of Mental Disorders, Fifth Edition* (DSM-V) (American Psychiatric Association, 2013).

As for pedagogical criteria, these center on low performance in standardized calculation tests and mathematical competency below what is expected for the chronological age, when this is combined with normal performance in other areas and the absence of socioeducational factors that could otherwise explain the problem. Thus, a specific learning deficit is initially defined as a discrepancy between the student's performance and the objectives determined by the curriculum, without any mention of underlying cognitive processes (Inserm, 2007). This "conceptually empty" criterion can certainly serve as an indicator of the need to assess a particular child, but it is not enough for adequate diagnosis or effective intervention.

Math anxiety is another widespread condition with well-known adverse emotional and cognitive effects; however, it is not officially recognized in mathematics remediation programs.

When a case of learning difficulties is detected, the relevant professionals (teachers and/or pedagogical teams) must administer a battery for general intellectual performance and some specific mathematics measures. If the MLD is confirmed, then the school, via the teachers (as in Portugal or Greece) or the corresponding pedagogical team (as in France or Spain), provides the necessary support for a category included as children with special educational needs (not the same as "low achievers," who are not considered to have any specific learning difficulty).

Currently, two different approaches reflect the ongoing debate in the numerical cognition literature between those claiming that dyscalculia is a domain-specific deficit and those attributing it to general factors. According to this view, educators should be attentive to young children who show difficulties in recognizing small quantities of elements without counting them (i.e., subitizing), have problems when counting manageable sets, or make errors when comparing numbers and quantities. These kinds of indicators relate to the concept of *number sense*, which denotes our intuitive ability to "quickly understand, approximate, and manipulate numerical quantities" (Dehaene, 2001).

Along these lines, a document published by the French Pediatric Society (Société Française de Pédiatrie, 2009) includes a brief explanation of the milestones of logical–mathematical competency acquisition (see Table 11.2).

The same document (Société Française de Pédiatrie, 2009) indicates alert signs for two school stages:

- *Maternelle* (3–5 years old): Difficulties in accessing numerical symbols; very little or no imitation game; drawings with no representative level; no numerical chain and mistakes when counting; difficulties when enumerating

Table 11.2 Acquisition of logical–mathematical competencies

Grade (age)	Counting	Calculation (addition)	Problem solving
GSM (5–6 years)	Spontaneous comparison of two different collections ($n < 10$) with the same kind of object	Based on rhymes	
CP/CE1 (6–8 years)	Conservation of numbers: comparison of two collections with the same numerosity but different kinds of object	Addition of two numbers < 10, adding the smaller to the bigger, one by one $(3 + 1 = 4 + 1 = 5 + 1 = 6 \ldots)$	By combination (how many more, how many less)
CE2 (8–9 years)	Mastering counting above 100	Notions of commutativity $(4 + 3 = 3 + 4)$ and associativity $(4 + 3 + 2 = 4 + 5)$	By combination (how many more or less than); by choosing the proper operation

Adapted from Société Française de Pédiatrie (2009)
CE cours élémentaire, *CP* cours préparatoire, *GSM* grande section de maternelle

- *Primaire* (6–10 years old): Difficulties in memorizing arithmetical facts or multiplication and addition tables; problems when operating with borrowing or transcoding; poor comprehension of arithmetical problems

Similarly, two regions in Spain have provided indicators of mathematics difficulties (see Table 11.3).

A second, much more pervasive, approach claims that all learning difficulties are due to problems in general cognitive abilities. In line with this approach, mathematics-specific difficulties are most commonly assessed through tests of

Table 11.3 Indicators provided to teachers and counselors in two regions of Spain—Catalonia and Navarra—to detect mathematics difficulties

Region	Age group	Difficulty
Catalonia	Kindergarten (children up to 6 years old)	Symbol access difficulties Counting difficulties No cardinality principle Difficulty in understanding basic notions of quantity Difficulty in recognizing/reproducing basic geometric figures Difficulty in putting elements into an ordered series Persistent mistakes when reciting the number sequence
	First cycle of primary school (first and second grades, 6–7 years of age)	Difficulty in learning basic arithmetic Mistakes when performing calculation exercises Problems in choosing the correct operation to execute Problems with large magnitudes Reading and transcription errors Frequent errors in operations No number sense Difficulty understanding arithmetic situations Performance discrepancy with other academic subjects
	Middle cycle of primary school (third and fourth grades, 8–9 years of age)	Missing strategies for addition and approximate calculation Difficulty recalling basic arithmetic facts Difficulty counting backward and learning to divide Anxiety/math aversion Frequent errors in operations Slow mental calculation Difficulty in calculating large magnitudes Errors in understanding and solving problems Difficulty in understanding positional notation (e.g., the decimal system) Performance discrepancy with other academic subjects
	Higher cycle of primary school (fifth and sixth grades, 10–12 years of age)	Anxiety, low motivation, apparent lack of interest Secondary behavioral issues High engagement not corresponding to the final result Lack of a solid base in calculation taught in previous years Difficulty in complex geometry Lack of calculation strategies Need for support material (e.g., a calculator)

(continued)

Table 11.3 (continued)

Region	Age group	Difficulty
	Secondary and high school (12–18 years of age)	Difficulty in understanding basic concepts in physics, chemistry, finance, etc. Lack of a solid base of skills taught in previous years Anxiety/math aversion Slow processing, not remembering multiplication tables Performance discrepancy with other academic subjects
Navarra (CREENA, 2012)	Up to 5 years of age	Difficulty in reciting the number sequence Errors when representing numerals with their fingers Problems in reading or writing numbers seen at school Errors when enumerating small magnitudes Difficulty in subitizing (i.e., rapidly identifying quantities <5) Errors when comparing numbers and magnitudes
	Approximately 5–8 years of age	Difficulties in automatizing calculation (e.g., problems in performing addition or subtraction with a result less than 10, rapidly and without counting) Immature strategies such as finger counting Alignment errors in multidigit addition or subtraction Problems when working with numerical series
	Approximately 9–18 years of age	Persistent difficulty in memorizing multiplication tables, basic errors in multiplication Memory problems during mental calculation (e.g., forgetting quantities while performing operations) Frequent arithmetical errors when calculating or carrying Immature strategies for problem solving (e.g., “I don’t know if I must add or subtract”)

memory, perception, or cognitive styles, and mathematical ability is evaluated mainly through results obtained in mathematics exams.

As for the prevalence of these learning difficulties, none of the studied countries either specifies or publishes its own statistics, and there are no epidemiological studies of learning difficulties in school populations. At most, international references (about 5–7% of the student population, Inserm, 2007) or national estimations (2.5–3.5% of the Italian student population, Ministero della Salute, 2011) are considered, although not in a generalized way.

In general, teachers are encouraged to use batteries devoted to general intellectual abilities, such as WISC, K-ABC, Raven, etc. (all of which have adaptations for some of these countries). Also, other specific measures for mathematics have been adapted to or from other Southern European countries’ languages. For instance, the Nucalc battery, also known as Zareki (Deloche et al., 1993), has been adapted for French and Greek populations (Koumoula et al., 2004), and a new version, Zareki-R, has been adapted for the French population (von Aster & Dellatolas, 2006). Tedi-Math (Test Diagnostique des Compétences de Base en Mathématiques [Diagnostic Test of Basic Mathematics Competencies] (Grégoire, Noël, & Van Nieuwenhoven, 2001)), besides its original French version, has been adapted for Spanish (Sueiro & Pereña, 2005) and Italian (Girelli, Bizzaro, Krinzinger, & Willmes, 2015) populations.

Assessment of Mathematics Learning Difficulties in Italy

Dyscalculia in Italy cannot be diagnosed before the end of classe terza (8–9 years of age). From this age on, if any alert sign is detected, an external team must complete neuropsychological and intellectual assessments including attention, reading and writing, praxis, and visual–spatial abilities. After this evaluation, an intervention program must be suggested by the team and planned together with the child's school center. The intervention should include personalized plans, compensation strategies, specific materials, and the participation of a support teacher.

Although tests for neuropsychological or cognitive measurements are not indicated, there are a number of specific tests for mathematics difficulties designed originally in Italian. Some of these are:

- *AC-MT*—Test di Valutazione delle Abilità di Calcolo e Soluzione di Problemi [Test of Abilities of Calculation and Problem Solving]: Developed from the theoretical background of the ABCA test (Lucangeli, Tressoldi, & Fiore, 1998) as a response to recent official regulations, this is a revision of its original version but presents a shorter administration delay. This new version can be administered individually or collectively, and it takes around 20 minutes. It includes a series of arithmetical problems presented on paper, which must be answered on a response sheet. The evaluation includes the total number of responses, accuracy, errors, and time needed for completing the test.
- AC-MT is divided into two age groups. AC-MT 6–10 (Cornoldi, Lucangeli, & Bellina, 2002) for primary education includes tasks of written calculation, numerosity judgment, digit transformation, and numerosity order; and an individual part also including mental and written calculation, enumeration, dictation of numbers, and arithmetical fact recall. AC-MT 11–14 (Cornoldi & Cazzola, 2004) is oriented to lower secondary school; it has no numerosity order but instead adds tasks on arithmetic expression, series completion, approximate calculation, and numerical facts and procedures.
- *BDE-2*—Batteria Discalculia Evolutiva [Developmental Dyscalculia Battery] (Biancardi & Nicoletti, 2004): This test is designed to diagnose MLD and calculation difficulties in children from 8 to 13 years old, based on the triple code model described by Dehaene and colleagues (Dehaene, 1992; Dehaene, Piazza, Pinel, & Cohen, 2003). The tasks are distributed into three areas: number elaboration (counting and transcoding), calculation, and number sense. It also includes a test of logical–mathematical problem solving.
- *Dyscalculia Test* (Lucangeli, Molin, Poli, Tressoldi, & Zorzi, 2009): This battery, constructed using evidence from recent research, evaluates five categories of numerical cognition: number sense, number line, numerical facts, dictation of numbers, and mental calculation. It can be administered with a computer or in paper-and-pencil format.
- *ABCA Test*—Test delle Abilità di Calcolo Aritmetico [Arithmetic Calculation Ability Test] (Lucangeli et al., 1998): This test is based on a theoretical model of modular magnitude representation for 8- to 10-year-olds, described by McCloskey, Caramazza, and Basili's (1985).

Assessment of Mathematics Learning Difficulties in Greece

The DeDiMa battery (Karagiannakis & Baccaglini-Frank, 2014) was created as a proposed classification model for MLD and is standardized for grades 5 and 6 (ages 10–12). It distinguishes between four domains: core number, visual–spatial, memory, and reasoning. This can also serve to outline the pupil's strengths, thus orienting an eventual intervention. Administration is computer based. The battery includes 13 tasks: subitizing–enumeration, number magnitude comparison, dot magnitude comparison, addition fact retrieval, multiplication fact retrieval, number lines 0–100 and ordinality, number lines 0–1000, math terms, calculation principles, mental calculation, equations, and word problems. There is a French adaptation in preparation.

Assessment of Mathematics Learning Difficulties in Spain

- Evamat—Batería para la Evaluación de la Competencia Matemática [Battery for the Evaluation of Mathematical Competency] (García-Vidal, González-Manjón, García-Ortiz, & Jiménez-Fernández, 2010): Evamat, created to assess numerical abilities, can be administered individually or collectively in about 60 minutes and is aimed at children from the end of the first year of primary school (6 years of age), to the beginning of the third year of secondary school (15 years of age). It includes eight booklets, one for each educational level, and is composed of tests of basic number knowledge (writing numbers, composing and decomposing quantities, completing series, or placing numbers on a number line); calculation (mental or written calculation, relating multiplication and division, calculating doubles and halves); geometry (identifying a missing part in a drawing, differentiating figures); and problem solving involving different operations.
- *Mathematikoi II* (Camarero, Santos, García, et al., 2003): This was designed and standardized by an Asturian educational orientation and psychopedagogical team with the aim of individually or collectively assessing curricular achievement in mathematics. It is aimed at children in the second cycle of primary school (8–10 years of age) and is composed of 32 multiple-choice items with an increasing level of difficulty, including questions on geometry, volumes and measures, chart interpretation, Arabic digits, seriation, arithmetical calculation, fractions, and problem solving.

There are other more general tests, created in Spanish, that include numerical subtests:

- *BADyG*—Batería de Aptitudes Diferenciales y Generales [Battery of General and Specific Aptitudes] (Yuste & Martínez, 2012): This is a general performance test designed to evaluate 4- to 16-year-old children. It includes five global factors (general intelligence and logical reasoning, as well as verbal, numerical, and spatial factors). The subtests are specific for each age and standardized for every academic level. As for the numerical subtests, they include quantitative numerical concepts (for 4- to 6-year-olds), calculation and numerical–verbal problems (for 6- to 12-year-olds), and completion of numerical series (for 12- to 16-year-olds). The test can be administered on a computer or in a paper-and-pencil format, either collectively or individually.

- *Evalua* (García-Vidal & González-Manjón, 2003): This is a battery assessing general intellectual abilities in 6- to 12-year-olds, standardized by age groups (6–8, 8–10, and 10–12 years). Administration can be either individual or collective. The battery includes two numerical subtests: calculation and numeration, and problem solving.

Assessment of Mathematics Learning Difficulties in France

A compulsory general assessment of pupils is administered at two specific points: at the beginning of CE2 and in the sixth grade of *collège* (at 8 and 11 years of age, respectively). When a learning difficulty is detected, the effects of the intervention must be evaluated every 6 months, after which its contents and goals must be adjusted to the child's needs (article L2325-1 of the Public Health Code; see Legifrance, 2016).

To perform this general assessment, the Ministry of Education, via the Assessment Board (Direction de l'Évaluation) has created a computer program called J'Ade, which can be administered individually or collectively. It is designed not only to detect learning difficulties but also to evaluate general performance and to orient the intervention program in detected cases. As for sixth-grade mathematics, the items are distributed into five areas: space and geometry (15 items), numerical data utilization (18 items), sizes and measurement (eight items), numerical knowledge (27 items), and calculation (33 items). The assessment has been administered nationwide since 2005.

Some of the mathematics evaluations originally developed in French include:

- *ECPN*—Epreuve Conceptuelle de Résolution des Problèmes Numériques [Conceptual Test of Numerical Problem Resolution] (CIMETE, 1995; Duquesne, 2003; Noël, 2007): This is a short test (it takes 10–30 minutes to be administered) comprising nine tasks divided into four parts: comparison, equilibrating collections, creating a difference, and adding and subtracting. The test is standardized for children between 4 and 9 years old.
- *Numerical* (Gaillard, Segura, & Taussik, 2000): This is oriented to 8- to 9-year-olds and inspired by the EC301 assessment battery for adults (Deloche et al., 1993); it can be used to detect dyscalculia.
- *UDN-II*—Batterie sur L'Utilisation du Nombre [Battery for Number Use] (Meljac & Lemmel, 1999): This battery is based on the Piagetian theory of development and includes tests on elementary logic (classification, inclusion, and seriation), conservation, number use, spatial tasks, and numerical and operational knowledge and comprehension. It can be administered to children aged from 4 to 11 years.

Intervention: Theories, Research, and Educational Practice

All the countries studied here require the use of evidence in policy-making processes. Each ministry of education works with its own (France, Italy, and Spain) or external independent (Greece, Portugal) research institutes or international organizations. The Eurydice Agency (2017) has recently published a detailed description of the sources for each country and how the flow of information between evidence providers and policy makers works.

As for remediation methods for MLD, however, there is no national policy providing central guidelines apart from the diagnostic pedagogical criteria and the supports for the intervention, as far as we know. Nevertheless, the pedagogical services, along with schools (and families, in the case of France), are in charge of the intervention content for each case.

In Spain, the National Center for Innovation and Educational Research (http://educalab.es/cniie/) provides resources complementary to curricula. Among them, a publication compiles some recent evidence and measurements for MLD, along with some online intervention programs (Martínez-Berruezo, 2015). Likewise, the Portuguese government furnishes digital resources for teachers via the School Gateway program (Portal das Escolas; http://portaldasescolas.pt), and Italy offers an online learning scheme specifically for low achievers—SOS Studenti (http://puntoedu.indire.it/pon_sosstudenti/iscrizione/index.html).

In addition, some professionals or researchers in each country have proposed a number of programs, which are discussed below.

The Number Race (La Course aux Nombres; http://www.lacourseauxnombres.com/nr/home.php) (Wilson et al., 2006) aims to strengthen brain circuits for representing and manipulating numbers in children between 4 and 8 years old (with or without learning difficulties) and has been translated into French, Spanish, and Italian (see http://www.lacourseauxnombres.com/nr/nr_download.php?lang=en). Its activities include number presentation (sets of digits or number words), counting (from 1 to 40), and basic calculation (addition and subtraction).

Also, the same authors who created Tedi-Math have developed two associated intervention proposals—Ad-Math 1 and 2 (Cornet, Goerlich, Vanmuysen, & Van Nieuwenhoven, 2001)—consisting of four complementary games, each designed for 4- to 10-year-olds to improve numerical representation through games, and requiring the participation of sight, touch, and hearing.

In Portugal, an interesting proposal is a program for intervening in metacognitive procedures and numerical reasoning (Dias & Santos, 2009) for lower secondary school students. Verbal instructions are presented, both for self-assessment in calculation and for justifying procedures in solving a problem. The intervention also records the pupils on audio in order to complement feedback.

Similarly, in Spain, Miranda-Casas, Arlandis, and Soriano (1997) and Miranda-Casas, Marco, Soriano, Meliá, and Simó (2008) have developed two intervention programs using metacognitive strategies for problem solving in children with mathematical difficulties. The first training program, designed for 10- to 12-year-olds

(Miranda-Casas et al., 1997), requires students to reflectively engage in their problem-solving process while graphically representing problem data. After answering some reflective questions, students conduct a final self-correction. The second program, Escuela Submarina [Submarine School] (Miranda-Casas et al., 2008), has been developed for 8- to 10-year-olds and uses similar training in self-instruction combined with computer use.

Another program is Infopitagoras (www.infopitagoras.com), which allows teachers to create curricular adaptations by providing them with internet materials to use in their reinforcement programs. This program is available for specific learning difficulties in reading, writing, and arithmetic for 6- to 7-year-olds.

Some materials originally developed in Italian include:

- *Dyscalculia Trainer* (Molin, Poli, Tressoldi, & Lucangeli, 2009): This consists of a book and software designed for 8- to 12-year-olds. Several games are grouped into four categories: number sense, dictation of numbers, mental calculation, and number facts.
- *I Numeri e lo Spazio con la Lim* [Numbers and Space with Lim (an interactive whiteboard)] (Poli, Molin, & Lucangeli, 2017): This involves material presented in a book or on a computer, and includes manipulative material (a magnetic tablet, unity tiles) for children aged between 5 and 7.
- *La Linea del 20* [The 20 Line] (Bortolato, 2011) and *La Linea del 100* [The 100 Line] (Bortolato, 2008): The first consists of a book and manipulative material for practicing numerosities up to 20 with addition and subtraction games, and is specially designed for 5- to 8-year-olds with MLD; the second features a similar book and manipulative material for 6- to 8-year-olds, involving the numerical line up to 100. Both can be used independently.
- *L'Intelligenza Numerica* [Numerical Intelligence] (Lucangeli, Poli, & Molin, 2003): This program aims to improve the cognitive base underlying number processing for children from 3 to 11 years old, distributed into four age groups.
- *Memocalcolo* (Poli, Molin, Lucangeli, & Cornoldi, 2006): This is designed to work on number facts and mental calculation, paying special attention to memory, in 7- to 11-year-olds.
- *Potenziare le Abilità Numeriche e di Calcolo* [Potentiating Numerical and Calculation Abilities] (Biancardi, Pulga, & Savelli, 2008): This is designed for 6- to 11-year-olds. It consists of a book and software, with 12 activities in counting, transcoding, and calculation. It is specifically useful for children with dyscalculia but also as a support for normal learning.
- *Tabelline e Difficoltà Aritmetiche* [Tables and Arithmetic Difficulty] (Riccardi-Ripamonti, 2014): This is designed for the 5 years of primary school. It includes four sections: learning of tables and combinations; counting forward, backward and at speed; single, direct and inverse tables; and retrieval and velocity.
- *Numelline* (Riccardi-Ripamonti & Ripamonti, 2007): This is designed for 8- to 11-year-olds. It comprises a series of ten games (Memory, Countdown, etc.) designed to help children to memorize multiplication tables.

Some materials developed in French are:

- *CalculaTICE*: This is a website (http://calculatice.ac-lille.fr/calculatice/) designed by the Ministry of Education, the Department of the North, and the Sésamath Association (http://www.sesamath.net/) devoted to the public utilization of information and communication technologies (ICTs) in mathematical learning (this association has also developed related apps for the iPad). It features exercises adapted for each educational level from cours préparatoire (CP) to CM2 (ages 6–11), such as memorizing addition and multiplication tables; adding and subtracting unities, tens, or hundreds; division; magnitude estimation; calculating halves, doubles, thirds, triples, fourths; problem solving, etc.
- *L'Attrape-Nombres* [Number Catcher] (designed by Dehaene and colleagues; http://www.attrape-nombres.com/an/home.php): This precursor of the Number Race is aimed at 5- to 11-year-olds, especially (but not necessarily) those with mathematics difficulties. In this case, the content is focused on two-digit numbers. The activities include basic calculation, number presentation, and activities using base-10 and multidigit numbers.

In general the use of ICTs is encouraged by national guidelines, but so far their use has not been generalized, even for MLD remediation (Eurydice, 2011).

Conclusions

The European Community was built as a strong socioeconomic frame within which the member countries share similar concepts, references, and guidelines.

However, recent policy changes and the decentralization of these countries have probably contributed to the current absence of an overall policy, national or European, for dealing with specific difficulties in mathematics, although national plans do exist for the promotion of other essential skills such as reading.

Considering this situation, Europe, and especially the Southern European countries, have some work to do: teachers and other educational staff should have specific regulated training in MLD, and there should be a well-known, generally recognized chronology of mathematical milestones to rapidly detect atypical developmental trajectories. Further, it is essential for health and education professionals to work together with families. Given its importance, math anxiety should also be included in policies and in generalized assessment tools.

At the same time, all the participants in this process should be guided by comprehensive and updated policies and provided with the necessary support and materials. Guidelines should be based on long-term decisions with a clear final goal—to always encourage optimal development for every person, whatever his/her personal conditions are—and this objective should not be subordinated to gender, economic circumstances, or geographical circumstances.

There also should be a generalized aim (which does not currently exist) to decrease inequalities within and between countries. For this purpose, education is the best tool for providing basic skills and integrating migrants, refugees, and people at different socioeconomic levels. Every person can make a valuable contribution to society, and any educational difficulty without treatment will have a long-term cost, both for the person and for society.

Research and evidence should be the cornerstones for both professionals and policies, and should be shared among educational professionals. Similarly, families should have access to information on normal development and how to detect alert signs.

To conclude, more effort is still needed to promote further research and specifically to facilitate the transfer of such findings to new assessment tools and intervention programs, as well as to common policies aimed at improving mathematical achievement in Southern European countries.

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Chapter 12
Mathematical Learning and Its Difficulties in the United States: Current Issues in Screening and Intervention

Nancy C. Jordan, Luke Rinne, and Nicole Hansen

Mathematical Learning and Its Difficulties in the United States: Best Practices for Screening and Intervention

Results from recent cross-national comparative studies indicate that despite spending more per student than many other countries, the United States performs below average in mathematics, ranking in the bottom half of countries in the Organization for Economic Co-operation and Development (OECD, 2012). This result, however, does not provide a complete picture of US education. There are significant socioeconomic differences across and within states, which explain about 15% of variation in student performance (OECD, 2012).

Figures 12.1 and 12.2 depict the percentage of fourth- and eighth-grade students, respectively, who performed below the basic level on the US National Assessment of Educational Progress (NAEP), overall and broken down by selected states and public versus private schools. Although the percentage of students who are struggling has gone down since 1992, there are substantial achievement differences, depending on state and geographic region. In Massachusetts, for example, there is less poverty than in Mississippi; in 2015 80% of the students in Massachusetts met standards versus only 50–60% in Mississippi.

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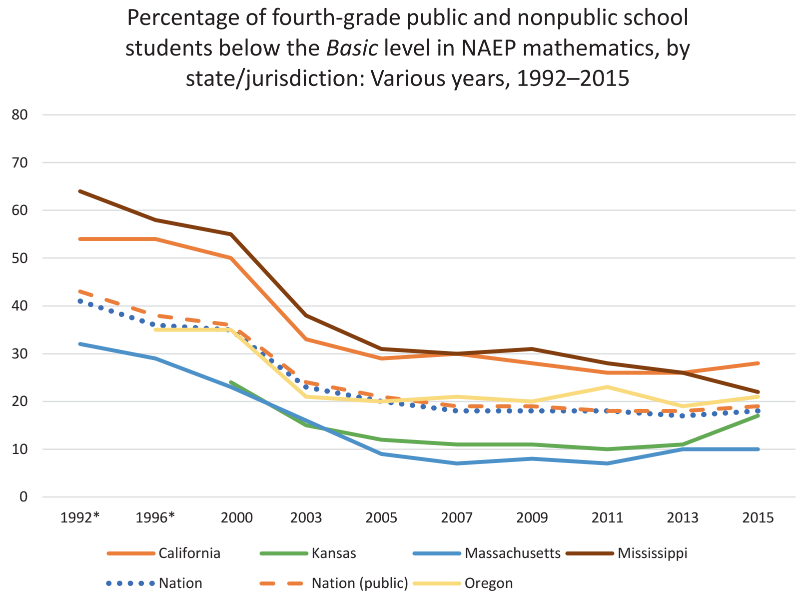
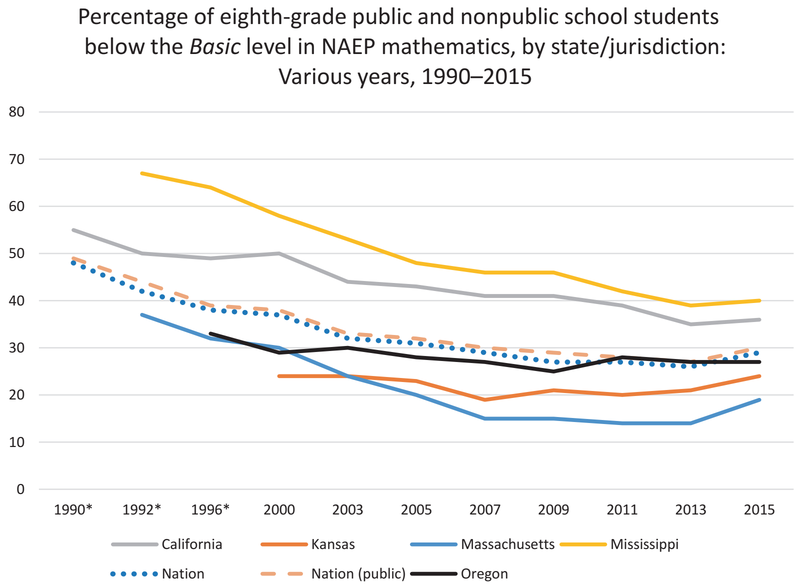


* Note. 1992-1996 Accommodations not permitted

Fig. 12.1 The percentage of fourth-grade public and nonpublic school students below the *Basic* level in NAEP mathematics. *Note: 1992–1996 accommodations not permitted. (Source: The National Assessment of Educational Progress (NAEP))



* Note. 1992-1996 Accommodations not permitted

Fig. 12.2 The percentage of eighth-grade public and nonpublic school students below the *Basic* level in NAEP mathematics. *Note: 1992–1996 accommodations not permitted. (Source: The National Assessment of Educational Progress (NAEP))

Historically, the United States has differed from other developed nations in that control of education policy—including that related to mathematics—has tended to be highly decentralized (Woodward, 2004). There is no US national curriculum, resulting in a high level of state and local control over what is taught in school. As such, there is significant variation in instruction, both across and within states. However, 42 of the 50 states and the District of Columbia have now voluntarily adopted the national Common Core State Standards (CCSS; Council of Chief State School Officers & National Governors Association Center for Best Practices, 2010), which specify in relative detail the mathematical content to be covered as well as standards for student learning. The CCSS, however, are controversial, and to date, their long-term impact on student achievement remains uncertain.

In terms of special education, US federal law, under the 1975 Education of all Handicapped Children Act, mandates that all children and youth with disabilities, including those with learning disabilities in mathematics, receive a free and appropriate education, including nondiscriminatory evaluation and an individual education plan. The Individuals with Disabilities Education Improvement Act (IDEIA) of 2004 eliminated the law's original requirement to consider whether children exhibit a severe discrepancy between achievement and intelligence, leading to the broad implementation of alternative response to intervention (RTI) approaches. RTI approaches screen broadly for academic problems and then provide evidence-based interventions aimed at helping individual students, tracking progress along the way to gauge effectiveness. Still, specific methods for assessment are typically established in a localized manner at school or school district levels, meaning that there is a high degree of variation in screening procedures and types of interventions provided to children with or at risk for disabilities.

In a widely cited article, Gersten, Jordan, and Flojo observed in 2005 that research on early screening for mathematics difficulties and disabilities in the United States was in its "infancy" (p. 293). In contrast, extensive research had already been conducted on early screening for reading difficulties, which produced reliable measures that could accurately predict which students would have trouble learning to read. The reading screeners helped US schools provide research-based literacy support and intervention for kindergarten and first-grade students and, to a large extent, drove the RTI movement in US special education. On the other hand, there was far less research on screening for potential mathematics difficulties and a relatively small corpus of evidence-based mathematics interventions.

Since that time, however, the field of mathematics learning difficulties in the United States has advanced significantly through various theoretical studies that identify the most powerful predictors of and influences on mathematics learning difficulties (MLD; e.g., Berch & Mazzocco, 2007; Geary, Hoard, Byrd-Craven, Nugent, & Numtee, 2007). Further, studies have validated screeners for detection of potential difficulties in mathematics (e.g., Jordan, Glutting, Dyson, Hassinger-Das, & Irwin, 2012), and rigorous intervention studies have helped determine best practices for young students with or at risk for MLD (e.g., Fuchs et al., 2008; Gersten, Jordan, & Flojo, 2005).

In addition to developing early screeners and interventions to help students acquire whole number competencies or number sense, recent studies have also focused on learning rational numbers (e.g., fractions) in later grades. Typically, fractions are introduced in US mathematics in third grade (Council of Chief State School Officers & National Governors Association Center for Best Practices, 2010). Both whole number and rational number knowledge are crucial aspects of mathematics education and are necessary for later success in mathematics as well in everyday life (Gersten et al., 2009).

In the present chapter, we highlight key contributions from relatively recent studies related to whole number understanding in the early grades and fraction understanding in the intermediate grades. Although not comprehensive, the contributions reflect research-based findings related to MLD that are currently influencing educational practice in the United States.

Early Number Competencies

Early Number Competencies Predict Future Mathematics Success, and Deficiencies in Number Concepts Underlie Many Mathematical Learning Difficulties

Early mathematics skills correlate with long-term outcomes. Independent of cognitive ability and social class, kindergarten mathematics concepts predict later learning outcomes not only in mathematics but also in reading (Duncan et al., 2007). Most US benchmarks (e.g., Council of Chief State School Officers & National Governors Association Center for Best Practices, 2010) for kindergarten and first grade primarily concern knowledge of number, including number relations and operations, forming a foundation on which later mathematics content is built (National Research Council, 2009). Mathematics delays as early as kindergarten and first grade put students at risk for difficulties in acquiring mathematics concepts in subsequent grades, including fractions and algebra (Mazzocco & Thompson, 2005; Milgram, 2005; Wu, 1999). Poor number sense also leads to dependence on rote memorization, which in turn makes it harder later on for students to develop meaningful problem-solving skills (Locuniak & Jordan, 2008; Robinson, Menchetti, & Torgesen, 2002).

Kindergarten number sense performance and growth, in particular, predict mathematics achievement in elementary school (Jordan, Kaplan, Ramineni, & Locuniak, 2009; Jordan, Glutting, & Ramineni, 2010; Locuniak & Jordan, 2008). Unfortunately, many children from low-income communities in the United States enter kindergarten showing delays in core number knowledge relative to their middle-income peers (Jordan, Kaplan, Olah, & Locuniak, 2006; Jordan, Kaplan, Locuniak, & Ramineni, 2007; Starkey, Klein, & Wakeley, 2004); additionally, they are four times more likely than middle-income children to show little to no growth in number knowledge between kindergarten and first grade (Jordan et al., 2006, 2007). Jordan et al. (2007) found that number sense performance in kindergarten and rate of number sense growth from kindergarten to early first grade accounted for about two thirds of the variance in general mathematics achievement at the end of first grade. Importantly, income status did not add explanatory variance after controlling for performance and growth in number knowledge. That is, the poor mathematics achievement of low-income learners was largely accounted for by their weak number knowledge. This finding is significant in that number competencies can potentially be changed through intervention, unlike income status, which is relatively immutable.

Core Number Competencies for Early Screening Involve Knowledge of Number, Number Relations, and Number Operations

A wide variety of number competencies have been targeted for early screening (Jordan & Dyson, 2016; Jordan, Resnick, Rodrigues, Hansen, & Dyson, 2016; Malofeeva, Day, Saco, Young, & Ciancio, 2004; National Research Council, 2009; Rittle-Johnson & Jordan, 2016). In US prekindergarten, kindergarten, and first-grade classrooms, screening has often focused on verbal number sense, that is, abilities related to the symbolic representation of numbers, as opposed to more fundamental nonsymbolic numerical representations (e.g., ANS or approximate number system), which appear to develop without much verbal input or instruction (Feigenson, Dehaene, & Spelke, 2004; Jordan & Levine, 2009). Each screening area is discussed next.

Number. Young children recognize small quantities through subitizing (Baroody, 1987; Baroody, Lai, & Mix, 2006), which involves apprehending and labeling the numerical value of two or three objects without having to count them. Counting, in turn, expands the child's quantitative understanding beyond small sets. Before formal schooling, many children can easily recite the count sequence to ten and higher. Later, children learn to enumerate sets in one-to-one correspondence with counting numbers, recognizing that the last number counted indicates the number of objects in the set (i.e., cardinality principle; Gelman & Gallistel, 1978). Children discover that they can count any set presented in any configuration, so long as they count each object once in numerical order (Gelman & Gallistel, 1978). Children also learn to recognize and produce written number symbols (Arabic numerals 1, 3, 5, etc.) (National Research Council, 2009). In kindergarten, many US children become familiar with the decade words and learn that two-digit numbers represent tens and ones. Persistent difficulties with counting are a characteristic of older children with MLD (Geary, 2004).

Number relations. Understanding the magnitudes of numbers is a key developmental achievement (Case & Griffin, 1990; Griffin, 2002, 2004; Siegler, Thompson, & Schneider, 2011). Recognizing that four objects is more than three objects—or that two objects is fewer than five—reflects understanding of magnitude relations early in development. Later in prekindergarten, children can make judgments about quantities in the absence of physical objects, through mental counting or external representations, such as the number line. Children learn that as they move to the right on the line, numbers represent larger quantities, while moving left is associated with decreasing quantities. Eventually, children learn that each number in the count list is exactly one more than the previous one. Linking abstract representations to observed numerical magnitudes is critical for the development of mathematical ability; deficits in the ability to draw such connections are associated with MLD (Rousselle & Noël, 2007).

Number operations. Many preschoolers successfully solve simple addition and subtraction problems using physical representations (Levine, Jordan, & Huttenlocher, 1992). Even children with limited counting facility can solve problems with sums or minuends of four or less (Huttenlocher, Jordan, & Levine, 1994). Early on, counting (e.g., counting fingers) is a key strategy for solving addition and subtraction problems with sums and minuends of five or more. Knowing that the next number in the count sequence is always one more than the preceding number enables children to compute the value of $n + 1$ (Baroody, Eiland, & Thompson, 2009). By the end of kindergarten, many children can count on from the first or larger addend to find the sum of two numbers (e.g., for 4 + 3, the child counts 5, 6, 7 to get 7). This approach is more efficient than counting out both addends (Baroody et al., 2006). Kindergartners who use counting principles to evaluate number combinations develop calculation fluency earlier in school (Jordan et al., 2009).

Children must also learn that whole numbers can be decomposed into sets of smaller numbers. For example, 4 can be broken into either 1 and 3 or 2 and 2. Along with quantity discrimination, number line estimation, counting, and number word comprehension, kindergartners' ability to identify different combinations that equal a given sum predicts growth in mathematics achievement from kindergarten through second grade (Fuhs, Hornburg, & McNeil, 2016). Children with strong mathematics skills use their knowledge of number sets to derive solutions for new combinations (e.g., if 1 + 3 = 4, then 2 + 3 = 5). However, young children with or at risk for mathematics difficulties have trouble counting on from a number, decomposing numbers, and deriving solutions from known combinations to help them calculate totals of 5 or more. These difficulties lead to poor addition and subtraction skills (Jordan et al., 2006).

Deficits in Number Sense Can Be Reliably Identified Through Early Screening, and Interventions Based on Screening Lead to Improved Mathematics Achievement in School

Gersten et al. (2012) evaluated the predictive validity of early number screeners developed by researchers. Screeners assessing number relations (e.g., Clarke, Baker, Smolkowski, & Chard, 2008; Jordan et al., 2008; Seethaler & Fuchs, 2010) and number operations (e.g., Jordan et al., 2010; Seethaler & Fuchs, 2010) have been especially effective in predicting later mathematics performance. These screening measures demonstrate high classification accuracy (Geary, Bailey, & Hoard, 2009; Jordan et al., 2010; Seethaler & Fuchs, 2010), accurately identifying children who will later need additional help in mathematics (Gersten et al., 2012). Moreover, measures assessing numerical magnitudes are sensitive diagnostic tools for identifying children with dyscalculia, a severe form of MLD (Reigosa-Crespo et al., 2012).

Importantly, there is clear evidence that core number competencies can be improved in most US children (Frye et al., 2013). At the prekindergarten level, experimental studies reveal meaningful effects for interventions that emphasize number sense (Baroody et al., 2009; Clements & Sarama, 2007, 2008; Dobbs, Doctoroff, & Fisher, 2003; Klein, Starkey, Sarama, Clements, & Iyer, 2008). Jordan and colleagues (Dyson, Jordan, & Glutting, 2011; Jordan & Dyson, 2016; Jordan et al., 2012) developed and tested a kindergarten number sense intervention that specifically targets skills with number, relations, and operations—competencies that underlie mathematics difficulties, as described in the previous section. Study participants were at-risk kindergartners who were from low-income communities and/or performed poorly on a number screener. Results from a series of randomized experiments showed that children in the intervention group consistently exhibited greater improvement in terms of both a proximal measure of number sense and a general mathematics achievement test compared to control children who received a language intervention or business-as-usual instruction (Jordan & Dyson, 2016). Of particular significance was the finding that many of the intervention gains held over time, and the achievement gap between intervention children and their normally achieving counterparts decreased substantially. Clarke et al. (2016) report comparable findings from a kindergarten intervention focused on whole number knowledge.

In sum, recent research has highlighted the importance of early number competencies or number sense for future mathematics success and has identified useful targets for intervention that are being used in US schools, such as skill with number relations and operations. Compared to basic cognitive abilities or socioeconomic status, number sense appears to be relatively malleable, and interventions targeting children identified through early screening lead to improved mathematics achievement. Current US practices in early mathematics education are continuing to be revised in concert with what researchers have learned about the sources of early

MLD risk and the effectiveness of early screening measures and interventions. Many research-based early number interventions are being incorporated under RTI models for assessment and intervention.

Fractions

Fraction Knowledge in the Intermediate Grades Predicts Algebra Success in Secondary School, and Weaknesses with Fractions Characterize Middle School Students with Mathematical Learning Difficulties

Whereas having a good sense for whole numbers is central in primary mathematics education, competency with fractions is the hallmark mathematics achievement in intermediate grades in the United States (Council of Chief State School Officers & National Governors Association Center for Best Practices, 2010). Fraction knowledge in middle school predicts subsequent performance in algebra, over and above socioeconomic status, IQ, and whole number abilities (Siegler et al., 2012). Relative to research on whole number knowledge, however, few studies have focused on the development of fraction competencies until recently.

Fractions typically afford students their first opportunity to learn about numbers with properties that differ from those of whole numbers (Siegler & Pyke, 2013). Many US students, especially those with MLD, struggle with basic knowledge of fractions (e.g., Bailey, Hoard, Nugent, & Geary, 2012; Ni & Zhou, 2005; Hansen, Jordan, & Rodrigues, 2017). These difficulties extend past the intermediate grades—students in middle and high school—and even some college students have trouble with basic fractions tasks, such as ordering simple fractions from least to greatest and estimating sums of two fractions (Siegler & Pyke, 2013). For example, when asked to estimate the sum of 12/13 + 7/8 from the response options 1, 2, 19, and 21, 15% of college students at a major US university estimated the sum to be either 19 or 21 (Lewis & Hubbard, 2015). That is, students tended to add together either the numerators or denominators of the fraction, overgeneralizing whole number properties to fractions. Despite errors such as these, whole number knowledge is helpful for learning about fractions. In fact, many students who struggle with fractions have concomitant difficulties with whole numbers, particularly with respect to judging numerical magnitudes (Jordan et al., 2016). Understanding numerical magnitudes with whole numbers provides a foundational structure for thinking about fractions in terms of magnitudes (Case & Okamoto, 1996; Siegler & Lortie-Forgues, 2014; Siegler et al., 2011). As such, effective whole number sense interventions, such as those described previously, may be crucial for building a general understanding of numerical magnitudes that can later be applied to fractions.

Fractions Are Especially Hard for Children with MLD

As noted, many students with or at risk for MLD have poorly developed fraction knowledge (Fuchs et al., 2013). Because children with MLD tend to lack a sound understanding of number magnitudes, many are unable to move beyond the erroneous assumption that properties of whole numbers are true for all numbers in general (Ni & Zhou, 2005; Jordan, Rodrigues, Hansen, Resnick, & Dyson, 2017; Siegler et al., 2011). In contrast to whole numbers, which each directly correspond to one and only one magnitude, have unique successors, and are expressed as a single symbol, different fractions may have the same magnitude and therefore refer to the same location on a number line (1/4 is the same as 2/8 or 4/16). The magnitudes of fractions do not always change in consistent ways with the absolute values of their numerators and denominators (Schneider & Siegler, 2010). For example, 4 is greater than 2, and 12 is greater than 4, but 4/12 is a smaller fraction than 2/4.

When children first start learning fractions, a common misconception is that larger numbers produce larger fraction values in all cases, regardless of whether they appear in the numerator or the denominator (Rinne, Ye, & Jordan, 2017). For example, a child may erroneously think that 1/12 is larger than 1/5 because 12 is larger than 5. Instruction leads some students to develop a partial misconception that smaller values in both denominators *and* numerators decrease fraction magnitudes, but this is usually just a stepping stone on the way to a normative understanding. Eventually, successful students come to understand that numeral values can be inversely related to fraction magnitudes, but this is only true for the denominator. However, Rinne et al. further showed that children who come to fraction instruction with a poor understanding of whole number magnitudes are much less likely to move beyond the simple view that larger numerals always lead to larger magnitudes. Thus, for children with MLD, a lack of whole number magnitude understanding impedes the ability to grasp fraction concepts.

Further problems arise when struggling children begin to learn about fraction operations. For example, multiplication of two fractions may yield a product smaller than either multiplicand, while multiplication of whole numbers greater than one always produces a larger product. A failure to understand numerical magnitudes also produces fraction operation errors that *do not* appear to derive from overgeneralizations of whole number properties. For example, students often mistakenly apply the procedure for fraction addition to fraction multiplication problems and leave the denominator unchanged rather than multiplying across both the numerator and denominator (Siegler & Pyke, 2013). Significantly, the one property that bridges whole numbers with fractions—and might thereby serve as a touchstone for helping students overcome such difficulties—is that both fractions and whole numbers have magnitudes that can be represented on a number line (Case & Okamoto, 1996; Siegler & Lortie-Forgues, 2014; Siegler et al., 2011).

Because they Lack Magnitude Understanding, Students with MLD Struggle to Place Fractions on a Number Line

The implications of poor magnitude understanding are also evident in research on fraction number line estimation. Resnick et al. (2016) examined the development of fraction number line estimation on 0–1 and 0–2 number lines between fourth and sixth grade, uncovering three distinct growth trajectory classes: (1) students who are highly accurate from the start and became even more accurate, (2) students who initially are inaccurate but show steep growth, and (3) students who initially are inaccurate and show minimal growth. Growth class membership accurately predicted subsequent performance on a standardized mathematics achievement test at the end of sixth grade, even after controlling for mathematics-specific abilities, domain-general cognitive abilities, and demographic variables. Students falling into the minimal growth class tended to place both proper and improper fractions below one on a number line, suggesting they do not effectively consider the relation between numerator and denominator. Multiplication fluency, classroom attention, and whole number line estimation acuity at the start of the study predicted class membership, indicating these areas make important contributions to learning fractions, and deficits in these areas may impede learning.

Fraction Difficulties Can Be Reliably Identified by Fourth Grade

Rodrigues et al. (2016) evaluated the diagnostic accuracy of mathematics screening measures (starting in fourth grade) for predicting MLD at the end of sixth grade. Receiver operating characteristic (ROC) curve analyses showed that of a broad group of fraction and general mathematics ability measures, fraction number line estimation acuity and knowledge of fraction concepts emerged as the strongest predictors of who would go on to fail a mathematics achievement test at the end of sixth grade. These measures were significantly more accurate predictors of sixth-grade mathematics failure than were measures of fraction procedures and multiplication fluency, both of which typically receive much more attention in instructional settings.

Fraction Difficulties Can Be Improved Through Meaningful Interventions that Center on the Number Line

Referring to current mathematics instruction in the United States, Gersten and Jordan (2016) observe the following, despite between- and within-state variation in the United States:

> Perhaps the most profound change in contemporary mathematics instruction for students in the elementary grades has been a strong emphasis on mastery of concepts involving fractions. This change, reflected in virtually all contemporary state standards, involves not only a shift in the amount of time dedicated to teaching fractions but also a shift in emphasis. Mathematics instruction is now making fraction concepts, most notably fraction magnitude, take priority over fraction procedures (p. 1).

This change is having a significant effect on instruction for students with MLD, as evidenced by new research showing that interventions that focus on representing fraction magnitudes on number lines lead to improved mathematics outcomes. Fraction number line activities require students to think about proportionality and to reason multiplicatively; both skills represent important underpinnings of fraction conceptual knowledge (Hansen et al., 2015; Vukovic et al., 2014). Until recently, a part-whole interpretation of fractions has been a pervasive influence in the US mathematics curriculum (Siegler, Fuchs, Jordan, Gersten, & Ochsendorf, 2015). However, in a series of experimental studies that used the number line as a basis for helping students evaluate magnitude (sometimes referred to as a measurement approach), Fuchs et al. (2016) showed that low-performing fourth graders can learn to determine the magnitudes of fractions, and this knowledge transfers to other fraction skills, including arithmetic.

To date, our research team (Dyson, Jordan, Rodrigues, Barbieri, & Rinne, in preparation; Jordan et al., 2016; Rodrigues, Dyson, Hansen, & Jordan, 2017) has conducted several experimental trials of an intervention for sixth and seventh graders who persistently struggle with fractions even after several years of typical classroom instruction. Our “fraction sense” intervention, which is centered on the number line, aims to build fundamental understandings of (1) the meaning of a fraction (how the numerator and the denominator work together to determine a fraction’s magnitude), (2) fraction relations (how the magnitudes of fractions are ordered on the number line), and (3) fraction operations (how fractions are added, subtracted, multiplied, and divided). Thus, this model of instruction is partly analogous to the whole number sense model described earlier in this chapter.

To develop core fraction knowledge based on just few key ideas, the three topics described above are taught using fractions with a narrow range of denominators. For example, we start with denominators of 2, 4, and 8 and gradually expand to include denominators of 3, 6, and 12. In addition, the intervention anchors ideas in a meaningful story line to help struggling learners think about fraction concepts in a more concrete way (Bottge et al., 2014). Specifically, instruction takes place in the context of a “color run” race for charity during which runners have colored powder thrown at them at regular intervals during the race. The race context facilitates thinking about fraction magnitudes using a measurement interpretation (e.g., finding fractions of a mile), and the number line helps students see relations between fractions with both different and equivalent magnitudes. Children are asked to compare the relative sizes of numerators and denominators and to think about fractions as being close to 0, close to 1, equal to 1, or greater than 1. The intervention also applies general learning principles from cognitive science by incorporating gestures that guide students' attention (Alibali, Spencer, Knox, & Kita, 2011), side-by-side comparisons of solution methods (Rittle-Johnson, Star, & Durkin, 2009), instructional explicitness (Gersten et al., 2009), and clear visual models to minimize cognitive load (Fuchs et al., 2009). Practice activities mix problems with more and less familiar fractions to develop fluency and improve retention (Carpenter, Fennema, & Romberg, 2012). Finally, fast-paced games help build both whole number and fraction fluency at the end of each lesson.

Although our intervention work is ongoing, preliminary findings have been positive. Participants (who were identified by their teachers as needing intervention or who performed below a predetermined cut-point on a reliable screener of fraction concepts) were randomly assigned to our intervention or a business-as-usual intervention contrast group. Children who received the intervention performed reliably better than controls, with large effect sizes on measures of fraction number line estimation, as well as more general fraction conceptual knowledge. For the most part, students maintained these gains on a delayed posttest administered 2 months after the conclusion of the intervention.

Overall, recent intervention work with fractions reveals that interventions that focus on fraction magnitude and that use the number line as a representational guide hold promise for helping all students learn fractions. The number line approach is likely to gain traction in US schools, including special education. In fact, the US benchmarks in math (i.e., CCSS) emphasize the use of the number line to teach fractions, starting in third grade. Future work is needed, however, to examine whether such interventions can help students succeed with respect to longer-term outcomes, such as algebra proficiency and using fractions in daily life.

Conclusion

In the early elementary years, the primary goal of mathematics instruction in the United States is to build children's number sense with whole numbers. Research shows that a good understanding of whole number magnitudes is critical for later facility with fractions, mastery of which is a key accomplishment in the intermediate grades. Failure to master fractions has severe long-term consequences for student success in mathematics, limiting eventual prospects for employment and leading to poor decisions in the increasingly number-rich environments of everyday life.

Acquiring both whole number and fraction knowledge is particularly challenging for students with MLD and thus a major educational concern in American schools, particularly in light of recent shifts in curriculum and standards (i.e., CCSS) toward deeper conceptual understanding of mathematics. One challenge that remains is how to balance the needs of students with MLD with these more rigorous standards; many US students with MLD have weak number sense and subsequent difficulty representing fractions as magnitudes on a number line, which prevents them from incorporating fractions and whole numbers into a coherent understanding of the rational number system. Fortunately, recent research suggests that both early difficulties with whole numbers and later difficulties with fractions can be remediated by helping students build solid magnitude representations, and interventions focused on representing fractions along with whole numbers on number lines lead to improved mathematics outcomes.

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Chapter 13
Mathematical Learning and Its Difficulties in Latin-American Countries

Beatriz Vargas Dorneles

Introduction

The aims of this chapter are to analyze the performance of Latin American students in mathematics and to describe some policies designed to identify and attend the needs of low achievers in mathematics in Latin America. As Latin America comprises 21 very different countries (not including the Caribbean countries)—with diverse languages, cultural traditions, educational systems, and policies—we summarize the situation of the eight Latin American countries (Argentina, Brazil, Chile, Colombia, Costa Rica, Mexico, Peru, and Uruguay) that participated in the Programme for International Student Assessment (PISA), especially in 2009, 2012, and 2015, the latest assessment available for inclusion in this chapter. This chapter only presents a brief summary of the subject due to the sheer volume of data available and limited space. Consequently, at times, we have highlighted data from the Brazilian context at the expense of data from other countries.

Approximately 22% of European students are low achievers in mathematics (EACEA/EURYDICE, 2011) and need some kind of special teaching to learn mathematics. Numerical difficulties are linked to lack of progress in education, increased unemployment, reduced job opportunities, and additional costs in mental and physical health (Duncan et al., 2007). This situation has been described in many countries. As an example, in 2011 a Department for Business, Innovation and Skills survey in the UK found that 49% of the adult population could only attain standards comparable to those of 11-year-old children in mathematics. Furthermore, 23.7% of adults only reached the mathematical standards typical for 9-year-old children, compared to 7.1% for literacy (Department For Business Innovation and Skills, 2012).

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Table 13.1 Evolution of student performance in mathematics and percentages of low performers in mathematics in eight countries in Latin America

Country	Student performance in mathematics (mean score)			Low performers in mathematics (% of students scoring below level 2)		
	2009	2012	2015	2009	2012	2015
Chile	421	423	432	51.0	51.5	49.4
Mexico	419	413	408	50.8	54.7	56.6
Uruguay	427	409	418	47.6	55.8	52.4
Costa Rica	409	407	400	56.9	59.9	62.5
Brazil	386	389	377	69.1	68.3	70.3
Argentina[a]	388	418	456	66.3	66.5	56.0
Colombia	381	376	390	70.4	73.8	66.3
Peru	365	368	387	73.5	74.6	66.2
OECD average	499	494	490	31.4	23.0	23.4

OECD Organization for Economic Co-operation and Development

[a] The coverage is too small to ensure comparability; it includes only Buenos Aires

We can infer that there is a worse situation in Latin American countries by analyzing the data from the latest PISA (OECD, 2014, 2016a). Some countries, such as Brazil and Peru, have only recently managed to introduce universal education for children aged from 7 to 14. Now the emphasis is on improving the quality of teaching systems. Documents prepared by the Organization for Economic Co-operation and Development (OECD) (2013a, 2013b, 2013c, 2014, 2016a, 2016b) about the situation, together with others based on that material, are analyzed in this chapter (for example, Rivas (2015)). We focus on the mathematical performance of children in the eight Latin American countries that participated in PISA, a test that evaluated 15-year-olds in mathematics, reading, and science in 65 educational systems in 2009 and 2012, and in 70 educational systems in 2015. These countries are very diverse in terms of culture, language, and customs. In terms of student performance in mathematics, the situation is summarized in Table 13.1.

Considering the OECD average scores in mathematics and the OECD average scores of low performers in mathematics, and analyzing the OECD documents about the situation of mathematics learning in the region (OECD, 2013a, 2013b, 2013c, 2014, 2016a, 2016b), we reached three particularly relevant conclusions. First, the situation varies across the Latin American countries; for example, in 2015 approximately 50% of students in Chile were very low achievers in mathematics (scoring less than 2 on the PISA scale), while in Brazil the figure was 70%. The results are more worrying if we remember that the OECD only considers students who achieve level 2 (out of 6 levels) or higher to be capable of full participation in modern societies. Students who achieve only level 0 or level 1 are incapable of using basic mathematical concepts, procedures, or rules to solve simple basic number problems. If we consider a longer time period, most of the Latin American countries have seen no significant improvement in mathematical performance in the last 12 years (OECD/CAF/ECLAC, 2016). The exceptions are Mexico and Brazil.

Secondly, most of the countries maintained their previous results until 2012, with the exceptions of Mexico and Uruguay. In 2015, there was a general decline in mathematical performance in almost all the OECD countries, from an average score of 494 in 2012 to 490 in 2015. In Latin America, there was an improvement in student performance in mathematics in Chile, Uruguay, Argentina, Colombia, and Peru, while the results from Brazil, Mexico, and Costa Rica worsened. If we only analyze the low performers, the same tendency is repeated: fewer students at levels 0 and 1 in Uruguay, Argentina, Colombia, and Peru; and more students in the countries with worse results. Thirdly, we can conclude that a large number of students are still failing to achieve level 2. It is a very different situation from that of the European and Asian countries. However, Rivas (2015) reminds us that comparing Latin American countries with European and Asian countries is not useful, considering that the gross domestic product (GDP), social conditions, and investment per student in Latin America are considerably lesser than in the other regions participating in PISA tests. Consequently, the results obtained in Latin America can be seen to reflect a "development debt in broader terms and may not be ascribed to a failure of education systems themselves" (Rivas (2015), p. 18). The achievements described in Table 13.1 are particularly remarkable if we consider the expansion of access to education and improved performance among part of the students in recent decades.

About the Region

Generally, the last decade has seen widespread economic and social growth in Latin American countries. The percentage of GDP spent on education jumped from an average of 4.04% in 2000 to 5.44% in 2011. Accordingly, Argentina, Brazil, and Uruguay saw the highest increases in investment in education, with Chile, Colombia, and Mexico increasing educational expenditure at lower rates. By contrast, in Peru the education budget increased significantly but remained in line with its GDP (Rivas, 2015). Despite the regional and educational differences existing among the countries, we can describe the last 20 years as being marked by a transition from decentralization, which was the prevailing tendency in the 1990s, to a recentralization of power in ministries of education in the 2000s. More prescriptive curricula with extensive content have been introduced along with concrete guides for teachers and provision of textbooks (Rivas, 2015).

In the following paragraphs we summarize some educational achievements of those countries and describe the consequences for mathematics education, specifically, in seven of the eight countries listed above. We exclude Argentina, in which data were collected only in the city of Buenos Aires.

While Chile has the best overall social indicators among the countries described, and Mexico has experienced economic and social stagnation since 2000, both achieved the best results in PISA in the region.

The Uruguayan education system is highly centralized, both in terms of distribution of responsibilities across levels of governance and in terms of geography.

Almost all the administrative and pedagogical decisions are taken at the central level (Llambi, Castro, Hernandez, & Oreiro, 2015). Uruguayan repetition rates in the primary and secondary educational levels are the highest among the surveyed countries, although they were reduced in primary education (Rivas, 2015). While the quality of educational resources in Costa Rica is among the highest in the PISA-participating countries and economies (OCDE, 2016), the country experienced an increase in the number of mathematical performers scoring below level 2.

There is nearly one computer for every student in Colombia—a higher ratio than that observed across OECD countries (on average), higher than that observed in Chile and Peru, and higher than would be expected given Colombia's level of spending on education. Colombian schools have less autonomy than the average school in OECD countries (OECD, 2016b). For instance, in Colombia, principals and teachers have 24% of the responsibility for selecting resources, compared to 42% of that responsibility across OECD countries, and they have 31% of the responsibility for student assessments, compared to 68% across OECD countries (OECD, 2016a).

Peru presented the worst results in mathematics in Latin America in 2012 but improved in 2015. There was a reduction in the number of low performers in 2015 despite the fact that teachers tend to focus on teaching numbers and arithmetic, usually assigning students mechanical exercises with low cognitive demand. Few exercises require students to solve problems (Cueto, 2013).

Brazil's per-capita GDP (USD 15,893 (OECD, 2016a)) is less than half the OECD average GDP (USD 39,333). Brazil is the most populous country in the region. Its educational structure is decentralized in 27 states and around 5565 municipalities. Brazil has introduced some policies aimed at improving teacher training and teaching quality in general, and developing programs for students with special needs in some cities in the country. This effort produced a significant result: Brazil obtained the greatest improvement in mathematics from 2000 to 2012, considering the 65 participating countries. At levels 0, 1, and 2 the improvement was 35 points, equivalent to two thirds of a year of schooling, and it especially affected the economically poorest and low achievers (Bos, Ganimian, & Vegas, 2014). The OECD indicators most related to the improvement observed in the Brazilian results from 2000 to 2012 were the improvement in teaching resources, the increased number of qualified teachers, and the increased use of assessment tests to make decisions, to compare the results between schools and students, and to evaluate teachers (Bos et al., 2014). We conclude that Brazil's improved results in mathematics were related to a general improvement in education rather than policies specifically designed to help students who are low achievers in mathematics. Despite this improvement, Brazil did not maintain this tendency in 2015, when there was a decrease in mathematical performance and an increase in the percentage of low performers in mathematics, reaching 70%, the worst performance in Latin America among the countries that participate in PISA. Even though the average mathematics score increased by 21 points between 2003 and 2015 (a significant increase of 6.2 points every 3 years), the most recent period (2012–2015) saw a decline of 11 points in the mean performance in mathematics (OECD, 2016a).

The tendency to consider national and international assessments when planning national educational policies has been described in various reports. It is worth mentioning that not all countries follow this tendency in the same way. While Argentina, Uruguay, and Peru scarcely consider these evaluations in the development of their educational policies and do not use the results to apply pressure on schools, Mexico and Chile have used student tests to develop economic incentives for schools and teachers, and to review the impact of those policies in recent years (Rivas, 2015).

Theories and Educational Practice

In the 1990s, the Nunes study (1992) analyzed informal mathematical practices conducted in specific out-of-school activities among Brazilian street venders that may be contrasted with "school mathematics." This research had a great influence on Brazilian and Latin American schools, and helped teachers understand what happens when children know how to use mathematical knowledge in everyday situations and fail to use the same knowledge in school. The dissemination of the results of this research is part of an expansion in the influence of the constructivist approach vis-à-vis the traditional way of teaching mathematics.

In many countries in the region, there has been a widespread tendency in educational theory to adopt teaching practices that combine the competence and constructivist approaches (Rivas, 2015). At the same time, there is some local theoretical debate about the relative value of adopting the constructivist or social interactionist approaches when teaching mathematics in specific groups that show unexpected mathematical learning difficulties, such as deaf students (Arnoldo Jr, Ramos, & Thoma, 2013; Barbosa, 2013). However, this theoretical academic debate has had little impact on schools.

Concerning the use of textbooks, Mexico, Chile, and Brazil have a long history of the state providing textbooks at the national level, and Mexico only produces one textbook for primary education (Rivas, 2015). By contrast, Brazil has expanded the supply and the variety of textbooks in recent years. Peru, Argentina, and Uruguay have also significantly increased the number of textbooks, while Colombia is an exception, because the state participates less in structuring the curricula and recommending books (Rivas, 2015). The expansion in the use of textbooks in Brazil has had consequences for learning mathematics, especially regarding the use of the textbooks for repeating mathematics exercises, a widely used and well-accepted practice in the country.

Mathematical Learning Disabilities in Latin American Countries

There are few prevalence and characterization studies on mathematical learning disabilities (MLD) in children in Latin America. A Latin American report (Dudzik, Elwan, & Metts, 2002) describing disabilities in general makes no reference to MLD.

Most of the Latin American countries failed to maintain all children in schools and to provide good-quality teaching. Hence, if they fail to keep students in schools, it is not reasonable to expect them to recognize students with MLD and help them. Some countries—like Uruguay, Argentina, and Brazil—have recently officially recognized the existence of students with learning disability, especially students with dyslexia, who are rarely identified in schools. Moreover, they have recognized the need to systematically provide resources for mainstream schools to integrate students with special needs (Santiago, Ávalos, Burns, Morduchowicz, & Radinger, 2016). However, there is no reference to MLD. The consequences of this policy are unclear, as few resources have been provided to help students with learning disability in schools.

Despite the fact that due to limited space, the Caribbean countries are not included in this chapter, it is worth mentioning that a research group from Cuba carried out a prevalence study on MLD. Reigosa-Crespo, Valdis-Sosa and colleagues (2012) defend the assumption that individuals with developmental dyscalculia (DD) could be a subset of a more extended arithmetical disfluency (AD) group, and support a definition of DD as a very selective deficit in basic numerical capacities, whereas their definition of AD includes a variety of cognitive disabilities related to inadequate counting-based and retrieval-based strategies from long-term memory (Reigosa-Crespo et al., 2012). The authors found an estimated AD prevalence of 9.4% in a sample of Cuban children from the second to ninth grades. In that sample, AD occurred three times more frequently than DD (frequency 3.4%) in the studied population. Despite the fact that the group does not use the MLD nomenclature, to our knowledge this is the only prevalence study on MLD in Caribbean countries.

In Brazil we found two recent prevalence studies that included MLD. The first (Bastos, Cecato, Martins, Risso, & Grecca, 2016) used the *Diagnostic and Statistical Manual of Mental Disorders, Fourth Edition* (DSM-IV) definition of DD as a specific disability in learning arithmetic, occurring in students with a normal intellectual level, without neurological, psychiatric, sensory, or emotional disorders, and found a DD prevalence of 7.8% in a sample of 2893 first-grade students attending public schools in the southeast region of Brazil. The second study (Fortes et al., 2016) adopted the DSM-5 criterion and definition of specific learning disability in mathematics (SLD-M) and found a global prevalence rate of 6.7% among a sample of 1618 children and adolescents attending schools in four different Brazilian regions. The difference in the prevalence rates could be due to the different tasks, ages, and cultural characteristics of the samples. If we consider these three prevalence studies in Latin America and Caribbean countries, we can conclude that the MLD prevalence varies from 3.4% to 7.8%. However, the tasks, samples, and criteria used to define MLD vary in the prevalence studies mentioned.

Mathematical Learning Disabilities in Brazil

Brazil is the largest and among the most culturally diverse countries in Latin America. Consequently, there are many different situations related to MLD. In public schools in big cities like São Paulo and Rio de Janeiro, some resources are available to help children with learning disabilities, and there are many private clinicians who help students from private schools. This is not the situation in most of the country. Despite the fact that the educational system recognizes some special needs in children and adolescents (BRASIL, 2013), in general there is no educational theory adopted to help them, no dedicated schools, and no diagnostic tests or special programs used in the schools. It is also worth mentioning that most teachers are unaware of MLD. Despite the fact that reading disabilities are more widely recognized than MLD, there are few resources to support children and adolescents with reading disabilities.

Research on Mathematical Learning Disabilities

At the beginning of the twenty-first century, a research group linked to a hospital in Brasilia, Brazil, participated in a comparative study in which Brazilian, French, and Swiss children were asked to perform 11 number-processing and calculation tasks involving knowledge of the written code of numbers, number comparison, mental calculation, problem solving, counting dots, counting backward, and estimation. The performance varied widely between the countries and within Brazil, the only participating country with two sample groups of children in the study. The authors' most important conclusion was that calculation and number processing are rather heterogeneous entities that suffer the influence of linguistic, cultural, and pedagogical "factors on different components of number processing and calculation, such as counting, literal number knowledge, calculation or estimation" (Dellatolas, von Aster, Willadino-Braga, Meier, & Deloche, 2000, p. 108). The inclusion of a Brazilian research group in this study is a rare example of Latin American participation in comparative research on the learning of mathematics.

There is some sparse research on reading disabilities (predominately dyslexia) in Latin America—especially in Chile, Uruguay, Argentina, Colombia, and Paraguay—which has been well described by Bravo-Valdivieso, Milicic-Müller, Cuadro, Mejía, and Eslava (2009). In that paper there is only one mention of MLD: a publication by Azcoaga, Derman and Iglesias (1979), which summarizes the knowledge on the subject at that moment. In fact, little specific research on MLD has been conducted in Latin America. Two exceptions in this scenario, both in Brazil, are the research groups based at the Federal University of Rio Grande do Sul (UFRGS) and the Federal University of Minas Gerais (UFMG). The former research group has dedicated considerable efforts to investigating a range of issues related to MLD. Dorneles (2009) looks at the kinds of MLD experienced in different groups of children, including children with attention deficit hyperactivity disorder (ADHD) and deaf students. The efficacy of a brief psychoeducational intervention program for increasing teacher awareness and knowledge about ADHD and learning disabilities, including MLD, is described by Aguiar et al. (2014). The lack of tests and the impact of the changes in the diagnostic criteria for MLD proposed by the DSM-5 on the prevalence of learning disorders is analyzed by Dorneles et al. (2014). Kieling et al. (2014) seek to identify the best approach for clinicians to get information about symptoms of ADHD from teachers in children with or without learning disorders. Recently the group applied working memory and arithmetical reasoning interventions in an effort to improve mathematical performance in children with ADHD (Sperafico, 2016).

The UFMG research group analyzed the impact of sociodemographic factors, psychosocial competencies, and math anxiety on mathematics and spelling performance in school children with and without mathematical difficulties (Haase et al., 2012). They developed a screening tool for students at risk of mathematical difficulties (Moura et al., 2015), an especially important achievement in a very large and diverse country. Some evidence of shared mechanisms in reading and writing words and numbers were found (Lopes-Silva et al., 2016) in Brazilian children, suggesting phonemic awareness is the cognitive variable that systematically predicts numerical and word learning abilities. This finding indicates that the ability might be shared by many learning tasks related to both reading and mathematics (Lopes-Silva et al., 2016). The results corroborate the idea of there being different groups of children with MLD, with diverse cognitive patterns and consequences for educational and clinical interventions (Júlio-Costa, Starling-Alves, Lopes-Silva, Wood, & Haase, 2015). This research adds to previous studies conducted by Robinson, Menchetti and Torgesen (2002), which together help us understand why mathematical difficulties are so frequently associated with reading difficulties.

Other groups have produced occasional related research; for example, Feldberg et al. (2012) used a multiple-case approach and found associations between neuropsychological deficits and poor mathematical performance. Moreover, there are many papers published in Portuguese (see Vargas and Dorneles (2013), Júlio-Costa, Lima and Haase, (2015), and Corso and Dorneles (2015) as examples).

The scarcity of research is reflected in the lack of recognition of MLD by teachers and schools. As far as we know, as with many special groups, there is no research available on the composition and profile of students with MLD in Latin American countries. Brazilian researchers have used the Arithmetic Subtest (AS) of the School Performance Test (SPT, or *Teste de Desempenho Escolar* (TDE)) (Stein, 1996)—a psychometric instrument that detects problems in the fundamental capacities for reasonable academic achievement of children and adolescents in the areas of reading, mathematics, and writing—when researching mathematics and MLD, and making clinical assessments. The test covers a wide range of levels of arithmetic and is widely used, despite the fact that it has only been validated for a small part of the country (see Fortes et al. (2016), Moura et al. (2015), and Aguiar et al. (2014) for examples).

We can conclude that there is an urgent need for more precise statistics in order to describe the extension of the problem and help make well informed decisions about MLD policies and programs. The difficulty in finding qualified people who can understand the situation of students with MLD is also a problem. In order to ensure good quality of disability-related educational programs, local technical expertise is required. However, well-qualified personnel with experience of MLD can be difficult to find, considering that this is a specific, little-known learning disability, about which there is scarce research available in Spanish or Portuguese, the predominant languages of Latin America. Another problem refers to the need to modify the attitude toward MLD. People with MLD suffer in society and schools because of negative cultural and social attitudes—for example, the idea that they are not intelligent or are lazy children/adolescents. We know that cultural conventions and deeply held beliefs such as these can be extremely difficult to change and lead to a lot of suffering for children/adolescents with MLD. So we must begin collecting data about MLD in Latin American countries in order to describe the phenomena and qualify teachers to identify and help these students.

Although there are many different programs designed to foster socioemotional skills and improve learning in some Latin American countries (for a recent review, see Cunningham, Acosta, & Muller, 2016), we found none that specifically target children with MLD. If we consider the aforementioned prevalence studies that indicate an MLD rate of between 3.4% and 7.8% in children and adolescents within the high rates (from 49.4% to 70.3%) of poor performers in mathematics, we must recognize the urgent need to improve the teaching and learning of the latter group while, at the same time, making efforts to meet the needs of the former group. This is particularly true given that many countries in the region are currently experiencing economic difficulties and limited resources. While some universities in almost all of the countries have offered teachers courses on learning disabilities, based on the neuroscience paradigm, in recent years there has been a lack of registered research associated with those courses.

Future of Mathematical Learning Disabilities in Latin American Countries

The future of MLD education in Latin America is uncertain because many countries are suffering from economic and political crises and instability, which affect their education systems. We have described some recent advances in the region such as the carrying out of prevalence studies and the official recognition of children with special needs and learning disabilities as a group, despite the fact that MLD is not identified as a specific disorder. Changes in educational policies are slow to take effect, and sometimes we can observe advances and regression. Within the region, there are many problems to solve through research and practice, two of which must be highlighted: the urgent need to use standardized tasks to identify children with MLD, and the need to raise awareness about MLD in schools and society.

Conclusions

To be numerate in the twenty-first century requires knowing how to think: to compare, reason, understand, analyze, make relations, and solve problems, which are all competencies that learning mathematics can improve. These are essential survival skills that, as shown above, few students in Latin America are developing. Despite advances in teaching mathematics in recent decades, the region has poor results in the performance of children in mathematics and no documented policies to specifically cater for children with MLD. The situation in the region as a whole is worse than that described in this chapter, considering that most of the countries in the region do not participate in PISA and have worse economic indicators than the countries included in this description. As a diverse continent, the situation in Latin America regarding learning mathematics is complex and there is a complete lack of systematic policies to cater for MLD. A special recommendation, previously proposed by OECD/CAF/ECLAC (2016), emerges from this chapter: there is an urgent need to improve schools and mechanisms for teachers to identify students who are low performers in mathematics and those who are struggling academically, economically, and socially, in order to help them and avoid failure and dropout.

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Chapter 14
Mathematics Learning and Its Difficulties: The Cases of Chile and Uruguay

Cristina Rodríguez, Ariel Cuadro, and Carola Ruiz

Introduction

Mathematics is one of the fundamental subjects of school education, and is proven to have a significant influence on the individual's personal and professional development, and on the society of which they are a part. It is therefore reasonable to expect academic and cognitive problems as a result of a specific mathematics learning difficulty (MLD), however these may also be accompanied by psycho-emotional problems which can have a more serious overall effect on the subject. The existence of "mathematical anxiety" has been demonstrated in relation to the performance of young children with MLD (Wu, Barth, Amin, Malcarne, & Menon, 2012), although definite answers have not yet been found as to the causation or directionality of the relationship. Coping with MLD has its costs in the long term; for example, adults with poor arithmetic skills are at a disadvantage in the employment market (Kaufmann & von Aster, 2012) compared to those with average levels of achievement. However, there is also a cost to society if appropriate action is not taken, and there is also evidence that improving poor numerical skills in the population amounts to significant expense for countries (Butterworth, Varma, & Laurillard, 2011; OECD, 2010). Despite all of the above, however, studies into semiotics,

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diagnosis and intervention in MLD have been overshadowed by studies on specific learning difficulties (SLD) in reading (Balbi, Ruiz, & García, 2017; Cirino, Elias, Fuchs, Schumacher, & Powell, 2015; Kuhn, 2015).

In Latin America, for example, studies on learning difficulties have only taken place from the 1950s onwards, and for the most part address SLD in reading (Bravo, Cuadro, Mejía, & Eslava, 2009). The lack of consensus as to the definition of MLD in terms of the subjects that it should encompass, whether calculation or mathematical reasoning, provides a possible explanation as to why studies often yield contradictory results (Balbi et al., 2017; Murphy, Mazzocco, Hanich, & Early, 2007; Rodríguez & Jimenez, 2016; Shin & Bryant, 2015). Its conceptualisation is made yet more complex by the comorbidity between MLD and other disorders such as reading (Peake, Jimenez, Rodriguez, Bisschop, & Villarroel, 2015; Vucovic & Lesaux, 2013) and ADHD (Landerl & Moll, 2010). These problems in terms of research into the issue are also reflected in the reported lack of training for educators in addressing their students' difficulties with learning mathematics (Balbi et al., 2017; De Almeidas, de Medeiros, & Borsel, 2013; Jiménez-Fernández, 2016; Wadlington & Wadlington, 2008; Williams, 2006), an aspect that is of particular concern in Latin America considering the low achievement of students in mathematics that has been reported by international studies. Identifying learning difficulties in mathematics – whether instructional, socio-cultural or neurobiological – at an early stage provides greater opportunity for effective prevention, as well as changes in teaching practices. This chapter will describe the situation in Chile and Uruguay, covering performance in mathematics and the approach taken by each country to respond to specific educational difficulties associated with these skills.

Mathematics Learning Achievement

International Assessment

In recent years, Chile and Uruguay have taken part in a number of international studies, including the Programme for International Student Assessment (PISA) conducted by the Organisation for Economic Co-operation and Development (OECD), Trends in International Mathematics and Science Study (TIMSS) organised by the International Association for the Evaluation of Educational Achievement (IEA), and the Third Regional Comparative and Explanatory Study (TERCE) managed by the Latin American Laboratory for Assessment of the Quality of Education (LLECE). According to the most recent PISA report, Chile and Uruguay's performance in mathematics remained low between 2012 and 2015, with mean performance in PISA 2015 coming in at 423 and 418 points respectively, well below the mean of 490 points across the 35 OECD countries. PISA 2015 (OECD, 2016) identified six levels of proficiency in mathematics (as it had with PISA 2003 and 2012). Level 2, the category representing scores between 420 and 482, is considered the baseline level of proficiency that an individual requires in order to be able to function fully in modern society. Average Chilean and Uruguayan performance in

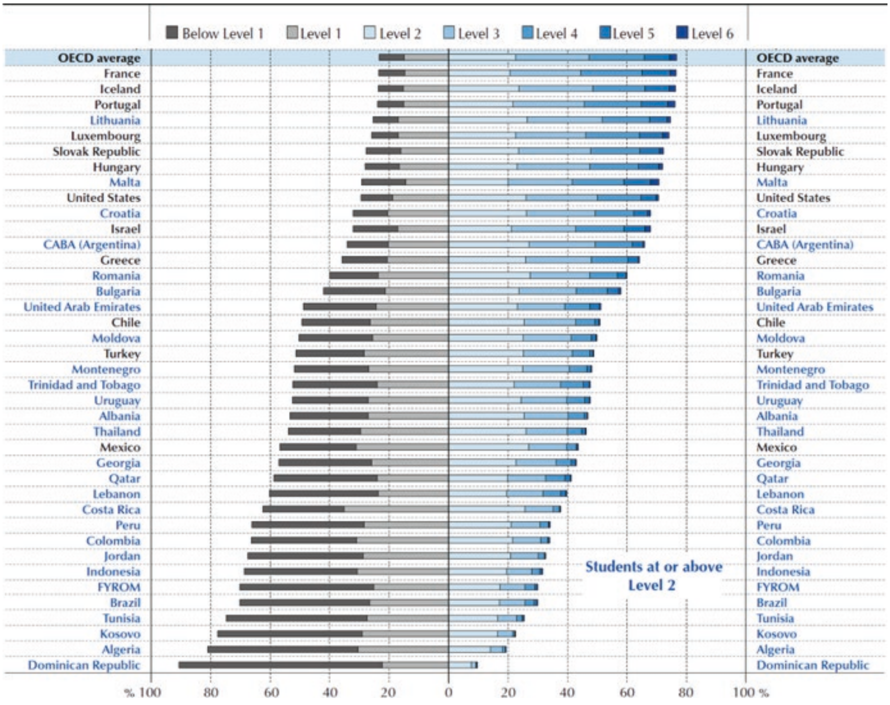


Fig. 14.1 Student proficiency in mathematics. (*Source*: 2016 PISA, mathematics performance in 15-year-olds according to their level of proficiency. *Notes*: The graph only includes countries scoring below the OECD average)

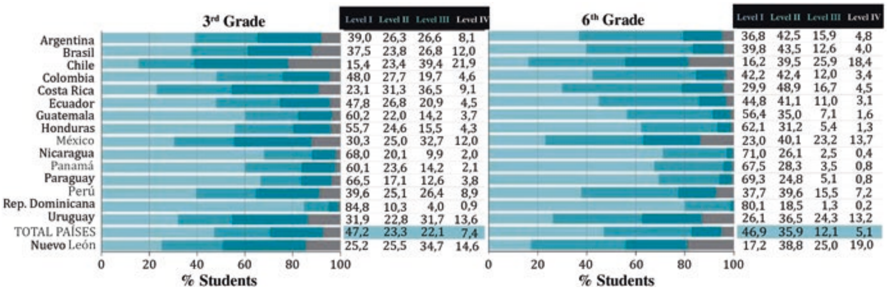
mathematics is therefore at or below the lower limit of Level 2. Performance scored at below 420 points (Level 1) implies that the individual's abilities are too low to effectively deal with situations encountered in the course of daily life. All of the countries participating in PISA have students who scored within Level 1; however, as can be seen in Fig. 14.1, the proportion of Chilean and Uruguayan students who scored at or below Level 1 is very high – approximately 50% – while a very low percentage of students achieved Level 5 or Level 6 (MINEDUC, 2015a; OECD, 2016).

The PISA results also illustrate this disparity. In the most recent assessment, 74% of students from very disadvantaged socio-economic backgrounds failed to reach the OECD's minimum proficiency threshold, as opposed to 26% from the highest socio-economic group. Only 1% of the former fell into the high achievement category. When the effect of the socio-economic level of educational institutions themselves – based on the socio-economic background of their students – is analysed, major differences can be seen between students from different establishments, a finding which is clear from both the PISA and TERCE (OECD, 2016; LLECE, 2015) studies, and places Uruguay and Chile among the countries with the least social inclusion in educational institutions in the region (INEEd, 2017; OECD, 2016).

Another feature of the Chilean and Uruguayan situation is the gender gap in mathematics performance; this gap is common to 28 countries, with boys outperforming girls. However, Chile, together with Austria, Brazil, CABA (Argentina), Costa Rica, Germany, Ireland, Italy, Lebanon and Spain, showed the most significant differences (OECD, 2016), and this disparity among Chilean students has remained stable over the past ten years (MINEDUC, 2012a, 2012b, 2015a).

Results from the TIMMS 2015 assessment support the PISA findings, with Chilean students (Uruguay did not participate in this study) showing poorer performance (478 points in 4th Grade and 454 points in 8th Grade) than the average level of achievement by coming in below the TIMSS scale centre point of 500. That said, a positive trend was reported between 2003 and 2011 in 8th Grade students (MINEDUC, 2013a). The report (Mullis, Martin, Foy, & Hooper, 2016) also showed that gender disparities in mathematics achievement increase across year groups; on average, the gender gap in 4th Grade was smaller than in 8th Grade, where there was a difference of 18 points in favour of boys. As with PISA 2015, when the students are classified according to SES, those from the poorest backgrounds show the weakest performance, and the proportion of students scoring less than 400 points (below the lowest level of proficiency) increases across year groups (30% for 4th Grade and 55% for 8th Grade) (MINEDUC, 2015a). Thus, disparities in mathematics performance according to gender and SES are a source of inequality in Chile, and indeed the two factors seem to interact with one another, with the gender gap being wider in the lowest socio-economic groups (MINEDUC, 2013a).

Although when compared to other OECD countries there is a clear need for Chile and Uruguay to improve the performance of their students in mathematics over the coming years, it is worth noting that according to the TERCE report (Flotts, Manzi, Jiménez, Abarzúa, Cayuman, & García, 2016), Chile presented the strongest performance of the fifteen Latin American countries assessed, followed by Costa Rica and Uruguay (see Fig. 14.2). In fact, in the 3rd Grade, Chile presented the smallest proportion of students at Level 1 – the lowest level of performance – and the largest proportion at the highest level of performance. Also, Uruguay showed an above-average proportion of students at the highest level compared to the other participating countries, and a lower than average proportion of students at Level 1. In the 6th Grade, Chile again presented the smallest proportion of students at Level 1, and



3rd Grade

	Level I	Level II	Level III	Level IV
Argentina	39,0	26,3	26,6	8,1
Brasil	37,5	23,8	26,8	12,0
Chile	15,4	23,4	39,4	21,9
Colombia	48,0	27,7	19,7	4,6
Costa Rica	23,1	31,3	36,5	9,1
Ecuador	47,8	26,8	20,9	4,5
Guatemala	60,2	22,0	14,2	3,7
Honduras	55,7	24,6	15,5	4,3
México	30,3	25,0	32,7	12,0
Nicaragua	68,0	20,1	9,9	2,0
Panamá	60,1	23,6	14,2	2,1
Paraguay	66,5	17,1	12,6	3,8
Perú	39,6	25,1	26,4	8,9
Rep. Dominicana	84,8	10,3	4,0	0,9
Uruguay	31,9	22,8	31,7	13,6
TOTAL PAÍSES	47,2	23,3	22,1	7,4
Nuevo León	25,2	25,5	34,7	14,6

6th Grade

	Level I	Level II	Level III	Level IV
Argentina	36,8	42,5	15,9	4,8
Brasil	39,8	43,5	12,6	4,0
Chile	16,2	39,5	25,9	18,4
Colombia	42,2	42,4	12,0	3,4
Costa Rica	29,9	48,9	16,7	4,5
Ecuador	44,8	41,1	11,0	3,1
Guatemala	56,4	35,0	7,1	1,6
Honduras	62,1	31,2	5,4	1,3
México	23,0	40,1	23,2	13,7
Nicaragua	71,0	26,1	2,5	0,4
Panamá	67,5	28,3	3,5	0,8
Paraguay	69,3	24,8	5,1	0,8
Perú	37,7	39,6	15,5	7,2
Rep. Dominicana	80,1	18,5	1,3	0,2
Uruguay	26,1	36,5	24,3	13,2
TOTAL PAÍSES	46,9	35,9	12,1	5,1
Nuevo León	17,2	38,8	25,0	19,0

Fig. 14.2 Distribution of 3rd and 6th Grade students by mathematics proficiency levels. (*Source*: 2015 TERCE, 3rd and 6th Grade mathematics achievement by proficiency levels. *Notes*: The graph combines figures from the original report for both the 3rd Grade and the 6th Grade)

the largest proportion at the highest level, although these figures are admittedly lower than those for the 3rd Grade. A similar pattern was seen in Nueva León (México), México and Uruguay. Once again, however, the report made it clear that Chile and Uruguay, along with Argentina, Costa Rica, Mexico and Peru present a very significant disparity in scores.

At the same time, the socio-economic background of the students and the type of educational institutions that they attend also have an influence on their general achievement, and on their performance in mathematics in particular. In the TERCE tests, a small proportion of Chilean and Uruguayan students from very disadvantaged socio-economic backgrounds achieved the highest performance category in mathematics, compared with the proportion of students from the highest socio-economic background; the reverse is also true. Furthermore, the disparity between average scores achieved by students from favourable socio-economic backgrounds and those achieved by disadvantaged students places Uruguay alongside Brazil and Peru as the countries with the widest divide in the region (INEEd, 2017).

National Assessment

All of the above is supported by studies carried out within each country. In fact, according to the results from the Education Quality Measurement System (SIMCE), which is Chile's national curricular objectives assessment mechanism for Mathematics, Spanish Language, Social Sciences and Natural Sciences, disparities as a function of SES are higher for mathematics than for reading. Moreover, between 2006 and 2014 a quadratic trend relationship between performance in mathematics and socio-economic status is reported. Between 2006 and 2010 the disparities were very stable, but between 2010 and 2014 there was a decrease at the 4th Grade level, especially when the highest and lowest SES groups were compared (MINEDUC, 2015b). At 8th Grade level, the differences across the groups are greater than at 4th Grade level, revealing that over time the effect of SES on performance in mathematics becomes stronger. SIMCE 2016 assessed students from Grades 4, 6 and 8, and the SES-based disparity decreased in the 4th Grade, but remained stable in Grades 6 and 8. The gender gap remained evident in the 8th Grade, although to a lesser degree than in previous years, while in the 4th and 6th Grades the performance of girls and boys was similar.

The standardised Uruguayan assessment of performance in mathematics began in 1996 with students from the final year of primary education, and there have been six to date, with the most recent taking place in 2013. The data obtained to date show that there has been no improvement either in performance or in the inequality generated by different socio-economic levels over the period (INEEd, 2017), has also reflected the shift, at least in legal terms, and Supreme Decree 170, concerning specific learning disorders (Decreto 170, 2009), regulates the diagnosis of and educational response to Chilean students with SLD. The document sets out the requirements for identifying students with special educational needs, who will be granted subsidies to receive special education, with the concept and identification criteria being largely in line with those proposed in DSM-5 (American Psychiatric Association, 2013). According to Article 23 of the Decree, a student is considered to present specific learning difficulties when the difficulty is "severe or significantly greater than that of the majority of students of the same age in learning to read, learning to write, and/or learning mathematics" (Supreme Decree 170, p. 8), and the measure is specified as being two standard deviations below the mean score for their age group in standardised tests, in spite of normal intelligence and scholastic opportunity. The Article also indicates that "it should be a difficulty that persists despite the application of pedagogical measures applicable to the previously specified subjects, according to the various styles, capacities and rhythms of learning of the students in the class" (Supreme Decree 170, p. 8). It also includes traditional exclusion criteria. SLD in mathematics is understood as being a variant within the general category of SLD, characterised by the presentation of difficulties in the acquisition and development of basic arithmetic knowledge (addition, subtraction, multiplication and division), difficulties with the concept of numbers or difficulties in solving prenumerical problems.

In conclusion, an overall improvement in mathematics achievement in Chile and Uruguay is very much needed, however particular focus should be put on students from a poorer socio-economic background, and on female students. If all of the above is to be considered applicable to students with an average level of performance, then for those presenting cases of MLD and who are therefore even more vulnerable, it is absolutely crucial.

Educational Policies Addressing MLD and Educational Practice

Chile

According to the Human Development Report (2016), Chile is ranked 38th in a group of 51 countries with a very high level of human development. The economic situation has been broadly stable in recent years, although on average the national gross income (NGI) per capita is far below that of other OECD countries. In fact, when NGI is adjusted for inequality per capita, Chile's figures are similar to those of Brazil and Botswana, showing strong income disparity and a high level of inequality (PNUD, 2014). This socio-economic context is a major source of segregation in schools, and the OECD (2012) reports that 81% of disadvantaged students in Chile attend schools with an overrepresentation of disadvantaged pupils. Chile's educational system is based on a system of vouchers received for each student enrolled, and there are three different types of school. Municipal schools (funded entirely by the state) presented the lowest performance levels in national and international studies, in comparison to private subsidised schools (which receive funding from the state as well as fees for attendance) and private non-subsidised schools, illustrating the fact that the socio-economic status of a family has a significant influence on the type of school that the children will attend (Taut, Cortés, Sebastian, & Preiss, 2009). Bravo Sanzana, Salvo Garrido and Muñoz Poblete (2015) found that the type of school attended is the second most relevant factor in explaining students' performance in mathematics, after the educational expectations of parents.

Special needs education is also influenced by socio-economic status and the type of institution attended. According to Law 20.422, which sets out regulations concerning equal opportunities and the social inclusion of people with disabilities (Normas sobre Igualdad de Oportunidades e Inclusión Social de Personas con Discapacidad), only those schools with a School Integration Programme (Proyectos de Integración Escolar, PIE) are permitted to accommodate students with special needs. These programmes ensure that all students receive the treatment (educational strategies, resources, diagnosis, intervention, etc.) that is prescribed by law according to their needs. However, only those schools funded by the State – that is, municipal and private subsidised schools – receive a "special education subsidy" for each student with special needs that is enrolled on the programme, and according to a report produced by the Ministry of Education in 2013 analysing the effect of the implementation of PIEs, of the total schools where a PIE had been introduced, 72% were municipals and 28% private subsidised. Assessing this distribution according to socio-economic status, 29.4% of the schools with a PIE presented the lowest level, 46.6% a medium-low level, 19.2% a medium level, and 4.8% a medium-high level (MINEDUC, 2013b). In other words, students with special educational needs are distributed disproportionately, being overrepresented in schools with low socio-economic status and in municipal schools. In order to fully understand the effect that the type of school has on the development of academic skills in children with special needs, an assessment of the intervention available – which can vary enormously from institution to institution – should be considered. Furthermore, the protocols for diagnosis are strictly regulated and highly homogeneous, at least in the case of SLD, and the law governing diagnosis is generally in line with current international trends, at least theoretically, as will be explored in the following paragraphs.

Over the past decade there has been a paradigm shift in the diagnosis of children with SLD, moving from a "wait to fail" model to one of preventive action (Al Otaiba, Wagner, & Miller, 2014). Research carried out in the USA into this new model known as Response to Intervention (RTI) places at its core the importance of early detection and intervention, constant monitoring of progress in learning using curriculum-based measures (CBM), and other factors such as the multilevel nature of the issue and the importance of data-based decision making, for the diagnosis and improvement of students at risk of presenting SLD (Crespo, Jiménez, Rodríguez, & Baker, 2018; Compton et al., 2010; Fuchs & Fuchs, 2007; Good, Simmons, & Kame'enui, 2001; Hill, King, Lemons, & Partanen, 2012; Tran, Sanchez, Arellano, & Lee Swanson, 2011; VanDerHeyden, Witt, & Gilbertson, 2007). This paradigm shift had an impact on an international level, with the subject's resistance to intervention over a period of at least six months being included as a diagnosis requirement in the latest edition of the Diagnostic and Statistical Manual of Mental Disorders (DSM-V, 2013).

Chile has also reflected the shift, at least in legal terms, and Supreme Decree 170, concerning Specific Learning Disorders (MINEDUC, 2009) regulates the diagnosis of and educational response to Chilean students with SLD. The document sets out the requirements for identifying students with special educational needs, who will be granted subsidies to receive Special Education, with the concept and identification criteria being largely in line with those proposed in DSM-V (2013). According to Article 23 of the Decree, a student is considered to present specific learning difficulties when the difficulty is "severe or significantly greater than that of the majority of students of the same age in learning to read, learning to write, and/or learning mathematics" (Supreme Decree 170, p. 8), and the measure is specified as being two standard deviations below the mean score for their age group in standardised tests, in spite of normal intelligence and scholastic opportunity. The article also indicates that "it should be a difficulty that persists despite the application of pedagogical measures applicable to the previously specified subjects, according to the various styles, capacities and rhythms of learning of the students in the class" (Supreme Decree 170, p. 8). It also includes traditional exclusion criteria. SLD in mathematics is understood as being a variant within the general category of SLD, characterised by the presentation of difficulties in the acquisition and development of basic arithmetic knowledge (addition, subtraction, multiplication and division), difficulties with the concept of numbers, or difficulties in solving prenumerical problems.

With regard to the diagnostic procedure, the first paragraph of Article 26 identifies two stages: "detection and referral" and "diagnostic evaluation". The first stage of assessment prescribes various pedagogical steps, including: 1) since the first year of Primary Education (5-6 years of age) the student must have been receiving a personalised intervention tailored to their needs; 2) implement a programme of continuous assessment based on the curriculum in order to evaluate the student's progress with the applied interventions; and 3) if the student does not show progress as a result of the measures taken, it will be necessary to move on to comprehensive diagnostic evaluation processes. Ultimately, although the law has not been formulated according to the paradigm of the RTI Model, it does contain all of the elements necessary for implementation of a preventive model based on student response.

However, there are various obstacles to early identification of students with MLD when following the proposed guidelines. Firstly, the law does not regulate the evaluation instruments to be used, and indeed these instruments are few in number, particularly those enabling early evaluation of numerical ability (Cerda et al., 2012). In fact, there are no CBM-like dynamic measurement instruments that allow for more than two annual measures, as is required in Chile, and that permit evaluation of progress throughout each year of study (Rodríguez et al., 2017). Secondly, training of Special Education teachers who are specialists in SLD diagnosis has not been sufficiently thorough when it comes to MLD, as the training puts particular emphasis on SLD in reading and writing. This could potentially lead to a lower rate of identification of students with this profile, and to the implementation of limited interventions. Thirdly, all of these preventive measures would be considered part of the first stage of diagnosis – “detection and referral” – meaning that schools would not receive subsidies to support their implementation, as this funding is only made available once a diagnosis is given. In conclusion, although the intention is to move to a preventive model, there are still many obstacles to be overcome before the law can be applied in terms of RTI model.

Uruguay

Uruguay’s current legislation on the subject, the General Education Law (2008), contains no information in respect to dealing with specific learning difficulties. Article 8 emphasises the importance of offering educational approaches that respect the diverse capabilities and characteristics of those being educated so as to develop their potential, and Article 72 states that those being educated are entitled to receive specific support in the presence of disabilities or illnesses as a result of which their learning process is affected. Furthermore, the protocol for inclusion of individuals with special needs in educational institutions (MEC-MIDES, 2017) provides a similar approach to that of the General Education Law. Here we find a series of measures to be taken in order that individuals with any kind of disability are able to enjoy their rights and liberties on equal terms. The guidelines include formulation of academic support strategies which have been validated by teams of integration support professionals, accessibility in terms of facilities, materials and study tools, and preventive, awareness and training measures for the different parties associated with educational institutions. There is also a repetition of the idea of promoting universal design in education, with the aim of catering effectively to the needs of all students.

In order to achieve all of these objectives, the National Administration for Public Education has put in place – in some cases with the support of other state bodies – a variety of educational programmes. The Mandela Network is among those programmes designed to address educational inclusion, and comprises a group of educational institutions – public preschools and schools – that are implementing a variety of inclusion projects with the aim of catering to their specific contexts (Consejo de Educación Inicial y Primaria, 2013). The aim is to promote empowerment and collaboration between educational entities, stimulating interchange and the spread of best practices, and the initiatives are supported by evaluations designed to manifest and standardise the achievements and learning brought about by the process. The Mandela Network is complemented by the Inter-In Project (ANEP-CEP-ASSE-INAU-MIDES-INFAMILIA, 2008), an initiative dealing with children who are just entering preschool all the way up to those in the second year of school, and is in place in many educational institutions. The main objective is the coordination of measures to detect different learning problems, such as specific learning difficulties, developmental disorders and problems of an emotional nature. Following detection, socio-therapeutic educational attention is given by interdisciplinary teams in order to encourage greater adaptation to the school environment.

At the secondary education level, the Integrated Student Department (DIE) – part of the Secondary Education Council – is responsible for dealing with learning difficulties, focusing particularly on curricular adaptations. Although the details of these adaptations are generally left to the discretion of subject teachers, the DIE website offers an adaptation prototype for various difficulties. In the case of difficulties with mathematics, the areas covered are reasoning and the use of strategies for problem solving, calculation and geometry. In 2016, a total of 5,500 curricular adaptations were registered, however in the same year, a study was carried out of public educational institutions across the whole country, which showed that only 60% of students who had difficulties with calculation reported a curricular adaptation. Besides this, the Ceibal Plan has been launched, with the general objective of promoting the integration of technology and using it to support and improve the quality of education, drawing on innovation, inclusiveness and growth. Initial studies into the impact of the Ceibal Plan on mathematics learning have shown mixed results. While Ferrando, Machado, Perazzo & Vernego (2011) found the effect to be positive, De Melo, Machado, Miranda & Viera (2013) found no evidence of positive impact. Nonetheless, over the past few years, this programme has been providing access to an adaptive digital platform called PAM (Adaptive Mathematics Platform), which offers tools for learning mathematics to both students and teachers, and allows each student's progress to be tracked with a view to tailoring activities according to their abilities. According to the Ceibal Plan, during 2016, 41% of students between the 3rd Grade of primary and 3rd Grade of secondary school accessed PAM (2017).

Besides the implementation of educational initiatives seeking to cater to all learning paces and styles, a number of health guidelines have been issued to tackle specific learning difficulties, issues relating to settling in at school, and chronic illnesses or disabilities which affect students' educational process (Ministerio de Salud Pública, 2012). However, these guidelines are directed towards the mental health of children and adolescents, and not specifically towards intervention for learning difficulties. In the vast majority of cases, evaluation and specific intervention for learning difficulties in mathematics occurs outside of the educational context, whether through provisions from the health or social security systems, in private clinics, or by independent professionals. While some private educational institutions have specialist technical teams that carry out evaluations, offer guidance for intervention and promote certain curricular adaptations for dealing with students with difficulties, others do not provide access to these benefits. Once again, this illustrates the lack of equality of access mentioned previously when it comes to mathematics learning.

Besides the need for defining and designing intervention proposals within the school system that are tailored to children with learning difficulties, it will also be necessary to increase the offering of programmes aimed both at teaching mathematics and teacher education. According to the current official programme, the teaching of mathematics at the preschool and primary education levels promotes the development of mathematical thought, enabling critical interpretation of reality, the ability to act upon this interpretation, and the ability to modify it (Administración Nacional de Educación Pública, 2008). The findings of national and international evaluations have led to the creation by the Preschool and Primary Education Council (CEIP) – the branch of the National Administration for Public Education that is in charge of provision of the country's preschool and primary education – of the Commission for Curricular Analysis of Mathematics Teaching in Schools. The purpose of this technical commission is to create a space for reflection on mathematics, and to make specific materials available to teachers for use from preschool to the end of primary school. Distribution of support materials for children from level 5 of preschool and the first three years of primary school began in 2017, with delivery of materials for the rest of the primary school period anticipated for 2018. These materials are designed to contextualise certain mathematical knowledge, solve problems by communicating in mathematical language, and apply individual procedures and strategies (Consejo de Educación Inicial y Primaria, 2016). Some private educational institutions have been applying the so called "Singapore Method" of learning mathematics, which has been introduced mainly through the "Thinking without boundaries" books from publishers Marshall Cavendish Education (Ho Kheong, Ramakrishnan & Choo, 2011). The approach aims for learning mathematics to be effective, measurable and open to evaluation.

All this being said, teachers demonstrate very little knowledge of learning difficulties in mathematics. In a national survey of teachers carried out by the National Institute of Educational Evaluation (INEEd, 2015), 84% said that they had received little or no training in learning difficulties, and similar results were obtained in interviews with primary education Masters graduates upon completion of their studies (INEEd, 2016). In the public sector, training opportunities beyond seminars and short courses are scarce, and postgraduate courses in learning difficulties, particularly in mathematics, are available only at private universities and institutes.

Research into MLD

Chile

It is well known that MLD is substantially under-researched in comparison with SLD in reading (Gersten, Clarke, & Mazzocco, 2007; Wilson et al., 2015). In fact, as shown by Moeller, Fischer, Cress and Nuerk (2012), as of 2009 the number of publications found on Web of Knowledge when searching for "dyslexia" came to 10,880, whereas the term "dyscalculia" yielded only 599 results. Over the past five years, there has been a considerable increase in the number of studies published about MLD; however it will take several years to accumulate the body of knowledge required to bring these two topics into line.

Chile reflects this situation, although with a certain degree of delay. The coup d'état saw the country suffer a period of stagnation in the advancement of neuropsychology between 1973 and 1990, and although the period did see growth in research into SLD, particularly in reading, Chile did not begin to follow international trends until after 1990, reaching a peak around 2001 (Rosas, Tenoria, & Garate, 2010). It is therefore only in recent years that research has begun to emerge (e.g. Cerda et al., 2015; del Río, Susperreguy, Strasser, & Salinas, 2017), and given this early stage, the practical implications of new discoveries will require time to develop, meaning that educators in training still lack sufficient knowledge of the subject (Friz Carrillo, Sanhueza Henríquez, & Sánchez Bravo, 2009). One reliable indicator of the application of research in practice is technological transfer, and while there are projects financed by Chile's Fund for the Promotion of Scientific and Technological Development (FONDEF) for the teaching of mathematics (e.g. Projects FONDEF – CONICYT D09 I1023, Resources for basic training of Primary Educators – REFIP), it will be a few years before similar results are seen in the area of MLD. There are currently no national-level prevalence studies being undertaken, only smaller ones at the regional level, however there are some research projects funded by Chile's National Fund for Scientific and Technological Development (FONDECYT) whose focus is on MLD (e.g. Proyecto FONDECYT REGULAR 1161213, Identification of explanatory factors in the occurrence of comorbidity between learning difficulties in mathematics and reading from a longitudinal perspective). To summarise, Chile is currently in a phase of take-off in terms of research into this topic, and the application of results to educational practices in the form of development of teacher training, diagnosis and intervention will come with time.

Uruguay

Research at a national level on the learning of mathematics and related difficulties is limited and of little significance. There are two main research teams dedicated to mathematics and learning. They are associated with the Universidad de la República and the Universidad Católica del Uruguay, and their work has focused primarily on applied research on educational practices and evaluation.

The Universidad de la República is home to the Centre for Basic Research in Psychology, which has been working on a line of research called Numerical Cognition, which focuses on the study and analysis of the cognitive processes involved in learning mathematics. They have studied, for example, a) the relationship between spatial activities and skills, and mathematics performance in children, b) analysis of the influence of music on performance in mathematics and, c) the relationship between engagement in numerical activities at home and mathematics performance in children (Grupo de Investigación en Cognición Numérica., s.f.), as well as the effect of the Approximate Number System (ANS) on development of mathematical ability (González, Kittredge, Sánchez, Fleischer, Spelke & Maiche, 2016; Valle-Lisboa, Mailhos, Eisinger, Halberda, Gonzalez, Luzardo & Maiche 2017, Valle-Lisboa, Mailhos, Eisinger, Halberda, Gonzalez, Luzardo & Maiche, 2015). The group has taken advantage of the possibilities afforded by the Ceibal Plan by performing an intervention study into the approximate number system (ANS), assuming its role as an ancient mathematical ability that could be related to MLD.

Although they report positive effects of training on mathematical ability, their study contains certain methodological problems which currently do not allow for the generalisation of its results to educational practice. Another important aspect of the study that should be taken into consideration is that none of the children involved presented MLD (Valle-Lisboa et al., 2015).

The research group at the Universidad Católica del Uruguay is working on the acquisition of mathematical calculation skills and the problems relating to it, and among other things they have worked on a) the development of evaluation techniques for arithmetic calculation (Singer, Cuadro, Costa & von Hagen, 2014; Singer & Cuadro, 2014), b) the study of the relationship between arithmetic and reading performance during school years (Singer & Strasser, 2017) and, c) the importance of teachers' ability to identify calculation difficulties (Balbi, Ruiz & García, 2017).

Conclusions

According to international reports, Chile and Uruguay present similar levels of achievement in mathematics. In relation to other Latin American countries, both present above average levels, however when compared to the countries that participate in the PISA or TIMMS evaluations, average performance is less than satisfactory. Furthermore, in both cases, performance in mathematics is affected by the socio-economic status and gender of the students, which adds yet another problem to that of poor performance. However, the extent to which these trends affect those children with general SLD and those specifically with MLD is still understudied. In the case of Chile, the School Integration Programmes and the regulations set out by Supreme Decree 170 ensure similar diagnostic processes, however the way in which intervention is applied is perhaps more heterogeneous. Today, as part of the previously mentioned Proyecto Fondecyt Regular 1161213, the CBM method is being used to track the progress of around 1,000 pre-school students from different types of schools as they advance 2nd Grade. It is expected that at least 5% of them will present MLD, and if this is the case we would able to gauge the impact of different types of school upon their progress.

In terms of the education policies around dealing with and diagnosing students with MLD, Chile and Uruguay present differing levels of development. Chile has already taken the first step towards a preventive model at a legislative level for diagnosis of students with SLD, however, further work is required on reinterpretation of what the law prescribes, freeing up subsidies for preliminary studies to advance prevention strategies, improving the training of education professionals, providing co-teaching strategies to facilitate collaboration between general educators and special educators, reducing student-teacher ratios, and strengthening intervention programmes. In the case of Uruguay, evaluation and intervention in specific difficulties with mathematics are managed, for the most part, within the health and social security systems, and the education system has not yet reached the point of assuming learning difficulties as one of its concerns. However, change is on its way, for example, the fledgling proposals within secondary education that seek to promote adaptations of the curriculum as a whole, instead of simple changes to specific evaluations, and the involvement of specialist professionals in educational institutions and bodies in charge of education. Another promising development is broader access to and use of new technology, which is bringing about the delivery of interventions through IT solutions.

In terms of research into MLD, both countries are seeing an increase in research projects, although given this early stage in the process, these are still relatively few in number, meaning that application of their findings is still some time away. However, it is important not to lose sight of the fact that improvements in pedagogical practices which have the potential to offer more effective solutions to learning difficulties in mathematics should always go hand in hand with research and findings based on empirical evidence. With this in mind, large-scale training programmes should be put in place immediately for specialists and for primary and secondary educators.

There is still a long way to go before children with general learning difficulties – and with difficulties with mathematics in particular – can be guaranteed access from an early age to the support they need to secure a successful education, and for this possibility not to be contingent upon the economic status of the family and on the educational institution that the child attends. It is in the hands of public institutions and government, in conjunction with researchers in the field, to ensure the best possible treatment of students with MLD. Collaboration between these two groups will lay the foundations for improving training for teachers who deal with students with MLD, make empirically tested intervention programmes available to schools, and develop suitable evaluation instruments for monitoring students' progress throughout their time at school.

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Chapter 15
Mathematical Learning and Its Difficulties in Southern Africa

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Introduction

This chapter opens with brief overviews of four Southern African countries' approaches to mathematics and inclusive education in Zimbabwe, Malawi, Lesotho and South Africa. Each outlines the general education context as well as the national policy approach to mathematics and inclusion and offers some insight in the enactment of this policy from the perspective of teachers and officials in primary schools.

Having sketched the regional context, a detailed case study of a 4-year intervention study (2012–2015) in a 'full-service school' (Department of Education, 2001) in South Africa is presented.

This chapter considers two of the domains relevant to special education: environment and cognition. The socio-emotional domain (of both learners and teachers), although of interest, and relevant, is not in focus. The environment is described in relation to the socio-economic and policy context of inclusive education in relation to mathematics in the Southern African region. The cognition is described in relation

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to early grade mathematics learning as measured through both standardised national assessments and provincially designed and administered assessments at Grade 3 and Grade 6 levels in the targeted primary school.

There are many factors which impact on mathematical exclusion. At the systemic level the resourcing of education, the population demographics and historical experiences of marginalised groupings, the policy frameworks in relation to access to mathematics, the mathematics curriculum, and special educational needs and language policies all play a role. In the resource constrained context of this low-income region, there are very limited choices for teachers and government officials in how to support learners experiencing mathematical learning difficulties. With very few options for specialised intervention, the quality of whole-class teaching and how ordinary teachers better support all learners – and particularly those with mathematical learning difficulties – must necessarily become a focus. The main thesis of the case study is that inadequate knowledge for teaching mathematics ought to be seriously considered as a barrier to mathematics learning and one of the key levers for making mathematics more inclusive.

It is hoped that this chapter provides a glimpse first at a macro level of the conditions under which teachers and government officials in this region work and then at a micro level of one primary school. The Foundation Phase teachers (Grades 1–3 and including the pre-school year of Grade R) in the focal school of this intervention study were supported to improve their knowledge of and approach to mathematics teaching through establishing a professional learning community of mathematics (Karin Brodie, 2013; Brodie, Molefe, & Lourens, 2014). The effect this intervention had on learner cognition (as measured in standardised assessments of learners on mathematics) was determined. So alternating the environment (improving the quality of whole-class teaching) was conjectured to potentially lead to changes in individual mathematical cognition.

Theoretical Framing

There is apparent tension/divide between the cognitive and environmental domains of research in learning difficulties related to mathematics education. Some more cognitive-focused research concentrates on core skills and competencies that are internal to the cognition of a particular child (Butterworth, 2015; Dehaene, Piazza, Pinel, & Chen, 2003; Pirjo & Räsänen, 2015). There is simultaneous acknowledgement that context matters and that changes to the child's environment (particularly their school environment) can affect their cognitive development (in this case their individual mathematical learning trajectory). Researchers in the more didactic or pedagogic school of mathematics education research focus on teaching, and the requisite knowledge for mathematics teaching (such as Rowland, Huckstep, & Thwaites, 2004; Ball & Bass, 2003; Wright, Martland, Stafford, & Stanger, 2002). There have been some efforts to bring these two schools together when research attention is placed on potential remediation pathways for mathematical learning difficulties (see, e.g., Wright et al., 2002; Wright, Stanger, Stafford, & Martland, 2006; Fritz-Stratmann, Ehlert,

& Klüsener, 2014). These bridges are not new between the domains, as much has been written which refers to learners with special educational needs (LSEN) as requiring adaptive changes to their learning environment in order to better support their cognitive development in schools. Some of these adaptations refer to whole-class interventions (which may include in-class or caring adult support) and/or pull-out support or after-school intervention making use of small-group or one-on-one interventions to remediate identified mathematical learning difficulties.

Such a suite of 'special needs' intervention options (whole class, small group and individual) are current in contexts where public schools have resources for such learning support opportunities and/or where the families of children diagnosed with particular learning difficulties may have the resources to seek out additional after-school intervention. However, in other contexts such learning support options are absent from the majority of public schools and therefore cannot be considered as possible special needs interventions.

This chapter outlines the environmental constraints in relation to mathematics and primary schooling in four Southern African countries. The intervention study reported on is situated in an environmental context of a South African full-service school community of very low socio-economic status. Here almost all opportunity for learning mathematics takes place during 'ordinary whole-class mathematics lessons' in a public school, with no or very little access to out-of-class, or out-of-school opportunity for small-group or individual specialist intervention, and with very few or no opportunities to learn mathematics in out-of-school time. Within such contexts the quality of whole-class learning opportunities becomes a key factor in individual mathematical cognitive development (or lack of development). The study makes use of the theoretical constructs of a professional learning community focused on teacher knowledge for mathematics teaching as a promising means to improve the whole-class teaching of mathematics.

Identified Problem and Research Questions

As a region sub-Saharan Africa has the lowest human development index (HDI) in comparison with other geographic regions. The greatest attention of the state is therefore necessarily on basic needs (including health, electrification, access to clean water, sanitation, housing and educational access). In terms of education, the main focus has been on increasing enrolment.

Having situated work on mathematical inclusion, within a broader regional context, a single school in South Africa is described. The focal school for the intervention which is considered as a detailed case is located in an urban South African community which is described on the website of an independent trust operating in the community as

> 'one of the poorest townships in the Western Cape[…]situated 30 kilometres South of Cape Town – the area is home to 40,000 inhabitants, many of whom are migrants from across Africa. Residents live in shack-like dwellings and healthcare, educational and recreational facilities are in short supply.'

The primary school in this community caters for 700 children (Grade R to Grade 7). A problem of systemically poor performance was identified in the focal school as the majority of learners were not meeting the grade level requirements of the curriculum for mathematics. This was evident in both standardised Annual National Assessments (ANAs) set annually by the national Department of Basic Education in Grades 1–6 for all primary school learners and administered and marked by classroom teachers, as well as in systemic assessments of mathematics at Grade 3 and Grade 6 set, administered and marked annually by the Western Cape Education Department (WCED).

To respond to this identified problem, the *Focus on Primary Maths* project based on a 3-year grant ($37,000 per annum) from a philanthropic trust was established. The project was a focused research and development intervention involving a three-way partnership between a consulting company with expertise in mathematics education and the school leadership and teaching staff at Foundation Phase level in two government schools in Cape Town. One primary school (the 'suburban school') was an old affluent, suburban school with an established track record of excellent attainment in mathematics and language, and the other school (the 'full service township school') was relatively new located in a poor township community which was performing below the provincial average in standardised assessments. This partnership was formalised with a contractual agreement outlining roles and responsibilities. It was supported by a small reference group comprising the project leader (author), school leaders from each school, three experts in primary mathematics education as well a representative from the Western Cape Education Department (WCED). Reference group meetings were convened biannually to agree on annual plans and report on progress against these plans and their related annual budgets.

While the research and development intervention was loosely framed at the outset, it included several key ingredients focused on creating a professional learning community relating to mathematics knowledge for Foundation Phase teaching. The grant was ring-fenced to pay for the professional consulting time, professional development tuition fees, incentives for teachers to share their teaching approaches in collegial forums, mathematics learning support materials and equipment as well as travel and catering for seminars and conferences.

The research question at the macro level was: What is the policy context for access to mathematics and mathematical inclusion in each country, with some reflections on how this enabling environment is experienced in practice for a mathematics teacher in a primary school. At the micro level of the focal school in the intervention study, the following research questions were posed:

- Over the 3-year period (2012–2014), how were Foundation Phase (Grades 1–3) teachers supported to improve their knowledge for mathematics teaching?
- Were there any shifts in mathematics learning attainment (as measured in standardised systemic assessments administered by the province) from 2012 to 2015?

Methods

The broad context to mathematical inclusion in the four countries in sub-Saharan Africa was developed drawing on expert knowledge of each national context and desk research.

The case study findings of the focal school were developed drawing on multiple sources of data. The implementation of the intervention was monitored making use of teacher feedback questionnaires at the outset and close of the intervention, teacher surveys following each professional development intervention (seminar or formal course), reports and papers on reflective sharing sessions, research output, detailed design experiment research interventions for expert interventions (the focus of PhD study) and a series of mini-lesson study cycles of reflection of co-teaching interventions and collection and analysis of learners scripts for annual national assessment data.

The data used to answer the micro level research question draws on project documentation, feedback obtained from teachers through survey and questionnaires, experiences of the project leader (who is the corresponding author) and learner attainment data as measured in standardised provincial assessments.

Results and Discussion of Findings

Sub-Saharan Africa is organised into a region comprising 15 countries in the Southern African Development Community (SADC): Angola, Botswana, Democratic Republic of Congo, Lesotho, Madagascar, Malawi, Mauritius, Mozambique, Namibia, Seychelles, South Africa, Swaziland, United Republic of Tanzania, Zambia and Zimbabwe. Each country has its own approach to education, mathematics and educational inclusion. We offer brief overviews of four of these countries in order to sketch the environment and policy frameworks in which teachers of mathematics and the learners in primary schools operate.

Lesotho

In the year 2000, the government of Lesotho introduced free and compulsory primary school education. This initiative made it possible for children from economically challenged backgrounds to have access to education. Children in Lesotho graduate from reception class (Grade R) at the age of 5 and start Grade 1 at the age of 6. According to the language policy, learners at Grades 1–3 are to be taught in home language (Sesotho). From Grade 4 upwards, the medium of instruction is officially English. Children attend public primary school for 7 years and then continue for a further 5 years to secondary school where they complete a Lesotho General Certificate of Secondary Education (LGCSE) (school leaving certificate).

The Lesotho gross enrolment ratio (GER) in pre-school enrolment is 33.95%, in primary school is 105.52% and in secondary school is 53.77% (Unesco, 2014, http://uis.unesco.org/country/Lesotho). In 2008, Lesotho was spending 24.7% of its GDP on education, and the expenditure per learner in primary school was $401.1 (in PPP$, UNESCO, 2014). Current statistics on how much the country spends on education are not available.

Mathematics is part of the curriculum from Grade 1 (age 6) on and is integrated into other subjects. Mathematics forms part of the integrated curriculum from Grade 1 up to Grade 6 (age 11) (known as a 'numerical and mathematical learning area') and emerges as a standalone subject in Grade 7. The Curriculum and Assessment Policy (2009) states that the numerical and mathematical learning area is aimed at promoting 'application of numerical and mathematical skills in solving everyday problems' (p. 32) amongst others. All learners must take mathematics until Grade 10 (age 15). From Grade 11 to Grade 12 (age 17), learners are free to choose between Mathematics Core and Mathematics Extended. Mathematics Core is taken by students who are not strong in mathematics or choose not to take mathematical-related careers beyond secondary school.

The government of Lesotho committed itself in providing equal access to quality and relevant education and training opportunities to all Basotho children (Strategic Plan, 2005, 2005–2015). In order to achieve this, the government developed an Inclusive Education Policy in November 2016 though still in draft form to date. The main objective of the policy is 'to ensure that Learners with Special Educational Needs (LSEN) or disability receive quality education which is accessible and efficient, education that promotes inclusion, non-discriminatory and caters for individualization to ensure participation and progression with high performance at all levels' (p. 13).

In relation to mathematics inclusion issues, the policy highlights temporary learning difficulties such as dyslexia, dysgraphia and dyscalculia and argues that if appropriate measures are put in place, these could be corrected.

The government aspires to build up a complete inclusive education in Lesotho, but there are serious challenges that teachers are faced with. Within the inclusive education policy context, a typical teacher in a primary school would be responsible for 80 learners in her class. As such it might be extremely difficult for a teacher to identify learners with special needs. A large number of teachers never got training on issues relating to inclusive education. Because of lack of resources, there are no supporting structures in place for teachers such as teacher development programmes. In most old schools, the buildings remain inaccessible for learners with special physical needs. Lack of teaching and learning materials remains a great challenge for most schools. Up to this day, what seems to be functioning well are a few schools in Maseru and one school in Leribe district that have qualified teachers in special education and have resources for learners with special needs. We therefore see a need for more teacher empowerment with skills and knowledge to handle learners with special needs.

Malawi

Children in Malawi start public primary schooling at age 6 in Standard 1. They attend public primary school for 8 years which is comprised of infant section (Standards 1–2), junior section (Standards 3–4) and senior section (Standards 5–8). At the end of the formal primary cycle (Standard 8), learners sit for Primary School Leaving Certificate of Education (PSLCE) examinations which are administered by Malawi National Examinations Board (MANEB), and only those that pass with a higher aggregate normative score are selected by the government to public secondary school (MoEST, 2014a). Some learners however continue for a further 4 years to secondary school where they sit for Malawi School Leaving Certificate examinations to obtain a Malawi School Leaving Certificate. Prior to public schooling, some children attend early childhood development programmes in form of pre-school, kindergarten or child-based care centres (CBCCs). However, access to such centres is limited to children in urban areas. The language of instruction in public primary schools is the common language or mother tongue of the area for the first four classes of primary school and English from Standard 5 on. However, according to the government of Malawi (2012:42), 'the medium of instruction in schools and colleges shall be English'. But this policy shift is yet to be implemented.

In Malawi, about 70% of eligible children do not access any form of ECD (NER = 32%) (MoEST, 2014b). The GER in primary school in 2014–2015 academic year was 133% and in secondary school was 24.3% (MoEST, 2016). The GER for primary school is greater than 100% because of the enrolment of underaged and overaged children. In the 2014–2015 financial year, Malawi spent 23% of the national budget on education and 49% of the education budget on primary education (MoEST, 2016).

Mathematics is part of the curriculum from 6 years and taught alongside other subjects. All learners must take mathematics for the whole of primary and secondary school. This is due to the fact that in Malawi, mathematics is considered to be one of the core subjects.

Malawi has ratified some essential UN declarations as Universal Declaration of Human Rights (1948), International Convention on Civil and Political Rights (1966) (ratified in 1994), Convention on the Elimination of All Forms of Discrimination Against Women (1979) (ratified in 1987), Convention on the Rights of the Child (1989) (ratified in 1991), African Charter on Human Peoples Rights (1981) (ratified in 1989), World Programme of Action Concerning Disabled Persons (1982) and UN Standard Rules on the Equalisation of Opportunities for Persons with Disabilities (1993) (Salmonsson, 2006). The commitment to international declarations resulted into production of inclusion/special needs education policies and infusion of such issues in several policy documents. Some of these national policy documents include National Policy on Special Needs (Ministry of Education and Vocational Training Malawi, 2007) with its corresponding Implementation Guidelines for the National Policy on Special Needs Education (Ministry of Education, 2009). Some of the objectives of the special needs education policy include:

- Provide the education and training to learners with special needs.
- Ensure for all learners with special needs equitable access.
- Provide educational facilities with needed supportive provisions.
- Ensure accommodating learning environments for all learners with special needs.
- Increase provisions of SNE services by all education stakeholders.
- Improve coordination and networking amongst school and related personnel.
- Enforce adherence to standards and ethical practices in providing SNE services (Ministry of Education and Vocational Training, 2007).

Further, Malawi had a broad education policy which considered special educational needs reflected in the following priorities/strategies: promote early detection, intervention and inclusion for children with special health and education needs, develop appropriate tools for special needs such as sign language and Braille and move towards the recommended teacher-pupils ratio of one teacher to ten pupils in special schools and one teacher to five pupils in resource centres by the end of the plan period (MoEST, 2008:6, 11–12).

However, recently (July 2017), the Malawi government has launched a National Inclusive Education Strategy aimed at giving quality education to the marginalised children who are mostly excluded and secluded in mainstream education.

Yet, there are no subject specific approaches to inclusive education. As such, when learning mathematics, learners are treated in the same way as when learning other subjects. This is also reflected during national examinations where extra time is given to some learners with special educational needs like visual impairment.

Within this policy context, a qualified teacher in primary school would on average be responsible for 75 learners in his/her class (MoEST, 2016). However, in some schools in rural areas, the enrolment is much higher with more than 100 learners per class in the early grades (Saka, in progress). In Malawi, learners with special needs are dealt with differently depending on the type and severity of the impairment. Learners with severe impairments like deafness or blindness are mostly taught in designated special schools by SNE teachers, while those with mild impairments are integrated in mainstream schools. In such cases, there is a special teacher at the school who provides support to such learners. One such teacher normally serves up to 15 schools (Salmonsson, 2006). In some schools where these learners are integrated in main stream classrooms, a resource centre unit is usually established that the special/itinerant teachers use after pulling out the learner(s) with special educational needs.

South Africa

Children in South Africa start compulsory public schools at age 5 (the year they turn 6) with Grade R. They attend public primary school for 8 years which is comprised of Foundation Phase (Grade R-3), Intermediate Phase (Grades 4–6) and the final year or primary school Grade 7 which is a transition year and part of Senior Phase

(Grades 7–9). Learners may leave school at Grade 9 and either continue with schooling in Grades 10–12, enrol in a further education and training college in the technical and vocation education sector or enter the world of work or unemployment. There is a school leaving certificate at Grade 12 level referred to as the National Senior Certificate. Prior to public schooling, there is a range of early childhood development centres (pre-school or kindergarten) although this is not universally accessible. The language of teaching in public primary schools most commonly involves home language teaching (in 1 of 11 official languages) for the Foundation Phase and transition into teaching in English or Afrikaans from Grade 4 onwards.

The South African gross enrolment ratio (GER) is 77.37% in pre-schools, 99.72% in primary schools and 98.82% in secondary schools (UNESCO, 2014, http://uis.unesco.org/country/ZA). South Africa spends 6.06% of its GDP on education, and the expenditure per learner in primary school is $2,270.81 (in PPP$, UNESCO, 2014).

Mathematics is part of the South African curriculum from age 5. In the Foundation Phase, mathematics is taught alongside language (comprising both home language and first additional language) and life skills. From Grade 4 onwards, mathematics is a compulsory subject and taught alongside other subject areas. All learners must take mathematics until Grade 9 level (age 15). Thereafter, in Grades 10–12, learners are offered a choice between mathematics and mathematical literacy. While the former is preparation for higher learning in mathematics-related degrees, the latter is preparation to be a numerate citizen and scholar.

The South African constitution includes a provision of the right education and the right to be free from discrimination including disability. This right is legislated in the South African Schools Act of 1996 (RSA 1996) which refers to learners with ‘special educational needs’ being served in the mainstream with the provision of relevant support ‘where this is reasonably practicable’ and physical amenities being made accessible to disabled learners. This was extended in 2001 with the Department of Education (DoE) *White Paper 6: Special Needs Education.* Walton (2014) indicates that this 20-year strategic plan aims to:

- Reach more of the children and young people who are not in the school system.
- Improve special schools and convert them into resource centres.
- Convert 500 ordinary primary schools to be full-service schools that are capable of responding to the full range of learning needs.
- Establish district-based support teams providing support service to schools.

As such the South African policy recognises educational inclusion as a human right and makes available special schools offering a continuum of inclusive education practices for learners with moderate to high support needs. At the same time, South Africa aims to improve the services at ordinary public schools so that learners with barriers to learning are adequately supported through mainstream/regular schools. Resources have been targeted at 500 ‘full service’ schools which are designated ordinary schools but are expected to become examples of good inclusive practice (ultimately paving the way for all schools to become inclusive).

While mathematics is compulsory in all grades of the South African curriculum, neurological barriers to learning (including mathematical calculations and numeracy skills) are recognised. 'Learners with special needs' are identified in policy frameworks where it is expected that 'measures should be taken in ordinary as well as special schools'. Provision is made at the Grade 12 exit level examination to assist with learners with special needs. In relation to mathematics, there is explicit mention of learning difficulties relating to mathematics in the promotion requirements for the national curriculum for Grades 10–12: 'Learners who have been diagnosed to have a mathematical disorder such as dyscalculia may be exempted from the offering of Mathematical Literacy or Mathematics' (Department of Basic Education, 2013, p.39).

Aunio et al. (2016) report that the public schools in their South African study were supported by only few professionals in the local school district office, who have to serve more children than they can accommodate properly. They explain that within school-based learning support teams, there are difficulties in 'guaranteeing that all of these professionals have the necessary background to assist children in public schools sufficiently'.

Within this policy context, a typical teacher in a primary school would be responsible for about 35 learners in an urban context, although class sizes may be far higher (up to 60 or extreme cases of 80 learners per class) in some rural areas. Mathematics will be taught in English or Afrikaans after Grade 4, and in many contexts these languages dominate mathematics in earlier grades. The teacher in an ordinary school would be likely to have very limited knowledge of mathematical learning difficulties, and although there would be access to district support, this would not be sufficient for meaningful engagement about particular children. In a full-service school, a teacher may be supported by a learning support teacher who has more specialised knowledge (however much of this would be in relation to language learning difficulties and other learning barriers, with little awareness of barriers specific to mathematics learning). In some provinces, learners not meeting grade expectations in languages and/or mathematics would be included in after-school 'intervention' lessons convened by the class teacher. The other option for addressing learners not meeting grade expectations would be to allow the learner to repeat a grade (which may occur only once in each phase).

Zimbabwe

The ordinary school education in Zimbabwe is divided into four main levels, namely, pre-primary, primary, secondary and tertiary education. Pre-primary education is offered to children aged 3–5 through the early childhood development system (ECD) which is provided at primary schools. Ninety-eight percent of primary schools have ECD centres for ages 4–5, while 60% of primary schools have ECD centres for ages 3–4 with trained teachers. Primary education is offered for children aged 6–12 for 7 years encompassing Grades 1–7. For example, some primary schools use English as language of instruction throughout the primary years;

township and rural primary schools use home language instruction (predominantly Shona and Ndebele) from Grade 1 to 2, and then from Grade 3 onwards the language of instruction changes to English. At the end of Grade 7, learners sit for a national examination in mathematics, English, Shona or Ndebele, the General Paper broadly covering the following learning areas: social sciences, environmental science and religious education. Secondary education is offered for children aged 13–18 and is made up of two cycles, namely, the General Certificate of Education, or Ordinary Level, encompassing Forms I–IV, and the General Certificate of Education Advanced Level, or Advanced Level, encompassing Forms V and VI. This structure was adopted from the British system of education. Progression from ordinary to advanced level is not automatic; it is by merit; and many learners exit the school education system after Form IV to enter the competitive world of work or vocational training, often competing with advanced level graduates who do not attain qualifying grades for university entrance. Tertiary education is offered to children aged 18–23 at various universities, technical, polytechnic and teacher training colleges and various vocational training centres. Pre-primary and primary education in government schools is subsidised by the government, while secondary and tertiary education is not.

Zimbabwe's school age population by education level is as follows: pre-primary school, 1,456,907; primary school, 2,902,600; secondary school, 2,069,539; and tertiary level, 1,607,620 (UNESCO, 2014, http://uis.unesco.org/country/ZA). Zimbabwe spends 8.43% of its GDP on Education, and the expenditure per learner in primary education is $385.36, in secondary education is 612.28 and in tertiary education is 4443.08 (in PPP$, UNESCO, 2014).

Mathematics is part of the Zimbabwean curriculum from age 5 in the ECD and is taught alongside literacy and life skills. From Grades 3–7 of the primary school, mathematics is compulsory, taught alongside English, Shona or Ndebele and General Paper. In the secondary school (up to Form IV), mathematics and English are compulsory and taught alongside a home language and a variety of subjects chosen from commercials, humanities (Ex Model C schools also offer foreign languages like French), practical subjects and sciences depending on the selection of subjects offered at the particular school. Generally, mathematics is a compulsory subject up to the ordinary level, and it is one of the selection criteria for accessing education beyond this level. However, recent examination statistics also show that many learners do not fare well in mathematics, as the pass rate is almost always low. In 2014, only 24% of the students who sat for the 'O' Level Mathematics examination passed. The Ministry of Primary and Secondary Education is currently revamping the mathematics curriculum to improve Zimbabwe's quality of mathematics teaching and learning and test scores through the Education Sector Technical Assistance (TA). The TA is intended to support the government's efforts to improve test scores and spending efficiency, strengthen capacity and update practices in specific policy areas aimed to address gaps in education (World Bank, 2016, http://www.worldbank.org/en).

To support the declaration of education as a basic right to every Zimbabwean school-going age child, relevant legislation was put in place. Although Zimbabwe does not have an inclusive education-specific policy, the country has a number of legislations with inclusive education-related policies. These legislations include The Constitution of Zimbabwe, Article 2 of the United Nations Convention on the

Rights of the Child (1989), the Convention on the Rights of Persons with Disabilities (2008), World Declaration for Education for All (1990), the Education Act (1996), the Zimbabwe Disabled Persons Act (1996), the UNESCO Salamanca Statement and Framework for Action (1994) and the Dakar Framework for Action (2000). Of these legislations, the Education Act of 1996 and the Zimbabwe Disabled Persons Act of 1996 advocate for non-discriminatory provision of education and non-discrimination of people with disabilities in Zimbabwe. In spite of these legislations, special education in Zimbabwe still lags behind the entire educational system (Chitiyo, 2006).

In an assessment of primary and secondary schools conducted by the National Education Advisory Board, Chakanyuka, Chung, and Stevenson (2009) estimate that, in Zimbabwe, as many as 469,000 children may require special needs education. They further estimated that only 30% of these children were able to access special needs education by 1998. Thus, 70% of children with disabilities and other special educational needs were and possibly still are being denied access and their right to education (Ncube & Hlatywayo, 2014). On the backdrop of the fact that these statistics fall within the period when the country's overall literacy levels were pegged at over 97%, the picture painted is a cause for serious concern.

According to Ncube and Hlatshwayo (2014), special needs education provision in Zimbabwe has until recently been double thronged, curing and segregating learners with special needs. In curing, special educational needs are effectively diagnosed and cured, while in segregation children with particular special challenges are taken out of mainstream schools and placed in special classes, special departments within the school or in special institutions. The curing is considered a short-term intervention, while the segregation is considered a long-term intervention. Although these curing and segregation provisions are still widely practised in many schools in Zimbabwe, numerous challenges arise from their implementation. It has only recently dawned on some schools in Zimbabwe that, to counter these challenges, there is a need for combining the short- and long-term arrangements to better serve children with special needs. To accommodate these new developments, it is imperative that new legislation repositions which redefines the purpose of special needs education in ordinary schools and moves towards inclusive education. Thus, in line with provision of quality education to all, it is inadequate for ordinary schools in Zimbabwe to plan for programmes that only cure short-term difficulties with the hope of passing on the long-term ones and expecting them to be solved by other institutions elsewhere (Ncube & Hlatywayo, 2014).

Zimbabwe's first 10 years of independence focused on increasing the number of schools so that the majority of learners could easily access formal education within their communities (Kapungu, 2007); however, the newly established schools did not have qualified staff to work in them. This resulted in the recruitment of unqualified personnel as temporary teachers to serve in the increased number of schools (Mutambara, Phoshoko, & Nyaumwe, 2016). In the following 10 years, policymakers' drastic changes were put in place to improve teacher quality in the country. These changes included the Zimbabwe-Cuba teacher education programme where school leavers were sent to Cuba, a 4-year teacher education programme in mathematics and sciences.

To supplement the Cuban trained teachers, local initiatives, like the Degree in Mathematics Education, increased enrolments in the Graduate Certificate in Education and the Science Education In-service Teacher Training (SEITT) were offered at the University of Zimbabwe. In addition, a university was established in each of the country's ten provinces to complement the existing teachers' colleges for the purposes of increasing university graduates teaching in schools. The Bindura University of Science Education replaced the Cuba-Zimbabwe programme and became a university mandated with the responsibility of providing the nation's mathematics and science teachers in the country, while the University of Zimbabwe transformed itself into a postgraduate centre offering postgraduate diplomas, masters and PhD studies in order to produce academics to staff the new universities (Mutambara et al., 2016).

Albeit all these commendable efforts to improve mathematics education in Zimbabwe; particular integration of special needs education still requires attention. A study conducted by Chireshe (2013) revealed that the implementation of inclusive education in Zimbabwe was perceived to be presently affected by lack of resources. Previous studies conducted in Zimbabwe (Chireshe, 2011; Mpofu, 2000; Mpofu, Kasayira, Mhaka, Chireshe, & Maunganidze, 2007; Peresuh, 2000) cited the shortage of resources as a huge challenge towards the implementation of inclusive education. The lack of resources is aggravated by the high teacher-pupil ratio (1 to 40) in many Zimbabwean primary schools. Because of this high teacher-pupil ratio, teachers have limited room to effectively cater for children with disabilities. In addition, the negative attitudes towards children with disabilities still prevailing in the country also negatively affect the provision of resources to them. The funding provided for education in the country in general and inclusive education in particular is insufficient (Chireshe, 2013).

Therefore, if quality mathematics education is currently inadequate for the mainstream learners in Zimbabwe, the situation is even direr for learners with special needs. By extrapolation if mathematics is used as a selection criterion for accessing higher levels of education, learners with special needs are distanced even further from this access.

Case Study of Mathematical Inclusion in a Full-Service School in South Africa

In order to support teachers in this focal, the *Focus on Primary Maths project* had the following aims:

1. Support 'focus on maths' teams, involving interested staff at each school, which focused attention on numeracy teaching and learning.
2. Provide staff professional development in mathematics education through:
    (a) Offering mini professional development seminars on topics of interest related to mathematics education.

   (b) Seeking out appropriate professional development interventions for interested staff.
   (c) Collaborative co-teaching interventions which allow the realisation in classroom practice of the groundwork laid though training/professional development.

3. Identify areas of particular weakness as evident in the Annual National Assessment (ANA) and systemic assessment results from each school and work on classroom level interventions for these problem areas.
4. Share the lessons learnt and approaches used in trying to improve numeracy results in particular case study schools with a wider evidence drawn from project documentation as well as teacher feedback on particular interventions to outline the project inputs in creating a professional learning community focused on Foundation Phase mathematics.

What Was Done to Support Teachers?

The original intention to work with interested staff was changed by school leaders to incorporate all Foundation Phase staff. As such, each school used their grade structure to create small groups of teachers (four teachers in each grade). Grade meetings were scheduled weekly, and all the teachers met to discuss their weekly plans and experiences for all of the subjects. A mathematics-focused grade meeting (referred to as mini-seminar) was held approximately once per term for the first three terms of the year, was attended by the project leader and lasted approximately 40 min to 1 h. The focus of discussion was on how mathematics was being taught across the grade and sharing ideas for teaching identified areas of difficulty. In 2012 and 2013, approximately 36 mini seminars per year were held (18 at each school). In 2014 the grade meetings continued at each school; however, the project leader no longer attended these sessions. The teachers in the full service school did not offer comments on the grade meetings, but for the majority of teachers in the suburban school, these meetings were the most useful component of the intervention:

> ‘Many discussions happened after meeting with [the project leader] on a specific topic. Very good for all teachers in grade.’ (suburban school teacher, 2014)

Once per term (in terms 1–3), the Foundation Phase staff from both schools were brought together for a 2-hour seminar session. These sessions were facilitated by the project leader and focused on topics of relevance to Foundation Phase mathematics (e.g., ‘functions patterns and algebra in the early grades’ or ‘addition and subtraction’). These seminars opened with a short mental math activity, followed by some expert input which included facilitation of mathematical problem-solving tasks. The seminars were intended to engage the participating teachers in thinking mathematically themselves and to support the grade teams in discussing and sharing their ideas:

> ‘Enjoyed all seminars and little discussions with [the project leader] - most valuable of all. Enjoyed working through issues experienced when trying out what [the project leader] had shared previously.’ (suburban school teacher, 2014)

> ‘Enjoyed the interaction with FOPM [the project] and [the suburban school]. Definitely made me more confident in teaching mathematics.’ (full service township school teacher, 2014)

Through the seminar discussions, the lack of a productive learning environment was frequently identified as hindering mathematics learning at the full service township school. Teachers complained of disruptive behaviour from learners that was exacerbated when attempting to use concrete materials and mathematics equipment. The deputy principal at the full service township school therefore innovated to allocate one of the CPS classrooms to be a dedicated ‘maths hub’. This venue was staffed by a teaching assistant who was tasked with managing all the mathematics equipment and stock for Foundation Phase. Concrete materials used daily in classrooms (such as number lines, bead strings and plastic bottle top counters) were kept in each classroom, but all specialised equipment (metre rulers, bathroom scales, bottles and cups for measuring fluids, maths-related games such as snakes and ladders, puzzles, card games, etc.) were all centrally stored in the maths hub. From 2013 this venue was timetabled to allow each class an hour session once a week, within the maths hub with their class teacher. Half of the class came to the maths hub for a ‘practical’ mathematics lesson, and the other half were occupied in class (often with departmental numeracy workbooks), supervised by a teaching assistant. The full service township school teachers commented on the equipment being useful:

> ‘Materials donated really helped with lessons and teaching. Resourcing our maths hub. Thanks. Teachers make use of it in the classes as well.’ (full service township school teacher, 2014)

From 2014 this maths hub venue was also used for afternoon maths clubs at the full service township school. The top 10 attaining learners in Grade 2 were invited to join a once-a-week maths club, where they spent 40 min engaging in mental maths and game-based mathematics activities. This was led by the head of Foundation Phase of the suburban school who modelled a club session with a teaching assistant. A teaching assistant then replicated the model lesson with three other club groups.

Staff Professional Development

Over the 3 years, a total of 18 teachers (2 from the suburban school and 16 from the full service township school) completed an intensive mathematical thinking course facilitated by the African Institute for Mathematical Sciences School Enrichment Centre (AIMSSEC). This course comprised an intensive 10-day residential course during the school holidays and was followed by a 3-month period during which two assignments were completed. The assignments directed teachers to plan, teach and then write a reflection on one mathematics lesson. The course focused on mathematical thinking skills, making use of content areas that were particular to the South African primary mathematics curriculum. The course included a diagnostic pretest as well as summative written test of primary school level mathematical content. Teachers on the course were supported in completing their assignments by the project through joint planning and reflection sessions facilitated by a project mentor who came to visit them at their schools. In addition all of the Foundation Phase staff in each school participated in a 2 h 'I Hate Maths' seminar facilitated by Prof Mike Askew.

For the suburban school, in 2014 where there were few teachers enrolled on the formal mathematical thinking course, the teachers in each grade engaged in two cycles of mini-lesson study process. At a seminar they identified an area of teaching need (making use of the ANA analysis) and then jointly planned, taught and reflected on their lessons on this topic. However, their lesson reflection was oral and not always formally written. Mini-lesson study topics included collaborative lessons on position in space, analogue time, grouping the tens, 'I am thinking of a number' and '1 more than/1 less than'.

In 2014 each teacher was paired with a 'buddy' from the other school who taught in the same grade. Collegial exchanges, where a buddy could observe and support a mathematics lesson in a different school environment, were arranged. There were mixed reactions to this initiative:

> 'Learned good techniques from my buddy which I implemented in my class as well.' (full service township school teacher, 2014)
>
> 'I really enjoyed having a [buddy] visit my classroom and to spend time there. I feel I had my eyes opened to a whole new world and learnt different teaching methods.' (suburban school teacher, 2014)
>
> '[Buddy exchange] was a great idea but was not beneficial as my visit to [the full service township school] did not involve Maths. My buddy did not come when she was scheduled to. When she did come it was for 20 min.' (teacher, 2014)
>
> '[Buddy exchange] was very beneficial & interesting; a different person's interpretation of the same curriculum.' (suburban school teacher, 2014)

Responding to Annual National Assessments (ANAs)

The ANA results were captured annually using a spreadsheet that captured question-by-question responses from each learner. The data capture of these ANA responses was conducted at each school with project data capturers, while the teachers marked the ANAs. The data capturers had academic backgrounds in mathematics education and were also available to support the internal moderation processes.

The ANA results were then analysed and the analysis circulated to all Foundation Phase staff and school managers. This process resulted in the identification of key topic areas, specific to each grade, which were prioritised for additional teaching and support during the fourth term, and identified as seminar topics.

> 'Thank you for the format of how to analyse question paper/answers.' (suburban school teacher, 2014)
>
> 'I liked the help with marking ANA's and making it easier for us to understand [areas of common difficulty].' (full service township school teacher, 2014)
>
> '[We are] Still using [the spreadsheet] programme for analysis of termly assessments for collecting data.' (suburban school teacher, 2014)

Several co-teaching interventions were undertaken relating to areas of particular need. These involved the project leader leading classroom interventions (over approximately 10 consecutive teaching days, with the normal classroom teacher present) to explore and test approaches to the teaching of particular 'hard to teach' topics. These co-teaching interventions included additive relations word problems, division, fractions and models of subtraction.

Sharing Lessons

There were several ways in which lessons emerging from the project were shared.

Firstly, a series of termly parent seminars were held at the suburban school, which were referred to as 'Keeping up with the Kids'. The first author facilitated these sessions to ensure that the approaches and methods being adopted in the school were understood and not resisted by parents.

Secondly, the teacher's assignments (completed for the mathematical thinking course) and/or mini-lesson study reports and/or discussions held during mini seminars were further polished and refined for presentation at Association of Mathematics Educators of South Africa (AMESA) conferences. Over the 3 years of the project, 15 *How I teach* papers or workshops were presented at AMESA national congresses, and 21 presentations were facilitated at AMESA regional conferences.

Thirdly, both schools responded to requests to facilitate sessions with Foundation Phase teachers organised either with their district (via the WCED curriculum advisor) or as arranged with other districts in the Western Cape by AMESA.

Finally, towards the end of the project (in September 2014), a mini-conference was held at the two project schools. The first afternoon of this mini-conference included series of presentations by teachers addressed to their school-based colleagues on how they approached the teaching of a particular mathematics topic. On the second afternoon of the mini-conference, colleagues from other schools in their districts joined the project teachers, and a series of presentations and workshop ideas on key Foundation Phase topics were offered by the teachers. The following comment captures a teacher's personal reflection on how their approach to mathematics teaching changed and integrates several of the project aims:

> 'I am more confident at teaching maths. Presenting at conferences wasn't my thing but got the confidence to present. Doing the AIMSSEC course also made me reflect all the time and research better ways to teach a topic. Number line: I never ever used a number line ever! So much more confident using it thoroughly in my lessons.' (full service township school teacher, 2014)

Were There Any Changes in Mathematics Learner Outcomes?

Considering ANA data from a low baseline of 25% of learners passing in 2011, the full service township school saw annual improvements in mathematics attainment in relation to the percentage of children passing, which culminated with 84% of learners passing in 2014. The improvements were also evident in the mean results for the ANAs. The full service school moved from passing percentages that were below the national average to well above it.

Similar evidence of improved mathematics attainment were evident in the WCED systemic results, from 25.3% of Grade 3 learners passing in 2011 to 67% of learners passing in 2014. Of particular interest were the sustained improvements within the school environment evident in 2015 and 2016, where both the ANA pass rates and the WCED systemic results were retained – despite the professional support to the school being withdrawn. Of further interest was the apparent impact of improving whole-class teaching of Foundation Phase learners (Grade 1–3) on the future learning trajectory of these learners in higher grades. This was evident in changes in Grade 6 standardised assessments. The provincial curriculum advisor in the department of education offered the following congratulations:

> 'Congratulations to all concerned for improving the [full service township school's] pass rate in Gr 6 Mathematics from 10% to 50% over three years while the provincial pass rate rose by 12%. The school is focused on getting the foundation right for these high-risk learners. It has drawn half of them into a space where success is possible. I am deeply moved by the evidence of substantial improvement from this independent test.' (Cameron, February 2016, curriculum advisor, personal communication)

It is worth noting that the school staff, learner intake, learner selection and school management team remained invariant over the intervention period and that the significant improvements in mathematics attainment were not as stark in language attainment (in either ANAs, or WCED systemic results) over the same period. This supports the conjecture that the improvements in mathematics were a result of the intervention and not of other changes in the school environment.

Conclusion

The regional context for mathematical inclusion in the four countries considered in this paper is significantly constrained by lack of resourcing, access to mathematics learning (in terms of enrolment) large class sizes, language issues as well as an identified lack of teacher knowledge for teaching mathematics and particularly for teaching children with mathematics learning difficulties. Within such a context, it is imperative to target resourcing so that approaches and models for more inclusive practices can be developed and scaled. A South African example of prioritising investments and capacity building at only 500 schools – to create 'full-service schools' which can model more inclusive pedagogies – is one possible response.

This intervention study in the South African full-service primary school provides an example of working on a context where there are very limited options available relating to special needs intervention for mathematics. The high proportions of learners not meeting grade-level expectations, along with the lack of resourcing for small-group and / or individual intervention, meant that the key opportunity for change was in addressing whole-class Foundation Phase teaching as a barrier to learning.

There remain many challenges within this environment – some of which may only be changed with more resourcing to allow for deeper intervention using the wider suite of available special needs interventions (including small-group and individual interventions). However, the teacher development intervention through a professional learning community focused on knowledge for mathematics teaching at Foundation Phase appears to have succeeded in putting the school onto a new trajectory with improved attainment in mathematics.

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Chapter 16
Mathematical Learning and Its Difficulties in Australia

Robert A. Reeve

Australia: The Big Picture

The purpose of this chapter is to describe current perspectives on math learning difficulties in Australia. It draws on official government documents and research papers. In June 2016 Australia had a population of 25 million, of which 30% were born overseas, and many immigrants do not speak English on arrival (Australian Census, 2016). Moreover, 180 separate Aboriginal languages are spoken by different groups of Australian indigenous people, the original inhabitants of the continent. Nevertheless, English is the language of instruction in all schools for all students.

Australia comprises six states and two territories. Education is primarily the responsibility of the states and territories because the federal government in Canberra does not have constitutional powers to enact laws on education. However, the federal government's Department of Education and Training helps fund independent and private schools. The Australian Curriculum, Assessment and Reporting Authority (ACARA, 2017), a federally funded, independent statutory authority, however, is responsible for crafting a national curriculum, including a mathematical curriculum. Nevertheless, the states and territories may have different curricula, education policies, and terminology. As noted below, these differences are evident in the math curricula.

The UN's Human Development Education Index (2013) indicates that Australia has the second highest number of students who complete secondary education in the world. Reports on mathematical achievement of Australian students are somewhat mixed. The Programme for International Student Assessment (PISA) data, for example, suggests there has been a decline in Australian students' math literacy over the last 12 years (Thomson, De Bortoli, & Underwood, 2016). Students

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achieved an average score of 494 points in mathematical literacy, slightly higher than the OECD average (490 points). According to the PISA data, only 44% of students achieved the national proficient standard. In contrast, the Australian National Assessment Program – Literacy and Numeracy (NAPLAN, 2016) data shows an overall pattern of stability in students' numeracy ability, with approximately 95% of students meeting the national minimum standard (ACARA, 2017). Indeed, the Trends in Mathematics and Science (TIMMS, n.d.) data shows math abilities have remained relatively stable over time. However, 21% of Grade 4 students and 25% of Grade 8 students achieve at the low international benchmark (Thomson, Wernert, & O'Grady, 2016). And 9% and 11% of Grade 4 and 8 students, respectively, perform at or below the low international math benchmark.

While the TIMMS data could be interpreted as showing 10% of Australian students have MLD, this inference should be treated with caution since it is based on a test cut point score, rather than a definition of MLD per se. In fact, there has been little emphasis on identifying or defining the meaning of learning disabilities in Australia (Elkins, 2001). Australia has only recently begun to collect data on LDs, including MLD (Education Services Australia, 2016). The first Nationally Consistent Collection of Data on School Students with Disability occurred in 2015. Initial findings suggests about 10% of students have some form of disability. As yet however, there is no specific information about the prevalence of different LDs, including MLD, in Australia (Education Services Australia, 2016).

Australia: Educational Policies and MLD

While some attempt has been made by state and religious education authorities to estimate the prevalence of different forms of learning disabilities in children, these estimates vary significantly. For example, the state of New South Wales estimated 4.7% of their students receive indirect funding for teaching specialists and a further 7.3% require support from specialists at some point in their schooling (NSW DEC, 2011). In contrast, the Catholic Schools Diocese of Sydney, NSW, indicated that approximately 18% of children likely have some specific learning difficulty. Interestingly, it has been suggested Catholic schools tend to have the lowest prevalence of students with disabilities, compared to government and independent schools (Education Services Australia, 2016; Senate Standing Committee on Education and Employment, 2016). The latter committee noted that it is likely these difference are attributable to differences in LD definitional criteria (the committee itself, however, did not offer a definition of LDs).

With some exceptions (e.g., children diagnosed with autism), schools do not receive direct funds for children with special learning needs. Schools, however, may assign funds to specialist teachers (e.g., reading specialists) from their own budget. In addition, schools most often refer children with learning difficulties, including MLD, to a state-funded educational psychologist, attached to the school or groups of schools in an area.

Educational psychologists in Australia complete a 4-year undergraduate degree in psychology, followed by a 2-year certificated professional master's degree in educational psychology. The latter degree provides specialist training in the identification and treatment of developmental and behavioral disorders more generally (from anxiety disorders to some learning difficulties). Educational psychologists may work with parents and/or classroom teachers to implement remedial math programs. As far as can be determined, math intervention programs are pragmatic in nature, tend to be based on practice and do not appear to be based on a programmatic or specific model of MLD learning difficulties. For example, no distinction is made between math learning deficits (including dyscalculia), and developmental differences due to developmental delays in designing intervention programs.

While relatively little attention has been paid to MLD or dyscalculia in Australian educational psychology training programs, this is beginning to change, albeit somewhat slowly. Several specialist groups (e.g., the Learning Differences Convention, (n.d.) AUSPELD) have invited renowned international specialist in math difficulties (e.g., Butterworth and Chinn) to give public lectures on MLD/dyscalculia and its treatment. These presentations have led to a high inclease of interest in MLDs and have been extremely well attended.

There is a growing awareness in Australia of a need to differentiate students with LDs from those experiencing difficulties in learning. It is acknowledged that there are many reasons for students being poor at math (see Butterworth, 2005; Reeve & Waldecker, 2017). And it is occasionally noted that most assessment methods may detect students with low achievement, rather than students with a LD. With respect to MLD, it has long been argued Australian education authorities need to distinguish dyscalculia from math learning difficulties per se (Peard, 2010). It is worth noting that AUSPELD, a highly respected, nationwide, independent organization that supports people with learning disabilities, estimates that at least 20% of Australian children currently experience some form of learning difficulty, of which 80% have dyscalculia (AUSPELD, 2015). Interestingly, AUSPELD is one of the few national groups in Australia to recognize dyscalculia as a specific learning disability and has begun to offer workshops on dyscalculia assessment and treatment on this from of MLD (see below).

Australia: Theories and Educational Practice

Curricula and state-based policies form the basis of teacher training, educational practices, and ipso facto teacher knowledge about mathematics in Australia. Insofar as "learning theory" guides educational practices, pedagogical theory appears to be implicit than explicit in curricula documents.

As noted above, the Australian Curriculum, Assessment and Reporting Authority (ACARA, 2017) is responsible for a national curriculum and assessment program. ACARA characterizes numeracy as understanding the role of mathematics in the world and having the capacities to use mathematical knowledge and skills purposefully (2013, p. 31). And mathematics is described as providing students with the skills and knowledge in number and algebra, measurement and geometry, and statistics and probability. These descriptions distinguish a content and a functional view of numeracy as the capacity to interact with situations involving mathematics. This distinction is illustrated in Table 16.1 which describes abilities to be acquired at the first level (within a year of beginning school) of numeracy and mathematics (ACARA, 2016).

The behaviors described in Table 16.1 may be classified as the ability to follow an instruction. Some behaviors, for example, could occur in play (e.g., following the actions of pouring liquid into two containers), while others require use of specific mathematical language or symbols (i.e., sorting numbers into ascending order). However, most behaviors require a receptive understanding of numeracy language. The skills listed in the mathematics column (column two) of Table 16.1 rely on students using expressive language (e.g., providing the names of days, using counting words, describing fractions, explaining a classification system). In the latter case, the use of language is more specific to mathematics content and is less likely to be used in everyday life and thus potentially more challenging for the young student.

Table 16.1 First level of numeracy (ACARA, 2016)

Numeracy capability	Mathematics learning
Estimating and calculating whole numbers Sorting numbers into ascending order; showing anticipation that something will happen on the count of 1, 2, 3; and recognizing that a pile of books is getting bigger when adding to it	Number and algebra (including money) Count by naming numbers in sequence to and from 20, connect number names to numerals, subitise small collections, recognize, describe, and order Australian coins according to their value
Recognizing patterns and relationships Recognizing patterns in games or music, continuing an alternating pattern	Number and algebra Copy, continue, and create patterns with objects and drawings, sort and classify familiar objects, and explain the basis for these classifications
Using fractions, decimals, percentages, ratios, and rates Folding a piece of paper into equal parts, pouring liquid into two containers	Number and algebra Recognize and describe one-half as one of two equal parts
Using spatial reasoning (including shape) Sorting objects by features of shape, size, and color, grouping 2D shapes	Measurement and geometry Describe position and movement, sort and name 2D shapes and 3D objects in the environment
Interpreting statistical information (and chance) Recognizing that it might or might not rain tomorrow, follow actions to a song or dance	Statistics and probability Answer yes/no questions to collect information, identify outcomes of familiar events involving chance, and describe them using everyday language such as “will happen”, “won’t happen”, or “might happen”
Using measurement (including time) Comparing the length of two objects and indicating which one is longer, associating familiar activities with times of the day or days of the week using pictures or technology	Measurement and geometry Connect days of the week to familiar events and actions, measure and compare the lengths and capacities of pairs of objects using uniform informal units

From a theory perspective, it could be argued that the focus in the early years is consistent with an embodied model of cognition which suggests knowledge is derived from action; however, the curriculum appears more pragmatic than informed by theory. ACARA focuses on three broad strands of mathematics (number and algebra, measurement and geometry, and statistics and probability). These strands are rooted in historical education practice and have been interpreted in the context of different theoretical perspectives (Piaget and Vygotsky), rather than motivated by developmental theory. As yet, there is little evidence that contemporary research from the neurosciences on the basis of mathematical understanding has impacted the ACARA math curriculum or other curricula in Australia.

As noted earlier, education is the responsibility of the Australian states, each of which may have a different educational perspective to ARCARA. As an example, the Victorian State Curriculum (VCAA, 2017) has adapted ACARA's curriculum for its own purposes. While the content of the Victorian curriculum is similar, VCAA combines numeracy and mathematics content into a single content area. Moreover, VCAA (2017) explicitly describes the learning skills of diverse learners and students with disabilities. The focus progresses from a pre-intentional level (i.e., where students are reliant on support of their teachers) to intentionally engage in learning with decreasing teacher support to become a more independent learner (VCAA, 2017). VCAA describes levels of increasing participation expected in the early years of schooling (see Table 16.2).

It could be argued the VCAA's focus is more consistent with a Vygotskian model of learning. Further, the Victorian curriculum levels were written specifically to support the educational needs of students with disabilities. The VCAA explicitly states evidence for improved learning outcomes through early, supportive and interactive numeracy practices for students with disabilities.

The influence of developmental theory is perhaps more explicit in the state of Victoria's early education curriculum documents. The Victorian Department of Education and Early Childhood Development (2014) document "Mathematics learning pathways for children from birth to five year," for example, draws explicitly on the work of Clements and Sarama (2007, 2009) to illustrate expected progressions in preschool numeracy skills (see Table 16.3). While typical developmental changes from birth to primary school entry are described, the authors acknowledge developmental changes and developmental pathways are highly variable. Moreover, the document implies math ability rests on early understanding of cardinal and magnitude concepts. This implication is consistent with a neurocognitive model of mathematical development (Gelman & Butterworth, 2005).

Definitions in MLD in Australian States and Territories

There is no consistent definition of learning disabilities across the Australian States. As Skues and Cunningham (2011) note: "Identification of students with learning difficulties is often left to the individual classroom teachers and other professionals

Table 16.2 Mathematics content from the A–D levels of the Victorian Curriculum (VCAA, 2017)

Sub-strands	Statements of achievement		
Number and place value	Respond to objects being counted and distributed	Explore the concept of "none", "one", and "more"	Compare and order two collections according to their quantity
Money and financial mathematics	React to everyday financial situations involving money	Respond to everyday financial situations involving money and match notes and coins	Use money in everyday financial situations and match coins to two-dimensional images
Patterns and algebra	Respond to the identification of objects	Pair identical objects from a small collection, and recognize simple repeated patterns	Identify repeated routines and sequences in everyday events
Using units of measurement	Respond to objects based on length	Compare objects using direct comparison	Compare two objects based on measurement attributes of length
Shape	Respond to familiar everyday shapes and objects	Identify whether two shapes or objects are the same sort or not	Match two familiar two-dimensional shapes and three-dimensional objects
Location and transformation	Respond to the movement of an object	Respond to a simple statement about location or direction	Locate familiar three-dimensional objects in the classroom when they are named
Data representation and interpretation	Respond to objects relevant to the given context	Experience data display being interpreted	Experiencing data being used for decision-making in everyday situations

(e.g., speech pathologists, psychologists), who are encouraged to select from a range of informal and formal test methods [to identify a learning difficulty]." Moreover, as is evident from the following examples, the Australian states differ markedly in reference to MLD and or dyscalculia in official documents.

The Australian Capital Territory's Department of Education and Training Taskforce on Students with Learning Difficulties Final Report (2013) reflects the concerns about LD more generally in Australia:

> The use of inconsistent terminology added to the complexity of the work of the Taskforce. Nationally and internationally, learning disability, specific learning disability, learning difficulty and to a lesser extent learning disorder and learning difference are used to describe the same things and also different things. Prevalence rates therefore vary due to differing definitions. There has been little debate around the definition in Australia and the inconsistent use of the terms learning disabilities and learning difficulties is a significant issue for educators and families. Until the issue of terminology is resolved, ambiguity and resultant implications for support remain, as does clarity concerning the definition in respect to the Disability Discrimination Act 1992 (DDA) and the Disability Standards for Education 2005.

The Victoria Department of Education and Training Review of the Program for Students with Disabilities (2016) does not define learning disabilities but nevertheless refers to programs for students with learning disabilities, specifically dyslexia.

Table 16.3 Math pathway progressions (DEECD, 2014)

Learning pathway	Milestones along the learning pathway			
Number skills	Subitising	Learning number words, first the order to ten, then beyond	Enumerating objects	Mastery of the cardinality principle
Comparing	Puts objects, words, or actions in one-to-one or many-to-one correspondences	Puts objects in rigid one-to-one correspondence	Compares and selects the largest of two differently sized collections	Puts objects into one-to-one correspondence but may not fully understand that this makes equal groups
Counting	Names some number words, no sequence	Verbally counts with separate words, not necessarily in the correct order over five	Verbally counts to ten, with some correspondence to words	Keeps one-to-one correspondence between counting words and objects
Pattern and structure	Recognizes a simple pattern	Fills in missing element in a pattern, first with ABAB patterns	Duplicates ABABAB pattern in a different location	Extends AB repeating patterns
Length measurement	Identifies length as an attribute	Physically aligns two objects to determine which is longer	Uses a third object to measure and compare the length of two objects	
2D shape	Comparing real world objects	Matching familiar shapes	Recognizing circle and square	Recognizing and comparing a wider variety of shapes of different sizes

Victoria offers a language support program for schools to assist supporting students with language disorders. MLD and/or dyscalculia is not mentioned.

The South Australia Department of Education and Child Development (2016) refers to learning disabilities as applying to a small group of learners who have more difficulty with schoolwork than expected for their age and ability. Factors which imply a learning disability include dyslexic-type confusion of letters or numbers, or confusion of letters or numbers within a sequence, as well as sequencing and short-term memory difficulties, both auditory and visual. The document explicitly mentions dyscalculia as a difficulty understanding or using mathematical concepts and symbols. Nevertheless, while there are general policy/practice recommendations for dyslexia, there are no similar policy documents or set of recommendations for dyscalculia.

New South Wales and Queensland support students with disabilities (see http://www.schools.nsw.edu.au/studentsupport/programs/disability.php; and http://education.qld.gov.au/students/disabilities/adjustment/pdfs/eap-handbook.pdf, respectively); however, neither state classifies MLD as a learning disability.

The above review should not be regarded as a lack of interest in LD more generally. The federal government's Senate Education and Employment References Committee Report (Australian Senate Standing Committee on Education and Employment, 2016) discussion on the needs of students with LD highlights this point (even though MLD is not mentioned explicitly):

> One dimension of the problems with data[…] is that current funding models have failed to adequately fund the education of students with disabilities because they have taken too narrow a definition of disability. If a student's disability is not recognised as such in the funding model, that model clearly cannot provide the financial assistance necessary to properly assist that child's access to education.

> Due to the inadequate support in the current school environment families are required to repeatedly advocate for their child's needs. This is particularly true for students who do not qualify for a diagnosis of intellectual disability (or any of the other specific funding categories) and hence must attend mainstream classes without teacher's aide support. The impact of this in practical terms is that even if an Individual Education Plan (IEP) is formulated by the school based on the child's individual needs, it is not always possible to implement recommendations due to lack of support staff. The end result is a maintenance or worsening of the child's behaviour and a stagnation of the learning process, resulting in unsatisfactory outcomes for all.

> The committee was concerned by evidence suggesting that many students have fallen through funding cracks because of limited information or narrow definitions of disability used in school systems, resulting in a failure to recognize need. An appropriate level of funding for students with additional needs in schools begins with adequate data on those students.

Of particular interest is that the committee recognized the lack of research on the education of students with disabilities and its relationship with school practices. They acknowledged that while research exists on the best practices in teaching mainstream students, little research has focused on how best to teach students with disabilities.

Moreover, it has long been recognized that many Australian teachers view numeracy and literacy difficulties as reflecting a common underlying deficit (Milton, 2001). While some schools identify numeracy difficulties through state-wide tests or national assessments, others used their own measures. Indeed, the report of the Numeracy Education Strategy Development Conference (AAMT, 1997) highlighted a lack of suitable assessment measures for assessing math difficulties, as well as a need to identify potential numeracy difficulties early in a child's education.

More recently, Skues and Cunningham (2011) noted that Australian teachers receive little or no formal training in identifying and treating children with LDs more generally. Indeed, White and Elkins (2000) examined the content of preservice primary education programs across Australia and showed that, while such programs provided limited literacy training, very few programs included any information about other LDs. This observation is consistent with Rohl and Greaves' (2005) findings that only one-quarter of beginning teachers felt prepared to teach students with LDs and only 10% of experienced teachers felt beginning teachers had sufficient training to teach students with LDs. The question of how students with LDs should be taught was not addressed.

The relative absence of pre-service and in-service teacher training on LDs is evident in the limited understanding among Australian teachers about the causes and characteristics of LDs. Watson and Bond (2007), for example, found over half of the teachers interviewed were unaware that students with LDs' intelligence is not below average. This limited understanding has implications for the support of students with LDs in mainstream classrooms.

Neuroscience and MLD/Dyscalculia in Australia

The last 20 years has seen a burgeoning interest worldwide in the neuroscience of mathematical cognition. This growth has been spurred by many factors (e.g., from an interest in neurobiological genetic origins and development of quantitative abilities, to claims about the domain specificity and/or generality of mathematical abilities, to the neurocognitive factors associated with atypical numerical cognitive development, and to the availability of brain scanning methodologies). With some exceptions, this contemporary neuroscience focus has had relatively little impact on perspectives on MLD in Australia or indeed the assessment of MLD (excepts see AUSPELD, 2015; Reeve & Gray, 2015, Reeve & Waldecker, 2017).

AUSPELD (2015) characterizes MLD and dyscalculia as a neurodevelopmental disorder and suggests clinicians (preferably with educational and/or developmental training) take into account a student's educational experiences, as well as his performance on a range of standardized tests. They recommend following the DSM-5 guidelines for assessing a learning disability, which specifies that learning disorders, including impairment in reading (dyslexia), and/or impairment in written expression, and/or impairment in mathematics (dyscalculia), are diagnosed through (1) a clinical review of the individual's developmental, medical, educational, and family history; (2) reports on test scores and teacher observations; and (3) evaluation of the response to academic interventions.

AUSPELD's assessment approach is consistent with recent changes to the definition of specific learning disabilities presented in the DSM-V (American Psychiatric Association, 2013), which has broadened its definition of LDs to include deficits associated with persistent difficulties in mathematical reasoning, arithmetic skills, reading, and writing. Markers of math learning impairment include, among other issues, (1) difficulties remembering number facts, (2) inaccurate mathematical reasoning, and (3) the speed with which individuals solve math problems.

Moreover they recommend a conventional clinical phased approach to identifying MLD, which they distinguished from general math learning difficulties. In particular, they recommend (1) assessing a child's response to math instruction within a classroom setting over an extended period (e.g., 6 months), (2) assessing performance on the DSM-V (American Psychiatric Association, 2013), and (3) using a specific assessment measure of MLD. They also suggest using the "Dyscalculia Screener" (Butterworth, 2003) for identifying a specific math learning disability. The screener is based on neurocognitive research findings, and its authors claim that dyscalculia is a persistent congenital condition (see Butterworth, 1999). Butterworth argued research strongly points to the existence of a brain-based number module that is based in the parietal lobe of the brain (Butterworth, 1999), specialized for dealing with numerical representations. He proposed that dyscalculia is probably associated with a dysfunction in this system. The screener comprises three timed tests: (1) dot enumeration, (2) number comparison, and (3) addition and multiplication (performance on these tests is standardized and age adjusted to provide a mean of 100 and standard deviation of 15). Difference in dot enumeration and number comparison abilities have been found to be related to math differences and, indeed, neurological functions.

Reeve, Reynolds, Humberstone and Butterworth (2012) analyzed Australian data for Dyscalculia Screener tests. They tracked 250 children, beginning at the age of 5, over the 6 primary/elementary school years on the screener and other cognitive and math measures. Because of large within-age and between-age variability in children's performance on the dot enumeration and number comparison measures, Reeve et al. used latent class analysis to identify different patterns of performance in measures. This analytic approach allowed the authors to identify the predictive significance of different dot enumeration patterns over time. On the basis of reaction times to different dot arrangements, Reeve et al. found three different patterns of performance could be identified, differences which persisted over primary/elementary school years. Moreover, these patterns were strongly linked to math performance and unrelated to standard cognitive measures (IQ, working memory, etc.). One of the dot enumeration patterns was specifically related to persistent poor math performance. Reeve et al.'s research confirm the validity of Butterworth's (2003) Dyscalculia Screener and show that it is possible to identify children with MLD on the basis of their dot enumeration profiles.

In a series of recent papers, Reeve and colleagues have confirmed the significance of different dot enumeration profiles in predicting preschool children's emerging math abilities (Gray & Reeve, 2014, 2016) and to use the identification of different profiles in a clinical assessment regime (Gray & Reeve, 2014; Reeve & Waldecker, 2017). Similar to AUSPELD, Reeve and colleagues suggest contemporary neuropsychological accounts on the origins and development of MLD have much to offer clinician and educators. Reeve and Waldecker (2017) in particular review the effectiveness/ineffectiveness of commonly used assessment techniques in identifying deficits known to be associated with developmental MLD/dyscalculia. They also outline the components of effective interventions that might be included in an effective assessment of MLD.

The question of what constitutes an effective intervention for children with MLD is difficult to answer, and, indeed, there might be different solutions, depending of the form of MLD. Reeve and Waldecker (2017) suggest an assessment regime, similar to the AUSPED regime, outline above. Reeve and Waldecker argue math interventions must be fit for purpose. In their review, they note many interventions involve practice on math tasks children find difficult in the hope children will improve. Evidence suggest brute force intervention methods rarely work and may, indeed, be counter-productive. More precise math diagnostic methods are required if we are to offer evidence-based intervention programs. As worthy as this goal might be, changing conceptualizations of the basis of math difficulties poses challenges for those interested in the diagnoses and treatment of students with MLD.

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Chapter 17
Mathematical Learning and Its Difficulties in Taiwan: Insights from Educational Practice

Ting-Ting Chang, Jun Ren Lee, and Nai-Shing Yen

Introduction

Mathematics is a cognitive domain crucial for academic and professional success, and hence it is highly emphasized in formal education worldwide (Geary, 2013; Ko, 2005; Richland, Zur, & Holyoak, 2007). However, a large proportion of the world population—including school-aged children, adolescents, and even young adults—have suffered from severe problems of mathematical learning difficulties (MLD) (Butterworth, Varma, & Laurillard, 2011; Geary, 2004; Shalev, Manor, & Gross-Tsur, 2005), a serious psychiatric disorder characterized by specific deficits in numerical and mathematical abilities while their intelligence and other cognitive skills remain intact or even beyond normal (Butterworth et al., 2011; Geary, 2004). Estimates of the prevalence rate of MLD are comparable to those regarding reading difficulties (RD) (Geary, 2004; Shalev, 2007). However, much less attention has been paid to MLD relative to RD, particularly in East Asian countries such as Taiwan, possibly due to limited knowledge on the part of parents, teachers, and educators. Given that mathematics is a learned product highly affected by cultural factors, understanding the cross-national differences can provide valuable insights into characterizing the cognitive and educational profile of MLD. In this chapter, we review the main issues of mathematical learning, including students' academic achievement, learning difficulties, MLD identification, and placement in East Asia,

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with a specific focus on Taiwan, taking a distinctive educational perspective. We first briefly discuss the cultural background of Taiwan, which deeply influences parents' perception of raising children (Tzeng, 2007). We then address the national differences in students' mathematical performance by summarizing the statistics on international comparative data—in particular, the recently released Trends in International Mathematics and Science Study (TIMSS) data. Next, because the current local educational authorities only recognize the general category of learning difficulties (LD), we introduce the educational policies used for diagnosis and determining children identified with LD. Finally, we close the chapter by summarizing the assessment tools predominantly used to diagnose MLD in Taiwan. Within each section, we raise issues crucial for researchers and educators to consider when conducting local MLD research. Through this chapter, we seek to not only emphasize the specific national profile of MLD but also highlight the consequences of the divergent cultural factors that have a bearing on the shaping of human learning and cognition.

The Cultural Background

Over the long course of Asian culture, parents have believed that it is necessary for their children to study ultra-diligently to achieve future success (Phillipson & Phillipson, 2007; Shek & Chan 1999). There is a strong belief that the ultimate personal achievements are dependent on having well-paid jobs or a high academic degrees. This has contributed to a unique Asian parenting style such that parents expect their children to study extremely hard during their school years and continue to pursue higher education as much as possible. In Taiwan, the compulsory education policy used to require only 9 years of school, from 7-year-olds in elementary school to 15-year-olds in middle high school (Primary and Junior High School Act, 1979). Not until 2014 did the government provide three extra years of tuition and entrance exam–free senior high school education (The Senior High School Education Act, 2013). However, more than 80% of students continue to pursue higher degrees (Ministry of Education, 2015b). This has led to a percentage of 73% of 20-year-old young adults being enrolled in colleges or universities—a much higher proportion than the rates of 51.8% in the USA, 38.7% in the UK, and 37.7% as the OECD average (Ministry of Education, 2015b). Having a higher degree—even better, from a top school—profoundly fulfills Asian parents' expectations.

Those lofty expectations on the part of Asian parents also affect children in choosing their major fields. As expertise in science, technology, engineering, and math (STEM) is typically considered to lead to a higher chance of obtaining the best-paying jobs, Asian parents tend to encourage children to choose these fields as majors in college. This trend is international in scope. According to one survey conducted by Georgetown University (https://cew.georgetown.edu/cew-reports/whats-it-worth-the-economic-value-of-college-majors/), nine out of the top ten major fields of undergraduate degrees for Asian Americans were in STEM. Asians also constitute a great proportion of the individuals in STEM-specialized schools and industry positions. Given that expertise in STEM fields greatly requires mathematics and reasoning skills, mathematics has become one of the subjects to which Asian parents and students devote most of their efforts.

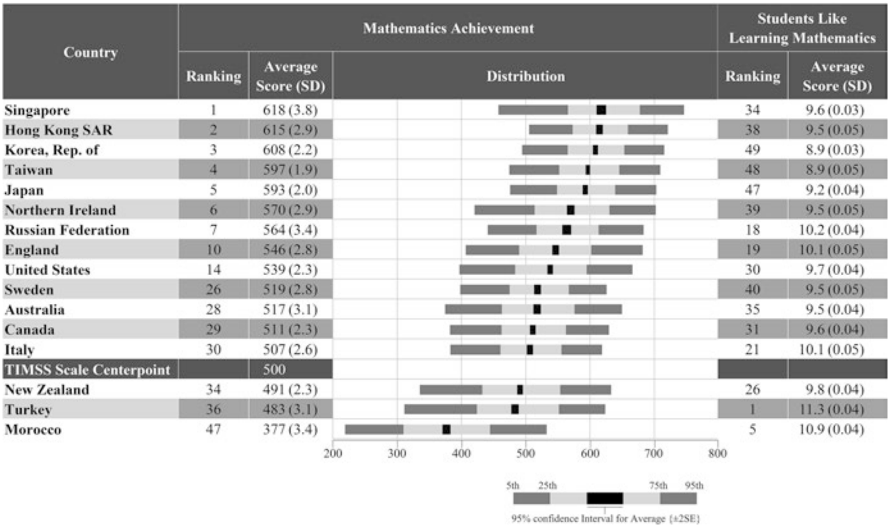
The unique Asian parenting style has led to a learning environment filled with high pressure and extreme endeavor. Under the stress and demand from the whole society, Asian students are required to strive throughout their school years. In Taiwan, there are more than 10,000 licensed cram schools providing academic training for language and mathematics after regular school hours (Education Bureau, 2016). This number is comparable to that of the formal schools and is still growing each year, suggesting that most of the students spend their after-school time on studying or supplementary learning. How parents evaluate these cram schools and teachers is generally based on how many students succeed in college and high school entrance.

Given that parental perception has such an impact on Asian culture, it is worth noting how much parents' expectations can influence children's academic performance. In one empirical study of 158 Hong Kong parents of primary school students, Phillipson and Phillipson found a strong positive correlation between parents' expectations and students' language and mathematics achievements. This suggested that those students with higher parental expectations do perform better at school (Phillipson & Phillipson, 2007). Parent perception is indeed a robust and unique predictor of students' academic achievement. Consistently, over the long trend of national comparative studies, such as TIMSS and the Programme for International Student Assessment (PISA), East Asians typically outperform their Western counterparts in mathematics (Mullis, Martin, Foy, & Hooper, 2016; OECD, 2016).

The Asian parent power does not lead to a perfect environment for learning. Rather, it produces an atmosphere of high pressure and competition in East Asian countries. In the TIMSS results, the East Asian students consequently displayed low learning motivation and low self-confidence, and placed a low value on mathematics even though their mathematical performance generally outperformed that of their non-Asian peers (Mullis et al., 2016). In the next section, we review the results of the recently published TIMSS and discuss the possible reasons contributing to the unique Asian profile.

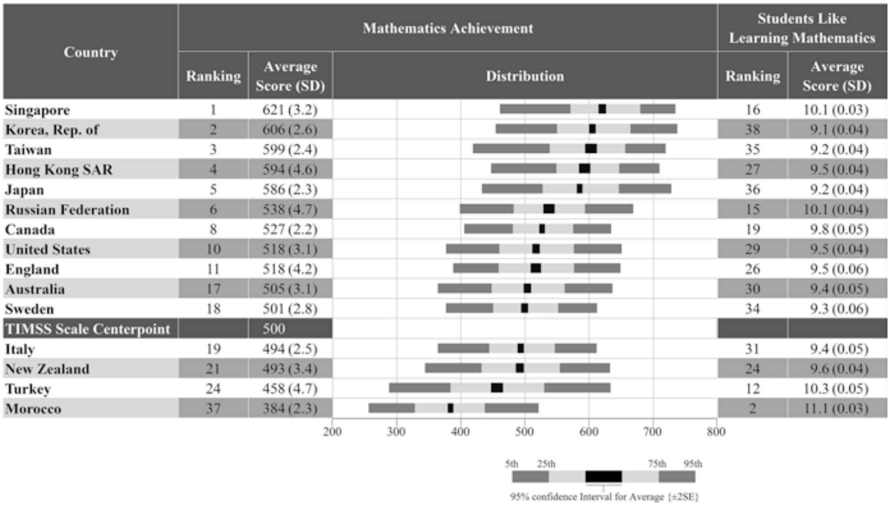
National Differences in Mathematical Learning

In past decades, cross-cultural comparison of mathematical skills has been of great interest to psychologists, cognitive scientists, and educators (Campbell & Xue, 2001; Mullis et al., 2016; Tang et al., 2006). Normative global surveys evaluating the classroom performance of 10- to 15-year-old adolescents every 3–4 years, such as PISA and TIMSS, have provided a close look at the cross-national differences in students' mathematical learning worldwide. According to the newly published TIMSS results, the countries in the top 5 chart of fourth grade (Fig. 17.1) and eighth grade (Fig. 17.2) mathematical achievements are Singapore, South Korea, Hong Kong,



Country	Mathematics Achievement			Students Like Learning Mathematics	
	Ranking	Average Score (SD)	Distribution	Ranking	Average Score (SD)
Singapore	1	618 (3.8)		34	9.6 (0.03)
Hong Kong SAR	2	615 (2.9)		38	9.5 (0.05)
Korea, Rep. of	3	608 (2.2)		49	8.9 (0.03)
Taiwan	4	597 (1.9)		48	8.9 (0.05)
Japan	5	593 (2.0)		47	9.2 (0.04)
Northern Ireland	6	570 (2.9)		39	9.5 (0.05)
Russian Federation	7	564 (3.4)		18	10.2 (0.04)
England	10	546 (2.8)		19	10.1 (0.05)
United States	14	539 (2.3)		30	9.7 (0.04)
Sweden	26	519 (2.8)		40	9.5 (0.05)
Australia	28	517 (3.1)		35	9.5 (0.04)
Canada	29	511 (2.3)		31	9.6 (0.04)
Italy	30	507 (2.6)		21	10.1 (0.05)
TIMSS Scale Centerpoint		500			
New Zealand	34	491 (2.3)		26	9.8 (0.04)
Turkey	36	483 (3.1)		1	11.3 (0.04)
Morocco	47	377 (3.4)		5	10.9 (0.04)

Fig. 17.1 Average scores and rankings of achievement and students' liking for mathematical learning among fourth graders worldwide. (Data retrieved from Mullis et al. (2016))



Country	Mathematics Achievement			Students Like Learning Mathematics	
	Ranking	Average Score (SD)	Distribution	Ranking	Average Score (SD)
Singapore	1	621 (3.2)		16	10.1 (0.03)
Korea, Rep. of	2	606 (2.6)		38	9.1 (0.04)
Taiwan	3	599 (2.4)		35	9.2 (0.04)
Hong Kong SAR	4	594 (4.6)		27	9.5 (0.04)
Japan	5	586 (2.3)		36	9.2 (0.04)
Russian Federation	6	538 (4.7)		15	10.1 (0.04)
Canada	8	527 (2.2)		19	9.8 (0.05)
United States	10	518 (3.1)		29	9.5 (0.04)
England	11	518 (4.2)		26	9.5 (0.06)
Australia	17	505 (3.1)		30	9.4 (0.05)
Sweden	18	501 (2.8)		34	9.3 (0.06)
TIMSS Scale Centerpoint		500			
Italy	19	494 (2.5)		31	9.4 (0.05)
New Zealand	21	493 (3.4)		24	9.6 (0.04)
Turkey	24	458 (4.7)		12	10.3 (0.05)
Morocco	37	384 (2.3)		2	11.1 (0.03)

Fig. 17.2 Average scores and rankings of achievement and students' liking for mathematical learning among eighth graders worldwide. (Data retrieved from Mullis et al. (2016))

Taiwan, and Japan (Mullis et al., 2016), all of which are East Asian countries. Notably, there is a massive gap—as large as 23 points—between the top 5 and the next highest countries (see Figs. 17.1 and 17.2), suggesting that East Asians significantly surpass other parts of the world. The fact that East Asian students excel in mathematical performance is a long-term effect (Mullis, Martin, Foy, & Arora, 2012). The possible reasons contributing to East Asians' superiority can be discussed, ranging from the educational environment to cognitive mechanisms.

One issue on which there is a consensus is that Asian teachers and parents are well-known for their high expectations of children (Aunio, Aubrey, Godfrey, Luejuan, & Liu, 2008; Leung, 2006; OECD, 2013a), as described in the previous section. Asian parents genuinely have high demands, causing students to fear failure to achieve success in school. In Western culture, in contrast, parents are more prone to teach their children to learn individualism and self-discipline. They typically show less control over their children's education. The cultural differences in parents' perception have been considered critical factors in the worldwide diversity of students' learning and performance. However, the price that Asian students pay for their mathematical superiority is overly low learning motivation. Across all TIMSS participating countries, both fourth graders (see Fig. 17.1) and eighth graders (see Fig. 17.2) in East Asian countries—including Hong Kong, Japan, Korea, and Taiwan—show tremendously low scores for students' self-reported liking for learning mathematics, even at the very end of the ranking chart (Mullis et al., 2016). These results have suggested that the major mathematical learning difficulty of Asian students is possibly their learning motivation, rather than their actual performance and competence. Lack of a self-initiated incentive for learning can likely cause great harm to school success and can result in students feeling reluctant to spend time on advanced math study.

A key interrelated question is the negative emotion toward schoolwork elicited by the highly demanding learning environment. Across individuals and countries, there is a trend for specific negative emotional reactions toward math—i.e., math anxiety (Ashcraft, 2002)—to impair students' math performance (Ashcraft & Krause 2007; Foley et al., 2017; Passolunghi, 2011). This tendency holds true in East Asian countries such that the negative relationship between nation-wise math performance and math anxiety is salient (Foley et al., 2017; OECD, 2013b). Within a country, several studies have attempted to investigate student math anxiety. In one large-scale behavioral study of 968 elementary and middle high school students in Taiwan, Wei confirmed that students do display high math anxiety (Wei, 1991). Furthermore, individual differences were observed in these tested students. Specifically, upper-level elementary school students, including fifth and sixth graders, exhibited higher levels of math anxiety than students at junior high school, i.e., seventh and eighth graders. Math anxiety was also found to differ between genders. Consistently with the findings in other countries (OECD, 2013b), girls in Taiwan experienced greater math anxiety than boys of the same age (Wei, 1991). Although these previous efforts have provided useful knowledge for preliminary understanding of math anxiety in Taiwan, still very little is known about how emotional factors actually influence mathematical cognition. Further studies of how math anxiety impacts math performance, along with longitudinal tracking, are needed.

Another educational factor the countries topping the PISA and TIMSS ranking charts have in common is a high standard of teacher selection and training (Jerrim, 2015; OECD, 2013a). Considering Taiwan as an example, even certified primary and high school teachers who are officially trained by a teacher training program have to pass a stringent recruitment process to obtain a tenure-track position. According to the latest *Yearbook of Teacher Education* published by the Taiwan Ministry of Education (2015a), out of the 40,000 certified teachers who apply for teacher recruitment for public primary and secondary schools each year, fewer than 10% pass the recruitment examination and are employed. Those who fail in the yearly recruitment can only work as substitute teachers and continue applying in subsequent years. The rigorous teacher selection processes have made teacher employment highly competent and have likely enhanced the quality of newly recruited teachers.

Asian students do not surpass their non-Asian peers only in conventional school-setting learning. Numerous cognitive–behavioral studies have shown that Asian students respond faster and more accurately to a basic level of numerical problems, such as naming digits and solving single-digit arithmetic problems (Campbell & Xue, 2001; Huntsinger, Jose, Liaw, & Ching, 1997; Miller, Smith, Zhu, & Zhang, 1995), even at the preschool age when formal education has not yet been introduced (Aunio et al., 2008; Cheng & Lorna, 2005). One factor often implicated is the language structures of number naming, which favor Asian students (Göbel, Shaki, & Fischer, 2011; Zhou & Boehm 2001). Unlike many of the alphabetic languages, such as English and French, most East Asian languages have systematic mapping between number words and number concepts using the base-10 system. For example, the number “twelve” is represented as “ten–two” in Chinese and Japanese. The number naming is more complicated in Western languages such as German and Dutch, in which the number “forty-five” is named “five-and-forty.” The regular number word system is helpful for Asian students to learn counting and retrieve number facts, leading to less error-prone and faster calculation (Aunio et al., 2008), even before children are able to understand the base-10 system and place value (Cheng & Lorna, 2005). The linkage between fraction naming and the fraction concept is also more transparent in East Asian languages. For example, the fraction “3/5” is called “three fifths” in English, whereas the Chinese name is “out of five parts, three,” which directly conveys the part–whole relationship between the numerator and denominator (Siegler, Fazio, Bailey, & Zhou, 2013). In this framework, when being taught to identify fraction numbers using “out of five parts,” three US children were able to perform as well as their Korean peers (Paik & Mix, 2003). These results support the concept that a transparent naming system is indeed helpful for young children to comprehend abstract numerical concepts that are not intuitive.

The phonological structure is another characteristic that gives East Asians an advantage in processing numbers. For example, Chinese has a distinctively monosyllabic structure. Each digit, therefore, has a one-syllable name. This means that Chinese-speaking students can utter more numbers than multisyllabic language speakers within a certain period of time. This suggests that Chinese language has a built-in reduced working memory load, which is a unique predictor for children in the early stage of learning arithmetic (De Smedt et al., 2009) and even adults (Imbo & LeFevre, 2010). Consistently, Chinese are less vulnerable to extra phonological and executive loading while solving two-digit addition problems in comparison with Belgians and Canadians (Imbo & LeFevre, 2009). These results have suggested that Asians seem to have a number retention advantage, inherited from their language system, over Westerners in mathematical learning.

Although the national populations have shown general superiority in comparison with other countries in mathematical content domains, there are individual differences within countries. The latest PISA and TIMSS results have demonstrated that across all the participating countries, the mathematics achievement of East Asian students—such as those in Taiwan, Hong Kong, and Singapore—demonstrated large standard deviations (see Figs. 17.1 and 17.2) (OECD 2016). This means that there is a huge gap between the high and low achievers, suggesting that not every student undergoes effective learning. A high percentage of Taiwan students did not acquire mathematical skills at an adequate level of proficiency. Take the actual TIMSS test items for example: when given a three-dimensional object, 26% of Taiwanese eighth graders are not able to make a two-dimensional drawing of it from a specific viewpoint. One out of ten eighth graders is not able to solve two-place plus three-place decimal addition problems (Mullis et al., 2012). Even worse, nearly 30% of fourth graders cannot figure out the correct number following the sequence of 6, 13, 20, and 27 (Mullis et al., 2016). The Taiwan local examination used for school admission within the country has provided even more insights into the profile of low-achiever students. In the yearly Comprehension Assessment Program (CAP)—a standardized test administered to all ninth grade students for high school entrance in Taiwan—30% of the participants only responded to multiple-choice problems, which allow random guessing, and left constructed response questions completely blank; 33% of testees were officially categorized as "below the basic level." These findings suggest that in countries with superior mathematical skills, the key issues of mathematical learning should be a focus at the lower end, i.e., those with particularly low achievement or classified as having disabilities.

Educational Policies for Learning Difficulties in Taiwan

In this section, we discuss the local education policy and special education resource. Because only general LD are officially recognized by educational authorities, we will emphasize the government-issued classification and placement of LD in this section. Although there is a well-established system for identifying and locating students with LD in Taiwan, very few teachers and parents realize that learning difficulties can occur specifically in a single academic domain, such as reading or mathematics. In the past few decades, many researchers have actively advocated understanding the core deficits and intervention in dyslexia and RD. MLD, in contrast, has received very limited attention in the local community, even though the prevalence rate is reported to be equivalent to or even higher than that of dyslexia in the Western literature (Butterworth et al., 2011). Immediate action should therefore be taken to promote public knowledge of MLD.

In Taiwan, the government provides a very organized hierarchical system for LD assessment, identification, and placement. Each local city and municipality are required to set up a special education consultation committees formed by researchers, school teachers, and parents to provide planning and development of special education services. LD identification in Taiwan is very tightly connected to that in the Western literature. According to *The Disabilities and Gifted Students Diagnosis Criteria Regulations* issued by the Ministry of Education (MOE, 2013), diagnosis of individuals with LD is manifested by severe deficits in listening comprehension, verbal expression, word identification, reading comprehension, writing, or arithmetic calculation, and must meet all of the following criteria: (1) IQ normal or above normal; (2) discrepancy between aptitude and achievement; and (3) persistence of the deficits even after provision of interventions in formal school settings; the interventions shall be 3 hours per week and last for at least 6 months. The diagnosing process typically starts from parents and teachers identifying suspicious students with learning problems based on their academic performance (Hung, 2014). These students then go through general remediation provided by general teachers or volunteers to ensure that their learning difficulties have not been elicited by an inappropriate learning or teaching environment. If this does not succeed, the students then continue with a successive diagnosing process.

The formal diagnosing processes are executed by a group of special education teachers, who are specifically trained to administer neuropsychological tests, interpret results, and make decisions according to whether the suspected cases are LD or not (Tzeng, 2007). The most commonly used neuropsychological assessments for identifying LD include the following three assessments: the Chinese Character List (Hung, Wang, Chang, & Chen, 2008), the Reading Comprehension Screening Test (Ko, 1999b), and the Basic Arithmetic Skill Test (Ko, 1999a).

The Chinese Character List tests phonological and lexical knowledge at the single character level. The most basic unit of Chinese script is character rather than word. A character can be a word in itself, but it can also be combined with other characters to form different words. During the assessment of the Chinese Character List, participants are first required to write down the pronunciation of each test item, using *Zhuyin*, a phonetics-based alphabet system predominantly taught to children before they learn to read and write Chinese characters. Participants are also required to generate a multiple-character word using the given test item. The numbers of correctly answered items are used to estimate how many Chinese characters each participant knows.

The Reading Comprehension Screening Test measures participants' reading skills in semantic integration and text inference at passage levels. During the assessment, participants are required to answer questions based on a short passage.

Mathematical skills are assessed using the Basic Arithmetic Skill Test. During the assessment, participants solve numerical problems, such as judging number magnitude and simple arithmetic operations. Except for the Basic Arithmetic Skill Test, all the other tests provide national norms for each grade of the academic year. At the diagnosing stage, these assessments are administered in a one-to-one manner so that the assessors can closely observe the students' behavioral performance and response strategies to make further decisions and suggestions accordingly. Finally, the diagnosis report is sent to the special education consultant committee to close the case.

Once diagnosed as having LD, identified students are reported to the government and are mandatorily treated with special education care (The Special Education Act, 1984). Up to senior high school level, all educational institutions are required to set up on-site special education classes either by teaching all of the students with LD in the same designated classroom or by distributing them into regular classes (The Special Education Act, 1984). These students then undergo remediation tutorials such as extra practice using regular school materials, training in cognitive strategies, multisensory learning, or digital learning. The responses to the given interventions are monitored throughout the remediation.

Although the government provides a well-established system for LD referral, diagnosis, and placement, the current system identifies LD in fewer than 1% of students each year (Ministry of Education, 2016)—a much lower rate than those in other countries, such as the reported 5% rate in the USA (Cortiella & Horowitz, 2014). One of the likely reasons is that Asian parents feel reluctant for their children to be categorized as low achievers or as having disabilities (Tzeng, 2007). Some parents may send children to cram schools for special training in cognitive skills tests such as reading, mathematics, and even IQ tests. Heavy academic training and practice using a matured test-taking technique might keep students from becoming low achievers. Another possibility is that LD could respond to effective intervention (Iuculano et al., 2015) and the low prevalence of LD in Taiwan likely reflects potential LD responses to intervention (Tzeng, 2007).

Another major problem in Taiwan is the inadequacy of diagnosing LD subtypes. In the past several years the Ministry of Education has requested the local LD classification committee to notify which subtypes the LDs are on their diagnosing reports. However, there is no standard for which subtypes should be considered, and no clear cutoff criteria for identifying LD subtypes have been provided. Since each specific LD is found to show distinct deficits in cognitive and neural mechanisms (Ashkenazi, Black, Abrams, Hoeft, & Menon, 2013), it is necessary to better characterize the distinct profile of each specific LD subtype so that schools and teachers can provide appropriate placement and remediation for those who are in need of special education resources. In the next section, we illustrate the current MLD diagnosis and assessment tools recently developed by local researchers.

Diagnosis and Assessment Tool for Mathematical Learning Difficulties

As mentioned above, identifications of LD, particularly in students who show deficits in mathematical skills, were never required to specify the subtypes until the past few years. Even now, only in some cities and municipalities is it required to specify the LD subtype in the current diagnosis protocol. The MLD classification is tightly connected to the Western literature. Take Taipei City for example: government-recognized MLD is characterized by severe deficits in the following cognitive behavior: (1) concepts of number magnitude; (2) simple arithmetic problem solving; (3) relying on insufficient problem-solving strategies, such as finger counting; (4) overly slow calculating speed; (5) ability to solve math problems in the daily life context but not to transform them into mathematical formulas; and (6) difficulties in learning simple geometry and visuospatial skills.

Currently there are two tests with national norms that are mainly used for diagnosing MLD: the aforementioned Basic Arithmetic Skill Test (Ko, 1999a) and the Basic Mathematical Core Skill Test (Hung & Lian, 2015). The Basic Arithmetic Skill Test provides a comprehensive measure of mathematical abilities from the second to the sixth grade in school-aged children. This test has been widely used to assess Taiwan children's mathematical skills and for diagnosis of children with mathematical disabilities. The test battery involves items that cover distinct cognitive components for each grade, from basic mathematical knowledge of number magnitude comparison to simple arithmetic calculation, including addition and subtraction with and without carries. Multiplication problems are added for testees in the third grade and beyond. Beyond the fourth grad, the test items include arithmetic problems with mixed operations and word problems. Coupled with discrepancy criteria, this test together with other reading assessments have identified a 3.18% rate of LD among school-aged children in Chiayi, a city in South Taiwan, where the development of socioeconomics, politics, and education is not as advanced as in the north. Among these identified students with LD, 28% were classified as having RD, 12% were identified as having MLD, and 60% were comorbid with both disabilities (Ko, 2005). The researchers then continued to remediate the MLD-specific students. They found that although the intervention can increase accuracy performance in these students, their response times for math problems continue to show severe deficits (Ko, 2005). These results have suggested that with valid assessment, LD can be classified into distinct subcategories, and each subtype is in need of specialized tutorial remediation.

The Basic Arithmetic Skill Test has recently been revised and updated with the latest Taipei and New Taipei City norms (Lee & Hsieh, 2016). In the revised version, more schools have participated and hence provided a larger scale of recent norms. The revised version mainly focuses on simple arithmetic operation problems that require procedural calculation and exclude number magnitude comparison, as well as word problems. In order to identify children who show real deficits, performance below 3% has been used as the cut-off criterion for identifying suspected MLD. This assessment has been used as an official assessment tool for MLD classification in many of the local municipalities.

The Basic Mathematical Core Skill Test (Hung & Lian, 2015) is another recently published test item with national norms. This test includes a broader range of mathematical knowledge, including number identification, number comparison, and serial order, as well as vertically and horizontally presented calculation problems.

Word problems are included in the test items for third graders and beyond. In order to provide a more appropriate diagnosis, this test requires the assessors to write down the testee's problem-solving strategies by observing the testee's physical indications. If counting during the assessment is overt (such as counting verbally or with fingers) or if the testee takes time to respond, the assessor will record this as a counting strategy. If immediate responses are given, the assessor will mark it as a retrieval strategy. After completing each session, the assessor will randomly choose a few test items to ask participants how those problem answers were derived. Previous studies have indicated that the combination of participant self-reporting and experimenter observation can provide a useful measure of children's arithmetic problem-solving strategies (Cho et al., 2012; Geary, Hamson, & Hoard, 2000), in which children with MLD have shown severe deficits (Geary, 2004).

Summary and Conclusion

Numeracy is a learned skill undergoing complex and experience-dependent changes throughout the schooling years, resulting in proficiencies over time. This process vastly depends on interactions between the individual and society. Cultural factors can therefore highly influence mathematical learning—for example, learning motivation (Mullis et al., 2012; Mullis et al., 2016), parent perception (Phillipson & Phillipson, 2007; Shek & Chan, 1999; Tzeng, 2007), cognitive skills such as arithmetic problem strategies (Campbell & Xue, 2001), and even digit reading (Tang et al., 2006). Scrutinizing cultural constraints is therefore crucial for advances in understanding the learning mechanism of building specialized long-term mathematical knowledge. In this chapter, we have reviewed how cultural specificity might impact mathematical learning. The existing evidence reveals a dynamic interplay between individual cognitive mechanisms and external environmental features, suggesting that cultural factors should be taken into account when conducting research in understanding MLD. Within the East Asian part of the world, such as Taiwan, research on LD has undergone significant progress in the past few decades (Hung, 2006). The focus of research on LD has shifted from laying out the general cognitive profile of LD toward systematic understanding of identifying each specific subtype. Among each subcategory, mathematical learning has steadily attracted more public attention. Although the identification, classification, and remediation of MLD is not yet fully mature in Taiwan, emerging endeavor and advocacy have been increasingly devoted to the field. Future development and progression in better classifying and characterizing children with MLD concerning the features shaped by culture in the East Asian culture are needed.

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Chapter 18
Mathematical Learning and Its Difficulties in Israel

Sarit Ashkenazi, Hannah Restle, and Nitza Mark-Zigdon

Introduction

Adults and children with low numerosity experience disadvantages both in the classroom and in day-to-day life. They have trouble making financial and medical decisions and evaluating risks (Agarwal & Mazumder, 2013; Gerardi, Goette, & Meier, 2013; Reyna, Nelson, Han, & Dieckmann, 2009). Their career choices are limited by their weakness in math, and their chance of being unemployed is increased (Henik, Rubinsten, & Ashkenazi, 2011). However, not until recently has the educational and academic field recognized math difficulties as a stand-alone learning disability. Until now, math learning disability (MLD) has been neglected both in educational and academic fields compared with other learning disabilities such as reading disability (Ashkenazi, Black, Abrams, Hoeft, & Menon, 2013).

There is a debate in numerical cognition research on the nature of the cognitive weaknesses underlying MLD (also known as developmental dyscalculia) and the most effective diagnostic tools for identifying MLD (Träff, Olsson, Östergren, & Skagerlund, 2017). While the core deficit approach suggests that school math is strongly influenced by innate preverbal number sense ability (the ability to intuitively understand approximate quantity and relations between quantities) and, hence, MLD originates from weakness in number sense

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(Butterworth, Varma, & Laurillard, 2011; Dehaene, Piazza, Pinel, & Cohen, 2003; Halberda & Feigenson, 2008; Halberda, Ly, Wilmer, Naiman, & Germine, 2012; Halberda, Mazzocco, & Feigenson, 2008), other theories suggest that MLDs originate from cognitive abilities that are not unique to math, such as impairments in working memory or executive functions (Ashkenazi, Rosenberg-Lee, Metcalfe, Swigart, & Menon, 2013; Szűcs, 2016; Szucs, Devine, Soltesz, Nobes, & Gabriel, 2013).

Due to this worldwide debate regarding MLD (Träff et al., 2017), locally in Israel, there are inadequate definitions and a deficit of acceptable diagnostic tools for assessing MLD. As a result, currently there is not even a single standardized normed tool to diagnose MLD in children in Israel.

This chapter aims to examine the Israeli case of MLD and mathematics education. First, we will briefly discuss the cultural background of Israel, which has deeply influenced the educational system. We will then describe the mathematics education policy in Israel. Afterward, we will address the international differences in math abilities tested by the Programme for International Student Assessment (PISA). Next, we will look at the definition of MLD in the primary and secondary educational system of Israel, future changes in the policy of diagnosis, and remediation of MLD in children and current remediation programs for children with MLD. Lastly, we will discuss the diagnosis of MLD in institutes of higher education in Israel.

General Description: Population and Diversity

The state of Israel is relatively young, 70 years old (it was founded in 1948). It is defined as a Jewish and democratic state, and it has been constantly changing (Masri, 2017). One of the most significant and ongoing changes is the immigration of new Jewish population into Israeli society (Zangwill, 2017). This resulted in a large number of immigrants from different socioeconomic backgrounds who immigrated to Israel during the years 1948–2000. Most of the founders of Israel arrived from European countries. However, during the years 1948–1960, the population of Israel increased 9.2% and diversified due to immigration. Over half of the new immigrants arrived from Africa and Asia (53%), and the remainder arrived from Europe and the United States. During the years 1990–1995, another wave of immigrants arrived from the former Union of Soviet Socialist Republics yielding a population increase of 3.2% (Eckstein & Weiss, 2004).

Another important change in Israeli society is related to the relatively high religious diversity in Israel. The Israeli population is comprised of religious minorities including Arabs (Muslim, Christian, and Druse) and the Jewish majority. For example, during the first years of the country, the Arab population was 18% of the general population and decreased to 11% during the 1960s. Currently (2017), the religious composition is 74.9% Jews, 20.9% Arabs, and 4.5% others. Hence, the Israeli population is very diverse, both culturally and socioeconomically. The complexity of

Israel's diversity poses great challenges to its education system ("Statistical Abstract of Israel 2017," 2017).

Due to the common observance of traditional customs, the average fertility rate of Israel is one of the highest in all of the Organization for Economic Cooperation and Development (OECD) countries. Specifically, in 2015, 45.7% of the population of Israel was younger than 15 years of age; in a comparison of 185 countries, Israel ranks 110 in fertility. The high fertility rate stems from the observant religious sectors (ultra-Orthodox and Arab). In line with the heterogeneous fertility rates in Israel, there is a significant wealth gap. Generally, Israel's gross domestic product (GDP – a score representing the economic performance of an entire country) was 83.2, ranking 48 out of 155 countries. However, in contrast to the heterogenic socio-economic status in Israel, the government expenditure on education (% of GDP) is medium to high. Specifically in 2013 the government expenditure on education was 5.9, ranking 17 out of 75 countries (see Table 18.1). The government expenditure on education was higher than 6 between the years 1990 and 2000, decreased to 5.5 in

Table 18.1 *Government expenditure on education (GEOE, % of GDP) by country

Country	GEOE*	Rank
Malawi	7.7	1
Sweden	7.7	2
Finland	7.2	3
Ukraine	6.7	6
Mozambique	6.5	7
Malaysia	6.1	11
Ghana	6	12
Honduras	5.9	15
Israel	*5.9*	*16*
United Kingdom	5.7	18
Austria	5.6	19
Bhutan	5.6	20
Netherlands	5.6	21
Barbados	5.5	22
Australia	5.3	25
Rwanda	5	29
Colombia	4.9	30
Niger	4.9	32
Saint Lucia	4.7	33
Benin	4.6	34
Chile	4.6	35
Japan	3.8	48
Mauritius	3.7	49
Albania	3.5	50
Philippines	3.4	53
Iran (Islamic Republic of)	3.2	56

Notes. Data from the United Nations Development Programme (2016)

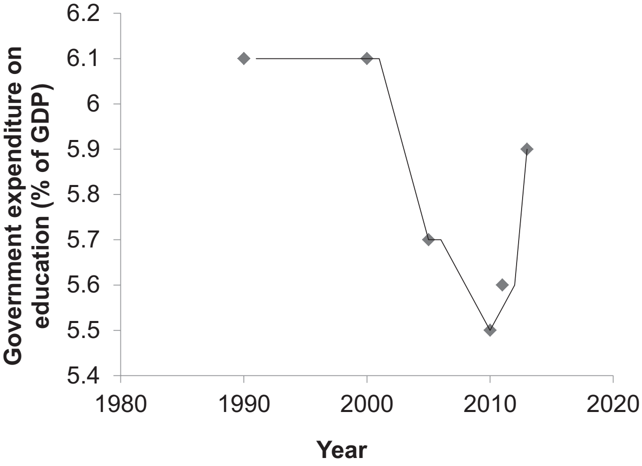

Fig. 18.1 Change in government expenditure on education in Israel 1990–2015. (Data from the United Nations Development Programme (2016))

2010, and since then has been increasing (see Fig. 18.1) (United Nations Development Programme, 2016). These statistic data further emphasize the heterogeneity in Israel (cultural, religious, and socioeconomic). Providing equal educational opportunities in such a heterogeneous society is one of the greatest challenges facing the Israeli educational system. The relatively high expenditure on education in Israel aims to achieve this goal.

General Education and Mathematics Education in Israel

From the early years of the country, due to the cultural diversity and the high number of immigrants in Israel, the local education system has dealt with two main contrasting principles: (1) the education system of Israel should serve as a melting pot, a tool to educate new immigrants and assimilate them to the foundational culture, and (2) acknowledging the cultural diversity of minorities and providing autonomy to schools to make their own educational choices (e.g., particular textbooks for each minority) (Zameret, 2012). In the first years of the country, the education system in Israel emphasized the first principle (i.e., melting pot) and mostly ignored the second one (i.e., cultural diversity) (Zameret, 2012). However, currently the two principles (i.e., melting pot and cultural diversity) are being emphasized simultaneously in the education system of Israel (State of Israel, Ministry of Education, 2003). The Ministry of Education developed a core educational program (including a detailed curriculum for each subject) that should be applied to all subjects in the schools in Israel while providing flexibility to minorities to help achieve specific additional educational goals (State of Israel, Ministry of Education, 2003).

Today there is a compulsory education law in Israel. The law determines that every child in Israel must be in the framework of education (kindergarten or school) from the age of 3 through kindergarten until the 12th grade. This law obliges the child's parents to enroll in the educational institution and to ensure the regular attendance of the child. The duty of the state is to provide free education from kindergarten to the end of high school (The Knesset, 2007). The Israeli educational system can be classified into schools that are under the full supervision of the Ministry of Education and schools that are not. This latter category consists mainly of the ultra-Orthodox Jewish school system and comprises approximately 20% of all students in the Israeli school system. According to governmental regulations, these unofficial schools must also uphold a set of standards, including administrative, pedagogical, physical, and social that are similar to the standards of official schools (State of Israel, Ministry of Education, 2003).

Math education in Israel begins at the age of 3 and continues throughout primary and secondary schooling. Each age group has a specific math curriculum. The math curriculum details what should be taught in each class level and how many hours each subject should be taught (see Table 18.2 for the curriculum for elementary school) (State of Israel, Ministry of Education, 2003). In general, there are five to six math lessons a week in the main educational system. However, due to the guiding principle of cultural diversity, the core educational program obligates the study of math for 3, 4, or 5 h a week in schools that are not under the full supervision of the Ministry of Education, in effect defining a standard but allowing variance based on school administrative choices (State of Israel, Ministry of Education, 2005). National tests are conducted by the Ministry of Education which monitors students' level of achievement in different areas of the country and in the various population sectors.

Various teaching methods are allowed by the Ministry of Education, as illustrated by the number of authorized math textbooks in Israel. Each school can choose their preferred textbooks. All textbooks are written in Hebrew and Arabic and include digital versions with hyperlinks to additional math activities.

Table 18.2 Math curriculum in elementary schools in Israel by grade

	1st grade	2nd grade	3rd grade	4th grade	5th grade	6th grade
Natural numbers	X	X	X	X	X	X
Operation in natural numbers	X	X	X	X	X	X
Fractions	X	X	X	X	X	X
Decimals					X	X
Percentages					X	X
Ratio						X
Units of measurements	X	X	X	X	X	X
Data research		X	X	X	X	X
Measurements	X	X	X	X	X	X
Geometry	X	X	X	X	X	X

International Educational Tests in Math in Israel

One way to evaluate math ability in Israel compared with other countries is with international educational tests. Israel has participated in a large number of these international tests including Trends in International Mathematics and Science Study (TIMSS) and PISA. On these early tests (1960s), Israel was ranked among the first 12 countries in mathematical achievements. However, only part of the Israeli population was permitted to take part in these tests: the Arab pupils as well as new Jewish immigrants from underdeveloped countries were excluded (Cohen 2007, taken from Feniger, Livneh, & Yogev, 2012), improving the relative performance of Israel on those international tests. To this day some Israeli students are still excluded from these international tests, however, to a lesser extent than previous years: (1) The strictly Orthodox Jews, which are not obligated to participate in the general math curriculum, and (2) the Arab students in Jerusalem, of whom approximately 90% are studying according to the Palestinian curriculum. These two categories of students could reach 20% of the students in Israel, yielding an overestimation of Israel's national average on these international tests.

One of the most updated published scores of international tests in Israel is the PISA. The PISA is administered to 15-year-old students from all the OECD countries, and it tests mathematics, among other subjects. In 2015, Israel received a score of 470 points in PISA in the mathematics section, a score that is 20 points lower than the average OECD score, ranking Israel 42 out of 70 countries (see Table 18.3) (OECD, 2017). Questions on the PISA mathematics section can be categorized into Levels 1–6. Level 1 proficiency identifies students that can answer questions with familiar contexts (e.g., about money) where all relevant information is present and the questions are clearly defined. Level 6 proficiency includes students that can draw on a range of interrelated scientific ideas. 32% of the students scored at Level 1 on the PISA (considered low achievers), compared with 23% on average for OECD (see Table 18.3). Israel's mathematics scores on the PISA have improved every year since 2006 (OECD, 2017). However, mathematical achievement in Israel still remains below average and a large percentage of students still exhibit weakness in mathematics. This can partially be explained by the low economic status of Israel and the high fertility rate in Israel (Feniger et al., 2012).

Diagnosis of Mathematical Learning Disabilities in the Israeli School System

The current diagnosis of MLD in the Israeli school system is founded on the definition of the *Diagnostic and Statistical Manual of Mental Disorders* (DSM) IV (1994). MLD should be diagnosed if one can prove two observed learning disparities: (1) within students, a gap between the expected level of math abilities according to intellectual level and actual performances, and (2) between students,

Table 18.3 PISA Scores on 2015, by country, mean score, and rank of scores and percentage of low achievers in math (Below Level 2) and rank of low achievers

Country	Mean score			Percentage of students below Level 2	
	Mean	S.E.		Percentage of students	Rank
Hong Kong (China)	548	(3.0)	2	9.0	3
Macao (China)	544	(1.1)	3	6.6	1
Korea	524	(3.7)	7	15.5	11
Switzerland	521	(2.9)	8	15.8	12
Canada	516	(2.3)	10	14.4	9
Denmark	511	(2.2)	12	13.6	8
Finland	511	(2.3)	13	13.6	7
Belgium	507	(2.4)	15	20.1	25
Germany	506	(2.9)	16	17.2	17
Poland	504	(2.4)	17	17.2	18
Norway	502	(2.2)	19	17.1	16
Sweden	494	(3.2)	24	20.8	26
Australia	494	(1.6)	25	22.0	32
France	493	(2.1)	26	23.5	37
European Union total	493	(0.8)	27	22.1	33
United Kingdom	492	(2.5)	28	21.9	31
Czech Republic	492	(2.4)	29	21.7	29
Portugal	492	(2.5)	30	23.8	39
OECD average	490	(0.4)	31	23.4	36
Italy	490	(2.8)	32	23.3	35
Iceland	488	(2.0)	33	23.6	38
Spain	486	(2.2)	34	22.2	34
Hungary	477	(2.5)	40	28.0	43
Slovak Republic	475	(2.7)	41	27.7	42
Israel	470	(3.6)	42	32.1	48
United States	470	(3.2)	43	29.4	46
Greece	454	(3.8)	46	35.8	50

a gap of 2 or more years between math abilities that are expected from the student according to grade level and actual performances. There is a fundamental problem related to the second gap in Israel: currently, there is no normed diagnostic tool for assessing MLD for Israeli children. Therefore, a reliable gap between observed math level and expected math level according to grade level cannot be accurately tested in Israeli children. Instead, most of the MLD diagnostic tests in Israel are local curriculum-based tests including grade level math test, similar to those administered in schools. Curriculum-based tests are examinations of math subject matter that should be learned in each grade level, according to the Israeli Ministry of Education. Please see Table 18.2 for the subject matter in the mathematical curriculum in Israel according to grade level. Table 18.4 presents an example for an Israeli curriculum-based test that was developed by e.g. the authors of this section

Table 18.4 Israeli curriculum-based test subdivided into subtest and content

Subtest	Content	Example
Part A. Knowledge of numbers		
Number-word sequence	Counting forward Counting backward	Count from 793 to 801 Count from 506 to 498
Numerical system	Understanding of the base-ten system	Build biggest/smallest number from a given set of written digits (e.g., 3, 7, 4, 8)
	Equation transformation from horizontal to vertical position	Write: 340 + 3+ 5706 = 6049
	Recognition of a numerical place value within a written number	What is the value of the digit 5 in the number: 1252?
Series of numbers	Number series completion	Complete the following series: 463, 473, 483, ___, ___, ___
Part B. Knowledge of number operations		
Equation	Addition Subtraction	200 + ___ = 550 ____ – 100 = 600
Simple multi-digit arithmetic	Addition Subtraction Multiplication Division	20 + 50 = 70–30 = 40 × 30 = 90/30 =
Written word problems	Arithmetical operation presented in verbal format	In each class there are 25 children. How many children are there in 5 classes?
Arithmetic algorithm	Addition	24 +*37*
	Subtraction	56 −*43*
	Multiplication	45 ×*3*
	Division	*94* 6
Estimation of written math problems	Multiplication	Is 32 × 19 bigger or smaller than 400?

(Ashkenazi, Mark-Zigdon, & Henik, 2009, 2013). Although Israel's population is highly heterogeneous, the math curriculum serves as a guideline for each grade level. However, due to the various permutations of the curriculum used across different schools, the use of non-normed curriculum-based tests is insufficient. Alternatively, normed tests from other countries have been applied. This method is also flawed due to international differences in math curricula. Therefore, a normed tool is urgently needed to correctly diagnose MLD.

Students in Israel who are diagnosed with MLD are entitled to receive appropriate school accommodations, including special assistance in school (personalized teaching methods and out-of-classroom interventions) and testing accommodations. Currently the Ministry of Education places an emphasis on the latter, i.e., testing accommodations (see the next section for a full explanation). In order to receive testing accommodations throughout schooling and during the final high school exam, two MLD diagnostic evaluations are authorized: (1) didactic evaluation, including testing of learning processes, such as math, and the building blocks of these learning processes, such as numerical comparison, and (2) psycho-didactic evaluation, including the didactic assessment, in addition to testing cognitive abilities and emotional difficulties that may impair learning. While a specialized psychologist is allowed to perform a psycho-didactic evaluation, the law regarding how didactic diagnosis is to be performed is less clear. However, there are very clear rules in the educational system about the evaluation (didactic or psycho-didactic) required to receive specific testing accommodations especially during the final high school exam.

Testing accommodations were divided into three levels in Israel according to their potential to benefit students without learning disabilities. Level 1 – testing accommodation does not change the content of the test and will not benefit children without learning disabilities (such as providing an extended formula page during the math test). Level 2 – testing accommodation might change the content of the test. Level 3 – testing accommodation will change the content of the test; hence they will be granted with extra caution and only to students with very severe cases of learning disabilities. For example, students diagnosed with severe MLD, accompanied by weakness in understanding basic numerical quantity mechanisms, receive the opportunity to replace the final high school math exam with another scientific subject such as biology or chemistry. While the testing accommodations of Levels 1 and 2 are approved by an in-school committee based on a didactic or psycho-didactic diagnosis, Level 3 testing accommodations are approved by an out-of-school district committee, according to the recommendations of a psycho-didactic evaluation only.

Please note that the diagnosis of MLD in Israel, based on the DSM-IV, has not been updated following the release of the new DSM-V in 2013. In the DSM-V, two major changes have been made in relation to the DSM-IV. Firstly, the in-student gap (discrepancy between intelligence level and math performance) is no longer necessary for the diagnosis of learning disability. Secondly, the separated diagnoses of dyslexia, dyscalculia, and dysgraphia are all united as one category of specific learning disabilities (one can specify particular weakness in an individual's reading, writing, or mathematical performance). Hence, major changes should be made in the diagnosis of learning disabilities in Israel and across the world. The next section will address the future plan, announced by the ministry of education, to change the diagnostic process.

Current Changes in the Diagnosis and Treatment of MLD in Israel

The percentage of students that are diagnosed as suffering from learning disabilities in the Israeli educational system increases every year. During the year 2000, 14% of the students in the education system were diagnosed with learning disabilities. The percentage of diagnosed students increased each year, and during 2013, it reached 41.3% (Psychological Consultation Services, 2017); this percentage is much higher than expected in reference to different studies examining prevalence rates of learning disabilities. Specifically in Israel, in 1996, the prevalence rate of MLD was found to be 6.4% (Gross-Tsur, Manor, & Shalev, 1996), the prevalence rate of MLD worldwide is between 3 and 6% (Reigosa-Crespo et al., 2012; Shalev, Manor, Amir, & Gross-Tsur, 1993; Von Aster & Shalev, 2007), and similar prevalence rates can be found in dyslexia (between 5% and 17% in different studies (Shaywitz, 1998)) and a similar prevalence rate of ADHD (the ADD/HD worldwide-pooled prevalence was 5.29%) (Guilherme, Maurício, Bernardo, Joseph, & Luis, 2007). Due to the inconsistency between the expected prevalence rate and the observed number of students that receive a diagnosis of learning disabilities, and as a result receive test accommodations, in addition to worldwide changes in the view that related to learning disabilities (moving to the diagnosis criterion of DSM-V, see the above section), a professional committee was called by the Israeli Ministry of Education during 2014 to discuss current changes in the policy toward students with learning disabilities: "Margalit Committee II" (Margalit et al., 2014). This committee included both professors from the Israeli universities investigating learning disabilities and policy-makers from the Ministry of Education. The main recommendation of the committee was to increase the resources devoted to remedial teaching in the school system while decreasing the resources devoted to diagnosis. Accordingly, the Ministry of Education created a new program that is now being piloted; according to the new program, the diagnostic process of learning disabilities will follow the model of response to intervention (RTI). Hence, the model of diagnosis of MLD will focus on a few in-school processes:

1. Mapping the math achievements of the entire school population by administering an in-class test. The children with the lowest achievements on the test will then be tracked and provided with in-class personalized teaching strategies.
2. Follow-up assessment of math achievement consisting of an in-class test and identifying students that still exhibit low achievement in class. These students will be provided with evidence-based remedial teaching in small groups, by a teacher who is specifically trained for that purpose.
3. Follow-up assessment of math level using a special tool (assessing MLD) and tracking the lowest achieving students. These students will be given personalized intervention by educators specialized in learning disability.
4. If a student did not reach the needed level after the personalized intervention, then a full diagnosis will be given to the student outside of school by a professional examiner with knowledge of learning disability (Psychological Consultation Services, 2017).

Teaching Accommodations for Children Suffering from MLD in Israel

The current theoretical approach of the ministry of education in Israel is to increase the resources devoted to remedial teaching in the school system while decreasing the resources devoted to diagnosis. Hence, it is important to set guidelines to create a customized intervention program according to the individual abilities and needs of each student. The intervention program should follow the general math curriculum. Teachers can create accommodations for each student by modifying the mathematical material: reducing or expanding it according to the child's strengths and weaknesses identified by the in-class diagnosis or providing an alternative math topic that requires application of the same mathematical principles (see Ministry Protocol, November 1, 2005, Amendment 3.1). To facilitate this process, the "Accommodations Document of the General Math Curriculum for Special Education Students" was published (Ministry of Education, 2014). This document is organized by mathematical topics and notably not by grade level. Each topic is broken down into specific math sub-topics and goals, recommending the appropriate teaching methods and didactic clarifications related to typical difficulties in the teaching of mathematics for students with MLD.

For example, for the goal of "acquisition of the multiplication tables," three sub-goals were suggested, aimed at students with different cognitive styles of learning:

1. Understanding-based multiplication table learning. The recommended activities for this sub-goal require that the children rely on an understanding of the connection between addition and multiplication or on mathematical laws such as commutative and distributive laws and use known facts to solve new problems.
2. Memorization-based multiplication tables learning. Here, games, songs, patterns, and cards were suggested.
3. Combination-based multiplication tables learning.

In order for the teachers to choose the appropriate goals for the learner's characteristics, they should bear in mind: (1) the assessment results, which provide information on the student's numerical knowledge and cognitive skills and (2) awareness of the required math knowledge for the student's age and grade level curriculum. Based on these considerations teachers may choose the appropriate goals and sub-goals to initiate. By referring to the relevant documents, they can administer the appropriate accommodations and activities to use. This program was designed for students in the special education system. Students in the general education system with the diagnosis of MLD are expected to learn according to the curriculum's requirements (see Table 18.2).

Diagnosis of MLD in Universities in Israel

On 2008, a law determining the rights of students with learning disabilities that are studying in post-secondary educational institutions was signed by the Knesset (Israel's parliament). The law included a few main requirements: every post-secondary educational institute should build a support center for students with learning disabilities and provide customized testing accommodations for students with learning disabilities and follow the recommendation of an approved diagnosis tool (derived from the law for the rights of students with learning disabilities that are studying in above high school institutes (Olmert, Tamir, Yishai, Peres, & Itzik, 2008). However, most of the diagnostic tools in Israel at that point were inadequate. For example, there was not even one single standardized norm tool to test math abilities. To fill this void, the MATAL was developed by Israel's National Institute for Testing and Evaluation. The MATAL assesses proficiency in mathematics, reading, writing, and English (as a second language), as well as ADHD (Ashkenazi & Danan, 2017; Ashkenazi & Silverman, 2017). The diagnostic battery includes 2 questionnaires and 20 tasks to test reading, writing, math abilities, and English as a second language. The diagnostic battery includes three mathematical or numerical tasks:

1. Calculation automaticity. The goal of the task was to measure retrieval of arithmetic facts. The task included 80 simple arithmetic equations (e.g., $2 \times 3 = 6$) that were presented sequentially on the computer screen; the participant needed to answer if the equation was correct or incorrect by keypress. The equations are divided equally into addition, subtraction, multiplication, and division problems.
2. Procedural knowledge. The participants needed to ascertain if the equation was correct or incorrect by keypress. The equations included numbers that ranged from one to four integers. All the equations required logarithmic, simple calculations. The equations were divided equally into addition, subtraction, multiplication, and division and evenly into correct and incorrect solutions within each category (e.g., $45 + 25 = 70$ or $1850 - 350 = 1550$).
3. Number line knowledge. This task measured understanding of the mental number line. For each trial, different values appeared at the anchors of the number line, below the line. Two target points marked with the same value are presented on the number line, and the participant needs to ascertain which of the target points were marked correctly. The distance between the target points was 20% or 40% of the length of the whole line. The number lines included natural numbers, fractions, and negative numbers.

Based on the result of these tests, computerized algorithms will determine whether a student is suffering from MLD and, if so, its severity: light, moderate, or severe. According to the evaluation results, the support center for students with learning disabilities might suggest specific testing accommodations and specialized academic assistance to the student.

Conclusion

This chapter introduced the complexity of the Israeli educational system and MLD. Confronted by a diverse influx of new immigrants (e.g., European and North African) and composition of residents (e.g., Muslim, Christian, and Jewish), the educational system mediated between two contrasting principles of assimilation to uniform educational standards and providing autonomy to schools to make their own educational decisions based on their unique cultures. To this day, this dichotomy exists (with greater emphasis on autonomy), and the burden on the educational system is further compounded by a growing young population.

Regardless of the type of schooling, math is one of the core subjects in Israel. The Israeli math curriculum starts at the age of 3 and includes a very detailed plan accompanied by national evaluation of math ability in each grade level. However, in comparison with other OECD countries (using international performance tests), Israeli students receive relatively low math scores. One of the explanations for these lower scores may be the high socioeconomic diversity in Israel. This indicates a need to examine new ways to promote math education in Israel.

Two areas of future growth for math education in Israel are the diagnosis of MLD and the development of appropriate interventions. There is still worldwide debate regarding the most effective diagnostic and remediation processes for MLD. In Israel, no standardized normed tool has been developed to assess MLD. Instead, local curriculum-based tests are used to test MLD. Difficulties in MLD diagnosis may be related to the assessment process of learning disabilities in Israel in general. Current diagnostic procedures have resulted in a relatively high percentage of Israeli students who receive test accommodations on their final high school exams, in relation to the expected prevalence rates found in literature. Future plans by the Ministry of Education to diagnose learning disabilities according to RTI are expected to greatly change MLD assessment in Israel, among other learning disabilities, and impact intervention plans. This will further the current Education Ministry's aim to reduce the resources dedicated to diagnosis and increase the resources devoted to remedial teaching.

Notably, MLD diagnosis in Israel is most advanced in institutes of higher learning. By law, every post-secondary educational institute is obligated to establish a support center for students with learning disabilities. In these support centers, a standardized normed tool was developed to assess MLD, among other learning disabilities.

The development of a standardized normed tool to diagnose MLD in institutes of higher learning, but not yet in primary and secondary schools, demonstrates one of the complexities and discrepancies in the case of MLD in Israel. However, current plans are underway in the creation of a standardized normed tool for assessing MLD in students under the age of 18.

Another weakness in the assessment process of learning disabilities in Israel is the assessment and treatment of MLD on the basis of outdated definitions. Following the new definition of specific learning disability in the DSM-V, the Ministry of Education in Israel has created a detailed plan to change the diagnostic process and treatment of MLD. This program will promote tailored in-school interventions. In parallel, the out-of-school diagnostic process will be postponed to allow students remediation through in-school interventions. Delayed assessment will be carried out by authorized out-of-school diagnostic centers in the case of lack of improvement following in-school remediation.

As reflected by this chapter, the most appropriate assessment of MLD and subsequent remedial plans are still under debate. Plethora of new endeavors, including emphasis on attempts at in-school early remediation and development of standardized normed assessment tools, indicate current and future progress in the treatment of MLD in Israel.

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Chapter 19
Learning Difficulties and Disabilities in Mathematics: Indian Scenario

S. Ramaa

Introduction

The diversity in India is unique. It is a large country with a huge population. According to provisional census data, the total population of India was 1,210,193,422 in 2011. India has more than 50% of its population below the age of 25 and more than 65% below the age of 35. It is expected that in 2020 the average age of an Indian will be 29 years. India has varied physical features and cultural backgrounds. It is a country with many languages. In short, India can be considered the "embodiment of the world."

The salient feature of Indian heritage is the coexistence of many languages, races, and religions. However, all these factors influence the learning and performance of students in different academic subjects especially at the school level. Linguistic diversity has a greater impact than other factors.

As per the census of 1961, as many as 1652 languages and dialects are spoken by the people of India. Since most of these languages are spoken by very few people, the subsequent censuses regarded them as spurious. The eighth schedule of the constitution of India recognizes 22 languages. Hindi in Devanagari script is recognized as the official language of the Indian Union by the constitution. Hindi and English are used for official purposes such as parliamentary proceedings, judicial proceedings, and communications between the central government and state governments. Further complexity is contributed by the great variation that occurs across this population in social parameters such as income and education. In India, the medium of instruction in the schools may be English, Hindi, or the respective states' official languages. Private schools usually prefer English, and government (primary/secondary education) schools tend to go with one of the last two.

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Education in India—New Initiatives

There are many initiatives in the Indian education system that are influencing the development and learning of students. They have given importance to the education of all children including those with special needs. The national policy for education (1986), which is in force even now, recommends a common core component in the school curriculum throughout the country. In the school curricula, science and mathematics have been incorporated as compulsory subjects till school grade X. Since many students were out of school, to realize the “Education for All” mission, certain educational schemes were implemented from 2000 onward initially at elementary level and later at secondary level. As a result, all children and youth, including those with disabilities, have been brought into the school system. The central government and all state governments had the responsibility of retaining all the children in the schools and creating a conducive learning environment. Under these schemes, regular in-service training has been provided to all government school teachers to improve the quality of education.

Initiatives for the Education of Children with Special Needs

Many initiatives have been taken in the country toward the education of children with special needs (CWSNs).They are important milestones in the history of Indian education. The Rehabilitation Council India Act of 1992, Persons with Disabilities Act of 1995, and National Trust for Welfare of Persons with Autism, Cerebral Palsy, Mental Retardation and Multiple Disabilities Act of 1999 developed awareness among teachers, parents, and the community about the rights and welfare of individuals with disabilities. Many welfare measures were taken to give a dignified life to persons with disabilities. This also had a positive impact on the education of CWSNs. Education of these children was given mainly in residential schools or special day schools. Later emphasis has been given to mainstreaming. The Right to Education Act of 2009 emphasized inclusive education for all children. This has resulted in conducting regular in-service teacher training programs on inclusive education for government school teachers. The acts mentioned above did not include all disabilities. Learning disabilities were not covered by any of them. It was only in 2016 that the Rights of Persons with Disabilities (RPWD) Bill was introduced. Under this bill the types of disabilities have been increased from the existing seven types of disability to 21 types of disability. Speech and language disability, as well as specific learning disabilities (SLD), have been included as benchmark categories of disabilities for the first time.

Since SLD was not recognized as a separate disability in India, the efforts to provide appropriate education and other support services have been very much limited. Whatever efforts were made to help these children so far were mainly because of the concern and determination of some of the parents, parents’ self-help groups, and some of the professionals interested in the field. Only now has it become a professional obligation.

Definition of Specific Learning Disability

In the RPWD Act of 2016, “specific learning disabilities” means a heterogeneous group of conditions wherein there is a deficit in processing language (spoken or written) that may manifest itself as a difficulty to comprehend, speak, read, write, spell, or do mathematical calculations, and it includes such conditions as perceptual disabilities, dyslexia, dysgraphia, dyscalculia, dyspraxia, and developmental aphasia. However, separate definitions are not given for different types of SLD.

As a consequence of the RPWD Act of 2016, the central government has issued guidelines for the purpose of assessing the extent of specified learning disabilities. A battery of tests developed by the National Institute for Mental Health and Neurosciences (NIMHANS) in Bangalore shall be applied for diagnostic tests for SLD. The teachers at public and private schools shall carry out the screening in grade III or when the child is 8 years of age, whichever comes earlier. The child should be referred for further assessment if required as per the guidelines. The medical superintendent, chief medical officer, or civil surgeon—or any other equivalent authority as notified by the state government—shall head the certification authority. The certification will be done for children aged 8 years and above only. The child will have to undergo repeat certification at the age of 14 years and at the age of 18 years. The certificate issued at 18 years will be valid for life.

At present, parents and students with SLD have to face a lot of hurdles to determine whether a child has a learning disability (LD) or not, as there are only a few certification centers in the country. In fact, linguistic diversity is a major hurdle for assessment of SLD.

Prevalence of Children with Special Needs in India

The National Sample Survey (NSS) released a report saying that the incidence of disability in India is declining. It gives reason for optimism with the overall decline in disability, but there is also a reason for concern about rural areas where the incidence of disability has been found to be higher than in urban areas. More than 10% of the country’s population suffers from some form of disability. The NSS, for the purpose of the survey, categorized disabilities into locomotor (lame/crippled), visual, hearing, and mental disabilities. The results show that locomotor disability is the most prevalent, followed by hearing and visual disabilities. Mental illness, including retardation, is seen among many people (Tables 19.1 and 19.2).

This shows so far in India that no national-level surveys have been conducted to estimate the prevalence of SLD.

Table 19.1 Proportions of disabled population by type of disability in India, 2011

Serial No.	Type of disability	Persons (%)	Males (%)	Females (%)
1	Total	100.0	100.0	100.0
2	In seeing	18.8	17.6	20.2
3	In hearing	18.9	17.9	20.2
4	In speech	7.5	7.5	7.4
5	In movement	20.3	22.5	17.5
6	Mental retardation	5.6	5.8	5.4
7	Mental illness	2.7	2.8	2.6
8	Other disability	18.4	18.2	18.6
9	Multiple disabilities	7.9	7.8	8.1

Source: Punarbhava (2011)

Table 19.2 Differences in enrollment according to the type of disability

		Grades (%)		
Serial No.	Type of disability	I–V	VI–VIII	I–VIII
1	In seeing	20.79	32.87	24.02
2	In hearing	11.69	11.04	11.52
3	In speech	13.04	8.28	11.77
4	In movement	27.28	32.09	28.56
5	Mental retardation	19.68	8.62	16.73
6	Other disability	7.51	7.10	7.40
7	% of total enrollment	0.79	0.80	0.80

Source: Compiled from the District Information System for Education (DISE) database, 2006–2007 (Nidhi Singal, 2009)

Teacher Preparation Courses in the Area of Learning Disabilities

In India the Ministry of Social Justice and Empowerment has established five national institutes and their regional centers for mental retardation, visual handicap, hearing impairment, orthopedic handicap, and multiple disabilities. There is an All India Institute of Speech and Hearing, and a National Institute of Mental Health and Neurosciences. Some of these institutes and centers offer services to students with learning disabilities and conduct research and training programs. The National Council of Educational Research and Training (NCERT) and its constituent units—regional institutes of education (RIEs)—provide pre-service and short-term in-service training for teachers and teacher educators on learning disabilities as part of an inclusive education program. The NCERT and RIEs prepare instructional materials on learning disabilities. Ramaa (1989, 1992) has prepared handbooks and manuals on SLD for the benefit of teacher educators, teachers, other related professionals, and parents. Some of the state institutes of education have also prepared instructional materials on LD in their regional languages.

For preparing teachers, regular and distance-mode diploma, bachelor's degree, and master's degree (D.Ed., B.Ed., and M.Ed.) courses in special education are offered by various institutions, which are recognized by the Rehabilitation Council of India (RCI) (http://www.rehabcouncil.nic.in/writereaddata/List22May2013.pdf)

In total there are 456 institutes offering RCI-recognized courses in India, out of which only three institutions offer courses for bachelor's and/or master's degrees in special education for learning disabilities [B.Ed.Spl.Ed.(LD) and/or M.Ed.Spl.Ed.(LD)]. This clearly indicates that the numbers of trained teacher educators and teachers with specialization in SLD in India are very much limited. Hence, preparation of teacher educators and teachers with specialization in SLD should be the topmost priority in India if the RPWD Act's directives are to be realized.

Management of Specific Learning Disability in Schools in India

There are no specific interventions provided in schools. However some provisions have been made. In 1996, the Maharashtra government was the first in India to formally grant children with SLD the benefit of availing themselves of the necessary provisions ("accommodations") to enable them to complete education in regular mainstream schools. These provisions comprised (i) extra time for all written tests, with spelling mistakes being overlooked; (ii) employment of a writer for children with dysgraphia; (iii) exemption from the need to study a second language (Hindi or Marathi in an English-medium school) and substitution with a work experience subject; and (iv) exemption from the need to study algebra and geometry and substitution with a lower grade of mathematics (standard VII) and another work experience subject. These provisions were made initially only for the grade IX and X examinations. Later they were extended to grades I to XII and even to college courses. Seats were reserved for these adolescents in the handicapped category.

It has been noticed that students with SLD who are availing themselves of the benefit of these provisions are showing a significant improvement in their academic performance at secondary schools.

National Institute of Open Schooling

The National Institute of Open Schooling (NIOS), formerly known as the National Open School (NOS), was established in November 1989 as an autonomous organization in pursuance of the National Policy on Education of 1986 by the Ministry of Human Resource Development (MHRD), Government of India. The NIOS provides a number of vocational, life enrichment, and community-oriented courses besides general and academic courses at secondary and senior secondary levels.

Students who benefit from this initiative include sportspersons who have to train and travel all through the academic year, students with a physical handicap, and students with chronic medical illness. In addition to that, a large number of candidates with learning disorders or psychiatric conditions register with the NIOS board. Regular schools require students to study all the subjects in the curriculum—two/three languages, science, mathematics, and social studies—and each subject may include more than one paper. The students have to complete them within an academic year.

In contrast, the NIOS requires students to study a minimum of five and a maximum of seven subjects. They choose from a very wide range of subjects, which are offered according to their ability to attempt an examination. Students are also allowed to change subjects midway through the course if they are not satisfied with their choice. Thus, if a student has a mathematical disability he need not take mathematics at secondary level. A lenient time limit of 5 years is given to students of the NIOS to complete each of the secondary and senior secondary courses (which are either academic or vocational, or both), with as many as nine possible attempts. This makes it much easier for students with learning disabilities to complete the courses. There are study centers in the country that prepare students with SLD for NIOS examinations.

Table 19.3 shows that the numbers of students with SLD who have enrolled in the NIOS are increasing every year. However, the numbers of students utilizing the benefit are a lot smaller (Table 19.3).

Students with SLD also receive services from the National Institute of the Mentally Handicapped (in Secunderabad), the All India Institute of Speech and Hearing (in Mysore), the National Institute of Mental Health and Neurosciences, and child guidance clinics run by professionals. They provide remedial instruction, counseling, and mental health services. Since comorbid disorders are common among students with SLD, they are taken care of by these centers and by psychiatric hospitals and clinics.

Thus, supportive services for students with SLD are available outside the schools, but most of these services are available only in the urban areas and some of them are very expensive. Students hailing from suburban and rural areas are deprived of these support services. Even in urban areas there are practical difficulties. The remedial services are provided by these centers during school hours only. The students are permitted to go to them by the schools, but the parents have to arrange private vehicles for that purpose. This may be difficult for some parents. Moreover, they should be free during those times or some other family members should be able to attend to this responsibility. Hence, the best option is to equip and empower the schools to provide the support services for students with SLD.

Table 19.3 Enrollment of students with specific learning disabilities (SLD) in the National Institute of Open Schooling from 2007 to 2012

Year	2007–2008	2008–2009	2009–2010	2010–2011	2011–2012
Number enrolled	770	992	1199	2406	2083

In most of the state-run schools, students with SLD will reach grade IX without much difficulty. With the introduction of continuous and comprehensive evaluation in the schools, students can get marks through a variety of tasks and through many formative tests. During grade X, there is a board examination. Hence their promotion to the next grades depends upon their performance. Considerable percentages of students fail in English, science, and mathematics due to various reasons; one of them may be SLD.

The National Achievement Survey conducted by the NCERT in 2017 helps in understanding the difficulties experienced by students at different grades of elementary and secondary schools.

Learning Indicators/Outcomes and National Achievement Survey

For the first time, the NCERT has come out with an exhaustive list of learning indicators for students in grades I to VIII. The indicators aim to standardize the parameters used for measuring the learning curves of students. To start with, learning indicators have been finalized for eight subjects: English, Hindi, Urdu, mathematics, environmental science (EVS), science, social sciences, and art education. Though it is a common practice globally, this is the first time learning indicators have been used in India to assess children.

The National Achievement Survey (NAS) was conducted throughout the country in November 2017 for grades III, V, and VIII in government and government-aided schools. The survey tools used were multiple-choice test booklets with 45 questions in grades III and V and 60 questions in grade VIII in mathematics, language, science, and social sciences. The competence-based test questions that were developed reflected the learning outcomes developed by the NCERT, which were recently incorporated into the Right of Children to Free and Compulsory Education (RTE) Act by the Government of India. Along with the test items, questionnaires pertaining to students, teachers, and schools were also used.

The learning levels of 2.2 million students from 110,000 schools across 701 districts in all 36 states/union territories (UTs) were assessed. The findings of the survey are helpful to guide education policy, planning, and implementation at national, state, district, and classroom levels for improving learning levels of children and bringing about qualitative improvements.

Tables 19.4, 19.5, and 19.6 give the percentages of students who demonstrated the selected learning outcomes correctly. They are from four randomly selected states. The names of the states are not disclosed here. The results of only those learning indicators that were attained by 75% or less than 70% of students in all four states are given in Tables 19.4, 19.5, and 19.6.

In India there are unaided schools. Though they charge more fees, upper middle class and high–socioeconomic status (high-SES) parents prefer them to government

Table 19.4 Percentages of students from four states of India (states A–D) who demonstrated learning outcomes (LOs) of grade III mathematics

Serial No.	LO of grade III	Students (%)			
		State A	State B	State C	State D
1	Reads and writes numbers up to 999, using place values	72	60	73	49
2	Solves simple daily life problems by using addition and subtraction of three-digit numbers with and without regrouping	70	54	67	43
3	Analyzes and applies an appropriate number operation in the situation/context	73	45	74	46
4	Explains the meaning of division facts by equal grouping/sharing and finds it by repeated subtraction	69	59	68	48
5	Fills a given region, leaving no gaps, by using a tile of a given shape	55	49	65	39
6	Estimates and measures length and distance by using standard units like centimeters or meters, and identifies relationships	39	26	36	33
7	Extends patterns in simple shapes and numbers	67	54	68	47

Table 19.5 Percentages of students from four states of India (states A–D) who demonstrated learning outcomes (LOs) of grade V mathematics

Serial No.	LO of grade V	Students (%)			
		State A	State B	State C	State D
1	Applies operations of numbers in daily life situations	66	40	64	31
2	Explores the areas and perimeters of simple geometrical shapes (triangle, rectangle, square) in terms of a given shape as a unit	60	42	61	46
3	Reads and writes numbers bigger than 1000 being used in his/her surroundings	74	66	70	50
4	Estimates the sums, differences, products, and quotients of numbers, and verifies the same by using different strategies like using standard algorithms or breaking a number and then using an operation	69	50	63	39
5	Finds the number corresponding to part of a collection	59	51	74	51
6	Identifies and forms equivalent fractions of a given fraction	48	40	56	36
7	Classifies angles into right angles, acute angles, and obtuse angles, and represents the same by drawing and tracing	70	41	58	47
8	Relates different commonly used larger and smaller units of length, weight, and volume, and converts larger units to smaller units, and vice versa	74	37	66	42
9	Estimates the volume of a solid body in known units	49	33	44	30
10	Applies the four fundamental arithmetic operations in solving problems involving money, length, mass, capacity, and time intervals	64	42	55	34
11	Identifies the patterns in a triangular number and a square number	63	42	59	38

Table 19.6 Percentages of students from four states of India (states A–D) who demonstrated learning outcomes (LOs) of grade VIII mathematics

Serial No.	LO of grade VIII	Students (%)			
		State A	State B	State C	State D
1	Solves problems involving large numbers by applying appropriate operations	55	29	48	28
2	Solves problems in daily life situations involving addition and subtraction of fractions/decimals	54	29	48	29
3	Finds out the perimeters and areas of rectangular objects in the surroundings like the floor of the classroom, surfaces of a chalk box, etc.	50	31	48	30
4	Arranges given/collected information in the form of a table, pictograph, and bar graph, and interprets them	52	28	48	24
5	Interprets the division and multiplication of fractions	40	43	49	32
6	Solves problems related to daily life situations involving rational numbers	50	36	53	27
7	Uses exponential forms of numbers to simplify problems involving multiplication and division of large numbers	47	24	37	19
8	Adds/subtracts algebraic expressions	60	35	53	37
9	Solves problems related to conversion of percentages to fractions and decimals, and vice versa	38	21	40	21
10	Finds out the approximate areas of closed shapes by using a unit square grid/graph sheet	36	31	43	31
11	Finds various representative values for simple data from his/her daily life contexts like means, medians, and modes	67	47	57	41
12	Interprets data, using a bar graph, e.g., consumption of electricity is greater in the winter than in the summer	52	28	48	24
13	Generalizes properties of addition, subtraction, multiplication, and division of rational numbers through patterns	35	25	32	27
14	Finds rational numbers between two given rational numbers	52	37	48	29
15	Proves divisibility rules of 2, 3, 4, 5, 6, 9, and 11	61	40	42	41
16	Finds squares, cubes, square roots, and cube roots of numbers by using different methods	56	32	45	28
17	Uses various algebraic identities in solving problems of daily life	61	49	52	44
18	Verifies properties of a parallelogram and establishes the relationship between them through reasoning	36	27	50	28
19	Finds the surface areas and volumes of cuboidal and cylindrical objects	35	32	46	23
20	Draws and interprets bar charts and pie charts	50	44	54	34

and aided schools. Thus, in urban and suburban areas, children with a low SES or a lower middle SES only attend government and government-aided schools. A sizable percentage of the children attending these schools are first-generation learners and are from socially disadvantaged sectors of the society. These are some of the reasons for the results shown in Tables 19.4, 19.5, and 19.6. In addition to these, there may be children with learning disabilities, mild intellectual disability, below-average intelligence, attention deficit hyperactivity disorder (ADHD), underachievement, or other disabilities. It is required to assess all these children who have specific problems or who are at risk of developing problems. It is essential to understand the reasons for learning difficulties and employ proper interventional strategies for overcoming the difficulties in the respective grades, otherwise the gaps in knowledge will be increased, which will hinder further learning. As in the case of the achievement tests in the NAS, grade-appropriate learning outcomes have to be tested among the students, but during the diagnosing process the learning outcomes of lower grades are also needed to be tested to identify the gaps and difficulties.

The listings of grade-appropriate learning outcomes (National Achievement Survey, 2017) are good initiatives in identifying the learning difficulties and disabilities in the country. They are helpful in constructing diagnostic tools.

Research on Learning Disabilities in India

As discussed in the opening paragraphs of this chapter, India is a vast country with linguistic diversity. Conducting research on learning difficulties and disabilities depends upon the availability of achievement and diagnostic tests in all the official languages of the country. Nonavailability of assessment instruments is a major hurdle in carrying out the research. In addition, the number of professionals with adequate knowledge and understanding about SLD is limited in the country. Hence, in India the number of studies in the area of SLD is limited.

However, some researchers have attempted to conduct research studies on SLD with different objectives and research questions. Since special schools/classes/remedial education centers for SLD are not common in India, these studies were conducted in general schools. A brief account of the research studies is given below.

On the basis of an intensive review of current practices in India, Thapa, van der Aalsvoort and Pandey (2008) commented that the entire area of learning disabilities in India is confronting fundamental and basic issues pertaining to assessment and interventions. Further, they remarked that many promising leads and initiatives have been taken in India.

Identification of the Prevalence of Learning Disabilities in Mathematics in India

In a review of studies on learning disabilities in India, conducted from 1980 to 2000 by Ramaa (2000), a 3–10% prevalence of different types of SLD was reported among the elementary school population. This review included major findings of studies conducted by different investigators to diagnose elementary and secondary school students with dyslexia, dyscalculia, language disabilities, and writing disabilities, and to try out remedial instruction programs, in the states of Karnataka, Kerala, and Tamil Nadu. Students with dyscalculia/mathematical disabilities may also have dyslexia and writing disabilities. Hence, studies on other types of SLD also help in conducting research on dyscalculia/mathematical disabilities. The paper based on such a review described a range of research studies relating to learning disabilities in India from 1980 to 2000. Attention is called to the existence of many different languages within India. Standardized and teacher-made tools have been developed for assessment and remediation purposes. The paper ends by making some suggestions for future research.

Mogasale, Patil, Patil et al. (2012) attempted to study the prevalence of SLD such as dyslexia, dysgraphia, and dyscalculia among primary school children in a South Indian city. A cross-sectional multistaged stratified randomized cluster sampling study was conducted among children aged 8–11 years from grades III and IV. The observed prevalence of SLD was 15.17%, while 12.5%, 11.2%, and 10.5% of the children had dysgraphia, dyslexia, and dyscalculia, respectively. On the basis of this study the investigators inferred that the prevalence of SLD is on the higher side compared to previous estimations in India.

Priti Arun et al. (2013) conducted a study to find out the prevalence of a specific developmental disorder of scholastic skills (SDDSS) in students in grades VII to XII and to assess the feasibility of a screening tool in Chandigarh, India. They observed an SDDSS prevalence of 1.58% in 12- to 18-year-old school students. The investigators attributed the apparent low prevalence of SLD to nonavailability of standardized psychological tests in the vernacular language. Further, they noted that information from parents is crucial in studies pertaining to academic problems in view of the fact that many causes of scholastic backwardness require a complete work-up including social, emotional, and physical factors.

In a review of research work done on SLD in the Indian context Annie, Akila et al. (2013) noted that while there have been studies on different aspects of SLD, there has been no sustained, rigorous research done on LD with other comorbid disorders like ADHD among Indian children and adolescents. Suresh and Sebastian (2003) attributed the limited epidemiological data on the prevalence of LD to many of the inherent difficult situations in India.

Ramaa (1985), Srimani (2000), Gowramma (2000), Prema (2002), and Jagathy (2006) conducted doctoral-level studies on dyslexia, language disabilities, dyscalculia, writing disabilities at elementary level, and writing disabilities at secondary level, respectively. They attempted diagnosis and remediation of SLD. The details are available in Gowramma (2005), Ramaa, Miles and Lalithamma (1993), Ramaa (1993), Ramaa (2000), Ramaa (2017), and Srimani (2012).

Research on Learning Difficulties and Disabilities in Mathematics in India

As in the global scenario, there is much less research on mathematical disability than onreading and writing disabilities in India. A brief account of this research is given below.

Ramaa (1990) conducted a study on neuropsychological processes and logicomathematical structure among students with both dyscalculia and reading/writing problems. Since there was no appropriate diagnostic tool in arithmetic for primary school children, she constructed such a tool. The test is diagnostic and criterion referenced in nature, and it is available in both English and Kannada versions. The Kannada version was used in this study. The test covers three major areas of arithmetic: number concepts, arithmetic processes (fundamental operations)—addition, subtraction, multiplication, and division—and also arithmetic problem solving. In each of these areas, a series of basic understandings and skills (criterion measures) expected to be mastered by children in grades I to IV are covered. The items helpful in assessing different criterion measures are varied in number. The numbers of items for grades III and IV are more or less the same; the difference is only marginal. The list of criterion measures are available in Ramaa (2015). Students with dyscalculia who were free from dyslexia and dysgraphia were identified among 251 children studying in grades II, III, and IV in primary schools by using a set of exclusionary and inclusionary criteria. Out of 251 children, 15 students (5.98%) were diagnosed as having dyscalculia. This figure is exactly the same as that noted by Kosc (1974). For details of the identification procedure, see Ramaa and Gowramma (2002).

In order to identify the difficulties in each of the criterion measures appropriate to the grades, the relevant items in the Arithmetic Diagnostic Test developed by the investigator were administered to all 15 children. The presence or absence of mastery by the students with reference to each of the grade-appropriate criterion measures were analyzed and tabulated. The items/criterion measures that were not attempted by the students were considered as not achieved (NA). Thus, the data obtained for each of the students was analyzed qualitatively and separately. The errors committed by the students while solving different arithmetic problems were also analyzed. The numbers of students who committed such errors were also calculated. Multiplication and division sums were found to be more difficult for children with dyscalculia even in grades III and IV.

In the study it was attempted to find out whether the children who had only dyscalculia were deficient in the specific neuropsychological processes of auditory sequential memory (memory for auditorily presented digits) and visual sequential memory (memory for shapes in sequence) and in the different components of logicomathematical structure—seriation, conservation, and classification. For this purpose, visual sequential memory and auditory sequential memory subtests of the Illinois Test of Psycholinguistic Abilities (ITPA) (Kirk, and McCarthy, & Kirk, 1968) and the Mysore Cognitive Development Status Test (Padmini and Nayar, unpublished) were administered to all 15 children with dyscalculia. The data obtained from each student were analyzed qualitatively. It was noticed that in all the cognitive abilities tested in the study, children with dyscalculia had moderate to severe deficiency. For a detailed description of the procedure and results, see Ramaa (2015).

Gowramma (2000) conducted a doctoral-level study (under the supervision of the author of this chapter) on development of a remedial instruction program for children with dyscalculia studying in grade III and IV at primary schools. She attempted to identify children with dyscalculia from a sample of 1408 children by using a set of inclusionary and exclusionary criteria. She diagnosed 78 students (5.54%) as having dyscalculia and further classified them into different categories as follows: dyscalculia without reading and writing problems (24 out of 78 (30.77%)), dyscalculia without a writing problem (14 out of 78 (17.95%)), and dyscalculia with reading and writing problems (40 out of 78 (51.28%)). Detailed descriptions of the procedure are available in Ramaa and Gowramma (2002) and in Gowramma (2005).

Gowramma (2000) analyzed the difficulties faced and errors committed by the students in her study by administering the Arithmetic Diagnostic Test for primary school children (Ramaa, 1990, 1994). In order to overcome these difficulties she planned a remedial instruction program and experimentally validated it. The remedial instruction program was found to be effective in general. She has given suggestions for further study (Gowramma, 2005).

In India, people who are deprived of basic social rights and security because of poverty, discrimination, or other unfavorable circumstances are called "socially disadvantaged." This group includes Scheduled Castes (SC), Scheduled Tribes (ST), Other Backward Classes (OBC), and Minorities (linguistic/religious). Among them the SCs and STs are the most disadvantaged due to social and geographical/cultural exclusions, respectively. Nearly 25% of the Indian population belong to these categories.

Shukla and Neerja (1994), in trend reports on research in mathematics education, indicated that certain factors were responsible for higher rates of failure in mathematics achievement, particularly at secondary level. The major contributing factors noted were intelligence and the socioeconomic background of the students. The authors stressed the need for continued research in mathematics education, keeping in mind the diversities in Indian schools, and for development of special strategies for teaching first-generation learners, children from backward classes (sectors), and children with sensory handicaps and intellectual disabilities, as well as children from tribal and hilly areas. There is a need to understand the specific difficulties faced by different categories of children in order to meet their special needs. Sawant and Athwale (1994) observed that the proportions of literate individuals are lower among those who belong to the SCs and STs. The main reason for this is deprivation of educational facilities for generations. Severe underachievement among tribal students at the primary level has been noted by Sujatha (1998), Shukla (1997), and Prakash (1997).

Ramaa and Gowramma (2001) conducted a study to identify the arithmetic difficulties faced by 138 socially disadvantaged students who were studying in grade V at five government primary schools. They administered the Arithmetic Diagnostic Test (Kannada version) developed by Ramaa (1994) in order to identify difficulties in arithmetic among the students in the study. The percentages of the students who were masters, partial achievers, and non-masters with reference to each criterion measure were calculated. The students experienced difficulties in almost all the criterion measures. Since the tool administered was meant for grades I to IV, a greater number of masters were expected, followed by partial achievers and non-masters. However, it was noted that the percentages of partial achievers and non-masters were greater than the percentages of masters in the majority of the criterion measures.

Ramaa (2015) did a qualitative meta-analysis of the data obtained from the 1990 and 2001 studies on children with dyscalculia and socially disadvantaged children, respectively. She compared the percentages of masters, partial achievers, and non-masters among children with dyscalculia with those of socially disadvantaged children in each of the criterion measures covered in the Arithmetic Diagnostic Test (Kannada version) (Ramaa, 1994). It was found that there was a considerable overlap in the difficulties experienced by both groups of students.

The results are discussed in detail, highlighting the educational implications in the chapter (Ramaa, 2015).

In the study conducted by Kaur, Kohli, and Batani Devi (2008), the investigators verified the comparative efficacy of various strategies for basic mathematical skills in children with learning disabilities. The students in the study were randomly assigned to multimedia, cognitive, eclectic, and control conditions. All the strategies employed in the study significantly enhanced the basic mathematical skills of learning-disabled children. The investigators concluded that multimedia, cognitive strategy, and eclectic approaches can be used to enhance the mathematical skills of children with learning disabilities.

Nagavalli (2015) studied the effectiveness of different intervention strategies for learners with dyscalculia at the primary school level. The study was carried out in three phases. The first and second phases focused on administration of screening and diagnostic tests to 2180 students studying in grade V at primary schools in the Salem district. The test results revealed that 50 students had learning difficulties in mathematics and were identified as students with dyscalculia. The cognitive deficiencies among them were analyzed. The investigator attempted to classify the children with dyscalculia identified in the study into different categories. The third phase of the study involved administration of remedial intervention programs to children with dyscalculia. The post-test scores of the boys and girls were better than their pre-test scores.

Conclusion

The Government of India, professionals, researchers, and service providers have started showing interest in learning disabilities in general and mathematical difficulties and disabilities in particular. There should be more intensive and extensive efforts in this direction in the country.

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Chapter 20
Adding all up: Mathematical Learning Difficulties Around the World

Vitor Geraldi Haase and Helga Krinzinger

Authors of the chapters in Section 2, "Math learning difficulties across the world," were invited to provide a description of math education and achievement in their respective countries, with an eye toward math learning difficulties. The following questions were proposed: (a) how are special needs in mathematics education defined and recognized?; (b) what kind of support do children get at school for severe math learning difficulties?; (c) who gives the support and what qualifications do they have for this work?; (d) are evidence-based assessment tools and intervention methods available?; (e) what are the key issues and current trends in math learning difficulties at the moment?

In this discussion, we consider if it is possible to delineate a global picture of math learning difficulties and which is the format eventually assumed by such a picture emerging from the diversity and local specificities. Is it possible to detect global trends in the recognition and support of individuals with math learning difficulties?

From the outset, it is possible to verify a widespread recognition of the importance of math education and concern from governments, educators, and parents for youngsters' math achievement on international tests. In the next section, we delineate in very broad strokes the state of math achievement across different countries.

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Math Achievement Around the World

From the results reported on youngsters' math achievement around the world, we believe it is possible to identify at least four situations. This is, admittedly, a gross simplification, neglecting specificities of distinct contexts. The first is represented by the top achievers, i.e., some Northern European and East Asian countries (Chang, Lee, & Yen, 2018; Räsänen et al., 2018). These two sets of countries seem to follow different paths to high math achievement. In these countries, both problems of universal access and quality of schooling have been largely solved. The path toward excellency in the Northern European countries seems to be related to the advances of the welfare state. Northern European countries present high indexes of "human development," related to affluence, heavy taxation, strong state regulation, political democracy, juridical stability, low corruption indices, economic and gender equality, etc. Investment in education and teacher training is massive, and, at the same time, management is decentralized, with schools presenting a considerable degree of autonomy and accountability.

The route taken by Eastern Asian countries is different. Economic affluence in Eastern Asian countries is a relatively new phenomenon, and political democracy and socioeconomic and gender equality are not uniformly widespread across these countries. Math achievement success in Eastern Asian countries seems to be associated with (a) efficient school systems; (b) relative transparency of verbal numeral systems in Eastern Asian languages, conferring a head start in learning arithmetics; and (c) motivational factors. Motivational factors are rooted in ancient cultural traditions related to respect for parents and teachers and diligence associated with the recognition of the importance of math education in the current global context.

In both Northern European and Eastern Asian countries, rates of low numeracy, incompatible with effective functioning in the contemporary knowledge society, are extremely rare. For example, rates of youngsters not achieving at or above PiSA Level 2 have oscillated around 10% in several of these countries. It seems that such countries improved their educational systems to such a degree that the rates of very low performance roughly correspond to the prevalence of the most severe forms of math learning difficulty. Some Northern European countries are moving one step ahead of universal and good quality school education. The challenge goes beyond mandatory education to mandatory learning (Räsänen et al., 2018). It is supposed that all children should learn, including those who present some sort of inherent difficulty, such as developmental dyscalculia.

The second situation is exemplified by most other Northern Hemisphere and Oceania countries (Baffaluy & Puvuelo, 2018; Csikos, András, Rausch, & Shvarts, 2018; Desoete, Dowker, & Hasselhorn, 2018; Jordan, Rinne, & Hansen, 2018; Reeve, 2018). These countries have a rich scientific and pedagogical tradition allied to high human development indexes. With a few exceptions, mean math achievement levels in these countries are generally average to high. However, intranational variability is also high in these countries, with a significant proportion of around 15% to 25% of individuals not attaining the minimal numeracy requirements to effectively navigate in the contemporary knowledge society. The relative high rates of very low performers, much higher than the prevalence of developmental dyscalculia, in the presence of well-organized school systems, suggest a role for socioeconomic and cultural factors. The discrepancy between general high indices of human development in the face of low math achievement is probably related to the drastic social transformations occurring in these countries in the last decades. These include massive migration and the challenge to educate children from diverse cultural contexts, who need to learn in a foreign language. Eastern European countries face the challenge of rebuilding their educational systems after many decades under oppressive political regimes, followed by devastating wars and social chaos in many cases.

The third situation refers to the so-called "developing" countries such as several countries in Latin America, North Africa, and the Middle East (Dorneles, 2018; Rodríguez, Cuadro, & Ruiz, 2018). These countries present reasonable indexes of human development and have, to a large degree, solved the challenge of enrolling all children in school. The next challenge is to improve the quality of education. As a consequence of poor educational quality, percentages of very low attainers may vary from 50% to 70%. Children attend school, but they simply do not learn. Such extremely high rates of math learning failure are at least one order of magnitude larger than the prevalence of developmental dyscalculia. Reasons may also be related to socioeconomic factors. The main culprits could be related to political and juridical instability, populist regimes, and corruption associated with a general inefficiency of the state in implementing educational policies. Poor quality of teacher training; ineffective, ideologized, and scientifically unsubstantiated pedagogical orientations (e.g., Freire, 2000), together with corporate disputes over wages (sometimes associated with frequent strikes and teacher absenteeism) (Sowell, 2010); and lack of valorization of the teaching profession must also be mentioned. This is even more unfortunate when one considers that many of these countries formulated reasonable curricular guidelines, following, for example, guidelines suggested by UNESCO and other international organizations. A recent report of the World Bank indicated that efficiency in education is more dependent on the quality than on the amount of public expenditures (World Bank, 2017).

It is important to keep in mind that, even in these developing countries plagued by political and socioeconomic problems, poverty is not the only risk factor for poor math achievement. Some children may be simultaneously poor and present inherent math learning difficulties. Not recognizing the full nature of these children's problems may prove to be a double handicap. They may be simultaneously disadvantaged because of poverty and because of some condition predisposing to specific math learning difficulties (Carvalho & Haase, 2018; Haase & Carvalho, 2018).

The fourth situation is related to very poor educational systems and lack of universal enrollment in school. This situation describes difficulties encountered by many sub-Saharan and some Latin-American countries, especially in the Caribbean (Roberts, Tshuma, Mpalami, & Saka, 2018). The challenge here is to build a school system in the first place.

The picture described of math achievement around the world is supported by several data reported in Section 2. In broad strokes, the following generalizations could be advanced:

(a) Although rates of low math achievers may be somewhat greater than those of reading low achievers, in general, math achievement is correlated with attainment in other curricular domains.
(b) Across countries, math achievement is correlated with the human development index.
(c) Intranational socioeconomic differences in math performance are lower in the top-performing countries.
(d) Cross-national variability is observed in the low-performing and not in the high-performing strata of youngsters.
(e) A country's overall average achievement and the rate of low-performing students in that country are correlated.
(f) In the top-performing countries, rates of low achievement are more or less similar to that of developmental dyscalculia.

The big picture is, thus, of huge cross- and intranational variability associated with human development and socioeconomic welfare in the case of top-performing countries and with political instability and socioeconomic apartheid in the low-performing countries. The overall picture is sometimes colored by regional hues. In the next section, we proceed to the discussion of four aspects that vary across countries: (a) gender issues; (b) historical heritage, such as from the former Soviet regime; (c) intranational diversity; and (d) paradox of high math achievement in the face of low motivation.

Gender Issues

Apart from socioeconomic status, another topic regarding equality in education was surprisingly only mentioned in the chapter by Rodríguez, Cuadro, and Ruiz, although it is a worldwide issue, namely, the gender gap in math education. An overview about prevalence studies of dyscalculia shows that older studies using a simple cutoff criterion commonly report higher prevalence rates in boys (Badian, 1983, 2.2:1; Badian, 1999, 1.2:1; Share, Moffitt, & Silva, 1988, 1.8:1), whereas studies with any kind of discrepancy criterion (either in comparison with IQ scores or literacy scores) show either no gender differences (Lewis, Hitch, & Walker, 1994) or even higher-risk ratios in girls (Hein, 2000, 0.2:1; Von Aster, Kucian, Schweiter, & Martin, 2005, 1:2.28; Gross-Tsur, Manor, & Shalev, 1996, 1:1.1; Klauer, 1992, no explicit rates reported). The same effect could be observed in a study using different criteria in the same sample of over 1000 children: no prevalence differences in girls and boys regarding dyscalculia were found if no discrepancy measures were used (Devine, Soltész, Nobes, Goswami, & Szűcs, 2013). However, a discrepancy of one standard deviation to a standardized reading score yielded 55 girls vs.

24 boys and a discrepancy of 1.5 standard deviations resulted in 110 girls vs. 78 boys identified as dyscalculic. So, contrary to most other developmental disorders, girls seem to be more often affected by developmental dyscalculia (e.g., Von Aster et al., 2005), whereas boys with mathematical learning difficulties combined with other cognitive deficits are found at least as often as girls with the same problems.

In general, no or only small performance differences in math between boys and girls are reported for aptitude tests, school achievement tests, or math grades (Kimura, 2000; Lindberg, Hyde, Petersen, & Linn, 2010). In some countries, like Finland, girls even typically outperform boys in mathematics grades at school. However, breaking down mathematics to more clearly defined cognitive subcomponents reveals a male advantage from adolescence onward for complex word problems (Kimura, 2000). A male advantage is also observed during primary school years (Kaufmann et al., 2009; Krinzinger et al., 2012; Krinzinger, Wood, & Willmes, 2012; Zuber, Pixner, Moeller, & Nuerk, 2009), and even in kindergarten (Weinhold Zulauf, Schweiter, & von Aster, 2003), for the acquisition of multi-digit number understanding (Krinzinger, 2011). These differences could be related to better spatial skills in boys (Voyer, Voyer, & Bryden, 1995), probably leading to an advantage in using spatial cognitive strategies (e.g., Carr, Hettinger Steiner, Kyser, & Biddlecomb, 2008; Van Garderen, 2006).

Alternatively, recent international studies (PISA, TIMMS) have shown that gender differences in the mathematical domain were related to the respective national "gender gap index" (GGI; e.g., Hausmann, Tyson, & Zahidi, 2006), mirroring economic, academic, and other fields of (in)equality between women and men (e.g., Else-Quest, Hyde, & Linn, 2010; Guiso, Monte, Sapienza, & Zingales, 2008). One respective study by Penner (2008) showed that gender inequality in the job market for the parent generation was the main factor explaining gender differences in the children. The cognitive mechanism behind this is most likely "stereotype threat" in the sense of a self-fulfilling prophecy (e.g., Osborne, 2001), meaning that girls and women score worse on math tests only because they were told they are less able. It has been shown that the correlations between self-concepts and attitudes toward math and math achievement grow stronger during puberty (Denissen, Zarrett, & Eccles, 2007). Hopefully, gender gaps and the negative stereotypes regarding females and mathematics (as well as sciences in general) will continuously decrease and finally vanish in the future, so a development like that found in Finland can be observed in more and more countries.

Studies trying to disentangle the effects of socially influenced factors (such as math self-concepts) and more biologically based factors (such as spatial skills) on gender differences in math usually find that both play a role (Casey, Nuttall, & Pezaris, 1997, 2001). It may even be senseless to discuss about the differential effects of biology and environment, as their effects influence each other and are thus "as inseparable as conjoined twins who share a common heart" (Halpern, 1997, p. 1097). However, as a society, we should definitely focus on social factors which may negatively impact math achievement in a substantial subgroup of children, whether it be poor socioeconomic background or gender gaps.

Heritage of the Soviet Regime

Countries belonging to the sociopolitical and economic sphere of influence of the former Soviet Union have a strong cultural, scientific, and pedagogical heritage dating back centuries (Csikos et al., 2018). Thus, the extreme variability in math achievement displayed by these countries deserves attention. Some countries are among the top performers, while others are situated among the very lowest rankings. This variability is certainly related to the social chaos and in some cases, warfare, occurring in many of these countries after the fall of the Soviet Union.

Csikos and coworkers call attention also to the pedagogical heritage of the Soviet regime: (a) political interference in the curriculum and textbooks; (b) centralization of decisions and lack of autonomy of schools; (c) emphasis on math competitiveness, favoring outstanding performance in selected individuals; (d) misuse of the math achievement of a few individuals for political propaganda; (e) apartheid between those who could and those who could not do math, against the background of an officially egalitarian ideology; and (f) emphasis on math and disregard for pedagogy in math teacher training. In some of these countries, the result was a relative disregard for the needs of children who did not follow the pace of most of their colleagues in learning mathematics. The authors explicitly advise against ideologization of math teaching: "any kinds of ideologically set mathematics education necessarily dismiss the integrity of mathematical knowledge." Unfortunately, ideologization of education is a danger that did not go away with the fall of the Soviet Union.

Intranational Diversity

It is important to recognize that the remarks presented above are gross simplifications, when one considers the huge socioeconomic, cultural, ethnic, linguistic, and religious differences faced by many areas such as sub-Saharan Africa, Caribbean, Brazil, India, and Israel (Ashkenazi, Restle, & Mark-Zigdon, 2018. Dorneles, 2018, Ramaa, 2018, Roberts et al., 2018). The situation regarding math achievement is not so clear in areas such as India and sub-Saharan Africa, which do not participate in international surveys. However, there are reasons to assume a heavy burden of inequality (Drèze & Sen, 2013; Rosas & Santa Cruz, 2013; Tooley, 2001).

According to Ashkenazi and coworkers, governments in these and other countries must solve the challenge of implementing a minimum curriculum and improving general level of achievement (the "melting pot" approach) and, at the same time, acknowledging diversity and preserving traditions. Additionally, intranational diversity frequently precludes participation of certain population groups in the international surveys of math achievement, potentially leading to distortions (Ashkenazi et al., 2018; Dowker, 2018).

Achievement-Motivation Gap

The last regional color is illustrated by the gaps between math achievement and motivation. For example, math motivation in several high-performing countries is, on average, lower when compared with that of individuals from low-performing countries (Chang et al., 2018; Räsänen et al., 2018). Low math motivation despite high math achievement has elicited the greatest interest in the context of East Asian countries (Lee, 2009; Stankov, 2010). In the case of East Asian cultures, the motivational paradox has been speculatively attributed to some cultural traits such as a more collectivistic orientation, high expectations, fear of losing face, and extreme conscientiousness. Anyway, pressure from parents and teachers is a widespread phenomenon that negatively impacts on emotional well-being and motivation to learn mathematics (Batchelor, Gilmore, & Inglis, 2017; Beilock, Gunderson, Ramirez, & Levine, 2010). Traditional teaching, characterized by high demands for correctness and little cognitive and motivational support, is associated with negative emotional experiences (Bekdemir, 2010, Meece, Wigfield, & Eccles, 1990, Turner et al., 2002, see also Haase, Guimarães, & Wood, 2018).

Finally, it is important to comment on the flip side of the motivation paradox. On average, youngsters from some developing countries report high levels of math self-concept, in spite of low math achievement, when compared with other countries (Lee, 2009). It seems that these youngsters are unaware of their difficulties and unpreparedness.

The variability of math achievement around the world and its increasing importance would suggest that governments should not spare efforts in improving math education as well as in recognizing and providing interventions for kids with math learning difficulties. In the next section, we will focus on the difficulties associated with these two last tasks.

Definition of Special Needs in Math

Research on math learning difficulties has traditionally fallen behind research on reading learning difficulties (Gersten, Clarke, & Mazzocco, 2007). Research on reading has also been an important source of motivation for research in math learning and its difficulties. Recognition of learning difficulties is not a simple matter. Evidence indicates, for example, that even nowadays developmental dyslexia is under-recognized (Barbiero et al., 2012). It is no surprise then that recognition of math learning difficulties is considerably less frequent than recognition of reading difficulties (Balbi, Ruiz, & García, 2017).

Virtually all authors in Part 2 reported that early recognition of math learning difficulties in their countries is clearly insufficient. Legislation in most countries foresees recognition and intervention for individuals with special educational needs. But lawful recognition of specific learning difficulties, and math learning difficulties in particular, is less common. In certain countries, guidelines have been formulated, stimulating teachers to identify math learning difficulties (Baffaluy & Puvuelo, 2018). In most West European and North American countries and elsewhere in the world, validated diagnostic instruments are available (Baffaluy & Puvuelo, 2018; Desoete et al., 2018; Jordan et al., 2018; Räsänen et al., 2018; Rodríguez et al., 2018).

A very interesting move is the development and slow but steady increasing rate of implementation of the response to intervention (RTI) approach (Jordan et al., 2018, see also Fuchs, Fuchs, Seethaler, & Zhu, 2018). The RTI approach foresees early diagnosis by teachers, using screening procedures to identify kids at risk for developing math learning difficulties. At-risk kids are then referred within the school environment to a series of interventions of increasing intensity if difficulties persist. There is cumulative evidence indicating the effectiveness of the RTI approach (Fuchs et al., 2018; Jordan et al., 2018).

The greatest advantage of RTI is limiting diagnosis and intervention to resources available in the schools without referral to specialists. This is in accordance with the growing evidence of the heterogeneous nature of math learning difficulties (Haase & Carvalho, 2018; Rubinsten & Henik, 2009) and, at the same time, avoids unnecessary labeling effects (Lauchlan & Boyle, 2007). Around the world, a clear move can be identified away from diagnostic categories based exclusively on a medical model.

The main hazard associated with RTI is delaying the referral of kids with genetic conditions to proper, specialized diagnosis and care (Haase & Carvalho, 2018). Implementation of the RTI approach is also associated with several logistic problems such as trained personnel shortage, treatment quality, and adherence. At the moment, although there seems to be clear movement toward RTI, this approach is foreseen in the legislation of some countries and research circles in a few others. Clearly, there is a gap between research results and real-life implementation.

Support at School for Children with Severe Math Difficulties

In recent decades, education has been inspired by the goal of inclusion, i.e., of not leaving any child behind. This is also important in the case of math education, considering the complexity inherent in the subject (Mazzocco, Hanich, & Noeder, 2012) and its socioeconomic importance (Parsons & Bynner, 2005). The experience of the top-performing countries clearly shows that this is possible (Chang et al., 2018; Räsänen et al., 2018). This is the case, at least for the vast majority of children, with the possible exception of those presenting more severe and genetically based forms of math learning difficulties (Carvalho & Haase, 2018; Haase & Carvalho, 2018). It remains to be investigated whether proper early intervention can prevent these more severe forms of math learning difficulties. This is the challenge currently posed to the top-performing countries.

The results of the top-performing countries suggest that decentralized policy making, autonomy, and accountability on the part of schools and teachers, teacher training, and adoption of the RTI approach described above are promising measures. Accumulating evidence suggests that more severe difficulties in math learning are associated with basic numerical processing impairments and that these impairments may be ameliorated when properly recognized at an early age (Siegler & Braithwaite, 2017). However, this is far from fully corroborated, and most countries are far from the ideal of investigating this hypothesis and, eventually, implementing effective early interventions. It would be fair to say that a reasonable proportion of countries is in different degrees of transition toward the model of early recognition and interventions in the school environment.

An interesting observation is that those countries which seem to take the best care of children with special educational needs, within the educational system (including high quality teacher education), are those which do not use international diagnostic classification systems (DSM-5 or ICD-10) and their criteria for the identification of learning difficulties. This is not surprising, as the diagnostic classifications are created for use in the health systems and not in the educational systems. Additionally, clinical diagnoses may be avoided by implementing proper pedagogical interventions. In general, countries with less specific teacher education and less elaborate special educational needs programs rely more heavily on clinical diagnoses, sometimes to provide affected children with help from outside the educational system.

Teacher Training

Judging from the experience of the top math performing countries, it seems that teacher training is one of the ingredients of effective math education (Chang et al., 2018, Räsänen et al., 2018). This is more easily said than done. We have already commented on the need for a qualitatively improved and not only quantitatively increased educational expenditure (World Bank, 2017). We mentioned also the worldwide ideological influences on education (e.g., Freire, 2000) and corporate interests as obstacles to a better education (Sowell, 2010). In this section, we comment on two possible obstacles to better teacher preparation.

The first one is related to lack of primary teacher training in mathematics. Primary teachers widely report feeling uncomfortable and unprepared to teach numeracy, compared to literacy (Bekdemir, 2010; Meece et al., 1990). Additionally, evidence indicates that teacher self-doubt regarding math teaching competence can negatively impact on the pupils (Beilock et al., 2010). It seems that, contrary to what happened in the former Soviet Union, teacher training in recent years has paid more attention to the pedagogical than to the math aspects of mathematical education. The result is that primary teachers feel unsure about teaching math to their pupils.

The second one is related to the lack of pedagogical training. The old math pedagogy emphasized and decontextualized math performance (Klein, 2003).

The new pedagogy of mathematics, as illustrated by the definition of the PISA levels of mathematics knowledge, emphasizes math competencies in real life (Brazil, 2016). The emphasis on contextualized competencies is associated with curricular goal implementation problems. As these competencies are rather loosely defined, teachers simply do not know what and how to proceed to implement the curricular goals. This problem is compounded by the fact that the competency model is usually implemented through a constructivist framework, as it will be discussed in the next section.

Toward Evidence-Based Education

Reading the chapters in Section 2, it is possible to infer that everywhere in the world, there is a gap between scientific knowledge and its practical implementation. It seems that lab results have a long and tortuous path until they reach the classroom. What varies cross-nationally is the width of the gap. The state of research is more advanced in Northern Hemisphere and Pacific Rim countries (Baffaluy & Puvuelo, 2018; Chang et al., 2018; Desoete et al., 2018; Jordan et al., 2018; Räsänen et al., 2018; Reeve, 2018). However, even in these countries, classroom practices lag behind scientific and legislative advances. The need for evidence-based education is widely recognized by several authors.

The difficulties several countries encounter in implementing public policies are classic. It seems, however, that some pedagogical choices may also interfere with the building of more effective and evidence-based education. As mentioned above, the competency model currently adopted is ill-defined and lacks objective implementation guidelines. Additionally, the competency goal model is usually implemented within a constructivist framework of discovery and cooperative learning (Christodoulou, 2014). Undoubtedly, the constructivist approach stimulates creativity (Bonawitz et al., 2011; Lee & Anderson, 2013). However, it also imposes heavy demands on the pupils in terms of cognitive abilities, intentionality, mind reading, and other social skills (Tomasello, Kruger, & Ratner, 1993). This option alone may not be the best way to promote math learning in children who present specific difficulties in this subject. Exclusive learning by discovery in an ill-structured setting overloads the working memory capacity and may not be the best way to teach children with special needs (Kirschner, Sweller, & Clark, 2006; Mayer, 2004). This contrasts with the efficiency of mixed methods, sharing at least some instructional components (Hattie, 2009; Lee & Anderson, 2013). A cross-generational and cross-national comparative study indicates, for example, that both math reasoning and calculation fluency decreased decade after decade in the USA, while these same competencies improved dramatically in China (Geary et al., 1997). This time epoch roughly corresponds to the implementation of the constructivist framework in the West. A more balanced pedagogical framework, combining discovery and cooperative learning with classical instruction, development of both conceptual and procedural knowledge, etc. could be in order.

If the analysis presented above is correct, math pedagogy is desperately in need of an evidence basis. The first move toward an evidentiary foundation of math pedagogy would be adopting an open mind approach, searching for evidence and not only basing classroom practices on teachers' ingenuity or undemonstrated theoretical assumptions, some heavily charged with ideological implications, despite their popularity in some quarters of the world.

Key Issues and Trends

In a nutshell, if a general trend is to be identified in math education and support for individuals with learning difficulties, the current models adopted in Northern European and advanced Eastern Asian countries are regarded as a standard toward which all eyes are directed. This model could be characterized in broad strokes as strong state intervention, efficiency of public expenditure, excellent teacher training, decentralization of the decision process, and school and teacher accountability. Ethically, this model is guided by the ideal of inclusion and demedicalization, limiting diagnosis and intervention to the school environment as much as possible. Theoretically, it is based on the RTI model of early diagnosis and intervention. The efficacy of this model is demonstrated by both research evidence and its success in restricting math learning difficulties to a tiny proportion of the school population. Some countries have even adopted, and are striving toward, the goal of learning for all children.

One assumption that is shared, at least implicitly, by all authors is state intervention. The authors share the assumption that education is a human right and a state obligation. The problem is that not always, and not everywhere, does the state fulfill its duties. This has led to the building, in some developing and underdeveloped countries, of an informal private school system for the poor (Tooley, 2001). As an educational phenomenon, private schools for the poor have received almost no attention from researchers and international agencies. Physical and human resources in these schools are very poor. Little is known about their efficacy. What is certain, however, is that there is a market for such schools. That is, when parents can afford to, even in the poorest countries of the world, they prefer to send their kids to private schools.

We end this discussion with more questions than answers. The difficulty in generalizing across countries suggests there is no one-size-fits-all solution for math educational needs. The current state-regulated and competency model seems to be adequate for some countries. The question must also be posed: are there alternatives that could better fit specific local needs? Definitely, more information is missing regarding math educational needs of poor performing countries. Research should also look for alternative models and sources of evidence.

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Part III

Mathematical Learning Difficulties and Its Cognitive, Motivational and Emotional Underpinnings

Chapter 21
Genetics of Dyscalculia 1: In Search of Genes

Maria Raquel S. Carvalho and Vitor Geraldi Haase

Introduction

Living at the beginning of the twenty-first century requires being numerate; this means dealing with numbers appropriately. Effective numerical abilities are increasingly important in our modern information societies and, in particular, with respect to topics involving science, technology, engineering, and math (STEM). The ability to reason numerically is critical for individual life and career prospects (e.g., Butterworth, Varma, & Laurillard, 2011; Dowker, 2005). In particular, math abilities have been associated with higher wages, employability, and mental health indexes; inverse effects are observed to be associated with low math achievement, even after excluding the effects of low literacy (Auerbach, Gross-Tsur, Manor, & Shalev, 2008; Parsons & Bynner, 2005).

Low math achievement affects everyday life profoundly. Affected persons may face difficulties in using money, identifying bus lines, understanding maps and deciding routes, counting blocks, understanding hours, culinary recipes, etc. Therefore, not only researchers but also policy makers and public opinion are increasingly sensitive to the economic and social importance of different national performance levels on international school achievement tests, such as PISA (Budd, 2015; Sturman, 2015).

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While some individuals excel in math achievement, others struggle with persistent and severe difficulties in learning the most basic aspects of arithmetic such as number reading and writing, performing the four operations, and learning the basic facts. These difficulties present a major obstacle to progressing across the math curriculum and deserve the diagnosis of developmental dyscalculia or math learning disability (Butterworth et al., 2011; Shalev, Manor, & Gross-Tsur, 2005).

At both ends of the population distribution, math abilities run in families (Landerl & Moll, 2010; Shalev et al., 2001) are moderately heritable (Docherty et al., 2010; Petrill, Kovas, Hart, Thompson, & Plomin, 2009). However, the investigation of the genetic mechanisms underlying math learning difficulties is hampered by the complexity of the phenotype. The phenotype is influenced by multiple cultural, linguistic, socioeconomic, cognitive, emotional, and motivational factors.

Mathematics is a cultural acquisition, something learned in school. However, we do need a brain that is specifically wired to learn math. As discussed below, dealing with math involves innate abilities, which have been selected in an evolutionary scenario that largely predates humans (Dehaene, 2011). Therefore, we share some components of math ability with other animals, but we also must have acquired some new abilities that they have not. Most of these new abilities reflect cultural evolution during the last 10,000 years. In terms of evolution, this is just a blip. How has it been possible to develop math to the point that we have it today, considering that the cultural evolution is so recent? Might better math ability itself has been a selective advantage? It is possible. Natural selection acts by changing allelic frequencies. Therefore, genes may provide answers to these questions, at least in part.

The investigation of such complex (in the sense of multicomponent), and somewhat ethereal, phenotypes such as math ability may be tricky. For example, there is a wide range of math abilities within human populations. However, most of this variability reflects environmental, particularly socioeconomic status (SES), effects. Disentangling the genetic components of variability found in the normal range of the distribution is usually hard.

In this context, studying dyscalculia may provide us with useful questions, such as: If dyscalculia is a consequence of an abnormal brain development, how can we discover the etiology of these abnormalities? How can we use genetics to characterize brain development, both normal and abnormal? If the main question posed by children with dyscalculia and their parents is *why*, will we ever be able to answer it? Will knowing the causes of dyscalculia help us to evaluate intervention programs and to establish prognoses? Are there red flags which would help us to identify children at risk, who would benefit from intervention, before math difficulties emerge? Will genetics help us deal with such red flags, if they exist? These are ambitious questions and most of them are beyond our grasp, for now. However, some important first steps have already been taken. Here, and in the companion chapter (Haase & Carvalho, 2018, this volume), we will review the state of the art of the genetics of dyscalculia.

The text is divided into three sections. First, we review the clinical epidemiology of developmental dyscalculia. Second, we review the evidence for genetic susceptibility in developmental dyscalculia. Third, we review the main gene-finding strategies currently available, concentrating the discussion on genome-wide association studies (GWAS), genome-wide linkage analysis (GWLA), and association studies conducted on comorbidities. Characterization of cognitive endophenotypes and searching for genotypic-phenotypic correlations in genetic syndromes associated with math learning difficulties are other important gene-finding strategies. This will be reviewed in the companion chapter (Haase & Carvalho, 2018, this volume).

Clinical Epidemiology of Developmental Dyscalculia

Difficulty in acquiring and retrieving the arithmetic fact tables is the central symptom of dyscalculia (Butterworth et al., 2011). Other manifestations include difficulties in estimating and comparing set sizes; counting, reading, and writing numbers; performing the four basic operations; etc. Impairments are severe and persistent and affect the most basic aspects of number processing and arithmetic (Shalev et al., 2005; Wong, Ho, & Tang, 2014). Children with dyscalculia are delayed in beginning to finger count, are slow to discover that finger counting may be used for calculation, and persist in finger counting for longer. Associated symptoms include difficulties in telling time and spatial/geographic orientation.

Defining dyscalculia for diagnostic and research purposes is complex. There are many labels in use such as math learning disability/disorder/difficulty, arithmetic disability, dyscalculia, and developmental dyscalculia. There are no theoretically or functionally grounded diagnostic criteria. Hale et al. (2010) reviewed the pros and cons for distinct diagnostic approaches (see also Wong et al., 2014). Three main diagnostic approaches are usually considered:

(a) Discrepancy criterion. The performance of the child on standardized arithmetic tests is low, when compared with his or her own performance in IQ or reading/spelling achievement tests. This approach is open to criticism because IQ and achievement are highly correlated, introducing statistical distortions and predisposing it to false-negative results. Studies have also failed to find any cognitive or intervention relevant distinctions between samples defined by the discrepancy or by the absolute threshold criteria (Ehlert, Schroeders, & Fritz, 2012).
(b) Absolute threshold. A cutoff score is set, and the individual is defined as having dyscalculia if his or her standardized achievement test score falls below this threshold. Cutoff scores used in the literature vary widely, so Mazzocco (2007) proposed to set a cutoff at the 25th percentile for math difficulties and a cutoff at the fifth percentile for developmental dyscalculia or math learning disability, respectively. The main advantage of the absolute threshold method is the easy operationalization. The main problem is related to the lack of characterization of the impairment as specific and restricted to one cognitive domain, since individuals with varying IQ levels will be included in the sample.
(c) Response to intervention (RTI). This method consists of applying the diagnosis of dyscalculia only after the response to a series of pedagogical interventions has been analyzed. The RTI approach is sound, in the sense that it restricts the diagnosis to just those individuals with severe and persistent impairments. RTI

> is difficult to implement and time- and resource-consuming and makes diagnosis dependent on the quality of and the adherence to the interventions implemented. RTI may also delay the access of children with dyscalculia, associated with additional medical conditions and/or genetic syndromes, to proper health care.

In summary, the longitudinal approach of RTI has the main advantage of selecting cases with persistent and severe impairments. Cross-sectional approaches, such as discrepancy and absolute criteria, are required to recruit large enough samples for research purposes. In this last case, motivational factors may also act as confounders. Difficulties in dealing with arithmetic are frequently attributable to motivational and environmental factors, such as socioeconomic status (SES) and quality of education. Children presenting a stronger motivational component may catch up after proper stimulation (Wong et al., 2014). Currently, most studies use one or the other of the cross-sectional definitions.

As math learning does not occur in a vacuum, social and linguistic contexts must also be considered (Gamboa & Waltenberg, 2012; Oliveira-Ferreira, Costa, Micheli, Pinheiro-Chagas, & Haase, 2012). Low SES is a risk factor for developmental dyscalculia (Gross-Tsur, Manor, & Shalev, 1996). The role of psychosocial specificities in the development of math abilities is a subject of increasing scientific interest. This may potentially explain both SES and cross-national differences in math achievement. As an example, 75% of Brazilian youngsters perform below level II in mathematics in the PISA examination (Brazil, Ministry of Education, 2016). This figure is 8% in Finland. Correspondingly, heritability estimates vary according to the sample sociodemographic composition, being higher in populations with fewer social disparities and lower in unequal societies (Bishop, 2015; Turkheimer, Haley, Waldron, d'Onofrio, & Gottesman, 2003).

The prevalence of developmental dyscalculia has been estimated to range from 3.4% (Reigosa-Crespo et al., 2012) to 6.2% (Gross-Tsur et al., 1996). Obviously, these figures change according to the cutoff used to define dyscalculia in different studies, and prevalence estimates as high as 10% have been described (reviewed by Devine, Soltész, Nobes, Goswami, & Szűcs, 2013). A preponderance of females has been reported in some studies (Reigosa-Crespo et al., 2012). Gender differences could be related to diagnostic criteria. Prevalence is higher in females when a reading-math discrepancy criterion is used (Devine et al., 2013). This may reflect the higher linguistic abilities in females.

Dyscalculia is frequently comorbid with dyslexia, attention deficit hyperactivity disorder (ADHD), autism spectrum disorder, and language impairment (Landerl, Göbel, & Moll, 2013; Shalev, Auerbach, & Gross-Tsur, 1995; Stefansson et al., 2014). The comorbidity among these conditions is much higher than it would be expected by chance, suggesting a common underlying impairment. As an example, some 60% of children with dyscalculia present with dyslexia and/or dysorthography, and some 30% of children with dyslexia and/or dysorthography present with dyscalculia (Landerl & Moll, 2010). The prevalence of ADHD in a sample of individuals with dyscalculia was estimated to be 26% (Gross-Tsur et al., 1996) and of dyscalculia in a sample of individuals with ADHD to be 18% (Capano, Minden, Chen, Schachar, & Ickowicz, 2008). Twin studies indicate considerable genetic correlation between these conditions (Hart et al., 2010). As a consequence, candidate genes/chromosomal regions and biochemical pathways, identified for several neurodevelopmental disorders, are also candidates for dyscalculia, as discussed below.

The cognitive foundations of number processing and arithmetic are also extremely complex (Hohol, Cipora, Willmes, & Nuerk, 2017). Basic arithmetic represents the most fundamental aspect of mathematics, and its acquisition requires the successful interplay between several neurocognitive systems associated with general intelligence; working memory; verbal abilities, including phonological processing; visuospatial and visuoconstructional abilities; basic numerical concepts and processing abilities; as well as motivational and emotional self-regulation. Wilson and Dehaene (2007) proposed a theoretical model of dyscalculia subtypes, comprehending (a) basic numerical processing impairments, (b) phonological processing impairments, (c) visuospatial/visuoconstructional impairment, and d) working memory/executive function impairments. As discussed in the companion chapter (Haase & Carvalho, 2018, this volume), these subtypes are useful to identify cognitive endophenotypes explaining comorbidities and characterizing math learning difficulties in genetic syndromes.

Considering the complexity of the underlying mechanisms, research or diagnostic protocols for dyscalculia usually include (a) SES measures; (b) standardized math and reading/spelling achievement tests; (c) general cognitive measures such as IQ, attention/executive functions, and working memory; (d) handedness and finger gnosia; (e) visuospatial and visuoconstructional abilities; (f) reading-related abilities (regular and irregular word and nonword reading, word reading fluency, phonemic awareness, etc.); and (g) specific math ability tasks (Arabic number dictation, number line, nonsymbolic and symbolic numerical magnitude comparison, estimation, single digit addition, subtraction and multiplication, word problem solving, math anxiety and beliefs, etc.). Not surprisingly, large, well-characterized samples are difficult to obtain. Not considering the cognitive heterogeneity of developmental dyscalculia reduces the power of molecular genetic studies.

Genetic Susceptibility to Dyscalculia

Usually, first-line evidence for a genetic component comes from reports of pedigree analysis, which may suggest Mendelian segregation. To date, no large Mendelian pedigree of a pure form of dyscalculia has been described. For many common conditions, such Mendelian pedigrees are rare, and the evidence for a genetic component comes from familial aggregation (but without a typical Mendelian segregation pattern) or heritability studies. More recently, molecular genetic evidence has become increasingly useful.

Familial Aggregation in Dyscalculia

Moderate to strong familial aggregation has been described for developmental dyscalculia (Shalev et al., 2001). In a sample composed of 39 affected children, they found that 66% of the mothers, 40% of the fathers, 53% of the siblings, and 44% of second-degree relatives had developmental dyscalculia. The frequency among sibs of affected children was ten times greater than the population frequency. No effects of IQ and attention were detected. Familial aggregation was also described in a sample of siblings of dyscalculia participants, in a study using number sense tasks as phenotypes (Desoete, Praet, Titeca, & Ceulemans, 2013).

Landerl and Moll (2010) investigated the frequency of comorbidity among arithmetic, reading, and spelling disorders and reported results for the arithmetic disorder in first-degree relatives of children with each one of these phenotypes. Recurrence rates in the families of children with pure dyscalculia were 30% for dyscalculia *plus* dyslexia, 22% for pure dyscalculia, and 15% for pure dyslexia. These results suggest that dyslexia and dyscalculia share some common genetic components but also have some specific ones.

Heritability of Dyscalculia

Another important approach to characterizing the biological underpinnings of numerical processing and arithmetic abilities is seen in behavioral genetic studies. Genetic research suggests considerable genetic continuity between typical and atypical development of numerical and arithmetic abilities (Asbury & Plomin, 2013).

Haworth, Kovas, Petrill, and Plomin (2007), studying a sample composed of 2178 twin pairs, described heritability for math achievement in the normal and low achievement ranges. In the low achievement range (198 MZ and 198 DZ same-sex pairs), the heritability values obtained were *using and applying* (0.70); *numbers and algebra* (0.69); shapes, spaces, and measures (0.74); and composite (0.75). High heritability values were also obtained for achievement in the normal range.

Pinel and Dehaene (2013) used fMRI to estimate heritability for brain activation while performing calculation tasks in a sample composed of 19 MZ and 13 DZ twin pairs. Heritability values in the 0.52–0.66 range were obtained for activation of the brain areas involved in calculation tasks. There are also some studies on math *a*bility showing similar results (Krapohl et al., 2014). These results suggest that math ability and some of its components are influenced by moderate to strong genetic factors.

However, methods for assessing the genetic susceptibility to dyscalculia do not provide all the needed answers. Familial aggregation studies do not distinguish between genetic and environmental influences. Heritability estimates help to partition variance among distinct environmental and genetic components but apply only to the population and not the individual level. Additionally, heritability alone does not help in finding genes, and specific research is necessary to bring the field forward. Current genetic-molecular approaches offer straightforward ways to address the genetic underpinnings of dyscalculia.

Gene-Finding Strategies

Currently, there is a routine for investigating the genetic basis of a complex condition, which includes (a) collecting evidence for the existence of a genetic component (familial aggregation, heritability, co-occurrence with genetic syndromes); (b) searching for candidate genes in a genome-wide approach, e.g., genome-wide association studies (GWAS), genome-wide linkage analysis (GWLA), genome-wide microdeletion and/or microduplication screening, and exome or whole genome sequencing; and (c) searching for confirmatory evidence, using additional approaches such as replication of association studies, mutation screening in affected persons, or functional studies (RNA and protein expression patterns, animal or cellular models). Over the last three decades, the genetic bases of a large number of complex phenotypes have been characterized, pointing to new genes and pathways, which would not have been discovered without such straightforward strategies. However, the success obtained with this routine for investigating depends on the phenotype complexity, as we will see below. In the remaining of this section, we will focus on two additional strategies for finding genes, genome-wide association studies and candidate genes associated with phenotypes that co-occur with dyscalculia.

Genome-Wide Association Studies

Genome-wide association studies (GWAS) are extremely powerful tools for detecting genetic components of a phenotype. The assumption is that, if there are genetic predisposing alleles, they are located somewhere in the genome. There are many possible approaches to GWAS. In all of them, single nucleotide polymorphisms (SNPs) are genotyped throughout the genome. For example, allele frequencies obtained for each SNP in a sample of affected persons are compared to the allele frequencies in a control group. Alleles presenting different frequencies between the groups are considered associated with the phenotype. Considering the large number of SNPs tested, correction for multiple testing is necessary. Nevertheless, false-positive results are frequent and independent replication is mandatory. A very small number of GWAS for math ability/disability have already been published and will be considered in some detail here.

Docherty et al. (2010) developed a three-phase study using two samples extracted from a twin study. Children were rated using an online test, evaluating “understanding number,” “computation and knowledge,” and “nonnumerical processes” or teacher rating, evaluating "using and applying mathematics," "numbers and algebra," and "shapes, space, and measures." Three different scores were constructed, one for the online test, one for the teacher ratings, and one combining both. In the first phase, two groups of 300 individuals (10-year-old children), representing the low and the high ends of the math ability distribution with a cutoff at the 16th percentile, were genotyped. In the second phase, two groups of 300 individuals (10-year-old children), representing the low and the high ends of the math ability distribution with a cutoff at the 20th percentile, were genotyped. In the genotyping, a 500 k SNP array was used in pools of 30 individuals of both sexes. In the third phase, 46 SNPs, which reached significance in the first and second phases, were genotyped individually in a sample of over 2000 children, representing the complete range of math ability. After the third phase, significant (Bonferroni corrected) associations were detected for three genes: *MMP7*, *GRIK1*, and *DNAH5*.

MMP7 encodes a protein that degrades components of the extracellular matrix. It has been associated with cancer, including brain tumors, among other conditions. In mice neurons, high levels of MMP7 protein inhibit NMDA-stimulated calcium flux, affecting synapsis function (Szklarczyk et al., 2008). Also in mice, *MMP7* mutations reduce the ability of cochlear cells to deal with acoustic trauma (Hu et al., 2012). No evidence of a role in learning has been published. *GRIK1* encodes the neuronal glutamate receptor subunit GluR-5 (or kainate receptor subunit 1) and has been implicated in the releasing of both glutamate and gamma-aminobutyric acid and has been associated with psychiatric conditions, including schizophrenia (review by Choi, Zepp, Higgs, Weickert, & Webster, 2009) and early-onset obesity (Serra-Juhé et al., 2017). This gene is expressed in the prefrontal cortex in the first 5 years of life (Choi et al., 2009). *DNAH5* encodes the dynein axonemal heavy chain 5 protein. Mutations in *DNAH5* cause primary ciliary dyskinesia. *DYX1C1 and DCDC2*, two genes associated with dyslexia, also cause primary ciliary dyskinesia. These genes are involved in the function of the primary cilium. During the embryonic development, specific groups of cells act as organizer centers (or nodes) regulating the fate and differentiation of the neighbor cells. These cells secrete signaling molecules which are set in movement by the primary cilia. The primary cilium is responsible for axon and cell migration (Kere, 2014; Tammimies et al., 2016; Tarkar et al., 2013). Another gene involved in the function of the primary cilium, *PCSK6*, is implicated in hemispheric lateralization and dyslexia (Paracchini, Diaz, & Stein, 2016). No other study has implicated *MMP7*, *GRIK1*, or *DNAH5* in dyscalculia.

Ludwig et al. (2013) reported a GWAS for math difficulties. They used five samples of approximately 200 children each, with two samples of dyslexics and one of controls from Germany and one sample of dyslexics and one of controls from Austria. Evaluated phenotypes included number judgment and mathematical calculation and the depth of the intraparietal sulcus (IPS), assessed using structural MRI. One significant p-value ($p = 8.8 \times 10^{-10}$) was replicated among the samples. Under the dominant model, rs133885 yielded an estimated effect size of approximately 5%. The gene is *MYO18B*. rs133885 mediates an amino acid substitution in the MYO18B protein. Carriers of the GG genotype presented lower performance in the number judgment task and displayed a significantly lower depth and volume of the right IPS. Mutations in this gene have been associated with nemaline myopathy (Malfatti et al., 2015) and malignancies (Nishioka et al., 2002).

In a subsequent study, no association of rs133885 and dyscalculia or math ability was detected (Pettigrew et al., 2015). Pettigrew and coworkers' study included large samples from five cohorts, some composed of children with dyscalculia, one composed of children with specific language impairment, and some composed of unaffected children.

Baron-Cohen et al. (2014) developed a GWAS using a sample composed of high school and university students. There were 419 students who obtained grades of A or A* in the British General Certificate Standard Examination (GCSE) for mathematics and 183 students who obtained grades of C or lower. Genotyping was carried out using a 900 k SNP platform and same-sex pools of 30 individuals, composing high and low performance groups. No SNP reached genome-wide significance. The authors selected 15 SNPs, which reached $p < 1.5 \times 10^{-5}$, to be individually genotyped in a confirmatory sample composed of 375 high and 167 low math groups. Only one SNP (rs789859) was significantly associated with math ability. This SNP maps within the promoter of *FAM43A* (family with sequence similarity 43 member A) and also close to *LSG1* (large 60S subunit nuclear export GTPase 1). These genes map to 3q29. Microdeletions and microduplications in this region have been associated with autism, schizophrenia, and intellectual disability (Nava et al., 2014; Sagar et al., 2013; Willatt et al., 2005). In *S. cerevisiae*, *LSG1* is a GTPase required for the export of the 60S ribosomal subunit from the nucleus to the cytoplasm. *FAM43A* encodes for a protein of unknown function.

Chen et al. (2017) described a GWAS developed using three independent cohorts, with 494 and 504 individuals in the discovery and 599 individuals in the confirmatory phases. Participants were Chinese children (7–13 years old), and the phenotypes were the grades on midterm and final exams. Children with IQ below the 25th percentile were excluded. Genotyping was carried out using a 1.2 million SNP platform and samples were genotyped individually. SNPs with p-values $<1.0 \times 10^{-5}$ were selected for confirmation. A meta-analysis was conducted combining significant results from the three cohorts. After the meta-analysis, five SNPs in the gene *SPOCK1* reached $p < 10^{-9}$. *SPOCK1* maps to 5q31.2 and encodes testican-1, a protein associated with tumor progression and prognosis, as well as neurogenesis. SPOCK1 protein is associated with epithelial-mesenchymal transition, a change in cell behavior needed for migration in both metastasis and neurodevelopment. Using exome, a mutation in *SPOCK1* was detected in a female patient presenting intellectual disability, dyspraxia, dysarthria, partial agenesis of the corpus callosum, prenatal-onset microcephaly, and atrial septal defect with an aberrant subclavian artery. These findings support the hypothesis that subtle variations in *SPOCK1* may underlie a condition such as dyscalculia.

In conclusion, there are few studies and no SNP-candidate gene association has been replicated. Most of these studies do not have ideal designs, considering diagnostic criteria, sample size, and sample homogeneity. DNA pooling was used to reduce costs. However, not knowing the per-person SNP frequencies prevents the use of the most powerful strategies available for GWAS, e.g., combining neighboring

SNPs into haplotypes, which work as barcodes for each region. This approach was used successfully by Chen et al. (2017).

Independently of technical issues, some phenotypes are harder to investigate, and dyscalculia/math learning ability may be among them. Dyscalculia is etiologically heterogeneous, including, in some cases, strong environmental effects. Therefore, samples also tend to include subjects with low genetic susceptibility. In addition, precise diagnosis of developmental dyscalculia depends on extensive neuropsychological examination. Large, well-characterized population samples, enriched for subjects with strong genetic susceptibility, as needed for GWAS, are hard to obtain. Although difficult, the search for the genetic components of dyscalculia is fundamental to understanding the underlying cellular/molecular processes.

Candidate Genes from Comorbidities

Dyscalculia has been a difficult phenotype to deconstruct, because it includes large numbers of components or endophenotypes. In addition, developmental dyscalculia presents frequently in comorbidity with common conditions such as dyslexia, specific language impairment, ADHD, autism, intellectual disability (ID), schizophrenia, and others.

One important question raised by Ashkenazi, Black, Abrams, Hoeft, and Menon (2013) deals with the partial overlap of the neurobiological underpinnings of developmental dyscalculia and dyslexia. Extensive research on dyslexia has implicated a circuit that includes the left cortical areas in the infero-lateral occipitotemporal transition, the temporoparietal junction, and the frontal operculum. These areas largely overlap those recruited by symbolic numerical processing and verbal calculation operations. Considering the partially overlapping neural substrate and the high frequency of co-occurrence of dyscalculia and dyslexia, the number of studies specifically investigating the genetics of the mechanisms underlying this association is low.

Some studies have been conducted investigating the contribution of copy number variations (CNVs) to learning disabilities. CNVs include microdeletions and microduplications which may occur throughout the genome. They are common causes of intellectual disability, autism, and schizophrenia. The 15q11.2 (BP1–BP2) microdeletion confers a four times higher risk for the dyslexia and dyscalculia phenotype. This deletion is also associated with smaller volumes of both gray and white matter structures, as well as reduced activation of brain regions important for reading and arithmetic, observed using MRI (Stefansson et al., 2014; Ulfarsson et al., 2017).

Some of the genes associated with dyslexia have been investigated for pleiotropic effects on math learning. For example, association between *DCDC2* and "numerical facts" was detected in a sample of 85 informative families, with a genetic effect = 0.57%. Association between *DYX1C1* and "mental calculation" was detected in a sample of 40 informative families, with a genetic effect of 0.65% (Marino et al., 2011). As referred above, *DCDC2* and *DYX1C1* are involved in the functioning of the primary cilium. The cilium is involved in many developmental functions, including the establishment of the left-right body axis and axon and neuronal migration (reviewed by Kere, 2014, Brandler & Paracchini, 2014, Trulioff, Ermakov, & Malashichev, 2017). During development, groups of cells secrete signaling molecules that diffuse to regulate differentiation of other cells in the neighborhood. These molecules are set in motion by the beating of the cell cilia. Left-right asymmetries are also fundamental to the specialization of the brain areas required for number processing and calculation. Cilia defects have been described in dyslexia, among many other conditions. Not surprisingly, “numerical facts” and “mental calculation,” the phenotypes associated with these genes, are situated in the interface between number and word processing.

Perspectives

Comparing with other topics related to education (e.g., intellectual disability, autism, dyslexia, and ADHD), the progress in understanding dyscalculia has been slow. Patients with pure forms of dyscalculia are difficult to find. In addition, families, in which the phenotype segregates, which have been useful in mapping dyslexia genes, seem to be less common in dyscalculia.

However, the main point concerns to the phenotype. As discussed in the companion chapter (Haase & Carvalho, 2018, this volume), dyscalculia has subtypes and endophenotypes. In addition, there are non-genetic forms of dyscalculia. Heterogeneity is frequent in most phenotypes. Here, a very useful concept is trait architecture. Different phenotypes will have different trait architectures. For example, one trait may be produced by one or a small number of major effect genes. Others may be produced by a large number of minor genes, each one with a small effect. Heterogeneity also implies that different individuals have the same phenotype with different trait architectures, e.g., in one child dyscalculia is caused by a microdeletion, in another by a single gene mutation, and in another by no detectable gene.

Therefore, we need to move our attention to more specific components or endophenotypes of dyscalculia, intermediate between the etiologic and the phenotypic level. An ideal endophenotype should be specific (not common to other conditions and in unaffected persons), early in manifestation, relatively independent of formal education, easy to score, and as close as possible to the biological substrate of the phenotype in investigation. The cognitive impairments underlying the subtypes of dyscalculia proposed by Wilson and Dehaene (2007) are serious candidate endophenotypes. Examining their role in the math difficulties of genetic syndromes could be an efficient approach.

Dissecting connections between genes, endophenotypes, and phenotypes will help to reduce the analytic complexity; but it will certainly not solve the problem for one and for all. This situation is analogous to the discovery of the DNA structure or the genome projects. Great expectations were associated with these scientific advances. To date, we still have not solved all the mysteries related to the definition and origins of life and its diversity. However, we have learned so much that life is no longer considered a mysterious process but, rather, a subject perfectly amenable to scientific investigation. In the end, endophenotypes may prove not to be the key to dissecting all existing connections between genotype and phenotype. However, throughout this process, we will continue to learn a lot. We are already learning. Advances in genetics allow us to diagnose and discover an increasing number of conditions associated with math learning difficulties, to discover and characterize specific patterns of impairment, and to plan more efficient educational interventions.

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Chapter 22
Genetics of Dyscalculia 2: In Search of Endophenotypes

Maria Raquel S. Carvalho and Vitor Geraldi Haase

Introduction

Molecular genetic approaches developed over the last two decades will help in the identification of genes underlying developmental dyscalculia. So far, progress has been limited by the heterogeneity of the phenotype and patterns of comorbidity (Carvalho & Haase, 2018, this volume). In this chapter, we review the co-occurrence of math learning difficulties in genetic syndromes as an approach to characterize specific endophenotypes involved in problems with math. Cognitive endophenotypic analysis is discussed as a gene-finding strategy. First, we discuss the concept of endophenotype and its relevance to the phenotypic characterization of developmental dyscalculia in genetic studies. Second, we discuss five syndromes caused by different genetic mechanisms as potential models of the endophenotypes implicated in dyscalculia. Finally, we discuss educational implications, potentially relevant to the classroom.

Cognitive Endophenotypes of Dyscalculia

One way to reduce the complexity, in order to disentangle genotypic-phenotypic correlations underlying math ability and disability, is to use the concept of endophenotypes (Bishop & Rutter, 2009). As implicit in the name, endophenotypes are

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intermediate phenotypes arising in the complex epigenetic pathways between the interaction of genetic and environmental experiences and the phenotypic expression. Endophenotypes can be characterized at multiple levels through cognitive, neural, and molecular analyses. According to this view, the phenotypic expression could be understood as resulting from a network of interacting endophenotypes. Dissecting the molecular genetic bases of the endophenotypes could facilitate the discovery of the genotypic-phenotypic links. In this vein, genetic research on math abilities and disabilities would aim at characterizing the relevant endophenotypes.

Research on the endophenotypes of math ability is relevant to the understanding of the subtypes and the most frequent patterns of comorbidity of developmental dyscalculia. In an influential theoretical work, Wilson and Dehaene (2007) identified four cognitive endophenotypes corresponding to four subtypes of developmental dyscalculia: basic number processing, phonological processing, visuospatial processing, and working memory/executive function processing. Next, we discuss the four cognitive endophenotypes potentially underlying the heterogeneity of manifestations of dyscalculia.

Basic Number Processing

Research on numerical cognition goes under the assumption that basic numerical processing is an important precursor to arithmetic learning (Siegler & Braithwaite, 2017). Numerical processing refers to the ability to quantify the numerosity of sets and to transcode between different numerical notations. Numbers are represented using two main notations: nonsymbolic and symbolic. Accurate nonsymbolic numerical representations up to four are implemented by visual attentional processes in a parallel individuation, object tracking, or object file system (OFS, Hyde & Spelke, 2011). The ability to accurately quantify sets up to four elements is called subitizing and seems to depend on visual attentional rather than numerical processes. Numerosities larger than four are represented approximately in analogical format by means of a spatially oriented mental number line implemented in an approximate number system (ANS, Dehaene, 1992, Dehaene & Cohen, 1995). The symbolic numerical notations comprise phonological and orthographic verbal notations and the visual Arabic system. Accurate quantification is possible by means of the OFS up to numerosities of four and by means of verbal counting with larger sets. The ANS allows for approximate estimation of larger numerosities.

ANS has attracted much attention as a precursor to arithmetic because it represents a core conceptual system, effective from infancy on, and shared with other animal species. ANS obeys the basic psychophysical laws of Weber and Fechner, explaining several effects in numerical processing (Dehaene, Dupoux, & Mehler, 1990). The distance and ratio effects correspond to the Weber law: it is increasingly more difficult to discriminate numerosities as the numerical difference between them decreases. This corresponds to a scalar or ratio distribution with a just noticeable difference corresponding to a constant, the Weber fraction. The size effect corresponds to the Fechner law: it is increasingly more difficult to discriminate between larger than between smaller numerosities. This is explained by the logarithmic compression characteristic of the function that fits between the stimuli numerosities and their mental representations. The distance, ratio, and size effects suggest an analogic representation of numbers on a spatially oriented mental number line. In addition, it is easier to react to small digits with the left hand and to larger digits with the right hand, corresponding to the spatial-numerical association of response codes or SNARC effect (Dehaene, Bossini, & Giraux, 1993).

The role of nonsymbolic over symbolic numerical processing as a precursor of arithmetic ability and as a marker of disability is controversial. According to one perspective, the ANS (Piazza, 2010) or a number module (Butterworth, 2010) is a powerful source of influence. Another influential perspective reinforces the role of linking rote verbal counting skills with numerosities in the subitizing range as a decisive factor in the development of the number concept (Le Corre & Carey, 2007). Impairments of number processing in the subitizing range have been observed in some studies (Bruandet, Molko, Cohen, & Dehaene, 2004; Koontz & Berch, 1996; Landerl, Bevan, & Butterworth, 2004). Other evidence points to an association between ANS accuracy and both typical (Halberda, Mazzocco, & Feigenson, 2008) and atypical math achievement (Mazzocco, Feigenson, & Halberda, 2011; Piazza et al., 2010; Pinheiro-Chagas et al., 2014). This association between math achievement and ANS accuracy has not been confirmed by other studies (De Smedt & Gilmore, 2011; Rousselle & Noël, 2007). Other evidence indicates that symbolic over nonsymbolic number processing is the crucial predictor of math achievement (Geary, Bailey, & Hoard, 2009; Nosworthy, Bugden, Archibald, Evans, & Ansari, 2013; Vanbinst, Ceulemans, Peters, Ghesquière, & De Smedt, 2017). Extant meta-analyses indicate that both nonsymbolic and symbolic number processing accuracy are weakly correlated with math achievement (Chen & Li, 2014; Fazio, Bailey, Thompson, & Siegler, 2014; Schneider et al., 2017). Correlations are slightly stronger for symbolic processing.

An interesting pattern of dissociation was observed by Rousselle and Noël (2007). Children with dyscalculia presented impairments in symbolic but not in nonsymbolic number processing (see also De Smedt and Gilmore 2011). This led to the formulation of the access hypothesis of dyscalculia (De Smedt & Gilmore, 2011; Noël & Rousselle, 2011; Rousselle & Noël, 2007). According to this hypothesis, number processing impairments in dyscalculia could be ascribed either to a representational inaccuracy in the ANS or to difficulty in automatizing connections and accessing nonsymbolic quantitative representations from symbolic numerals.

Assessment of ANS accuracy is plagued by several methodological constraints, such as covariation between nonsymbolic discrete and continuous magnitudes (Leibovich, Katzin, Harel, & Henik, 2017). Studies differ widely in their measures, designs, and samples. So, it is difficult to draw definite conclusions. However, additional evidence indicates that it is discrete rather than continuous nonsymbolic magnitude representations that associate with math achievement (Anobile, Castaldi, Turi, Tinelli, & Burr, 2016). It is also noteworthy that single case studies have been published in which an ANS impairment was the most probable determinant of math learning difficulties (Davidse, de Jong, Shaul, & Bus, 2014; Haase et al., 2014; Júlio-Costa, Starling-Alves, Lopes-Silva, Wood, & Haase, 2015; Ta'ir, Brezner, & Ariel, 1997).

Phonological Processing

Several arithmetic abilities are heavily dependent on verbal processing, such as number reading and writing, arithmetic facts learning, and word problem solving. In the case of Arabic number dictation, the verbal numerals must be temporarily stored and decoded in the phonological format (Barrouillet, Camos, Perruchet, & Seron, 2004). It has been proposed that phonological processing impairments could be an underlying mechanism explaining math learning difficulties in individuals with dyslexia (Simmons & Singleton, 2008). Evidence suggests that phonological processing – encompassing rapid automatized naming, phonological short-term memory, and phonemic awareness – could be a critical cognitive mechanism shared by the reading and writing of words and numbers (Lopes-Silva et al., 2016; Lopes-Silva, Moura, Júlio-Costa, Haase, & Wood, 2014). Phonological processing abilities are thus important candidates for an endophenotype explaining both math learning and the connection between arithmetic and reading/spelling disorders. Research should, ideally, elucidate the molecular genetic basis of each component of phonological processing.

Visuospatial and Visuoconstructional Abilities

Visuospatial and visuoconstructional abilities are crucial for several arithmetic tasks. Multidigit calculation, for example, depends on the ability to spatially organize the execution of the algorithm, maintaining the alignment of the columns and rows (Raghubar et al., 2009). Spatial working memory also plays an important role in the carrying and borrowing operations between columns (Mammarella, Caviola, Giofrè, & Szűcs, 2017; Passolunghi & Mammarella, 2012). Spatial forms of acalculia in adults are well established (Benavides-Varela et al., 2017; Granà, Hofer, & Semenza, 2006; Hartje, 1987). Characterization of a visuospatial form of dyscalculia remains elusive (Barnes & Raghubar, 2014; Wilson & Dehaene, 2007). It is noteworthy that a condition referred to as nonverbal learning disability (NLD) presents both arithmetic and visuospatial impairments (Cornoldi, Mammarella, & Fine, 2016). Some genetic syndromes such as Turner syndrome (TS), velocardiofacial syndrome (VCFS), and Williams syndrome (WS) are considered causes of NLD (Cornoldi et al., 2016).

Working Memory

Impairments in working memory, most notably in the central executive component, are an important cognitive marker of developmental dyscalculia (Bull & Lee, 2014; Raghubar, Barnes, & Hecht, 2010). Every new acquisition in arithmetic learning imposes heavy demands on the storing and controlled processing of information in working memory, such as verbal counting (Camos, Barrouillet, & Fayol, 2001), single-digit operations and facts learning (De Visscher & Noël, 2014), transcoding (Camos, 2008; Moura et al., 2013), multidigit calculation (Trbovich & LeFevre, 2003), and word problem solving (Andersson, 2007; Costa et al., 2011).

The construct of working memory is extremely complex. The term working memory refers to the limited capacity to temporarily store information in order to consciously process it or to control behavior. The influential multistore model assumes the existence of several distinct mechanisms, related, respectively, to short-term store of phonological, visual, and visuospatial information and with the control or executive processes (Baddeley, 2000; Baddeley & Hitch, 1974). The newer version of the model assumes an episodic buffer related to binding of information represented in multiple formats and interactions with long-term memory. The episodic buffer has not been researched in the context of numerical cognition. Mechanisms implemented by the central executive are related to dual-task processing, information updating, concurrent inadequate response inhibiting, and set-shifting (Miyake, Friedman, Emerson, Witzki, & Howerter, 2000).

The multistore model is complex, and virtually every single syndromic and non-syndromic neurodevelopmental disorder presents with working memory deficits (Johnson, 2012). These are major obstacles for considering working memory as a possible endophenotype of developmental dyscalculia. Johnson (2012) suggested that a learning disability results when an individual has a specific deficit in one cognitive domain and general processing resources in working memory are insufficient to compensate for the difficulties. In this vein, dyscalculia would be explained by a deficit in some specific endophenotype such as numerical, phonological, or visuospatial processing that cannot be compensated by general cognitive resources. Accordingly, executive impairments in working memory are conceived as an endophenotype shared by distinct neurodevelopmental disorders.

We now turn our attention to evidence of a genetic basis for dyscalculia obtained from the analysis of some genetic conditions, such as chromosomal abnormalities, genomic disorders, and single-gene diseases.

Chromosomal Abnormalities

Two chromosomal aneuploidies will be discussed, the Turner and Klinefelter syndromes.

Dyscalculia in Turner Syndrome

Turner syndrome is a chromosomal disorder that affects females and refers to the set of signs and symptoms caused by the complete or partial deletion of the second sexual chromosome. Chromosomal mosaicisms are also frequent and may include lineages with a Y-chromosome, e.g., in a 45,X/46,XY karyotype. Additional mechanisms are X-autosomal translocations, ring X-chromosome, X-isochromosome, and a wide range of microdeletions.

Fine mapping of X-chromosomal microdeletions has implicated the pseudoautosomal region on X (PAR1; Xp22.3) as the candidate region for the Turner syndrome cognitive phenotype. The pseudoautosomal regions are good theoretical candidates because the genes in these regions have homologues on the Y-chromosome. This suggests that two copies of these genes are necessary. The PAR1 region includes at least 32 genes, but clear evidence of a specific association for one of them and for the Turner syndrome cognitive phenotype has not yet been published (Ross, Roeltgen, Kushner, Wei, & Zinn, 2000; Zinn et al., 2007). Considering the complexity of the X-chromosome and autosomal interactions, the phenotypes described here may reflect a direct effect of genes on the X-chromosome or the deregulation of genes elsewhere.

As a consequence of the diversity of karyotypes, clinical manifestations vary among affected individuals. The most frequent clinical manifestations are heart congenital malformation and/or aortic dilatation, short stature, hearing impairment, and infertility due to premature ovarian insufficiency. Quality of life can be effectively improved by proper care (Gravholt et al., 2017). The frequency of Turner syndrome is estimated to be from 1:2000 to 1:4000 females. Several studies have described the cognitive profile of children with Turner syndrome. Intelligence is usually in the normal range (Mazzocco, 2007). They may present attention deficit hyperactivity disorder (ADHD), specific learning disorders, social communication disorder, autism spectrum disorder, and developmental coordination disorder.

Except for the absence of phonological processing deficits (Temple & Shephard, 2012), the Turner syndrome cognitive phenotype encompasses all the other postulated endophenotypes of developmental dyscalculia. Females with Turner syndrome present a 50% prevalence of dyscalculia (Murphy, Mazzocco, Gerner, & Henry, 2006). Impairments in executive (Kirk, Mazzocco, & Kover, 2005) and visuospatial functions (Mazzocco, Singh Bhatia, & Lesniak-Karpiak, 2006; Temple & Carney, 1995) are also common. A discrepancy between high verbal and low performance IQ is characteristic (Mazzocco, 2007). Dyscalculia in Turner syndrome is related to deficits in speeded processing and calculation (Baker & Reiss, 2016), subitizing in some (Bruandet et al., 2004) but not all studies (Simon et al., 2008), and symbolic number processing, also in some (Brankaer, Ghesquière, De Wel, Swillen, & De Smedt, 2016; Simon et al., 2008) but not all studies (Baker & Reiss, 2016; Zougkou & Temple, 2016). Results are also discrepant regarding the association (Brankaer et al., 2016) or dissociation (Mazzocco et al., 2006) between number and visuospatial processing impairments. In summary, there is evidence for impairment of three dyscalculia endophenotypes in Turner syndrome. They are executive function, number processing, and visuospatial processing. It is not known whether and how these endophenotypes are related in Turner syndrome.

Dyscalculia in Klinefelter Syndrome

Klinefelter syndrome is a genetic condition having, in most cases, a 47,XXY karyotype; more complex (48,XXXY, 48,XXYY), isochromosome (47,iXq,Y) and mosaicism (e.g., 47,XXY/46,XY) are seen in 10–20% of cases (Bonomi et al., 2017). The frequency of Klinefelter syndrome ranges from 0.1% to 0.2% of newborn males but is higher in specific groups, such as azoospermic men. The phenotype is highly variable and most cases remain undiagnosed. From puberty on, symptoms become more prominent and reflect direct effects of an extra X-chromosome and/or low testosterone levels including small testis, eunuchoid skeleton, gynecomastia, sparse body hair with female distribution, impaired sexual desire, impaired erectile function, language and speech disabilities, and low but normal IQ.

Klinefelter syndrome is proposed as a model of the phonological endophenotype underlying math difficulties associated with dyslexia, although evidence is not always compelling. Executive dysfunction is characteristic of Klinefelter syndrome (van Rijn & Swaab, 2015), but this kind of impairment is shared with almost all other forms of neurodevelopmental disorders. Visuospatial abilities are spared in Klinefelter syndrome (Bender, Harmon, Linden, Bucher-Bartelson, & Robinson, 1999; Ross et al., 2008). The most salient cognitive feature is the impairment in verbal abilities. Earlier reports called attention to the importance of Klinefelter syndrome as a cause of developmental dyslexia (Bender, Puck, Salbenblatt, & Robinson, 1986; Pennington, Bender, Puck, Salbenblatt, & Robinson, 1982). More recent investigation has confirmed the impairment not only of reading but also of arithmetic achievement (Ross, Zeger, Kushner, Zinn, & Roeltgen, 2009; Rovet, Netley, Keenan, Bailey, & Stewart, 1996). The cognitive profile of math impairment in Klinefelter syndrome remains unknown, as reports have focused on standardized and not on theoretically grounded tasks. In one study, phonemic awareness was not impaired (Bender, Linden, & Harmon, 2001). Klinefelter syndrome is characterized by low achievement in reading and math. It remains to be investigated if there is an underlying mechanism explaining the co-occurrence of these impairments in Klinefelter syndrome.

Genomic Disorders

Copy number variations (CNVs) in specific regions of the genome tend to recur and, if associated with recognizable patterns of congenital malformations, allow the identification of specific microdeletion and/or microduplication syndromes. CNVs are common in genomic regions enriched for segmental duplications, which are long repetitive elements that mediate nonhomologous pairing. Nonhomologous pairing favors unequal crossing over, resulting in nonallelic homologous recombination and causing gain or loss of the chromosomal segments between them. Dyscalculia has been described in several genomic disorders, e.g., Williams-Beuren, 22q11.2 and 15q11.2(BP1–BP2) deletion syndromes.

Dyscalculia in 22q11.2 Deletion Syndromes

Dyscalculia has been particularly well characterized in 22q11.2 deletion syndrome (22q11.2DS) or velocardiofacial syndrome (VCFS). This chromosomal region is divided according to the segmental duplications present, which are named LCR22 (for low copy repeats in chromosome 22). In approximately 90% of cases, VCFS is caused by the deletion of the segment between LCR22-2 (also known as LCR22-A) and LCR22-4 (or LCR22-D). This interval is also referred to as the 22q11.2 typically deleted region (TDR). In 8% of cases, the deletion spans the segment between LCR22-2 and LCR22-3a (or LCR22-C). Only a small percentage of the cases do not coincide with these boundaries (for a review, see Carvalho et al. (2014)). Main symptoms include congenital heart malformations, developmental delay (particularly speech delay), intellectual disability, submucous cleft palate, hypernasal speech, and schizophrenia in the late teens, although none of them is obligatory (McDonald-McGinn et al., 2015; Shprintzen, 2008).

Approximately, 20 patients have been reported with deletions spanning the segment between LCR22-4 (or LCR22-D) and LCR22-5 (or LCR22-E). These deletions do not overlap the LCR22-2 to LCR22-4 interval and do constitute a newly described syndrome, characterized by prematurity, congenital heart defects, subtle facial dysmorphisms, and developmental delay (particularly speech delay), among others. Intelligence was normal in 2 of the 20 cases, and they both presented with dyscalculia (Carvalho et al., 2014). Nine additional patients with this microdeletion have been reported subsequently (Lindgren et al., 2015; Mikhail et al., 2014). Developmental delay, intellectual disability, and psychiatric conditions including autism, bipolar disorder, and schizophrenia are frequently seen. In synthesis, we have two different syndromes caused by deletions of two contiguous and nonoverlapping regions on 22q11.2. Both are associated with dyscalculia.

Genotypic-phenotypic correlations have been established for some phenotypes in 22q11.2, particularly concerning the 22q11.2 TDR. There is a good amount of evidence associating congenital heart malformations with *TBX1* and psychiatric disorders with *COMT* and *PRODH*. *COMT* has also been associated with ANS accuracy and dyscalculia (Júlio-Costa et al., 2013). ANS accuracy (usually assessed by nonsymbolic number magnitude comparison of dot sets) is associated with polymorphisms of the *COMT* gene, regulating the bioavailability of dopamine in the synapses of the prefrontal and parietal cortices (Júlio-Costa et al., 2013). The COMT val158met polymorphism has also been proposed as a mechanism underlying working memory performance (Dumontheil et al., 2011). The *COMT* gene maps to the 22q11.2 region deleted in the velocardiofacial syndrome. The specific working memory components associated with the val158met polymorphism remain to be elucidated (Karayiorgou, Simon, & Gogos, 2010; Karlsgodt, Bachman, Winkler, Bearden, & Glahn, 2011).

Intelligence is in the normal range in 50% of individuals with VCFS (De Smedt et al., 2007). Those individuals with normal intelligence present a higher prevalence of the nonverbal learning disability profile, characterized by math, motor, visuospatial, and social-pragmatic impairments (Schoch et al., 2014). Visuospatial impairments are a hallmark of VCFS (Antshel et al., 2008; Simon et al., 2005; Simon, Bearden, McDonald-McGinn, & Zackai, 2005). Phonological short-term memory is spared (De Smedt et al., 2009).

Math impairments have been observed in single-digit, multidigit, and word problem-solving abilities in VCFS (De Smedt et al., 2008). Both symbolic (De Smedt et al., 2009) and nonsymbolic number comparisons (Attout, Noël, Vossius, & Rousselle, 2017; Oliveira et al., 2014) are impaired in VCFS. Several studies have indicated that basic number processing impairments in VCFS could be reduced to an underlying visuospatial deficit (Simon, Bearden, et al., 2005, Simon, Bish, et al., 2005; see review in Simon (2008)). Control of visuospatial abilities in VCFS did not attenuate number processing impairments in one study (Brankaer et al., 2016). In another study, an ANS accuracy impairment was observed in the visuospatial modality but not in the auditory modality (Attout et al., 2017). The relative importance of symbolic vs. nonsymbolic number processing for math achievement and the connection to visuospatial processing in VCFS remain important research questions.

Dyscalculia in Williams Syndrome

The Williams syndrome (or Williams-Beuren syndrome) is characterized by distinctive facial features including wide forehead, puffy eyes, short nose with broad tip, full cheeks, wide mouth with full lips, and small, widely spaced teeth. Congenital heart malformations are frequent, and the most common defects are supravalvular aortic stenosis and peripheral pulmonary stenosis. They may present hypertension and endocrine abnormalities (hypercalcemia, hypercalciuria, hypothyroidism, and early puberty) and short stature. Hyperacusis is frequent. Williams syndrome is a highly heterogeneous and complex condition from the neuropsychological point of view. IQ is normal in half of the individuals (Pitts & Mervis, 2016). The cognitive-behavioral phenotype in individuals with normal IQ includes hypersociability, anxiety proneness, interest in music, and impairments in motor, executive, visuospatial, syntactic, and pragmatic-discursive abilities, in addition to a relatively spared phonological-semantic lexicon (Vandeweyer, Van der Aa, Reyniers, & Kooy, 2012).

Williams syndrome is caused by recurrent deletions in 7q11.23. LCRs are also common in this region, and the most common deletions in the Williams syndrome region are 1.55–1.8 Mb. The size of the deletion depends on which LCRs are involved in nonallelic homologous recombination. At least 28 genes map to the Williams syndrome region, including the elastin (*ELN*) gene, which is associated with the congenital heart malformations and also with other symptoms of WS. *ELN* mutations cause autosomal dominant supravalvular aortic stenosis in some families, without typical Williams syndrome, suggesting that other genes in the Williams syndrome typically deleted region cause the other clinical manifestations of this syndrome. Small 7q11.23 deletions helped in the search for genotypic-phenotypic correlations, and some candidate genes emerged: *GTF2IRD1* and *BAZ1B* (craniofacial features), *STX1A* and *MLXIPL* (diabetes mellitus), and *GTF2IRD1* (cognitive symptoms) (reviewed by Vandeweyer et al. (2012)). Most cases are sporadic, but some autosomal dominant families have been described.

Molecular mechanisms underlying Williams syndrome are being investigated in cell cultures of Williams syndrome neurons obtained by reprogramming mature cells into self-renewing, induced pluripotent stem cells (iPSCs). For example, one of the genes in the Williams syndrome region, *BAZ1B* (also known as Williams syndrome transcription factor), has been associated with both neurogenesis and neuron differentiation. *BAZ1B* deletions have induced a transcription dysregulation in Williams syndrome-induced neurons, with over 700 downregulated and over 1000 upregulated genes. Dysregulation altered the transcription profiles of genes implicated in cognition, synaptic transmission, and intellectual disability (e.g., *CACNA1C, GABRG2, GRIN3A, NLGN3)* and genes implicated in axon guidance and formation of neuronal projections; delta-catenin (*CTNND2*) and *KANSL1* are associated in the literature with the expression "conspicuously happy disposition," meaning that *BAZ1B* may be implicated in the personality traits of persons with Williams syndrome. *BAZ1B* deletion explains almost 42% of the transcriptional deregulations observed in Williams syndrome cells (Lalli et al., 2016).

Duplications of the Williams syndrome region have been associated with autism. Crespi and Procyshyn (2017) reviewed the evidence suggesting that the behavioral manifestations in deletions and duplications in the Williams syndrome region parallel those observed in individuals with high and low oxytocin levels, respectively. Indeed, patients with Williams syndrome present higher oxytocin levels. Social behavior patterns and visuospatial difficulties (low performance on the mental rotation test) are similar in people with Williams syndrome and high oxytocin levels. These authors suggested that the higher levels of oxytocin observed in Williams syndrome are mediated by *GTF2I*, a gene in the Williams syndrome region.

Math learning impairments are an important characteristic of Williams syndrome (O'Hearn & Landau, 2007). Impairments of enumeration in Williams syndrome have been observed in the subitizing range (O'Hearn, Hoffman, & Landau, 2011). Deficits have been observed in the numerosity estimation of sets of dots up to 11 (Ansari, Donlan, & Karmiloff-Smith, 2007) and in positioning numbers on the number line (Opfer & Martens, 2012). Difficulties in numerical magnitude comparisons have been observed both in symbolic (Krajcsi, Lukács, Igács, Racsmány, & Pléh, 2009) and nonsymbolic modalities (Rousselle, Dembour, & Noël, 2013). In another study, Libertus, Feigenson, Halberda, and Landau (2014) observed a dissociation characterized by sparing of symbolic and impairment of nonsymbolic number processing in Williams syndrome. Some studies did not obtain clear-cut results regarding impairments in basic number processing in Williams syndrome (Paterson, Brown, Gsödl, Johnson, & Karmiloff-Smith, 1999; Paterson, Girelli, Butterworth, & Karmiloff-Smith, 2006; van Herwegen, Ansari, Xu, & Karmiloff-Smith, 2008). However, these studies did not distinguish between OFS and ANS. Some questions remain open. The relationship between visuospatial and numerical processing in Williams syndrome is not clear. Deficits are not restricted to nonverbal mechanisms, as phonological awareness is also impaired in Williams syndrome (Menghini, Verucci, & Vicari, 2004).

Monogenic Conditions

Dyscalculia has been described in some monogenic conditions, e.g., in neurofibromatosis type 1 (Mazzocco, 2001), phenylketonuria (Chang, Gray, & O'Brien, 2000), and girls with the fragile X syndrome (Hagerman et al., 1992). Here, we will focus our attention on the fragile X syndrome.

Dyscalculia in Fragile X Syndrome and FMR1 Premutations

Compelling evidence of a specific association with dyscalculia has been published for fragile X syndrome (FXS). FXS is caused by the expansion of a cytosine-guanine-guanine (CGG) repeat in the 5′-untranslated region (5′-UTR) of the fragile X mental retardation (*FMR1*) gene. Depending on the number of CGGs, alleles are classified as normal (6–44 CGGs); intermediary, also known as gray zone (45–54 CGGs); premutation (55–200 CGGs); and full mutation (>200 CGG) (Bassell & Warren, 2008; Fu et al., 1991; Lozano, Martinez-Cerdeno, & Hagerman, 2015; Maenner et al., 2013). *FMR1* full mutations cause the FXS, characterized by intellectual disability, autism, ADHD, and working memory (WM) deficits in boys and borderline to normal IQ and autism associated with obesity in girls. ADHD and dyscalculia are also frequent in girls with FXS (Abbeduto, McDuffie, & Thurman, 2014; Bailey, Raspa, Olmsted, & Holiday, 2008; Brown et al., 1982; Ciaccio et al., 2017; Jäkälä et al., 1997; Lozano, Rosero, & Hagerman, 2014).

In contrast to virtually all males, who present intellectual disability (Bailey Jr, Hatton, & Skinner, 1998), 50% of females with FXS present an IQ in the normal to borderline range (Rousseau, Heitz, Tarleton, et al., 1994). High prevalences of autism (50–65% in males and 20% in females) and ADHD (80% in males and 30% in females) are usually reported (see review in Grigsby, 2016). Eighty-seven percent of girls with FXS meet diagnostic criteria for dyscalculia (Murphy et al., 2006). Females with FXS present an uneven cognitive profile of assets (verbal memory and analytic visual perception) and deficits (visuospatial and executive function, see review in Murphy, Mazzocco, and McCloskey (2010)). In girls with FXS (6–16 years old), a relative strength in verbal and processing speed abilities, with weaknesses in visuospatial-constructional and working memory abilities, was observed by Quintin et al. (2016). Girls with FXS preserved number reading/writing and rote counting abilities with deficits in magnitude judgments, mental line judgments, and understanding of counting principles and basic addition (Murphy et al., 2006). Impairments in basic numerical processing are masked by rote verbal skills in older female children with FXS (Murphy & Mazzocco, 2008). Differently from Turner syndrome, visuospatial perceptual abilities were correlated with math performance in FXS (Mazzocco et al., 2006).

Individuals with *FMR1* premutation may present deficits in cognitive functions such as working memory, executive function, visuospatial perception, phonological processing, and reaction time (Bodega et al., 2006; Bretherick, Fluker, & Robinson, 2005; Debrey et al., 2016; Fernandez-Carvajal et al., 2009; Hall, 2014; Kenna et al., 2013; Liu, Winarni, Zhang, Tassone, & Hagerman, 2013; Sullivan et al., 2005). Females with a premutation allele may present difficulties in basic numeric comprehension and numerical transcoding of mathematical questions and calculations (Lachiewicz, Dawson, Spiridigliozzi, & McConkie-Rosell, 2006; Murphy & Mazzocco, 2008; Roberts et al., 2005; Semenza et al., 2012). Currently, *FMR1* premutation and full mutations in females provide the best evidence of a monogenic component for dyscalculia. As with the other syndromes discussed here, the role of nonsymbolic vs. symbolic number processing and their connections with visuospatial abilities remain to be more thoroughly investigated.

From the Lab to the Classroom

We have reviewed the concept of cognitive endophenotype as a key for understanding the genetic bases of math learning abilities and disabilities. The cognitive endophenotype was postulated as a complexity-reducing strategy that could help to disentangle the relationships between the genetic and environmental etiologic levels and the phenotypic expression of math achievement. From this perspective, severe and persistent difficulties in learning arithmetic – corresponding to the diagnosis of developmental dyscalculia – result from the interaction of a host of genetic and environmental factors from which math achievement is expressed at the phenotypic level. Interactions between genetic and environmental factors are complex, encompassing multiple crisscrossing chains of events with distinct intermediate steps. These steps correspond to intermediate or endophenotypes that can be characterized at the neurochemical, neurofunctional, and neurocognitive levels. The phenotypic expression of math facility or difficulty would then result from the interaction of a network of such endophenotypes. The task of investigating the genetic basis of arithmetic achievement is thus reduced to the identification of relevant endophenotypes. What have we learned that could be relevant to the classroom?

First, we have learned that cognitive mechanisms underlying math abilities vary widely, albeit in systematic ways. Klinefelter syndrome was used as an example of a verbally mediated cognitive endophenotype eventually useful in explaining the pattern of math difficulties associated with developmental dyslexia. Other models of verbally mediated endophenotypes could include Down syndrome (Naess, 2016) and Noonan syndrome (Pierpont et al., 2010). These are all conditions in which a predominantly verbal pattern of impairment was uncovered to date.

Second, we have learned that in other syndromes such as Turner syndrome, velocardiofacial syndrome, Williams syndrome, and fragile X syndrome, the mechanisms implicated seem to be nonverbal. Their difficulties seem to be related to impairments in basic numerical processing and/or visuospatial processing.

Third, we have learned that impairments in working memory and/or executive functions lack the specificity to be good candidates for an endophenotype implicated in dyscalculia. All reviewed syndromes present one kind or another of impairments in working memory/executive functions. Impairment in these processes is also observed in virtually all other forms of neurodevelopmental disorders. Teachers should be aware of this fact and address executive problems of their pupils by using proper behavioral and cognitive self-management techniques.

Fourth, we have learned about the extreme phenotypic variability that characterizes each genetic syndrome. In some syndromes, such as velocardiofacial, Williams, and fragile X syndrome in females, the median split of intelligence is at the cutoff point for intellectual disability. Therefore, half of the affected individuals will present intellectual disability, and in the other half, intelligence will be in the normal range. This has important implications for diagnosis. Increasingly efficient molecular diagnostic methods make it possible to identify specific genetic causes of learning problems. Such diagnoses have important therapeutic implications and may be established in individuals who have not been previously identified because of mild phenotypic expression, lack of severe life-threatening malformations, and/or normal intelligence.

Fifth, we have also learned about the extreme genetic heterogeneity underlying arithmetic abilities. Multiple genetic mechanisms may be implicated in similar phenotypic expressions. For example, adjacent and nonoverlapping microdeletions in the 22q11.2 region may be associated with impairments in the accuracy of nonsymbolic numerical representations. Otherwise, the same genetic mechanism such as the COMT val158met polymorphism may be implicated in different phenotypic expressions such as anxiety, impulsivity, working memory, accuracy of nonsymbolic numerical representations, etc.

Finally, we have learned that early recognition of genetic syndromes, referral for proper diagnosis and treatment, and planning of customized interventions are essential. In this way, math learning difficulties may function as a kind of red flag, pointing to possible genetic etiologies. General red flags for genetic syndromes, which teachers can observe, are short or tall stature, congenital malformations, hypotonia, poor motor coordination, anomalous handedness, and history of developmental delay. "Funny face" is an important red flag. These children have no facial malformation but, rather, small, subtle dysmorphisms such as a low nasal bridge, markedly upslanting or downslanting palpebral fissures, small or prominent chin, low set ears, etc. (Huang et al., 2010). Normal people may have one or two such dysmorphisms, but they are not enough to characterize a "funny face." However, it is important to consider that most children with math learning difficulty will have a perfectly normal constitution and no genetic syndrome.

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Chapter 23

Neurobiological Origins of Mathematical Learning Disabilities or Dyscalculia: A Review of Brain Imaging Data

Bert De Smedt, Lien Peters, and Pol Ghesquière

Introduction

Mathematical skills constitute basic competencies that children need to acquire during elementary school. These skills are quintessential to our daily life, as we use numbers every day and early mathematical competencies in young children are the most stable predictors of their later academic outcome (Duncan et al., 2007) as well as of their future income and socioeconomic status (Ritchie & Bates, 2013). On the other hand, approximately 5–8% of the children experience lifelong difficulties in acquiring and executing these mathematical skills: children with mathematical learning disabilities or dyscalculia (American Psychiatric Association, 2013; Butterworth, Varma, & Laurillard, 2011; Geary, 2011). Critically, these difficulties are not merely explained by intellectual disabilities, uncorrected sensory problems, mental or neurological disorders, or inadequate instruction (American Psychiatric Association, 2013). In the most recent version of Diagnostic and Statistical Manual of Mental Disorders or DSM-5 (American Psychiatric Association, 2013), this specific learning disorder is categorized into the section of *neuro*developmental disorders, which indicates that abnormalities in brain structure and function underlie the behavioral manifestations of the disorder. Despite its assumed neurobiological origin, there are only but a handful neuroimaging studies that have investigated the neural basis of these mathematical learning disabilities, which we review in this chapter.

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The first time the term dyscalculia was coined in the scientific literature (Kosc, 1974), it was already stipulated that this disorder was brain-based and that it was the consequence of an “impairment in the growth dynamics of brain centers, which are the organic substrate of mathematical abilities” (p. 166). At that time, there were no brain imaging techniques available to investigate brain function and structure in a noninvasive way, such as magnetic resonance imaging or MRI, and these only became routinely available since the early 1990s. Knowledge about the brain at that time was largely based on (adult) neuropsychological case studies. These were in-depth studies of patients with brain damage (e.g., due to injury or hemorrhage), which were not able to calculate any more after the damage, even though they were able to do so before the onset of the damage; these mathematical impairments were termed “acalculia.” These case studies revealed that damage to the parietal cortex resulted in different types of mathematical difficulties (e.g., Dehaene & Cohen, 1997). Already in 1919, Henschen (1919) described an association between a lesion in the left parietal cortex and acalculia, and in 1940, Gerstmann (1940) described a syndrome, caused by a lesion in the left angular gyrus (AG), that included finger agnosia (= difficulties in the ability to distinguish, recognize, or name fingers when touched), left-right disorientation, agraphia (= difficulties in writing), and, most relevant in this context, acalculia. It is important to emphasize that the results of these neuropsychological case studies cannot be readily generalized to dyscalculia. This is because patients with acalculia have an *acquired* disorder: They had typical mathematical development before the onset of the disorder and difficulties only emerged after specific brain damage had occurred. Dyscalculia, however, is a *developmental* disorder, in which the development of mathematical ability, and its associated brain networks, shows a different developmental trajectory from a very early age on.

The availability of MRI methods to study brain function and structure has resulted in a continuously increasing knowledge of the brain networks that support number processing (e.g., Sokolowski, Fias, Bosah Ononye, & Ansari, 2017) and arithmetic (e.g., Arsalidou & Taylor, 2011; Menon, 2015; Peters & De Smedt, 2018). The vast majority of this research has been carried out in healthy adults, and the generalization of these findings to the developing and atypical brain needs to be done with great caution (e.g., Ansari, 2010; Karmiloff-Smith, 2010): Studies in healthy adults only reveal something about the end state of mathematical development, yet they do not inform us on how these skills, and their underlying brain correlates, develop. Findings from these adult studies mistakenly suggest that brain structures and functions are static. However, it is becoming increasingly clear that the developing brain is highly plastic and that its structure and function change dramatically throughout development into adulthood, a process that is highly driven by environmental factors, such as (math) education (Johnson & de Haan, 2011). Against this background, the current chapter takes a developmental perspective in describing the brain networks associated with mathematical ability and how these are impaired in dyscalculia.

It is unlikely that our brains are predestined to perform mathematical operations. For example, we use specific symbols to perform these operations, such as Arabic digits, and these have only been invented rather late in the evolution of mankind (e.g., Arabic digits only emerged in the Early Middle Ages; Ifrah, 1998). As is the case in other academic abilities, such as reading, mathematical skills are culturally transmitted, and developing brains change as they are learning these cultural skills (Dehaene & Cohen, 2007), a process called experience-dependent cortical plasticity, pointing to how brains change and brain areas gradually acquire their function, as children learn new skills, such as arithmetic (Johnson & de Haan, 2011). On the other hand, animals and human infants are able to process numerical magnitudes, albeit in a nonsymbolic way (for reviews see Christodoulou, Lac, & Moore, 2017; Smyth & Ansari, 2017), and this nonsymbolic number processing consistently activates the intraparietal sulcus (IPS) and prefrontal cortices (Nieder & Dehaene, 2009; Sokolowski et al., 2017). Interestingly, similar areas are being activated during the processing of symbolic numbers and during arithmetic, and these areas might provide a basis via which the brain network for performing mathematical operations is gradually constructed. The connections between these nonsymbolic and symbolic number skills continue to be an area of very intense debate in cognitive and brain imaging studies (e.g., De Smedt, Noël, Gilmore, & Ansari, 2013; Leibovich, Al-Rubaiey Kadhim, & Ansari, 2017; Merkley & Ansari, 2016) as well as in studies on dyscalculia (De Smedt & Gilmore, 2011; De Smedt et al., 2013; see also Schwenk et al., 2017). Because the processing of numerical magnitudes is critical to children's mathematical development (e.g., Schneider et al., 2017) and a poor understanding of numerical magnitudes might constitute a key deficit in dyscalculia that cascades into impairments in arithmetic (De Smedt et al., 2013; Schwenk et al., 2017), we first describe the brain networks that subserve the processing of numerical magnitudes and subsequently summarize how the networks underlying arithmetic are gradually constructed. In both cases, we will elaborate on how these networks are altered in children with dyscalculia.

Brain Activity During Numerical Magnitude Processing and Arithmetic

As signals that indicate brain activity can only be meaningfully interpreted if they are linked to a cognitive model (e.g., Cacioppo, Berntson, & Nusbaum, 2008; De Smedt, Holloway, & Ansari, 2011), we provide a succinct description of the development of numerical magnitude processing and arithmetic and its impairments in dyscalculia. These cognitive models serve as a lens through which the subsequent neural data in typically developing children and children with dyscalculia are discussed.

Numerical Magnitude Processing

The ability to understand and process nonsymbolic numerical quantities already emerges at a very young age: Human infants are able to discriminate between numerosities (Feigenson, Libertus, & Halberda, 2013; Smyth & Ansari, 2017), and toddlers can identify the larger of two dot arrays (Barth, Landsman, & Lang, 2008). On the other hand, toddlers also learn to represent number in a symbolic way via the acquisition of the numerical meaning of number words and, later on, Arabic digits (Merkley & Ansari, 2016). These nonsymbolic and symbolic representations of number are gradually becoming more interconnected during development. The dominant view posits that symbolic representations are being mapped on the earlier developed pre-existing nonsymbolic representations of quantity (e.g., Dehaene & Cohen, 1997; Piazza et al., 2010). This dominant unidirectional view has been challenged against the background of developmental and brain imaging data (e.g., Leibovich et al., 2017; Leibovich & Ansari, 2016), suggesting that the developmental trajectory of how numerical symbols acquire their meaning is less straightforward than originally thought. In any case, it is clear that the understanding of the meaning of numbers, i.e., their quantity, is critical for successful mathematical development.

The understanding of the meaning of numbers is commonly investigated with number comparison tasks in which children have to identify the larger of two presented dot arrays, number words, or Arabic numerals. Performance on these tasks correlates with individual differences in mathematics achievement (De Smedt et al., 2013 for a systematic review; Schneider et al., 2017 for a meta-analysis: $r = 0.28$, 95%CI [0.24, 0.32]). These associations, cross-sectional as well as predictive, are robustly observed for symbolic comparison tasks, yet the associations with nonsymbolic measures are significantly smaller and far less consistent (De Smedt et al., 2013; Schneider et al., 2017), indicating that symbolic magnitude processing is a more powerful predictor of mathematical performance. Studies in children with dyscalculia (De Smedt et al., 2013 for a systematic review; Schwenk et al., 2017 for a meta-analysis) have consistently revealed that these children show impairments in numerical magnitude processing. More specifically, children with dyscalculia show robust deficits on symbolic comparison measures, particularly when reaction times are measured. On the other hand, when nonsymbolic measures are considered, the picture is far less conclusive, as some studies have shown deficits in nonsymbolic number processing, while others have not.

Various studies have examined brain activity during the processing of nonsymbolic and symbolic numerical magnitudes via comparison, ordering, or passive viewing paradigms (Ansari, 2008; Arsalidou & Taylor, 2011; Dehaene, Piazza, Pinel, & Cohen, 2003). A recent meta-analysis in healthy adults (Sokolowski et al., 2017) revealed that regions in the prefrontal cortex (PFC) and parietal lobes (including IPS and posterior superior parietal lobe or PSPL) are consistently active when healthy adults are processing nonsymbolic and symbolic numerical magnitudes. This meta-analysis also revealed that there were differences when processing nonsymbolic and symbolic numbers, such that left-lateralized parietal regions were more active for symbolic number processing, while the processing of nonsymbolic number showed increased activity in the right (superior) parietal lobe.

Developmental imaging studies have pointed to both communalities and differences between children and adults in this brain network. For example, an fMRI study in 4-year-olds showed that during the passive viewing of nonsymbolic number, children recruited the (right) IPS in a similar way as healthy adults

(Cantlon, Brannon, Carter, & Pelphrey, 2006). Other studies that compared children and adults have highlighted important differences. Children showed a larger engagement of frontal regions than adults during number comparison, which suggests that they need more working memory and attentional resources to perform this type of task. Adults, on the other hand, exhibited increased and more specific activity in the bilateral (intra)parietal cortex than children (Ansari & Dhital, 2006; Ansari, Garcia, Lucas, Hamon, & Dhital, 2005; Holloway, Price, & Ansari, 2010; Kaufmann, Kucian, von Aster, Cohen Kadosh, & Dowker, 2014). These data indicate that the brain networks that support the processing of number are not static but evolve over time. This development is characterized by a frontal-to-parietal shift in brain activity and an increasing functional specialization of the parietal cortex. A similar development has been observed in the context of learning arithmetic (Rivera, Reiss, Eckert, & Menon, 2005), as we will review below. Such evolution from widespread networks to more focused activity is common and has been described in many other cognitive domains (Johnson, 2011). They nicely illustrate the interactive specialization account of children's development of brain function, which posits that brain networks are gradually constructed over development (via interaction with the environment) and evolve from widespread to more specific functional networks.

A handful of studies have compared the brain activity of typically developing children and children with dyscalculia during a numerical magnitude processing task. These studies observed that children with dyscalculia showed significantly less brain activity in the IPS compared to age- and IQ-matched controls during nonsymbolic comparison (Price, Holloway, Räsänen, Vesterinen, & Ansari, 2007), symbolic comparison (Mussolin et al., 2010; Soltész, Szucs, Dékány, Márkus, & Csépe, 2007), and symbolic ordering (Kucian et al., 2011). These findings suggest that the abovementioned functional specialization of the parietal cortex for the processing of number might be delayed or disturbed. On the other hand, some studies have failed to observe group differences during a nonsymbolic comparison task (Kovas et al., 2009; Kucian et al., 2006). This may not be unexpected in view of the above-reviewed accumulating behavioral evidence that deficits in nonsymbolic number processing in dyscalculia are much less consistent than originally thought, while symbolic number processing deficits are more reliably observed (De Smedt et al., 2013; Schwenk et al., 2017). Future imaging studies with symbolic number processing tasks are needed in order to verify whether deficits in symbolic number processing in dyscalculia can also be reliably observed at the neural level.

Arithmetic

Decades of cognitive developmental research have investigated the acquisition of arithmetic, and this development involves a change in the mix of strategies that are used to calculate the answer to a particular problem (Geary, 2011; Jordan, Hanich, & Kaplan, 2003; Siegler, 1996). Already before the start of formal schooling, children use (finger) counting to solve simple sums, such as 2 + 3. The repeated use of these counting routines allows children to develop associations between problems and their answers, arithmetic facts, which are stored in long-term memory. The acquisition of these facts is important because fact retrieval is more efficient, and it consumes less working memory than the more cognitively demanding and error-prone procedures, such as counting. The availability of arithmetic facts also allows children to use these facts to decompose problems into smaller problems, such as $7 + 8 = 15$, $7 + 3 = 10$, and $10 + 5 = 15$. These decomposition strategies usually occur in problems with larger numbers (typically when they cross 10). They are used more often during addition and subtraction – albeit more frequently in subtraction than in addition (Barrouillet, Mignon, & Thevenot, 2008) – but they are much less used in multiplication, in which fact retrieval is the most dominant strategy from an early point on in development (Imbo & Vandierendonck, 2007; Lemaire & Siegler, 1995). The development of these strategies is not an abrupt shift from one strategy to the other but rather a change in the frequency distributions of strategies children use, the so-called overlapping wave theory (Siegler, 1996). This theory posits that strategies remain available over development, even into adulthood, but that the frequency in their use changes at different time points, with the more efficient strategies, such as fact retrieval, becoming more dominant. Difficulties in this development of strategy use have been considered to be the hallmark of dyscalculia (Geary, 1993, 2011): Children with dyscalculia are known to have problems in understanding and executing procedural strategies, and they show persistent deficits in the retrieval of arithmetic facts from memory. Are these difficulties related to functional abnormalities in the brain networks that show increased activity during calculation?

A considerable body of fMRI studies in healthy adults has revealed that a large, whole-brain network is active when they perform arithmetic (Arsalidou & Taylor, 2011; Menon, 2015), as is depicted in Fig. 23.1. This network includes the bilateral posterior parietal cortex (comprising the IPS, PSPL, AG, and supramarginal gyrus or SMG), inferior and superior prefrontal cortex (PFC), and occipitotemporal regions (such as the fusiform gyrus). Activity in this network is modulated by the arithmetic operation (Rosenberg-Lee, Chang, Young, Wu, & Menon, 2011), strategy use (Grabner et al., 2009; Tschentscher & Hauk, 2014), expertise (Grabner et al., 2007), and training (Zamarian, Ischebeck, & Delazer, 2009). Consistent across these data is the activation of the bilateral IPS during arithmetic, potentially reflecting the role of numerical magnitude processing during calculation (Arsalidou & Taylor, 2011). The activity in this area appears to be higher for subtractions, large problems, and during the execution of procedural strategies. Activity in the temporoparietal cortex (AG and SMG) has been typically associated with the retrieval of arithmetic facts from long-term memory. Increases in brain activity in this area are usually observed in multiplication and correlate with mathematical expertise (Grabner et al., 2007). Originally, this temporoparietal activity was thought to reflect the involvement of phonological processes in fact retrieval and multiplication. This interpretation has been questioned (Menon, 2015), and recent data by De Visscher, Berens, Keidel, Noël, and Bird (2015) suggest that it rather reflects the automatic mapping between an arithmetic problem and its answer in long-term memory. Increases in activity in the lateral PFC have been typically attributed to the involvement of auxiliary cognitive

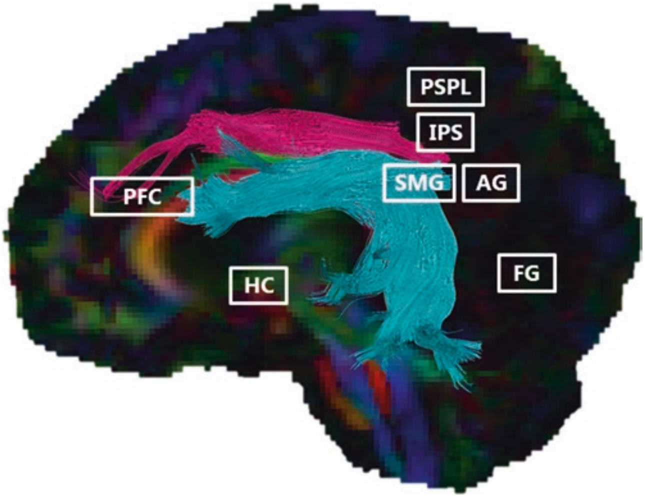

Fig. 23.1 Sagittal slice showing the brain networks for number processing and arithmetic. The white boxes indicate the most relevant areas implicated in number processing and/or arithmetic, including PFC prefrontal cortex, HC hippocampus, PSPL posterior superior parietal lobe, IPS intraparietal sulcus, SMG supramarginal gyrus, AG angular gyrus, and FG fusiform gyrus. The colored tracts represent the relevant frontal-to-parietal white matter connections as revealed via spherical deconvolution analysis of DTI data. Pink superior longitudinal fasciculus (SLF), blue arcuate fasciculus (AF)

functions that are crucial during calculation, such as working memory, inhibitory control, and attentional processes (Arsalidou & Taylor, 2011), and these regions are typically recruited more during more demanding problems, such as larger problems, and during the execution of procedural or backup strategies, when the answer cannot be retrieved from long-term memory. Finally, occipitotemporal regions, including the fusiform gyrus, are involved in the visual processing of symbolic numerical information, given that arithmetic stimuli represent visual symbols (Arsalidou & Taylor, 2011), but the specific role of this region in arithmetic has not been studied in much detail (Peters, De Smedt, & Op de Beeck, 2015).

Training studies in adults have tried to simulate the abovementioned developmental process of strategy change in arithmetic, in particular the development from procedures to arithmetic fact retrieval (Zamarian & Delazer, 2015, for a review). These studies offer a window to our understanding of how arithmetic networks change across skill acquisition. These data revealed, as a function of training, a decrease in activity in PFC coupled with an increase in the posterior parietal cortex. At the same time, activity in the posterior parietal cortex shifts from the IPS to the AG, potentially reflecting the increasing reliance on retrieval strategies and a decreasing reliance on procedural backup strategies, such as counting or decomposition, which is in line with the overlapping wave model of strategy development (Siegler, 1996). These data offer insights into how brain activity changes as a function of learning, but these studies in adults are not necessarily directly transferable to children, because children's brains are not merely smaller versions of a highly skilled adult brain (Ansari, 2010).

fMRI studies on arithmetic in typically developing children have substantially increased over the last 5 years (Peters & De Smedt, 2018, for a recent review), revealing, as in adults, a widespread bilateral (frontoparietal) network of areas that show increased brain activity during arithmetic (Fig. 21.1). Is the activity in this network also modulated by the same factors as has been observed in adults?

De Smedt et al. (2011) investigated the effects of problem size (small vs. large problems) and operation (addition vs. subtraction) on brain activity during arithmetic in 10–12-year-olds. They reasoned that the large vs. small and subtraction vs. addition contrast would reveal those brain networks that showed an increased involvement in procedural strategies, whereas the reverse contrasts would unravel those networks that are more relevant to fact retrieval. Commensurate with the abovementioned adult data, children showed increased activity in a widespread frontoparietal network that included the bilateral IPS and PFC during the solution of procedural problems (see also Polspoel, Peters, Vandermosten, & De Smedt, 2017). Different from the adult data, fact retrieval problems, which in adults were accompanied by increases in the AG, showed increased activity in the medial temporal lobe, specifically the (left) hippocampus (HC), an observation that has been confirmed in more recent studies (Menon, 2016, for a review). This role of the HC might be related to the formation of long-term memories of arithmetic facts, a hypothesis that has gained increased attention in the last years (Menon, 2016). The differences between children and adults might be explained by the time-limited role of the HC in long-term memory (Smith & Squire, 2009): Its role appears to be crucial in the early consolidation of (arithmetic) facts, while in later stages of more automatization, posterior parietal systems, including the AG, become more relevant. This again emphasizes the importance of a developmental perspective and illustrates that brain imaging findings of adults are not merely applicable to children.

Differences between adults and children clearly indicate that the brain networks that are activated during calculation change over time. Rivera et al. (2005) examined these age-related changes in children aged 8–19 years old, by investigating which brain areas showed negative (i.e., age-related decrease) and positive (i.e., age-related increase) associations with chronological age. They observed that activity in the PFC decreased with age, potentially reflecting the decreased involvement of working memory and attentional resources. On the other hand, activity in the (left) inferior parietal cortex increased (including IPS and AG) with age. This points to an increasing functional specialization of the (inferior) parietal cortex with age, and a similar development in the brain networks supporting the processing of numerical magnitudes has been observed (as reviewed above). Similar age-related changes have been observed in more recent studies (Chang, Metcalfe, Padmanabhan, Chen, & Menon, 2016; Qin et al., 2014; Rosenberg-Lee et al., 2011), and they echo the observed training-related changes that have been found in adults (Zamarian & Delazer, 2015).

Only but a few studies have examined the brain activity during arithmetic in children with dyscalculia, and their findings remain mixed (Peters & De Smedt, 2018, for a review). Some studies have reported increased brain activity in children with dyscalculia in the abovementioned frontoparietal network that is active during arithmetic (Davis et al., 2009; Rosenberg-Lee et al., 2015), particularly during more difficult problems as in subtraction, suggesting some compensatory mechanisms, which are, however, poorly understood. Other studies have observed decreased brain activation in children with dyscalculia, together with the observation that no brain area showed increased activity in dyscalculia compared to age-matched controls (Ashkenazi, Rosenberg-Lee, Tenison, & Menon, 2012; Berteletti, Prado, & Booth, 2014; De Smedt et al., 2011). These decreases in brain activity have been observed in prefrontal (Ashkenazi et al., 2012; Berteletti et al., 2014) as well as posterior parietal, including IPS, areas (Ashkenazi et al., 2012; Berteletti et al., 2014; De Smedt et al., 2011). A common observation in these studies is that typically developing children showed a difficulty-related modulation of the frontoparietal network, whereas children with dyscalculia did not. For example, typically developing children showed increased brain activity in the IPS during the solution of more complex (i.e., large) problems than during easier (i.e., small) problems, whereas children with dyscalculia recruited the IPS to the same extent for both types of problems (De Smedt et al., 2011; see also Ashkenazi et al., 2012). This might reflect that children with dyscalculia continue to rely on (more immature) procedural strategies for easy as well as complex problems, while their typically developing peers already shifted to the use of fact retrieval strategies for solving the easy problems, as has been observed in behavioral data (Geary, 2011).

Structural Brain Imaging

Various studies have used structural MRI (voxel-based morphometry) to investigate the anatomical characteristics (in particular gray matter) of the abovementioned networks of number processing and arithmetic in children with dyscalculia. These studies have observed that children with dyscalculia have significantly less gray matter in the posterior parietal cortex, including the IPS (Isaacs, Edmonds, Lucas, & Gadian, 2001; Rotzer et al., 2008; Rykhlevskaia, Uddin, Kondos, & Menon, 2009), in prefrontal cortex (Rotzer et al., 2008) and in hippocampal areas (Rykhlevskaia et al., 2009), compared to typically developing children.

Connectivity

The above-reviewed brain imaging data indicate the involvement of multiple brain areas in the processing of number and arithmetic that are distant from each other. These areas are connected via white matter tracts, which allow for communication between these brain areas. This implies that in order to fully understand the neural basis of dyscalculia, one also needs to consider the connections between these areas, rather than only focusing on isolated brain regions (Uddin et al., 2010, for a discussion). There is now an increasing interest in studying these structural and functional connections (for a review see Matejko & Ansari, 2015; Peters & De Smedt, 2018).

Structural connections between different brain areas, i.e., white matter tracts, can be investigated by means of diffusion tensor imaging or DTI, which examines the properties of these white matter tracts. This technique also allows one to investigate how the quality of these tracts is correlated with individual differences in performance, such as reading (e.g., Vandermosten, Boets, Wouters, & Ghesquière, 2012) or arithmetic (e.g., Matejko & Ansari, 2015). For example, children with higher arithmetical skills show stronger connections between the frontal and parietal areas of the arithmetic network than those with lower arithmetical skills (Tsang, Dougherty, Deutsch, Wandell, & Ben-Shachar, 2009; Van Beek, Ghesquière, Lagae, & De Smedt, 2014). To our knowledge, only one study has used DTI to examine white matter tracts in children with dyscalculia (Rykhlevskaia et al., 2009). This study revealed a reduced white matter integrity of the superior longitudinal fasciculus, a tract that connects the prefrontal cortex and the posterior parietal cortex (Fig. 21.1) in children with dyscalculia. Future studies are, however, necessary to replicate and further consolidate this finding.

It is also possible to investigate the functional connections between the areas of the abovementioned brain networks via fMRI. There are two approaches to investigate this. Task-based connectivity studies investigate the temporal correlations in brain activity in distant but connected areas during the execution of a particular task. Resting-state connectivity studies examine the temporal correlations between distant brain areas during the brain at rest, assuming that these networks at rest correlate with how they are functionally coupled during the execution of a particular task. One study examined task-based functional connectivity in children with dyscalculia during the execution of an arithmetic task (Rosenberg-Lee et al., 2015). This study observed increased parietal-frontal functional connectivity in children with dyscalculia compared to age-matched controls. This type of connectivity has been linked to working memory systems that are recruited during arithmetic (Menon, 2016), yet the connectivity differences between children with and without dyscalculia were not so easy to interpret as they might reflect compensatory effects as well as inefficient use of working memory resources. One study investigated resting-state connectivity in children with dyscalculia and age-matched controls (Jolles et al., 2016). This study observed that children with dyscalculia showed increased interhemispheric IPS connectivity and increased connectivity between the IPS and (dorsal) frontoparietal regions.

In summary, there is evidence to suggest that there are differences in the (frontoparietal) connectivity between the areas of number and arithmetic networks in children with dyscalculia. The number of structural and functional connectivity studies in dyscalculia is currently too few to draw strong definitive conclusions. Future research is needed to further examine this.

Effects of Remedial Interventions on Brain Activity

Is it possible to change brain activity during number processing and arithmetic in children with dyscalculia? Two studies have addressed this question by investigating the effect of a number line (Michels, O'Gorman, & Kucian, 2018) and an arithmetic (Iuculano et al., 2015) intervention on the brain activity in children with dyscalculia.

Michels et al. (2018) examined the effect of the computer-based number line training *Rescue Calcularis* in children with dyscalculia and a control group of age-matched typically developing children. *Rescue Calcularis* is a 5-week program and consists of number lines in which children have to position numbers as well as the outcomes of calculations. Michels et al. showed that children with dyscalculia had increased functional connectivity between the IPS and parietal, frontal, visual, and temporal regions before the training but that this hyperconnectivity disappeared after training.

Iuculano et al. (2015) investigated the effect of an 8-week one-on-one math tutoring intervention, which focused on learning increasingly efficient counting strategies and arithmetic facts and which has been shown to improve performance in children with mathematical difficulties (Fuchs, Compton, Fuchs, Bryant, & Davis, 2008). Iuculano et al. (2015) studied brain activity during single-digit addition before and after the tutoring in children with dyscalculia and an age-matched control group, who also underwent the training. Before the training, children with dyscalculia showed increased activity in frontal, superior parietal, temporoparietal, and hippocampal areas, compared to age-matched controls. After training, the brain activity of children with dyscalculia did not differ anymore from the control children, suggesting that normalization of the brain activity had occurred in children with dyscalculia after training.

The findings of these intervention studies, although preliminary, indicate that it is possible to change brain activity in children with dyscalculia via specific interventions. Future studies are however needed to fully elucidate the effects of remedial interventions on the brain function, structure, and connectivity in children with dyscalculia.

Discussion

The number of neuroimaging studies on dyscalculia has steadily increased over the last decade, yet it left us with a rather scattered picture of findings. There are various methodological reasons for this. First, studies differ in the populations under study. For example, they collapse children of different ages, and the criteria studies use to define dyscalculia vary greatly, with differences in cutoff criteria, the lack of taking persistency of the mathematical impairments into account, and the in- or exclusion of accompanying reading impairments. Second, studies differ in the tasks they use in their fMRI design or they use to correlate with brain structure and function. One particular issue is the type of control condition in an fMRI design that is subtracted or compared to the condition of interest, as this is a critical determinant of observed findings and group differences (Menon, 2016, for a discussion). Third, studies also differ in their data-analytic methods of imaging data, such as the choice of preprocessing parameters, the selection of the normalization template (pediatric or not), and the decisions to correct for multiple comparisons. Each of these methodological reasons (and their interactions) can explain the inconsistencies between studies. The effect of these between-study differences is even exacerbated when the pool of studies on a given topic, as is the case in dyscalculia, is small. The only way to address this issue is to conduct more studies.

One possibility to address this issue of inconsistency is to perform meta-analyses to statistically extract communalities across studies. Although Kaufmann, Wood, Rubinsten, and Henik (2011) conducted a meta-analysis on studies in dyscalculia, the current body of studies is simply too small to reliably perform such an analysis. Eickhoff et al. (2016) recommended to include in an imaging meta-analysis at minimum 17–20 experiments to have enough power to detect a moderate effect. This number of studies increases if one wants to test moderators of the differences between typically developing children and children with dyscalculia, such as the type of task (number processing vs. arithmetic). We currently lack a critical mass of studies on dyscalculia that would allow us to do such an analysis.

An important limitation of the existing body of evidence is that there are no longitudinal studies that have examined brain function or structure in dyscalculia at multiple time points. The available data do not allow us to determine whether the observed brain abnormalities are the cause or the consequence of their difficulties in number processing and arithmetic. It is possible that the observed abnormalities are simply the consequence of less experience with number and arithmetic in children with dyscalculia compared to typically developing children and that they do not represent the etiology of the disorder. For example, studies in dyslexia, which compared children with dyslexia to age-matched controls and to children who had similar reading level and experience but were younger in age (reading-level-matched controls), revealed that some of the observed brain abnormalities were explained by their reduced reading experience (Vandermosten, Hoeft, & Norton, 2016, for a review). To the best of our knowledge, such comparisons with ability-level-matched children are nonexistent in dyscalculia, and they clearly represent an area for future research.

We also do not know whether the abovementioned brain abnormalities in children with dyscalculia are already present before they learn to calculate and hence may represent a neurobiological cause of their disability. In dyslexia research, there is now an increasing number of brain imaging studies that have investigated children before they learn to read, including children who are at risk for developing dyslexia (see Vandermosten et al., 2016, for a review). Such studies are also needed in the context of number and arithmetic as they will allow us to further unravel the neurobiological cause of atypical mathematical development.

From a practical point of view, it is important to emphasize that the abovementioned brain abnormalities in children with dyscalculia are very subtle. They can only be observed at the group level, and this does not necessarily imply that such abnormalities can be found at the level of the individual child. Stated differently, most children with dyscalculia will not show abnormalities on a clinical brain scan. As a result it is (currently) not possible to determine via a brain scan whether a child has dyscalculia or not. Related to this, there is an active interest in the possibility of neurobiological measures, or biomarkers, to predict which children will develop learning disorders and how they will respond to interventions (Black, Myers, & Hoeft, 2015). To the best of our knowledge, there are currently no studies in dyscalculia that have investigated such biomarkers, but this might be an interesting avenue for further research.

Conclusion

The first scientific reports of dyscalculia suggested that these impairments originated from abnormalities in brain structures or functions related to mathematical processing, yet the study of the brain networks that support number processing and arithmetic in children is only a very recent endeavor. There are emerging trends in our understanding of these networks, but the research on this topic is still nascent. Studies in typically developing children indicate that a frontoparietal network is consistently active during number processing and arithmetic. This network shows both communalities and differences with what is being observed in adults. Only but a few studies have investigated these networks in children with dyscalculia, showing that these children have functional as well as structural abnormalities in these networks. In the absence of longitudinal data, it is currently unclear whether these abnormalities are a cause or consequence of this learning disorder, and future studies are needed to unravel this.

Acknowledgments This paper has some text in common with De Smedt and Ghesquière (2016) and Peters and De Smedt (2018).

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Chapter 24

Comorbidity and Differential Diagnosis of Dyscalculia and ADHD

Helga Krinzinger

Introduction

It has been well-known for a long time now that working memory, attention and executive functions are some of the main predictors of mathematical abilities (Geary, 2005; Passolunghi, Cargnelutti, & Pastore, 2014) and that children with attention deficit/hyperactivity disorder (ADHD) are frequently impaired in these cognitive domains (Barkley, 1997). Hence, it is no surprise that children with ADHD often score worse on mathematical tasks than children without ADHD (Ackerman, Dykman, & Peters, 1977). Comorbidity rates for ADHD and mathematical learning disability (MLD) or dyscalculia are reported to lie between 25% (Gross-Tsur, Manor, & Shalev, 1996; Silva et al., 2015) and 42% (Desoete, 2008). These comorbidity rates are much higher than expected given that the prevalence rates for both developmental disorders are not much higher than 5% (Lindsay, Tomazic, Levine, & Accardo, 1999). This already shows that MLD and ADHD may not be unrelated disorders – but what are possible reasons for this?

What Is Comorbidity?

The term comorbidity was first (and still is) used to describe patients presenting with more than one medical condition and was not used before the 1980s (Arcelus & Vostanis, 2005). In the meantime, several distinctions of comorbidity types have

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been proposed (see Angold, Costello, & Erkanli, 1999, for in-depth descriptions): (a) homotypic vs. heterotypic comorbidity, meaning comorbidity between disorders from the same vs. from different diagnostic groupings; (b) concurrent vs. successive comorbidity; and (c) primary vs. secondary comorbidity, which is usually conceptualized as the secondary condition being caused by the primary condition. The second condition can run concurrently with the primary disorder, but this is not required for this distinction (Samet, Nunes, & Hasin, 2004).

Considering MLD and ADHD, according to DSM-V, they both belong to the category of neurodevelopmental disorders. Comorbidity between these two disorders should thus be seen as homotypic (Moreno-De-Luca et al., 2013). Usually they will occur concurrently (at the same time), although the diagnosis of one disorder may precede the other. With respect to the last distinction, impairments in mathematics can clearly be caused by ADHD (and should then be considered a secondary problem: see Rubinsten & Henik, 2009), whereas MLD cannot lead to generalized attentional problems as seen in ADHD.

Why Are Comorbidity Rates for Neurodevelopmental Disorders So High?

In general, comorbidity rates between psychiatric and developmental disorders occur much more frequently than could be accounted for by the respective prevalence rates, which are usually explained by covariation between disorders (Angold et al., 1999). However, several methodological artefacts can lead to systematic overestimation of comorbidity rates. Among them are referral bias (children with more than one disorder are more likely to be referred to specialized institutions), coding single behaviours (and most importantly "nonspecific symptoms") as symptoms of multiple disorders with overlapping diagnostic criteria, and the related issue of diagnostic boundaries between disorders in the classification systems (see Angold et al. (1999) and Arcelus and Vostanis (2005) for more extensive overviews). It is important to note here that both the DSM-V and the ICD-10 classify disorders by the observation of clusters of behavioural symptoms, without regard to their biological causes (Rapin, 2014).

However, none of these possibilities fully explains the above-mentioned phenomenon (Angold et al., 1999). True comorbidity in the field of neurodevelopmental disorders (homotypic comorbidity) may arise because different disorders share the same risk factors (Arcelus & Vostanis, 2005). Heterotypic comorbidity may also share certain causes (Angold et al., 1999). Among the most probable shared risk factors for MLD and ADHD are genetic factors. Wilcutt et al. (2010) have shown that correlations between biological siblings are moderate to high on dimensional measures of math, reading and ADHD symptoms. The relative risk of first-degree relatives compared to relatives of individuals without the disorder (or familiarity) of ADHD is 4–8 times higher, and of MLD it is estimated to be 5–10 times higher. Furthermore, they summarized the results of published twin studies of reading, math and ADHD symptom dimensions (mostly inattention) and reported moderate to high shared heritability, whereas shared environmental influences did not explain additional variance for the ADHD symptom dimensions. Moreover, they suggested that comorbidity between specific learning disabilities and ADHD are primarily explained by common genetic influences (see Moreno-De-Luca et al., 2013, for a general overview of genetic studies in neurodevelopmental disorders). These high heritability rates have led to initial optimism that single genes with major effects would be identified for each disorder. However, results of different types of genetic studies (candidate gene studies, linkage studies and association studies) all suggest that the aetiologies of specific learning disabilities and ADHD are complex and polygenetic, with multiple genetic and environmental risk factors contributing to the total phenotypic variance in the population. Also, they suggest rather than being unique and specific neurodevelopmental deficits, learning disorders and ADHD may be distinguished by subtle differences in cognitive profiles (Wilcutt et al., 2010). This notion has led the National Institute of Mental Health to propose new biologically based Research Domain Criteria (RDoC) classifications, e.g. for working memory, to cut across traditional diagnostic categories and identify neurodevelopmental disorders rather as dimensional than categorical dysfunctions of specific capacities (Rapin, 2014; see also the Research Domain Criteria approach RDoC of the NIMH, http://www.nimh.nih.gov/research-priorities/rdoc/index.shtml).

To this end, genetic studies and epidemiological data suggest that neurodevelopmental disorders should rather be thought of as different patterns of symptoms or impairments of common underlying continua than being considered as causally and pathophysiologically distinct (Moreno-De-Luca et al., 2013). The authors argue for a conceptual framework of developmental brain dysfunction; however, they are very explicit in stating that this should not be considered a final diagnosis, since "categorical diagnoses and specific impairments must be identified to guide treatment" (Moreno-De-Luca et al., 2013, p. 410). Rapin (2014) also argued that conventional DSM or ICD diagnoses are still needed to inform decisions about kinds of interventions one should try and other questions, despite their flaws and lack of classification rigour. Similarly, Arcelus and Vostanis (2005) argue that in clinical terms it may not matter whether a child is diagnosed as suffering from one single, two distinct or one mixed disorder, as long as all behavioral symptoms are recognized and treated.

What Can Be Causes for Difficulties in Mathematics?

Rubinsten and Henik (2009) explicitly postulated that different aetiologies can lead to the same problem, namely, difficulties in mathematics. They differentiate between "mathematical learning disability" on the one hand, which they see as secondary deficiencies in mathematics due to general cognitive impairments such as inattention or a working memory deficit (related to frontal lobe dysfunction). On the other hand, they conceptualized "developmental dyscalculia" as a primary disability in mathematics caused by a core deficit in numerical magnitude representation (related to parietal lobe dysfunction; see Wilson & Dehaene, 2007; Landerl, Chap. 2, this volume; De Smedt, Peters & Ghesquière, Chap. 21, this volume). Similarly, a recent opinion article by 14 of the leading researchers on this topic (Kaufmann et al., 2013) also differentiates between primary and secondary dyscalculia, with the latter being entirely caused by nonnumerical impairments such as – but not limited to – ADHD.

Interestingly, three studies found that within the group of MLD children, large subgroups mostly had difficulties associated with inattention, such as omitting numbers while counting or failing to add and subtract carried digits (Badian, 1983; Shalev, Manor, & Gros-Tsur, 2005; von Aster, 2000). These children also had great difficulties remembering multiplication tables and other number facts. Badian suggested already in 1983 that many children make arithmetic errors because of attentional problems rather than due to a specific mathematical deficit.

In more detail, several specific mechanisms linking domain-general cognitive impairments to deficits in different aspects of arithmetic have been postulated. Problems with the carry procedure have been linked to poor attentional and working memory skills for a long time (Geary, 1993). Furthermore, Ackerman, Anhalt, and Dykman (1986) stated that children with ADHD have problems in mathematics due to interference during automatization processes. As a consequence, arithmetic fact retrieval is often impaired in children with ADHD even without specific learning disabilities (Zentall, 1990). Recently, de Visscher and Noël (2014, 2015) have presented empirical evidence for a specific link between impaired arithmetic fact retrieval and hypersensitivity to interference.

In addition to a core deficit in numerical magnitude processing and to domain-general cognitive impairments, affective disorders may indirectly interrupt successful arithmetic performance. In a recent meta-analysis, Moran (2016) confirmed a negative impact of anxiety on a variety of working memory tasks. More specifically, math anxiety may impair mathematical performance due to a decreased working memory capacity (for a specific review, see Eden, Heine, & Jacobs, 2013). Depression also disrupts cognition in children (mostly concerning memory and retrieval: Günther, Holtkamp, Jolles, Herpertz-Dahlmann, & Konrad, 2004), which can have a negative impact on arithmetic performance. This means that affective disorders cannot only be caused by problems in mathematics but that they can themselves lead to secondary MLD.

Why Is It Important to Distinguish Between Primary and Secondary MLD?

Mathematical learning disabilities have severe negative consequences on mental health (Auerbach, Gross-Tsur, Manor, & Shalev, 2008) as well as employment rates and wages (Paglin & Rufolo, 1990; Parsons & Bynner, 2005). However, not all children and adolescents struggling with mathematics suffer from developmental dyscalculia (Rubinsten & Henik, 2009) or primary MLD (Kaufmann et al., 2013). In order to be able to provide the best help for affected individuals, it is necessary to know about the respective reasons for deficits in mathematics.

As described in the section above, a substantial number of children will fulfil most diagnostic criteria for the diagnosis of a specific learning disability (SLD) in arithmetic, but do not present with a core deficit in numerical magnitude representation. The DSM-5 (American Psychiatric Association, 2013) does not require evidence for a core deficit as a diagnostic criterion for a SLD in arithmetic but rules out MLD if a developmental disorder (such as ADHD), poor intelligence or poor education is the cause of the impairment in mathematics. Consequently, secondary MLD should not be diagnosed as an SLD in mathematics.

For the families of affected children, the question of how to treat the problems is of course much more important than that of any terminology. However, it is important to know that the treatment possibilities for ADHD (behavioural therapy, medication and possibly also neurofeedback therapy) and for internalizing disorders such as anxiety or depression (psychotherapy) are in most countries not only much easier accessible than a specific learning therapy for dyscalculia but also provided for by healthcare systems (see chapter on international differences in this volume).

The most widely and for decades successfully used pharmaceuticals for the treatment of ADHD are based on methylphenidate (MPH). MPH shows general positive effects on performances in standardized mathematics tests (for an overview, see Elia, Welsh, Gullotta, & Rapoport, 1993, and Lindsay et al., 1999). Furthermore, an early and successful ADHD treatment leads to a good prognosis if a hasty and impulsive performance style was the cause for most errors in the numerical domain (von Aster, 2000). It has also been shown that treatment of math anxiety alleviated poor performance in mathematics (Hembree, 1990; Ramirez & Beilock, 2011). To this end, treating the underlying causes of secondary MLD might suffice.

On the other hand, primary MLD can only be diminished by an individualized learning therapy and not by any medication or psychotherapy known of. In most countries, parents will have to pay for an individualized learning therapy privately as there are no public resources. In some countries (such as Belgium), the school system is supposed to take care of all children with MLD. In some other countries (such as Germany), there are very limited respective public resources. In these cases, too many false-positive diagnoses may systematically prevent children with primary MLD from getting the help they need if parents cannot afford to pay for it privately.

Recapitulatory, treating the unspecific causes of secondary MLD first would not only prevent many affected children from the need for an often lengthy and costly learning therapy but also save limited respective public resources for children with primary MLD if available.

What Are Difficulties for a Respective Differential Diagnosis?

As outlined above, choosing an optimal treatment for MLD depends on knowledge about the underlying causes. However, it is not an easy task to differentiate between primary dyscalculia with a comorbid ADHD and a secondary MLD because of ADHD (or anxiety, depression, etc.). There are rather several difficulties complicating this issue.

Firstly, the high comorbidity rates between developmental disorders in general due to their shared genetic causes (Moreno-De-Luca et al., 2013) lead to a high number of children with attentional and working memory deficits even in the group with primary MLD.

Secondly, the development of numerical cognition and arithmetic abilities relies to some extent on executive functions, which are themselves related to attentional processes (for a meta-analysis, see Friso-van den Bos, van der Ven, Kroesbergen, & van Luit, 2013). Recent findings suggest that even tasks considered as cognitive marker for a numerical core deficit such as the accuracy of nonsymbolic magnitude representation are influenced by children's capacity for inhibition (Bugden & Ansari, 2015; Gilmore et al., 2013).

Another problem is that tasks for testing a numerical core deficit such as magnitude comparison of dots are usually developed for research questions only, based on reaction times, and not standardized for clinical use. Some dyscalculia tests based on neuropsychological models include number line tasks or estimation tasks, which should also tap a core deficit in number magnitude processing. However, the respective subtests are mostly comprised of only very few items or show other constructive deficits and thus suffer from poor reliability (Krinzinger & Günther, in press). The existence of a numerical core deficit in a child would provide strong evidence for a primary MLD (even if not required for the clinical diagnosis). However, practising psychologists usually do not have the means for testing this.

Furthermore, standardized dyscalculia tests are often not constructed to optimally tap specific numerical deficits. For example, one of the most widely used tests in German speaking countries is the ZAREKI-R (Von Aster, Weinhold Zulauf, & Horn, 2009). In this test, primary school children are asked to solve orally (and not visually) presented additions, subtractions and word problems. Children with working memory deficits frequently do not remember the respective numbers correctly and score much better if the same items are presented to them visually later on. What is more, the ZAREKI-R includes two obligatory working memory tasks, which account for 24 of overall 119 possible raw score points. To this end, it is very likely that this test (and other similarly constructed ones) is highly sensitive to domain-general working memory deficits and does not have the desired specificity for primary MLD.

Last but not least, almost nothing is known so far about possible qualitative performance differences (such as different error types) between children with primary MLD and children with secondary MLD. Many more studies have investigated calculation error types and strategies in children with ADHD without MLD. As these children present with average mathematical performance, any problems they have must be considered unspecific and should not be regarded as conclusive cognitive marker for primary MLD. The next section will describe the respective findings.

Which Error Types Are Not Specific to Primary MLD?

It is well-known that even adults with ADHD have difficulties with written multi-digit calculations (Seidman, Biederman, Weber, Hatch, & Faraone, 1998). In children with ADHD, inattention has been shown to predict overall accuracy, the number of systematic procedural errors (i.e. the same procedural error type made at least twice), and math fact errors in this task (Raghubar et al., 2009).

One of the most comprehensive studies concerning the question of calculation error types and strategies in 7–11-year-old children with ADHD but average mathematical ability was conducted by Benedetto-Nasho and Tannock (1999). Compared to healthy controls matched for age, IQ, and mathematical ability, ADHD children used more immature strategies including finger counting. Finger counting reduces the working memory load during calculation (Costa et al., 2011), explaining its higher occurrence in ADHD children. The above-described group also made significantly more trading errors, with the majority of errors involving a misunderstanding of the concept of borrowing. Most of these errors resulted from consequently subtracting the smaller number from the larger number, irrespective of its position in the calculation problem (i.e. 688–259 = 431; "subtract small from large").

Another study in 115 8–10-year-old children born preterm showed that they had worse mathematical abilities than control children without any deficits in magnitude representation or other basic numerical abilities (Simms et al., 2015). The specific deficits in this group lay – similar to the study cited above – mostly in the more frequent use of immature strategies such as counting even for easy arithmetic problems, as well as in deficits in counting forwards and backwards in the higher number ranges if tens, hundreds or thousands had to be crossed (i.e. 2995–3004, 325–317). This group also had lower visual-spatial abilities and a lower working memory span compared to control children. For both deficits, the group differences vanished as soon as they were controlled for visual-spatial and working memory skills. Consequently, the experimental group could be categorized as presenting with secondary MLD.

Interestingly, the MLD subgroup with the highest number of comorbid ADHD and often presenting with a hasty, impulsive working style cited above (von Aster, 2000) also exhibited multi-digit counting errors and fact retrieval deficits as most prominent problems.

More direct evidence for specific error types or immature strategy use being secondary to attentional and working memory deficits comes from studies investigating respective MPH impact. MPH-induced improvements in the mathematical domain have been shown for general improvement concerning speed and accuracy; less use of finger counting in multi-digit subtraction (both: Benedetto-Nasho & Tannock, 1999); higher number of correct additions and subtractions in a certain amount of time, especially including carry procedures (Elia et al., 1993); faster and less error-prone fact retrieval for addition (Carlson, Pelham, Swanson, & Wagner, 1991); and faster and more accurate fact retrieval for all four basic arithmetic operations (Douglas, Barr, O'Neill, & Britton, 1986). Even more specific, a study in small, but very well-controlled, groups of children with ADHD without any mathematical deficits, with mathematical deficits and with comorbid primary dyscalculia showed that MPH had a general positive effect on subtractions with borrow procedure, but no impact on the deficit in easy, small subtraction in the group with comorbid dyscalculia (Rubinsten, Bedard, & Tannock, 2008). The authors explained these findings with a positive effect of MPH on working memory, which is the most important factor for the performance in subtractions including borrowing. On the other hand, a deficit in small subtractions should be caused by a numerical core deficit, which will not be improved by MPH.

In summary, the following calculation error types and strategies are consistently found in ADHD children without MLD and therefore should not be considered as specific for dyscalculia: (i) finger counting and immature counting strategies; (ii) problems with crossing tens, hundreds and thousands in counting; (iii) fact retrieval deficits; and (iv) inaccurate trading procedures in addition and subtraction, including an error type called “subtract small from large” and others implying insufficient conceptual understanding of the trading procedures.

Objectives of the Current Study

As described so far, the differential diagnosis between a primary MLD with comorbid ADHD (or working memory deficits due to, i.e. an anxiety disorder) and a secondary MLD is a very difficult task. Most importantly, we are missing clear cognitive markers which are specific to primary MLD on the one hand and usable by clinicians on the other hand.

To this end, the present study was conducted to analyze differences in qualitative error patterns between children and youths with primary versus secondary MLD. The main goal was to identify error types which are made significantly more often by children with primary MLD. Another aim was to validate a previously suggested clinical cut-off (Krinzinger, 2016) for the discrimination between primary and secondary MLD using the German dyscalculia test Basis-Math 4–8 (Moser-Opitz et al., 2010; see also below).

Materials and Methods

The current study consists of ex post facto analyses of a clinical population tested for dyscalculia at the Child Psychiatry Department of the RWTH Aachen University Hospital in Germany. The analysis of the convergent and discriminant validity of the suggested clinical cut-off in Basis-Math scores as well as differences in error patterns made by the resulting separated groups will be presented in this chapter.

Participants

Data of all children and youths tested between 2011 and 2015 as inpatients or outpatients at the Child Psychiatry Department of the RWTH Aachen University Hospital in Germany with the dyscalculia test Basis-Math 4–8 (see below) because of the suspicion of MLD were eligible. Most of them had never been tested for dyscalculia. They were included if they attended a regular secondary school and both an IQ measure (HAWIK-IV, the German version of the WISC-IV, Petermann & Petermann, 2007) as well as a protocol of self-corrections (see below) were available.

Exclusion criteria were an unpathological Basis-Math result (>72 raw score points), below-average IQ (<85), undertaking a dyscalculia learning therapy in the past and changing the language of math lessons due to moving to Germany from the Netherlands or Belgium. The resulting sample consisted of 51 children (39 girls) of which only 4 had been diagnosed with dyscalculia according to ICD-10 criteria (Dilling, Mombour, & Schmidt, 1993: rather used in Germany but very similar to the DSM-V criteria for SLD in arithmetic) in the past.

Consisting of a psychiatric clinical population, all of them had psychiatric comorbidities (all classified according to ICD-10 criteria). The highest numbers of comorbidities were found for ADHD ($n = 31$; with 9 children taking MPH at the time of the assessment) and depression ($n = 36$). Twenty children suffered from some anxiety disorder, 17 from a somatization disorder and 20 of any other psychiatric disorder and/or absence epilepsy. A comorbid reading and spelling disorder was present in five children and an isolated reading disorder in only two children.

Assessment

The Basis-Math 4–8 (Moser-Opitz et al., 2010) was in 2011 the only available dyscalculia test in Germany for children and youths in Grade 7 or higher. It usually takes no longer than 45 minutes and does not pose high demands on working memory capacity, as all problems are presented in written format and can be solved using written calculation, if necessary. The Basis-Math was constructed in regard to the content of the ICD-10 criteria for dyscalculia, namely, deficits in the basic arithmetic operations and numerical understanding and not in higher domains such as fractions or other topics of secondary school mathematics. The maximum overall raw score is 83. Children achieving at or above 73 raw score points are considered unpathological, raw score points between 72 and 68 constitute a tolerance zone and children achieving below 67 raw score points are considered to have MLD. The test authors reported high sensitivity but an insufficient specificity of only 42% in a German sample using the strict cut-off (Moser-Opitz et al., 2010). As described above, the published part of the current study (Krinzinger, 2016) suggested a more specific cut-off of ≥50 Basis-Math raw score points for secondary MLD, a tolerance zone of 45–50 and only children with a resulting score of less than 50 as being candidates for primary MLD. To this end, a cut-off of ≥50 in the Basis-Math was used to differentiate a group with secondary MLD and a group with possible primary MLD for the current study.

For the number line task used in a large subsample, children were asked to mark where 20 numbers should go on respective horizontal lines (all 10 cm long) with the endpoints 0 and 1000. The sequence of the items was 500, 3, 432, 600, 287, 400, 743, 800, 173, 300, 95, 200, 314, 822, 989, 700, 565, 900, 651 and 100 for all children. The fits for the linear and the logarithmic functions were calculated for each participant in EXCEL, with their difference used as a measure for the exactness of their number magnitude representation in this number range. A negative number in this difference score means a better fit of the logarithmic function, which is considered a valid marker for an immature number magnitude representation and is typically found in second graders, but not any more in fourth graders (Booth & Siegler, 2006).

Error Categories

Calculation error types were coded for all (complete and incomplete) additions, subtractions, multiplications, divisions and word problems of the Basis-Math 4–8 (Moser-Opitz et al., 2010; see above).

The following error types could occur in all problems: counting error (+/− 1 or 2), trading error (+/− 10, 100 or 1000), counting and trading error combined (+/− 9, 11, 90, 110, 900, 1100), split-5 error (+/− 5 or "a full hand"; see Domahs, Krinzinger, & Willmes, 2008), wrong procedure (i.e. additions instead of multiplication or subtraction), complex procedural (more than one procedural error per item but other than counting and trading combined) and "don't know" as answer.

Error types occurring only in subtraction were "subtract small from large" (i.e. 688–259 = 431; see Benedetto-Nasho & Tannock, 1999), resulting in a solution larger than the minuend and zero as result.

Error types only possible in multiplications and divisions were table errors. Furthermore, due to the choice of item sequence (18:2, 180:2, 108:2; 160:4, 160:40), wrong generalizations (i.e. 160:4 = 4 ➔ 160:40 = 40) or consecutive errors (i.e. 18:2 = 6 ➔ 180:2 = 60) were possible for divisions.

Only in the problem type 100,000–100 error types were possible which were only explainable by a faulty understanding of the decimal system. These errors were comprised of too few digits (i.e. 100,000–100 = 9.900) and too many digits (i.e. 100,000–100 = 999,900) or started with the digit 1 (i.e. 100,000–100 = 100,900) in the calculation results.

Calculation errors which did not fit into any of these types were categorized as "unidentifiable".

Counting errors were categorizable within six types (multiple coding possible): omissions (all three problems), early stop (all three problems), parity change (counting in twos), change of the unit digit to zero (counting in tens backward), change to any other unit digit (counting in tens backward) and change counting hundreds to counting thousands or tens (counting in hundreds).

Analyses

The convergent and discriminant validity analyses of the suggested clinical Basis-Math cut-off score for the discrimination between secondary MLD (sMLD) and possible primary MLD (pMLD) was conducted using Pearson correlations between the Basis-Math overall scores, the differences between the fit for the linear and the logarithmic function for the number line estimates as a measure for convergent validity, and the IQs as a measure for discriminant validity. Furthermore, group comparisons for these two measures were carried out between these two groups (using T-tests) as well as between the three groups of children without ADHD, unmedicated ADHD children and medicated ADHD children (univariate ANOVA) to support the interpretation of convergent and discriminant validity.

Chi-square tests were carried out for the analyses of different distributions of error types between pMLD and sMLD. In order to foster the specificity of respective differences, chi-square tests for the analyses of different distributions of error types between a group of children without ADHD, a group of unmedicated ADHD children and a group of medicated ADHD children were carried out as well.

Results

Descriptive Statistics

The descriptive statistics concerning age, grade, IQ, Basis-Math overall raw score (BM score) and the difference between the fit of the linear and the logarithmic function of the number line estimates can be obtained from Table 24.1.

Convergent and Discriminant Validity of the Postulated More Specific Clinical Cut-Off

As expected, the Basis-Math overall scores (BM scores) correlated significantly with the differences between the fit for the linear and the logarithmic function of the number line estimates ($n = 34$, $r = 0.415$, $p = 0.015$; see Fig. 24.1 for the respective scatterplot), but not with IQs ($n = 51$, $r = 0.254$, $p = 0.073$; see Fig. 24.2 for the respective scatterplot).

The two experimental groups differed significantly in the number line difference score (pMLD, mean = 0.15, sd = 0.11; sMLD, mean = 0.20, sd = 0.04; $t = 2.16$, $p = 0.039$), but not in IQ (pMLD, mean = 94.9, sd = 10.2; sMLD, mean = 98.3, sd = 8.3; $t = 1.21$, $p = 0.231$). On the other hand, the three groups of children without ADHD, unmedicated ADHD and medicated ADHD did not differ significantly in these two measures or in the Basis-Math overall score (all $F < 0.81$, all $p > 0.45$).

Table 24.1 Descriptive statistics (means, standard deviations, minima, maxima) for age, grade, IQ, Basis-Math overall scores (BM score) and differences between the fit of the linear and the logarithmic function of the number line estimates (Fit lin-log) for the clinical sample ($n = 51$)

	Mean	Standard deviation	Minimum	Maximum
Age	13.16	2.072	10	18
Grade	7.35	1.787	5	11
IQ	97.47	8.812	85	123
BM score	55.45	12.960	21	71
Fit lin-log	0.18	0.08	−0.06	0.26

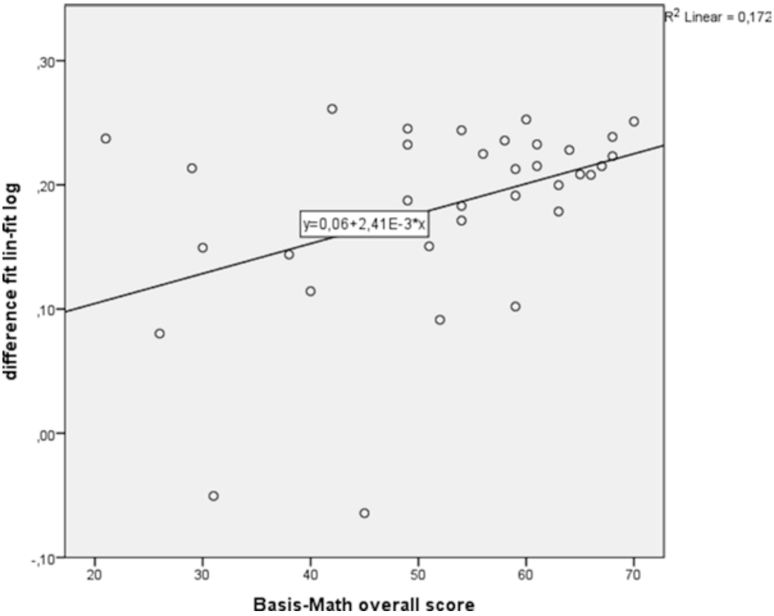

Fig. 24.1 Scatterplot of Basis-Math overall scores (*x*-axis) and difference between the fit of the linear and the logarithmic function for number line estimates (*y*-axis; $n = 34$)

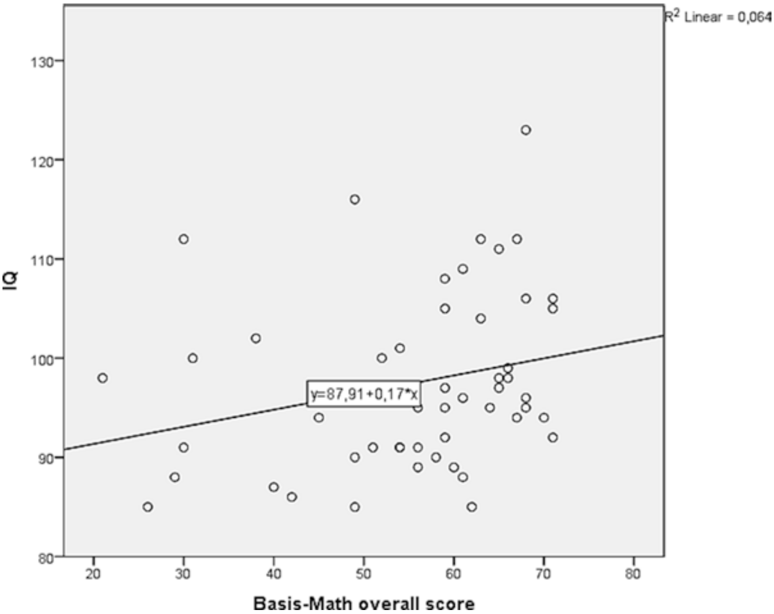

Fig. 24.2 Scatterplot of Basis-Math overall scores (*x*-axis) and IQ scores (*y*-axis; $n = 51$)

Furthermore, only two children who had a BM score of less than 50 showed a better fit of the logarithmic function (negative difference score), thus presenting with an immature magnitude representation for numbers in the range 0–1000.

Differences in Calculation Error Types Between Secondary and Possible Primary MLD

For six calculation error types, significant differences in their frequencies between children with secondary MLD compared to possible primary MLD could be observed (percentages of children per group *not* making the respective error type, χ^2 and p-levels can be obtained from Table 24.2).

The use of a wrong procedure (mostly addition instead of multiplication) and wrong generalizations occurred in more than half of the children with possible primary MLD (pMLD) and in a minority (not more than 16%) of children with secondary MLD (sMLD).

Unidentifiable errors (which fit in no other category) were made by 77% of pMLD and 63% of sMLD children. If the two items eliciting the most unidentifiable errors (1000/8 and a word problem comprising the calculation 7.2/3) were excluded from this analysis, this error type was still made by a majority of pMLD (54%) but only by a minority of sMLD children (16%; $\chi^2 = 19.9$, $p = 0.001$).

Table 24.2 Relative number (%) of children with a BM score of <50 points (possible primary MLD/pMLD) vs. ≥50 (secondary MLD/sMLD) who never made specific calculation error types, ordered by significance between differences in frequencies (*p*-levels for chi-square tests)

Error category	% of pMLD ($n = 13$) not making error type	% of sMLD ($n = 38$) not making error type	χ^2	*p*-level
Wrong procedure	39	90	14.6	0.001
Unidentifiable	23	37	16.5	0.011
Decimal understanding	23	61	11.2	0.024
Wrong generalization	46	84	7.4	0.025
Counting error	31	55	9.1	0.028
Complex procedural	0	13	13.8	0.032
Zero as subtraction result	92	100	3.0	0.084
Split-5 error	85	97	2.8	0.092
“Don’t know”	31	50	9.3	0.097
Consecutive error	69	87	2.1	0.150
Table error	62	71	3.1	0.216
Trading error	54	71	7.0	0.224
Counting and trading error combined	92	92	3.9	0.267
“Subtract small from large”	92	84	0.5	0.464
Result larger than minuend	92	95	0.1	0.748

Errors only explainable by a faulty understanding of the decimal system ("decimal understanding") and counting errors also occurred in more than half of the children with pMLD but in less than half of the children with sMLD. All children with pMLD but only 87% of the children with sMLD made complex procedural errors (more than one procedural error per item, excluding counting and trading error combined).

The distributions of three error types were marginally significantly different between the two groups: no child from the sMLD group came up with zero as a subtraction result, but one child from the pMLD group did. Split-5 errors (exactly 5 of a correct addition or subtraction result) occurred in 15% of the children with pMLD but only in 3% of the children with sMLD. A "don't know" answer was given by more than two thirds of the children with pMLD but only by half of the sMLD children.

The distributions for consecutive errors, table errors, trading errors, counting and trading errors combined, the error type "subtract small from large" and subtractions for which the result was larger than the minuend (for error category descriptions, see above) were not significantly different between the two groups.

No significant differences for any calculation error-type distribution were found between children without ADHD ($n = 20$), unmedicated ADHD ($n = 22$) and medicated ADHD ($n = 9$) children.

Differences in Counting Error Types Between Secondary and Possible Primary MLD

Out of six possible error categories which were made during the three counting items (multiple coding was possible), only two were distributed differently between the two groups (percentages of children per group *not* making the respective error type, χ^2 and p-levels can be obtained from Table 24.3).

Table 24.3 Relative number (%) of children with a BM score of <50 points (possible primary MLD/pMLD) vs. ≥50 (secondary MLD/sMLD) who never made specific counting error types, ordered by significance between differences in frequencies (*p*-levels for chi-square tests)

Error category	% of pMLD ($n = 13$) not making error type	% of sMLD ($n = 38$) not making error type	χ^2	*p*-level
Change of unit digit to other digit (excluding zero)	69	100	12.7	<0.001
Change counting hundreds to tens or thousands	54	90	7.9	0.019
Parity change	69	87	2.1	0.150
Early stop	85	92	3.0	0.225
Change of unit digit to zero	100	97	0.3	0.555
Omission	62	58	0.1	0.818

Changing the unit digit from 7 to another digit (excluding zero) while counting backwards in tens occurred in 21% of the pMLD group, but in no child of the sMLD group. Changing from counting hundreds to counting tens or thousands happened to almost half of the pMLD children but only 10% in the sMLD group.

No significant differences were found for the distributions of parity change in the counting in twos item, early stops, changing of unit digit to zero in the counting in tens backwards item and omissions.

No significant differences for any error-type distribution were found between children without ADHD (n = 20), unmedicated ADHD (n = 22) and medicated ADHD (n = 9) children.

Discussion

The current study had two main aims. First, a previously suggested clinical cut-off of 50 raw score points (Krinzinger, 2016) for the dyscalculia test Basis-Math 4–8 (Moser-Opitz et al., 2010) which does not provide standardized norms was validated in a clinical population study. Furthermore, in the same sample, qualitative error analyses for calculation and counting error types were carried out comparing two groups differentiated by this cut-off as having secondary MLD vs. possible primary MLD. The results will be summarized and discussed in the following sections.

Validation of the Postulated Clinical Cut-Off for the Basis-Math Overall Score

In a large subgroup of the current sample (34/51 children), a number line task was presented to children in which they had to provide estimates for numbers between 0 and 1000. For the resulting estimates, the fit for the linear and the fit for the logarithmic function were calculated in each child. The respective differences were used as a measure for the children's number magnitude representations, with higher values meaning a relatively better fit for the linear (exact) function and values smaller than one indicating a better fit of the logarithmic (immature) function.

As expected, this difference measure correlated significantly with the Basis-Math overall score, and the two experimental groups (secondary MLD vs. possible primary MLD) differed significantly in their respective values. Furthermore, only two children presented with a negative value in this difference measure, and both of them belonged to the group with possible primary MLD. These results confirmed the convergent validity of the postulated cut-off, as indicators for a numerical core deficit were related to (and only observed in) the group with possible primary MLD.

On the other hand, neither did the Basis-Math overall score correlate significantly with IQ nor did the two experimental groups differ significantly in their

IQs. Furthermore, the number line difference measure was not different between the three groups of children without ADHD, unmedicated ADHD and medicated ADHD. These results confirmed the discriminant validity of the postulated cut-off.

To this end, the discrimination between children with secondary MLD (sMLD) caused by attentional or working memory deficits (due to ADHD and/or depression or other psychiatric disorders) with at least 50 raw score points in the Basis-Math and children with a possible primary MLD (pMLD) caused by a numerical core deficit seems to be valid.

Specific and Unspecific Error Types

One of the main results of this study is the confirmation of various error types not being specific for primary MLD but also frequent in ADHD children without any differences in mathematics (Benedetto-Nasho & Tannock, 1999; see also 3.7.1.5 for an extensive overview). Unspecific calculation error types which have already been described before were table errors in multiplication, trading errors (+/− 10, 100, 1000), counting and trading errors combined (+/− 9/11, 90/110, 900/1100), and the error type "subtract small from large" (e.g. 688–259 = 431). Children with secondary problems in mathematics have also been found to present with problems in counting in steps. Respective unspecific errors in the current study were a parity change in the counting in twos item, early stops, changing of unit digit to zero in the counting in tens backwards item and omissions. These should be regarded as most likely being caused by attention and/or working memory deficits while carrying out calculation or counting procedures (procedural errors) and not by a deficient understanding of numbers.

On the other hand, changing the unit digit while counting backwards in tens (except to zero) was only observed in the pMLD group, and changing the unit of counting (i.e. from hundreds to tens or thousands) was significantly more frequent in the pMLD group. Regarding calculation errors, zero as a subtraction result was only observed in one child of the pMLD group. Furthermore, complex procedural errors (more than one procedural error per item, excluding counting and trading errors combined), counting errors, errors only to be explained by a faulty understanding of the decimal system (e.g. 100,000–100 = 9.900; 100,000–100 = 999,900; 100,000–100 = 100,900), wrong generalizations which were possible due to a specific item sequence (i.e. 160:4 = 4 ➔ 160:40 = 40), using the wrong calculation procedure (mostly addition instead of multiplication) and errors fitting in none of the described categories or unidentifiable errors were significantly more frequent in the pMLD group. Most of these error types can only be ascribed to deficits in conceptual understanding of calculation procedures or of numbers themselves (conceptual errors).

These findings are a first step in bridging the long-vacant gap concerning cognitive markers to distinguish primary from secondary MLD. In sum, conceptual errors which cannot be explained by deficits in attention or working memory should be considered specific for primary MLD. On the one hand, if a child makes hardly any conceptual and mostly procedural errors (which can happen to anyone in certain situations like fatigue), clinicians should suspect domain-general deficits (in attention and/or working memory, caused by ADHD or a psychiatric disorder like depression or anxiety) to lie at the core of the deficits in mathematics.

Limitations of This Study

This study was conducted out of a clinical motivation, namely, to distinguish between secondary school pupils with primary and secondary MLD with only a dyscalculia test at hand which does not provide standardized norms (luckily, this situation has changed in the meantime in German-speaking countries). This means that the small sample size, its clinical nature, the utilized test itself, the ex post facto study design and the fact that no primary school children were included in this study pose serious threats as to how far it is possible to generalize the current results. In this spirit, the current study is only seen as a means to raise awareness for the important and difficult task clinicians have with the differential diagnoses of dyscalculia and/or ADHD and/or math anxiety (or other psychiatric disorders causing a working memory deficit) and as first steps towards the identification of cognitive markers helping in this task.

Conclusions

Whenever there is a choice of different treatment options for children and youths struggling with mathematics, it is of highest importance to understand the respective reasons in each individual case. There is a broad consensus in the field that the only promising treatment option for children with primary MLD (presenting with a core deficit in number magnitude representation) is an individualized learning therapy. In the absence of a core deficit, for most children with an average IQ who have difficulties in mathematics, their problems stem from domain-general deficits like inattention or working memory deficits. The most frequent reasons for these are again either ADHD or internalizing psychiatric disorders like depression or (math) anxiety. It has been shown that both treating ADHD (Elia et al., 1993; Lindsay et al., 1999; von Aster, 2000) and math anxiety (Hembree, 1990; Ramirez & Beilock, 2011) alleviated poor performance in mathematics. To this end, if mathematical difficulties are secondary to another problem, this should always be treated first. Furthermore, if you are testing a child for dyscalculia and suspect ADHD and/or an internalizing psychiatric disorder (even if as a comorbidity to primary MLD), you should not be afraid to initiate a respective thorough assessment and provide parents with treatment options if necessary and possible.

However, the results of a dyscalculia test alone are often insufficient to disentangle the possible reasons for a respective bad outcome. A thorough case history and behavioural observation during the test are always mandatory in such cases, but may not be enough either. As assessments testing a possible numerical core deficit are usually lacking clinical practice (but see above for a number line task suitable for secondary school pupils), qualitative error analyses are a way to provide information needed for the differential diagnoses in question.

The present study was carried out in a clinical sample of secondary school pupils as a first step in this direction. The main results were on the one hand that a group with possible primary MLD did not differ from a group with secondary MLD (due to ADHD or internalizing psychiatric disorders) in a variety of procedural errors (e.g. trading errors) and in multiplication table errors. Anyone may be prone to such error types in situations like fatigue, which is also known to diminish attention and working memory. These results confirm earlier studies analyzing the behaviour of ADHD children without MLD (e.g. Benedetto-Nasho & Tannock, 1999).

On the other hand (and more importantly), several error types which can only be explained by faulty conceptual understanding of calculation procedures or the decimal system of numbers were made significantly more often by children with possible primary MLD.

To this end, the more conceptual errors a pupil makes in basic arithmetic tasks, the higher is the need for an individual learning therapy.

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Chapter 25
Working Memory and Mathematical Learning

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Introduction

An increasing number of students show severe mathematical difficulties. Between 5% and 10% of children and adolescents experience a substantial learning deficit in at least one area of mathematics (Barbaresi, Katusic, Colligan, Weaver, & Jacobsen, 2005). The identification of these mathematical difficulties is fundamental if we consider the negative widespread drawbacks determined by math difficulties. Basic mathematical skills are regularly used in everyday life, and their deficiency affects both employment opportunities and socio-emotional well-being. In addition, results of recent studies show how mathematical abilities predict financial and educational success, particularly for women (Geary, Hoard, Nugent, & Bailey, 2013). It is therefore crucial to promote an early identification of children at risk for mathematical learning difficulties at preschool level and develop effective evidence-based mathematics curricula considering all the cognitive processes involved in the development of mathematical skills.

In the last decades, various studies investigated the cognitive factors, defined as *precursors*, that underlie the development of mathematical abilities. The identification of these cognitive markers of mathematical learning plays a key role in the early identification of children that may develop math difficulties and disabilities. Competencies that specifically predict mathematical abilities, such as digit recognition, magnitude understanding, and counting, may be considered domain-specific precursors. General cognitive abilities, such as working memory, processing speed, and intelligence, which may predict performance not only in mathematics but also

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in other school subjects, can be considered as domain-general precursors (Passolunghi & Lanfranchi, 2012).

In this chapter, we will not discuss in detail the domain-specific precursors of mathematical learning, but we will focus on a key general precursor, the working memory, and its influence on the mathematical learning processes.

Working Memory (WM): A Domain-General Precursor of Mathematical Learning

Domain-general cognitive abilities such as memory, attention, or processing speed are important precursors of school learning. Of all these general cognitive skills, several studies demonstrated that working memory is a key predictor of mathematical competence. The term "working memory" (WM) refers to a temporary memory system that plays an important role in supporting learning during the childhood years because its key feature is the capacity to both store and manipulate information. Various models of the structure and function of working memory exist, but in the present chapter we will refer to the multicomponent model of working memory proposed by Baddeley and Hitch in 1974 and revised in succeeding years (Baddeley, 2000). Baddeley's models consist of three main parts. The two "slave" systems of working memory (i.e., the phonological loop and visuospatial sketchpad) are specialized to process language-based and visuospatial information, respectively. The central executive, which is not modality-specific, coordinates the two slave systems and is responsible for a range of functions, such as the attentional control of actions. The distinction between the central executive system and specific memory storage systems (i.e., the phonological loop and visuospatial sketchpad) in some way parallels the distinction between the working memory, involving storage, processing, and effortful mental activity, and the short-term memory, typically involving situations in which the individual passively holds small amounts of information (Swanson & Beebe-Frankenberger, 2004).

In this multicomponent model, the central executive is responsible for control and regulation of cognitive processes in which executive functions are involved. Miyake et al. (2000) identified three main executive functions in working memory: inhibition, updating, and shifting. Inhibition involves the ability to suppress dominant responses, shifting involves the ability to shift strategies when attending to multiple tasks or mental processes, and updating involves the ability to replace outdated and irrelevant information by maintaining only a restricted set of elements in working memory.

More recently, Baddeley (2000) added a fourth component to his model, the episodic buffer, which is a limited-capacity system that both integrates and provides temporary storage of information from the two subsystems and long-term memory. Developmental research related to this fourth component is very limited, so in this chapter we will focus on the first three components of Baddeley's working memory model.

Verbal short-term memory is traditionally assessed using tasks that require the participant to recall a sequence of words (e.g., word span task forward) or numbers (e.g., digit span task forward). On the other hand, tasks such as the visual pattern test are designed to assess visuospatial short-term memory. In the visual pattern test, participants are presented with matrix patterns of black and white squares and are required to memorize patterns of increasing complexity. All these tasks designed to assess short-term memory skills require individuals to recall a sequence of verbal or visual information in the same format of presentation. Differently, working memory capacity is reliably assessed by dual tasks in which the individual is required to store and, at the same time, process increasing amounts of information. An example of verbal working memory tasks is the listening span task (Daneman & Carpenter, 1980). Participants are presented with an increasing number of sentences, they are required to judge whether the sentences are true or false, and then at the end of each set, they have to recall the last word of each of the sentences of the set.

A long-standing body of research suggests that there is a direct influence of working memory on mathematical skills (De Smedt et al., 2009; Passolunghi, Mammarella, & Altoè, 2008; Passolunghi, Vercelloni, & Schadee, 2007). Longitudinal studies show that working memory performance assessed in preschool years predicts mathematical achievements several years after kindergarten (Gathercole, Brown, & Pickering, 2003; Mazzocco & Thompson, 2005). These results support the hypothesis that working memory is a distinct and significant correlate of early numerical abilities. However, the same cannot be said of either verbal or visuospatial short-term memory (Passolunghi, Lanfranchi, Altoè, & Sollazzo, 2015). Indeed, there is substantial evidence for separating the involvement of short-term and working memory as correlates of mathematical learning (Shah & Miyake, 2005; Swanson, 2006), with active working memory skills having an essential influence on early numerical abilities and later mathematical performance. Indeed, even the simplest mathematics calculations require WM processes: temporary storage of problem information, retrieval of relevant procedures, and processing operations to convert the information into numerical output. These same processes are needed even for simple number comparison tasks: the child needs to map the different number symbols onto the corresponding quantities, store them into memory, and then integrate this with the incoming information to performing the task.

Despite the growing evidence that WM plays a fundamental role in the development of mathematical abilities, there is still an absence of shared consensus about the relative extent of the involvement of domain-specific and domain-general precursors in the development of mathematical abilities. Some authors, for example, highlight the importance of domain-specific precursors such as the approximate number system (ANS) in the development of mathematical learning (Halberda, Mazzocco, & Feigenson, 2008). The ANS is an innate system, which allows the manipulation of quantities and magnitudes in an approximate way. A typical example of ability underlying ANS consists in approximately estimating computation results or in comparing two or more sets of elements to identify, without counting, which could be the most numerous. The involvement of ANS in mathematical learning is nevertheless very much debated. Indeed, while some studies account for its significant role, many others do not. Moreover, while some authors report deficits associated with ANS in children with or at risk for mathematical learning disability, others highlighted impairments in making comparisons between quantities, but only when quantities are represented by symbols and not when using nonsymbolic, approximate numerosities. In order to further investigate the relation between domain-specific and domain-general precursors of mathematical development, we conduct a wide assessment of memory components and domain-specific factors, such as the ANS (Passolunghi, Cargnelutti, & Pastore, 2014). A large sample of first grade typically developing children was tested at both beginning and end of their Grade 1. Both general (working memory and intelligence) and specific (ANS) precursors were evaluated by a wide battery of tests and put in relation to concurrent and subsequent mathematical skills. Results demonstrated that working memory and intelligence were the strongest precursors in both assessment times. ANS had instead a milder role, which lost significance by the end of the school year. Some authors argue that the relationship between ANS performance and mathematics achievement may in fact be an artefact of the WM (inhibitory control) demands of some trials of the numerosity comparison task (e.g., Gilmore et al., 2013; Soltész, Szűcs, & Szűcs, 2010).

Contribution of WM Components to Mathematical Learning

With regard to the contribution of the three core components of working memory to the development of mathematical skills, many studies showed a direct association between executive function and children's early emergence and development of mathematical abilities across a wide age range. For example, dual-task studies suggest that central executive resources are implicated in children's arithmetic performance (e.g., Imbo & Vandierendonck, 2007), and longitudinal data found that inhibitory control predicted later mathematical outcomes (Blair & Razza, 2007; Mazzocco & Kover, 2007). On the other hand, children who are poor in mathematics also have a poor performance in central executive tasks, especially in tasks that require the inhibition of irrelevant information and updating (Passolunghi, Cornoldi, & De Liberto, 1999; Passolunghi & Siegel, 2001, 2004; St Clair-Thompson & Gathercole, 2006).

Spatial skills and visuospatial working memory were also found to be related to children's early counting ability and general mathematical competence (e.g., Passolunghi & Mammarella, 2012). Indeed, the visuospatial sketchpad appears to support the representation of numbers in counting, arithmetic calculations, and especially mental calculation (McKenzie, Bull, & Gray, 2003). This component is also fundamental in the process of problem-solving, because it allows the individual to build a visual mental representation of the problem (Holmes & Adams, 2006). Moreover, visuospatial WM abilities assessed in the preschool years predict complex arithmetic, number sequencing, and graphical representation of data in primary school (Bull, Espy, & Wiebe, 2008).

The results of studies that considered the role of the phonological loop in children's mathematical processing have been unclear. Dual-task studies showed that 8–9-year-old children (but not younger children) use a verbal approach supplemented by visuospatial resources during online arithmetic performance (McKenzie et al., 2003). In the field of learning disabilities, some studies found no differences in phonological loop abilities between children with and without mathematical difficulties, especially when differences in reading ability were controlled (Passolunghi & Siegel, 2001, 2004). Other authors suggest that the phonological loop is involved in basic fact retrieval (Holmes & Adams, 2006).

The role of each working memory component in mathematical cognition must be considered to vary with expertise and development (Meyer, Salimpoor, Wu, Geary, & Menon, 2010), with an increasing involvement of the phonological loop in mathematical cognition from the age of 7 onward (Rasmussen & Bisanz, 2005).

Working Memory, Word Problems, and Calculation

One of the main goals of mathematical education is to develop students' ability to solve mathematical word problems. This ability is important both for academic success and for problem-solving in everyday life. However, mathematical word problem solution is very demanding and difficult for many students.

In the school setting, mathematical word problems are typically presented as a short story that includes relevant numerical information, the "problem data," and a question (e.g., John bought 4 pizzas with 8 slices each. He and his friends Bruce ate 12 slices of the pizzas. How many slices were left?). The solution of the problem requires the use of arithmetic operations (i.e., addition, subtraction, multiplication, or division) and the execution of several different cognitive processes. Initially, in the understanding phase, children must formulate a cognitive representation of the information drawn from the text of the problem. This initial cognitive representation requires discriminating relevant from irrelevant information. Subsequently, in the solution phase, they need to formulate a plan for solving the problem. Devising a plan involves choosing appropriate sub-goals for the solution and consequently includes the choice of the correct arithmetic operations and algorithms. Finally, they have to correctly perform the calculations.

A more strict focus on word arithmetic problem-solving suggests that working memory can be critically involved even when the written text is still available. Indeed, text comprehension requires incoming information to be integrated with previous information maintained in the working memory system. Furthermore, the complete comprehension of the problem requires that the solvers build up a mental representation of the problem, which involves the capacity of the working memory system. According to Baddeley's three component model, the central executive is probably more specifically and strongly involved in this process than the articulatory loop. In fact, problem-solving does not simply involve the maintenance of given information, but it requires its control, i.e., that this information is examined for relevance, selected or inhibited according to its relevance, integrated, used, and so on. Baddeley (1990) also suggested that reading comprehension involves the central executive more than the articulatory loop. This suggestion seems to apply even more to written word arithmetic problem-solving which requires not only text comprehension but also additional operations on it.

Arithmetic calculation is another important academic skill that children learn when they start formal education. Basic addition skills are fundamental milestones for the development of multiplications skills and increasingly complex arithmetic abilities. The substantial body of research focused on identifying the cognitive processes that underlie arithmetic calculation stresses once again the important role played by working memory. For instance, to perform an addition (e.g., 13 + 9) without being able to use a pen and paper, we must temporary retain the phonological representations of the numbers. The next step would be to employ one or more procedures (e.g., counting) to combine the numbers and produce an answer. Alternatively, employing carrying or regrouping strategies involves maintaining recently processed information while performing other mental operations. First of all, we have to retain the 2 from adding 3 + 9. Next we add the 1 from the tens column of the 13 to the 1 from the tens column of the 12 produced from adding the 3 + 9. Finally, we would need to add the products held in working memory, resulting in the correct solution.

These examples show clearly how the cognitive processes involved in performing arithmetic calculations are embedded within the working memory system. For instance, even the simplest mathematics calculations require the temporary storage of problem information, retrieval of relevant procedures, and processing operations to convert the information into numerical output. Studies have also shown that the different working memory components (e.g., visuospatial sketchpad, phonological loop, and central executive) play specialized and unique roles in arithmetic calculation (Geary, Hoard, Byrd-Craven, Nugent, & Numtee, 2007). Higher working memory capacity is associated with higher accuracy in solving complex arithmetic problems in adults as well as in children. In particular, children with higher working memory abilities tend to use more sophisticated strategies such as decomposition instead of less sophisticated strategies such as finger counting (Geary, Hoard, Byrd-Craven, & DeSoto, 2004).

Research studies in this field emphasize that the central executive plays a greater role in mental calculation compared to the phonological loop (e.g., De Rammelaere, Stuyven, & Vandierendonck, 2001). In particular, the phonological loop plays a major role when calculation involves storing temporary information, whereas carrying operations put a major demand on the central executive processes (Fuerst & Hitch, 2000). Only a limited number of studies examined the role of the visuospatial component of the WM model. These studies showed that visuospatial WM is related to performance in written calculation. In particular, it is important during the initial stages of arithmetic calculation for encoding arithmetic problems presented visually.

Executive Functions of Central Executive Component of WM and Their Role in Mathematics

Within the Baddeley's WM model, the functions of the central executive can be fractionated into at least three separate functions: inhibition, updating, and shifting (Miyake et al., 2000). Executive processes, and in particular inhibition, appear to be particularly important for successful solutions of mathematical word problems (Passolunghi & Siegel, 2001, 2004). Previous research has demonstrated a strong relationship between inhibitory processes and reading comprehension. Specifically, children with reading disabilities perform poorly on working memory tasks that require inhibition of irrelevant information (Chiappe, Hasher, & Siegel, 2000). These findings show how poor comprehenders' performance on working memory tasks is impaired because they are unable to inhibit irrelevant information adequately. The negative consequence of this situation is an overload of their working memory capacity.

The ability to inhibit irrelevant information is also related to the success in problem-solving tasks. Indeed, in both text comprehension and problem-solving, it is necessary to process a great number of information units. Some of these must be rejected in order to maintain only those that are relevant. In particular, in the problem-solving process, the integration of the relevant information into a coherent structure allows a correct and complete mental representation of a text of problem. Passolunghi and Siegel (2001, 2004), for example, found that poor problem-solvers had a deficit in their ability to reduce accessibility of nontarget and irrelevant information (see Passolunghi et al., 1999). These findings are compatible with Engle's (2002) suggestion that individual differences in working memory capacity are not related to how many items can be stored in memory but in the difference in ability of controlling attention and maintaining information in an active, quickly retrievable state. Moreover, he argues that attentional control is related to inhibitory deficits, that is, individuals who have difficulty maintaining attentional focus on the task-relevant information are likely to make intrusion errors.

Another executive function associated with the central executive is the updating of information. Updating is a complex activity that requires attributing different levels of activation to the items presented and maintaining a restricted set of elements activated continuously. A typical measure of updating ability is Morris and Jones' updating task (Morris & Jones, 1990), which requires participants to listen to several lists of letters of varying length (4 to 10). Participants are asked to recall only the final four letters of each list. Since the length of each series is unknown, they are forced to update the information maintained in their WM continuously in order to remember the final four letters only. Updating skills are involved in resolving arithmetic word problems. Indeed, in order to understand word problems, children have to process all information derived from texts. Some information will be inhibited very early because it is not relevant to the solution. Other information will be connected in a coherent model that will be enriched successively by new information. This model will be complete when all the information relevant to solving the question has been integrated. Further information concerning other questions will then be processed and structured in different models. In short, a child who has to update information during a problem-solving task has to select relevant information, to inhibit information already processed but no longer relevant, and to substitute the no longer relevant information with a new one (Passolunghi & Pazzaglia, 2004). Shifting from one model to another requires individuals to update information in working memory, in a fine modulation of the mechanisms of enhancement and inhibition (Passolunghi & Pazzaglia, 2004).

It is widely assumed that updating processes are important also in calculation, in particular during the early development of arithmetical skills. Indeed, arithmetic calculation requires the storage and manipulation of intermediate results, by updating the results of operations such as carrying and borrowing. In line with this view, research shows that children with low updating skills had a poorer performance in solving word problems and calculation, compared to children with higher updating skills (Passolunghi & Pazzaglia, 2004). Recent studies show that updating deficits in children with ADHD may be a further source of their difficulty when solving mathematical problems (Re, Lovero, Cornoldi, & Passolunghi, 2016). In this respect, their impairment in the ability to update information should not differ from the updating difficulties of children with learning difficulties in mathematics showing difficulties in the recall of relevant information and controlled use of problem procedure.

Another executive function is the ability to shift back and forth between multiple tasks, operations, or mental sets. Among the typical complex tests usually used in cognitive and neuropsychological studies to assess executive function, the Wisconsin card sorting task (WCST) involves testing of the shifting processes. The WCST requires matching a series of target cards, presented individually, with any one of four reference cards. The participants are aware that the sorting criterion would change during the task, but they are not explicitly told the exact number of correctly sorted cards to be achieved before the criterion shifts. This test is often conceptualized as a set-shifting task because of its requirement to shift sorting categories after a certain number of successful trials. It is worth noting that some researchers view this task as requiring inhibitory control to suppress the current sorting category before switching to a new one. There is very little research on shifting and mathematical ability, and further research is necessary to clarify this issue. Bull and Sherif (2001) found that the WCST percentage of perseverative responses was negatively correlated with mathematical ability in typically developing primary school children. That is, children with higher mathematics ability made a lower percentage of perseverative responses in this task. These results suggested that the main difficulty for children with mathematical learning difficulties in performing arithmetic tasks is to inhibit a learned strategy and to switch to a new one.

Interestingly, the results of Espy et al. (2004) showed that shifting or mental flexibility did not contribute to mathematical skills in preschool children. They assessed shifting ability by tasks that require rule-based learning and shifting (e.g., spatial reversal task), similar to WCST. It is possible that mental flexibility may contribute more to mathematical abilities in older children, allowing the child to flexibly apply different mathematical procedures in problem-solving and calculation (e.g., borrowing, carrying) to obtain correct mathematical solutions.

Working Memory Training

As extensively described in the previous paragraphs, results from most of the studies conducted up to date suggest working memory abilities influence children's performance in mathematical achievement. Indeed, different mathematical tasks, such as performing mental arithmetic and understanding mathematical word problems, require the storage of information, while it is being processed or integrated with information retrieved from long-term memory. Given the important role played by WM abilities in the development of children's mathematical skills, in the last 15 years, different studies have explored whether mathematical learning problems can be overcome by training specifically designed to enhance working memory. WM was traditionally considered a genetically fixed cognitive ability (Kremen et al., 2007). Therefore, in the past the possibility to enhance WM skills by acting on an individual's environmental experiences and opportunities was not considered. Recently, a growing set of studies with children with typical development and adults has shown that WM skills can be improved through training (e.g., Alloway, Bibile, & Lau, 2013; Kroesbergen, van't Noordende, & Kolkman, 2014; St Clair-Thompson, Stevens, Hunt, & Bolder, 2010).

The debate regarding the effects of WM training is still open: some studies show positive effects of WM training on arithmetic abilities in primary school children using computerized or school-based training procedures (Alloway et al., 2013; Dunning, Holmes, & Gathercole, 2013; Holmes, Gathercole, & Dunning, 2009; St Clair-Thompson et al., 2010). Other authors questioned the effectiveness of WM training concluding that there is no convincing evidence of the generalization of working memory training to other skills (Melby-Lervåg & Hulme, 2013). However, the possibility should be considered that cognitive training applied to younger individuals tends to lead to significantly more widespread transfer of training effects (Wass, Scerif, & Johnson, 2012).

Holmes et al. (2009) provided the first evidence of the efficacy of the computerized "Cogmed" training in overcoming common impairments in working memory and associated learning difficulties in 10-year-old children with low working memory skills. They proposed different training tasks that involve the temporary storage and manipulation of either sequential visuospatial information, verbal information, or both. Children in the training group engaged in the Cogmed program for 35 min a day, for at least 20 days in a period of 5–7 weeks. The majority of the children who completed the program improved on tasks tapping the central executive and the visuospatial sketchpad components of WM. Moreover, a significant increase in mathematics performance assessed with the mathematical reasoning subtest of the Wechsler objective number dimensions (WOND; Wechsler, 1996) was also found, 6 months after the training. St Clair-Thompson et al. (2010) showed the effectiveness of a computerized working memory training ("Memory Booster") in typically developing children aged 5–8 years. The computer program used teaches memory strategies to children, over a period of 6–8 weeks, and resulted in significant improvements in tasks that assess the phonological loop, the central executive, mental arithmetic, and following instructions in the classroom. Enhancing mathematical abilities in 9- to 10-year-old typically developing children is also possible using individual school-based working memory training (Witt, 2011). The WM training program developed by Witt (2011) was carried out over a period of 6 weeks, the children in the intervention group were seen individually, and each training session lasted approximately 15 min. This study suggested that children who underwent working memory training made significantly greater gains in the trained working memory tasks, as well as on an untrained visuospatial working task, compared to a matched control group. Moreover, the training group also made significant improvements in mental arithmetic.

Only a few studies have explored the possibility of enhancing working memory abilities in kindergartners using a specific working memory training. In a recent study (Passolunghi & Costa, 2016), the authors of this section systematically investigated the effects of a training program focused on the enhancement of working memory and a second training program focused on the enhancement of early numeracy. The participants were 48 5-year-old typically developing preschool children. Both the working memory and early numeracy training programs were implemented for 5 weeks, twice weekly, each session lasting 1 h. The working memory training included different paper-and-pencil tasks designed to enhance all three components of Baddeley's working memory model (Baddeley, 1986). On the other hand, the early numeracy training included different paper-and-pencil tasks designed to enhance early numerical abilities such as counting, number-line representation, one-to-one correspondence between quantities and numerals, and quantity comparison. The results of this study showed that the early numeracy intervention specifically improved early numeracy abilities in preschool children. On the other hand, the working memory intervention improved not only verbal and visuospatial working memory abilities but also general early numeracy skills assessed with the early numeracy test (Van Luit, Van de Rijt, & Pennings, 1994). Interestingly, the early numeracy gain obtained in the working memory training group did not differ significantly from the gain obtained in the early numeracy training group. These findings stress the importance of performing activities designed to train working memory abilities, in addition to activities aimed to enhance more specific skills in order to support mathematical development. This kind of activities could be particularly important for those children who are considered to be at risk for developing learning disabilities later on in life. Indeed, WM training seems to be effective in improving math performance also in young children with low early numeracy abilities. The results of a study by Kroesbergen et al. (2014) showed that preschoolers with low numerical skills who participated in a working memory intervention program for 4 weeks significantly improved their working memory and early numeracy skills. The training program consisted of eight 30-minute sessions with hands-on activities, which were implemented in small groups of five children. The positive results of this study suggest that WM training activities can be used with low-performing preschool children, in order to minimize the future learning difficulties that result from WM deficits.

Conclusion

Individual differences in working memory capacity appear to have a strong influence on children's ability to acquire knowledge and new abilities. The great importance of WM in a range of cognitive skills including mathematics has been supported by different studies (see Cowan & Alloway, 2008). Moreover, several researches corroborate a network view of mathematical abilities where domain-general cognitive abilities as working memory sustain the development of mathematical abilities over and above the role of more domain-specific abilities (e.g., Geary, 2011; Passolunghi et al., 2014; Szűcs, Devine, Soltesz, Nobes, & Gabriel, 2014). In addition, training studies support the same view and suggest that timely action to prevent children from developing early difficulties in mathematical learning should focus both on domain-specific variables, such as number competence, and on more general abilities.

The hypothesis that WM training should improve not only working memory but will also have a transfer effect on early numeracy skills is supported by studies dealing with WM training and transfer effects on math abilities in primary school children and kindergarten (Alloway et al., 2013; Holmes et al., 2009; Kuhn & Holling, 2014; Passolunghi & Costa, 2016; St Clair-Thompson et al., 2010). However, the possibility that the role of working memory training could vary with development should be considered. Most of the studies investigating the effects of WM training focused on school-aged children, while only a few studies have explored the possibility of enhancing working memory (and related early numeracy abilities) in younger children. It is entirely possible that the effects of WM training might be stronger in younger children when the neural system is more malleable to experience (Wass et al., 2012). These results regarding the positive effects of WM training could have interesting implications for classroom practice in preschool and primary school. Performing hands-on activities as well as computerized training tasks designed to boost WM performance may help children to improve cognitive precursors fundamental in future school learning, encouraging the prevention of learning difficulties. Future research should focus on the investigation of the effects of WM training in children who are considered to be at risk for developing learning disabilities. In fact, these kinds of WM training activities could be particularly appropriate for low-performing children in order to minimize the future learning difficulties that result from WM deficits.

One final consideration regards the role played by emotional and motivational aspects in mathematical cognition (Cargnelutti, Tomasetto, & Passolunghi, 2016). Despite the clear importance of domain-general and domain-specific cognitive precursor of mathematical learning, future studies should consider a unitary a complete model which includes also emotional factors such as math anxiety. Even if emotional factors have clearly a potential (negative or positive) impact on a child's development of math skills, very few studies tried to provide a global profile, including both cognitive and emotional factors. This important and topical issue will be further discussed in Chap. 3 of this book.

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Chapter 26
The Relation Between Spatial Reasoning and Mathematical Achievement in Children with Mathematical Learning Difficulties

Ilyse Resnick, Nora S. Newcombe, and Nancy C. Jordan

Introduction

Although it is now widely recognized that mathematics learning difficulties (MD) stem from a range of general- and domain-specific cognitive competencies (Jordan et al., 2013; Rousselle & Noel, 2007), many investigators stress the primacy of basic weaknesses in understanding number and number relations (Clarke & Shinn, 2004; Jordan, Fuchs, & Dyson, 2015; Mazzocco & Thompson, 2005). For example, it is argued that core deficits in numerical magnitude understanding underpin dyscalculia (e.g., Butterworth, 1999, 2005; Butterworth & Reigosa-Crespo, 2007; Landerl, Bevan, & Butterworth, 2004). Dyscalculia is a severe type of MD that occurs across social classes and language groups (e.g., Butterworth, 1999, 2005). At the behavioral level, children with dyscalculia perform much more poorly compared to their typically developing peers on tasks requiring them to identify which of two numerals is larger and to map symbols to quantities (Butterworth & Reigosa-Crespo, 2007; Landerl et al., 2004; Rousselle & Noel, 2007). At the neural level, individuals diagnosed with dyscalculia have lower gray matter density in the intraparietal sulcus (Rotzer et al., 2008; Rykhlevskaia, Uddin, Kondos, & Menon, 2009), an area associated with numerical magnitude reasoning (Pinel, Dehaene, Rivière, & LeBihan, 2001).

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Positing a core number deficit, however, may underplay the potential role of cognitive processes in allied domains, either as allied deficits or as reservoirs of strength. One such domain is spatial reasoning. Spatial reasoning is broadly defined as the ability to mentally manipulate and understand the spatial relations between and within objects. Evidence from typically developing children reveals a close connection between spatial reasoning and mathematics achievement (for review, see Mix & Cheng, 2012). The relation between spatial reasoning and mathematics achievement across development can be characterized as consistent, predictive, and strengthening over time. Spatial reasoning is consistently correlated with mathematics performance. Across a range of measures, adults and children with stronger spatial reasoning perform better at mathematics tasks (e.g., Ansari et al., 2003; Burnett, Lane, & Dratt, 1979; Casey, Dearing, Vasilyeva, Ganley, & Tine, 2011; Delgado & Prieto, 2004; Geary, 2000; Lubinski & Benbow, 1992; Robinson, Abbott, Berninger, & Busse, 1996). The connection between spatial reasoning and mathematics performance is evident even among young children (Geary & Burlingham-Dubree, 1989; Geary, Hoard, Byrd-Craven, Nugent, & Numtee, 2007; Gunderson, Ramirez, Beilock, & Levine, 2012; Lachance & Mazzocco, 2006; Rasmussen & Bisanz, 2005; McKenzie, Bull, & Gray, 2003; Holmes, Adams, & Hamilton, 2008; Alloway, 2007).

Spatial reasoning also predicts later mathematics outcomes while controlling for a range of variables; at age 3 spatial reasoning uniquely predicts 27% of the variance in mathematics skills at age 4 (Verdine, Irwin, Golinkoff, & Hirsh-Pasek, 2014) and remains predictive of later high school mathematics achievement (Wolfgang, Stannard, & Jones, 2003). In two longitudinal studies, one spanning over 50 years and following approximately 400,000 people (Wai, Lubinski, & Benbow, 2009) and the other spanning 20 years and following approximately 600 people (Shea, Lubinski, & Benbow, 2001), spatial reasoning predicted entry into STEM (science, technology, engineering, and mathematics) careers.

Finally, the relation between spatial reasoning and mathematics outcomes grows stronger over the course of development (Stannard, Wolfgang, Jones, & Phelps, 2001; Voyer, Voyer, & Bryden, 1995). For example, early block play (a kind of spatial reasoning, assessed using the Lunzer Five-Point Play Scale) was increasingly associated with mathematics performance over the intermediate grades (Wolfgang et al., 2003). In light of the strong and persistent links between spatial reasoning and mathematics in typical development, this chapter considers how the two domains might be related in students with MD.

In exploring the linkage, it is important to keep in mind that both spatial reasoning and mathematics are multidimensional constructs (Mix et al., 2016; Mix & Cheng, 2012). Spatial reasoning is a broad term that encompasses many kinds of distinct spatial skills. For example, being able to imagine an object rotating (mental rotation) is a dissociable skill from being able to imagine the viewpoint from different orientations (perspective taking; Hegarty & Waller, 2004) and being able to imagine an object breaking (mental brittle transformation; Resnick & Shipley, 2013). In a recent meta-analysis of spatial training, Uttal et al. (2013) suggest that the broad suite of individual spatial reasoning skills can be categorized along two basic dimensions: intrinsic (spatial relations among internal components of an object) versus extrinsic (spatial relations between two or more objects) and dynamic (the relation is moving) versus static (the relation does not move). For example, mental rotation involves dynamic intrinsic spatial reasoning (imagining how the internal structure changes as the object moves), whereas perspective taking involves dynamic extrinsic spatial reasoning (imagining how the relation between objects changes as you move to a different location in space). Spatial working memory, the ability to hold spatial information actively in mind, supports spatial reasoning by providing a limited capacity mental workspace to move and manipulate information. See Fig. 26.1 for a conceptualization of spatial reasoning as a two-by-two grid and its relation to spatial working memory.

Similarly, mathematics performance can be thought of as a collection of separable, but interconnected, skills (Fuchs et al., 2010). For example, different patterns of numerical skills and general cognitive abilities support solving word problems versus calculation problems (Fuchs et al., 2010). There are also distinct cognitive processes involved in enumerating small sets (i.e., four or less) versus larger sets (e.g., Dehaene & Cohen, 1994). An important broad distinction is that mathematical skills can be categorized in terms of procedural knowledge versus conceptual

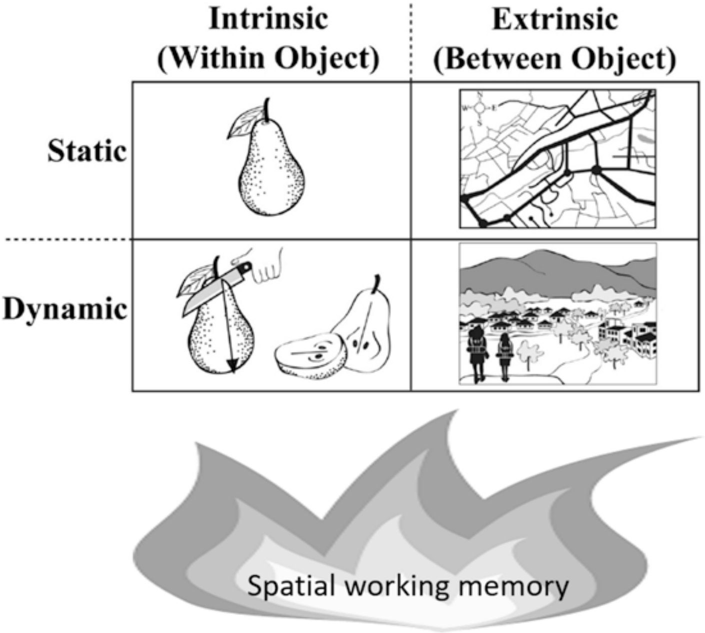

Fig. 26.1 Spatial cognition is comprised of dissociable spatial skills that can be characterized as related to object manipulation (coding of the spatial structure of static objects and transformation of those relations) or to navigation (coding of the spatial structure of the environment and transformation of that structure). Spatial reasoning is supported (or fueled) by spatial working memory. (Adapted from Newcombe, 2018)

knowledge (Rittle-Johnson, Siegler, & Alibali, 2001). Procedural knowledge refers to knowing a sequence of actions (i.e., algorithm or computation) to complete a problem. Conceptual knowledge refers to understanding mathematics principles and how they relate to one another.

Given that spatial reasoning and mathematics performance are both comprised of somewhat dissociable skills, it should be no surprise that the strength of the relation between spatial reasoning and mathematics performance varies by task (e.g., Holmes et al., 2008; Mix et al., 2016). There are likely to be many different casual mechanisms and/or shared cognitive processes that connect individual spatial and mathematics skills (Mix & Cheng, 2012). For example, subtraction problems require spatial working memory, but multiplication problems do not (Lee & Kang, 2002). Even more specifically, working memory is required only when subtraction problems involve carrying (Caviola, Mammarella, Lucangeli, & Cornoldi, 2014).

How does spatial reasoning relate to numerical knowledge in students with MD? To address this question, we begin with a review of research with typically developing children and then consider spatial reasoning skills and spatial working memory in children with MD. In the final section, we explore educational implications of the evidence, focusing on a specific mathematics area – numerical magnitude. We suggest number line activities (i.e., spatial representations of magnitudes) to foster a connection between spatial reasoning and mathematical skills in children with MD.

Numerical Magnitude and Spatial Reasoning in Typically Developing Children

Reasoning about numerical magnitude is supported by a cognitive representation referred to as the mental number line (e.g., de Hevia & Spelke, 2010; Dehaene, Bossini, & Giraux, 1993; Dehaene, Izard, Spelke, & Pica, 2008; Pinel, Piazza, Le Bihan, & Dehaene, 2004). Typically developing children begin with a "compressed" representation of whole number magnitudes, dedicating relatively more space on their mental number line to smaller values and relatively less space to larger values (Siegler & Booth, 2004; Siegler & Opfer, 2003). With increasing age, children develop an increasingly linear representation for an increasingly wider range of whole numbers (Siegler & Lortie-Forgues, 2014; Siegler, Thompson, & Schneider, 2011). Eventually, they learn magnitudes of non-integer quantities, such as fractions and decimals. In fact, fraction magnitude estimation appears to be crucial skill; it is more predictive of later mathematics achievement than whole number magnitude estimation when controlling for general cognitive processes (Bailey, Hoard, Nugent, & Geary, 2012; Resnick et al., 2016; Siegler et al., 2011, 2012; Siegler & Pyke, 2013). Children initially represent all fractions as being less than one, irrespective of the relation between the numerator and denominator (Resnick et al., 2016). Most students sequentially develop an accurate representation of fractions with numerators smaller than the denominator (e.g., 2/3) and then fractions with numerators larger than the denominator (e.g., 3/2).

Children with MD are not as accurate in estimating whole number (Geary et al., 2007; Piazza et al., 2010; Van't Noordende & Kolkman, 2013) and fraction (Jordan, Resnick, Rodrigues, Hansen, & Dyson, 2016) magnitudes compared to their typically developing peers. For both whole numbers and fractions, children with MD rely more heavily on initial representations: "compressed" for whole numbers (Geary, Hoard, Nugent, & Byrd-Craven, 2008) and "less than one" for fractions (Resnick et al., 2016). Evidence from whole number line estimation studies suggests that core deficits in inhibition underlie difficulties understanding numerical magnitude in children with MD (Geary et al., 2008; Szucs, Devine, Soltesz, Nobes, & Gabriel, 2013) and in typically developing children (Friso-van den Bos, Kolkman, Kroesbergen, & Leseman, 2014; Kroesbergen, Van der Ven, Kolkman, Van Luit, & Leseman, 2009). Under this account, the executive control is required to inhibit children's initial "compressed" representation. Indeed, children with MD often have broad deficits in inhibition (Geary et al., 2007; McLean & Hitch, 1999).

Developing a linear representation of numerical magnitude involves a spatial reasoning skill called spatial visualization, which is the ability to mentally manipulate 2D, 3D, and 4D objects (including mental rotation and mental brittle transformation). A longitudinal study by Gunderson et al. (2012) examined the relation between the mental number line (assessed by the number line task), mathematics performance (assessed by an approximate symbolic calculation task), and spatial visualization (assessed by mental rotation) in a typically developing population. Number line estimation acuity at age 6 mediated the relation between spatial visualization at age 5 and mathematics performance at age 8. These findings suggest that spatial visualization supports representation of an accurate mental number line, which, in turn, supports mathematics performance. Spatial visualization may assist students in representing and comparing numerical magnitudes on their mental number line.

Spatial Reasoning in Children with MD

Although the question has yet to be fully explored, research suggests that students with MD may have similar levels of spatial reasoning skills as their typically developing peers (Butterworth, 1999, 2005; Landerl et al., 2004). For example, 9- to 10-year-old children with dyscalculia (defined here as falling under the 16th percentile on a standardized mathematics test and average on other standardized tests, such as reading and vocabulary) and typically developing children performed similarly on a test of mental rotation but not on tests of spatial working memory and inhibitory control (Szucs et al., 2013). Additionally, a study following 226 children from kindergarten through third grade found that overall performance on the *Developmental Test of Visual Perception—Second Edition* (DTVP-2) in kindergarten is not predictive of MD (defined here as having scores under the tenth percentile on the TEMA-2 and WJ-R Calculation) in grades two and three (Mazzocco & Thompson, 2005). The DTVP-2 includes items assessing understanding positions in space, disembedding (identifying shapes within more complex designs), visual closure skills (identifying an object with only partial information), and matching shapes (Hammill, Pearson, & Voress, 1993). However, more studies are needed to confirm specific areas of spatial strengths as well as potential weaknesses in children with MD. For example, while no differences were observed between typically developing 9- to 10-year-old children and children with MD on mental rotation (Szucs et al., 2013), other spatial visualization skills have not been assessed. Spatial visualization skills such as mental folding, form board, and block design have all been found to predict mathematics performance (see Mix et al., 2016 for review) and should be investigated more fully in children with MD.

Children with MD do appear to have weaknesses in spatial working memory relative to typically developing children (e.g., Geary, 2004; Hitch & McAuley, 1991; Keeler & Swanson, 2001; Passolunghi & Siegel, 2001). As noted earlier, spatial working memory is the ability to hold spatial information actively in short-term storage and functions as a mental workspace to hold and manipulate numbers. A meta-analysis (Swanson & Jerman, 2006) shows that children aged between 9 and 10 years old with MD (defined here as children with average intelligence that fall below the 25th percentile on a given mathematics assessment) have lower spatial working memory compared to typically developing students as well as to students with other kinds of learning difficulties (i.e., reading). Importantly, while their spatial working memory may be impaired, some children with MD seem to have relatively strong verbal working memory (Andersson & Ostergren, 2013; McLean & Hitch, 1999). This suggests general working memory impairments may not underlie MD but, rather, more specific weaknesses in *spatial* working memory.

Spatial Training to Support Children with MD

Spatial reasoning and mathematics achievement are related in the typically developing population. However, as noted, children with MD seem to have relatively strong spatial skills, with the exception of spatial working memory, which potentially can be leveraged to support their mathematics learning. For example, MD interventions could map content onto spatial and visually represented tools (e.g., the number line) to support understanding. However, learning activities for children with MD should also help students compensate for weaknesses in spatial working memory and inhibitory control. Below we discuss studies showing positive effects of spatial training on mathematics learning in the general population, which includes students with MD. We then focus on numerical knowledge, in particular, and discuss the use of number line activities to help students connect numerical magnitudes with spatial representations.

Relatively large bodies of evidence that suggest both spatial reasoning (Uttal et al., 2013) and number competencies (Frye et al., 2013) are malleable and can be developed in all or most children (including children with MD). Unfortunately, there are relatively few studies that assess the effects of specific types for spatial training on mathematics learning (Stieff & Uttal, 2015). One study, however, found that a single spatial training session on mental rotation improved accuracy in mathematics calculation in 6- to 8-year-olds relative to their peers who completed crossword puzzles (Cheng & Mix, 2014). In the mental rotation training session, children had to rotate two parts of a geometric object to identify which of four pictures showed the object as a whole. Improvements were associated with better performance on calculation problems with missing terms (e.g., 3 + ___ = 10). Spatial training may support performance on missing term problems, because the problems involve mentally moving values around the equal sign. However, a study extending Cheng and Mix's work to include training over a 6-week period found opposing results (Hawes, Moss, Caswell, & Poliszczuk, 2015). In this study, 6- to 8-year-old students either completed iPad games requiring the rotation of shapes or literacy training. The iPad training involved identifying which of four response options showed a target object rotated (and not its mirror image) and then completing puzzles that require the rotation of pieces. There were no observed differences between the spatial and literacy conditions on calculation problems with missing terms. Because Hawes et al. assessed mathematics performance 3 to 6 days after completing the program, compared to Cheng and Mix who assessed mathematics performance immediately after the training session, these contrasting results may raise questions about the durability of the effects of spatial training at a delayed posttest. However, three key differences between the two studies may also account for the conflicting findings. Hawes et al. used a game-based format for spatial training, had children rotate familiar shapes (e.g., animals and letters), and differentiate between rotated vs. mirrored images. Cheng and Mix used more direct spatial training format rotating geometric shapes. Consequently, it is not clear if the spatial visualization requirements were equivalent in the two studies.

Another study (Lowrie, Logan, & Ramful, 2017) examined the effects of 20 h of spatial training on mathematics achievement in 10- to 12-year-old children. Spatial training consisted of a variety of spatial visualization, mental rotation, and spatial orientation tasks. For example, students drew maps, read inverted maps, differentiated between mirrored and rotated images, practiced 2D rotation around a point and 3D rotation of objects, and so on. In a control condition, students completed a standard course curriculum. Students in both conditions were pretested within 2 weeks prior to starting the intervention and posttested within 2 weeks of completing the intervention. Students in the spatial training condition exhibited higher scores on a mathematics achievement test compared to their peers who did not receive the training. Using curricula emphasizing spatial skills in elementary school (Cunnington, Kantrowitz, Harnett, & Hill-Ries, 2014; Hawes, Moss, Caswell, Naqvi, & MacKinnon, 2017) and engaging younger kindergarten and first-grade students in spatial play (Grissmer et al., 2013) have also led to improved mathematics performance.

Although the above findings are sometimes conflicting, taken together, they suggest a potentially causal relation between spatial training and mathematics achievement. Thus, spatial training may help children with and without MD improve their mathematics achievement, although much more research needs to be conducted in this area before clear conclusions can be drawn.

Spatial training in the context of teaching mathematics is likely to be fruitful for students with MD. For example, given the central role of magnitude understanding in learning mathematics, the visual number line may be a key representational tool in developing core number concepts (Gunderson et al., 2012). Visual number line activities generally require young children to move left to right along a horizontal number line (e.g., 1 to 10) as part of a board game. Visual number line activities may be particularly effective for connecting spatial and numerical skills, including developing a mental number line that encompasses rationale numbers as well as integers. Visual number lines may also support spatial working memory, which, as noted previously, is a barrier for many children with MD (e.g., Swanson & Jerman, 2006).

Visual number line activities support understanding of whole number magnitudes in preschoolers (Ramani & Siegler, 2008; Whyte & Bull, 2008), as well as understanding of fraction magnitudes in older students (Fuchs et al., 2013, 2014; Saxe et al., 2007). Such activities encourage children to name numbers, count, and compare magnitudes (Ramani & Siegler, 2008). They provide a visual experience of placing numbers in a linear representation. Thus, students can see that each number is one more than the previous one and, for example, that the difference between one and nine is larger than the difference between four and six.

Numerical magnitude knowledge on the number line appears to be malleable through training. Only four 15-minute sessions with a linear board game led to improved and lasting understanding of whole number magnitudes among low-income preschoolers (Ramani & Siegler, 2008), and two 30-minute sessions with number lines improved symbolic fraction magnitude understanding among typically achieving second- and third-grade students (Hamdan & Gunderson, 2017). Intensive instructional approaches centered on the number line also improve fraction concepts in fourth graders with MD (Fuchs et al., 2014).

Conclusions

There is a robust and tight connection between elements of spatial reasoning and mathematics achievement. While children with MD appear to show weaknesses in some spatial processes (i.e., spatial working memory), they do not in others (i.e., mental rotation). Further investigation needs to confirm these results and to examine a wider range of spatial processes in relation to mathematics outcomes in children with MD. Relative strengths in some areas of spatial reasoning may be underutilized resources for fostering mathematics learning in students with MD. However, children with MD may need additional scaffolds to support their relative weaknesses in spatial working memory and inhibitory control. Further studies should address how visual/spatial tools, such as the number line, can reduce spatial working memory demands and encourage inhibitory control. Nevertheless, spatial representational tools, such as the visual number line, may be particularly helpful for children with MD to develop magnitude knowledge.

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Chapter 27
The Language Dimension of Mathematical Difficulties

Susanne Prediger, Kirstin Erath, and Elisabeth Moser Opitz

In this chapter (which is a slightly modified version of Prediger, Erath, & Moser Opitz, 2018), we briefly (1) report on theoretical backgrounds and empirical studies showing strong connections between language factors and mathematics achievement (as a *result* of learning), (2) explain in which way language is relevant in the *processes* of learning, and (3) present instructional approaches for enhancing students' language proficiency for supporting the learning of mathematics.

Language Factors on Different Levels and Their Connection to Mathematics Achievement

Many empirical studies have shown that students' mathematical difficulties are often tightly connected to language factors (Secada, 1992). But what exactly does *language* mean in these contexts? Whereas some researchers mainly refer to reading difficulties, other studies identify language factors on word, sentence, and text/discourse level, and all of them can contribute to mathematical difficulties.

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Differences Between Everyday and Academic Language on Word, Sentence, and Text/Discourse Level

Language gaps concern not only multilingual learners but also monolinguals who are fluent in the everyday language. This phenomenon can be explained by the difference between everyday language and academic language, which goes back to Cummins' (2000) distinction between Basic Interpersonal Communication Skills (BICS) and Cognitive Academic Language Proficiency (CALP). Whereas students from socially privileged families acquire CALP in their families, socially underprivileged or immigrant students sometimes have less access to it, although it corresponds to a crucial register (Halliday, 1974) in the context of schooling – in textbooks, exams, and learning tasks (Schleppegrell, 2004).

Researchers in education and linguistics have characterized the specificities of the school academic language similar to the technical languages of different scientific disciplines (Bailey, 2007; Jorgensen, 2011; Schleppegrell, 2004; Snow & Uccelli, 2009). Table 27.1 summarizes typical features on the word, sentence, and text level (referring to written texts) as well as the discourse level (referring to oral discourse practices and their structures beyond the sentence level; see next section).

The school academic register as well as the technical register of each subject has its own characteristics and challenges. Morgan, Craig, Schütte, and Wagner (2014, p. 845) emphasize that "language has a special role in relation to mathematics because the entities of mathematics are not accessible materially." Therefore,

Table 27.1 Typical features of the school academic and technical register (compared to everyday register)

Word level (lexical features)
Less familiar words with distinct meanings More complex word structures, nominalizations Relevance of other types of words (e.g., prepositions)
Sentence level (syntactical and morphological features)
More complex syntactical structures like prepositional phrases, more complex subordinate clauses Impersonal constructions like passive voice construction More subtle and precise use of morphological distinctions and connectives
Text level
Specific text genres which are not used in the everyday practices (e.g., protocol, report, interpretation) More explicit and more complex markers of cohesion and coherence …
Discourse level
Specific discourse practices (e.g., explaining why, arguing, etc.) with different norms of explicitness than in everyday practices. More abstract and more general talk ….

communication about mathematics requires symbols, drawings, and mathematical language and demands a high precision and abstractness, which are characterizing features of the academic language.

Disentangling Language Obstacles on Word, Sentence, Text, and Discourse Levels and Their Connection to Mathematics Achievements

Different research groups have developed different approaches to disentangle language factors and their connection to mathematics achievement: whereas some researchers exclusively refer to tests and differences on the item level (for all students or for those with mathematical learning difficulties), other researchers (mainly mathematics education researchers and linguists) also investigate the learning processes themselves (see next section).

Obstacles on the Word Level

Linguistic structure of number words Among young children, the linguistic structure of the number words has an influence on their acquisition of numbers. In a study of Miura, Okamoto, Kim, Steere, and Fayol (1993), first graders from Asian countries who spoke languages which are organized so that numerals are congruent with the base-10 number system had higher counting competences and a better understanding of the base-10 number system than children from countries with less congruent number words (e.g., France, Sweden). Moser Opitz, Ruggerio, and Wüest (2010) found similar results for Turkish-speaking kindergarten children (a language with number words congruent with the base number system) who had a higher counting competence than Italian- and Albanian-speaking children. In many languages (e.g., Arabic, Czech, Danish, Dutch, German), pupils especially struggle with the challenge of inversion by transcoding Arabic numbers from and into number words (e.g., Zuber, Pixner, Moeller, & Nuerk, 2009). In the inversion property, the order of basic lexical elements in their syntactical organization is inverted in symbolic and verbal notation (e.g., in German, the number word for 32 is “zweiunddreissig” [two and thirty], Zuber et al., 2009). Klein et al. (2013, p. 4 f.) conclude that the numerical development is moderated by language, especially with regard to two-digit numbers. Therefore, it seems to be important to explicitly discuss the irregularity of number words in the respective languages when working with young children.

Mathematical vocabulary Different categorizations of mathematical vocabulary are possible, also depending on the linguistic feature of a specific language. Riccomini, Smith, Hughes, and Fries (2015, p. 238) distinguish (for English) the following categories: “(a) meanings are context dependent (e.g., foot as in 12 inches vs. the foot of the bed), (b) mathematical meanings are more precise (e.g., product as the solution to a multiplication problem vs. the product of a company), (c) terms specific to mathematical contexts (e.g., polygon, parallelogram, imaginary number), (d) multiple meanings (e.g., side of a triangle vs. side of a cube), (e) discipline-specific technical meanings (e.g., cone as in the shape vs. cone as in what one eats), (f) homonyms with everyday words (e.g., pi vs. pie), (g) related but different words (e.g., circumference vs. perimeter), (h) specific challenges with translated words (e.g., mesa vs. table), (i) irregularities in spelling (e.g., obelus [÷] vs. obeli), (j) concepts may be verbalized in more than one way (e.g., 15 min past vs. quarter after), and (k) students and teachers adopt informal terms instead of mathematical terms (e.g., diamond vs. rhombus)."

Studies give evidence that such kind of lexical features can affect mathematics achievement. Haag, Heppt, Roppelt, and Stanat (2015) showed that increasing the difficulties of lexical features (e.g., more general and specialized academic vocabulary) in test items increases the item difficulties. Schindler, Moser Opitz, Cadonou-Bieler, and Ritterfeld (in press) found a significant correlation between students' mathematical vocabulary and arithmetical competence in a sample of fifth graders. With regard to different categories of mathematical vocabulary, the findings are inconsistent: Schindler et al. (in press) report that mathematical terms which have different meanings in the everyday language and the academic language (e.g., difference, product) and terms which are used in both languages with the same meaning (e.g., square, rectangle) seem to be especially difficult for fifth graders. In contrast, this does not seem to apply for tenth graders who seem to master polysemy without problems (Prediger, Wilhelm, Büchter, Gürsoy, & Benholz, 2018).

Obstacles on the Sentence and Text Level

On sentence and text level, several factors have an impact on mathematics learning. Reading difficulties on the sentence and text level have often been shown to influence the students' test achievement results (Abedi & Lord, 2001; Hirsch, 2003).

Syntactical complexities on the sentence level have to be taken into account for test and learning situations: complex prepositional clauses (Jorgensen, 2011), conditional clauses, the use of nominalization (e.g., double – doubling; Schleppegrell, 2004), and complex issues of cohesion on the sentence and the text level. In addition, text length and the number of noun phrases can appear as significant predictors for low achievement in mathematics for third graders (Haag et al., 2015), but not necessarily for tenth graders (Prediger, Wilhelm, et al., 2018).

However, it is important to realize that especially students with low language proficiency do not only encounter reading obstacles: A differential functioning analysis on a high-stakes test in Grade 10 has shown that all items with which students with low language proficiency had specific difficulties were items with high conceptual demands, not high reading demands (Prediger, Wilhelm, et al., 2018). Haag et al. (2015) investigated if the linguistic simplification of test items affects the mathematical performance of second-language learners. They found only a small effect with limited practical relevance. These results hint to difficulties in the learning processes rather than only test biases (see next section).

Language Factors in the Achievement of Specific Groups

Second-Language Learners

Since 25 years, research shows that language minority students often obtain lower test scores in mathematics than native speakers (Abedi & Lord, 2001; Haag, Heppt, Stanat, Kuhl, & Pant, 2013; Secada, 1992). They are often disadvantaged in school if their first language does not correspond to the language of instruction (Barwell, 2009; Schleppegrell, 2004). Whereas some researchers have investigated the language gaps between the students' home language and the language of instruction in more cognitive terms, e.g., problem of interferences between both languages for expressing number names (Krinzinger et al., 2011), other researchers have focused on cultural aspects and issues of identity and agency which apparently underprivileged language minority students more than the pure cognitive and communicative aspects (Norén, 2015; cf. Barwell et al., 2016, for an overview). That is why many researchers plead for avoiding deficit perspectives on multilingual students (cf. Barwell, 2009; Moschkovich, 2010).

Additionally, it is important to note very explicitly that the risk factor of second-language learners is not their multilingualism itself, because multilingualism can also provide cognitive benefits (e.g., shown by Cummins, 2000; Kempert, Saalbach, & Hardy, 2011; see Barwell et al., 2016, for an overview). In contrast, the major risk factor seems to be the proficiency in the language of instruction: Paetsch, Felbrich, and Stanat (2015) found a significant relationship between reading comprehension, vocabulary, and mathematics achievement of second-language learners (similarly Abedi & Lord, 2001). Also in a survey of tenth graders, the language proficiency was the factor with the strongest statistical connection to the mathematics achievement, stronger than multilingualism, immigrant status, or socioeconomic status (Prediger, Wilhelm, et al., 2018). That is why Moschkovich claims that "studies should focus less on comparisons to monolinguals and report not only differences between monolinguals and bilinguals but also similarities" (Moschkovich, 2010, p. 11).

However, second-language learners should not only be regarded as emergent language learners in the language of instruction. Their home language can be a resource which – if activated appropriately – allows a second access to mathematics (see last section).

Students with Learning Disabilities in Mathematics and Reading

Even if problems with learning mathematics are widely known, researchers have not yet been able to agree on a single definition (e.g., Gunn & Wyatt-Smith, 2011; Scherer, Beswick, Deblois, Healy, & Moser Opitz, 2016). In addition, different terms (learning disabilities, learning disorders) are used to describe the affected students, and different diagnostic approaches may lead to different results (Branum-Martin, Fletcher, & Stuebing, 2012). Without discussing this issue here, and also without discussing genetic factors (e.g., Petrill et al., 2012) or neurobiological underpinnings (e.g., Ashkenazi, Black, Abrams, Hoeft, & Menon, 2013), we use the term "learning disabilities" for referring to students with significant and long-lasting learning problems. For many years it was assumed that such problems in learning mathematics and reading are isolated impairments, and even nowadays, the ICD-10 manual defines comorbid learning disorders in mathematics and reading as a "poorly defined residual category" (Deutsches Institut für Medizinische Dokumentation und Information, 2015).

This often led to the consequence that affected students had access to support *either* in mathematics *or* in reading. However, empirical evidence shows that significant problems with reading and of mathematics often co-occur (Dirks, Spyer, van Lieshout, & de Sonneville, 2008). According to Mann Koepke and Miller (2013), 17–66% of pupils with learning disabilities in mathematics also have reading disabilities. The research from Willcutt et al. (2013) gives evidence that significant problems with reading and mathematics are distinct but related disorders that often co-occur because of shared neuropsychological weaknesses in working memory, processing speed, and verbal comprehension. Moll, Göbel, Gooch, Landerl, and Snowling (2016) found only verbal memory as a shared risk factor of pupils with reading and mathematics disorders and divergent other factors for reading impairment (slow verbal processing speed) and mathematics impairment (limitations in temporal processing, verbal, and visuospatial memory). Fuchs, Geary, Fuchs, Compton, and Hamlett (2016) conclude that pathway to calculation and word-reading outcomes are more different than alike.

These findings have important implications: first, it has to be acknowledged that a considerable part of students meets substantial problems in both domains, reading and math, and therefore needs special support. Second, it has to be considered that the different cognitive profiles of students require interventions tailored to the special needs of the affected students.

Students with Specific Language Impairment and Mathematics Learning

The relationship between specific language impairment (SLI) and mathematics development is only scarcely investigated. The few results, which are available, show that students with SLI have lower mathematical achievement in some areas, compared with other students. According to Fazio (1996, 1999), children with language impairment do understand the process of counting objects and the cardinality principle. However, they have difficulties acquiring the number sequences correctly. Other authors (Donlan, 2003, 2015; Donlan, Cowan, Newton, & Lloyd, 2007) report difficulties of SLI students in the production of number words, in calculation, and in understanding place value. In a study of Ritterfeld et al. (2013), students, who attended a special school for children with SLI and followed the normal curriculum, had a lower achievement in mental calculation than students in regular classrooms. However, they did not use more problematic counting strategies than pupils without SLI. Alt, Arizmendi, and Beal (2014) discuss multiple possible problem sources for the aforementioned difficulties of pupils with SLI: the manipulation of mathematical symbols, the use of working memory for patterns, and the combination of complex linguistic syntax plus mathematical symbols. Durkin, Mok, and Conti-Ramsden (2015) investigated the relationship of language factors and IQ in the core subjects language, science, and mathematics in a sample of students with SLI. Achievement in mathematics was predicted by IQ, but not by language factors. Other authors (Nys, Content, & Leybaert, 2013; Rhöm, Starke, & Ritterfeld, 2017) assume that deficits in working memory – especially in the phonological loop – influence the mathematics learning processes of students with SLI.

To sum up, SLI is a risk factor for a successful mathematical development, which seems to be caused by other factors than language. Schröder and Ritterfeld (2015) argue that that these SLI students are in need for qualitatively enriched interactions with their teachers for achieving a successful participation in mathematical conversations. In this way, they promote to transcend the word and sentence level and work on the discourse level also for the students with most serious language challenges.

Language Dimensions in Learning Processes

Language as a Learning Medium, Learning Prerequisite, and Learning Goal

Students with low language proficiency experience difficulties not only in test situations but – more importantly – in the learning situations themselves. This relates to the role of language as a *learning medium* in classrooms (Lampert & Cobb, 2003; Morgan et al., 2014): language in mathematics classrooms is at the same time a medium of knowledge transfer and discussion (*communicative* role of language) and a tool for thinking (*epistemic* role of language, Morek & Heller, 2012; Pimm, 1987). Research in mathematics education repeatedly emphasizes the intertwinement of language and mathematical thinking for all students, but especially for students still acquiring the language of instruction (e.g., Moschkovich, 2015). As not all students have the same level of academic language proficiency, the learning medium turns into an unequally distributed *learning prerequisite*. In order to compensate differential learning prerequisites more explicitly, language must be an explicit *learning goal.* Lampert and Cobb (2003, p. 237) stress that “Learning to communicate as a goal of instruction cannot be cleanly separated from communication as a means by which students develop mathematical understanding” linking the language as a learning goal to language in its epistemic role.

Discourse Practices as a Construct to Capture Language Demands on the Discourse Level

Whereas the lexical and syntactical features of academic language on the word and sentence level have been discussed in the first section, this section focuses on language demands on the discourse level which has been shown to be crucial for the meaningful learning of mathematics (e.g., Erath, Prediger, Heller, & Quasthoff, in press; Bailey, 2007; Moschkovich, 2015), especially for language learners.

Discourse is a construct frequently used in mathematics education (e.g., Erath et al., in press; Bailey, 2007; Barwell, 2012; Moschkovich, 2015) which is tied to different linguistic theories that can be united under the term “discourse analysis.” For example, Barwell (2012) refers to discursive psychology and conversation analysis for defining discursive demands; Moschkovich (e.g., 2015) refers to sociolinguistics in order to conceptualize academic literacy in mathematics for English learners; Erath et al. (in press) introduce the theory of Interactional Discourse Analysis in order to contribute to an empirically grounded theorization of academic language proficiency on the discourse level. Here, we refer to the definition of discourse practices from Interactional Discourse Analysis since it is compatible with other definitions used in mathematics education. In addition, it allows to differentiate between different subcategories of discourse. Explanations, arguments, descriptions, etc. are defined by the task they solve in a speech community (e.g., a mathematics classroom):

> Oral discursive practices are defined as multi-unit turns which are interactively co-constructed, contextualized and functionally oriented towards particular genres (Bergmann & Luckmann, 1995) such as narration, explanation or argumentation. By making use of conventionalized genres, discourse units in their joint achievement in interaction rely on patterns available in speech communities’ knowledge to, e.g., convey or construct knowledge (explanations) or negotiate divergent validity claims (argumentation). (Erath et al., in press).

In this perspective, *discourse practices* are patterns that can be observed repeatedly in a speech community (e.g., Cobb, Stephan, McClain, & Gravemeijer, 2001).

Discourse Practices and Discourse Competence in Mathematics Classrooms

In mathematics classrooms, the most important discourse practices comprise (Prediger, 2016):

- Reporting on procedures.
- Explaining the meaning of concepts and operations.
- Arguing about the validity of a claim.
- Describing patterns in a general way.

However, these most important practices appear with different frequencies in different classrooms (Erath et al., in press), and in addition, patterns are specific for different mathematics classrooms as Erath (2017a) shows for the case of explaining. Furthermore, it can be theoretically and empirically shown that talking about conceptual knowledge (that Hiebert, 1986, defined in contrast to procedural knowledge) is tightly linked to the shared discourse practice of explaining in whole class discussions (Erath, 2017a) and moderated small group work (Erath, 2017b).

Discourse analytic constructs allow mathematics education researchers to understand how students' low language proficiency is intertwined with restricted mathematical learning opportunities. In this context, learning is conceptualized as "a process of enculturation into mathematical practices, including discursive practices (e.g., ways of explaining, proving, or defining mathematical concepts)" (Barwell, 2014, p. 332; Vygotsky, 1978). When learning mathematics is linked to participation in classroom interaction, this specifically explains limits in acquiring conceptual knowledge in discourse practices like describing general pattern or explaining meanings of concepts (e.g., Erath et al., in press; Moschkovich, 2015): empirical studies on whole class discussions and moderated small group work (e.g., Erath, 2017b; Erath et al., in press; Barwell, 2012) show that students with low language proficiency rarely participate in the conceptually interesting moments of discourses, often due to missing linguistic resources on the discourse level.

In Interactional Discourse Analysis, these resources are specified by the concept of discourse competence which distinguishes three facets (Quasthoff, 2011):

- *Contextualization competence* refers to a student's ability to recognize if, for example, a question requires an answer with half a sentence or a longer utterance (i.e., a discourse unit). For a discourse unit, a student needs to recognize if a narration, an explanation, or an argument is required.
- *Textualization competence* refers to a student's ability to structure the discourse unit according to the different requirements of explanations, arguments, etc. in the case of explanations in mathematics classrooms, this, for example, involves stating the procedure of an algorithm stepwise.
- *Marking competence* refers to a student's ability to use language means that make the chosen discourse practice and the related textualization recognizable for the public. Language means for argumentations are, for example, "because" or "for this reason."

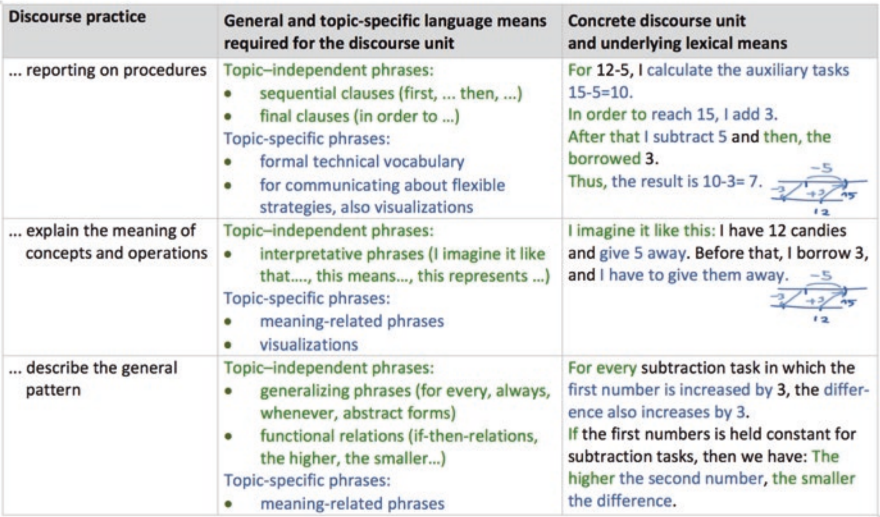
These three subcompetences are required by students for participating in discourse practices in mathematics education (further potential mathematical challenges are not explicated here): they need to "recognize *contextually* when to place which discourse practice (e.g., explanation instead of narratives), master the specific *textualization* patterns (e.g. explicating general procedures step-by-step) and use a specific lexical and grammatical repertoire to *mark* it (e.g., 'that's why' for explaining)" (Erath et al., in press).

General and Topic-Specific Lexical Means for Different Mathematical Discourse Practices

Beyond the *general* discourse competence and language means for *generally* marking specific discourse practices, topic-specific language means are required for each mathematical topic. We illustrate the differences for the case of a whole class discussion on flexible subtraction strategies (Grades 3–5) in which a student, Kevin, has suggested to calculate 12 – 5 by the auxiliary task 15 – 5 – 3: “If this were 15, then it would be 10, and then make 3 less.” As his ideas are not immediately understandable to the whole class, the teacher asks to explain Kevin’s ideas (Prediger, 2016). Table 27.2 shows three different discourse practices that might follow and the different general and topic-specific language means required for the discourse units.

The examples in Table 27.2 specifically illustrate the difference between a formal, technical vocabulary (subtracting, adding, result) and meaning-related vocabulary (giving away, borrowing, etc.). A second example can be given for percentages: the formal vocabulary comprises concepts like rate amount and base. To enable students to construct meanings for these concepts, it is important to engage them in the discourse practice of explaining meanings. For explaining, topic-specific meaning-related vocabulary is required such as “the old price,” “the new price,” “discount as a share of the new price,” etc. (Pöhler & Prediger, 2015). Scaffolding these meaning-related phrases allows students with low language proficiency to participate in these kinds of discourse practices.

Table 27.2 General and topic-specific lexical means for different mathematical discourse practices – examples for the auxiliary strategy 15 – 5 – 3 for calculating 12 – 5



Discourse practice	**General and topic-specific language means required for the discourse unit**	**Concrete discourse unit and underlying lexical means**
... reporting on procedures	Topic–independent phrases: • sequential clauses (first, ... then, ...) • final clauses (in order to ...) Topic-specific phrases: • formal technical vocabulary • for communicating about flexible strategies, also visualizations	For 12-5, I calculate the auxiliary tasks 15-5=10. In order to reach 15, I add 3. After that I subtract 5 and then, the borrowed 3. Thus, the result is 10-3= 7.
... explain the meaning of concepts and operations	Topic–independent phrases: • interpretative phrases (I imagine it like that...., this means..., this represents ...) Topic-specific phrases: • meaning-related phrases • visualizations	I imagine it like this: I have 12 candies and give 5 away. Before that, I borrow 3, and I have to give them away.
... describe the general pattern	Topic–independent phrases: • generalizing phrases (for every, always, whenever, abstract forms) • functional relations (if-then-relations, the higher, the smaller...) Topic-specific phrases: • meaning-related phrases	For every subtraction task in which the first number is increased by 3, the difference also increases by 3. If the first numbers is held constant for subtraction tasks, then we have: The higher the second number, the smaller the difference.

Approaches for Fostering Students' Language Proficiency in Mathematics

As language has turned out to be an important factor for different groups of students at risk, fostering students' academic language proficiency is demanded all over the world. The Council of Europe claims it as a major approach for achieving more equity (Thürmann, Vollmer, & Pieper, 2010), and many empirical studies show that it might positively influence the mathematics learning (see below). However, the practices of fostering language (e.g., DfEE, 2000) are often criticized for being too restricted to vocabulary training (Moschkovich, 2013) without taking into account the discourse level. In this section, we give some examples on more enhanced instructional approaches.

Enhancing Discourse Practices: Qualitative Output Hypotheses

Given the empirical results on the discourse level (see last section), enhancing discourse practices like explaining, arguing, etc. is an important instructional approach (Moschkovich, 2015). This is in line with the principle of pushed output formulated in second-language learning (Swain, 1995), according to which language learning requires the enforcement of oral and written language production. Pushing output can be reached by suitable tasks and activity structures and can be accompanied by materialized scaffolds. This may include language frames and teachers' continuous micro-scaffolding moves during the interaction (Bakker, Smit, & Wegerif, 2015).

Schröder and Ritterfeld (2015) emphasize the significance of enhancing discourse practices also for students with SLI on three dimensions: the dimension of the linguistic-interactive requirements and their supportive function, the didactical dimension of using materials and visualizations, and the dimension of mathematical knowledge, resp., knowledge acquisition.

Enhancing Conceptual Knowledge: Relating Registers and Representations

In mathematics education, pushed output can be successfully combined with continuous activities of relating registers (the everyday, academic, and technical register) and (symbolic, graphical, etc.) representations forward and backward, rather than sequencing through them once (cf. Fig. 27.1 from Prediger et al., 2016; similar in Moschkovich, 2013). These activities of translating between registers and finding coherences or differences offer very suitable, mathematically rich opportunities for

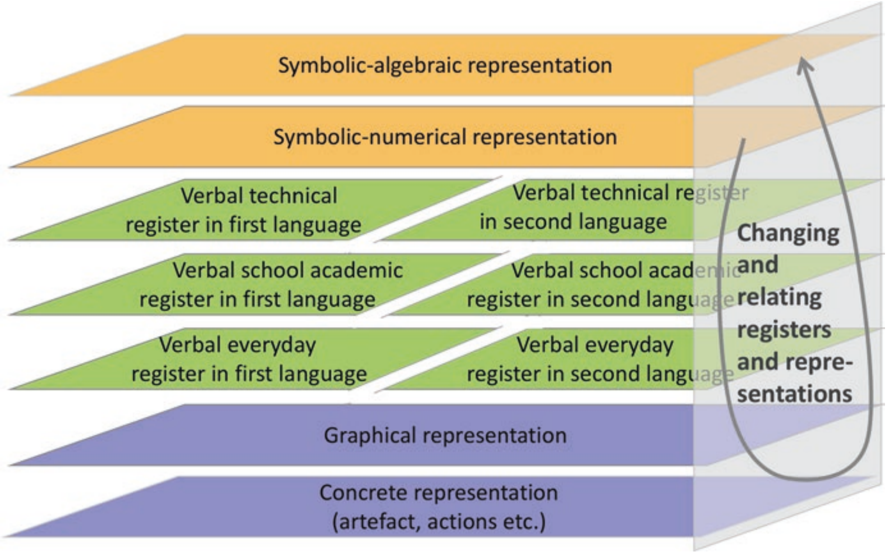


Fig. 27.1 Design principle of relating registers and representations (Prediger, Clarkson, & Bose, 2016)

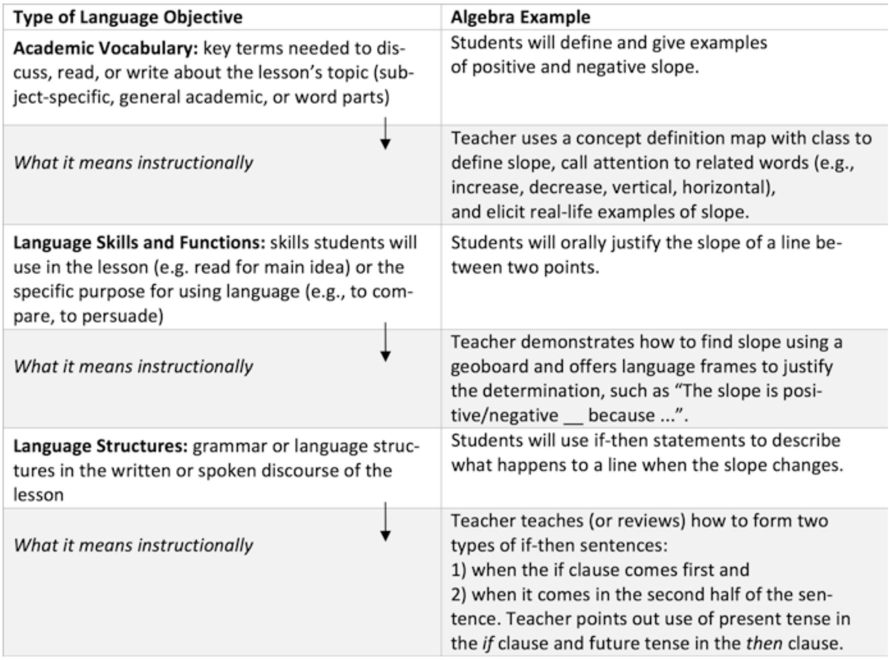
verbalizing, explaining, and arguing. At the same time, they have proven powerful for enhancing conceptual understanding, a critical point in the learning of students with low language proficiency.

Empirical evidence for the efficacy of this design principle has been provided, e.g., in a control trial in Grade 7 on an intervention designed for enhancing students' conceptual understanding of fractions based on this design principle (Prediger & Wessel, 2013).

Specifying Mathematical and Language Goals: The SIOP Model

One of the most widespread approaches for supporting language learners in subject matter education stems from the Sheltered Instruction Observation Protocol (SIOP model, Echevarria, Vogt, & Short, 2010). The SIOP model starts from specifying mathematical learning goals, deriving the discursive demands, and then figuring out the language objectives for realizing these discursive demands (Fig. 27.2). The necessary words and phrases are offered in language frames and trained by initiating the necessary discursive practices.

Empirical evidence for the efficacy of this model has been offered in various control trials (for a summary see Short, 2017).



Type of Language Objective	**Algebra Example**
Academic Vocabulary: key terms needed to discuss, read, or write about the lesson's topic (subject-specific, general academic, or word parts)	Students will define and give examples of positive and negative slope.
What it means instructionally ↓	Teacher uses a concept definition map with class to define slope, call attention to related words (e.g., increase, decrease, vertical, horizontal), and elicit real-life examples of slope.
Language Skills and Functions: skills students will use in the lesson (e.g. read for main idea) or the specific purpose for using language (e.g., to compare, to persuade)	Students will orally justify the slope of a line between two points.
What it means instructionally ↓	Teacher demonstrates how to find slope using a geoboard and offers language frames to justify the determination, such as "The slope is positive/negative __ because ...".
Language Structures: grammar or language structures in the written or spoken discourse of the lesson	Students will use if-then statements to describe what happens to a line when the slope changes.
What it means instructionally ↓	Teacher teaches (or reviews) how to form two types of if-then sentences: 1) when the if clause comes first and 2) when it comes in the second half of the sentence. Teacher points out use of present tense in the *if* clause and future tense in the *then* clause.

Fig. 27.2 Categories and examples of language objectives in the SIOP model (Short, 2017, p. 4246)

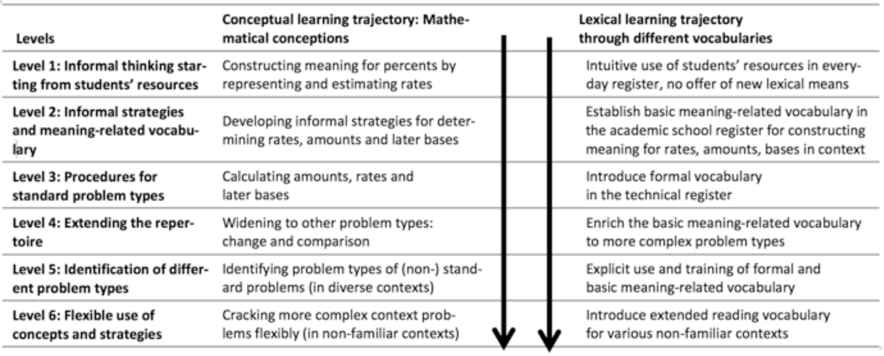
Combining Conceptual and Lexical Learning Trajectories: Macro-Scaffolding

Gibbons' (2002) approach of macro-scaffolding suggests to combine the conceptual and lexical learning opportunities in well-sequenced trajectories. Table 27.3 gives an example for a macro-scaffolding approach for the mathematical topic of percentages: the conceptual learning trajectory is sequenced in six steps starting from students' resources in informal thinking to informal strategies, formal procedures, and their flexible use. Each step requires other discourse practices and different vocabularies which are sequenced in the lexical learning trajectory. Empirical evidence has been given that this intertwinement can be effective for mathematics learning (Smit, 2013; Pöhler & Prediger, 2015 with quantitative evidence in Pöhler, Prediger, & Neugebauer, 2017).

Including Home Languages: Activating Students' Multilingual Repertoires

With respect to the group of multilingual students whose home language differs from the official language of instruction (which is the majority worldwide), another important instructional approach addresses the inclusion of home languages in order to activate the complete multilingual repertoire for mathematics learning.

Table 27.3 Combining conceptual and lexical learning trajectories: macro-scaffolding example for percentages (Pöhler & Prediger, 2015)



Levels	Conceptual learning trajectory: Mathematical conceptions	Lexical learning trajectory through different vocabularies
Level 1: Informal thinking starting from students' resources	Constructing meaning for percents by representing and estimating rates	Intuitive use of students' resources in everyday register, no offer of new lexical means
Level 2: Informal strategies and meaning-related vocabulary	Developing informal strategies for determining rates, amounts and later bases	Establish basic meaning-related vocabulary in the academic school register for constructing meaning for rates, amounts, bases in context
Level 3: Procedures for standard problem types	Calculating amounts, rates and later bases	Introduce formal vocabulary in the technical register
Level 4: Extending the repertoire	Widening to other problem types: change and comparison	Enrich the basic meaning-related vocabulary to more complex problem types
Level 5: Identification of different problem types	Identifying problem types of (non-) standard problems (in diverse contexts)	Explicit use and training of formal and basic meaning-related vocabulary
Level 6: Flexible use of concepts and strategies	Cracking more complex context problems flexibly (in non-familiar contexts)	Introduce extended reading vocabulary for various non-familiar contexts

The demand for including all multilingual resources has often been formulated, and qualitative empirical insights have been offered for its benefits (Barwell et al., 2016), such as participating in mathematical discourses activating everyday out-of-school experiences (Planas, 2014) or upgrading resources for meaning-making processes (Clarkson, 2006; Norén, 2015) and increasing agency (Norén, 2015). However, the quantitative evidence for its efficacy for mathematics learning is still too weak (Reljić, Ferring, & Martin, 2015). Therefore, further research is required for strengthening the quantitative evidence.

Conclusion

Language is a major *learning medium* used for communicative and epistemic purposes in mathematics classrooms. Hence, language proficiency is an important *learning prerequisite* without which participation in classrooms tends to be restricted. This applies to different groups of students:

- Second-language learners.
- Students with learning disabilities in mathematics and reading.
- Students with specific language impairment.
- Monolingual students (often of low socioeconomic status) which have not yet had sufficient learning opportunities especially for the academic language.

For all these students with limited language proficiency, language thus has to become a *learning goal*, also in mathematics classrooms (Lampert & Cobb, 2003).

The language dimension is crucial for students with difficulties in mathematics not only in test situations but particularly *during the whole learning process*: Students with low academic language proficiency regularly meet challenges on *word, sentence, text, and discourse levels*.

Therefore, instructional approaches to support language learners should not isolate the word level from the discourse level. Instructional approaches seem to become most effective for supporting mathematics learning when they provide learning opportunities especially for the discourse practices of explaining meanings of mathematical concepts and operations and for describing general pattern. The lexical support of meaning-related vocabulary is therefore equally important to the formal technical vocabulary and specifically fruitful when offered in structured phrases rather than isolated words (Moschkovich, 2013; Prediger & Wessel, 2013).

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Chapter 28
Motivational and Math Anxiety Perspective for Mathematical Learning and Learning Difficulties

Elke Baten, Silvia Pixner, and Annemie Desoete

Introduction

There are several arguments in favor of focusing on the predictors for mathematical learning and learning difficulties. Firstly, Duncan et al. (2007) demonstrated that mathematical achievement at the beginning of primary school is one of the strongest predictors of later academic success—stronger than early reading skills, even when important socioeconomic characteristics of the child and his or her environment are controlled for. In addition, pupils with high levels of mathematical performance were found to have a greater chance of later school success than pupils with low achievement levels in mathematics (Claessens & Engel, 2013). Moreover, Duncan and Magnuson (2009) revealed that children who kept having low scores in mathematics during elementary school had a 13% lower chance of graduating from high school and a 29% lower chance of starting college education. Finally, a lack of mathematical skills was found to affect people's ability to gain full-time employment and often restricted employment options to manual and often low-paying jobs.

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Several studies have provided evidence that mathematical learning is made up of multiple components and that mathematical learning disabilities (MLD) are not unitary (Dowker, 2015; Gifford & Rockliffe, 2012; Pieters, Roeyers, Rosseel, Van Waelvelde, & Desoete, 2015). Some authors propose a subtype characterized by a delay in the "accuracy" of procedures used to solve simple arithmetic problems and a subtype characterized by a lack of "fluency" and by errors in the speed of the retrieval of arithmetic facts (Pieters et al., 2015).

In this chapter we focus on motivation and math anxiety as consequences of mathematical learning and learning problems with math "accuracy" and "fluency" in addition to known cognitive predictors. We first describe the opportunity–propensity (O-P) model as a way of looking at more pieces of the puzzle of predictors of leaning and learning problems.

Opportunity–Propensity Model

Byrnes and Miller (2006) proposed the O-P model to examine predictors for learning processes in a comprehensive way. This model explains mathematical achievement as a result of not only getting the right opportunities to learn (opportunity factors) but also being able to take advantage of these opportunities (propensity factors; Byrnes & Miller, 2006, 2016). Analyses on secondary data using this model reveal that 58–81% of the variance in mathematical learning and learning problems can be explained by antecedent (A) factors, specific opportunity (O) factors, and propensity (P) factors (Byrnes & Miller, 2006).

Antecedent (A) factors include parameters such as socioeconomic status (SES), parent expectations, child expectations, parent values, and prior achievement, and they account for 28.8–43.0% of the variance in mathematical learning. In the case of MLD, mathematical learning problems appear to aggregate in families. The relative risk of MLD is substantially higher in first-degree relatives of individuals with these learning difficulties than in those without them (Desoete, Praet, Titeca, & Ceulemans, 2013). In addition, prematurity and very low birth weight increase the risk (Desoete & Baten, 2017).

Opportunity (O) factors include factors such as the home and school environment in which children are presented with content to learn or given opportunities to practice math skills, explaining between 11.2% and 44.1% of the variance in 10th- and 12th-grade mathematics. A longitudinal study revealed that not only the school environment but also the home environment mattered (Ceulemans, 2014). There was a significant relationship between the frequency of mothers spontaneously focusing on numerical cues at 24 months and the skills of children in solving arithmetic problems at 48 months (Ceulemans, 2014).

Propensity (P) factors add 21.9–27.6% to the prediction of mathematical learning in the model. Propensity factors include cognitive and noncognitive predictors of math achievement later on. Prerequisite knowledge such as magnitude comparison (Ceulemans et al., 2014; Vanbinst, Ghesquière, & De Smedt, 2015); counting, seriation, and classification (Stock, Desoete, & Roeyers, 2010); language (Praet,

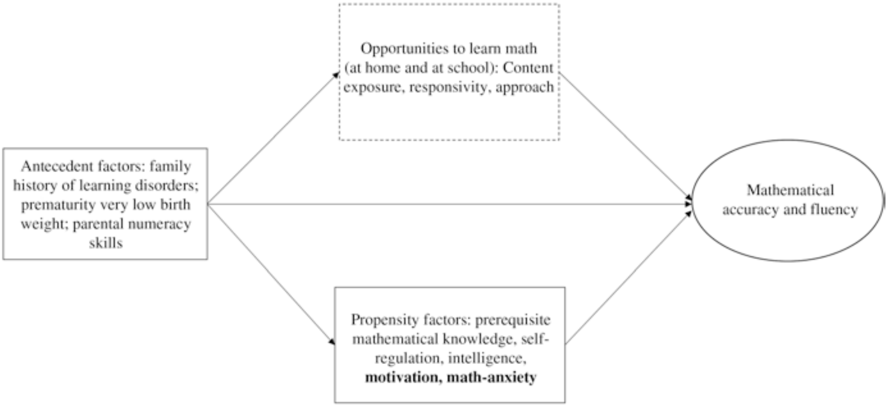

Fig. 28.1 Opportunity–propensity model. (Adapted from Desoete & Baten, 2017)

Titeca, Ceulemans, & Desoete, 2013); and intelligence (Desoete, 2008) as cognitive predictors explain some of the variance in math learning and learning problems (Desoete, 2014, 2015) (Fig. 28.1).

Also, affect and motivation are powerful factors that influence how students learn and master mathematics (Moore, Rudig, & Ashcraft, 2015). In the following sections, the research on these factors is reviewed.

Motivation

Definition of the Construct

> Motivation is the processes that accounts for an individual's intensity, direction, and persistence of efforts toward attaining a goal (Judge & Robbins, 2013).
>
> Motivation refers to psychological forces that move people, bring them into action, and keep them going (Lens, Vansteenkiste, & Matos, 2008).

From the *developmental perspective*, children begin formal education with a very positive view of mathematics and with good feelings about their own abilities. There appears to be a reciprocal nature of the development of achievement and motivation, with both interest and motivation declining as children grow older (Moore et al., 2015). The Programme for International Student Assessment (PISA) ranking of mathematical literacy of 15-year-olds in 2015 demonstrated that although Flanders was in eighth place, children in Flanders disliked mathematics and felt uncertain when solving math problems. So paradoxically students with a high ranking were more afraid of math and showed low self-assessment. Moore and colleagues suggested that affect and beliefs were crucial in learning and understanding math. Nevertheless, a longitudinal study by Garon-Carrier et al. (2016) demonstrated that mathematics predicted intrinsic motivation and not the other way around.

Studies using the O-P model revealed that motivation (as propensity) accounted for 10% of unique variance on top of other propensity, opportunity, and antecedent predictors.

Despite the overall decline, self-rated interest in mathematics has a strong positive relationship with math-related success. This high correlation between interest and achievement might be caused by the fact that the more people are interested in a subject, the more knowledge they gain. Preknowledge seems to be a strong predictor, surpassing the influence of intelligence. In higher grades, variance in motivation (in terms of interest) and performance might be confounded, partly explaining why high motivation had a positive relationship with performance, with continued engagement in math classes and, subsequently, better achievement (Moore et al., 2015).

Some researchers differentiate here-and-now task engagement and longer-term patterns of engagement, with some students developing interest and enjoying algebraic puzzles but avoiding further mathematics coursework. In addition, they differentiate situational interest and state-level preferences. Individual interest, exogenous instrumentality, goal orientations, and broader academic self-efficacy currently were found to lessen learners' enjoyment of and persistence in challenging mathematics as they grow older (Middleton, Jansen, & Goldin, 2017).

Next to interest and engagement, there are several other ways to operationalize motivation as a complex construct. One of these conceptualizations is to investigate *self-perceived abilities*. Spinath, Spinath, Harlaar, and Plomin (2006) demonstrated in 1678 children that motivation in terms of ability, self-perceptions, and intrinsic values predicted achievement in mathematics and English more than general intelligence (g). Although intelligence was the strongest predictor (25% of the explained variance), a substantial percentage of the variance could be explained by common variance and self-perceived abilities had an incremental validity (over g) of 5%. In 2010, researchers tried to replicate these findings in a population of 179 Chinese primary school pupils, but they could not confirm the incremental validity of motivation beyond intelligence; only marginal significances were reported. Several explanations such as cultural differences (Europe versus China) and different conceptualization of the construct of motivation were postulated (Lu, Weber, Spinath, & Shi, 2011). Kriegbaum, Jansen, and Spinath (2015) investigated the incremental role of motivation in academic achievement in 2003 and 2004 data from PISA and found extra explained variances in outcome of between 1% and 29% for several motivational factors. Task-specific self-efficacy was the strongest motivational predictor (Kriegbaum et al., 2015).

Another attempt to study motivation was *self-determination theory* (SDT), one of the leading theories in motivational psychology. SDT (Deci & Ryan, 1985) states that motivation is achieved by fulfilling three important needs, which are universal for every single person: autonomy, competence, and relatedness. *Autonomy* is the psychological concept of feeling free to make your own choices (Van Petegem, Soenens, Vansteenkiste, & Beyers, 2015), *Competence* is achieved when you attribute successful performance to your own capacities (Gagné & Deci, 2005). *Relatedness* is described as the experience of feeling loved by significant others (Vansteenkiste & Ryan, 2013). Satisfaction of these needs was found to result in

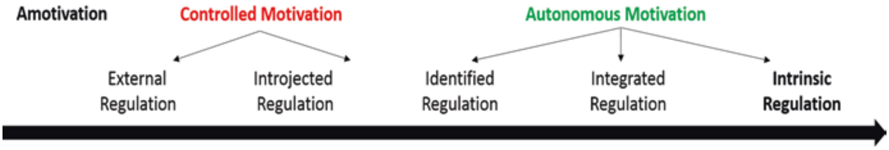

Fig. 28.2 The continuum of motivation. (Based on Vansteenkiste et al., 2009)

positive outcomes such as psychological growth (Vansteenkiste & Ryan, 2013) and well-being in terms of self-esteem (Chen et al., 2015), whereas frustration of these needs resulted in stress and ill-being such as depressive symptoms or externalizing problems (Vansteenkiste & Ryan, 2013). In addition, SDT claims that the quality of one's motivation is more important than the quantity. The more autonomous (versus controlled) the motivation is, the better (Vansteenkiste, Sierens, Soenens, Luyckx, & Lens, 2009).

On a continuum (see Fig. 28.2) from external regulation to intrinsic motivation or passion, different motivational categories are described that vary in terms of the quality of motivation and the level of self-regulation (Gagné & Deci, 2005; Vansteenkiste et al., 2009). Intrinsic regulation, at the end of the continuum, can be reached by fulfilling the three basic psychological needs described above. The distinction that is important for the purpose of this chapter is that between *controlled motivation* (CM) and *autonomous motivation* (AM).

Controlled motivation consists of external and introjected regulation. The force that drives you to fulfill a task in these types of regulation is hypothesized as completely outside yourself. For instance, external regulation means that you study mathematics because if you succeed on a test, your parents will give you a new bike. Introjected regulation is one step further on the continuum beyond external regulation because there you tell yourself you have to study for an external reason. For instance, you tell yourself to study mathematics for the entire afternoon, because afterward you will have time to go a night out with friends.

Autonomous motivation consists of identified regulation, integrated regulation, and intrinsic regulation. Integrated regulation means that you find some aspects of the task really valuable; for instance, you study mathematics because you see the relevance for your later academic career. At the end of the continuum you find intrinsic regulation or intrinsic motivation where you study mathematics for instance because you feel that the task on its own gives you feelings of pleasure (Gagné & Deci, 2005).

There has been a long tradition of research on SDT, both theory-specific research and research on applying the theory in several domains of life—for instance, work, sports, relationships, and education (Milyavskaya & Koestner, 2011). In the field of education, most research on SDT has focused on physical education and sport (Moreno, Gonzalez-Cutre, Sicilia, & Spray, 2010; Ntoumanis, 2001; Standage, Duda, & Ntoumanis, 2003). However, research has also indicated significant positive relations between the level of autonomous motivation and achievement in mathematics and reading (Grolnick, Ryan, & Deci, 1991). A meta-analysis of

18 studies on the relationship between SDT and general academic achievement revealed moderately strong positive relations of intrinsic motivation and introjected regulation with general school achievement. The different types of controlled motivation had significant negative relations with academic achievement (Taylor et al., 2014). In adults a significant positive correlation between autonomous motivation and the fluency of fact retrieval, but not between autonomous motivation and math accuracy, was found (Baten & Desoete, 2016), so the accuracy and speed of mathematics learning might be predicted by other (propensity) factors.

In addition, studies revealed that high motivation was negatively related to math anxiety (Zakaria & Nordin, 2008), serving as a strong moderator of the relationship between motivation and mathematics achievement. This construct is described in the next section.

Math Anxiety

Math anxiety is very common. Beilock and Willingham (2014) report a prevalence of 25–50% of students with math anxiety in the USA. Math anxiety is defined as the negative affective reaction of an individual in situations that include numbers, quantities, and calculating (Ashcraft & Moore, 2009). Negative consequences of math anxiety can acutely originate in impairment of the current performance, especially caused by impairment of working memory (Ashcraft & Krause, 2007; Krinzinger, Kaufmann, & Willmes, 2009). As is generally known, our working memory has a relatively limited capacity. Anxiety-provoking thoughts and worries about anticipated failure require a lot of the working memory's capacity (Ashcraft & Krause, 2007) and therefore leave little to no capacity for performing the current task. Furthermore, even long-term negative impairments in mathematical performance caused by math anxiety are kept up by a negative approach to mathematics and resulting avoidance behavior (Chinn, 2009). Avoidance behavior caused by math anxiety mostly leads to a vicious circle characterized by less calculation practice. With regard to the onset as well as its effects on the body, cognition, and behavior, math anxiety is comparable to other specific phobias.

On the student's propensity side, the dynamic between motivation and achievement is clearly stated in a study by Jansen et al. (2013). With an adaptive computer-based approach they manipulated the success rate for solving mathematical tasks. The more success the tested children experienced in solving the mathematical problems, the higher they rated their subjective mathematical competencies and the less math anxiety they reported. This pictures a possible dimension of comparison, regarding one's achievement: the more efficient I experience myself as being, the more I am willing to provide the performance. In school, however, children prefer to compare themselves to their peers (Maloney & Beilock, 2012). If their performance is generally better than or roughly the same as that of their peers, the children attribute their upcoming failures externally, knowing that others also have not shown better performances in the past. Children who solve significantly fewer tasks or produce more mistakes than their peers merely attribute their failures internally, since others have shown better performances, pointing to a complex interaction of motivation (self-perceived abilities, attribution) and affect (anxiety). This dynamic, identified by Jansen et al. (2013), could be an explanation for the phenomenon of increasing math anxiety according to the duration of formal schooling (Krinzinger et al., 2009). Not only does the number of experiences of failure rise with time, but also mainly the success rate decreases because of more complex issues. According to Maloney and Beilock (2012), two important factors—social influence (most notably the influence of the teachers) and cognitive predispositions—are of major interest in the development of math anxiety. In our opinion, also emotional dispositions have an important influence, since some people's personality and their availability of coping strategies make them react with anxiety faster and more intensely. To Maloney and Beilock (2012), cognitive predisposition includes not just general cognitive competencies like attention, intelligence, or memory, but also basic numerical competencies. Therefore, it is also postulated that children with dyscalculia are more likely to develop math anxiety due to their poor basic numerical competencies. However, we want to point out that not only children with dyscalculia suffer from math anxiety. When social factors are inspected more carefully, we find that especially teachers—but also parents in homework and other learning situations—play an important role in forming an important basis concerning the approach to arithmetic (Zhao, Valcke, Desoete, & Verhaeghe, 2012; Zhao, Valcke, Desoete, Verhaeghe, & Xu, 2011).

There is also evidence of environmental opportunity factors that might serve as risk or protective factors that influence the development of math anxiety. There seems to be a more general link between teachers' behavior and students' math performance (Beilock & Willingham, 2014). The higher the teachers' math anxiety is (most primary school teachers are female), the lower their female students' math achievement is. The second possible factor is the kind of feedback the teacher gives and also how the teacher reacts when the student struggles. This results from the teachers' epistemological beliefs. An epistemological belief indicates an opinion on the source of knowledge. When the source of the child's knowledge is considered innate and static, the feedback will go in that direction of skills and turn out to limit the student's achievement. On the other hand, when the teacher believes that mathematical knowledge can be learned and improved with the right strategies, the feedback will, rather, include those strategies. Gradually the student's epistemological beliefs are shaped through the teacher's belief (Brownlee & Bertelsen, 2008). Consequently, the student stabilizes his or her belief on the source of knowledge based on the teacher's belief.

The phenomenon of epistemological belief and the impact on achievement is also represented in the attribution theory (Kelley, 1973). Arkin, Kolditz and Kolditz (1983) showed that children with great math anxiety attribute failures rather internally and stably to their poor abilities. Shores and Shannon (2007) confirmed the importance of failure attribution for mathematics in sixth-grade students. This closes the vicious circle that accompanies math anxiety, even if we are not sure where it starts. Poor abilities; the belief that they are stable, innate, and not or hardly changeable; the epistemological belief that makes teachers alter their style of teaching and feedback to the student; the attribution over success and failure; or the math anxiety itself—which influences mathematical competencies not only acutely but fairly long term—could mark the beginning of the vicious circle that accompanies the phenomenon of math anxiety. This naturally influences the children's motivation to deal with arithmetic tasks and makes the situation difficult for students and their surroundings. The only ways to retrieve the child from the vicious circle of math anxiety are knowledge of this dynamic and active attempts to take action against it.

Conclusions and Implications

Our overview has demonstrated that noncognitive factors such as motivation and math anxiety should not be overlooked in the assessment of mathematical learning and learning problems. If we want more children to be motivated we should also provide chances for autonomy, relatedness, and competence in our math lessons. Only then will we maintain a positive view, autonomous motivation, and interest without anxiety in mathematics. In addition, math accuracy and fluency might be predicted by other propensity factors. Furthermore, it seems important to keep in mind that variables are mostly correlated. A comprehensive model can help us to overcome dichotomous interpretations as either consequences or causes, which make biased conclusions seem indicated. The opportunity–propensity model might help us to take into account the reciprocally interrelationship between predictors. Future research using this model is important to develop and study interventions that tackle the real sources of variance in different mathematical abilities and learning problems.

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Chapter 29
Mathematics and Emotions: The Case of Math Anxiety

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Introduction

Math is important, difficult, and emotion-arousing. Math achievement and level of mathematical training are demographically associated with employability and wages (Dougherty, 2003; Parsons & Bynner, 2005). In a knowledge-based society, math abilities constitute an important economic asset (Beddington et al., 2008). This is recognized by educators, policy-makers, and the wider public opinion (Budd, 2015). Cognitive capital (e.g., math ability) and social capital (e.g., a democratic and stable legal order) are determinants of human development (Newson & Richerson, 2009).

At all grades, math is perceived as the most difficult subject matter in the academic curriculum (Mazzocco, Hanich, & Noeder, 2012). Math contrasts with reading. Reading acquisition may be systematized in two hierarchical levels: word reading and text comprehension (Gough, 1996). It takes the child 3 to 4 years of hard work to automatize word reading (Dehaene, 2009). Text comprehension is a more complex, lifelong task (Oakhill, Cain, & Elbro, 2014). Math is a complex topic from the beginning, and its complexity increases steadily. The arithmetic curriculum is, at least partially, hierarchically organized in successive steps of increasing complexity (Clements & Sarama, 2009). Progression in the math developmental trajectory requires

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mastery of a succession of abilities. At all levels of math development, the child's processing capacity is challenged (Bull & Lee, 2014; McLean & Rusconi, 2014).

It is no surprise then that math arises (sometimes strong) emotions. Even highly accomplished mathematicians experience some degree of negative emotions, such as anger, frustration, and anxiety, when solving math problems. Accomplished mathematicians' attitudes and coping strategies differed widely from those of novices (McLeod, Metzger, & Craviotto, 1989). Experts perceive math problems as challenging and use a more flexible and varied problem-solving approach. Novices feel overwhelmed, use less varied and rigid strategies, and attribute failure to internal, negative traits. It seems that experts work at the left side of the Yerkes-Dodson inverted-U curve relating arousal do performance, while novices work on the right side.

Interactions between math and emotions are complex and bidirectional. Math activities may elicit both positive and (more often) negative emotions. High performance is associated with joy. Lower performance may cause frustration, anger, revolt, resentment, tension, dread, anxiety, shame, lower self-esteem, hopelessness, and emotional detachment. In the long run, persistent math difficulties are a risk factor for both externalizing and internalizing psychiatric disorders (Auerbach, Gross-Tsur, Manor, & Shalev, 2008; Parsons & Bynner, 2005). Therefore, both positive and negative emotions influence math performance.

In the last 60 years, research on math and emotions has focused mainly on math anxiety (MA) (e.g., Artemenko, Daroczy, & Nuerk, 2015, Dowker, Sarkar, & Looi, 2016. Moore, Rudig, & Ashcraft, 2014, Suárez-Pellicioni, Núñez-Peña, & Colomé, 2016). In this chapter, we will review the relevance of MA in the context of math learning and its difficulties, introducing the construct, its relations to motivation, its antecedents, consequents, relationships with math achievement, neurocognitive underpinnings, assessment, and interventions.

Math Anxiety as a Construct

MA has been classically defined as a "feeling of tension and anxiety that interferes with the manipulation of numbers and the solving of mathematical problems in a wide variety of ordinary life and academic situations" (Richardson & Suinn, 1972) and as a "feeling of tension, helplessness, mental disorganization and dread produced when one is required to manipulate numbers or to solve mathematical problems" (Ashcraft & Faust, 1994). According to Chinn (2009), MA definitions vary according to a focus on performance (feelings of tension, apprehension or fear that interferes with math performance) or on the self (state of discomfort, which occurs in response to situations involving mathematics tasks that are perceived as threatening to self-esteem).

The phenomenology of MA is highly similar to a specific phobia (Faust, 1992; Krinzinger, Kaufmann, & Willmes, 2009). MA is considered to be a stimulus- and situation-specific learned reaction in predisposed individuals, manifesting itself at different levels: cognitive (negative attitudes, worrisome rumination, feelings of helplessness, low self-esteem, self-efficacy, etc.), affective (dysphoria), behavioral (avoidance, hurry-up to finish math tasks, etc.), and physiological (sweating, trembling, high pulse rate, etc.). MA is thus a multidimensional construct. The structural validity of MA will be discussed in the section on Assessment of MA.

The construct of MA is not officially recognized as a psychiatric disorder (American Psychiatric Association, 2013). MA guards also similarity to other performance-related anxiety manifestations such as reading (Piccolo et al., 2017), test, and social anxiety. Increasing levels of manifestation of MA from childhood to adolescence and college-age as well as associated feelings of shame and fear of losing face suggest a connection with social phobia. Correlations between MA and other anxiety-related constructs are moderate to low (Hembree, 1990), indicating the existence of both shared and non-shared sources of variance.

MA is considered a significant educational and clinical problem although not as a psychiatric disorder. There is no external golden standard or diagnostic cut-off scores available, and its prevalence is difficult to establish in a reliable manner. Most studies are based on the application of self-report instruments to samples of students at different grades. Those students with scores higher than an arbitrary cut-off are considered to have MA. For example, if the chosen cut-off score is on the 75th percentile, prevalence will correspondently be situated around 25% of the population. MA is considered especially important in certain professional categories such as nurses and elementary school teachers (Beilock, Gunderson, Ramirez, & Levine, 2010; Hembree, 1990; McMullan, Jones, & Lea, 2012). Thus, prevalence rates vary widely from 2–6% to 68% according to the sample and diagnostic criteria investigated (Dowker et al., 2016).

MA is a relatively specific construct. MA is subject-specific, and correlations with other manifestations of anxiety are low to moderate, with r's between 0.35 and 0.52, according to the meta-analysis by Hembree (1990). Some data suggest that MA is associated with math performance but not with literacy performance or other forms of maladaptive behaviors and that its cognitive correlates are important from elementary school onward, distinguishing typical achievers from children with math learning difficulties (Haase et al., 2012).

Math Anxiety and Motivation

At the cognitive level of manifestations, MA is related to several motivationally relevant constructs. Cognitive perceptions associated with MA may focus on the school subject or the self (Chinn, 2009). Attitudes toward math refer to "a liking or disliking of mathematics, a tendency to engage in or avoid mathematics activities, a belief that one is good or bad at mathematics and a belief that mathematics is useful or useless" (Neale, 1969, p. 623).

Cognitive perceptions focused on the self are math self-concept and math self-efficacy. Math self-concept consists of one's perception of personal math accomplishments, and math self-efficacy refers to one's conviction or belief about their capability to engage successfully in math activities (Lee, 2009).

The distinction between self-concept and self-efficacy is problematic, as the two constructs considerably overlap (Lee, 2009). According to Bong and Skaalvik (2003), the self-concept refers to perceived competence or knowledge and perceptions about oneself in math achievement situations. The self-concept results from past reinforcement by significant others. The self-concept is domain- and context-specific, past-oriented, and stable. In contrast, self-efficacy consists of perceived confidence or convictions for successfully performing given math tasks. Self-efficacy results from previous personal experiences accomplishing math tasks. Self-efficacy is domain- and context-specific and future-oriented.

The construct of math self-efficacy is of great motivational significance. Self-efficacy expectations concern a person's beliefs about their ability to successfully perform a given math task. Self-efficacy is thus a major determinant of whether a person will engage in math tasks and indicates how much effort and persistence will be expended (Akin & Kurbanoglu, 2011). Self-efficacy beliefs are at least as predictive of math performance as intelligence (Pajares & Kranzler, 1995). Pajares and Graham (1999) found self-efficacy to be associated negatively with math anxiety. This is in accordance with the postulates of social learning theory that verbal persuasion and vicarious experience are weak and enactive mastery experience and physiological arousal are strong builders of self-efficacy (Babad, 2009; Bong & Skaalvik, 2003). Akin and Kurbanoglu (2011) observed that math attitudes act as a mediator between math self-efficacy and math anxiety. Moreover, these authors also showed that math self-efficacy positively predicts positive attitudes and negatively predicts negative attitudes. Math self-efficacy and positive attitudes predict MA in a negative way, and negative attitudes predicted MA in a positive way (Akin & Kurbanoglu, 2011). Finally, the positive influence of math self-efficacy on performance and its negative influence on MA can be used in the planning of interventions based on errorless learning or successful mastery experiences.

Antecedents of Math Anxiety

In this section, we will discuss the genetic, personal, and environmental antecedents of MA. In the following sections, we discuss cognitive and math achievement correlates of MA, as they are both antecedents and consequents.

Genetics

The diathesis-stress model widely used in psychiatry has been successfully applied to several forms of phobias (van Houtem et al., 2013). According to the diathesis-stress model, psychopathological manifestations arise from the interaction of individual vulnerabilities (diathesis) and negative experiences. However, some current models tend to emphasize the role of genetic influences over experience in the development of phobic disorders (Kendler, Myers, & Prescott, 2002).

To the best of our knowledge, there are only two behavioral genetic study of MA in twins reported in the literature (Malanchini et al., 2017; Wang et al., 2014). Wang and coworkers' study uncovered a heritability estimate of 40% and genetic correlations with other forms of anxiety such as general anxiety. Wang and coworkers' results suggest that MA emerges from the interaction between genetic influences on math and general anxiety. General anxiety, by its turn, emerges from the interaction between its own genetic and non-shared environmental influences. This is accordance with the diathesis-stress model. In this scenario, MA emerges in anxiety-prone individuals with math difficulties, who have more negative experiences in the context of math activities. Malanchini et al. (2017) obtained similar results, indicating a role for genetic and non-shared environmental factors and for both shared and specific genetic influences on spatial and math anxiety.

We were unable to find any research specifically investigating genetic influences on MA at the molecular level. This contrasts with other phobias, for which a role of genetic polymorphisms in neurotransmitters' metabolism and activity were reported. This also contrasts with other pediatric anxiety disorders for which a host of candidate genes and molecular mechanism were reported (see review in Sakolsky, McCracken, & Nurmi, 2012).

Age

Levels of MA grow up with age, as kids progress from childhood to adolescence (Hembree, 1990; Ma & Kishor, 1997). Two-thirds of 11-year-olds rate math as their favorite subject, but only a few 16-year-olds do so (Blatchford, 1996). Significant mathematics anxiety can be observed even among early primary school children (Haase et al., 2012; Krinzinger et al., 2009; Ramirez, Chang, Maloney, Levine, & Beilock, 2016; Ramirez, Gunderson, Levine, & Beilock, 2013; Thomas & Dowker, 2000; Wu, Barth, Amin, Malcarne, & Menon, 2012). Dowker et al. (2016) commented that this trajectory of growing MA in adolescence coincides with the greater incidence of anxiety disorders and to cumulative exposure to other people's negative attitudes, stereotypes, and a more demanding curriculum. MA associated with dyscalculia is present already in the very early school years. MA unaccompanied by math learning difficulties makes its debut later, around the third to fifth grades.

Gender

At all grades, MA levels are significantly higher in females than males (Hembree, 1990). Gender differences can be detected in young children and tend to increase with age (Dowker, Bennett, & Smith, 2012). MA is correlated with text anxiety in boys and girls. Both MA and test anxiety negatively correlate with math performance (Devine, Fawcett, Szűcs, & Dowker, 2012). Removal of the more general effect of test anxiety reduces the impact of MA on the scores of boys but not of girls. Higher levels of

MA in females contrast with absence or negligible presence of gender differences in average math performance (Lindberg, Hyde, Petersen, & Linn, 2010; Wai, Cacchio, Putallaz, & Makel, 2010). Several hypotheses have been raised to explain gender differences in MA, such as female proneness and willingness to admit anxiety symptoms (Chapman, Duberstein, Sörensen, & Lyness, 2007; McLean, Asnaani, Litz, & Hofmann, 2011), stereotype threat (Spencer, Steele, & Quinn, 1999), and social transmission of MA by female teachers (Beilock et al., 2010).

Stereotype threat is the dominant explanation in the literature. In situations where women are reminded of the stereotype that males are better at mathematics than females, their performance drops (Spencer et al., 1999). It was also observed that stereotype threat activates ventral cerebral areas associated with negative emotional processing and inhibits dorsal areas relevant for controlled and math processing (Krendl, Richeson, Kelley, & Heatherton, 2008). Stereotype threat as a dominant explanation of MA in females was questioned by Stoet and Geary (2012). These authors observed that studies only uncovered stereotype effects when prior math performance was statistically controlled. This may have attenuated the effects of previous math performance influences.

Other line of research indicates that cognitive differences may underlie MA gender proneness. For example, Stoet and Geary (2013) observed gender differences favoring boys at the extremes of the performance distribution. These subtle but significant differences may be cancelled out when averaged. Additionally, higher MA levels in girls and undervaluation of girls' math abilities by parents are independent of socioeconomic development and gender equity in a cross-national comparison (Stoet & Geary, 2016). In another study, lower MA levels in boys was mediated by better visuospatial processing abilities (Maloney, Waechter, Risko, & Fugelsang, 2012). These subtle but potentially relevant cognitive differences could originate from fetal testosterone levels (Stoet & Geary, 2016).

It seems safer then to conclude that gender differences in MA are related to a host of biological and cultural factors. Cultural influences such as stereotype threat and gender-specific transmission of MA are susceptible to psychosocial interventions with the goal of stimulating participation of women in math and science.

Culture

Cultural stereotypes and practices also exert an effect on MA: Math is generally considered to be a "difficult subject matter," "it is normal to have difficulties with maths," "girls do worse and Asians do better in maths," etc. (Aronson et al., 1999; Krendl et al., 2008). As an example, White male's performance is susceptible to the stereotype that Asians do better in math (Aronson & Lang, 2010). Harsh disciplinary practices and parental pressure on the belief that effort rather than ability is the primary source of success could also be associated with MA (Stankov, 2010; Tan & Yates, 2011). In general, higher economic development is associated with better math performance and lower MA levels (Stoet & Geary, 2016). However, some notable exceptions exist. Although math performance and MA usually negatively covary, dissociations also occur (Lee, 2009). Math self-concept is extremely low in some Asian countries with high math performance, and conversely, math self-concept is high in some developing countries with low math performance. This raises the questions of the price eventually paid in terms of stress and suffering for improvements in math performance, and of the lack of awareness regarding poor math teaching and/or performance in other settings.

Teachers

Negative experiences of embarrassment with teachers and colleagues caused by being socially exposed to math failure are recognized as an important risk factor for MA (Ashcraft, Krause, & Hopko, 2007). Higher MA levels are associated with negative experiences with teachers and with traditional teaching characterized by high demands for correctness and little cognitive and motivational support (Bekdemir, 2010; Meece, Wigfield, & Eccles, 1990; Turner et al., 2002).

Elementary school teaching is a career with a female preponderance. MA levels in teachers are high (Beilock et al., 2010; Hembree, 1990) and negatively correlated with teaching efficacy (Gresham, 2008; Swars, Daane, & Giesen, 2006). Teachers report feeling better prepared to teach literacy over math (Bursal & Paznokas, 2006). Then, it is no surprise that evidence indicates an intergenerational transmission of MA between female teachers and female pupils (Beilock et al., 2010). Awareness of the potential role of teachers in transmitting negative attitudes and emotions toward math to their pupils indicates the need of better preparing teachers to teach math or transforming math in a specialized subject matter assigned to mathematicians.

Parents

For better and for worse, the family is a major source of influence on math performance and interest (Chiu & Xihua, 2008). Some positive correlates are family SES, literacy level of parents, and possession of books. Some negative correlates are single parenthood, immigration, and speaking foreign language at home as well as resource dilution through many siblings. Boys may be especially susceptible to influences from maternal MA (Batchelor, Gilmore, & Inglis, 2017).

Parents influence MA through expectancy socialization, role models, overcontrol, reinforcement, and attachment (Batchelor et al., 2017). Parents influence children's mathematics achievement by calibrating expectations and reducing MA, particularly in more difficult math topics (Vukovic, Roberts, & Green Wright, 2013). Parents' role models in homework help to moderate effects of parents' MA on children's math performance and MA (Maloney, Ramirez, Gunderson, Levine, & Beilock, 2015). Overcontrol and punishment could explain the dissociation between high math achievement and low math self-concept observed in some countries by Lee (2009, see also Stankov, 2010). Finally, insecure attachment is correlated with low IQ and math performance and high MA (Bosmans & De Smedt, 2015).

Parental involvement in math activities at home has a beneficial effect on math performance and interest (Berkowitz et al., 2015). This relationship is moderated, however, by some parental and child characteristics, such as parents' own MA, educational background, and disciplinary style (Chiu & Xihua, 2008; Maloney et al., 2015). A child's disruptive and oppositional behavior may transform homework involvement in a source of conflict (Sibley et al., 2016; Wong & Goh, 2014).

Peers

Adolescents are especially sensitive to peer influences. Accordingly, one can expect that peers influence math performance and interest for better and for worse. A positive contagion influence on math interest was described in college students (Hazari et al., 2017). Evidence also indicates that, besides previous mastery experiences and self-efficacy beliefs, recognition by peers is an important precursor of math identity (Cribbs, Hazari, Sonnert, & Sadler, 2015). Correspondingly, Frenzel, Pekrun, and Goetz (2007) observed low to moderate associations between peer attitudes and adolescents' emotional involvement with math. Finally, segregation of adolescents in classes according to performance level has a positive effect on math interest for those on the higher-performing end and a negative effect for those on the lower-performing extreme of the distribution (Frenzel, Goetz, Pekrun, & Watt, 2010). These results can be interpreted under the social learning theory (Babad, 2009). Verbal persuasion is ineffective in increasing but extremely effective in lowering self-efficacy beliefs. Mixed classes with low- and high-performing students allow for both upward and downward social comparisons.

Math Achievement

Cognitive performance and math achievement are both antecedents and consequents of MA and manifest themselves as state and trait levels (Carey, Hill, Devine, & Szücs, 2016). The correlation between math achievement and MA is significant (Hembree, 1990). Poor math achievement raises the probability of MA. Low grades in math are an antecedent of MA in younger kids (Krinzinger et al., 2009) and in adolescents (Ma & Xu, 2004). Individuals with developmental dyscalculia are at risk of developing MA (Rubinsten & Tannock, 2010). Successful treatment of MA improves math performance (Hembree, 1990) even in individuals with developmental dyscalculia (Kamann & Wong, 1994).

Poor math achievement arises also because of MA. On a short-term range, effects of MA on performance are mediated by disruption of processing in working

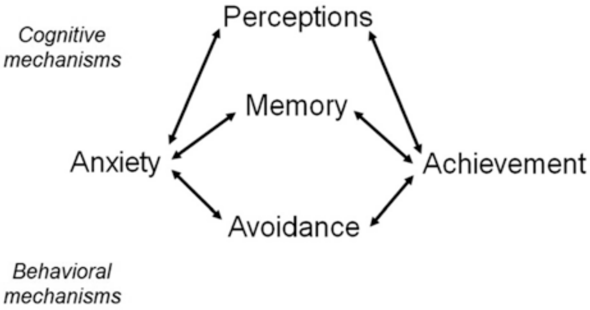

Fig. 29.1 Possible pathways between (math) anxiety and (math) achievement. In the cognitive pathway, the relationship between anxiety and achievement is mediated by perceptions (attitudes toward math and self-beliefs). In the behavioral pathway, avoidance behavior mediates between anxiety and achievement

memory (WM), as it will be discussed later. On the long run, MA interferes with performance through attitudes (Akin & Kurbanoglu, 2011) and avoidance behavior (Dew, Galassi, & Galassi, 1984). Negative attitudes toward math and avoidance of math activities reduce opportunities to learn mathematics and feed MA retroactively. Avoidance of math courses and heavily math-demanding careers may have lifelong consequences (Dew et al., 1984; Hembree, 1990; McMullan et al., 2012).

In Fig. 29.1, we systematize the possible relationships between MA and achievement. Two main pathways are postulated: a cognitive one, through which MA exerts short-term effects on math performance by disrupting WM processing, and a long-term one related to the development of negative perceptions toward math, such as negative attitudes and low self-efficacy. A behavioral pathway mediates effects of MA on performance by avoidance of math activities. A direct pathway between MA and math achievement and additional pathways between perceptions, WM, and avoidance should also be investigated.

Cognitive Mechanisms

A host of cognitive mechanisms has been implicated in MA. Although IQ is significantly correlated with math achievement, IQ is generally not associated with MA (Hembree, 1990). Interest on working memory impairments remains in the forefront, but basic numerical processing deficits also catch increasing attention.

Working Memory

WM has called most attention as a potential locus of cognitive impairments in MA. Six, not mutually exclusive, mechanisms have been characterized:

(a) Speed-accuracy trade-offs: In some circumstances, individuals with high MA solve calculation problems faster and less accurately compared to individuals without MA (Ashcraft & Faust, 1994). This speed-accuracy trade-off may result from the wish to terminate anxiety-eliciting situations as soon as possible.

(b) Competition for resources in working memory: According to the processing efficiency theory (Eysenck & Calvo, 1992), experiencing anxiety will draw on working memory capacities and therefore will compromise cognitive performance. Studies using the dual task paradigm have consistently shown that MA interferes more with complex tasks, heavily demanding on controlled processing, such as mental multi-digit calculation, than with more simple and automatized tasks, such as simple facts retrieval (Ashcraft & Faust, 1994; Ashcraft & Kirk, 2001; Faust, Ashcraft, & Fleck, 1996). The hypothesis is that anxious rumination competes for scarce processing resources in WM.

(c) Lack of inhibition: The inhibition theory predicts that performance should be proportionally reduced to the number of distractors present in the task (Hasher & Zacks, 1988). This mechanism was suggested in a study by Hopko, Ashcraft, Gute, Ruggiero, and Lewis (1998). In a dual-task paradigm, highly anxious MA individuals had their performance impaired as the number of distractors was raised.

(d) Attentional bias: Compared to individuals with low MA, those with high MA suffered more interference by mathematics-related distractors comparatively to neutral distractors (Hopko, McNeil, Gleason, & Rabalais, 2002; Suárez-Pellicioni, Núñez-Peña, & Colomé, 2014, 2015). This suggests the effect of an attentional bias toward emotionally negatively laden math information.

(e) Abnormal response to errors: In a series of studies, Suárez-Pellicioni, Núñez-Peña, and Colomé (2013a, 2013b) compared the performances of individuals with high and low MA levels in a neutral and in a numeric interference task. Evoked electrophysiological error negativities were enhanced in response to errors in the numerical in comparison to the non-numerical task. Emotional negativity to errors may thus represent a marker of vulnerability to MA and interference with WM processing.

(f) Arousal: Relationships between MA and WM processing are not straight linear. In some studies, individuals with high WM capacity are more susceptible to MA interference than those with lower WM capacity (Ramirez et al., 2013). This could be related to complex interactions between WM capacity and emotional arousal (Mattarella-Micke, Mateo, Kozak, Foster, & Beilock, 2011). Individuals with low WM capacity rely more on heuristics to solve arithmetic problems, and their performance is not affected significantly by individual differences in salivary cortisol. The situation is different for individuals with high WM capacity, who rely more on this ability to solve arithmetic problems. In individuals with high WM capacity, higher cortisol levels were associated with worsening of performance.

Numerical Abilities

Research initially suggested that numerical tasks susceptible to MA interference were those more demanding on WM resources. For example, Maloney, Risko, Ansari, and Fugelsang (2010) observed that MA impaired the performance in counting but not in subitizing tasks. Other evidence indicates that low-MA individuals use a plausibility strategy to discard large split errors in an arithmetic operation verification tasks and high-MA participants need to resort to controlled processing even in relatively easy items with large splits (Suárez-Pellicioni et al., 2016). This effect could be related to an impairment in some decision or evaluation stage of performance on the part of highly anxious participants. Otherwise, the impairments could rest on the low accuracy of numerical representations, predisposing these individuals to MA. The second hypothesis is suggested by studies uncovering impairments in both symbolic and nonsymbolic numerical processing tasks (Dietrich, Huber, Moeller, & Klein, 2015; Lindskog, Winman, & Poom, 2017; Maloney, Ansari, & Fugelsang, 2011; Núñez-Peña & Suárez-Pellicioni, 2014).

Visuospatial Abilities

Two studies indicate that impairments in visuospatial processing may be associated with risk of MA. Individuals with high MA reported a worse sense of direction and lower performance in behavioral tests of small- and large-scale spatial skills (Ferguson, Maloney, Fugelsang, & Risko, 2015). Gender differences in MA are moderated by visuospatial processing abilities (Maloney et al., 2012). As some forms of math performance are dependent on visuospatial processing, impairments in these abilities could predispose the individual to MA.

Neurobiological Underpinnings of Math Anxiety

The literature on the neurocognitive underpinnings of MA is increasing in size (Artemenko et al., 2015). Some patterns are emerging. Interpretation of results should consider that MA is related to test anxiety and both may be considered variants of social anxiety. MA inhibits the workings of dorsal hippocampus-derived regions associated with controlled and math-related processing such as the posterior parietal and dorsolateral prefrontal cortices (Young, Wu, & Menon, 2012). At the same time, MA activates ventral areas related to fear processing, such as the amygdala (Young, Wu, & Menon, 2012); regions such as the anterior insula, related to body discomfort and pain; and the cingulate gyrus, related to monitoring, social rejection, and psychological suffering (Lyons & Beilock, 2012a, 2012b). Activations in the insula and cingulate cortices are more salient in the anticipation than during the execution of math activities (Lyons & Beilock, 2012a, 2012b).

Functional neuroimaging studies suggest that MA is related to the activation of two emotional systems: monitoring and anticipation of psychological suffering and feelings of fear. These results fit to the extant knowledge on the neurofunctional bases of social anxiety. Hyperactivity of the cingulate cortex has been linked to feelings of rejection, embarrassment, and shame (Bastin, Harrison, Davey, Moll, & Whittle, 2016; Eisenberger, Lieberman, & Williams, 2003). Besides that, genetic relevance to social anxiety in the dopaminergic and serotonergic systems was associated with abnormal activations in the amygdala, insula, and cingulate areas (Eisenberger, Way, Taylor, Welch, & Lieberman, 2007; Frick et al., 2015; Hariri et al., 2002; Klumpp et al., 2014).

Assessment of Math Anxiety

The phenomenology of MA may be subtle, such as mild feelings of apprehension and dislike, or more intense and easily observable, such as genuine fear or dread (Ashcraft & Ridley, 2005). Manifestations may be antecedent, concomitant, or following the engagement in math activities. At the cognitive level, MA is associated with rumination and expression of negative thoughts and feelings related to math itself and to the individual's own math ability. Affectively, MA manifests as tension, annoyance, frustration, anger, revolt, despair, shame, etc. Physiological manifestations consist of palm sweating, nausea, tension in the stomach, heartburn, breathlessness, fever, etc. Behavioral concomitants are avoidance of engagement in math activity, lack of concentration, inability to follow teachers' instructions, etc.

Few studies have assessed the physiological components of MA, using, for example, determination of salivary and hair cortisol levels (Mattarella-Micke et al., 2011; Pletzer, Wood, Moeller, Nuerk, & Kerschbaum, 2010; Sarkar, Dowker, & Kadosh, 2014). For clinical, educational, and research purposes, MA is usually assessed using self-report scales. The psychometric properties of the most widely used scales for children, adolescents, and adults are generally good and reviewed, respectively, in Tables 29.1, 29.2, and 29.3.

MA self-report scales are built to assess the cognitive (attitudes, worry, and self-representations) and affective (emotions and distress) dimensions of the construct. These two-factor structure has been replicated across instruments (Dew et al., 1984), age ranges (Richardson & Suinn, 1972; Suinn & Edwards, 1982; Suinn, Taylor, & Edwards, 1988), and countries (Ho et al., 2000; Wood et al., 2012). Lack of reliable biological and cognitive markers and gold standard or criterion validity of MA scales is unfortunate. Decisions regarding intervention must ultimately rely on clinical judgment as to the academic and psychosocial impact of MA.

Table 29.1 Psychometric properties of some scales to assess math anxiety in children

Instrument	Suinn Mathematics Anxiety Rating Scale, Elementary Form (MARS-E)	Mathematics Anxiety Survey (MAXS)	Math Attitude and Anxiety Questionnaire (MAQ)	Scale for Early Mathematics Anxiety (SEMA)	Children's Anxiety in Math Scale (CAMS)	Child Math Anxiety Questionnaire (CMAQ) **Revised Child Math Anxiety Questionnaire (CMAQ-R)**	Math Anxiety Scale for Young Children (MASYC)	12-item mathematics anxiety scale
Authors	Suinn et al. (1988)	Gierl and Bisanz (1995)	Thomas and Dowker (2000)	Wu et al. (2012)	Jameson (2013)	CMAQ: Ramirez et al. (2013) CMAQ-R: Ramirez et al. (2016)	Harari, Vukovic, and Bailey (2013)	Vukovic et al. (2013)
Target population	4th–6th grades	3rd and 6th grades	1st–2nd grades	2nd–3rd grades	1st–5th grades	1st–3rd grades	1st grade	2nd–3rd grades
Number of items	26	14	28	20	16	CMAQ: 08 CMAQ-R: 16	12	12
Rating	5-point Likert scale	5-point Likert scale with drawings to help	5-point Likert scale aided by different pictures at each rate	5-point Likert scale aided by graded anxious and non-anxious faces.	5-point Likert scale aided by graded anxious and non-anxious faces.	5-point Likert scale aided by graded anxious and non-anxious faces	4-point Likert scale	4-point Likert scale

(continued)

Table 29.1 (continued)

Instrument	Suinn Mathematics Anxiety Rating Scale, Elementary Form (MARS-E)	Mathematics Anxiety Survey (MAXS)	Math Attitude and Anxiety Questionnaire (MAQ)	Scale for Early Mathematics Anxiety (SEMA)	Children's Anxiety in Math Scale (CAMS)	Child Math Anxiety Questionnaire (CMAQ) **Revised Child Math Anxiety Questionnaire (CMAQ-R)**	Math Anxiety Scale for Young Children (MASYC)	12-item mathematics anxiety scale
Constructs/ internal structure	Two factors: (1) mathematics test anxiety (worry about evaluation of math) and (2) mathematics performance adequacy anxiety (worry about doing well on math)	Level of nervousness to different situations involving mathematics Two subscales: (1) mathematics test anxiety and (2) mathematics problem-solving anxiety	Two factors: (1) general math-related attitudes (evaluation of mathematics) and negative emotions and (2) anxiety concerning mathematics (math anxiety)	Two factors: (1) numerical processing anxiety (the child is confronted with math problems and reports how they feel) and (2) situational and performance anxiety (the child reports how they feel in math-related class situations)	Three factors:(1) general math anxiety (generic situations), (2) math performance anxiety (worry about showing good performance), and (3) math error anxiety (worry about making mistakes)	Attitudes about problem-solving and specific situations. Internal structure not reported	Three factors: (1) negative reactions (physiological responses), (2) numerical confidence (think positively about mathematics) and (3) worry (math anxiety)	Mathematics anxiety internal structure not reported

Sample items	Factor 1: “Taking a big test in your math class.”factor 2: “Getting called on by the teacher to do a math problem on the board.”	Subscale 1: “How nervous and tense do you usually feel during math tests?” subscale 2: “How nervous would you usually feel reading your answer to math questions out loud in class?”	Four different types of questions (“how good are you at. ..?” “how much do you like. ..?” “how happy or unhappy are you if you have problems with. ..?” and “how worried are you if you have problems with. ..?”) for seven math-related situations (math in general, written calculations, mental calculations, easy calculations, difficult calculations, math homework, and listening and understanding during math lessons; e.g., “how much do you like math in general?”)	Factor 1: “Is it right 9 + 7 = 12” Factor 2: “You are in math class and your teacher is about to teach something new”	Factor 1: “When I solve math problems, I feel:”factor 2: “When the teacher calls on me to answer a math problem, I feel:” Factor 3; “when I make a mistake in math, I feel:”	(1) attitudes about problem-solving: “There are 13 ducks in the water, there are 6 ducks on land, how many ducks are there in total?”) (2) specific situations: “Being called on by a teacher to explain a math problem on the board”	Factor 1: “Math gives me a stomachache” factor 2: “I like doing math problems on the board in front of the class”factor 3: “I get nervous about making a mistake in math”	Positive items: “I like being called on in math” negative items: “I get nervous about making a mistake in math”
Reliability	Cronbach’s alpha = 0.89	Cronbach’s alpha for 3th grade: 0.75 Cronbach’s alpha for sixth grade: 0.76	Cronbach’s alpha = 0.83–0.91	Cronbach’s alpha = 0.87 Split-half reliability = 0.77	Cronbach’s alpha = 0.86	CMAQ’s Cronbach’s alpha = 0.55 CMAQ-R’s Cronbach’s alpha = 0.83	Cronbach’s alpha = 0.70	Cronbach’s alpha = 0.80

(continued)

Table 29.1 (continued)

Instrument	Suinn Mathematics Anxiety Rating Scale, Elementary Form (MARS-E)	Mathematics Anxiety Survey (MAXS)	Math Attitude and Anxiety Questionnaire (MAQ)	Scale for Early Mathematics Anxiety (SEMA)	Children's Anxiety in Math Scale (CAMS)	Child Math Anxiety Questionnaire (CMAQ) **Revised Child Math Anxiety Questionnaire (CMAQ-R)**	Math Anxiety Scale for Young Children (MASYC)	12-item mathematics anxiety scale
Some studies using the scale	Satake and Amato (1995)	No additional studies found in PubMed and Google scholar	Krinzinger et al. (2007, 2009)	Young et al. (2012)	Jameson (2014)	CMAQ-R: Maloney et al. (2015)	Ching (2017)	No additional studies found in PubMed and Google Scholar
	Wittman, Marcinkiewicz, and Hamodey-Douglas (1998)		Dowker et al. (2012)	Jansen et al. (2013)				
	Tsui and Mazzocco (2006)		Haase et al. (2012)	Supekar et al. (2015)				
	Wang et al. (2014)		Wood et al. (2012)	Wu et al. (Wu et al., 2017, Wu, Willcutt, Escovar, & Menon, 2014)				

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Table 29.2 Psychometric properties of some scales to assess math anxiety in adolescents

Instrument	Fennema-Sherman Mathematics Attitude Scale (FSMAS) **Fennema-Sherman Mathematics Attitude Scale Short Form (FSMAS - SF)**	Mathematics Attitude Inventory (MAI)	Math Anxiety Rating Scale for Adolescents (MARS-A)	Math Anxiety Questionnaire (MAQ)	Mathematics Anxiety Scale for Children (MASC)
Authors	Fennema and Sherman (1976) Mulhern and Rae (1998)	Sandman (1980)	Suinn and Edwards (1982)	Wigfield and Meece (1988)	Chiu and Henry (1990)
Target population	High school	7th–12th grades	7th–12th grades	6th–12th grades	4th–8th grades
Number of items	108 51	48	98	11	22
Rating	5-point Likert scale	4-point Likert scale	5-point Likert scale	7-point Likert scale	4-point Likert scale
Constructs/ internal structure	FSMAS subscales measure: (1) anticipate consequences of success in math; (2) math as a male domain; (3) perception of the teacher (4 and 5); mother's and father's confidence in the student's ability; (6) confidence in one's own ability; (7) anxiety related to doing math; (8) effectance as applied to mathematics; and (9) students' beliefs about the usefulness of mathematics FSMAS-SF six factors: (1) mathematics-related affect scale (confidence x anxiety in math), (2) parent's attitudes, (3) usefulness, (4) male domain, (5) success, (6) teacher	Six factors: (1) perception of the mathematics teacher, view regarding the teaching characteristics of the teacher, (2) anxiety towards mathematics, uneasiness a student feels in situations involving math; (3) value of mathematics in society, usefulness of mathematical knowledge; (4) self-concept in mathematics, perception of own competence; (5) enjoyment of mathematics, pleasure derived from engaging in mathematical activities; (6) motivation in mathematics, desire to do work in mathematics beyond the class requirement	Two factors: (1) numerical anxiety (situations involving numbers) and (2) mathematics test anxiety (situations involving math evaluation)	Two factors: (1) negative affective reactions (primarily strong affective reactions to mathematics) and (2) worry (cognitive concerns about doing well in math)	Four factors: (1) mathematics evaluation anxiety (evaluation of mathematics learning), (2) mathematics learning anxiety (activity or process of learning mathematics), (3) mathematics problem-solving anxiety (solving math problems in non-testing situations), and (4) mathematics teacher anxiety (mathematics teacher characteristics). Two items don't fit in any factor

(continued)

Table 29.2 (continued)

Instrument	Fennema-Sherman Mathematics Attitude Scale (FSMAS) **Fennema-Sherman Mathematics Attitude Scale Short Form (FSMAS - SF)**	Mathematics Attitude Inventory (MAI)	Math Anxiety Rating Scale for Adolescents (MARS-A)	Math Anxiety Questionnaire (MAQ)	Mathematics Anxiety Scale for Children (MASC)
Sample items	Not reported Factor 1: "Generally I have felt secure about attempting mathematics"; factor 2: "My mother thinks I need to know just a minimum amount of math"; factor 3: "Mathematics is of no relevance in my life"; factor 4: "Girls who enjoy studying math are a bit peculiar"; factor 5: "I'd be proud to be the outstanding student in math"; factor 6: "Getting a mathematics teacher to take me seriously has usually been a problem"	Not reported	Factor 1: "Playing cards where numbers are involved"factor 2: "Taking a final exam"	Factor 1: "Taking math tests scares me" Factor 2: "In general, how much do you worry about how well you are doing in school?"	Factor 1: 'Taking a quiz in a math class" Factor 2: "Getting a new math book" Factor 3: "Reading and interpreting graphs or charts" Factor 4: "Listening to the teacher in a math class" None of factors: "Picking up a math book to begin working on a homework assignment"
Reliability	Split-half reliability range 0.86–0.93 Cronbach's alpha = 0.93	Cronbach's alpha range 0.68–0.89	Spearman-Brown split-half = 0.90Guttman split-half method = 0.89Cronbach's alpha = 0.96	Cronbach's alpha for the negative affective reactions = 0.82Cronbach's alpha for the worry scale = 0.76	Cronbach's alpha = 0.92
Some studies using the scale	FSMAS: Sherman (1982) FSMAS: Iben (1991) FSMAS: Drisko (1993)	No additional studies found in PubMed and Google Scholar	No additional studies found in PubMed and Google Scholar	Meece et al. (1990)	Lai, Zhu, Chen, and Li (2015)

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Table 29.3 Psychometric properties of some scales to assess math anxiety in adults

Instrument	Numerical Anxiety Scale (NAS)	Math Anxiety Rating Scale (MARS)	Mathematics Anxiety Scale (MAS)	Revised Math Anxiety Rating Scale (MARS-R)	Short Math Anxiety Rating Scale (sMARS)	Abbreviated Math Anxiety Scale (AMAS)
		Math Anxiety Rating Scale 30-item (MARS 30-item)	Mathematics Anxiety Scale-Revised (MAS-R)			
Authors	Dreger and Aiken Jr (1957)	Richardson and Suinn (1972)	Betz (1978)	Plake and Parker (1982)	Alexander and Martray (1989)	Hopko, Mahadevan, Bare, and Hunt (2003)
		Suinn and Winston (2003)	Bai, Wang, Pan, and Frey (2009)			
Target population	College students	College students	College students	College students	College students	College students
Number of items	3	98	10	24	25	9
		30	14			
Rating	Nominal scale (yes/no)	5-point Likert scale	5-point Likert scale	5-point Likert scale	5-point Likert scale	5-point Likert scale
Constructs/internal structure	Syndrome of emotional reactions to arithmetic and mathematics Two factors: (1) negative math reaction and (2) nervousness in the presence ofmath	Single homogeneous factor "mathematics anxiety"	Feelings of anxiety, dread, nervousness, and associated bodily symptoms related to doing mathematics Internal structure not reported	Two factors: (1) learning math anxiety (activity of studying mathematics) and (2) math evaluation anxiety (evaluation of mathematics)	Three factors: (1) math test, (2) numerical task, and (3) math course anxiety	Two factors: (1) learning math anxiety (activity of studying mathematics) and (2) math evaluation anxiety (evaluation of mathematics)
		Two factors: (1) mathematics test anxiety (situations involving math tests) and (2) numerical anxiety (situations involving numbers)	Identify the bidimensional affects, positive and negative, toward mathematics Two factors: (1) positive affect factor and (2) negative affect factor			

(continued)

Table 29.3 (continued)

		Math Anxiety Rating Scale (MARS)	Mathematics Anxiety Scale (MAS)			
Instrument	Numerical Anxiety Scale (NAS)	Math Anxiety Rating Scale 30-item (MARS 30-item)	Mathematics Anxiety Scale-Revised (MAS-R)	Revised Math Anxiety Rating Scale (MARS-R)	Short Math Anxiety Rating Scale (sMARS)	Abbreviated Math Anxiety Scale (AMAS)
Sample items	Factor 1: “I was never as good in math as in other subjects” Factor 2: “I am often nervous when I have to do arithmetic”	“Adding two three-digit numbers while someone looks over your shoulder”	Positive items: “I have usually been at ease in math courses” Negative items: “I get really uptight during math tests”	Factor 1: “Starting a new chapter in a math book” Factor 2: “Thinking about an upcoming math test one day before”	We did not have access to sample items	Factor 1: “Watching a teacher work in a algebraic equation on the blackboard” Factor 2: “Taking an examination in a math course”
		Factor 1: “Taking an examination (final) in a mathematics course” Factor 2: “Figuring out your monthly budget”	Positive items: “I find math interesting” Negative items: “Mathematics makes me feel confused”			
Reliability	Not reported	Test-retest. 7 weeks later = 0.85 Cronbach’s alpha = 0.97	Split-half reliability = 0.92	Cronbach’s alpha = 0.98 Correlated 0,97 with full MARS	Coefficient alpha was 0.96 for factor 1, 0.86 for factor 2, and0.84 for factor 3	Cronbach’s alpha = 0.90 2-week test-retest reliability = 0.85
		Cronbach’s alpha = 0.96 1-week test-retest reliability = 0.90	Cronbach’s alpha = 0.85 9-week test-retest reliability = 0.71			

Some studies using the scale	No additional studies found in PubMed and Google Scholar	98-item: Clute (1984)	MAS: Pajares (1996)	Hopko (2003)	Kirk and Ashcraft (2001)	Maloney et al. (2010, 2011, 2012)
		98-item: Austin, Wadlington, and Bitner (1992)	MAS-R: Ariapooran (2017)	Buelow and Frakey (2013)	Baloglu and Kocak (2006)	Devine et al. (2012)
		98-item: Gresham (2007)		Mizala, Martínez, and Martínez (2015)	Beilock et al. (2010)	Jameson and Fusco (2014)
		30-item: Sarkar et al. (2014)				Andrews and Brown (2015)

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Interventions for Math Anxiety: From the Lab to the Classroom

Ideally, interventions tailored to mitigate MA should consider early recognition, prevention, classroom practices, new learning technologies, math tutoring, functional cognitive training, cognitive-behavioral therapy, and might include even neuromodulation, and so on. Interventions should also consider the four levels of MA manifestations and their interactions. For example, psychoeducation and cognitive restructuring act on attitudes and beliefs, potentially reducing avoidance behaviors. At the behavioral level, experiences of programmed success contribute to changes in motivation. Relaxation and emotional reassurance reduce the affective and physiological components, potentially reflecting on the other levels.

As MA is both an antecedent and a consequent of poor math performance, measures to promote math achievement should be effective in preventing MA. It has been shown, for example, that preschool math education is an important long-term predictor of future achievement (Melhuish et al., 2008). Data on the intergenerational transmission of attitudes toward math and MA indicate that psychoeducational measures promoting parents' math talk are effective (Berkowitz et al., 2015; Gunderson, Ramirez, Levine, & Beilock, 2012; Levine, Suriyakham, Rowe, Huttenlocher, & Gunderson, 2010; Maloney et al., 2015). Moreover, fostering primary teachers math self-efficacy (Beilock et al., 2010; Gunderson et al., 2012; Swars et al., 2006; Tooke & Lindstrom, 1998) could play an important role. Media campaigns may focus on social stereotypes and promote positive attitudes toward math education and STEM careers (Budd, 2015). Initiatives such as the Math Olympics yearly attract an increasing number of students and potentially stimulate math interest and achievement (Biondi, Vasconcellos, Menezes-Filho, & Cristia, 2012). Programs have also been developed focusing on gender stereotype threat (Johns, Schmader, & Martens, 2005).

Classroom practices also play an important role in the development and maintenance of MA. Traditional teaching based on high demand for correctness and little cognitive or motivational support is associated with MA (Bekdemir, 2010; Turner et al., 2002). Several pedagogical practices have been suggested to prevent or alleviate MA. In general, teachers should cope with their own fears and negative attitudes, promoting a positive climate toward learning math. Learning should be as fun as possible. As an example, block constructions and board games have been used to promote math achievement (Siegler & Ramani, 2009; Wolfgang, Stannard, & Jones, 2001). Some traditional resources such as math magics and math stories have gained renewed attention (Berkowitz et al., 2015; Budd, 2015; Hassinger-Das, Jordan, & Dyson, 2015; Tahan, 1993). Finally, new technologies in learning analytics (Baalsrud-Hauge et al., 2015) can be employed both to improve the math learning experience and to help identifying MA and mobilizing available psychosocial and cognitive resources to mitigate it.

Reducing working memory load is an important goal, as MA competes with task-related demands in working memory and leads to extremely aversive processing capacity overload. Approaches to reduce working memory load include use of concrete materials, explicit instruction and practice, worked examples, errorless learning, programmed learning, etc. (Sweller, Ayres, & Kalyuga, 2011). Timed tests should probably be avoided (Faust et al., 1996). Moreover, the use of game elements in math tasks may help enhance working memory capacity (Ninaus et al., 2015) by increasing the level of concentration on the task and away from maladaptive negative emotional reactions.

Moreover, feedback from teachers and parents is important. Negative influences may originate not only from harsh educational practices but also from teachers and parents trying to console for failure. Consolation for math failure may deliver the implicit message that the task is above the student's capacity (Beilock & Willingham, 2014; Rattan, Good, & Dweck, 2012). This supports a fixed mind set, according to which achievement is more dependent on a limited capacity than on effort. Thus, reinforcement should be conditional on effort and not on ability. Accordingly, a customized achievement assessment is in order. Teaching is an exercise in theory of mind (Strauss & Ziv, 2012). Teachers and parents should be aware of what is going on in their own and their students' and children's minds and of the hazardous potential associated with even inadvertent negative commentaries. The Hippocratic precept of *primum non nocere* could be generalized for math teaching.

Children with poor math performance are an important target group. Math learning difficulties are a risk factor for MA (Rubinsten & Tannock, 2010). Therefore, considering the bidirectional relationship between MA and math achievement, early recognition and intervention for poor math achievement are important to prevent spiraling positive feedback loops that worsen both math difficulties and MA. Data indicate that math tutoring is effective to alleviate MA (Hembree, 1990; Supekar, Iuculano, Chen, & Menon, 2015), and successful treatment of MA improves performance in children with math learning difficulties (Kamann & Wong, 1994).

Established MA can be addressed by psychotherapy. Several forms of psychotherapy, based on cognitive-behavioral approaches such as relaxation, desensitization, restructuring, and self-management, have proven efficacy, with effect sizes ranging from 0.30 to 0.60 on math achievement (Hembree, 1990). Expressive writing of math anxiety has been proposed as a technique that can be used in the classroom with adolescents as well as college students (Park, Ramirez, & Beilock, 2014; Ramirez & Beilock, 2011). The rationale is to reappraise anxiety and reduce ruminative thoughts, thus freeing working memory resources to focus on the task and not on the anxiety itself. Training to achieve mindful states is also effective to reduce MA on the short time range (Brunyé et al., 2013).

Finally, some new technologies have raised enormous educational and therapeutic interest (Kadosh, Dowker, Heine, Kaufmann, & Kucan, 2013). Interest in computer games to promote numerical and arithmetical abilities dates back to the 1960s, and effect magnitudes are in the order of $d = 0.30$ (Räsänen, 2015). It has been difficult, however, to integrate computer-assisted technologies in the classroom and in the therapeutic setting (Young et al., 2012). The playability of such games is low as compared to leisure videogames. Notwithstanding the lack of prescriptive evidence, videogames exert a powerful motivational effect and contribute to attitude changes regarding math and self-efficacy (Butterworth & Laurillard, 2010; Cezarotto & Battaiola, 2016; Ninaus et al., 2015; Räsänen, 2015). Enjoyable videogames could help to offload emotional and cognitive demands associated to difficult and/or repetitive tasks. Noninvasive micro-current brain stimulation has been also proposed as a technique to both improve arithmetic performance (Iuculano & Kadosh, 2014; Pasqualotto, 2016) and reduce MA (Sarkar et al., 2014). Evidence of efficacy is still preliminary, and effectiveness outside the experimental setting is unknown. Use of neurophysiological signals to control mental states by means of neurofeedback holds promise in the treatment of MA. Evidence for the efficacy of neurofeedback in the treatment of anxiety disorders is growing (Micoulaud-Franchi et al., 2015; Schoenberg & David, 2014). To the best of our knowledge, there are no studies reporting effects of neurofeedback procedures on MA. Home-based portable neurofeedback appliances could be used to augment ecological validity and generalizability of interventions for MA.

Conclusion

The complete reach of MA as an impairment of math learning still has to be determined in both basic and applied educational and neuroscientific research. However, data suggest it is an increasingly important problem in our knowledge society. MA also raises questions related to gender equity.

Progress has been made in the measurement and evaluation of the cognitive and emotional aspects of MA. In the near future, the agenda should include reliable assessments of the psychosocial impact of MA as well. The traditional approach of categorizing individuals in a sample according to their performance level on self-report scales does not provide information on the real impact of the condition in daily and academic life. Another limitation of current research is the reliance on self-report measures that assess only the cognitive and affective dimensions of this multidimensional construct. Research efforts should integrate the physiological and behavioral dimensions of MA.

The good news is that parents and teachers are more aware of the nature and extension of MA as a real and severe problem, which costs a huge amount of human capital and may become more active and influence policymaking and research funding. These good news were absolutely necessary. The increasing level of dependence on high technology in modern societies opens several new niches for the emergence of MA among individuals who would not be affected a century ago. The main challenge posed by our modern times is to understand how to mobilize the cognitive and emotional resources available to individuals with different degrees of interest and competencies before the demands of life in technological societies generate suffering and maladaptation. Relatively simple and effective behavioral and cognitive measures both at home and at school are available, helping to prevent and reduce the negative consequences of MA. Moreover, if they are barely sufficient to cope with the demands offered by our present technological stand, much more research is necessary to prepare for the ever-growing demands of mathematics abilities we will encounter in modern society in the near future.

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Obs. References marked with # refer to self-report questionnaires presented in Tables 29.1, 29.2, and 29.3.

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Chapter 30
Cognitive and Motivational Underpinnings of Mathematical Learning Difficulties: A Discussion

Mark H. Ashcraft

The nine chapters in Part III are all devoted to different facets of the same question: What factors are responsible for mathematical learning difficulties, or, as it is frequently called, *dyscalculia*? The answer to the question is important, quite obviously, in that identifying the factors responsible for the difficulties, even if each identified factor is only part of the answer, should give us improved methods for intervening with children (or adults) who experience the difficulty. (In this chapter, the terms MLD for mathematical learning difficulties and DD for developmental dyscalculia will be used interchangeably.) Equally important, of course, is the approach taken in investigating MLD. Uniformly, the overall approach presented in this section is a cognitive science approach.

Reflect for a moment on the importance of theoretical approach. To take a rather extreme example from psychology's past, imagine that your theoretical approach to learning was a behaviorist viewpoint; learning about number magnitude and arithmetic would be an issue of reinforcement, with rewards delivered repeatedly for correct performance. From such a standpoint, a child who experiences MLD would be "treated" with an enhanced regimen of learning trials – more reinforced learning – in order to strengthen correct responses and weaken incorrect responses. In practice, such an intervention would involve repeated drill, rote rehearsal, flash cards, repetition, and the like, with no consideration of any of the subtleties or different types of knowledge that are used to perform numerical tasks. There likely would be no difference in "treating" a child who makes occasional, non-systematic errors in subtraction like 13–7 = 5 and one who makes a systematic error like 13–7 = 14. The latter error is now widely acknowledged to be due to the common "smaller from larger" bug (e.g., VanLehn, 1990), a mistaken procedure in the child's knowledge of subtraction.

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In contrast, far more is known about numerical cognition today, about the mental processing involved in a variety of tasks that tap into our knowledge of number, and about the cognitive, affective, and biological factors that can influence learning and performance. These are the topics covered in Part III. In this chapter, I summarize the chapters in this section, seeking especially to identify common themes across chapters and areas of research. As appropriate, areas of consensus as well as of disagreement will be noted, and important gaps in the existing literature will be pointed out as well. An overarching sense of the chapters is that a tremendous amount of progress has taken place in our understanding of mathematics learning difficulties, much of that progress coming in the past 20 years or so.

Chapter 21: Carvalho and Haase

In the first of their two chapters, Carvalho and Haase discuss the role of genetics in developmental dyscalculia (DD) and the literature that reflects recent research that has searched for evidence of that genetic role. They note at the outset that math abilities tend to run in families, at both ends of the ability distribution, and mention that twin studies have shown at least moderate heritability of math abilities. A difficulty in all of this research, however, is the heterogeneous nature of the MLD or DD phenotype; there are many different ways in which an individual can be impaired in math ability, and samples of such individuals, for twin studies or genotype assays, surely reflect that heterogeneity. A second prominent reason for the slow progress in this field involves different criteria that have been used to define DD, with some studies using a discrepancy criterion, some a threshold criterion, and some a response-to-intervention criterion; samples with different inclusion criteria will, of course, be difficult to compare, and within-sample variation can still be large, depending on the criterion. A final source of sample variation is the degree to which DD is often comorbid with other disorders, e.g., dyslexia, ADHD, and language impairment, and how those comorbidities will compromise the results of genetic testing.

The chapter explains current methods for conducting genetic studies and portrays the results of the relatively few studies that have examined MLD or DD, noting that progress in this area has been unfortunately slow, due to the aforementioned heterogeneity of the phenotype of DD. The authors' second chapter proposes a possible solution to this dilemma.

Chapter 22: Haase and Carvalho

In their second chapter, Haase and Carvalho explore the concept of a cognitive endophenotype, an intermediate cognitive phenotype that might exist between the genetic/environmental etiologic level and the phenotypic level of DD. That is to say, our phenotype of DD, our category so to speak, is a broad, complicated, and heterogeneous one, and might be reconceived as having several subtypes or endophenotypes, each with its own separate set of characteristics or abilities. The search for genetic patterns underlying developmental dyscalculia might be advanced if meaningful subtypes – endophenotypes – can be identified, according to the authors' proposal.

To further this idea, the authors adopt the four-way set of abilities originally proposed by Wilson and Dehaene (2007): (1) basic number processing, including the quantification of sets and transcoding between different notations; (2) phonological processing; (3) visuospatial and visuo-constructional processing; and (4) working memory and executive functioning. After a discussion of these abilities, the authors turn to evidence about several genetic disorders known to affect mathematical abilities, to see how those effects map onto the four-way set of abilities. The syndromes considered include Turner, Klinefelter, Williams, and Fragile X. After reviewing these, the authors conclude that the genetic evidence to date supports all but the fourth category, the working memory and executive functioning set of abilities, as viable endophenotypes; disruptions in working memory and executive function appear to be common to all the reviewed syndromes, so lack the necessary specificity to be a useful candidate for an endophenotype in DD.

Chapter 23: DeSmedt, Peters, and Ghesquière

DeSmedt, Peters, and Ghesquière's chapter concerns the neurobiological evidence about dyscalculia, in particular the evidence we now have about dyscalculia afforded by modern brain scanning technology, functional magnetic resonance imaging (fMRI). This research is of course quite new; as the authors point out, MRIs only became available as research tools in the early 1990s. Nonetheless, a sufficient number of studies have been conducted on adults, typically developing children, and dyscalculic children, using tasks that assess knowledge of magnitude and arithmetic, to draw several important conclusions about dyscalculia's neurobiological signature, both in terms of brain structure and function.

Two types of numerical knowledge were examined in the research reviewed in DeSmedt et al.'s chapter: knowledge of numerical magnitude and arithmetic. For clarity, it seems important to be explicit about the methods used to test such knowledge in the behavioral research literature, and then how these methods are translated into the MRI environment.

The common way of testing magnitude knowledge is the magnitude comparison task. In the symbolic version of the task, participants are shown a pair of digits, say 5 and 8, and are asked to make a speeded judgment to indicate which digit is larger, the one on the left or on the right (we could just as easily ask for which digit is smaller). We typically would then organize the data by looking at reaction time (RT) as a function of the difference between the values, such that all pairs with a difference of 1 (the pairs 1 2, 2 3, 3 4, etc.) would be grouped together, and likewise for all pairs with differences 2, 3, etc. The standard effect, observed repeatedly, is that RT decreases significantly as the distance between the digits grows larger. In other words, the “numerical distance effect” is basically a classic discrimination effect; the more distant two digits are in terms of their magnitude, the more easily they are discriminated, and the easier it is to decide which is larger (smaller). This effect is robust and is routinely obtained both for adults and for children who know the values of the digits. Children with dyscalculia, as noted by DeSmedt et al., show significant deficits in this task when their timed performance is examined. This is critical, of course, since symbolic magnitude comparison is a strong predictor of later mathematical achievement.

The nonsymbolic magnitude comparison task is similar in that two stimuli are displayed on each trial, but the stimuli are not digit symbols. Instead, two arrays are displayed, typically arrays of dots, with one array containing more dots than the other. As before, the participant must indicate whether the left or the right display is larger (or smaller, depending on instructions). As in the symbolic version of the task, a numerical distance effect is also obtained here as well. The effect is easily obtained with typically developing children and adults. Interestingly, it is also obtained even with very young children, including those who do not yet know the number system, and if the ratios between the two displays are large enough, even lower animals can make these discriminations (see, e.g., Agrillo, 2015, and Brannon & Park, 2015, for reviews). Several theorists argue that this evidence for magnitude effects in nonsymbolic comparisons suggests that sensitivity to magnitude is an inherent ability, that is, a genetically provided ability of the human system, a part of our biology sometimes referred to as the approximate number system (ANS).

Results of nonsymbolic magnitude comparisons are less consistent with children with dyscalculia; some studies show deficits, and some do not. Overall, however, children show greater involvement of the frontal cortex than do adults, signifying greater need for attentional resources on the part of children, whereas adults showed greater involvement of the bilateral (intra)parietal cortex than children. Of concern, as DeSmedt et al. note, are the few studies showing that children with dyscalculia showed lower IPS (intraparietal sulcus) activity than their matched peers when doing nonsymbolic magnitude comparisons. Given the wealth of evidence showing IPS involvement in number processing (e.g., Dehaene, 1992), this would clearly be an area of research in need of further attention.

Tests of people’s knowledge of arithmetic generally involve showing participants arithmetic problems, most typically the so-called basic facts of addition, subtraction, multiplication, and division (division has seldom been tested), i.e., 0 + 0 up to 9 + 9 for addition and 0 × 0 up to 9 × 9 in multiplication, and the inverses of those for subtraction and division. Typically, one problem is shown per trial, and the participant states the answer to the problem, although sometimes the problem is shown with a proposed answer and the participant makes a true/false decision (especially in fMRI settings) via a simple button press response. We routinely time participants’ performance and use both RT and errors as indices of performance. The overwhelmingly common result, as DeSmedt et al. note, is called the “problem size effect,” that is, that RT (and errors) will increase as the size of the problem increases, whether size is indexed by the answer or the operands of the problem. This foundational effect is found across all four arithmetic operations, at all ages, and across all cultures that have been tested.

It is particularly important to note that, for children especially, a major part of the problem size effect is due to reliance on time-consuming strategies. The major change across schooling, in other words, is a change in the basis of performance, with a decreasing reliance on strategies like counting and an increase in fact retrieval; fact retrieval is widely viewed as the fastest and most efficient method for performing arithmetic and importantly the least demanding on working memory resources (e.g., Ashcraft, 1992).

DeSmedt et al. review evidence that indicates children with dyscalculia have difficulties in understanding and executing procedural strategies in arithmetic and also show fact retrieval deficits. That is, as normally developing children begin to abandon reconstructive strategies in favor of fact retrieval, children with MLD persist in relying on strategies for solving basic facts, despite the tendency for those strategies to generate errors (losing track in a sequence of steps, losing track during counting) and to drain the resources of working memory.

When fMRI scans are considered, DeSmedt et al. summarize by saying that the typical adult shows a frontoparietal network that is consistently active during number processing and arithmetic, frontal areas indicating attentional and executive control functions and parietal regions indicating especially active networks involving number. Comparisons of adults and typically developing children showed similar frontoparietal activations, but also greater hippocampal activity in the children, suggestive of formation of long-term memories for the arithmetic facts. There have been only a few fMRI studies with children with dyscalculia, unfortunately, so the results are not yet consistent. There is some evidence, however, that typically developing children show an increase in activation in the frontoparietal network when confronted with more challenging arithmetic problems, whereas there is no such change in activation for children with dyscalculia. As noted, this could be because the dyscalculic children continue to use the same immature strategies for all problems, simple as well as more challenging, whereas the normally developing children have switched to fact retrieval for the simple problems (see Geary, 2011, for details).

Chapter 24: Krinzinger

Krinzinger's chapter differs from the other chapters in this section in that it is not intended to be an extensive review of a particular area, in this case the area of ADHD research and its relationship to MLD. Instead, Krinzinger describes a critical dilemma in this area and then presents the results of a study aimed at addressing this dilemma. The dilemma is the issue of comorbidity of MLD and ADHD, and whether a child diagnosed with both should be considered (and treated) as suffering primarily from MLD or ADHD. She identifies this as pMLD for primary mathematical learning disability and sMLD for secondary mathematical learning difficulty due to ADHD (or other primary disorders such as depression). She then reviews attempts in the assessment literature to test for MLD, noting the deficiencies or difficulties that have been encountered. In particular, the more general cognitive effects of ADHD will impact attentional and working memory functions, affecting a child's math performance, even though the child's core numerical abilities may be unaffected. On the other hand, a child with pMLD should show evidence of core numerical deficits, regardless of attentional or working memory difficulties.

Krinzinger then presents new evidence from a study of adolescents, where the nature of their errors on a battery of math tests has been examined. The adolescents were asked to complete a number line estimation task, to perform some number counting tasks, and then to take a math test involving simple through complex arithmetic problems. Analysis of the error types that were made revealed some interesting patterns across the several groups of participants. Several types of errors did not discriminate between the pMLD and sMLD groups, e.g., making table errors in retrieval, making trading errors in subtraction, or making counting and trading errors in combination. These can all be attributed to attentional and working memory lapses, she reasoned. But changing the unit (1s) value in a counting backward by 10s task, or changing from 100s to 10s when counting backward by 100s, occurred only in the pMLD groups and was suggestive of conceptual errors that are unrelated to attentional or WM difficulties. Overall, she found that procedural errors (e.g., multiplying instead of adding) and incorrect generalizations occurred in more than half of the children in the pMLD group vs. only 16% of the children in the sMLD group.

The importance of such results, if confirmed with follow-up research, is that appropriate interventions, often provided by the schools, are often available for children diagnosed with ADHD, whereas school-provided interventions for MLD often are not. Thus, a child may be diagnosed with MLD when in fact the correct diagnosis is secondary MLD due to (primary) ADHD. Only when there is a core deficit in numerical magnitude representation, and corresponding errors in conceptual aspects of performance, would a primary MLD diagnosis be warranted.

The overarching issue of importance here, from a broader perspective, is the lesson that current assessments of MLD appear not to carefully distinguish between difficulties that can be attributed to the core numerical deficits that Kringinger discusses and depressed performance on math that might be attributable to nonnumerical difficulties like ADHD. The general issue, then, is one of the precision of assessments, and therefore the equivalence (or lack thereof) of individuals identified as having MLD. This is an especially important cautionary note for anyone with a stake in devising assessments or interventions based on assessments.

Chapter 25: Passolunghi and Costa

Passolunghi and Costa present a chapter on the topic of working memory (WM) and its relation to the learning of math. This is a topic that has been researched in ways very parallel to investigations of working memory's role in language comprehension, for example. That is, two common paradigms have been commonly used. In the first, we have participants perform a primary task, here one testing math performance, along with a secondary task, one involving a load on WM. Such a dual-task format then examines performance to see how much the math task is disrupted when it is deprived of sufficient WM resources. In the second paradigm, we pretest our participants on a WM task, to determine their WM span, then test them on the math task of interest, interpreting differences in performance as a function of the groups' differences in WM capacity.

As Passolunghi and Costa note, the research shows, both with normal adults and typically developing children, that arithmetic and math performance depends on the resources of WM. Adults are often found not to rely on WM resources for the simple basic facts of addition and multiplication, these being performed routinely by rapid fact retrieval. But any arithmetic or math involving more complexity than this is generally found to require the resources of WM, e.g., for carrying in two-column addition (e.g., Ashcraft & Kirk, 2001), for larger subtraction facts (Seyler, Kirk, & Ashcraft, 2003), and so forth (see Ashcraft & Krause, 2007, for further discussion).

Developmentally, the role of working memory is larger, of course, since children do not rely heavily on fact retrieval for simple arithmetic until well into their school years. All reconstructive and strategy-based performance, therefore, can be expected to depend on WM processing, and therefore to be affected by the child's available WM resources. Evidence that children with MLD have reduced WM capacity or resources, therefore, suggests that these children will experience learning difficulties for this reason.

More particularly, Passolunghi and Costa discuss three important types of executive functioning with WM that are especially important, and especially a concern in dyscalculia, those being inhibition, updating, and switching. Inhibition refers to inhibition of attention to irrelevant information or stimuli, for example, aspects of a word problem that are irrelevant to a problem's solution. Updating refers to the mental process of performing running calculations and updating a mental register with newly calculated values as part of the running calculations, as well as keeping track of now-completed or yet-to-be-completed steps in a multistep problem. And finally, switching refers to any kind of necessary switch, say from one strategy on one problem to a different one on a new problem when the new problem requires a different strategy. According to the authors, all three of these types of executive functions may be deficient to a degree in children with MLD.

The chapter concludes with a discussion of training studies, that is, studies that have implemented a training regimen in which WM functions have been subjected to a training program in order to improve the functioning of WM and hence improve mathematics performance. Although the authors concede there is some debate in the literature regarding the overall success of such training programs, with some studies showing positive effects of training while others show no such effects, they focus especially on several studies that have shown positive outcomes in young children with initially low numerical abilities. They conclude that such training programs can indeed work and that improvements in working memory functioning indeed can lead to improvements in math performance.

Given the nature of the current debate in the literature, and the extent of that debate, it is probably prudent to remain somewhat skeptical about studies showing positive training effects. A very thorough review of training programs, and the methodological issues that need to be satisfied for adequate evidence to support their conclusions, has concluded that there is currently no solid evidence that such training programs actually deliver genuine "far transfer" effects on working memory or cognitive processes in general (see McCabe, Redick, & Engle, 2016; Simons et al., 2016). That is, a program that provides training and practice on a selected WM task, say the n-back task, will show significant, long-term gains on the trained task itself, but little or no transfer to other WM tasks, much less to other cognitive tasks thought to rely on WM processes (i.e., math tasks). Long-term follow-ups on training gains, many of which failed to compare gains to appropriate controls, were often either missing in the literature or were nonsignificant when conducted. Significantly, many of the studies reviewed by Simons et al. overlap with those discussed by Passalunghi and Costa. Thus, there is reason to be very cautious in accepting the conclusion that WM training interventions can be successfully applied in cases of MLD to overcome deficiencies in children's WM.

Chapter 26: Resnick, Newcombe, and Jordan

Resnick, Newcombe, and Jordan review the area of spatial reasoning and mathematics achievement, especially for typically developing children and those with MLD. The rationale for this focus is the well-established finding that spatial reasoning is strongly predictive of children's math achievement. As the authors note, however, spatial reasoning is a rather broad, multidimensional category, composed of many different skills. Likewise, math achievement and performance refer to a multitude of skills, are measured in a variety of ways, and are also multidimensional. As such, different components of each would be expected to show varying degrees of relationship. Nonetheless, at the global level, there is a close relationship, in typically developing children, between spatial reasoning and math achievement.

One particular reflection of MLD children's difficulties is found in how they reason about magnitude, that is, how they represent numerical magnitude mentally. MD children are less accurate on number line estimation tasks than typically developing children, both for whole number and for fraction magnitudes. Typically, all children show evidence of a "compressed" representation of magnitude early on, but then later begin to estimate in a more linear fashion as they adopt a linear representation of magnitude. MLD children tend not to shift as readily from compressed estimates, possibly due to a failure to inhibit the compressed representation.

The authors then focus especially on one aspect of spatial reasoning, mental rotation, which has been found not to differ between typically developing children and children with MLD. If so, they reason that this relative strength on the part of MLD children could serve as a "reserve of strength" that could be used to help develop math achievement. That is, by presenting number lines visually, in instructional settings, and using number lines as a way of reinforcing notions about relative magnitudes of numbers, a teacher could convey magnitude knowledge in an accessible fashion to MLD children, taking advantage of their spatial reasoning strength involving mental rotation and spatial visualization.

Of course, this possibility needs further research, as the results on MLD children's spatial visualization skill are sparse. The authors do review some training studies that have attempted to improve MD children's math performance via spatial training, pointing out that considerably more research needs to be conducted before firm conclusions can be drawn (see also the caveat offered in remarks concerning Passolunghi and Costa's chapter). It is nonetheless an intriguing possibility that a relatively unnoticed "preserved" skill in MLD children might be leveraged to provide a somewhat customized way of teaching about number magnitude and, eventually, mathematical competence.

Chapter 27: Prediger, Erath, and Opitz

The Prediger et al. chapter focuses on the role language plays in children's mastery of mathematics, and in particular the difficulties that can arise, especially for children with math difficulties, due to features both of their own native language and in the "language of math." To begin with, they note the distinction between an individual's knowledge of everyday, conversational language, and what Cummins (2000) termed cognitive academic language proficiency (CALP), the more formal, precise, and abstract language we use in the context of schooling. Whereas children from relatively privileged families acquire CALP, and therefore face no great difficulties in school, less privileged children, or those struggling to learn a new language, may suffer and lag behind.

Discussion is also provided of the well-documented difficulties that arise from the language's structure for naming numbers. Many children benefit from a native language that has a regular number-naming system, i.e., has number words that are congruent with the base-ten number system. Such children acquire those number words rather easily, learn to count earlier, and understand the base-ten system earlier and more easily than children whose language involves irregularities, i.e., word inversions or other irregular number words. The standard example of a regular language for number naming is Chinese. After acquiring the 1–10 count string of novel words, children then encounter a regular system for two-digit count words; starting with 11 on, Chinese number words are (translated into English) ten-one, ten-two, ten-three, and so on. The word for 20 is two-ten, 21 is two-ten-one, and so forth. In contrast, French substitutes novel words for 11 through 16, novel but unique decade terms up through 60, but then for 70 uses sixty-ten; for 80 it uses four-twenty (as in 4×20), and for 90 it uses four-twenty-ten (as in $4 \times 20 + 10$). Word inversion is characteristic throughout German (e.g., 21 is one-and-twenty) and related languages. In all cases, departures from regular base-ten order disrupt children's learning and math achievement, for typically developing children and certainly for children struggling with number and math.

The chapter continues with a discussion of the comorbidity of language and math difficulties, and how these relate to overall learning difficulties. The focus turns especially to students struggling with language barriers (e.g., immigrant children who are just learning the language of instruction), and how their deficiencies appear in both reading and math. For such children, participation in the most meaningful aspects of classroom activities will be rather severely restricted, due to their lower language proficiency. As such, when examined at the discourse level, they cannot participate fully in the classroom conversation during which explanation, argument, and knowledge are transmitted. Thus, aside from not having mastered the more formal, school-based language (CALP), they also lack the more basic language proficiency to join the classroom discourse that transmits mathematical knowledge and understanding.

The chapter concludes with some indications of effective educational efforts to overcome language difficulties in the math classroom or to support language learners as they study various math topics. The point emphasized by the authors here is that for a large number of settings, true in many places, language proficiency must be the learning prerequisite for successful learning in any classroom; in other words, language has to become a learning goal, even in the math classroom.

Chapter 28: Baten, Pixner, and Desoete

Baten et al. pursue the interesting idea that math anxiety and motivation might be useful factors to explore in seeking a greater understanding of math learning difficulties. They approach this possibility by couching their discussion in terms of the Byrnes and Miller (2006) opportunity-propensity model. The model takes into account a variety of antecedent factors (e.g., SES, parent expectations), opportunity factors (e.g., home and school environment), and propensity factors (e.g., existing math knowledge, self-regulation, motivation, math anxiety) in order to explain math achievement. The focus for their chapter, in particular, involves the propensity factors of motivation and math anxiety.

The discussion of motivational factors covers the topics of interest in math, engagement with the topic of math, and self-perceived abilities in math and culminates in a careful discussion of the continuum from controlled motivation under external regulation to autonomous motivation under intrinsic regulation. This discussion then turns to math anxiety, and its relationship to motivation. The interplay between motivation, number knowledge, and other factors – from the standpoint of propensity factors – is quite useful in thinking about a child's trajectory in school. That is, the authors propose that lower number knowledge and fewer coping strategies may yield math anxiety reactions among some children and that children with dyscalculia may be more prone to develop math anxiety because of poor numerical competencies (hence lower desirable propensities). This scenario would be made even more dire if environmental opportunity factors were further risk factors, e.g., poor school environments. Desirable school environments, of course, would be protective factors.

The authors conclude by, in essence, imagining a somewhat ideal learning environment for children in school, by examining factors – propensity factors – that can be influenced and designing classroom features to support autonomy, relatedness and competence in math lessons.

Chapter 29: Haase, Guimarães, and Wood

Haase et al. present a thorough review of the literature on math anxiety in their chapter, examining the relevance of math anxiety to math learning and mathematical learning difficulties from the standpoints of motivation, antecedents of math anxiety, its relation to math achievement, its neurocognitive underpinnings, its assessment, and possible interventions. Noting that math anxiety matches the classic definition of a phobia and that it manifests itself at four different levels (cognitive, affective, behavioral, and physiological), the authors point out that math anxiety is widely viewed as both an educational and a clinical problem, though not a psychiatric disorder. Given that it involves issues of motivation and self-efficacy, it also has a strong association with math achievement and therefore is relevant to a discussion of MLD.

A variety of antecedent factors for math anxiety are discussed. There is some evidence for a genetic component to math anxiety, involving people who are genetically prone to anxiety who also have math difficulties, and also a possible role for genetic influences on spatial anxiety. Levels of math anxiety grow with increasing age, with likely earlier onset for children with dyscalculia. Females demonstrate higher levels of math anxiety than males, although the literature has not yet fully agreed on the reasons for this gender difference. Cultural factors are also associated with math anxiety, with complicated relationships found between cultural values for success, for disciplinary practices, and stress. Evidence on teacher and parent effects is now becoming quite clear in terms of transmission of math anxiety to children both in the classroom and at home. Peer effects, finally, are also discussed, including the unfortunate finding that segregating adolescents into different classes based on performance level yields little improvement in self-efficacy for the high-performance group but substantial declines in self-efficacy for the low-performance group.

The review then turns to cognitive factors, mechanisms that yield math anxiety effects due to various cognitive processes that are undermined by math anxiety. These include speed-accuracy trade-offs during performance, competition for working memory resources that are compromised by math anxiety, a lack of inhibition to irrelevant or distracting information, attentional bias or an increase in interference by math-relevant distracting information, abnormal responses to errors, and arousal factors, with greater disruption on the part of individuals with higher working memory capacity. There is also increasing evidence of visuospatial processing impairments playing a role in math anxiety, with a noted relationship to gender differences in visuospatial abilities. Finally, an increase in research on the neurobiological underpinnings of math anxiety is discussed, with the finding that math anxiety seems to activate two emotional systems in the cortex.

The chapter closes with a brief discussion of assessment of math anxiety, including a useful table that provides the psychometric properties of the most widely used math anxiety assessment scales. It also considers various intervention principles that have been tried in attempts to both alleviate math anxiety and improve math achievement. As noted elsewhere, the conclusion offered is that the "knowledge society," with its increasing reliance on technology, will require greater degrees of mathematical fluency from its citizens, thus increasing the pressure for our field to both understand and devise successful interventions for math anxiety.

Common Themes

Heterogeneity within MLD Note that the goal of the Haase and Carvalho's chapter, a refined set of subtypes of DD, is essentially the same goal articulated by Krinzinger in her chapter on the comorbidity of MLD and ADHD. Of course, the purposes of the two chapters are quite different; Krinzinger wants to refine the MLD category (yielding primary MLD vs. secondary MLD after primary ADHD diagnosis) in service of educational goals, whereas Haase and Carvalho have the goal of specifying subtypes of DD in order to advance research into genetic indicators of dyscalculia. But the efforts have an overwhelmingly similar starting assumption: we currently have an overly inclusive, heterogeneous category of individuals labeled with MLD or DD. The category contains individuals with widely varying skills and abilities. They have widely differing deficiencies, often differing qualitatively, making it inappropriate to study them as if they were all the same. And, as Resnick et al. noted in their chapter, some of these individuals also have some preserved (and underappreciated) strengths, which also needs to be taken into account in our studies.

Indeed, this common theme, the great variability of the category known as MLD or DD, occurs throughout many of the chapters in this section. Prediger et al. point to this variability from the standpoint of language, and the current situation that many children in today's world are struggling to learn math in a second language, and are probably mistakenly labeled as MLD for language-related reasons, rather than math-related reasons. Baten et al. broaden the basis for this variability within the MLD category by noting the likely contributions of motivational and emotional factors to MLD, and the likelihood that math anxiety may be a more likely outcome for children with MLD who have lower desirable propensities for normal achievement. It even seems plausible that some of the equivocal results reported in DeSmedt et al., e.g., that MLD children may be less affected in their nonsymbolic magnitude estimations than typically developing children, may also be explained by the heterogeneity problem.

Involvement of WM The second common theme, running through several of the reviewed chapters, is the central role of working memory to a full appreciation of math achievement and a better understanding of math learning difficulties. The neuroimaging results discussed in DeSmedt et al.'s chapter, with clear evidence of attentional and working memory involvement in number magnitude and arithmetic processing, are consistent with the discussions of the central role that WM plays in understanding MLD, for instance, in the Passolunghi and Costa review and the Haase et al. contribution on math anxiety, with the notable result that WM is compromised by math anxiety during math processing. Also telling is the review by Carvalho and Haase, finding that the fourth category of abilities, involving attention, executive function, and working memory, was a pervasive set of abilities throughout all the evidence on MLD. The central role of WM to math processing has been firmly established in the literature on adult performance, and on performance by typically developing children, so it is not surprising that WM and WM difficulties also play a prominent role in understanding MLD.

Concluding Remarks

For whatever kind of work is involved, whether basic research, intervention studies, or classroom practice, it would seem critical to be certain that a child labeled as MLD or DD has a true disorder in number and math, as opposed to a comorbid difficulty. This is the central message in Krinzinger's chapter, of course, that a child identified as MLD not be mistakenly identified due to a comorbid primary ADHD, for instance. Such a comorbidity would disrupt cognitive processes more generally, thus impairing learning and performance not only in math but also a variety of other school-based domains. Efforts to remediate the MLD would be relatively less effective, of course, because they would in essence be treating the wrong disorder. Other subtype possibilities of MLD would also appear to exist, e.g., little or no disruption in nonsymbolic magnitude tasks, i.e., in the ANS, but difficulties representing symbolic number and arithmetic procedures. As research on MLD continues, the field clearly needs to settle on a set of "best practices" for defining and assessing MLD, or different MLD subtypes, to enable comparisons across studies, meta-analyses, and consensus to emerge on the nature and characteristics of the disorder.

A second useful aim, it would appear, would be to broaden the research base to examine performance beyond the magnitude and arithmetic domains that have been examined so far. It was actually surprising, in the DeSmedt et al.'s chapter, to read that brain imaging studies have almost exclusively been limited to magnitude comparison and arithmetic tasks. Although this is somewhat characteristic of the research literatures on adults and typically developing children too, these latter literatures have also recently examined fraction magnitudes, number line estimations, and algebra problem-solving. The Resnick et al. chapter reminds us of the few studies showing apparently preserved abilities on the part of MLD children in the area of spatial reasoning, suggesting that other numerically related areas remain to be investigated with respect to MLD. Such research might possibly find additional areas of preserved abilities, but at the very least could reveal additional insights, and possible methods of intervention, for other numerical skills that are problematic for MLD individuals. As part of this effort to extend our understanding of higher math, additional efforts could be made to understand how to build motivation, especially among the MLD population, to persist in attempts at mastery of the critical life skills represented by numerical and mathematical processing.

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Part IV

Understanding the Basics: Building Conceptual Knowledge and Characterizing Obstacles to the Development of Arithmetic Skills

Chapter 31
Counting and Basic Numerical Skills

Emily Slusser

> You're enjoying a lovely day at the park with your 3-year-old nephew. A paddling of ducks waddles by and you start a conversation, *"Hey Charlie, look at the ducks! How many are there?"*
>
> A pretty straightforward question. Your nephew jumps at the opportunity to demonstrate his skills. Faithfully pointing to each duck, one-by-one, he responds, *"one…, two…, three…, four…, five!"*
>
> Ah, he's brilliant. You knew as much. Let's keep this conversation going. *"That's right!"* you say. *"So, how many ducks are there?"*
>
> He immediately responds, "*Eight!*"
>
> Right! Wait…what?

This narrative, having played out in countless situations, is likely familiar to any caretaker or educator. Indeed, the phenomenon is well documented: while most children appear to have learned to count by the time they are 2 or 2 ½ years old (Fuson, 1988), most often, they are simply demonstrating their ability to reproduce a counting routine. Consequently, their behavior is often difficult to interpret – it is not, as we would be inclined to presume, a reliable indicator of their number knowledge. This is similar (and not unrelated) to that other pre-scholastic phenomenon of reciting the alphabet without yet having developed an understanding of orthography or phonics.

In fact, even after a successful counting routine is achieved, children continue to face several underlying challenges on their way to acquiring early number concepts and basic counting skills. One of the core challenges follows from the fact that there is an important dissociation between conceptual and procedural knowledge of counting. In early phases of number acquisition, conceptual knowledge lags far

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behind that of procedural knowledge. Our nephew in the anecdote above has clearly learned some basic counting procedures (and recognizes that the question "how many" prompts these procedures) well before he will ultimately understand how this activity reveals the correct answer to this question. In fact, only over the next couple years will his incremental advances in both procedural and conceptual knowledge culminate in the ability to form and maintain precise representations of natural number (e.g., Carey, 2010).

Number Sense

While ubiquitous in discussions of early education and mathematics, the term *number sense* is often used to refer to a variety of abilities and behaviors. Early childhood curricula and assessments often use the term to broadly describe children's "fluidity and flexibility with numbers, the sense of what numbers mean and an ability to perform mental mathematics… and make comparison" (e.g., Gersten & Chard, 1999). The following review, however, will adopt the term's primary definition, referring specifically to the evolutionarily primitive ability to represent non-symbolic quantity (Dantzig, 1967; Dehaene, 2011). This definition includes the ability to subitize (i.e., the ability to recognize the exact number of items in a small set without counting[1]; Kaufman, Lord, Reese, & Volkmann, 1949), which manifests from our ability to represent and track individual items (e.g., Feigenson & Carey, 2003). This definition of number sense also includes the ability to represent rough estimates of magnitude and number (e.g., Xu, 2003).

Small Number Representations

> It's time for a snack. You offer your nephew two cookies but he immediately recognizes that you have given yourself three. He raises the alarm. "*How did he know?*" you think to yourself, "*didn't we just establish that he doesn't know how to count yet?*"

We can chalk this one up to the ability to represent and visually discriminate arrays of one, two, or three items, an ability available to even very young infants (Xu, 2003). Consider the following experiment: 10- to 12-month-old infants were presented with two adjacent buckets, one containing just 1 cracker and the other containing 2 crackers. When given the opportunity, the infants in this study consistently chose (crawled to) the bucket with 2 crackers over the bucket with 1 (wouldn't you?) (Feigenson, Dehaene, & Spelke, 2004). Similarly, the infants chose the bucket with 3 crackers when the other had just 2 or 1. However, with choices of 4 vs 6, 3 vs 4, 2 vs 4, and even 1 vs 4 crackers, infants chose at random. Taken together, these

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[1] The term "subitize" also enjoys many definitions across early childhood curricula and assessment. The present chapter, however, will adopt and adhere to the definition provided above.

results show that infants' preference for the greater number does not depend on the *relative* quantity, or the ratio of the two sets (infants consistently chose the bucket with 3 crackers to a bucket with 2 but seemed perfectly happy to go to either bucket when presented with a choice between 4 vs 6 crackers). Instead, their ability to make a meaningful choice is contingent upon *absolute* quantity (in this case the number of crackers), and their ability to represent these exact quantities is capped at three items. This limited (though impressive) ability has been demonstrated across a variety of experimental paradigms, each yielding similar results (e.g., Clearfield & Mix, 1999; Feigenson & Carey, 2003; Starkey & Cooper, 1980).

While greater number is generally correlated with greater continuous quantity (such as summed spatial extent or volume) in the natural world, these studies extensively control for continuous properties showing that these discriminations are based on number alone. Moreover, these representations are not limited to the visuospatial modality. Infants also assess exact quantities (up to 3) when presented with a series of temporal events and auditory sequences (e.g., puppet jumps and sounds; Wynn, 1996).

This representational system then allows us to easily identify small, exact quantities immediately, accurately, and without counting (cf., Cordes, Gelman, Gallistel, & Whalen, 2001). The signature limits of this system, however, remain relatively constant over the course of development (though older children and adults are often able to represent up to 5 or possibly 7 items in a set; Mandler & Shebo, 1982; Trick & Pylyshyn, 1993) such that subitizing does not present a viable pathway to the representation of large, exact numbers like 27 or 308.

Approximate Number Representations

> So we've righted our mistake. Both of us now have three cookies. *Phew*. Wait… your astute (and somewhat righteous) nephew notices that yours has more chocolate chips! It seems there are a gazillion chocolate chips in each cookie, so we are well beyond subitizing. And, he's not counting… Enter the Approximate Number System.

The ability to represent large approximate quantities and detect differences between two large sets is supported by the approximate number system (ANS), a cognitive resource that is also available in early infancy (e.g., Lipton & Spelke, 2003). Early access to this system is often demonstrated through the use of a habituation paradigm. For example, infants (as young as 6 months) are presented with a series of pictures, each with an array of 8 dots. Then, when presented with a picture with 16 dots, infants look longer at the novel array, showing that they discern the difference between sets of 8 and 16. While infants also respond to changes in overall spatial extent (e.g., summed area and/or contour length; Clearfield & Mix, 1999), several studies that have controlled for alternative dimensions of quantity have shown that infants are able to make judgements on numerosity alone.

Judgments supported by the ANS, however, are imprecise, and the threshold for a just noticeable difference follows Weber's law, such that numerical discrimination is a function of the ratio between the two magnitudes under comparison, and not their absolute difference (e.g., Halberda & Feigenson, 2008). Importantly, and unlike the small number representation system discussed above, ANS precision improves over the course of development (Halberda & Feigenson, 2008; Odic, Libertus, Feigenson, & Halberda, 2013). On average, 6-month-olds can reliably discriminate 1:2 ratios (such as was presented in the example above; Lipton & Spelke, 2003), 9-month-olds can discriminate 2:3 ratios (Xu & Spelke, 2000), 3-year-olds discriminate 3:4 ratios, 4-year-olds discriminate 4:5 ratios, and 5-year-olds discriminate 5:6 ratios (Odic et al., 2013); and adults can discriminate 10:11 ratios (Halberda & Feigenson, 2008).

Notably, individual differences in ANS acuity within these age groups are associated with math achievement. In fact, several studies have shown that individuals with more precise ANS acuity perform better on tests of formal mathematics (Libertus, Feigenson, & Halberda, 2011; Libertus, Odic, & Halberda, 2012; Lyons & Beilock, 2011). In one study, performance on the Test of Early Math Ability (TEMA-3; Ginsburg & Baroody, 2003) could be predicted from ANS acuity measured at 6 months (Libertus et al., 2011). In another, numerical acuity measured in 14-year-olds correlated with their performance on standardized math tests as far back as kindergarten (Halberda, Mazzocco, & Feigenson, 2008). Furthermore, there is evidence to suggest that ANS acuity is malleable and may be influenced by environmental factors (Tosto et al., 2014) and formal instruction (Halberda, Ly, Wilmer, Naiman, & Germine, 2012; Piazza, Pica, Izard, Spelke, & Dehaene, 2013).

Summary

Together, these two systems are considered core cognitive resources that serve as a foundation for the construction of natural-number concepts (Carey, 2010). Each is clearly necessary for the development of counting and basic number skills; however, neither is sufficient. The following sections will review how children's developing understanding of the verbal count list (e.g., individual number words such as *one, two,* and *three*) ultimately allows for the construction of natural-number concepts (i.e., the ability to represent exactly 27 or 308).

Number Language

As discussed above, the ability to represent small, exact numbers and large, approximate numerosity is available in early infancy, but mapping these representations to symbolic representations of number (e.g., number words) is no small feat. Whereas children as young as 2 years old have little difficulty mapping approximate quantifiers (such as *more* and *a lot*) to representations of quantity (Dale & Fenson, 1996), children can spend upward of 2 years sequentially assigning meaning to individual number words and figuring out how the verbal count list works.

While a long and protracted process, the acquisition of number language is a crucial milestone in children's quantitative development (Fuson, 1988; Gelman & Gallistel, 1978; Wynn, 1990, 1992). As the following section will discuss, the language system itself is largely responsible for the ability to represent large exact number. In fact, children who experience significant language barriers, such as those born deaf to hearing parents, show delays not only in their acquisition of individual number words but also in later math achievement (Kritzer, 2009). Moreover, individuals who grow into adulthood without learning to count proficiently demonstrate poorer performance on tasks assessing representations of exact number and cardinality (Frank, Everett, Fedorenko, & Gibson, 2008; Spaepen, Coppola, Spelke, Carey, & Goldin-Meadow, 2011).

Knower Levels

> "*The kid's really put one over on me,*" you think. When it comes to cookies, he clearly knows what he's talking about (*three* cookies is more than *two*, and don't even think about saving the cookie with more chocolate chips for yourself!). But you're not entirely satisfied so you decide to put it to the test…
>
> You give him the whole bag of cookies, but ask him if you can have just *one*. He happily obliges. One cookie, no problem. "*Can you give me two cookies?*" you ask. Sure, he hands you two. One last time for good measure – this time you ask for *three* cookies. "*Sure!*" he says as he hands over as many as he can grab. Not *three*, not *two*, but an entire handful!

While seemingly inconsistent and unpredictable, it turns out that our nephew's response is not unusual for a 3-year-old. In fact, it often takes 2 or more years to learn even a subset of number words, during which time children work out the cardinal meanings of each number word one at a time and in order (Le Corre, Van de Walle, Brannon, & Carey, 2006; Sarnecka & Lee, 2009; Wynn, 1990, 1992). Interestingly, as they go through this process, children appear to traverse a predictable series of knowledge states, or "knower" levels (see Sarnecka, Goldman, & Slusser, 2014 for review).

This incremental progression shows up on assessments such as the Give-N (or Give-a-Number) task in which children are asked to create sets in response to specific prompts (e.g., "Can you give *three* bananas to the puppet?") (Wynn, 1992; see Fig. 31.1). In such tasks 2- to 4-year-olds, who can generally recite the count list up to 10 or so without error, are often unable to give the correct number of items when asked for those same numbers in the Give-N task. In response to a Give-N trial asking for *six* bananas, for example, these children may simply grab a handful of items without counting, even when prompted to count or check their response (e.g., "Can you count and make sure you gave the puppet *six* bananas?" or "Can you fix it so that the puppet gets *six* bananas?") (e.g., Le Corre et al., 2006).

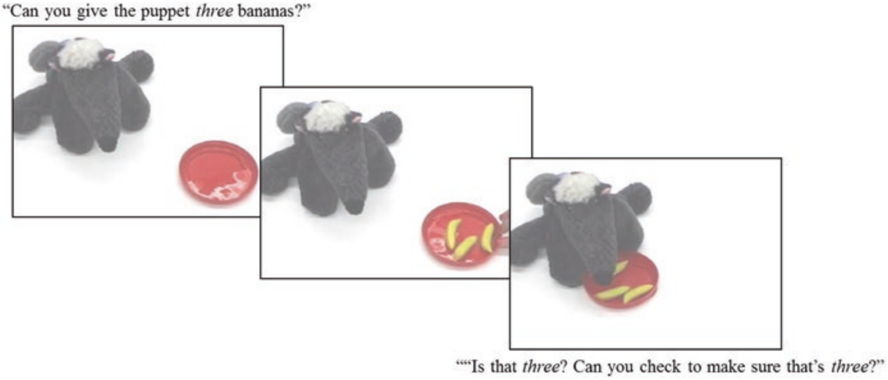

Fig. 31.1 The Give-a-Number task can be used to assess children's number-knower levels (e.g., Wynn, 1992). For this task, children are typically asked to create set sizes of 1 to 6 items. Children are given the opportunity to check and fix their responses after each trial

At the earliest knower level (often referred to as the "preknower" level; e.g., Slusser, Ditta, & Sarnecka, 2013), children's responses to any given prompt are generally unrelated to the number of items requested. These children may give just one item, or even a handful of items, regardless of the specific prompt. At the next level, children reliably give 1 item when asked for *one* but give 2 or more items when asked for any other number. Note that their responses seem to be simple guesses, not counting or estimation errors (Sarnecka & Lee, 2009), and these children appear to understand that number words that they do know are not used to refer to sets of any other size (i.e., they will not offer 1 item when asked for any number other than *one*; Wynn, 1990, 1992). The one-knower level is followed by the "two-knower" level, then the "three-knower" level, and sometimes the "four-knower" level. At each *N*-knower level, children demonstrate predictable and accurate performance up to *but not* beyond *N*. Eventually, around the time they reach the three- or four-knower level (often 2 years after they first entered the one-knower level), children realize that the final number word in their count sequence refers to the cardinal value of the set they are enumerating. At this point they may be said to have induced the "cardinality principle" (Gelman & Gallistel, 1978) and can henceforth employ counting procedures felicitously to create any set size within their count list (Sarnecka & Carey, 2008; Wynn, 1990; cf. Davidson, Eng, & Barner, 2012). It has been argued that, as children progress through these individual knower levels, they are incrementally assigning each of the first three or four number words to their representations of small, exact sets (Carey, 2010). Numbers exceeding the set size limit of 3 or 4 items must then be represented through counting. For this reason, we don't typically see children who would be characterized as "five-," "six-," or "seven-knowers" (cf. Wagner & Johnson, 2011).

The one- through four-knower levels are found not only for speakers of English but also for speakers of Japanese (Sarnecka, Kamenskaya, Yamana, Ogura, & Yudovina, 2007), Mandarin Chinese (Li, Le Corre, Shui, Jia, & Carey, 2003), and Russian (Sarnecka et al., 2007). Furthermore, bilingual children who have memorized the counting lists in both of their languages before learning the exact meanings of these words in either language show the same or similar knower-levels in both languages (Goldman, Negen, & Sarnecka, 2014).

There is, however, a notable variability across children with different learning backgrounds and experiences. For example, while children from relatively high socioeconomic backgrounds typically reach an understanding of cardinality sometime between 3 and 4 years old (see Sarnecka & Lee, 2009), children from less privileged backgrounds often do not reach this level of understanding until well after their fourth birthday (e.g., Dowker, 2008; Jordan & Levine, 2009).

While the cardinality induction is often recognized as a major conceptual achievement, we will put this aside for now (but revisit it in the "Counting Principles" section below). The following sections will instead explore what subset-knowers (a term used to describe children at the one-, two-, three-, and four-knower levels; Le Corre et al., 2006) know and have yet to learn about number.

Discrete Quantification

One piece of knowledge that is integral to understanding natural-number concepts is the idea that number is a property of sets and that sets are comprised of discrete individuals. Indeed, a conceptual dissociation between continuous substances (such as water and sand) and discrete objects (such as blocks and coins) is available in infancy (Hespos, Ferry, & Rips, 2009), and as children acquire language, they reflect this distinction through their appropriate use of linguistic morphology (i.e., the English singular/plural marking) to denote the difference between mass and count nouns (e.g., Barner, Thalwitz, Wood, & Carey, 2007).

To determine whether children with an incomplete understanding of number words (i.e., subset-knowers) understand that number words, in general, are used to refer only to sets of discrete individuals, we invited a group of subset-knowers (2–4 years old) to complete the Blocks and Water task (Slusser, Ditta, & Sarnecka, 2013; see Fig. 31.2). For this task, children watched as an experimenter placed five objects (e.g., blocks) in one cup and five scoops of a continuous substance (e.g., water) in another cup. Four trials asked children about a number word outside the range of numbers known by any subset-knower (e.g., "Which cup has five?"), and another four trials asked about a quantifier (e.g., "Which cup has more?").[2] For half of the trials, the cup with discrete objects was full; for the other half, the cup with the continuous substance was full. Results showed that, while children correctly chose the full cup when asked which cup has "more," they had to have reached the three-knower level before reliably choosing the cup with discrete objects as an example of "five." A series of follow-up experiments seem to indicate that one- and two-knowers have an emergent but tenuous understanding of this constraint but are,

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[2] Note that approximate quantifiers such as "more" and "a lot" can take a wide range of referents, with few constraints, while number words refer only to collections of discrete individuals.

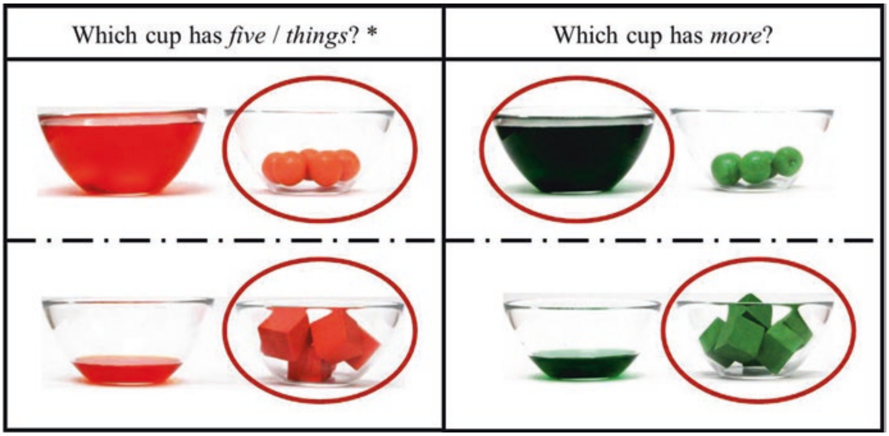

Fig. 31.2 The Blocks and Water task was used to determine whether and when children understand that number words reference discrete sets (Slusser, Ditta, & Sarnecka, 2013) and whether linguistic context (in the form of a count noun + plural marking in English or the general noun classifier, 個 [*ge*], in Mandarin) facilitates this understanding (Slusser, 2010) (Figure adapted from Slusser, Ditta, & Sarnecka, 2013). (* Prompt differed according to the experiment and trial type. *Note:* The cup with continuous substance is full for half of the trials. Red circles indicate the correct response)

in general, as likely to extend the word “five” to continuous substances as to sets of discrete objects.

Thus, it seems that children come to understand that number words are used for discrete quantification only after learning the precise meanings of at least a subset of number words. It is possible then that children use their understanding of the number words “one” and “two” to draw inferential connection between number words and discrete objects. Alternatively, children may use the linguistic context that generally occurs in natural speech to form this connection (Bloom & Wynn, 1997). This argument arises from the observation that number words reference nouns morphologically coded according to their conceptual category (i.e., count vs mass) – that is to say, count nouns take the plural marking, “-s,” whereas mass nouns do not. After first confirming that number words are in fact most often accompanied by an adjacent count noun and plural marking (e.g., “Look, five *ducks*!”) in both child and child-directed speech (Slusser, 2010), we tested whether children use this information to establish that number words reference count nouns, and consequently collections of discrete objects.

The 2- to 4-year-old children in this study completed the Blocks and Water task above, but in this iteration each test question was presented within a syntactically “rich” linguistic context (Slusser, 2010; see Fig. 31.2). For example, children were asked, “Which cup has five *things*?” rather than “Which cup has five?” Results show that English-speaking children connect number words to discrete quantification before learning the specific meaning of any number words *so long as* the number word is paired with an adjacent count noun and plural marking.

Similarly, Mandarin-speaking children demonstrate similar learning trajectories when presented with a number word in isolation and when accompanied by the noun classifier 個 (pronounced "*ge*").

Overall, this series of experiments shows that children use their emerging understanding of number words as well as linguistic cues that occur in natural speech to connect number words to discrete quantification. Moreover, these data constrain future hypotheses on how children learn number words: the fact that this process may involve generalization from certain exemplars and surrounding language provides evidence that number word knowledge is not entirely built upon a priori principles.

Numerosity

Connecting number words to discrete quantification is only one step in acquiring an understanding of natural numbers. Children must also understand that number words denote numerosity (and not, for example, some other characteristic of set, such as total volume or spatial extent). Setting out to address this question, Sarnecka and Gelman (2004) invited 2- to 5-year-old subset- and CP-knowers to complete the Transform-Sets task. For this task, the experimenter placed a certain number of objects in a box while saying (e.g.), "I'm putting *six* buttons in this box." The experimenter then performed some action with the box (either shaking it, turning it around, adding one object, or removing one object). The children were then asked (e.g.), "Now how many buttons are in the box? *Five* or *six*?" Results show that subset-knowers (and CP-knowers) do indeed understand that the number word should change when an item has been added or removed from the box (and that the number word does not change when a non-numerical transformation takes place, such as when the experimenter simply shakes the box). It seems that, while they still do not understand the precise meanings of the number words *five* and *six* (as illustrated through their performance on the Give-N task), subset-knowers do understand something about these number words – that they denote some aspect of quantity.

Note the use of the term *quantity*, not *numerosity*. Upon careful inspection, we see that the Transform-Sets task does not disambiguate number or numerosity from the broader dimension of quantity. Remember, children's intuitive number sense supports representations of both numerosity and continuous spatial extent (see section on "Approximate Number Representations" above). In the Transform-Sets task described above, the number of items in the box changed, but so did other dimensions of quantity (i.e., area, volume, weight). While subset-knowers clearly associate number words with quantity, it is not entirely clear whether they understand that number words refer specifically to numerosity.

To address this specific confound, we developed a Match-to-Sample task with careful controls and manipulations of continuous spatial extent (either summed area or contour length, depending on the trial) so as to pit dimensions of quantity directly against numerosity (Slusser & Sarnecka, 2011; see Fig. 31.3). For this task, children

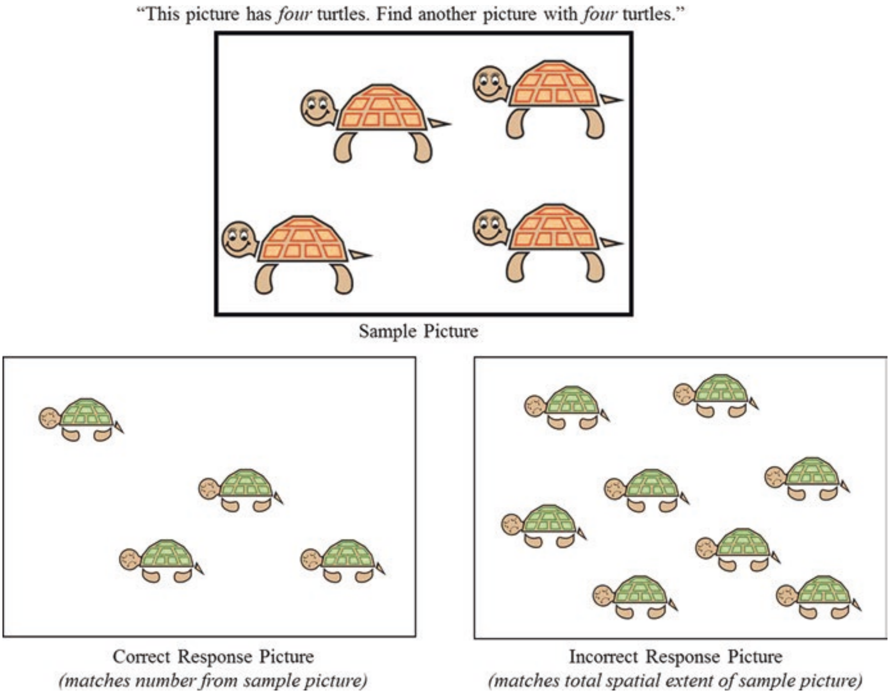

Fig. 31.3 A Match-to-Sample task was used to determine whether children understand that number words denote numerosity, rather than some other dimensions of quantity (e.g., summed spatial extent) (Slusser & Sarnecka, 2011) (Figure adapted from Slusser & Sarnecka, 2011). (*Note:* On this particular trial, there is no possible match on the characteristics of the individuals comprising the set (e.g., the color or mood of the turtles))

were presented with a sample picture as the experimenter said (e.g.), "This picture has *four* turtles." The experimenter then presented two additional pictures and said (e.g.), "Find another picture with *four* turtles." One picture had the same number of items as the sample but different overall spatial extent (e.g., 4 small turtles). The other had a different number of items, but the same overall spatial extent (e.g., 8 small turtles). Results showed that while CP-knowers understand that two sets of the same numerosity should be labeled with the same number word, subset-knowers are as likely to extend that number word (e.g., *four*) to other dimensions of continuous quantity (by, in this case, selecting a picture of 8 small turtles).

Summary

Taken together, these findings reveal that subset-knowers' understanding of numbers matures as they acquire the meanings of individual number words. In addition to enriching our understanding of how children's understanding develops over time, these studies highlight a series of additional conceptual and linguistic challenges that are often overlooked in the development of early childhood curricula and assessments.

Counting Principles

The previous section discusses how children learn each of the number words in their count list one-by-one and in order. The process appears to take upward of 2 years, and as they do this, they learn some of the fundamental properties of number (i.e., number words refer only to discrete sets and are used to denote numerosity, not continuous quantity). Whereas the counting routine, in and of itself, does not appear to be integral to this process, children are certainly gaining experience and learning about counting procedures over this period of time.

As Gelman and Gallistel (1978) pointed out in their seminal work on *Young Children's Understanding of Numbers*, in order to count productively, children (and adults) must at the very least (1) recite the count list in the same sequence every time (e.g., *one, two, three, four* and not *one, four, three, two*), (2) count each object in a set without skipping or double-counting, (3) understand that they can count the objects in any order (e.g., counting from left to right yields the same answer as when counting from right to left), and (4) understand that the last number word recited in the counting routine indicates the total number of items in the set. While the first three rules seem to unfold with experience and practice, the following sections will focus on the final counting principle in this list – the cardinality principle.

Cardinality Principle

> After your little experiment with the cookies, you think back to your conversation about the ducks in the park. Your nephew *did* recite the count list in order; he *did* count each duck in one-to-one correspondence, and he didn't seem too concerned with the order or arrangement of the ducks. But wait… there's just one thing missing. He did *not* seem to understand that the last word in his count list should indicate the total number of ducks. Well, jeez, that seems simple enough…

When considered a part of Gelman and Gallistel's (1978) list of counting principles, the cardinality principle (or "last word rule") simply stipulates that the last number word in a count sequence represents the cardinal value of that set. In reality, however, it seems children's understanding of this specific procedure is contingent upon a crucial conceptual induction – often referred to as the cardinality principle induction (Carey, 2010). As mentioned previously (section "Knower Levels"), prior to this induction, children progress through a series of intermediate knowledge states (knower levels), during which time they do not seem to understand how counting is used to generate or identify specific set sizes (e.g., Le Corre et al., 2006). Importantly, children who understand the cardinality principle (i.e., CP-knowers) perform differently from subset-knowers on a variety of tasks assessing early number knowledge. Some of these tasks explicitly involve counting. For example, on the Give-N task, CP-knowers use counting to generate specific set sizes and can fix their answers when they make mistakes. While subset-knowers often engage in counting behaviors (extensively abiding by the counting principles outlined above), they fail to use counting to generate specific set sizes. Some tasks, however, do not explicitly involve counting. Examples of these include the Blocks and Water and Match-to-Sample tasks discussed above, which reveal that subset-knowers do not yet understand the fundamental properties of number words (i.e., that they are used for discrete quantification and denote exact numerosities).

Another notable difference between subset- and CP-knowers is that only CP-knowers understand that any set with *N* items can be put into one-to-one correspondence with any other set labeled with the same number word (*N*) – an idea referred to as "equinumerosity" (Muldoon, Lewis, & Freeman, 2009; Sarnecka & Wright, 2013). Like many of the skills outlined above, children's understanding of equinumerosity seems to align closely with their induction of the cardinality principle. For example, if one child were to have a handful of grapes for a snack and the other were offered the same (both snacks are recognized to be "just the same" through one-to-one correspondence), then each snack should also be labeled with the same number word. Results on a task that evaluated children's understanding of this concept show that only CP-knowers know that sets that are "just the same" are labeled with the same number word (and if the sets are not the same, then a different number word should be used) (Sarnecka & Wright, 2013).

Furthermore, there is emerging evidence to suggest that children tap into ANS representations as they learn how counting represents number (Carey, Shusterman, Haward, & Distefano, 2017; Chu, van Marle, & Geary, 2015; Shusterman, Slusser, Halberda, & Odic, 2016; van Marle, Chu, Li, & Geary, 2014). One such study tracked 2- to 4-year-old's understanding of individual number words and counting procedures (through the Give-N task) as well as their ANS acuity over a 6-month period (Shusterman et al., 2016). Results show that children's acquisition of the cardinality principle is tightly linked to marked improvement in ANS acuity and that there is little evidence to suggest that ANS representations underlie advancements across subset-knower levels (e.g., moving from the one-knower to two-knower level) (see Fig. 31.4). These findings provide further evidence for the notion that the cardinality principle is not just a counting rule – it is essential to the creation and representation of natural-number concepts.

Importantly, children did not have an opportunity to count when completing any of the tasks introduced above (including the Block and Water and Match-to-Sample tasks discussed above), showing that children who understand the cardinality principle know more than the route counting procedures – they have developed deeper insight about numbers and number words. Thus the promotion from subset- to CP-knower seems to be far more profound than it initially appears.

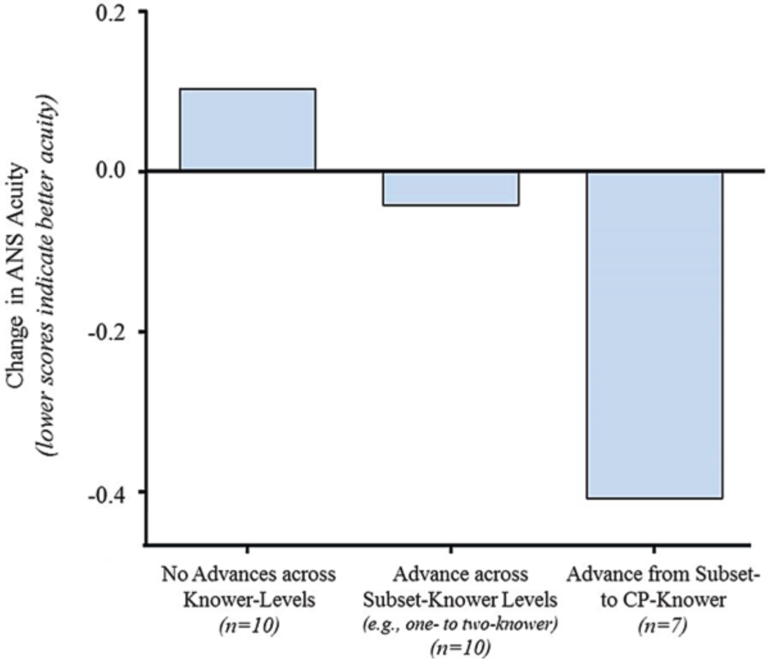

Fig. 31.4 A 6-month longitudinal study evaluating children's developing number knowledge, counting skills, and ANS acuity shows that the acquisition of the cardinality principle is tightly linked to notable increases in ANS acuity (Shusterman et al., 2016). Note that ANS acuity is not clearly linked to advances across number-knower levels. (Figure adapted from Shusterman et al., 2016)

Successor Function

With the cardinality principle comes an understanding of the successor function, which reflects another fundamental property of number – with each additional item in a set, we advance one step (i.e., word) along the verbal count list. In conjunction with the cardinality principle, an understanding of the successor function allows children to represent the cardinal meanings of every word in their count list (Sarnecka et al., 2014).

To explore children's understanding of the successor function, Sarnecka and Carey (2008) showed a group of 2- to 4-year-old children a box with 5 items inside. Similar to the Transform-Sets task described above, experimenters explained (e.g.), "There are *five* apples in this box," and then added an item to the box. In this task, however, the experimenter asked (e.g.), "Now how many are in the box? *Six* or *seven*?" As with the tasks reviewed above, only the CP-knowers seemed to understand that adding 1 item to a set moves the total count one step (word) forward along the count list (and adding 2 items moves the count two steps forward).

Together, children's understanding of the cardinality principle and successor function is often considered to be "the final piece of the puzzle" (Sarnecka et al., 2014) – the last thing that children must figure out in order to use counting to construct natural-number concepts.

Summary

While your 3-year-old nephew at the beginning of this chapter has clearly memorized several words in the verbal count list and has acquired at least some of Gelman and Gallistel's (1978) counting principles, it seems that this routine serves no meaningful purpose other than offering the expected response to the question "how many?". Gradually, however, over the next several months or years, he will come to realize that counting is used to determine the exact number of items in a set and that cardinality changes with each additional item.

Facilitating the Acquisition of Exact Number Concepts

The sections above outline several challenges that children inevitably face as they develop counting and basic numerical skills while presenting the argument that children must confront and conquer these challenges in order to construct and represent exact number concepts. Moreover, recent research has identified these achievements as central to children's eventual success in school (Aunio & Niemivirta, 2010; Bartelet, Vaessen, Blomert, & Ansari, 2014; Duncan et al., 2007; Göbel, Watson, Lervåg, & Hulme, 2014), with the unfortunate caveat that children who start school without these fundamental number concepts are at a serious disadvantage, both in the short and long term (Dowker, 2008; Jordan, Kaplan, Ramineni, & Locuniak, 2009):

> Even though you realize that your simple "judgement calls" on who has more chocolate chips will have to be supported with clear empirical evidence from here on out, you nevertheless decide to help your nephew out (that's what family's for, right?). Lucky for you, researchers' evaluations of both small- and broad-scale interventions have culminated in a collection of best practices that can be easily implemented even in informal settings.

Facilitating the Acquisition of Individual Number Words

In addition to the four counting principles outlined in the section "Counting Principles" above, Gelman and Gallistel (1978) noted that children must also understand abstraction – the idea that number is an inherent property of any set of discrete items and that a set of 10 apples, for example, shares something in common with a set of 10 oranges (who said that we can't compare apples and oranges?). Unfortunately (though interestingly), many researchers who have attempted to teach children the meaning of a new number word (e.g., teach a two-knower the exact meaning of the word *three*) find limited success. Whereas these children may come to recognize that the new number word can be used to label a set of, e.g., three marbles, they often do not understand that the word *three* can be applied or generalized to other sets of 3 (e.g., 3 blocks, 3 buttons, 3 meals) (Carey et al., 2017; Huang, Spelke, & Snedeker, 2010; Mix, Huttenlocher, & Levine, 2002).

To explore this phenomenon further, we introduced a group of two-knowers to the word *three* (Slusser, Stoop, Lo, & Shusterman, 2017) through one of three training conditions (Fig. 31.5). Children randomly assigned to the Number Word Only condition were presented with several pictures of 3 animals and were told, "This picture has *three*." Children in the Count Noun condition were presented with this same series of pictures but were told, (e.g.) "This picture has *three* dogs." And children in the Superordinate Category condition were told, "This picture has *three* animals." Following training trials with corrective feedback, two-knowers in the Count Noun and Superordinate Category conditions failed to extend the new number word (*three*) to sets of new animals (e.g., lions) or objects (e.g., shoes), while children in the Number Word Only condition succeeded. These findings suggest that the specificity of the linguistic context in which a number word is introduced influences children's ability to generalize newly acquired number words. Thus, while a rich linguistic context seems to facilitate children's understanding of number word semantics (see "Discrete Quantification"), when introducing a specific number word, it seems adults and educators should provide varied input and avoid coupling a number word with a specific noun or category label unnecessarily.

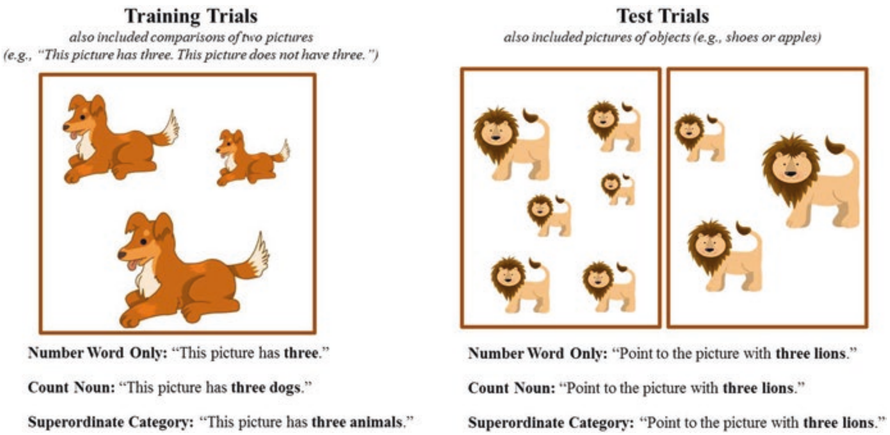

Fig. 31.5 Examples of training and test trials: To evaluate the role of linguistic context in children's acquisition of individual number words, we designed 3 training conditions. Children who were trained with the Number Word Only were more likely to generalize the newly acquired number word to new sets than children assigned to the Count Noun or Superordinate Category conditions

Facilitating the Acquisition of the Cardinality Principle

Efforts to teach children the cardinality principle over a short period of time have also been met with mixed success (e.g., Mix, Sandhofer, Moore, & Russell, 2012). Nevertheless, it seems there is growing evidence that adults can effectively scaffold children's understanding of the cardinality principle by presenting the counting routine in close temporal contiguity with an appropriate label of cardinality. Most recently, Paliwal and Baroody (2017) found that modeling a counting procedure that emphasizes the total number of items in a set facilitates children's understanding of the cardinality principle. For this study, 3- to 5-year-olds were randomly assigned to one of the three training groups. Children practiced counting 1–6 items with an experimenter several times over a 6-week period. Upon posttest (which included a measure similar to the Give-N task described above), children who practiced counting using a procedure that emphasized the total number of items in a set (e.g., "One, two, three. *Three*. There are *three* elephants!") outperformed children who simply counted the items (e.g., one, two, three) without repeating or emphasizing the cardinal value of the set.

Notably, however, adults often do not approach counting activities in this way (Mix et al., 2012). While they may count or provide a cardinal label, they do not often do both. This coupled with the observation that number talk, in general, is relatively rare in everyday interactions (Levine, Suriyakham, Rowe, Huttenlocher, & Gunderson, 2010) suggests that many children are not, on a daily basis, exposed to input that facilitates this understanding.

Broad-Scale Intervention

Following participation in "broad-scale" mathematics intervention programs (meaning that they include a multitude of both classroom- and home-based activities), children from low and middle socioeconomic backgrounds have consistently demonstrated improved performance on composite mathematical assessments (e.g., Arnold, Fisher, Doctoroff, & Dobbs, 2002; Starkey, Klein, & Wakeley, 2004). Not only do children's math scores improve, but other numerically related skills, such as measurement and problem-solving, also improve.

One notable demonstration of these benefits follows Greenes, Ginsburg, and Balfanz's (2004) evaluation of their Big Math for Little Kids program. This curriculum, designed to increase mathematical competency among 4- to 5-year-old children, includes a series of engaging number-based games that encourage and facilitate critical thinking related to number. The studies presented in the following two sections, however, suggest that meaningful experience and intervention need not take the form of established curriculum. Instead, it seems that parents and educators can facilitate children's counting and basic numerical skills by simply offering or creating numerically based games and toys and by incorporating "number talk" into daily conversations.

Numerically Based Toys

Over the last several years, researchers have begun to study the direct cognitive benefits associated with children's play with numerically based toys. One study linked cognitive benefits of play with numbered board games in preschoolers from low-income backgrounds (Siegler & Ramani, 2008). Children (ages 4–5) completed 4 sessions of play using a board game with squares labeled 1–10. Even though they initially struggled with math-related tasks as compared to their more affluent peers at pretest, these children consistently demonstrated improvements at posttest, suggesting that numerically based play can have profound effects on mathematical cognition.

More recently, in a study funded by the toy manufacturing giant Mattel©, 3- and 4-year-old children were randomly assigned to one of the four conditions, each with a specific toy predicted to support development within a particular cognitive domain (Slusser et al., 2013). Children in the Number Condition were given a set of ten small race cars (think Hot Wheels™) and a parking garage. Each car was labeled with a numeral from 1 to 10, and the parking garage included a series of parking spaces, each with an array of 1–10 dots. After a 1-month period (during which time children were encouraged to play with the toy but received no other specific instruction from the researchers), children's counting and basic numerical skills increased dramatically, significantly more than children assigned to any other condition[3] (see Fig. 31.6). Thus, simply playing with numbered toys appears to promote improvement in numerical understanding.

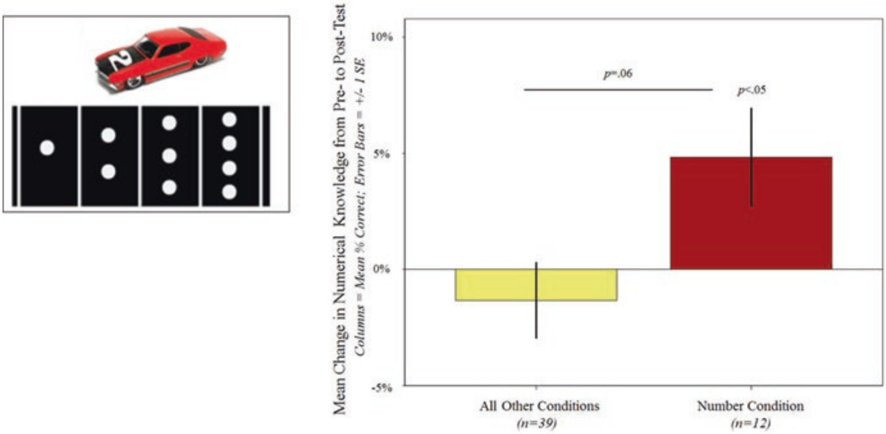

Fig. 31.6 Children's independent play with numerically based toys (left) over a 1-month period promotes their numerical understanding (right) (Slusser, Chase, et al., 2013)

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[3] Children in the other conditions received either a set of ethnically diverse dolls, dress-up clothes, or wooden blocks.

Number Language

Even without the use of games or toys, recent research has shown that exposure to number language facilitates children's acquisition of number word meanings. In fact, children's knower levels can be predicted by the quality and quantity of number-specific language at home (Gunderson & Levine, 2011; Levine et al., 2010), and interventions that help parents engage in meaningful number talk can facilitate children's progress toward understanding cardinality (Berkowitz et al., 2015).

This important link between number knowledge and early language exposure is further demonstrated through a recent study that evaluates and models the influence of parent education, general vocabulary, ANS acuity, and number word knowledge on children's early math achievement (Ribner, Shusterman, & Slusser, 2015). For this study, we first evaluated the receptive vocabulary, number-knower level, and ANS acuity of a diverse group of 3- to 5-year-old preschoolers. We then administered the TEMA-3 approximately 1 year later, as they entered kindergarten. We found that children's early language (general vocabulary and number word knowledge) fully mediates the relationship between parent education and math ability. Additionally, number word knowledge mediates the noted relationship between ANS acuity and early math (see Fig. 31.7).

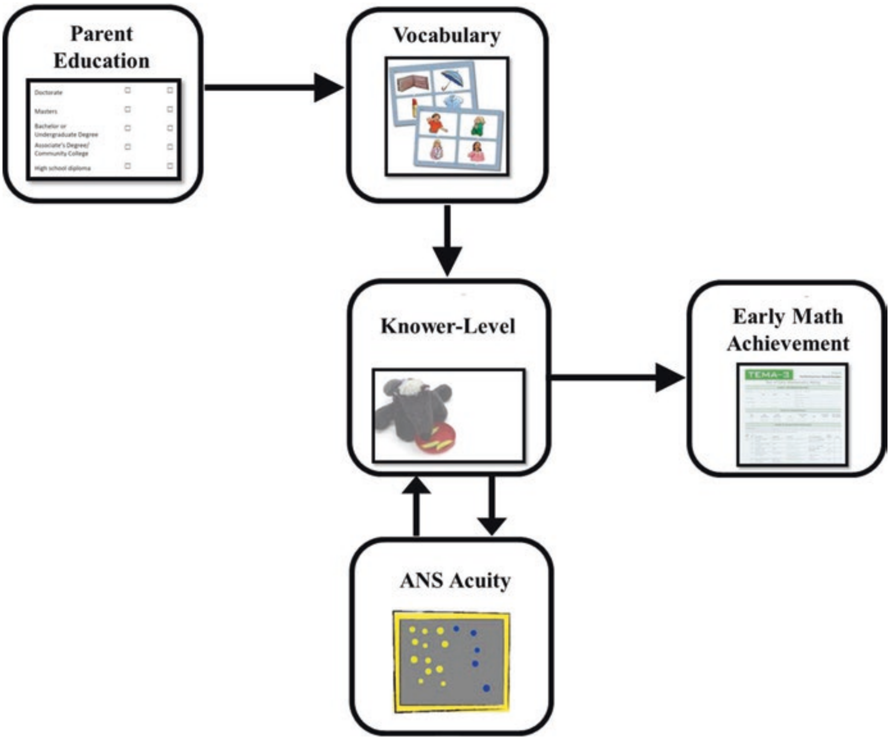

Fig. 31.7 A diagram that illustrates the relationship of parent education and early math. Results from a 1-year longitudinal study following preschoolers through kindergarten show that early language skills are linked to number word knowledge and these factors fully mediate the relationship between parent education and math ability (Ribner et al., 2015)

Even with a clear need for additional research, these findings carry implications for early education and intervention. For example, while proposals for early intervention to support children's developing number sense (ANS acuity; e.g., Wang, Odic, Halberda, & Feigenson, 2016) remain justified, these findings suggest that an increased focus on number language and general vocabulary may help to minimize disparities in math ability as children enter kindergarten.

Summary

In sum, a sampling of research across various disciplines (including early education and instruction, child development, psychology, and cognitive science) shows that children's intuitive number sense, their understanding of individual number words, and their procedural and conceptual counting knowledge serve as key building blocks for future math ability. While idiosyncrasies in each result in predictable developmental outcomes, researchers have identified a series of effective, low-cost, and practical interventions that can be easily adopted by parents and practitioners alike.

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Chapter 32
Multi-digit Addition, Subtraction, Multiplication, and Division Strategies

Marian Hickendorff, Joke Torbeyns, and Lieven Verschaffel

It has long been recognized that children's arithmetic is characterized by strategy variability. Children use a variety of different strategies to solve arithmetic problems. This variability is characterized by both interindividual variability, meaning that different individuals rely on different strategies to solve a given arithmetic task, and intraindividual variability, referring to one individual using different strategies to solve different tasks or even the same task at different moments and/or in different settings (e.g., Siegler, 2007). Furthermore, with increasing age and experience, children not only tend to develop from using less efficient to more efficient strategies but also become increasingly adaptive in their strategy choices, as described in Siegler's (1996) overlapping waves theory.

To optimally enhance children's arithmetic learning, it is important to know what strategies children use and what obstacles they encounter in acquiring these strategies. There are many studies on children's strategy use in single-digit arithmetic (for a review, see, for instance, Verschaffel, Greer, & De Corte, 2007), but research in the domain of multi-digit arithmetic is rather limited, in particular for multi-digit multiplication and division. This is problematic since the upper grades of primary school are usually devoted to instruction and practice in solving multi-digit arithmetic problems, and children may experience quite large difficulties in that domain.

The current chapter's aim is therefore to give an overview of what is known about primary school children's strategy use in multi-digit arithmetic, defined as

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addition, subtraction, multiplication, and division tasks in which at least one of the operands contains two or more digits. Furthermore, we aim to identify obstacles children encounter in developing, selecting, or executing these strategies, in the population of all learners as well as specifically in the group of children with mathematical difficulties.

Multi-digit Arithmetic Solution Strategies

Strategies for multi-digit arithmetic differ from those for single-digit arithmetic. In *single-digit arithmetic*, an important distinction is between computational strategies and retrieval. In computational strategies (also called backup strategies), the answer is computed in subsequent solution steps, for instance, by counting on from the larger integer (9 + 3 = 9, 10, 11, 12) or by reference to another easier or already known problem (derived facts: e.g., 9 + 3 = 10 + 3 – 1 = 13 – 1 or 12). Retrieval concerns recalling the answer from long-term memory as an arithmetic fact, without intermediate computational solution steps (e.g., 9 + 3 = (immediately) 12). Generally speaking, children's single-digit arithmetic development is characterized by the progression from concrete counting strategies via derived fact strategies to the final mastery of retrieval of the arithmetic fact (e.g., Verschaffel et al., 2007). By contrast, in *multi-digit arithmetic* retrieval of the outcome as an arithmetic fact is not feasible: the outcome needs to be computed. Hence, in multi-digit arithmetic the question is how the numbers are manipulated in order to find the answer. That is what we call a (solution) strategy.

An important characteristic of multi-digit strategies is how the numbers are operated on: respecting the place value the single digits of those numbers represent or not. This distinction yields two major types of strategies: number-based strategies and digit-based strategies (for reviews, see Fuson, 2003; Kilpatrick, Swafford, & Findell, 2001; Verschaffel et al., 2007).[1] In *number-based strategies*, the place value of the digits in the numbers is respected (e.g., the number 83 may be split into 80 and 3), whereas in *digit-based strategies*, the place value of the digits is ignored (e.g., 83 may be split in the digits 8 and 3, ignoring that the 8 actually stands for 8 tens = 80). The most common digit-based strategies are the written algorithms of long addition, subtraction, multiplication, and division, operating on single digits in a proceduralized way, usually from right to left. In the current chapter, we also dis-

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[1] Some authors use the terms mental computation strategies and written arithmetic instead of number- and digit- based arithmetic, where mental computation strategies may refer to either operating on numbers *with* the head or entirely *in* the head, whereas written arithmetic refers to the execution of digit-based algorithms usually with paper and pencil (for more details, see Verschaffel et al., 2007). Since the most important distinguishing feature between the different types of multi-digit strategies is operating on numbers versus on digits (rather than mental versus written computation), we prefer the terms number-based versus digit-based strategies.

cuss so-called column-based strategies: a specific form of number-based strategies that in some reform-based mathematics curricula, such as in realistic mathematics education (RME) in the Netherlands, are instructed as an intermediate strategy to make the transition between number-based strategies and the digit-based algorithm smoother and more insightful (e.g., van den Heuvel-Panhuizen, 2008; van den Heuvel-Panhuizen & Drijvers, 2014). These column-based strategies have elements of the digit-based algorithmic approaches, since they also involve a structured, vertical, notation. However, they operate on whole numbers instead of digits, and they proceed from left to right, which are two characteristics that clearly distinguish them from the digit-based algorithms. Some authors (e.g., Buijs, 2008) therefore call these column-based strategies *stylized mental computation strategies* (where mental refers to computing *with* the head instead of entirely *in* the head; see also Footnote 1). Similar approaches can be found in other innovative mathematics learning-teaching methodologies, such as the open calculation based on numbers in Spain (Aragón, Canto, Marchena, Navarro, & Aguilar, 2017).

Given that there are several possible strategies to solve multi-digit arithmetic problems, the question arises how children select a particular strategy from their repertoire. This question has intrigued cognitive psychologists already since the 1950s (e.g., Siegler, 2007) and is also relevant from a mathematics education perspective: an important goal of contemporary mathematics education around the world is that children acquire the competence to solve mathematical problems efficiently, creatively, and flexibly or adaptively with an array of meaningfully acquired strategies (e.g., Hatano, 2003; Star et al., 2015). Scholars use different definitions of flexibility and adaptivity. In the current chapter, we use flexibility and adaptivity interchangeably as selecting the optimal strategy for a given problem in a given setting for a given person. Verschaffel, Luwel, Torbeyns, and Van Dooren (2009) discuss that adaptivity can be conceptualized with respect to task characteristics (i.e., Does the child select the strategy that is best for that problem given a rational task analysis?), subject characteristics (i.e., Does a child select the strategy (s)he performs best with?), and contextual characteristics (i.e., Does a child select the strategy that is optimal given the circumstances, such as the value of speed over accuracy?). According to that conceptualization, a child behaves adaptively if (s)he chooses the strategy that is the optimal one, taking into account the features of the task at hand, his/her mastery of the various strategies available in his/her strategy repertoire, and the sociocultural setting wherein (s)he is confronted with the task (Verschaffel et al., 2009)

In the following, we will discuss the research literature on children's strategy competencies in the additive domain (i.e., multi-digit addition and subtraction) and the multiplicative domain (i.e., multi-digit multiplication and division). In both parts, we start with presenting a comprehensive framework of the different number-based and digit-based strategies, followed by a review of empirical findings regarding children's use of these strategies and ending with a discussion of the obstacles in developing these strategies.

Multi-digit Addition and Subtraction Strategies

Strategies Framework

Table 32.1 shows an overview of the number-based and digit-based strategies for multi-digit addition and subtraction, based on earlier categorizations (Peltenburg, van den Heuvel-Panhuizen, & Robitzsch, 2012; Torbeyns, De Smedt, Stassens, Ghesquière, & Verschaffel, 2009). A first dimension along which the strategies can be categorized is the operation that underlies the solution process: addition or subtraction. In multi-digit addition there is only way of carrying out the operation, as *direct addition*: one operand is directly added to the other. By contrast, in multi-digit subtraction, there are three different ways in which the operation can be carried out: as *direct subtraction* in which the subtrahend is taken away from the minuend, as *indirect addition* in which one adds on from the subtrahend until the minuend is reached (also called adding-on strategy), and as *indirect subtraction* in which one determines the difference by how much has to be taken away from the minuend to reach the subtrahend.

A second, complementary, dimension concerns how the numbers are dealt with. In *sequential* strategies (also called jump or N10 strategies), the numbers are primarily seen as objects on the (mental) number line and the operations as forward or backward movements along this number line. By contrast, in *decomposition* strategies (also called split or 1010 strategies; e.g., Beishuizen, 1993;

Table 32.1 Overview of solution strategies for multi-digit addition and subtraction

		Number-based strategies				Digit-based algorithm
		Sequential	Decomposition	Varying	Column-based strategy	
Addition e.g., 38 + 46	Direct addition	38 + 40 = 78; 78 + 6 = **84**	30 + 40 = 70; 8 + 6 = 14; 70 + 14 = **84**	38 + 50 = 88; 88–4 = **84**	38 46+ 70 14+ **84**	1 38 46+ **84**
Subtraction e.g., 82 – 69	Direct subtraction	82 – 60 = 22 22 – 9 = **13**	80 – 60 = 20; 2 – 9 = −7; 20 –7 = **13**	For example, compensation 82 – 70 = 12; 12 + 1 = **13**	82 69– 20 –7 **13**	7 12 82 69– **13**
	Indirect addition	69 + 3 = 72; 72 + 10 = 82; 3 + 10 = **13**	9 + 3 = 12 60 + 10 = 70 3 + 10 = **13**	69 + 1 = 70; 70 + 12 = 82; 1 + 12 = **13**		
	Indirect subtraction	82 – 10 = 72; 72 – 3 = 69; 10 + 3 = **13**	80 – 10 = 70; 2 – 3 = −1; 10 + 3 = **13**	82 – 20 = 62 62 + 7 = 69 20 – 7 = **13**		

Blöte, van der Burg, & Klein, 2001), the numbers are primarily seen as objects with a decimal structure, and the operations involve partitioning or splitting the numbers. The category of *varying* strategies includes diverse strategies that involve the adaptation of the numbers and/or operations in the problem, such as in the compensation strategy where one of the operands is rounded up to a near round number (e.g., subtracting 70 instead of 69 and compensating back the 1 that was subtracted too much). Besides these three types of strategies, we distinguish – in line with Dutch (RME-based) mathematics educators – a fourth number-based strategy in Table 32.1: the column-based strategy, which essentially consists of the same numerical approach as the decomposition strategy. In the Dutch RME, this strategy is explicitly instructed as a separate strategy, functioning as an intermediate strategy bridging the gap between number-based strategies and digit-based algorithms, by its "hybrid" nature of, on the one hand, operating on numbers rather than digits but, on the other hand, doing so in a standardized step-by-step sequence accompanied by a structured vertical notation.

In most countries the *digit-based algorithms* fall in the category of *direct* addition or subtraction.[2] The main difference with the number-based strategies is that the integers are dealt with as digits, ignoring the place value they represent. For instance, in the digit-based addition strategy, one starts by adding the unit integers 8 + 6 = 14, then writes down the 4 and holds the 10 in memory as a 1, and then adds the tens integers 3 + 4 + 1 = 8. It is not before the 8 is combined with the 4 that the 8 turns out to represent 8 tens. The digit-based addition and subtraction algorithms proceed from the right to left (i.e., starting with the units, then the tens, etcetera).

Children's Strategy Use: Empirical Findings

As discussed in Verschaffel et al. (2007), studies on children's number-based and digit-based strategy competencies conducted in the 1900s and early 2000s revealed that children rely on different types of number-based strategies before the standard digit-based strategies are introduced at school. The level of strategy variety tends to depend on the nature of the provided instruction: Children who received instruction that focused on the mastery of a given number-based decomposition or sequential strategy with hardly any attention for strategy variety tend to rely on only the instructed strategy and to demonstrate less strategy variety than children who experienced instruction that focused on strategy variety.

Furthermore, Verschaffel et al. (2007) discuss how children's use of number-based strategies is typically challenged by the introduction of the digit-based algorithms for multi-digit addition and subtraction at school: Once the digit-based

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[2] In some countries, such as Germany, the digit-based algorithm via indirect addition is used for subtraction; see Verschaffel et al. (2007) for an example. This is also called the Austrian algorithm.

algorithms are explicitly taught to the children, they tend to prefer these algorithms over the previously learnt number-based strategies, also on tasks for which the use of (specific) number-based strategies would be equally or even more efficient, such as 601 – 598 = _. However, children do not necessarily apply these newly learnt digit-based algorithms more efficiently, as illustrated by the frequent occurrence of errors due to the application of so-called “buggy procedures”–systematic erroneous procedures whereby one or more specific steps of the correct procedure are overlooked or executed wrongly (e.g., always subtracting the smaller from the larger digit when applying the digit-based subtraction algorithm, resulting in errors as 258 – 179 = 121).

Since Verschaffel et al.’s (2007) review, quite a number of researchers have continued to deepen our understanding of the variety, frequency, efficiency, and adaptiveness of children’s number-based and digit-based multi-digit addition and subtraction strategies. One particularly interesting way they did so was by using a more sophisticated research paradigm than before: the so-called choice/no-choice method developed by Siegler and Lemaire (1997); see also Luwel, Onghena, Torbeyns, Schillemans, and Verschaffel (2009). In this method, children solve problems in two different condition types: the choice condition where they are free to select their strategy, and in two or more no-choice conditions where they have to use a particular strategy. The choice condition allows investigating children’s strategy repertoire and variety, but strategy efficiency may be biased by selection effects. For instance, when a strategy is selected by weaker children and/or on more difficult problems, this strategy may seem less efficient than it actually is. The no-choice conditions overcome this because all children have to solve all problems with a particular strategy, allowing assessing the strategy’s efficiency (accuracy and speed) in an unbiased way. These unbiased strategy efficiency data can be used to address adaptivity to individual mastery of the strategies, by investigating the extent to which children select the strategy (in the choice condition) that is most efficient for him/her (based on data from the no-choice conditions).

A first series of studies addressed (mid and upper) primary school children’s number-based strategy competencies. Studies addressing children’s number-based decomposition, sequential, and varying strategy use on multi-digit additions and subtractions generally confirm the results discussed above. Children who received instruction with primary focus on the mastery of one specific type of (decomposition or sequential) number-based strategy tend to consistently apply the instructed (decomposition or sequential) strategy on different types of multi-digit addition and subtraction problems (Csíkos, 2016; Heinze, Marschick, & Lipowsky, 2009). By contrast, reform-oriented instructional approaches stimulate children’s efficient and adaptive use of different types of number-based strategies, including – although applied with limited frequency – varying strategies as compensation and indirect addition. Somewhat in contrast with this general finding, studies focusing on children’s use of the number-based indirect addition strategy for multi-digit subtractions indicated that 9–12-year-olds frequently, efficiently, and adaptively rely on this indirect addition strategy, despite the strong instructional focus on and the frequent practice of (only) direct subtraction strategies (Peltenburg et al., 2012;

Peters, De Smedt, Torbeyns, Ghesquière, & Verschaffel, 2013; Peters, De Smedt, Torbeyns, Verschaffel, & Ghesquière, 2014; Torbeyns, Peters, De Smedt, Ghesquière, & Verschaffel, 2017). Although children were hardly confronted with indirect addition during mathematics instruction, they tended to frequently and highly efficiently apply this strategy, with accuracy and speed of strategy execution being at least as high as for direct subtraction (Torbeyns et al., 2018). Moreover, notwithstanding the absence of instruction in this strategy, they even adaptively took into account the numerical characteristics of the subtractions when selecting indirect addition versus direct subtraction strategies (Peltenburg et al., 2012; Peters et al., 2013, 2014; Torbeyns et al., 2018) as well as their individual mastery of the different types of strategies (Torbeyns et al., 2018). Importantly, these findings were observed for children of all mathematical achievement levels, including the lower-achieving children (Torbeyns et al., 2018) and children with mathematical difficulties (Peltenburg et al., 2012; Peters et al., 2014).

Other studies focused on (middle and upper) primary school children's use of number-based versus digit-based addition and subtraction strategies in different countries: the Netherlands, Belgium, Spain, and Taiwan (Hickendorff, 2013; Karantzis, 2010; Linsen, Torbeyns, Verschaffel, Reynvoet, & De Smedt, 2016; Torbeyns, Hickendorff, & Verschaffel, 2017; Torbeyns & Verschaffel, 2013, 2016; Yang & Huang, 2014). Confirming the results of previous studies in the domain, once being taught digit-based algorithms, many children tended to prefer them over number-based strategies (even applying the mental version of the digit-based algorithm when required to compute entirely in the head). But, contrasting previous findings, they applied the digit-based algorithms remarkably efficiently, with an accuracy and speed level that was at least as high as for the (previously taught and highly frequently practiced) number-based strategies. Finally, children demonstrated adaptive strategy choices, using number-based versus digit-based strategies in relation to the numerical characteristics of the problems (Torbeyns et al., 2018) and their individual mastery of the different types of strategies (Torbeyns & Verschaffel, 2013, 2016) but not the format of the problem (word problem versus symbolic problem; Hickendorff, 2013).

Obstacles in Development

Cumulative evidence indicates that the acquisition of multi-digit addition and subtraction strategies is a real challenge for many children, especially these of lower mathematical achievement levels. As discussed in Verschaffel et al. (2007), previous investigations point to children's limited conceptual understanding of number as one of the major sources of their difficulties in the acquiring and application of number-based and digit-based strategies. Linsen et al. (2016) recently provided further support for this claim, by analyzing the relation between 9–10-year-olds' magnitude understanding (i.e., insight into the magnitude or value of the numbers) and number-based and digit-based strategy efficiency in the domain of multi-digit subtraction. Their results revealed strong associations between children's magnitude understanding and their efficiency in both types of strategies. But the observed associations were stronger for number-based than for digit-based strategy use, suggesting a larger involvement of children's conceptual understanding of numbers in the execution of the former than in the execution of the latter type of strategies. Moreover, children's arithmetic fact knowledge for single-digit addition and subtraction was strongly related to their multi-digit strategy efficiency, which points to a second possible obstacle for children's multi-digit strategy acquisition in the domain of addition and subtraction, namely, their mastery of single-digit facts.

In addition to children's conceptual understanding of multi-digit numbers and their fluency with single-digit arithmetic facts, Selter, Prediger, Nührenbörger, and Hußmann (2012) discuss another possible obstacle for the development of fluency in multi-digit addition and subtraction, namely, their understanding of the arithmetic operations and their corresponding symbols (see also Robinson, 2017). For instance, using indirect addition on multi-digit subtractions relies on a broadened interpretation of the minus sign as indicating not only "taking away" (resulting in direct subtraction: taking away the smaller from the larger number) but also "bridging the difference" (enabling indirect addition). Likewise, when applying indirect addition, children have to understand the complementary relation between the addition and subtraction operation (i.e., understand that $a - b = ?$ can be solved via $b + ? = a$). For an extensive overview of the research on the role of understanding of the operations of addition and subtraction and their various arithmetical principles, see Baroody, Torbeyns, and Verschaffel (2009) and Robinson (2017).

The limited number of studies addressing the strategy competencies of children of the lower mathematical achievement levels and of children with mathematical difficulties did not yet provide unequivocal results about specific difficulties and the related foundational obstacles in their strategy development in the domain of multi-digit addition and subtraction (Peltenburg et al., 2012; Peters et al., 2014; Torbeyns, Hickendorff, et al., 2017; Torbeyns, Peters, De Smedt, Ghesquière, & Verschaffel, 2017). Studies with children without diagnosed mathematical difficulties reported that children with higher general mathematical achievement level had higher levels of strategy variety, efficiency, and adaptivity (Torbeyns, Hickendorff, et al., 2017; Torbeyns, Peters, et al, 2017). However, the studies of Peltenburg et al. (2012) and Peters et al. (2013, 2014) indicated that children with mathematical difficulties are also able to frequently and adaptively apply various number-based strategies. Future studies in children of the lowest mathematical achievement levels, including children with mathematical difficulties, are needed to get a better view on the contribution of children's conceptual understanding of numbers, symbols, and operations (cf. Linsen et al., 2016; Selter et al., 2012; Torbeyns, Peters, De Smedt, Ghesquière, & Verschaffel, 2016), their arithmetic fact knowledge (cf. Linsen et al., 2016), and other child- and context-related characteristics to their strategy development in the domain of multi-digit addition and subtraction.

Multi-digit Multiplication and Division Strategies

Strategies Framework

There is much less consensus on the different types of strategies for multiplication and division than there is for addition and subtraction. Based on the existing frameworks (e.g., Buijs, 2008; Hickendorff, 2013; van Putten, van den Brom-Snijders, & Beishuizen, 2005; Zhang, Ding, Lee, & Chen, 2017), in the current chapter, we propose a comprehensive classification system with dimensions comparable to those for multi-digit addition and subtraction: one dimension characterizing which operation underlies the solution process (multiplication or division) and the other dimension how the numbers are dealt with; see Table 32.2. Regarding the first dimension, in multi-digit multiplication there is only *direct multiplication* in which the underlying process is multiplication. In multi-digit division one can start with dividend in *direct division.* An alternative way to solve division problems is by *indirect multiplication*, also called multiplying-on (van den Heuvel-Panhuizen, Robitzsch, Treffers, & Köller, 2009) or reversed multiplication (Ambrose, Baek, & Carpenter, 2003), where one starts with the divisor and determines how many times it has to be multiplied to reach the dividend.

With respect to the second dimension, within the number-based strategies, it is again possible to distinguish between sequential, decomposition, and varying strategies. *Sequential* strategies involve movements forward or backward on the (mental) number line. In multiplication and division strategies, the sequential strategies are repeated addition or subtraction strategies, based on additive reasoning (e.g., see Larsson, 2016). In repeated addition, the multiplication problem 23 × 19 is solved, for instance, by adding the number 23 for 19 times. Of course, it is also possible not to repeatedly add single 23 s but multiples of 23 (see Table 32.2). Repeated addition can also be used to solve division problems within the indirect multiplication approach. In repeated subtraction, a division problem is solved by subtracting the divisor repeatedly from the dividend until there is nothing left. Again, it is possible to do this with single divisors or multiples of the divisor. By contrast, in *decomposition* strategies the numbers are decimally split (one or both operands in multiplication and only the dividend in division – splitting the divisor leads to an incorrect procedure). These strategies are, according to Larsson (2016), based on two-dimensional multiplicative reasoning. *Varying* strategies involve the adaptation of number and/or operations, like in the compensation strategy examples in Table 32.2. As a final number-based strategy, we again distinguish the *column-based strategy*, inspired by Dutch (RME) mathematics educators. The column-based strategy is a vertically notated schematized version of the decomposition strategy in multiplication and of the repeated subtraction strategy in division (e.g., Buijs, 2008; Treffers, 1987; Van Den Heuvel-Panhuizen, 2008).

The digit-based strategies involve operating on the digits ignoring their place value. It is important to note that the digit-based multiplication algorithm proceeds from right to left, like the digit-based algorithms for addition and subtraction. By contrast, the digit-based division algorithm proceeds from left to right and does not work with only one digit at a time.

Table 32.2 Overview of solution strategies for multi-digit multiplication and division

		Number-based strategies				Digit-based algorithm
		Sequential	Decomposition	Varying	Column-based	
Multiplication, e.g., 23 × 19	Direct multiplication	23 + 23 + 23 + … + 23 = **437** *or* 5 × 23 = 115 4 × 23 = 92 115 + 115 + 115 + 92 = **437**	23 × 10 = 230 23 × 9 = 207 230 + 207 = **437** *or* 20 × 10 = 200 3 × 10 = 30 20 × 9 = 180 3 × 9 = 27 200 + 30 + 180 + 27 = **437**	For example, compensation 23 × 20 = 460; 460 – 23 = **437**	23 19× 200 30 180 27+ **437**	23 19× 207 230+ **437**
Division, e.g., 168 : 12	Direct division	168 – 12 = 156; 156 – 12 = 144; [subtracting 12s 14 times → **14**] *or* 168 – 120 (10x) = 48 48 – 48 (4x) = 0 10 + 4 = **14**	100 : 12 = 8,25 60 : 12 = 5 8 : 12 = 0,75 8,25 + 5 + 0,75 = **14**	For example, compensation 240 : 12 = 20 72 : 12 = 6 20 – 6 = **14**	168 : 12 = 120 – 10× 48 48 – 4× 0	12 / 168 \ 14 12 – 48 48 – 0
	Indirect multiplication	12 + 12 = 24; 24 + 12 = 36; [adding 12 s 14 times → **14**]	8,25 × 12 = 100; 5 × 12 = 60; 0,75 × 12 = 8 8,25 + 5 + 0,75 = **14**	20 × 12 = 240 6 × 12 = 72 20 – 6 = **14**		

Children's Strategy Use: Empirical Findings

Compared to multi-digit addition and subtraction, there is little research into children's solution strategies use in multi-digit multiplication and division. Verschaffel et al. (2007)'s summary of the (at that time) available studies showed that, as for addition and subtraction, children rely on different types of number-based strategies to solve multi-digit multiplication and division, before the digit-based algorithms were introduced at school. In multiplication, the use of number-based strategies seems to progress from sequential (i.e., additive) strategies to decomposition (i.e., multiplicative) strategies. In multi-digit division, children tend to progress from the sequential strategies repeated addition/subtraction with single divisors to more efficient approaches using multiples (also called chunks) of the divisor. There is some evidence that once the digit-based algorithm is instructed, children rely heavily on that, abandoning the number-based strategies they had been using before.

Since the review of Verschaffel et al. (2007), few studies addressed children's number-based strategy competencies in the domain of multi-digit multiplication and division. Buijs (2008) followed Dutch 9–10-year-olds' strategy development in multi-digit multiplication. At each measurement point, children used the decomposition strategies most often, and the use increased over time. The frequency of repeated addition strategies was rather low, contrasting with Larsson's (2016) findings that Swedish 10–13-year-olds multi-digit multiplication strategy use remained to be heavily based on the repeated addition strategy.

Recent studies addressing (upper primary school) children's multi-digit number-based and digit-based strategy competencies in multiplication and division have primarily been conducted in the Netherlands. One exception is the study of Zhang et al. (2017), investigating the strategy use across single-digit and multi-digit multiplication problems in 8–11-year-old children from the USA. They found three distinct strategy use patterns, resembling different developmental levels: children who primarily used direct retrieval or the digit-based algorithm with high accuracy, children who primarily used number-based strategies (unitary counting, doubling, repeated addition, sequential, and decomposition strategies) with medium accuracy, and children who primarily used an incorrect operation or skipped the problems.

Before discussing the findings of the studies with Dutch children, it is important to note that due to the large influence of RME, the vast majority of the Dutch mathematics textbooks abandoned the digit-based algorithm for division for a long period of time (roughly mid-1990s–2010), because it was deemed very time-consuming to attain procedural mastery and at the same time rather meaningless and error-prone for children (Treffers, 1987; van den Heuvel-Panhuizen, 2008). Instead, the column-based strategy served as the standard written procedure. More recently, the digit-based division algorithm has returned in the latest version of the most common textbooks in the Netherlands (Royal Dutch Society of Arts and Sciences, 2009). A series of studies addressing Dutch 11–12-year-olds' strategy use in multi-digit multiplication and division showed, first, that strategy use was much less dominated by the digit-based algorithm than in addition and subtraction;

second, that in division children tended to use the column-based strategy as the preferred written procedure instead of the digit-based algorithm, in line with the instructional approach; and third, that the digit-based algorithms were as least as successful as the column-based strategies (Fagginger Auer, Hickendorff, Van Putten, Béguin, & Heiser, 2016; Hickendorff, 2013; Hickendorff, Heiser, Van Putten, & Verhelst, 2009). When analyzing the types of number-based strategies the children used, in multiplication, the decomposition strategies in which one or both of the operands were decimally split were the most often used number-based strategy in multiplication, whereas repeated addition was hardly used (Hickendorff, 2013), resembling the findings of Buijs' (2008) in 9–10-year-olds. In division, the column-based strategy was the most frequently used number-based strategy; repeated subtraction without the structured vertical notation, repeated addition, and decomposition were used rather infrequently (Hickendorff, 2013). Very recently, Hickendorff, Torbeyns, and Verschaffel (2017) investigated cross-national differences between 9–12-year-old children from the Netherlands and Flanders (Belgium) in solving multi-digit division problems. Children's strategy profiles were generally in line with differences in instruction between the two countries, as, for instance, reflected by the absence of the column-based strategy in Flemish children's strategy repertoire, although large intra- and interindividual strategy variety remained.

The few results regarding the adaptivity of strategy selection showed that, with respect using varying strategies in response to task characteristics, sixth graders' use of the compensation strategy on problems suitable for compensation (e.g., 2475: 25 via 2500: 25) was modest at most (Fagginger Auer, Hickendorff, & van Putten, 2016; Hickendorff, van Putten, Verhelst, & Heiser, 2010) but somewhat higher in Dutch children instructed according to RME principles than in Flemish children being taught in a more traditional way (Hickendorff et al., 2017).

Obstacles in Development

As in multi-digit addition and subtraction, the number-based strategies require sufficient conceptual knowledge of the place value system, and understanding of the arithmetic operations and symbols is also essential (e.g., Larsson, 2016; Robinson, 2017). Furthermore, children need to have sufficient knowledge and skills in elementary arithmetic to solve multi-digit multiplication and division problems. The example strategies in Table 32.2 illustrate that in multi-digit multiplication mastery of the single-digit addition and multiplication facts are essential in a multi-digit division strategies (multi-digit), subtraction is also involved.

As in addition and subtraction, there are some common systematic errors ("buggy procedures"), for instance, in the digit-based algorithms N × 0 = N, errors with carries and errors in forgetting to write down zeros (Kilpatrick et al., 2001; Verschaffel et al., 2007). Larsson (2016) and Buijs (2008) identified a common error in number-based multiplication strategies: the incomplete factorization into partial products (e.g., 23 × 19 = 20 × 10 + 3 × 9). Larsson (2016) interpreted that “buggy” strategy as an overgeneralization of addition strategies forming a structural hindrance for the conceptualization of the two-dimensionality of multiplication.

The discussed research findings signal some specific obstacles children may encounter. Larsson (2016) found that children’s understanding of multiplication was robustly rooted in repeated addition (and the associated understanding of multiplication in terms of equally sized groups). While this was found to be beneficial for their understandings of calculations and underlying arithmetical principles such as distributivity, it hindered them in making further steps in their multiplicative reasoning, for instance, in the fluent use of commutativity and in the proper conceptualization of decimal multiplication. In multi-digit division Dutch 11–12-year-olds seem to have difficulties making a choice between when and when not to write down their procedure and/or calculation steps: Substantial numbers of 11–12-year-olds solved the multiplication and/or division problems without writing anything down, and these nonwritten strategies were less accurate than written ones (Fagginger Auer, Hickendorff, & van Putten, 2016; Hickendorff et al., 2009). Two follow-up studies in division showed that demanding children who used nonwritten strategies to write down their working increased their performance: in all children (Hickendorff et al., 2010) or only in the children with lower mathematical achievement levels (Fagginger Auer, Hickendorff, & van Putten, 2016). Importantly, children with lower mathematical achievement levels were found to use nonwritten strategies just as often, or even more often, than their higher-achieving peers. This suggests that lower mathematical achievers have difficulties selecting their strategies, and multi-digit division problem-solving may be improved by promoting the use of written strategies. This is supported by the ideas that writing things down may both free up cognitive capacities and sequence the actions by schematizing (e.g., Buijs, 2008; Ruthven, 1998).

To the best of our knowledge, there is hardly any research addressing multi-digit multiplication and division strategies in children with mathematical difficulties. Only Zhang, Xin, Harris, and Ding (2014) investigated the effectiveness of strategy training interventions for children struggling with multiplication in a small-scale study with three 8–9-year-old children. Their results imply that children may experience difficulties in multiplication because their strategy development lags behind and that targeting (strategy) instruction to the individual child’s current level of strategy knowledge may be beneficial.

Discussion

The current chapter focused on number-based and digit-based solution strategies for multi-digit addition, subtraction, multiplication, and division problems. Based on the strategy classifications used in the literature, we presented two similar, comprehensive frameworks for addition/subtraction and multiplication/division strategies. These frameworks are based on two complementary dimensions: first, the way the numbers are manipulated to compute the outcome, as whole numbers in number-based strategies or as single digits ignoring their place value in digit-based strategies, and second, the kind of operation that is underlying the strategy. Within the number-based strategies, we distinguished sequential strategies, in which the operation involves movements along a (mental) number line, from decomposition strategies, in which the numbers are primarily seen as objects with a decimal structure and split and processed accordingly. Varying strategies involve the flexible adaptation of numbers and/or operations. Finally, the column-based strategy is an intermediate strategy between number-based and digit-based strategies due to its hybrid nature and its position in the RME-based instructional pathway.

Starting from these two frameworks, we discussed the empirical findings on the (development of) children's solution strategies in multi-digit arithmetic. Compared to single-digit arithmetic, the research body is rather small, and in particular, studies addressing multi-digit multiplication and division remain remarkably scarce (see also Larsson, 2016). Further research addressing multiplication and division simultaneously is necessary, since these two operations and the relations between them are more difficult for children to understand and may require explicit instruction (Robinson, 2017). Relatedly it is interesting to note that the four multi-digit arithmetic operations are very rarely addressed simultaneously, see Hickendorff (2013) for an exception, whereas mathematically, psychologically, and educationally, the operations are clearly interrelated. For instance, the work of Larsson (2016) signals the overgeneralization of aspects of additive reasoning to multiplication. In order to increase our understanding of (the development of) multi-digit solution strategies, further research into children's multi-digit strategy competencies in the four operations and their interrelations is called for. Finally, an important remark is that the majority of the studies discussed were carried out in the USA or Europe, whereas cultural differences in preferred strategies have been reported which may be related to the curriculum (e.g., abacus instruction enhancing visualization strategies) as well as extracurricular culture-specific factors (e.g., language for numbers) (e.g., Campbell, Xue, & Campbell, 2001; Cantlon & Brannon, 2006). Future cross-cultural research would allow a broader perspective on children's strategy development in multi-digit arithmetic in different curricula and cultures.

The empirical findings show that children use a variety of number-based strategies efficiently and adaptively, before the introduction of the digit-based algorithms. The introduction of the digit-based algorithms seems a critical instructional event: children show a large tendency to use the digit-based algorithms once they are instructed, and recent findings indicate that they do so rather efficiently. Furthermore, in the Dutch RME approach, the column-based strategies are introduced as a smooth transitory path between number-based strategies and the digit-based algorithm, or even – more radically – as a more insightful, more conceptually based alternative for these digit-based algorithms. Studies show that Dutch children perform equally well with column-based strategies as with the digit-based algorithm in division. Moreover, Flemish and Dutch children with rather different instructional settings perform equally well in the domain of subtraction and division. These results may indicate that the column-based strategy may act as a fruitful stepping stone, or even alternative, to the digit-based algorithms. However, further research into the value of the column-based strategies, in particular for children with mathematical difficulties, is necessary.

All these results combined are relevant for the debate between proponents of different mathematics educational theories on the position and value of digit-based algorithms (e.g., Kamii & Dominick, 1997; Treffers, 1987). As noted before (e.g., Verschaffel et al., 2007), strategy efficiency may be at odds with other components of mathematical competence, such as insightful and adaptive computations. The focus of mathematics education on these latter aspects is not only because these are expected to increase computational efficiency but also because mathematics education targets other goals as well, such as conceptual understanding of mathematical operations and the disposition to choose flexibly from a repertoire of strategies. These elements in particular may form a challenge for children with mathematical difficulties.

The acquisition of multi-digit arithmetic strategies is a real challenge for many children, especially those with mathematical difficulties. The major obstacles these children may encounter in multi-digit arithmetic seem to be their conceptual understanding, procedural fluency, and adaptive/flexible strategy selection. Children's limited understanding of multi-digit numbers is likely one of the major obstacles in multi-digit arithmetic, since it is essential in the execution of both number-based and digit-based strategies. Moreover, children may have difficulties understanding the arithmetic operations and their corresponding symbols. Regarding procedural fluency, not having mastered single-digit arithmetic facts is an obstacle for children with mathematical difficulties in acquiring multi-digit strategies competence. Lastly, the research findings suggest that the adaptive selection of strategies from a repertoire of candidate strategies, and choosing when to write down the solution procedure instead of calculating in the head, may be challenging for children with lower levels of mathematical achievement.

Finally, this brings us to the issue of the effective strategy instruction for children with mathematical difficulties. Although the available studies show that at the group level there are differences in children's strategy use that can be related to differences in the instruction they received, at the level of an individual child there are a lot of variety and manifestations of strategy preference and use that do not coincide with the nature of the instruction received. Given this complex relation between strategy instruction and strategy development, we plead for instruction that (a) acknowledges that children develop their own strategies and stimulates children to use them, (b) diagnoses strategic development by ongoing assessment and progress monitoring, (c) assigns tasks based on children's current strategy level, (d) stimulates children to (self-)explain their strategies, and (e) provides explicit strategy instruction for struggling children (Zhang et al., 2014). Evidently, more research has to be done to optimize strategy instruction in the domain of multi-digit arithmetic for children with mathematical difficulties.

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Chapter 33

Development of a Sustainable Place Value Understanding

Moritz Herzog, Antje Ehlert, and Annemarie Fritz

Introduction

"0" is the Hindu–Arabic sign for the magnitude of nothing. "0" was the last numeral added to the Hindu–Arabic numbers. While other numbers have been represented for thousands of years, zero is relatively young: only about 1500 years old (Ifrah, 1998). What is the point of a symbol of something that does not exist? Why not just omit the nonexistent? The ancient Babylonians, considered to be the first culture that used a place value system, did not see any reason to invent a number sign for zero. They just left a little gap between digits. Although this practice is rather error prone, the Babylonian place value system worked well enough that the way we partition the hour (60 min) or a circle (360°) can be referenced back to the ancient Babylonians (Ifrah, 1998).

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Even though the Babylonians did not have a zero, they required some kind of placeholder to distinguish numbers such as 2017 from 217 or 2107, as we know them today. Therefore, it might be no coincidence that the adoption of the place value system in Europe, the Middle East, and Asia – which resulted in the worldwide use of the Hindu-Arabic number system – did not start until the ancient Indians invented a symbol for zero over 2000 years after the Babylonians invented a place value system. The fact that it took such a long time before zero joined the place value system illustrates how difficult its invention was (Ifrah, 1998). Accordingly, the place value system is difficult to understand for learners. Though research on learners' trajectories toward place value understanding is of great theoretical and practical significance, it is still rarely addressed.

Properties of Place Value Systems

The central concept of place value systems is to use symbols that carry information not only about the magnitude they represent but also about how many of a certain magnitude they hold. In additive number systems (e.g., the Roman system) a number like 2017 is written as MMXVII. Each symbol represents a certain quantity (M = 1000, X = 10, etc.) and the whole number is represented by adding up the values of all symbols: M + M + X + VII (= 1000 + 1000 + 10 + 5 + 1 + 1). How many of each bundle a number holds is represented by the quantity of the same symbols.

In contrast, in a place value system each digit provides information about the size *and* the quantity of the bundle it represents. While the quantity of the represented bundles is indicated by the face value of the digit, the bundles' size has to be derived from the position within the number. The parts of a number are added as in additive number systems, yet the partition is strictly decimally structured, i.e., digits from 0 to 9 in the decimal or base-ten place value system, e.g., $2017 = 2000 + 10 + 7 = 2\cdot1000 + 0\cdot100 + 1\cdot10 + 7\cdot1$.

The unique elegance and efficiency of the place value system allows the writing of infinitely big and (according to absolute value) small numbers with a finite set of symbols. This is facilitated by the following four properties (Ross, 1989):

- The *positional property* is the aforementioned additional information regarding the size of the bundles each digit holds by means of its position in the number.
- The *base-ten property* means that the bundles are powers of ten.
- The *multiplicative property* refers to the multiplication of the digits' face value and the bundles they represent to construct the decimal decomposition of a number.
- The *additive property* expresses the additive composition of the decimal decomposition of a number.

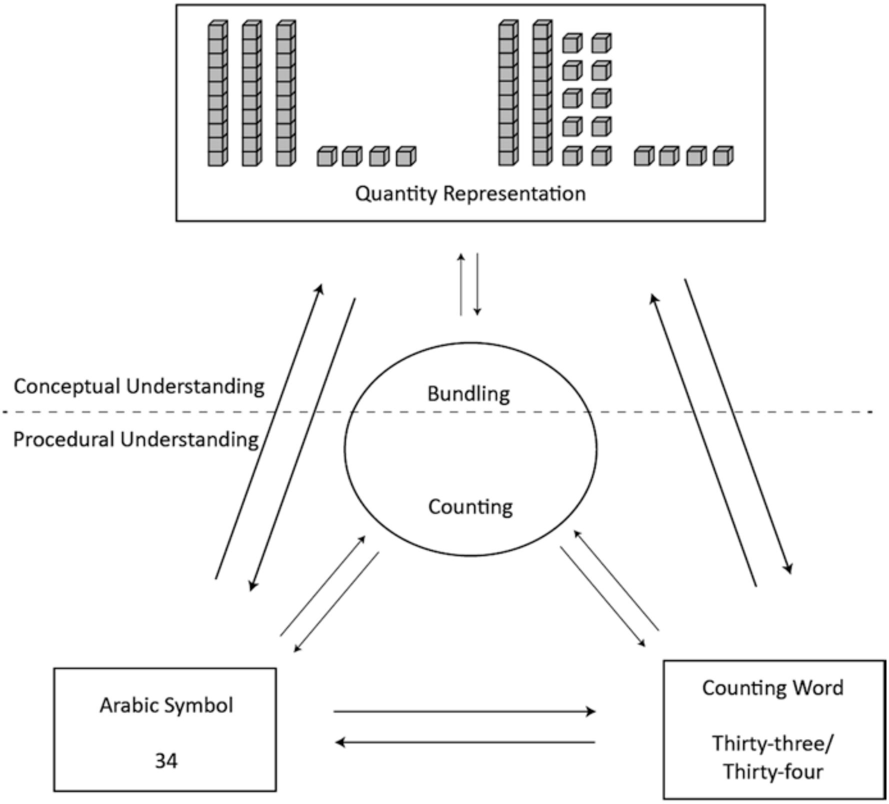


Fig. 33.1 Procedural and conceptual place value understanding (see also van de Walle et al., 2004)

Place Value Understanding

The place value system is of cultural origin. Its invention is a historical achievement that strongly affected the development of culture. It took about 30,000 years to evolve from the first tally marks to a fully developed place value system (Ifrah, 1998). Against this background it is no wonder that children do not understand the decimal structure of our number system automatically (Cobb & Wheatley, 1988; Fuson et al., 1997; Ross, 1989). Thus, place value understanding has to be attained via learning processes and begs the question: What are the necessary skills to understand the place value system?

Van de Walle, Karp, and Bay-Williams (2004) differentiate between conceptual and procedural aspects of place value understanding (see Fig. 33.1). *Procedural* place value understanding consists mainly of semiotic and linguistic insights into the structure of number words and reading skills regarding numerals, i.e., counting in decimal units (tens, hundreds, etc.) and multidigit arithmetic. The central element of the *conceptual* place value understanding is the relation between the decimal units, i.e., bundling and unbundling actions and different representations of decimal structures. The ability to integrate number words, numerals, and decimally structured representations of magnitude result in procedural and conceptual place value understanding (Fuson et al., 1997; van de Walle et al., 2004).

Procedural Place Value Understanding

Even though, as research suggests, numbers are initially perceived as nonstructured entities, Hindu–Arabic numerals are decimally structured, i.e., they consist of different digits that represent quantities of powers of ten, which increase from right to left. This decimal structure that is induced by the digits is the starting point for understanding of Hindu-Arabic numerals (Cobb & Wheatley, 1988; Fuson et al., 1997; Ross, 1989). As the digits are differentiated, their positions are named. For example, in the number 2017 the 2 is in the thousands place and the 1 is in the tens place.

Corresponding to the structure of the numerals, most European languages, most Asian languages, and several African languages structure number words in the same way (Miura, Kim, Chang, & Okamoto, 1988; Pixner et al., 2011; Zaslavsky, 1999). Each digit is expressed separately with power of ten and face values – except in some Asian languages and the numbers between 10 and 20. Number words are sorted in this way from the biggest to the smallest decimal unit, keeping the order of powers of ten in the verbal form.

There is some variance in transparency between the languages; however, the decimal structure in phrases for each place value seems to be relatively universal. The decimal structure of number words supports counting not only in steps of 1 but also in steps of bigger powers of 10. According to Fuson et al. (1997) a child's competence in counting in units, tens, etc., reflects its representation of the decimal units and thus its development of place value understanding. Flexible counting tasks are used to assess early place value understanding (Aunio & Räsänen, 2016; Chan, Au, & Tang, 2014).

Conceptual Place Value Understanding

While procedural aspects of place value understanding are in the first place necessary social conventions that children have to learn – i.e., *how* to write, read, and count numbers – place value conceptual aspects focus on children's representations explaining *why* these conventions hold. Thus, conceptual place value understanding underlies and supports procedural knowledge.

Central to the place value system is the relation between the decimal units: 10 units make up 1 ten, 10 tens make up 1 hundred, and so on. The significance of continued bundling for the understanding of place value is broadly recognized (e.g., Cobb & Wheatley, 1988; Ladel & Kortenkamp, 2016; van de Walle et al., 2004). The relation between the decimal bundling units allows more than one representation of a certain number (see Fig. 33.2). For example, 42 can be represented as 4 tens and 2 units (canonical), as well as 3 tens and 12 units or 2 tens and 22 units

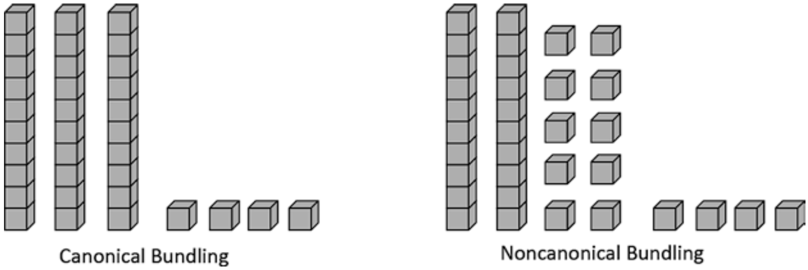


Fig. 33.2 Canonical and noncanonical place value bundling

(noncanonical). The ability to structure numbers canonically and noncanonically has to be considered a main place value concept (Ladel & Kortenkamp, 2016; van de Walle et al. 2004).

The relation between tens and units can be visualized by bundling and unbundling as well as trading acts. The most prominent visualizations are base-ten blocks, abaci, play money, or bundling sticks, which all support the bundling of ten units into one ten or at least trading in a similar way (Nührenbörger & Steinbring, 2008). By creating the bigger bundling unit from smaller units (e.g., combining ten little cubes into a tens stick), their equivalence becomes observable. Thus, decimally structured manipulatives are used in instructive settings and schooling (Fuson et al., 1997; Nührenbörger & Steinbring, 2008).

The aforementioned manipulatives all share a cardinal representation of place value. That means that numbers can be decomposed into subsets of tens and units. Cardinal representations of numbers are considered a crucial number concept that is not acquired independently of instruction (Dehaene, 2011; Fritz, Ehlert, & Balzer,. 2013). Besides the cardinal, also ordinal number representations are discussed in research—in particular, the mental number line on which numbers are aligned in increasing order (Dehaene, 2011). Symbolic magnitude comparison tasks suggest that digits might be processed separately, indicating an underlying base-ten structure of the mental number line (Nuerk, Moeller, & Willmes, 2015).

Difficulties in Place Value Understanding

The way learners attain place value understanding is a topic that is not often researched. The literature mostly addresses learners' difficulties in place value understanding and the role these difficulties play regarding mathematical learning difficulties. In particular, difficulties in procedural aspects have been given attention (e.g., Chan & Ho, 2010; Desoete, 2015; Nuerk et al., 2015). In contrast, difficulties in conceptual place value understanding have received relatively little focus.

Research has revealed that many children do not acquire a profound understanding of place value (Fritz & Ricken, 2008; Fuson et al., 1997; Kamii, 1986; Ross, 1989). Difficulties in place value tasks have been reported for nearly all grades, different countries all over the world, and learners from the whole performance range (Fritz & Ricken, 2008; Fuson et al., 1997; Gervasoni & Sullivan, 2007; Hart, 2009; MacDonald, 2008). Place value understanding is a basis for mathematics in middle school and beyond (Moeller, Pixner, Zuber, Kaufmann, & Nuerk, 2011; Van de Walle et al., 2004). Studies have revealed a positive relation between place value understanding and basic arithmetic problems and operations – in particular, regarding standard algorithms (Fuson & Briars, 1990; Gebhardt, Zehner, & Hessels, 2012; Nuerk et al., 2015).

Several authors describe how low skills and performance in place value can be seen as a predictor and a source of math difficulties (Cawley, Parmar, Lucas-Fusco, Kilian, & Foley, 2007; Chan & Ho, 2010; Desoete, 2015; Fuson et al., 1997). Intervention studies involving grouping and trading actions between the place values as well as counting in "tens" showed that training of second graders' place value understanding with help of base-ten blocks improved addition and subtraction performances (Fuson & Briars, 1990; Ho & Cheng, 1997). In particular, written standard algorithms are affected by place value training (Cawley et al., 2007).

Regarding the notation of Hindu-Arabic numerals, difficulties in mapping numerals and magnitude representations have been reported (Byrge, Smith, & Mix, 2014; Zuber, Pixner, Moeller, & Nuerk, 2009). Young children tend to write numbers according to their verbal form: "seven hundred fifty three" might appear as "700,503" or "70053" or "7100503." These children probably do not combine the decimal addends correctly and/or do not perceive the decimal addends as multiples of decimal units.

The transparency of number words affects the difficulties children have in learning them and mapping them onto numerals. For example, in German and Dutch, "27" is read as "seven and twenty." This is a less transparent way of speaking – known as tens-units inversion – and is more prone to errors, leading to lower performance in transcoding and number line estimation tasks (Dowker & Roberts, 2015; Pixner et al., 2011).

The contrast in the transparency of number words between European and Asian languages motivated several cross-cultural studies to research the effects of number word systems on place value understanding and early numeracy (for an overview see Okamoto (2015), Miller, Smith, Zhu, and Zhang (1995)) revealed that the development of number word chain repetition in Asian languages and English proceeds similarly except for the numbers 10 to 20. In this number range, Asian-speaking children outperform English-speaking children. Recent studies replicated this result and added the limitation that this initial advantage does not persist over the years (Mark & Dowker, 2015).

Development of Place Value Understanding

As described, place value understanding involves knowledge of various more or less complex concepts and procedures. However, knowing the components of place value only provides hints about how a sustainable place value understanding is developed – and how it can be taught. Competency models of place value understanding can structure the building blocks that a subject analysis provides and inform about possible

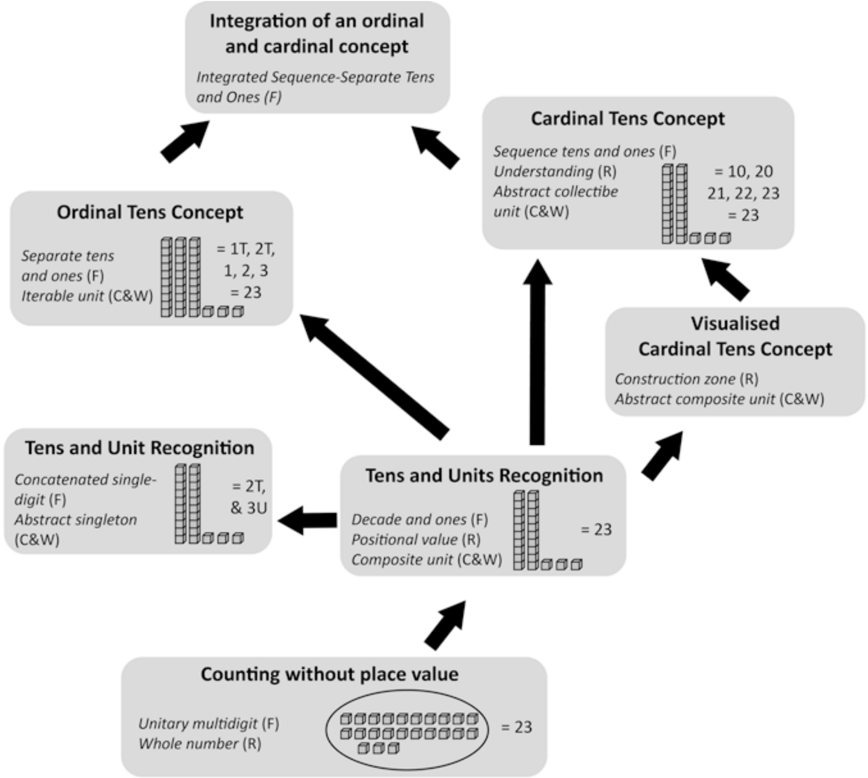


Fig. 33.3 Synopsis of theories on place value development (C&W=Cobb & Wheatley, 1988; F=Fuson et al., 1997; R=Ross, 1989)

difficulties and barriers a child encounters while learning. Naturally, such models need a sound theoretical foundation as well as empirical validation.

Over the last 30 years, quite a few competency models have originated. Below, the models of Cobb and Wheatley (1988), Fuson et al. (1997), and Ross (1989), which have had a great influence on research related to place value, are described and their similarities are emphasized. Figure 33.3 aims to provide an overview of the communalities of these models.

Nonstructured Numbers

All authors state that children initially do not perceive the decimal structure in numbers. According to Cobb and Wheatley (*ten as a numerical composite*), a ten "is structurally no different from the meaning given to other number words" (1988, p. 4). On this initial level, ten represents ten counting steps. Thus, children on this level focus on the individual steps rather than on the whole of the steps; "ten" has no property referring to the magnitude of ten. Fuson et al. (1997) (*unitary multidigit*) suppose that this results from a similar view of single-digit numbers that is maintained and transferred to two-digit numbers. Ross states that on the first level (*whole numeral*), children see numbers as unitary and not decimally structured, and thus the digits hold no information; rather, it is the whole numeral that represents a certain number.

Identifying Decimal Units

It is not until the second level that children pay attention to digits. According to Ross (1989) (*positional property*) and Fuson et al. (1997) (*decades and ones*), the first knowledge they gain is the names of the place values and their position within the numeral. Consequently, children on this level can name and show place value positions as well as partitions of a number decimally. For example, children know that "35" has a tens part (3) and a units part (5). Sometimes learners tend to write "305," keeping the 30 as they are not yet integrating both parts. This error is found also for three-digit numbers with simultaneously correct tens-and-units notation (e.g., "70053" instead of 753), indicating that hundreds as a bundling unit are not yet integrated into the existing concepts of a place value for two-digit numbers (Byrge et al., 2014).

Following Cobb and Wheatley (1988), children subsequently construct a type of unit called "ten". On this level (*ten as an abstract composite unit*), children can count in steps of ten from the middle of a decade on but are reliant on some kind of visual aid that is presented to the children. Some children are then able to detach from the presentation of material and coordinate the ten-structured counting as they keep track of the counting act with the help of their fingers.

Ordinal Aspect of Place Value Understanding

Cobb and Wheatley (1988) and Fuson et al. (1997) differentiate how the relation between tens and units is interpreted into a cardinal and an ordinal aspect on their third and fourth levels, which the authors derive from predominant counting strategies. Both state that these levels are acquired independently and in no specific order. Some children tend toward an ordinal understanding of place value, i.e., following Cobb and Wheatley (1988) (*ten as an iterable unit*) and Fuson et al. (1997) (*sequence tens and ones*) they tend to count in steps of tens, independent of the starting number. For example, 71 − _ = 39 is solved by counting 71, 61, 51, 41–40, 39, and keeping track of the counting steps taken. The aspect of ten as a bundling unit is less emphasized according to this understanding. Tens are rather understood as unitary counting intervals on the number line.

Ross (1989) suggests that the ordinal aspect precedes the cardinal aspect. According to her, children do not see the tens part as multiples of ten on this level (*face value*). Learners on this level know that, for example, in the number 53 the 5 represents 50; however, they do not interpret the 50 as 5 bundles of ten. Thus, children on this level focus rather on the numbers that the digits represent individually instead of relating them to bundling units.

Cardinal Aspect of Place Value Understanding

Regarding those that tend toward the cardinal place value aspect, Cobb and Wheatley (1988) and Fuson et al. (1997) report that these children focus on tens as bundling units. Children on this level are supposed to use a counting strategy that counts complete decades (one ten, two tens…) first before facing the units. They solve 37 + 24 by counting 37, 40, 50, 60, and 61. Cobb and Wheatley (1988) (*ten as an abstract collectible unit*) and Fuson et al. (1997) (*separate tens and ones*) interpret the separated counting of tens and units as cardinal understanding of the bundling units.

Ross (1989) describes that children on this level (*construction zone*) understand digits as representations of bundles of units, tens, etc. Although they know that tens are bundles of ten units, they do not know about the relation between tens and units at this time. Their knowledge allows them to solve tasks that involve canonical but not noncanonical partition. This stage "is characterized by unreliable task performances" (Ross, 1989, p. 49).

Integration of Cardinal and Ordinal Aspects

The integration of cardinal and ordinal aspects is seen as an important step in the development of place value understanding by Cobb and Wheatley (1988) and Fuson et al. (1997) (*integrated sequence-separate tens and ones*). The integrated place value understanding allows for flexible perspectives on tens and units, which facilitate using counting strategies fluently and reliably for two-digit arithmetic. According to Ross (1989) (*understanding*), children on this level know that the digits of a number represent a canonical partition into groups of units and tens. As they get insight into the relation between tens and units, children can also handle noncanonical partitions including bundling and unbundling flexibly by deriving the whole quantity from its parts as well as deconstructing the whole into partial quantities. How exactly level learners attain a conceptual place value understanding is not explained in this model.

Nonsustainable Concepts

Children do not always develop useful representations of place value. Cobb and Wheatley (1988) (*ten as an abstract singleton*) and Fuson et al. (1997) (*concatenated single digit*) describe such developments as concepts that separate tens and units. Tens and units are not viewed in relation to each other; they exist beside each other, i.e., "tens" do not have a "ten-ness" about them. Thus, children on this misleading level count tens while saying single-digit number words without knowing that they represent bundles of ten. Following the authors, this understanding inhibits the development of a sustainable place value understanding.

Both of the central counting strategies Fuson et al. (1997) assigned to their levels could be found in an empirical study. When presented with paper strips with various numbers of tens and units similar to base-ten blocks, Chinese children used both counting strategies (Chan et al. 2014). However, this is the only empirical validation of the models presented.

Our Own Model

Although for some time place value understanding has been referred to by many researchers, there is still a lack of empirical research to validate the competency or progression models. To be considered a valid model, the model should be based on a profound and broad theoretical framework, as well as on empirical validation.

The models described above mostly focus on counting strategies that are used as assessments of place value development and characterization of levels. Counting is surely an important learning step during the development of arithmetic competencies (Desoete, Ceulemans, Roeyers, & Huylebroeck, 2009; Fritz et al., 2013). Persistent counting as the main or only computation strategy, however, is related to poor performances in math and thus is considered a characteristic and indicator of math difficulties (Aunio & Räsänen, 2016; Fritz et al., 2013). For this reason, the conceptual development of place value deserves more attention.

In order to fill these gaps, through empirical validation and a focus on theoretical concepts, Herzog, Fritz, and Ehlert (2017) aimed to assess learners' conceptual and procedural place value understanding. A broad theoretical basis was reviewed, items to assess procedural and conceptual knowledge were constructed, and several pilot studies were conducted (in 2010–2016) with about 10,000 children from Grades 2 to 5 in total. Finally, a hierarchical sequence of levels describing the conceptual place value competencies was developed and proved.

The model contains 5 levels that are hierarchically designed—each level builds up on the previous levels. Since the model focuses on place value concepts, the described concepts are cumulative, too. However, the place value concepts of the model do not replace previous concepts. They overlap them and become predominant in most situations. Previous concepts remain available to children and might be linked to certain tasks where they are still mainly used (Rittle-Johnson, Siegler, & Alibali, 2001).

Predecadic Level

In accordance with earlier models, children start their place value development with no knowledge about place values at all (Cobb & Wheatley, 1988; Fuson et al., 1997; Ross, 1989; van de Walle et al., 2004). Multidigit numbers appear to them as unitary

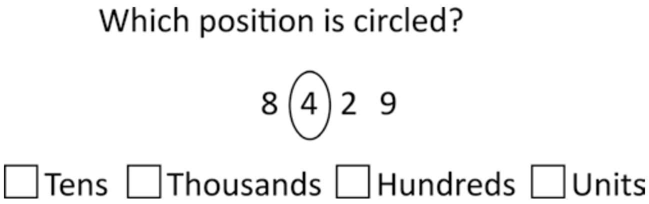


Fig. 33.4 Example item on Level I

entities that do not have a special partition into tens and units. Thus, children on this level might be able to decompose a number (e.g., 14 into 8 and 6) but would not detect decimal partitions as more relevant and related to number words or Hindu-Arabic numerals.

Level I: Place Values

The first step toward a sustainable place value understanding for children is when they start to distinguish between the bundling units, as previous research has revealed (Cobb & Wheatley, 1988; Fuson et al., 1997; Ross, 1989; van de Walle et al., 2004). Children on this level know the bundling units and can assign them to the positions of digits in Hindu-Arabic numerals (see Fig. 33.4).

The number range for this skill depends largely on instruction, i.e., children need to be taught the place values and their positions. However, the number range itself does not affect the difficulty of these tasks significantly, i.e., children only need to learn new names for digits in their respective position to enhance the number range. However, an understanding of the relation between the bundling units is not necessary.

Their knowledge at this stage enables them to construct and decompose numbers into their canonical partitions. Canonical partitions are not limited to a certain number, yet learners have to know the involved positions. However, they do not know how these partitions feature in the place value system. They derive them from the numerals and the number words. At this stage, the bundling units stay unrelated and cannot be transferred to each other. For this reason, addition tasks are feasible up to 1000, but only without carries.

Level II: Tens-Units Relation with Visual Support

On this level, children understand the relation between tens and units if they have some kind of visual support. According to Cobb and Wheatley (1988), children need visual aids to relate tens and units at the early stages. The concept of ten is not yet elaborated enough that it forms a kind of bundling unit, and there is no abstract

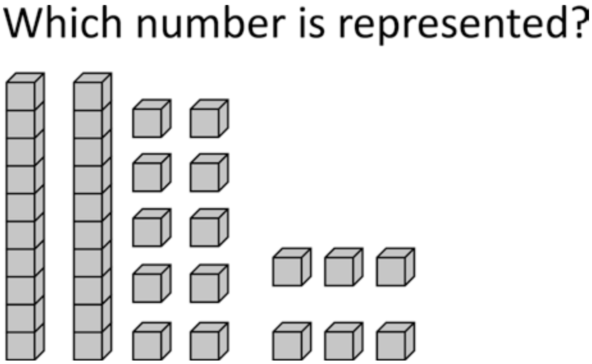


Fig. 33.5 Example item on Level II

property of "being ten units" in a ten. If children on this level have access to a decimally structured visual aid, they can handle noncanonical representations (see Fig. 33.5).

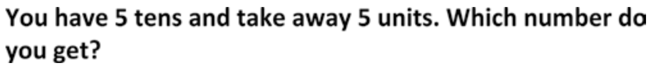
However, the relation between tens and units has to be verified by counting the units in a ten or comparing ten units and one ten. For example, when using base-ten blocks of different size, some children will pay more attention to the blocks' extension than to their amount. The concept of ten is closely linked to a symbol (base-ten blocks) or counting acts. Without such a supportive visual aid, children cannot handle noncanonical partitions. Children seem to understand the relation between tens and units in a functional way on this level. Bigger decimal bundling units are not interrelated on this level.

Arithmetic routines are internalized well enough for the child to be able to solve addition tasks without carries in an unlimited number range and subtraction tasks without carries up to 100. Those tasks can be solved by digit-separating strategies (e.g., tens + tens and units + units); thus, automatized concepts from Level I are sufficient. Addition tasks that require carries are feasible to them only for two-digit numbers; maybe counting routines and mental representations of structured manipulatives, as described by Cobb and Wheatley (1988), facilitate this.

Level III: Tens–Units Relation Without Visual Support

While children detach from visual aids and internalize the relation between tens and units, they acquire the concepts of Level III. At this stage, they do not have to verify the equivalence of ten units to one ten. To them, ten is no longer simply a symbol but a bundle of ten single units that are combined into an abstract unit. This concept enables them to handle noncanonical representations without a visual aid (see Fig. 33.6).

In contrast to Cobb and Wheatley (1988), we state that conceptual and sustainable knowledge of the tens-units relation is not facilitated by counting acts. Ten has a cardinal meaning that is sufficiently abstract to be independent from concrete representations and counting routines. Counting acts, of course, remain a necessary



You have 5 tens and take away 5 units. Which number do you get?

Fig. 33.6 Example item on Level III

skill in certain situations, e.g., when asked to determine which number is represented with base-ten blocks. However, children are no longer reliant on this strategy to interrelate tens and units. At this stage, the abstract knowledge of the tens-units relation remains limited to two-digit numbers and higher bundling units like hundreds and thousands therefore cannot be handled in an abstract way.

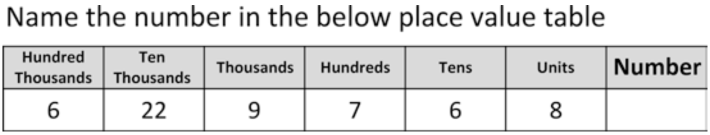
While the relation between tens and units is automatized and abstracted on this level for two-digit numbers, three-digit numbers are already slowly integrated into this concept. This means that for the bundling of tens and hundreds, learners will still rely on visual aids. Just as was the case for units and tens on the previous level, learners will construct and verify the relation between tens and hundreds via counting. Thus, concepts regarding tens and units are not automatically transferred and applied to bigger bundling units.

As Ross (1989) stated, children are often able to solve complex arithmetic tasks without having a sound conceptual basis. Although children on Level III only understand the relation between tens and units up to 100, they can solve additions and noncarry subtraction tasks in any number range; subtraction tasks with carries remain limited to the number range up to 1000.

Level IV: General Decimal-Bundling-Unit Relations

Although bundling and unbundling principles do not vary across the number range, children show difficulties in transferring the concepts of tens and units to three- and more-digit numbers (Herzog, Fritz, et al., 2017). The findings by Byrge et al. (2014) support these results, as similar errors can be found for two- and three-digit numbers (see above). On the fourth and final level, children expand the concepts of bundling and unbundling between the decimal units into higher number ranges and thereby further bundling units. They can handle abstract noncanonical partitions for multidigit numbers in general (see Fig. 33.7).

Once they understand the relation between decimal bundling and unbundling, children can apply this concept to higher numbers without limitation of the number range; only if they do not know certain place values (e.g., very high place values like billions) can their place value concepts not be linked to these bundling units. Also, subtraction tasks are feasible to them in any number range on Level IV. As bundling concepts and arithmetic performances are acquired to a maximal extent, the model is considered to be completed with this level.



Name the number in the below place value table

Hundred Thousands	Ten Thousands	Thousands	Hundreds	Tens	Units	Number
6	22	9	7	6	8	

Fig. 33.7 Example item on Level IV

Empirical Research

As stated, the model was developed on the basis of several pilot studies that used a Rasch analysis. In a Rasch analysis, item difficulties and person abilities are represented on the same scale. This allows the comparison of items regarding their difficulty. Items that share the same conceptual requirements are expected to be of a similar difficulty. In other words, the hierarchy of levels described in the model is supposed to be reflected in the items' difficulties, too. Items of lower levels should be easier than items of higher levels (Dunne, Long, Craig, & Venter, 2012).

In a study with about 1300 learners from Grades 2 to 5 the model was underpinned by means of a Rasch analysis (Herzog, Fritz, et al., 2017). The results of this study were recently able to be replicated in a further study with over 500 fifth graders (Herzog et al., in preparation).

A comparison of learners from different grades revealed that younger learners in general are on lower levels than older learners. A cross-sectional study revealed a significant improvement toward higher levels as the age of learners increased (Herzog et al., in preparation). This result suggests that learners rise through the described levels as they develop place value concepts.

The model hierarchy seems to be valid not only in German but also in English. In a study with about 200 South African learners from Grades 2 to 4, the level hierarchy was replicated in an English test version (Herzog, Ehlert, & Fritz, 2017). This suggests that language and cultural backgrounds have little impact, if any, on the conceptual understanding of the place value structure.

Conclusion

Competency models allow us to identify and specify barriers in mathematical learning. These barriers indicate which concepts really facilitate place value understanding and which learning steps are particularly difficult for learners. Knowledge of these learning barriers gives us the ability to design tools for learning in the classroom as well as individual fostering (Fritz et al. 2013).

Several studies reveal that many leaners have difficulties grasping place value, yet empirically evaluated information regarding the prevalence is rare (e.g., Chan & Ho, 2010; Desoete, 2015; Fritz & Ricken, 2008; Kamii, 1986; Nuerk et al., 2015; Ross, 1989).

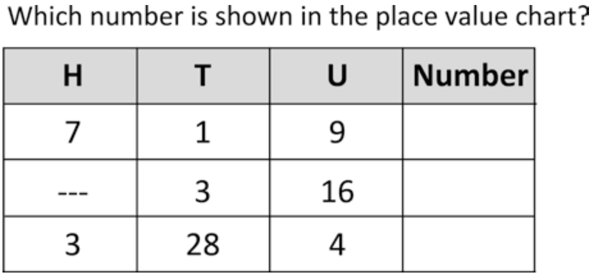
Barriers in the Development of a Sustainable Place Value Understanding

A sustainable place value understanding consists of various skills that children have to learn. Not all of them appear to be really challenging and result in severe difficulties. Research is mostly focused on procedural place value understanding difficulties, i.e., the mapping of number words, numerals, and canonical magnitude representations have been discussed as the main learning barriers in place value understanding (Cobb & Wheatley, 1988; Fuson et al., 1997; Schulz, 2014; van de Walle et al., 2004). This refers, in particular, to transcoding between the elements of the triple-code model (Dehaene 1992; Dehaene & Cohen, 1995). Different levels of conceptual place value understanding, like knowledge of number partitions and canonical and noncanonical representations in varying number ranges, have been mostly neglected.

In our empirical studies, however, procedural transcoding did not appear to be the crucial skill for most learners. In a study with 404 German fourth and fifth graders, which included students with all socioeconomic backgrounds represented equally and covered the whole performance range, about 6% of learners failed to even reach Level I. These children had difficulties with procedural place value understanding and did not perceive the decimal structure of numbers. Only the very poorest-performing learners at the end of primary school had difficulties in transcoding tasks. Another 24% had attained the concept of decimally structured numbers and reached Level I. These learners showed serious difficulties in place value understanding as they could not relate the bundling units with or without visual aids (Fuson et al., 1997), although they could solve procedural place value tasks. In conclusion, procedural place value understanding is a learning barrier only to a small percentage of learners; the larger number of learners with poor place value proficiency tend to get stuck on the important concepts regarding the relations between the bundling units.

Therefore, it is obvious that the efficiency and elegance of the place value system relies on a conceptual understanding. The use of efficient computing strategies is only facilitated if the relation of the bundling units is internalized and carrying, borrowing, and trading acts can be conducted instantly. If these concepts are not internalized, the advantages of the place value system that arise from its properties are hindered.

Place value concepts are not transferred and applied automatically to bigger numbers. Although it is obvious that the system of bundling and unbundling is the same for carries at any position, children seem to stick to tens and units initially. This corresponds to recent research results (Byrge et al., 2014). In this respect, learners often apply “old” concepts to “new” tasks, notwithstanding that they are not appropriate. For example, children below Level III predominantly failed the last task in Fig. 33.8 (“3284”) because they do not know how to handle noncanonic representations without visual aids. Instead, on these tasks, they apply the routine of canonical representations (i.e., lining up all parts of the place value chart in the



Which number is shown in the place value chart?

H	T	U	Number
7	1	9	
---	3	16	
3	28	4	

Fig. 33.8 Example item on different levels

given order). In contrast, the most frequent wrong answer from children on Level III is "332." These learners know how to relate tens and units in noncanonical representations and apply that concept (i.e., combining tens and units) to this task.

Educational Implications

At the end of primary school, learners are expected to have achieved a sustainable place value understanding. Learners are supposed to know the decimal structure and digit names of multidigit numbers (Level I), understand how tens and units are related with visual aids (Level II), internalize the tens-and-units' relations independently from manipulatives (Level III), and handle bigger bundling units (Level IV). These concepts are important requisites for subsequent mathematical contents – for instance, decimal fractions or units of measure.

In our aforementioned study, 30% of the learners were far from a sufficient conceptual place value understanding (i.e., below Level II). This corresponds to earlier findings that revealed missing place value understanding in learners from different grades, too (Fritz & Ricken, 2008; Gervasoni & Sullivan, 2007; Kamii, 1986; Ross, 1989).

A reason might be that in assessment and instruction, place value is often addressed rather superficially. To the best of our knowledge, there is no specific test or training for procedural and conceptual place value understanding. Place value is often tied to procedure-oriented applications of place value understanding, like addition and subtraction with carries or written standard algorithms (e.g., Dowker & Morris, 2015; Lonnemann & Hasselhorn, 2018). Those tests allow for differentiation of easy and difficult tasks. For instance, addition tasks without carries are easier than those with carries, while the number and position of the carries affect the task difficulty, too.

However, this only gives limited information regarding *why* certain tasks are easy and others are difficult. What makes subtraction tasks with carries more difficult, and why does it matter if the carry is at the units or the hundreds position? Against the background of our model, subtraction tasks without carries require knowledge of the digits (Level I). Subtraction with carries requires at least procedural unbundling of tens in units (Level II) or, depending on the task, unbundling of bigger bundling units (Level III) is necessary. The variance in task difficulties is often trivialized rather than based on specific place value theories.

Future Perspectives

Empirical research suggests that children pass through the reported level sequence as they develop a conceptual place value understanding. This hypothesis has to be confirmed in a longitudinal study in the future. Results from such a study could add to the existing data and underpin the level sequence in the sense of a progression model.

Recent research revealed that poor performance in place value-related tasks contributes to mathematical difficulties. It might serve as an empirically validated theoretical framework for the influence of conceptual place value understanding on arithmetic. This applies, in particular, to task performances and the use of strategies that rely on place value understanding.

The results of our study show that a substantial number of learners need specific place value training. Empirically validated theories provide information about the concepts that allow derivation of specific instruction. Thus, place value training should be designed on the basis of the level sequence. Such intervention would match learners' trajectories and pay attention to important milestones in the development of sustainable place value understanding.

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Chapter 34
Understanding Rational Numbers – Obstacles for Learners With and Without Mathematical Learning Difficulties

Andreas Obersteiner, Kristina Reiss, Wim Van Dooren, and Jo Van Hoof

Introduction

An understanding of rational numbers is important not only for the daily life/day-to-day life but also for further mathematical development (Bailey, Hoard, Nugent, & Geary, 2012; Booth, Newton, & Twiss-Garrity, 2014; Siegler et al., 2012; Siegler, Fazio, Bailey, & Zhou, 2013; Torbeyns, Schneider, Xin, & Siegler, 2015). However, research over the past decades has provided broad evidence that learning of rational numbers, particularly of fractions, is a great challenge for many children (Behr, Wachsmuth, Post, & Lesh, 1984, 1985; Carraher, 1996; Cramer, Post, & delMas, 2002; Hart, 1981; Siegler & Pyke, 2013; Stafylidou & Vosniadou, 2004; Vamvakoussi & Vosniadou, 2004). Most of these studies in mathematics education concentrated on typical misconceptions or on mistakes that learners make when working with rational numbers. More recently, cognitive psychology and neuroscience contributed to this research and investigated the cognitive and neural mechanisms that underlie the mental processing of rational numbers (DeWolf, Chiang, Bassok,

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Holyoak, & Monti, 2016; DeWolf, Grounds, & Bassok, 2014; Gómez, Jiménez, Bobadilla, Reyes, & Dartnell, 2015; Ischebeck, Schocke, & Delazer, 2009; Matthews & Chesney, 2015; Siegler et al., 2013). It appears that learning rational number concepts requires the extension of properties that all numbers share (Siegler, Thompson, & Schneider, 2011) but also the reorganization of the previously acquired concept of number. Emphasizing learners' challenges, learning of rational numbers has been considered an instance of *conceptual change* (Stafylidou & Vosniadou, 2004; Vamvakoussi & Vosniadou, 2004). In addition to conceptual change, there seem to be subtle cognitive mechanisms that make working with rational numbers difficult even if learners have understood the concept of rational number. As working with natural numbers is strongly automatized, intuitive response tendencies related to natural numbers can interfere with more demanding cognitive processes required by rational number problems. Researchers have used *dual process theories* to describe the cognitive mechanisms underlying the systematic bias caused by natural numbers when working with rational numbers (Vamvakoussi, Van Dooren, & Verschaffel, 2012).

In the following two sections, we elaborate on the conceptual change approach and on dual-process theories as theoretical frameworks for understanding learners' difficulties with rational numbers.

Learning of Rational Numbers: Learning a New Concept

Rational numbers can be defined as those real numbers that can be represented as the quotient a/b of two integers, a numerator a and a denominator b, with b unequal to zero. Rational numbers share many properties with natural numbers, which children are typically familiar with long before they learn about rational numbers. For example, rational and natural numbers can both be ordered according to their sizes and represented on the same number line. However, rational numbers also differ in important ways from natural numbers, and these differences are potential obstacles for learners (see Prediger, 2008). Before we review empirical studies that documented learners' difficulties, we elaborate on four important differences between natural and rational numbers.

A first difference concerns the way symbols for natural numbers and rational numbers represent numerical *sizes* or *magnitudes*. Understanding the magnitude of a given natural number requires some understanding of the magnitudes of its digits and of the base-ten system. Comparing the sizes of two natural numbers is straightforward because it can be done in a digit-by-digit manner. By contrast, understanding the magnitudes of fractions requires an additional understanding of the multiplicative relationship between two natural numbers. For instance, understanding the magnitude of 5/11 requires an understanding that 11 is a bit more than 2 times 5. Comparing two fractions is straightforward only if they have the same denominator or the same numerator, because in these cases, comparing the fractions requires only comparing natural numbers: 4/7 is larger than 2/7 because 4 is larger than 2; 4/7 is smaller than 4/5 because 7 is larger than 5. Comparing two fractions is, however, difficult if the fractions do not have common components. For example, to compare the fractions 2/3 and 4/7, reasoning about the natural number components alone is not sufficient. In fact, the fraction with the smaller components (2/3) represents the larger number. Comparing two decimal fractions is also more difficult than comparing two natural numbers because the number of digits does not necessarily align with overall number magnitudes as it does for natural numbers. For example, 1.6 is larger than 1.452 although the latter has more digits.

Another conceptual difference between natural and rational numbers is *density*. Natural numbers have a unique predecessor (provided the number is larger than 1) as well as a unique successor, and between any two natural numbers, there is a limited number of other numbers. Rational numbers, however, are dense, which means that an infinite number of numbers lies between any two different rational numbers. This property implies that rational numbers do not have unique predecessors or successors.

A third difference can be labelled as *representation*: While natural numbers have a unique symbolic representation (within the base-ten system), there are infinitely many different symbolic representations for any rational number. For instance, the symbols 1/2, 2/4, 3/6, and 0.5 all represent the same number.

Finally, rational numbers differ from natural numbers with respect to the effect that *operations* have on these numbers. While addition and multiplication (by a number other than 1) among natural numbers always increase the initial number, this is not generally true for rational numbers. Multiplying by 1/2 makes a number smaller. Similarly, subtraction and division (by a number other than 1) among natural numbers always decrease the original number. Again, this is not generally true for rational numbers. For instance, dividing by 1/2 makes the initial number larger. In addition to the differences in the effect that operations have on numbers, it is more difficult to explain the meaning of these operations with rational numbers than it is for natural numbers. For example, among natural numbers, multiplication can easily be explained as repeated addition (e.g., $3 \times 2 = 2 + 2 + 2$), and division can be explained as equal sharing or partitioning. However, these explanations are not as meaningful for rational numbers. For example, it is not so clear how to add 5/8 times the number 3/4, which would be the corresponding explanation for the multiplication $5/8 \times 3/4$.

When children first learn about rational numbers, they have already worked intensively with natural numbers and consequently developed a concept of what numbers are and how operating with numbers works. As rational numbers differ in important ways from natural numbers, learners have to modify their concept of number. Conceptual change theories propose that acquiring a new concept does not only require integrating new information into existing knowledge structures but also requires reorganizing the initial concept (Stafylidou & Vosniadou, 2004; Vamvakoussi & Vosniadou, 2004).

There is empirical evidence that each of the four conceptual differences between natural and rational numbers described above are challenging when children learn about rational numbers. Concerning the aspect of size, there is empirical evidence that when children compare the numerical values of fractions, they initially often choose the one with the larger components (Clarke & Roche, 2009; Rinne, Ye, & Jordan, 2017; Stafylidou & Vosniadou, 2004). When comparing two decimals, children tend to think that more decimals represent the larger number (Desmet, Grégoire, & Mussolin, 2010). There is also evidence that learners struggle with understanding the density concept of rational numbers (Merenluoto & Lehtinen, 2002; Vamvakoussi, Christou, Mertens, & Van Dooren, 2011; Vamvakoussi & Vosniadou, 2004, 2010). Vamvakoussi and Vosniadou (2010) found that although learners' responses became more sophisticated with increasing grade level, about a third of 11th-graders still responded that there was only a finite number of numbers between any two rational numbers. Studies also documented children's (and even adults') difficulties in the aspect of representation, which means that they struggle with accepting that fractions and decimals can represent the same number (DeWolf, Bassok, & Holyoak, 2015; Vamvakoussi & Vosniadou, 2010). Finally, there is evidence that people of varying age groups make fewer mistakes in judging the effect of operations when this effect is in line with the effect these operations have on natural numbers (e.g., judging that multiplication makes a given number larger), rather than when this is not the case (i.e., judging that multiplication can also make a number smaller) (Siegler & Lortie-Forgues, 2015; Vamvakoussi, Van Dooren, & Verschaffel, 2013).

The fact that not only young learners but also educated adults can struggle with rational numbers suggests that understanding the concept of rational number is not an all or nothing issue. Rather, learners develop a deeper understanding of the various aspects of rational numbers over the course of their learning career, and a more or less advanced understanding of various facets of the concept can coexist, so that people might rely on more or less advanced concepts or strategies depending on a specific problem situation (see Siegler, 1996). In addition to these phenomena captured by the conceptual change approach, there is evidence that even learners who have acquired a sound concept of rational numbers can be biased in their problem-solving process.

Dual Processes in Rational Number Problems: The Natural Number Bias

The dual-process account is a theoretical framework to describe reasoning processes. It can be valuable in addition to the conceptual change approach because it can be applied to reasoning processes that occur after learners have acquired an initial – or advanced – understanding of a specific concept. Thus, while the conceptual change approach is useful to understand why learners struggle with fully understanding a certain concept, the dual-process approach is particularly helpful to understand why they make typical mistakes although they actually understand the concept, and more specifically to unravel the cognitive mechanisms that take place at the moment when they solve a specific task. The dual-process account has been applied to mathematical reasoning (e.g., Gillard, Van Dooren, Schaeken, & Verschaffel, 2009; Leron & Hazzan, 2009), and several authors have also used it to describe the cognitive mechanisms underlying the processing of rational number tasks (Vamvakoussi, 2015; Vamvakoussi et al., 2012; Van Hoof, Lijnen, Verschaffel, & Van Dooren, 2013).

The dual-process account distinguishes between superficial, effortless type 1 processing and analytic, effortful type 2 processing. The assumption is that people are by default inclined to type 1 processing, which allows a quick response that comes to mind with little effort and with a subjective feeling of certainty, while some tasks require more effortful type 2 processing. In the case of rational number problems, when people engage in type 1 processing, an answer based on their intuitive natural number knowledge comes to mind easily. This kind of processing will lead to a correct answer on problems that are congruent (i.e., relying on natural number knowledge leads to the same response as relying on rational number knowledge). For instance, relying on the number of digits will yield the correct response when comparing the magnitudes of 1.65 and 1.4. However, it will lead to an incorrect answer on problems that are incongruent (i.e., relying on natural number knowledge leads to a different response than relying on rational number knowledge), unless type 2 processing inhibits the intuitive response tendency. For instance, relying on the number of digits will yield an incorrect response when comparing the magnitudes of 1.6 and 1.45. Using paper-pencil tests, previous studies found that learners were in fact more accurate on congruent than incongruent items (Vamvakoussi et al., 2011; Van Hoof, Janssen, Verschaffel, & Van Dooren, 2015). This performance pattern is an indicator of the "whole number bias" or "natural number bias" (Ni & Zhou, 2005).

Further studies included computer-based experiments that allowed recording response times in addition to accuracy (e.g., DeWolf & Vosniadou, 2011; Obersteiner, Van Dooren, Van Hoof, & Verschaffel, 2013; Vamvakoussi et al., 2012, 2013; Van Hoof et al., 2013). According to the dual-process account, type 2 processing is much slower and draws heavily on working memory. Therefore, if people show equally high accuracy on both congruent and incongruent items, they will still need more time for solving incongruent items than congruent items correctly.

There is recent evidence that learners show a natural number bias with respect to the four conceptual differences between natural and rational numbers described above. Addressing the aspect of size, Van Hoof et al. (2013) asked learners in secondary school to choose the larger of two fractions as fast and accurately as possible. The larger fraction was either composed of the larger component(s) (congruent problem) or of the smaller component(s) (incongruent problem). Although the participants were highly accurate on all problems, they solved incongruent problems significantly slower than congruent ones, indicating that automatized activation of the fractions' number magnitudes affected their performance. Obersteiner et al. (2013) found the same effect even in expert mathematicians. These participants performed equally well and extremely high on both congruent and incongruent fraction comparison problems. However, they solved incongruent problems significantly slower than congruent ones, suggesting that even these experts needed to inhibit their initial (biased) response tendency. It is noteworthy that the bias was present only in problems that had either the same numerators or the same denominators but not in problems that had no common components. Presumably, the latter type of problems required different strategies that relied more strongly on the overall fraction magnitudes rather than on the fraction components alone (see Obersteiner & Tumpek, 2016) and were thus less prone to be affected by a natural number bias. When comparing decimals (e.g., 0.25 vs. 0.3), the (combined) natural numbers after the decimal point (25 and 3) seem to impact children's and even adults' response tendencies (DeWolf & Vosniadou, 2011, 2015; Stacey et al., 2001; Van Hoof et al., 2013). That is, there is the tendency to erroneously consider 0.25 as being larger than 0.3, because 25 is larger than 3.

Concerning operations, studies demonstrated that learners in grades 8, 10, and 12, but also university students, show a systematic bias in terms of accuracy (Siegler & Lortie-Forgues, 2015; Vamvakoussi et al., 2013; Van Hoof et al., 2015). Results from reaction time research are, however, mixed. In a study by Obersteiner, Van Hoof, Van Dooren, and Verschaffel (2016), learners at secondary school showed a clear bias on problems that required reasoning about the effect of multiplication and division, while expert mathematicians did not show such a bias in terms of accuracy or response times.

Together, the results from these studies suggest that learners, and even adults, can be biased by their natural number knowledge when reasoning about rational numbers. Although people with high mathematical skills are able to use reasoning processes that allow them to be unaffected by the bias, the natural number bias seems to play an important role in less experienced adults and learners who are just learning about rational numbers (see Alibali & Sidney, 2015).

Obstacles for Learners with Mathematical Learning Difficulties

As described above, learning of rational numbers is challenging for many learners, not only for those with mathematical learning difficulties (further abbreviated with "MLD"). While it seems obvious that learners with MLD (further abbreviated with "LMLD") also struggle with understanding the rational number domain, we know little about whether they experience specific challenges – different and additional to the challenges they experience with natural numbers – when they learn about rational numbers, and if so, what kind of challenges these would be (Hecht & Vagi, 2010; Mazzocco & Devlin, 2008).

A more fine-grained understanding of LMLD's rational number understanding is of great importance when offering adaptive (remedial) instruction in order to increase learners' understanding specifically of the rational number domain (Hecht & Vagi, 2010). A study by Mazzocco and Devlin (2008) is a rare example of research that investigated LMLD's rational number understanding. Based on comparison tasks with fractions, decimals, and a combination of both, they concluded that the rational number understanding of learners with dyscalculia was lower, not only compared to typically developing learners but even compared to learners with low general mathematics achievement (but without dyscalculia). An open question remained, however, whether the learning trajectories of children with MLD are just delayed compared to typically developing children, or whether their learning trajectories differ in a qualitative manner. Moreover, children with MLD are a group that largely has been ignored in research on the role of the natural number bias (see 3.) in learners' rational number understanding. Given that LMLD are known to have low inhibition capabilities and are especially known to struggle with number inhibition tasks (Szucs, Devine, Soltesz, Nobes, & Gabriel, 2013; Wang, Tasi, & Yang, 2012), one could argue that they will suffer more from the natural number bias because they are less able to inhibit their natural number knowledge in rational number tasks.

Van Hoof, Verschaffel, Ghesquière, and Van Dooren (2017) conducted a study with the aim to examine whether LMLD's rational number understanding is characterized by a "delay" or a "deficit" compared to learners without dyscalculia. They also assessed whether LMLD show a stronger natural number bias. Three different groups of participants were included in this study: grade 6 students with an official clinical diagnosis of dyscalculia, grade 6 students who formed a chronological age match group, and grade 4 students who formed a group matched in terms of general mathematical ability. The intention of including both an ability match group and an age match group was to investigate whether there was a "deficit" or a "delay" in LMLD's rational number understanding (Torbeyns, Verschaffel, & Ghesquière, 2004). If LMLD's rational number understanding was significantly lower than that of the chronological age match group, but not significantly different from the ability match group, this would indicate that LMLD's rational number understanding is characterized by a delay rather than a deficit. However, if LMLD's rational number understanding was not only significantly lower than that of the age match group but also lower than that of the group matched on mathematical ability, this would suggest that LMLD's rational number understanding is characterized by a deficit (Torbeyns et al., 2004). Participants' performance data surprisingly showed that while there was no significant age difference between the LMLD and the age match group, the math achievement level of the (fourth-grade) ability match group was much higher than of the LMLD group, so that the fourth-graders could actually not be considered to be truly matched to the LMLD in terms of mathematical ability. To account for this unexpected outcome, the authors included learners' mathematical ability as a control variable in their analyses, allowing them to examine a difference between both groups' rational number understanding that could be explained by having dyscalculia and not by a difference in mathematics achievement level. Next to a general IQ and reading achievement test, which were used as control variables, learners solved a rational number test (see Van Hoof et al., 2015) that included both congruent and incongruent items of the aspects of size, density, and operations. Examples are given in Table 34.1.

When comparing LMLD with the sixth-grade control group, the analyses indicated that even after controlling for age, IQ, and reading achievement, the sixth-grade

Table 34.1 Examples of items of the rational number test

	Congruent	Incongruent
Size	Choose the largest number: 4.4 or 4.50	Choose the largest number: 3/2 or 9/8
Density	Write a number between 1/4 and 3/4	Write a number between 3.49 and 3.50
Operations	Is the outcome of 50 × 3/2 smaller or larger than 50?	Is the outcome of 40 × 0.99 smaller or larger than 40?

control group still significantly outperformed the LMLD group on congruent and incongruent rational number tasks. The accuracy difference was much higher in incongruent compared to congruent rational number items, which suggests that LMLD were affected considerably stronger by the natural number bias than sixth graders, while they differed to a lesser extent on other items addressing rational numbers. When comparing LMLD with the fourth-graders, there was no significant difference between both groups' accuracy on congruent and incongruent rational number tasks.

Since LMLD' s rational number understanding was significantly lower than that of learners of the same age without mathematical learning difficulties, but not significantly different from younger learners, one can conclude that the development of LMLD's rational number understanding is characterized by a delay rather than a deficit. This finding has an important implication for mathematics education, since it indicates that LMLD can develop a better understanding of the rational number domain at a later time. Therefore, more (remedial) instructional attention should aim to enhance LMLD's rational number understanding. Additionally, the different accuracy levels between LMLD and their peers of the same age were larger on incongruent compared to congruent items, suggesting that LMLD were affected more severely by the natural number bias compared to their peers. While more attention should be payed on supporting learners in making the transition from natural to rational numbers in all classrooms, the inappropriate reliance on natural number knowledge might deserve special attention when teaching LMLD.

How to Support Learners: Evidence from Intervention Studies

The research described above has identified a number of difficulties that children face when learning about rational numbers. These difficulties do not first occur in learning complex rational number arithmetic. Rather, they already occur when learning about very fundamental ideas of what number symbols, especially fractions, mean. Studies suggest that one of the most crucial factors for further mathematical development and yet a great stumbling block is an understanding of the numerical size or magnitude of rational number symbols (Rinne et al., 2017; Siegler et al., 2011; Siegler et al., 2012). Accordingly, intervention programs aimed to support rational number learning have often focused on an understanding of fraction magnitudes. To that end, studies used visual representations of fractions, because linking visual and symbolic representations can effectively support learning (e.g., Butler, Miller, Crehan, Babbitt, & Pierce, 2003; Cramer et al., 2002; Fazio, Kennedy, & Siegler, 2016; Fuchs et al., 2013, 2016; Gabriel et al., 2012; Mack, 1993; Moss & Case, 1999; Rau, Aleven, & Rummel, 2013). While a great variety of intervention studies can be found in literature, few studies have evaluated the effectiveness of their intervention in a controlled experimental design. In the following, we describe selected studies that are among the few exceptions.

Gabriel et al. (2012) delivered an intervention that was focused on the development of fraction magnitudes to children in grades 4 and 5. The intervention consisted of games that involved cards with different representations of fractions as well as wooden disks that children used to represent and manipulate fractions. The activities included comparison of fractions and matching symbolic fractions with nonsymbolic fraction representations. There were two 30-min intervention sessions per week, over a period of 10 weeks. Before and after the intervention, children took tests that measured their conceptual and procedural understanding of fractions. Children of the intervention group showed stronger improvements of their conceptual understanding of fractions compared to children of a control group who received regular classroom instruction. Children of the control group, by contrast, showed stronger improvement of fraction arithmetic procedures, suggesting that typical classroom teaching focusses on procedures more strongly than on concepts.

Fuchs et al. (2013) contrasted an instructional approach that was thought to be conventional with a more innovative one. While the conventional approach focused on the part-whole aspect of fractions and on fraction procedures, the innovative approach focused on the measurement aspect of fractions and emphasized fraction magnitudes. The intervention was designed for at-risk children and included compensational training of general-cognitive abilities. The participants in the intervention study were fourth graders who performed below the 35th percentile on an arithmetic test and were therefore considered to be at-risk of low mathematical achievement. Each session lasted 30 min, and the intervention was carried out for 12 weeks, 3 times per week. The results showed that children of the intervention that focused on the measurement aspect outperformed children of the control group with conventional teaching on conceptual as well as procedural measures. The authors found that the higher gains of the intervention group could indeed be attributed to the particularly large gains in children's understanding of the measurement aspect of fractions. In another study, Fuchs et al. (2016) could replicate the positive learning effects with a similar version of their intervention program in another sample of at-risk fourth-graders.

Fazio et al. (2016) also focused on enhancing children's understanding of fraction magnitudes. In their experiments, children of grades 4 and 5 played a computerized game where they had to place fractions on the correct position of a number line. The authors found positive learning effects after a remarkably short intervention period of just 15 min. They also found that the intervention was most effective when children received feedback after each problem they had worked on. Hamdan and Gunderson (2017) also found that the use of number lines during a brief intervention enhanced second and third graders' learning of fractions. Moreover, learners who used number lines were more able to transfer their knowledge to novel problems than learners who used area models during the training phase.

To summarize, there is empirical evidence that supporting learner's understanding of rational number magnitudes can enhance their conceptual and procedural rational number knowledge. Promising activities are those that emphasize the measurement aspect of fractions or require reasoning about relative magnitudes (such as placing numbers on number lines). The use of visual representations seems to be particularly powerful.

Conclusions and Perspectives

Understanding rational numbers is an important aim of the mathematics classroom, and it is important for further mathematical development. Yet, many learners have difficulties in understanding very basic concepts of rational numbers. Over the last decades, research from mathematics education and cognitive psychology has contributed to our understanding of when and why learners struggle with rational numbers. The conceptual change approach emphasizes that learning of rational numbers requires to some degree the reorganization of learners' existing knowledge about numbers, which is predominantly knowledge about natural numbers. Dual-process theories suggest that automatized knowledge of natural numbers can in some cases interfere with reasoning about rational numbers – even when learners have developed a basic concept of rational numbers.

Rational numbers are a challenge for many learners. Those with mathematical learning difficulties seem to experience some specific challenges but overall, their learning trajectories seem to be similar to those of typical learners. Learning time seems to be a key factor when learning about rational numbers, and learners with mathematical learning difficulties probably just need more of it. Classroom teaching should take this into account and strive for learning settings that can be adapted to the individual learner. Intelligent computerized learning environments (e.g., Rau et al., 2013) could assist in reaching that goal.

Proficiency with natural numbers is certainly a prerequisite for learning about rational numbers. On the other hand, many learners' difficulties seem to stem from overgeneralizing knowledge about natural numbers. More research is certainly needed to better understand the relation between previously acquired knowledge about natural numbers and learning of rational numbers. To that end, it is necessary to assess the same group of learners over a longer period of time in a longitudinal study design. Although initial studies have used such designs (e.g., Braithwaite & Siegler, 2017; Mou et al., 2016; Resnick et al., 2016; Rinne et al., 2017), these studies focused on very specific aspects of rational number development, such as understanding of number magnitudes. Further research with a broader perspective could assess how a variety of variables contribute to rational number development.

For example, studies could take into account cognitive and noncognitive variables but also school-related and socioeconomic factors.

To support learning of rational numbers, classroom instruction should focus more on core concepts of rational numbers (such as magnitudes) and less on procedures than it is currently done. Activities with rational numbers should include appropriate visualizations and problems that require reasoning about numerical magnitudes such as placing numbers on number lines (e.g., the Common Core State Standards Initiative, 2010). To account for individual differences, further studies could use controlled intervention designs to find out which learning activities are most effective for learners with varying needs.

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Chapter 35

Using Schema-Based Instruction to Improve Students' Mathematical Word Problem Solving Performance

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Problem solving is a central focus of educational reforms in mathematics around the world (EACEA/Eurydice, 2011; National Council of Teachers of Mathematics, 2000). Word problem solving, a key component in learning mathematics, is a vital component of mathematical school tasks beginning in early grades. However, problem solving in context, algebra, and mathematical communication are considered problematic areas for many students as well as teachers (EACEA/Eurydice, 2011; OECD, 2010). Furthermore, many secondary school students have difficulties solving algebra word problems (e.g., Bush & Karp, 2013; Carpraro & Joffrion, 2006; Van Amerom, 2003). The purpose of this chapter is to describe an evidence-based instructional program, schema-based instruction (SBI), which provides support in word problem solving for students who have mathematical learning difficulties (MLD). First, I describe mathematical word problem solving and the critical components linked to the ability to understand and solve word problems. Next, I describe the theoretical framework for SBI, including a discussion of its unique features and how SBI contributes to word problem solving performance. Then, I summarize previous research on SBI to understand the instructional conditions that need to be in place to support mathematical word problem solving for students with MLD. Last, I conclude with a discussion of challenges yet to be addressed.

Mathematical Word Problem Solving

Mathematical word problem solving plays a prominent role in school mathematics curricula. Given that word problems are "typically composed of a mathematics structure embedded in a more or less realistic context" (Depaepe, De Corte, &

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Verschaffel, 2010, p. 152), opportunities to solve word problems can help to develop students' understanding of the meaning of operations and consequently proficiency with whole numbers. Consider the following two problems: (1) Keisha had 6 stickers. She gave 4 stickers to Zora. How many stickers did Keisha have left? (2) Keisha had 6 stickers and Zora had 4 stickers. How many more stickers does Keisha have than Zora? Both situations can be represented with the same number sentence: 6–4 =? However, the interpretations of these problems are different in that the first involves a "separate or take away" interpretation and the second denotes a "comparison" interpretation. Many children will solve these problems by modeling the actions and relations described in them such that they may find the answer to the first problem by starting with a set of 6, removing 4, and then counting the remaining stickers. In contrast, they may solve the second problem by starting with a set of 4 (referent or smaller amount), adding some more as they count on to 6 (the compared or bigger amount), and then finding the answer by counting those added (the difference amount). How one thinks of the relations between the quantities in this problem can lead to either an addition or subtraction number sentence; one can add on to 4 to get to 6 or one can subtract 4 from 6. In sum, word problems help children connect the many different meanings, interpretations, and relationships to the mathematical operations (Van de Walle, Karp, & Bay-Williams, 2013).

Word problem solving is a complex, multifaceted process that is primarily composed of two phases – problem representation and problem solution (Mayer, 1999). Critical components of the problem representation phase include (a) reading the word problem with the aim of understanding and defining the problem situation and (b) identifying the relevant numerical and linguistic elements and the relations between elements to construct a coherent representation of the problem situation. The problem solution phase of the word problem solving process involves: (c) planning how to solve the problem, (d) executing the plan, (e) interpreting the solution in relation to the original problem situation, and (f) checking the reasonableness of the mathematical outcome (Depaepe et al., 2010; Mayer & Hegarty, 1996).

Some children, particularly those with MLD who not only have difficulties with the abstract formal structures of mathematics but also have difficulties with reading comprehension or the language of mathematics, may find word problem solving to be considerably more challenging than solving no-context problems (Andersson, 2008; Fuchs et al., 2010). Furthermore, research indicates that despite competence in computational skills for solving word problems, many children with MLD experience difficulties in problem comprehension or understanding the problem text, identifying the relevant quantities and the relations between them, and generating an adequate visual representation of the problem situation (Boonen, van der Schoot, van Wesel, de Vries, & Jolles, 2013; Hegarty & Kozhevnikov, 1999; Schumacher & Fuchs, 2012; van Garderen, 2006). As such, providing support in the comprehension phase is important in enhancing these students' word problem solving performance. However, for children with both computational and word problem solving difficulties, there is no effect or less robust effect of word problem solving intervention (Jitendra et al., 2013; Schumacher & Fuchs, 2012). It is likely that for these children, word problem solving intervention alone is not sufficient. Rather, word problem solving instruction integrated with knowledge of foundational mathematics content (e.g., understanding the base-ten system to represent numbers, strategies for addition and subtraction categories) will be necessary for gains in word problem solving.

Theoretical Framework for Understanding How Schema-Based Instruction Is Beneficial to Word Problem Solving Performance

Conventional instructional practices often focus on the problem solution phase and have had limited success in improving the word problem solving performance of students with MLD (see Fuchs et al., 2008; Jitendra et al., 1998; Jitendra et al., 2007; Xin et al., 2011). The theoretical framework that guides this chapter about teaching word problem solving derives from a model of word problem solving based upon schema theories of cognitive psychology (Briars & Larkin, 1984; Carpenter & Moser, 1984; Kintsch & Greeno, 1985; Riley, Greeno, & Heller, 1983). Consistent with schema theory, recognition of the semantic structure of the problem is critical for understanding the problem text (Kalyuga, 2006) and is an essential feature of SBI. Schemata are hierarchically organized, cognitive structures that are acquired and stored in long-term memory. When multiple elements of information are grouped into and conceptualized as a single schema, there is a reduction in working memory load that allows for more efficient and effective problem solving (Kalyuga, 2006).

In addition, the SBI model of word problem solving is guided by research on expert problem solvers and cognitive models of mathematical problem solving (Mayer, 1999). SBI integrates essential processes that expert problem solvers engage in when solving problems, such as distinguishing "relevant information (related to mathematical structure) from irrelevant information (contextual details), perceiving rapidly and accurately the mathematical structure of problems and in generalizing across a wider range of mathematically similar problems" (Van Dooren, de Bock, Vleugels, & Verschaffel, 2010, p. 22). Another aspect of the SBI model is its emphasis on knowledge about problem solving procedures (e.g., problem representation, planning) for a given class of problems (see Marshall, 1995; Mayer, 1999). What is most relevant to this chapter on word problem solving instruction is the need for teaching to ensure that instructional practices (e.g., guided questions to engage students in conversations about their thinking and problem solving) support students in the problem solving process, such as recognizing common underlying problem structures, representing problems using appropriate diagrams, planning how to solve problems, and solving and checking the reasonableness of answers.

Consistent with the research on expert problem solvers, SBI supports the development of students' metacognition skills such as planning, checking, monitoring, and evaluating their performance (e.g., Rosenzweig, Krawec, & Montague, 2011).

Guided by their teachers, students "think about what they are doing and why they are doing it, evaluate the steps they are taking to solve the problem, and connect new concepts to what they already know" (Woodward et al., 2012, p. 17). Giving students time to think through problem situations by asking questions (e.g., What type of a problem is it? Is it similar to or different from others that they solved before?) can facilitate word problem solving.

What Are the Unique Features of SBI and How Does It Contribute to Word Problem Solving Performance?

The SBI program described here is based on work with elementary school students with and without MLD and is unique in several ways (see Jitendra, 2007). First, SBI incorporates four components identified as essential to improving students' word problem solving performance. These components (described in the next section) are cited as main recommendations in What Works Clearinghouse's research syntheses on improving students' mathematical problem solving performance and assisting struggling students with mathematics (Gersten, Beckmann, et al., 2009; Woodward et al., 2012).

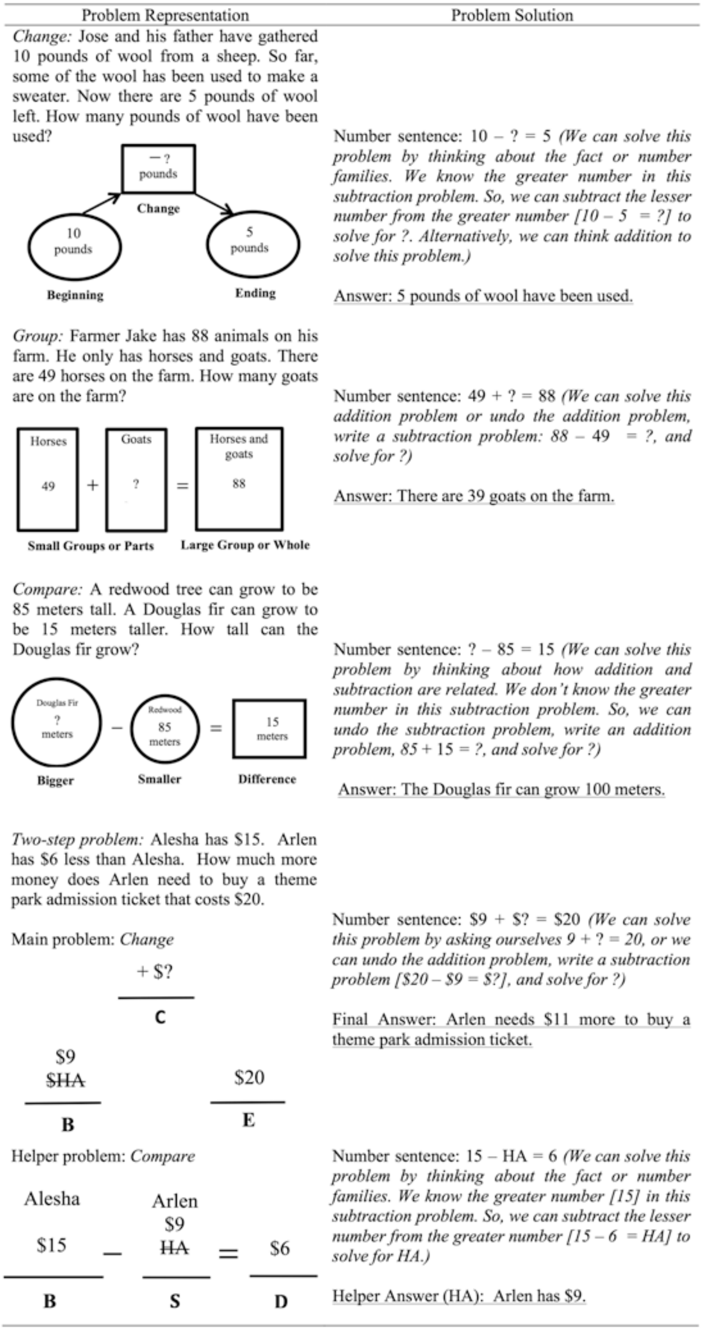
Problem Structure In the domain of arithmetic word problems, researchers have identified basic types of problems (i.e., *Change*, *Combine*, *Compare*, *Equal Groups*, *Multiplicative Compare*; see Carpenter, Fennema, Franke, Levi, & Empson, 2015; Greer, 1994). Problems such as *Change*, *Combine*, and *Compare* belong to the additive field in that the solution operation is either addition or subtraction, whereas *Equal Groups* and *Multiplicative Compare* problems belong to the multiplicative field, because the solution operation is either multiplication or division (Christou & Philippou, 2001). There is strong evidence of the benefits of word problem solving instruction that focuses on identifying problems involving the additive structure (e.g., Fuchs et al., 2008; Fuson & Willis, 1989; Jitendra et al., 1998; Jitendra et al., 2007; Jitendra, Dupuis, & Zaslofsky, 2014) and emerging evidence for problems involving the multiplicative structure (e.g., Xin et al., 2011).

The primary focus of SBI is on providing students with problem categories and teaching them to recognize the underlying mathematical structure (*Change*, *Group*, and *Compare*). The assumption is that clarifying the problem structure should facilitate understanding of the problem situation, which consequently can result in improved problem solving performance especially when quantitative reasoning is used to associate relationships (e.g., part-part-whole) between quantities in the problem with the appropriate operation (e.g., addition or subtraction) rather than relying on relational keywords (e.g., "have left") that are sometimes misleading and can result in incorrect solutions. For the two word problems described earlier, the underlying problem schemata are *Change* (Problem 1) and *Compare* (Problem 2). A *Change* problem is one that has a starting quantity, and a direct or implied action causes either an increase or decrease in that quantity. The three features of a *Change* problem are the beginning, change, and ending quantities. The object identities (e.g., pounds of wool) for beginning, change, and ending are the same in the *Change* schema (see Fig. 35.1). In contrast, a *Group* problem involves the joining of two distinct small groups to form a large group, and the relationship between quantities in the problem is a part-part-whole relationship. This relationship is static (i.e., no action is evoked). A *Compare* problem involves the comparison of two disjoint sets (compared and referent), and the relationship between the two sets is static.

Visual Schematic Representations Visual schematic representations serve a variety of purposes: (a) organize and summarize problem information, (b) make abstract relationships concrete, and (c) reason about story situations (Diezmann & English, 2001; Presmeg, 2006; Zahner & Corter, 2010). Training children in the process of using visual schematic representations to meaningfully represent word problems can result in improved word problem solving performance (Yancey, Thompson, & Yancey, 1989). Such representations can lead to deep understanding of the problem and transfer of learning to novel problems when used as instructional tools to make sense of word problems (Goldin, 2002; Zahner & Corter, 2010).

However, word problem solving instruction in many mathematics textbooks uses approaches that do not contribute to problem comprehension, such as having students directly translate the elements of the problem into corresponding mathematical operations (arithmetical representations) or having them create their own representations, which are often pictorial representations or images depicting the "visual appearance of the given elements in the word problems" (Boonen et al., 2013, 272). Instead, teachers need to provide instruction on how to represent problems using a few types of visual schematic representations (e.g., bar models, visual schematic diagrams) that effectively link the relationships between the relevant quantities in the problem (Woodward et al., 2012). Seeing those quantitative relationships and connecting them to operation meanings can result in the identification of the computations to be performed. Research indicates that in contrast to pictorial representations, visual schematic representations positively influence the word problem solving process (Hegarty & Kozhevnikov, 1999; van Garderen & Montague, 2003).

Visual schematic diagrams are an integral component of the SBI approach (e.g., Jitendra et al., 2007; 1998). The SBI program ensures that teachers discuss the structure of the schematic diagrams and connect them to quantities in the problem and to operation meanings. Consider the following problem: *A redwood tree can grow to be 85 m tall. A Douglas fir can grow to be 15 m taller. How tall can the Douglas fir grow?* (Jitendra, 2007, p. 117). Using a visual schematic diagram, the key components of the *Compare* problem structure are made visible – the bigger (compared), smaller (referent), and difference sets. Further, SBI facilitates translating contextual information (nonmathematical) in the problem text to meaningfully represent the problem by showing how quantities in the problem are related. Firstly, students are cued to the linguistic expression "taller" [translated to mean "taller" than the redwood tree] in the relational sentence (*A Douglas fir can grow to be*



Problem Representation	Problem Solution
Change: Jose and his father have gathered 10 pounds of wool from a sheep. So far, some of the wool has been used to make a sweater. Now there are 5 pounds of wool left. How many pounds of wool have been used?	Number sentence: 10 – ? = 5 *(We can solve this problem by thinking about the fact or number families. We know the greater number in this subtraction problem. So, we can subtract the lesser number from the greater number [10 – 5 = ?] to solve for ?. Alternatively, we can think addition to solve this problem.)* Answer: 5 pounds of wool have been used.
Group: Farmer Jake has 88 animals on his farm. He only has horses and goats. There are 49 horses on the farm. How many goats are on the farm?	Number sentence: 49 + ? = 88 *(We can solve this addition problem or undo the addition problem, write a subtraction problem: 88 – 49 = ?, and solve for ?)* Answer: There are 39 goats on the farm.
Compare: A redwood tree can grow to be 85 meters tall. A Douglas fir can grow to be 15 meters taller. How tall can the Douglas fir grow?	Number sentence: ? – 85 = 15 *(We can solve this problem by thinking about how addition and subtraction are related. We don't know the greater number in this subtraction problem. So, we can undo the subtraction problem, write an addition problem, 85 + 15 = ?, and solve for ?)* Answer: The Douglas fir can grow 100 meters.
Two-step problem: Alesha has $15. Arlen has $6 less than Alesha. How much more money does Arlen need to buy a theme park admission ticket that costs $20. Main problem: *Change*	Number sentence: $9 + $? = $20 *(We can solve this problem by asking ourselves 9 + ? = 20, or we can undo the addition problem, write a subtraction problem [$20 – $9 = $?], and solve for ?)* Final Answer: Arlen needs $11 more to buy a theme park admission ticket.
Helper problem: *Compare*	Number sentence: 15 – HA = 6 *(We can solve this problem by thinking about the fact or number families. We know the greater number [15] in this subtraction problem. So, we can subtract the lesser number from the greater number [15 – 6 = HA] to solve for HA.)* Helper Answer (HA): Arlen has $9.

Fig. 35.1 Solving one-step and two-step problems involving *Change*, *Group*, and *Compare* schemata from Jitendra (2007, pp. 32–33, 78, 117, 158). (Copyright by Pro-Ed)

15 m taller.) to understand that the problem involves a comparison of two sets. Secondly, instruction emphasizes that the relational sentence not only describes the difference (i.e., 15 m) between the two things compared (i.e., height of the Douglas fir and height of the redwood wood) but also helps identify the compared or bigger set (height of the Douglas fir) and the referent or smaller set (height of the redwood tree). Thirdly, students use the information in the remaining verbal text to identify the known or smaller quantity (85 m) and the unknown or bigger quantity (? m) to represent in the *Compare* diagram (see Fig. 35.1). In sum, the *Compare* diagram helps students represent the quantities and their relationships for the word problem given above: (a) the difference between the Douglas fir and the redwood tree is 15 m, (b) the Douglas fir represents the bigger set (? m), and (c) the redwood tree represents the smaller set (85 m). The relationship between quantities in the diagram can be seen as a comparison of the bigger and smaller quantities so that the operation required is subtraction, or the relationship can be translated to mean that the bigger quantity is the result of joining the difference and smaller quantities, and the numerical expression is 15 + 85.

Problem Solving Procedures Although Pólya's (1990/1945) four-step approach to problem solving (i.e., *understand the problem*, *devise a plan*, *carry out the plan*, *look back and reflect*) has been used in traditional mathematics instruction for decades, it has come under scrutiny primarily for not leading to improved problem solving outcomes (Lesh & Zawojewski, 2007; Schoenfeld, 1992). On the other hand, a prescriptive problem solving procedure such as the keyword approach (e.g., use subtraction whenever a word problem includes "have left") has limited value (e.g., "have left" is in the problem but subtraction is not the needed operation). In contrast, teaching problem solving procedures that are connected to classroom-related conditions such as how they are taught or integrated into the mathematics curriculum is known to be effective (Gersten, Chard, et al., 2009).

SBI integrates the problem solving process with the critical mathematics content (e.g., word problems involving addition and subtraction). The focus of problem solving in SBI is to get students to think systematically about solving problems and the emphasis is on four separate, but interrelated problem solving phases (problem schema identification, representation, planning, and solution, Marshall, 1995; Mayer, 1999). As such, problem solving is a process grounded on reasoning. The four phases of problem solving in the SBI program (i.e., find the problem type, organize information in the problem using a schema diagram, plan how to solve the problem, and solve the problem) correspond with Mayer and Hegarty's (1996) problem representation and problem solution phases (see Fig. 35.1). Consider the following problem in Fig. 35.1: *Farmer Jake has 88 animals on his farm. He only has horses and goats. There are 49 horses on the farm. How many goats are on the farm?* To solve this problem, SBI emphasizes reasoning to (a) find the problem type by reading and paraphrasing the problem, as well as using the word problem context (e.g., two small groups, horses, goats; a large group, animals or horses and goats) to understand the problem situation (a *Group* problem), (b) organize and represent information in the problem using the *Group* diagram by identifying the known (a total of 88 animals, 49 are horses) and unknown quantities (? goats) and identifying statements in the problem that express relationships (part-part-whole) between quantities, (c) plan how to solve the problem by translating relationships between quantities given in the problem to numerical representations (e.g., 49 +? = 88 or 88–49 =?) that also illustrate the connections between and among operations, and (d) solve the problem (e.g., one way to solve 49 +? = 88 is as follows: 49 + **1** = 50; 50 + **30** = 80; 80 + **8** = 88. So **1** + **30** + **8** = 39) and check the solution (e.g., the answer, "There are 39 goats on the farm," seems reasonable. If 88 is the large group amount, then 39 representing **one of the two** small group amounts seems reasonable.).

Metacognitive Strategy Knowledge Instruction Another critical feature of effective problem solving instruction is metacognitive strategy knowledge (De Corte, Verschaffel, & Masui, 2004; Rosenzweig et al., 2011). Metacognitive strategy knowledge is strongly correlated with successful problem solving (e.g., Desoete, 2009; Fuchs et al., 2003; Hegarty, Mayer, & Monk, 1995; Schoenfeld, 1992).

SBI embeds metacognitive activities such as analyzing the problem, monitoring strategy use, and evaluating the outcome within the word problem solving context. Teachers use deep-level questions to encourage students to monitor and reflect on the four phases: (a) problem comprehension (e.g., How do you know it is a *Change* problem?), (b) problem representation (e.g., What schematic diagram best fits this problem type to represent information in the problem?), (c) planning (e.g., How can you solve this problem? What are the solution steps or operations needed?), and (d) problem solution (e.g., Are your calculations correct? Is the answer reasonable given the question asked?).

A second unique feature of the SBI program is that it includes effective instructional practices (e.g., systematic and explicit instruction, opportunities for student response and feedback) to support the learning of students with MLD (Clarke et al., 2011; Gersten, Chard, et al., 2009). For example, the SBI program incorporates scaffolding to support student learning in the following ways: (a) teacher-mediated instruction (making instruction explicit and visible using teacher think-alouds) is followed by paired partner learning and independent learning activities, (b) tasks begin with story situations with no unknown information followed by word problems, and (c) external visual schematic representations are replaced by student-constructed diagrams.

Throughout the SBI program, teacher-mediated instruction entails using think-alouds to make instruction explicit and visible as well as guided questions to engage students in conversations about their thinking and problem solving. Partner learning activities in the SBI program provide many opportunities for students to practice solving word problems. Students use a think-pair-share model to first think about the problem type independently and then work with their partner to model the problem situation using a visual schematic representation and solve it before sharing their solutions and explanations with the whole group. This practice of verbalizing the strategy steps during partner work and with the whole group is important as it not only allows the teacher to monitor student understanding and provide instructive feedback to support students (Hattie & Timperley, 2007) but also enables students with MLD to express their own thinking and listen to the ideas of their peers.

It is well documented that many students, especially those with MLD, jump immediately into calculating the answers when solving word problems without understanding the problem situation or reasoning whether the answer is meaningful (see Verschaffel, Greer, & De Corte, 2000). As such, the SBI program introduces story situations with no unknown information to ensure that students focus on the relevant numerical and linguistic elements and the relations between elements to understand and reflect on the problem situation. Because problem comprehension is particularly difficult for many students with MLD, the SBI program provides external visual schematic diagrams as they translate and integrate information in the problem into the representation before they are taught to construct representations (see Fig. 34.1 for student-constructed diagrams in solving two-step problems).

In addition, the SBI program incorporates consistent, systematic practice in solving word problems involving the different problem types. Furthermore, student progress or response to instruction is monitored using research-validated measures of word problem solving performance to inform instruction (Jitendra et al., 2014; Jitendra, Sczesniak, & Deatline-Buchman, 2005; Leh, Jitendra, Caskie, & Griffin, 2007).

Teaching Word Problem Solving Using SBI: Empirical Evidence from Intervention Studies

Multiple studies have been conducted to evaluate the effectiveness of SBI for improving students' problem solving performance. In the review below, I discuss four randomized controlled studies that targeted arithmetic word problems involving addition and subtraction for children with MLD and for typically developing children.

Studies 1 and 2: Supporting Evidence for SBI Compared to Traditional Instruction

Two studies have demonstrated that SBI is more effective than a general strategy instruction (i.e., typical textbook instruction) for improving word problem solving performance of students with and without MLD. Jitendra et al. (1998) reported that elementary school students with MLD (identified by their teachers to possess adequate addition and subtraction computational skills, but to be poor word problem solvers) and those with school-identified disabilities, who received about 45 min daily of small group (3 to 6 students), pull out tutoring in solving one-step addition and subtraction word problems for about 4 weeks from researchers outperformed students in the GSI condition on outcome measures of word problem solving. Following treatment and 1–2 weeks later, results indicated moderate to large effects ($d = 0.65$ and 0.81) favoring the SBI group. Importantly, only students in the SBI group performed at the same level as average-achieving students on the word problem solving posttests. Further, the results revealed a large effect size favoring SBI on a transfer measure of novel problems derived from curricula not used in the treatment ($d = 0.74$).

Additionally, the SBI intervention was successfully implemented by third-grade classroom teachers with students with and without MLD in Jitendra et al. (2007). Students in both SBI and GSI conditions participated in a 10-week program that consisted of focused lessons targeting both one- and two-step word problems using the assigned intervention. Students in the SBI group on average outperformed students in the GSI group on word problem solving in the posttest ($g = 0.52$) and maintained the effects 6 weeks later ($g = 0.69$). The advantage for the SBI group was also evident on the Pennsylvania System of School Assessment mathematics test ($g = 0.65$). This study also provided evidence of the effectiveness of SBI for students with MLD and English language learners in these inclusive classrooms.

Studies 3 and 4: Supporting Evidence for SBI Compared to Standards-Based Instruction

Two studies explored whether SBI enhances word problem solving compared to standards-based instruction. Standards-based curricula and SBI are similar in their theoretical underpinnings (i.e., emphasize meaningful learning to develop conceptual understanding); however, they differ in terms of their instructional practices. Standards-based instruction is characterized by an inquiry-based student-directed approach, whereas SBI incorporates a teacher-mediated approach that relies on think-aloud procedures to make the content explicit and guided questions to engage students in thinking through the problem situations.

In Jitendra, Rodriguez, et al. (2013), tutors (e.g., parents, instructional assistants, undergraduate students) implemented the fully developed SBI program (Jitendra, 2007). Tutors, who were randomly assigned to SBI and control (standards-based textbook instruction) conditions, provided all instruction (daily 30 min of small group tutoring sessions for 12 weeks) to third-grade students with MLD (scored <40th percentile in mathematics and > the beginning of second grade level in reading on their district accountability assessment). SBI students received tutoring in solving one-step and two-step word problems; the control group received instruction in whole number concepts and procedures, including word problem solving using the school-provided standards-based practices. For word problem solving outcomes, there were significant interaction effects indicating that SBI students with higher pretest scores outperformed students in the control group with higher pretest scores, whereas students with lower pretest scores in the control group outperformed SBI students with lower pretest scores. Jitendra et al. (2014) reported that many students who entered the study without mastering the basic computational skills did not benefit from SBI word problem solving tutoring only.

Jitendra, Dupuis, et al. (2014) extended the focus of SBI content to also include foundational concepts (e.g., understanding the base-ten system to represent numbers). The methods and procedures were the same as in Jitendra et al. (2013). Students in the SBI condition on average outperformed students in the control condition on a word problem solving posttest ($g = 0.46$). The effect of SBI proved to be equivalent for students in both high at-risk (scored at or below the 25th percentile) and low at-risk subgroups (scored between the 26th and 40th percentile on a standardized mathematics achievement test). On a district-administered mathematics achievement test, SBI students scored significantly higher than control students ($g = 0.34$); however, there were no significant effects on an 8-week retention test.

Remaining Challenges

The studies reviewed provide evidence that applying SBI to solve word problems results on average in greater learning compared to alternative approaches. Despite the positive evidence for SBI, two formidable challenges remain. The first concerns the fact that although there were significant positive effects relative to comparison conditions, many students do not respond adequately and remain impaired following word problem solving intervention (see Jitendra et al., 2013). Ensuring that classroom instruction includes explicit word problem solving instruction that is integrated in meaningful opportunities to connect with the mathematical operations while also emphasizing foundational mathematics content is critical. Schools must find ways to ensure that all students who have persistent MLD, even those in middle school, receive interventions of sufficient quality and intensity to accelerate their progress so they can access grade level materials. Teachers are likely to need not only professional development but also instructional materials that are feasible to implement and result in observable progress in their students and consequently reduce the incidence of MLD.

The second challenge involves the ability to solve more complex or nonroutine word problems, which are common in contemporary mathematics classrooms (Boonen et al., 2013). Although visual representations that make visible the problem structure and the relations between quantities in the problem alleviate the difficulties of problem comprehension, ensuring that teachers understand what is involved in using representations and when they are appropriate to use is critical. Professional development is needed to ensure that teachers are aware of multiple representations and link different representations to each other when solving multistep problems and understand that "visual-schematic representations should be used to support the first phase of the word problem solving process (i.e., problem comprehension) and that arithmetical representations are only appropriate in the problem solution phase" (Boonen et al., 2013, p. 60).

In closing, evidence indicates that improving word problem solving is no small task, but requires systematic and explicit instruction beyond reading and comprehending the words and doing the needed calculations. Explicit attention to problem solving as a process grounded in reasoning and understanding the meanings of operations is needed to prepare students, including students with MLD, for learning advanced mathematics content (e.g., algebra). To bridge the gap between arithmetic and algebra, word problem solving instruction should emphasize not only understanding and "applying the arithmetical operations in numerical and algebraic expressions" (Jupri & Drijvers, 2016, p. 2482) but also understanding the different meanings of the equal sign (calculation vs. sign of equivalence) and the notion of a variable. Further, creating questions and tasks that encourage "languaging" (e.g., speaking, writing, using representations to explain) in a mathematics classroom can promote problem solving and generalizations.

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Chapter 36
Geometrical Conceptualization

Harry Silfverberg

Characterizing School Geometry

It could be said that arithmetic and algebra provide a pupil with mathematical tools for understanding and utilizing number relations. Similarly, *geometry provides mathematical tools for comprehending and managing spatial relations*. One could consider this a sort of working definition for school geometry. According to this theory, geometric knowledge manifests itself in a variety of ways. Understanding spatiality, i.e. the development of comprehensive spatial thinking, is central to this. Sometimes it remains unclear how much of this process of understanding is attained through learning and how much is dependent on the perception of visual reality culturally characteristic for humans. Even very young children can differentiate between a closed line and an open one, a straight line and a jagged one, a circle and a polygon, etc. This kind of direct visualization can be considered to have a similar function as the subitizing phenomenon, in which simple calculations involving small numbers are completed as a process of direct visualization, without involving any actual calculation processes (Markovits & Hershkovicz, 1997). Part of the visualization process, however, is clearly the result of supervised activity and development. Communal interaction and especially the teaching of geometry offered by school will contribute to a child eventually adapting methods of preferring certain visual properties over others, as befitting of their operational culture. This type of controlled change in perspective constitutes one of the bases of geometrical conceptualization.

In school, pupils learn to construct spatial relations through a surprisingly small set of well-established concepts. The concepts and relations that geometry uses to operate, such as points, lines, planes, plane and space figures, parallelism, perpendicularity, etc., have been carefully selected over time, proving their usefulness as

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compared to the true abundance of the shapes and spatial relations that the reality surrounding to us displays. True mastery of spatial relations also requires observation and processing of physical quantities, such as length, angles, surface area, volume, etc. using computational methods. External representations developed to express various spatial relations, such as geometrical patterns, models, maps, diagrams, verbal depictions, etc., support spatial thinking. The level of spatial and geometrical thinking an individual possesses is e.g. apparent in their competence to interpret, produce and utilize such representations.

Visual, computational, conceptual and algebraic viewpoints/perspectives on geometry have been evident at different times, with different emphases (Dieudonné, 1981), and during the last few years particularly the methods of the teaching of the geometry have changed radically. This chapter focuses largely on the issues of geometrical conceptualization, mostly from the perspective of Euclidean plane geometry. This choice is not meant to imply that teaching geometry with computational methods is somehow less important, even regarding conceptualization. Due to the limitations, this article will not, for instance, compare the learning outcomes of geometry with the other areas of mathematics, or compare the changes in the level of learning geometry to the past. Internationally comparative surveys such as PISA and TIMSS, as well as national learning performance inquiries, offer more information on these topics.

It is obvious that geometry is not taught in school simply for its own significance but also for transfer of learning – geometrical education offers plenty of knowledge that can be transferred to other types of mathematical education and understanding. Geometry provides exceptionally a versatile learning environment when it comes to practicing generalizations, deductive reasoning and concept development. The validity of hypotheses and the consequences of various definitions can be tested in a variety of ways. There are various tools that can be utilized, including tangible models, experimental inductive reasoning, information gathered through calculations as well as deduction. Dynamic geometry has contributed to the field by adding a new aspect, that of movement, and the transformability of figures. These facts provide the grounds for geometry's exceptional potential to achieve learning with understanding, grasping the relations between things, and the forming of knowledge structures. In the early stages, understanding is founded on the observations of the real world, and the formed concepts are imprecise. As understanding develops, it is increasingly based on the structure of the geometrical system and concepts that gain more precise meaning through defining.

Three Approaches to School Geometry

At different stages of education, the starting point for examining geometrical relations as well as the justifications for observations and conclusions differ. Naturally, these differences in perspectives are also apparent in learning objectives and practical applications. In accordance with Houdement and Kuzniak's (2003) thoughts, I will differentiate between three expressions of school geometry, G1–G3, which will largely correspond with the basic approaches that are adapted for teaching geometry in our educational system at different levels.

G1. The Geometry of Concrete Objects

This approach is based on the assumption that nature is to a large extend geometrically ordered, and geometric relations and concepts can, in a sense, be found in reality. Geometrical truths are explained with direct observations and measurements. The principal aims of teaching are the understanding, memorization and identification of geometrical figures, estimating and measuring proportions and calculation.

G2. The Geometry of Graphically Justified Ideal Plane Figures and Solids

Geometrical concepts are defined and technically gain meaning through this approach, though the interpretational foundation of concepts is still strongly rooted in observations of reality. Geometrical "facts" or theorems are presented as part of the mathematical system but explained or made plausible with concrete observations, experiments and measurements. The principal aims of teaching are examining geometrical concepts as mathematical objects rooted in perception, presenting quantitative correlations as formulas and applying these to geometrical and practical contexts.

G3. Quasi-axiomatic Geometry

Basic concepts and relations are expected and found to match our intuition. Other emerging concepts are defined with the help of basic concepts or other, already defined concepts. Geometrical "facts", e.g. theorems, are proven deductively using information that has been proven previously. The principal foundations for building up understanding are comprehending the role of definition in mathematics, deductive reasoning and proving (proofs) as well as applying knowledge to geometrical calculations.

These perspectives are constructed to overlap so that the previous view will always provide basis for interpretation in the following one (Houdement & Kuzniak, 2003; Parzysz, 2003; Silfverberg, 1999). The overlapping quality of perspectives G1 and G2 is cemented in geometrical figures. On the one hand, they can be interpreted as drawings or images (perspective G1); on the other hand, they can be seen as representations, models of ideal figures and solids (perspective G2).

The van Hiele Theory about the Stages of Development in Geometrical Thinking

The old but still frequently quoted and in general outline acknowledged van Hiele theory was developed by a married Dutch couple, Pierre van Hiele and Dina van Hiele-Geldof, during the 1950s, using their accumulated observations gained through years of educational work as its basis (van Hiele, 1957; van Hiele-Geldof, 1957). The van Hiele theory describes the developmental stages of geometrical thinking. Since then, various other researchers have expanded on the theory, adapting it to new fields of study and modernizing it to match the new goals of school geometry (e.g. Atebe, 2008; Blair, 2004; Clements et al., 1999; Guven & Baki, 2010; Patsiomitou & Emvalotis, 2010; Silfverberg, 1999). Though the current prevalent theories differ in their interpretation of school geometry, the key hypotheses of the van Hiele theory are still widely accepted. The descriptive part of the van Hiele theory is the most well-known. This includes the hypothesis of the existence of levels of development characteristic to the growth of geometrical thinking. These five levels, which have come to be known as the *van Hiele levels*, describe the qualitative shifts apparent in geometrical thinking that are largely analogous and occur in the same sequence between different individuals, though not always at the same pace. The key characteristics of geometrical thinking typical of the different van Hiele levels, as described by Silfverberg (1999, 27–28), are as follows:

Level 1 (Visualizing)

At this basic level, figures are viewed holistically as a part of the visual field of perception. Identifying, naming, categorizing, comparing, describing, etc. figures is achieved through the holistic appearance of the figure rather than its properties. Typical examples of basic geometrical figures can be recognized and classified. These prototypes can be visualized and drawn. At this level, classifications, such as a rectangle, gain meaning through an example, e.g. “a rectangle is like a window or the classroom whiteboard”.

Level 2 (Analyzing Properties)

At the second van Hiele level, figures are interpreted as the “bearers” of their properties. At this level, figures are examined in the light of their characteristics. Properties are considered individually in the sense that their logical relationships are ignored. Figures can be analyzed and compared based on their properties and not just on their visual similarity or difference. At the level of analyzing properties, all common characteristics of a figure belonging to a particular category are discovered and utilized. At this level, shapes such as a rectangle are still specified by the visual image of a conventional example, but the figure can also be described through its properties, e.g. "A rectangle has two long sides and two short sides, and all of its angles are equal". When a pupil tries to define a concept it is typical that he attempts to place in the definition all she or he knows about the concept and not just what actually is sufficient from the point of view of the definition.

Level 3 (Ordering Properties)

At this level, the properties of figures have an innate hierarchy created by logical relationships. Deductions can be followed and applied to brief instances of deductive reasoning. Definitions can be formed, and the sufficient and fundamental defining properties of figures can be identified. Defining properties can be utilized when determining whether a figure class is part of another class or not. At this level, a square is recognized as a rectangle, and the reasons for this can be explained.

Level 4 (Formal Deduction)

At the level of formal deduction, the mindset required for systematic, deductive geometry is developed. At this level, consequences can be inferred based on the information given and geometric proofs can be formed independently. Information regarding a problem and the information that needs to be proven can be identified. The difference between definitions, axioms and theorems as well as the distinction between a theorem and its converse theorem, along with essential and sufficient conditions, is grasped. At this level, a pupil understands why the sentence "A rectangle's diagonals are equal" requires proofs from Euclidean geometry, and why the reasons cannot simply be inferred from the figure. Their geometrical knowledge will be structured enough to allow for independent construction of formal proofs.

Level 5 (Understanding Axiomatic Systems)

At the highest van Hiele level, different geometries can be compared by observing their differences and similarities as axiomatic systems. At this level, an individual can, among other things, research equivalents to Euclidean geometry's concept of a rectangle in other geometrical spaces, such as the taxi cab geometry or the spherical geometry.

The van Hiele theory contains the following basic premises with regard to the progression of geometrical thinking, notable for their importance in developing the curricula and didactics for geometry (Silfverberg, 1999, 31–32):

1. The development of geometrical thinking is discontinuous, which is apparent in the consecutive nature of the van Hiele levels. The order of the levels is fixed, and none of them can be skipped completely. Advancing to the next level always requires understanding of the previous levels.
2. Progression moves from the implicit to the explicit. Each level's functions are the object of the next level's analysis; i.e. what was implicit in the thought processes of the previous level becomes explicit on the next one.
3. Each level has its own symbolical structure, and there is a gap of understanding between individuals on different levels. Each level has its own specific linguistic symbols and the network of relationships between them. If the pupil's thinking is on a different van Hiele level from what the teaching is aiming at, the desired development will be hindered. This is especially the case when the teaching is on a higher level than the van Hiele level of the pupil, making him/her unable to properly understand what they are being taught. The level reduction, in other words teaching the content with methods intended for a lower than the actual van Hiele level (e.g. by memory), is possible, but will not lead to true understanding or raise the van Hiele level of the pupil's geometrical thinking.
4. The van Hiele levels primarily evolve out of the learning process. The progression of a pupil's geometrical thinking is more reliant on the content covered and the quality of teaching than the pupil's age or biological maturity.
5. Geometrical thinking can be supported and furthered with a method of teaching that takes the levels of development into account.

The structural nature of the van Hiele theory is apparent in the way geometrical understanding is seen as a process where the central elements are sequential and located at different levels of abstraction yet still parallel paths of learning. At each stage, the paths of learning consist of getting to know the objects under examination first graphically and comprehensively, and then by analyzing their properties and the relationships between them. As noted before, each level's functions are the target of the next level's analysis; i.e. what was implicit in the thinking of the previous level becomes explicit on the next one (see Fig. 36.1).

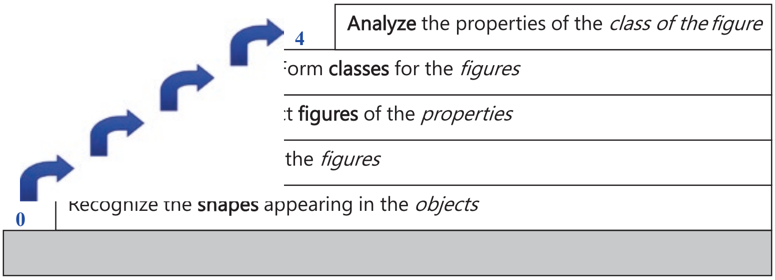


Fig. 36.1 The demonstration presented by Gawlick (2005, 70) on the sequential progression of geometrical thinking according to the van Hiele levels

About the Characteristics of Geometric Concept Formation

According to Fischbein (1993), the tension between the visual and the conceptual components marks geometrical concepts in an especially clear and unique manner. Fischbein emphasizes that the objects studied in school geometry are *figural concepts* by nature, meaning that they cannot be considered as purely conceptual constructions, since they are always accompanied by both a mental image born from figures and a conceptual interpretation born from verbal descriptions and definitions (see Fischbein & Nachlieli, 1998). Naturally, it is possible to consider geometrical concepts as a combination of required properties fixed by verbal definitions, but this type of method will not encompass all the implicit meanings of concepts. In the elementary geometry, the concepts that are studied involve, in addition to the meaning content given through definition, an empirical and visual idea of a concept's interpretative background ("image"), inseparably and perhaps even primarily attached to its concept (consider, e.g. the shape of a figure or the straightness of a line).

Figural concepts are often also characterized by the dimension of typicality, related to examples. All the cases belonging to the domain of a concept are not equally pertaining to the meaning of the concept; rather, some of the cases within a concept's domain are more representative than others. The dimension of typicality and the irregularity of class distinctions caused by it have been tentatively accounted for with the fact that a concept is, in fact, not defined by the combined properties of all of the objects contained in its domain, but rather by *family resemblance*. Objects belonging to the same class do not necessarily require collective properties at all. However, the more properties an object has in common with others, the more typical it is considered (Rosch & Mervis, 1975). Several studies (see, e.g. Okazaki & Fujita, 2007; Silfverberg, 1999) have proven that both primary and secondary school students tend to understand the basic concepts of geometry as prototypical, vaguely defined constructions. As an example, pupils around the age required for comprehensive school will often accept a figure with curved lines in the shape of an iron as a triangle and parallelograms with almost straight angles as rectangles. Besides the excessive allowance described above, attachment to prototypical definition can also be seen as a limitation of interpretations in exclusive classification. Squares are not considered rectangles and rectangles are not parallelograms. Similarly, the only real polygons are the ones that have at least five angles, etc. (Silfverberg & Matsuo, 2008b).

Learning a geometrical concept involves three central components, namely the concept itself, its examples and the properties or attributes that describe it. Conceptualization as such requires synthesis through association, abstraction and differentiation between properties before analysis. In the theory of hypothesis testing, learners are seen to abstract examples of a concept by analyzing its defining properties and constructing a concept as their synthesis. The process of conceptualization can then be illustrated with a diagram (Fig. 36.2).

According to this interpretation, concept learning requires a learner to be able to make inductive generalizations based on the cases studied, as well as the ability to

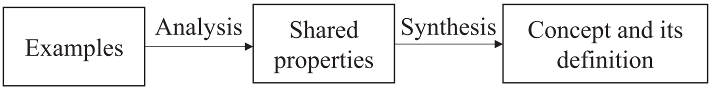


Fig. 36.2 Conceptualization according to the theory of hypothesis testing

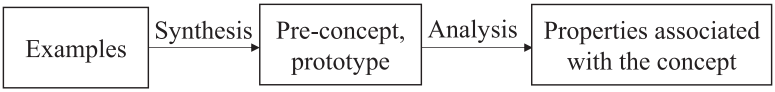


Fig. 36.3 Prototypical conceptualization

make logical deductions in order to combine collective, defining properties into a description of a concept.

In the *prototype theory*, the concept, or "pre-concept", is considered to be the result of a synthesis of examples united by context before the properties characteristic of the concept have even reached the conscious mind. This type of learning process can be illustrated with a diagram (Fig. 36.3).

Since the concepts studied in school geometry generally have a very concrete background, it is natural to assume that concept learning will often result in the creation of prototypical concepts, particularly in the beginning. At some point in learning geometry, the teacher will attempt to clarify the intuitive prototypical (pre-) concept into a defined concept. Since in most cases both processes for concept learning guide conceptualization, the geometrical concept adopted by the pupil will often contain both traits of a well-defined concept and the traits of a vaguely defined concept.

An analysis by Trzcieniecka-Schneider (1993), however, demonstrates how both inductive and deductive processes of concept learning can lead to perceiving a concept prototypically through the properties of its isolated, typical examples, leaving the properties relevant to forming an actual definition beyond the grasp of the learner. It is possible to learn the core content of a concept without learning the kind of variation the core of the concept allows in examples.

Basic Skills in Geometry

Classifying and Designating Figures

One of the manifesting characteristic for the way geometrical concepts gain meaning through typical examples is the avoidance of hierarchical classification in conceptualization – i.e. the exclusive classification. This phenomenon, familiar to experienced teachers from practice, has also systematically received verification in research (e.g. Silfverberg & Matsuo, 2008a, b). Squares are not considered

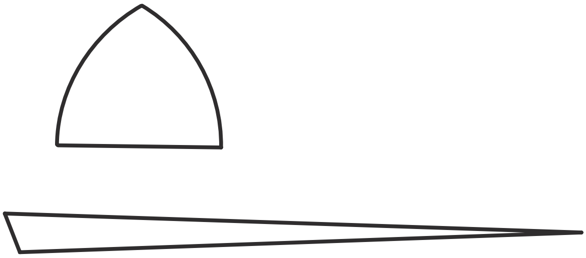


Fig. 36.4 Triangles or not?

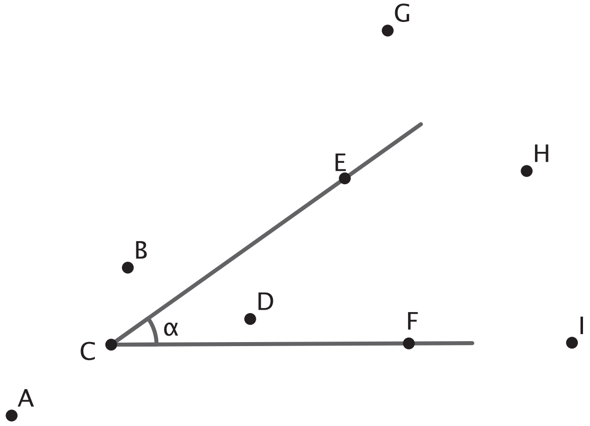
rectangles, because they do not look like typical rectangles. Likewise, rectangles are not considered parallelograms, and quadrilaterals are not considered polygons, because they lack the necessary number of angles. Similarly, a figure that bears close enough resemblance to a triangle can be considered a triangle, even if its "sides" are not straight, while an actual triangle will not always be accepted as such if its shape is elongated enough, etc. (Fig. 36.4).

The exclusive classification derived from prototypical conceptualization appears to be connected to the context that serves as the basis for classification (Silfverberg & Matsuo, 2008a, b). In the study by Silfverberg and Matsuo, there was a surprising distinction between the classifications of the Finnish and the Japanese pupils. In unrestricted classification, in which an individual is allowed to group example figures the way they feel is most fitting, exclusive classification appeared to be more common among the Japanese students than it was in the forced classification, in which the pupils are asked to group the examples in accordance with the previously agreed classifications. Generally, it can be said that learning and teaching hierarchical conceptualization is a demanding task (see, e.g. Okazaki & Fujita, 2007), since it requires intentional detachment from how things appear to be according to their visual appearance and acting against the individual's immediate visual perception. Naturally, hierarchical structures can be learned as simple, memorized facts, but explaining them mathematically requires understanding the idea of definition; in other words, understanding that something is this way in mathematics, because it has been (intentionally) defined as such, and not because it (or rather, its representation) visually appears like something.

The Skills of Definition and the Clarification of Concepts

After discovering how surprising primary school students' understandings of what is meant with the size of an angle can be, Silfverberg and Joutsenlahti (2007) built on the above mentioned research by testing how uniform the interpretations of students studying the subject itself and those studying to teach in primary school were regarding the well-known concept of angles, utilizing, among others, the following test:

Which of the points *A*, *B*, *C*, *D*, *E*, *F*, *G*, *H* and *I* are part of the angle α?



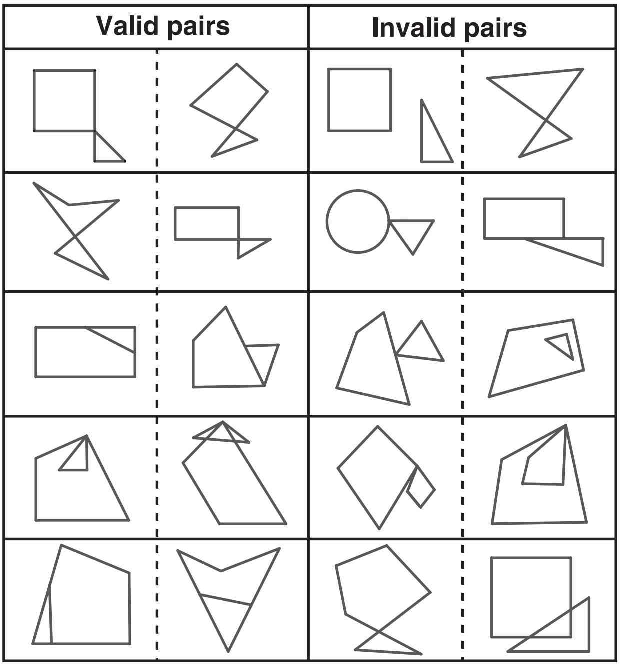
Surprisingly, the results clearly indicated that even prospective teachers can interpret the concept of angles in many different ways. There was diversity in whether to think an angle as a line construct consisting of two line segments or rays or think it as the area defined by two line segments or rays. On the other hand, there were different interpretations on whether the angle continues in the direction indicated by the drawing, beyond the limits of the section that were drawn. The angle was also interpreted as a rotation which made the actual question seem poorly worded. The results also proved that, in comprehensive and upper secondary school education, and in several cases also in university education, students could retain surprisingly diverse understandings (beliefs) about basic mathematical concepts, such as angles, for a very long time, without even suffering from any larger conflicts in communication. The explanation for this may lie in the teaching practices of school mathematics. Task assignments typical of school mathematics, such as “calculate...”, “draw…”, “classify…”, “determine the area…”, etc., appear to be of a type that allows for communication over the objects of study and achieving the desired result even while basic concepts, such as the angle, are understood in fundamentally different ways. In the spirit of socio-constructivist learning, highlighting differences in interpretations by using tailored assignments that reveal differences and discussing and negotiating the possibilities for interpretation, it is possible to aim for more uniform and accurate concepts.

Several of our studies indicate that the process of defining concepts and, most importantly, understanding the idea of definition are extremely challenging for students of both comprehensive and upper secondary schooling and even those at the university (Matsuo & Silfverberg, 2011; Silfverberg, 1999; Silfverberg & Joutsenlahti, 2007; Silfverberg & Matsuo, 2008a, b). This is partially because of the readily offered definitions in most study materials and the lack of interest in or the time for a deeper analysis. Regardless, understanding the significance of definition in mathematical conceptualization is more important than remembering the actual definitions themselves.

Relative to research, understanding the idea of definition is a complex task. Analysis can be directed at, for example, (1) how well the definition provided by the student defines the concept, as compared to the standard interpretation, (2) how well the form of the provided definition corresponds with the criteria set for mathematical definitions (de Villiers, 1995; Hershkowitz, 1990; Leikin & Winicki-Landman, 2000a, 2000b), (3) how well the definition provided by the student corresponds with the concept image that they appear to hold, based on its application (Tall & Vinner, 1981; Vinner, 1991; Vinner & Dreyfus, 1989) or (4) what type of linguistic form the definition text provided by the respondent holds (Barnbrook, 2002).

With regard to practical teaching, the problem is easy to solve, at least in principle: Definitions are learned by devising definitions, exploring different options and pondering the type of a definition the creator of the definition is aiming for, and why. Most of all, study materials are in need of assignments that provide the opportunity to practice definition. Below, you will find an example of a similar task,[1] which is intended to spark conversation about the possibility of differing definitions and the hierarchical superiority of definitions formed in distinct ways.

Example 1 Based on the pictures of ten examples and ten non-examples of the concept "Duo" you can deduce the properties of the concept Duo. Form a definition for a concept "Duo". Compare the definitions provided by the class. Which of the suggested definitions do you think is the best, and why?



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[1] Source: Textbook Silfverberg, Viilo & Pippola. Matematiikan Taito 3. Geometria, Weilin+Göös, s. 48

Each new concept introduced in the classroom offers an opportunity for conversation about how to limit the concepts, describe them verbally and define them, and why this is the way it is done. Below, you will find one more example[2] of an assignment in which the definition is provided, and the task is to explore the consequences of its stipulations.

Example 2 A kite can be defined as a quadrilateral with two pairs of equally long, adjacent sides.

(a) Draw three kites that differ from each other.
(b) Which of the following properties apply to all the kites?

  (i) The diagonals are perpendicular.
  (ii) Two vertices are at equal distances from the diagonal defined by two other vertices.
  (iii) No two sides are parallel.

(c) Complete the sentence so that it is true by filling in the blank with one of the options *is always*, *can be* and *is never*.

A square ___________ a kite.
A rectangle ___________ a kite.
A rhombus ___________ a kite.
A parallelogram ___________ a kite.
A trapezoid ___________ a kite.

The Skills of Proving

Euclid already utilized the axiomatic method when compiling geometrical knowledge of his time. The starting point were certain basic concepts, i.e. basic objects such as the point and the line, which are not defined, and the statements or axioms regarding them, which are accepted as true. Based on the basic geometrical objects, new concepts can be defined, which are then used in defining yet more new concepts. With the help of axioms and definitions, new propositions i.e. theorems can be proved, which will then help to prove yet more theorems. Theorems present the properties of concepts being studied, as well as the relationships between properties, which are then validated by a proof.

In geometry, we have got used to geometrical theorems being proved through a chain of reasoning, in which what is already known takes us step by step from a given to a statement. From experience, we know that learning the method of proof is a difficult task for every student and one that takes significant time to learn. Understanding a ready-made proof and more so constructing a valid proof requires

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[2] Source: Textbook Silfverberg, Viilo & Pippola. Matematiikan Taito 3. Geometria, Weilin+Göös, s. 47

structured understanding of the composition of all the geometrical knowledge covered up to the point, as well as correct understanding of how a proof works. This goal will be easier to reach if the students have, over time, been familiarized with presenting reasoning for their claims in school mathematics and taught the skills of argumentation in a less demanding context than creating a valid proof. Harel and Sowder (1998) accentuate that being convinced of the truth of something and proving it are done in different contexts, with different grounds and criteria. Overall, Harel and Sowder differentiate between the 12 different grades of methods for being convinced, ranging from belief in authority to proof by deduction in an axiomatic system.

In a similar manner, Mason et al. (1982) presented a three-phase method for testing the plausibility of an argument, in the form of didactic instructions for practicing reasoning: Convince yourself – convince a friend – convince an enemy. I have personally found that these steps are especially useful for situations requiring co-operative teamwork and experimental work with dynamic geometry in the form

Be convinced yourself (testing observations)

Does your friend believe you? (presenting to a small group)

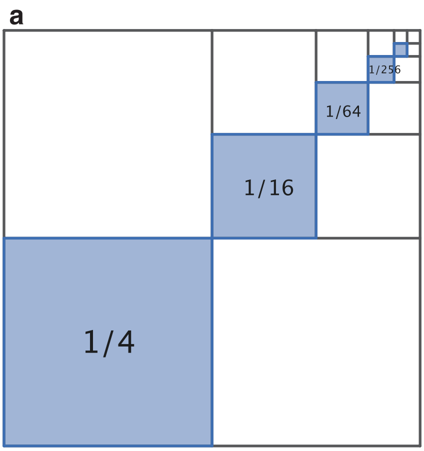
Does everyone believe you? (presenting to the class and the teacher)

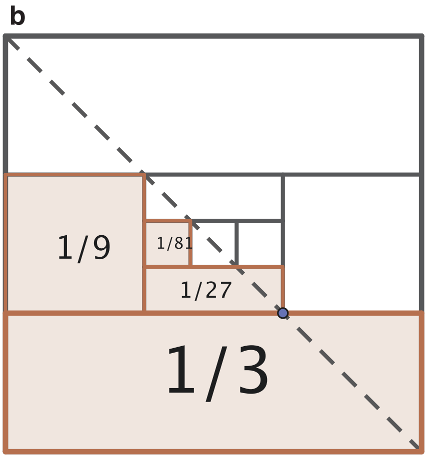
In comprehensive education, rather than actual proof, arguments are generally tested by making plausible, inductive generalizations from several explored cases or comparing, for example, the results achieved by measurement. Another conventional option is to present a demonstration in which, for instance, by altering a figure or figures, the existence of an assumed property, such as invariance, can be established. Of course, the property being studied does not have to be geometrical. Geometrical models are often used to explain algebraic relationships. For example, the Pythagorean theorem has hundreds of such established "visual proofs" based on the geometrical properties of figures. Some also claim that understanding a demonstrative model can often be harder than understanding the issue itself. The skill of understanding demonstrations can be practiced, however. Below, there are some examples of assignments that link algebraic and geometrical perspectives and allow the reader to ponder the extent to which geometrical demonstration helps to understand algebraic content, and to what extent it presents auxiliary challenges for understanding. This kind of a dialogue between the visual and algebraic thinking is a central method when mathematical fact is tried to be justified without formally proving the statement. "The proofs without words" both in a static and in a dynamic format help the learner see why a particular mathematical statement is true and perhaps also how one might begin to formally prove it true (Bell, 2011; Nelsen, 1993, 2000, 2015; Sigler, Segal & Stupel, 2016).

Task 1 (a) Explain how the following Figure a demonstrates the formula

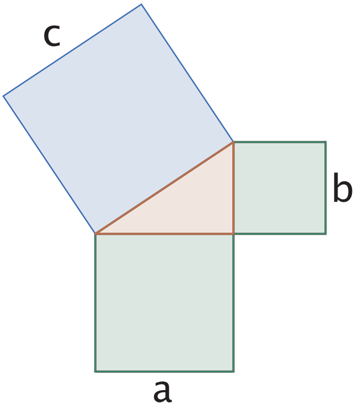
$$\left(\frac{1}{2}\right)^2+\left(\frac{1}{4}\right)^2+\left(\frac{1}{8}\right)^2+\left(\frac{1}{16}\right)^2+\ldots=\frac{1}{3}$$

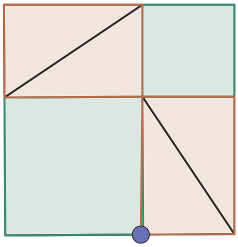
(b) Explain on which principle the rectangles in Figure b have been drawn. Form an equation (formula), which the Figure b demonstrates.

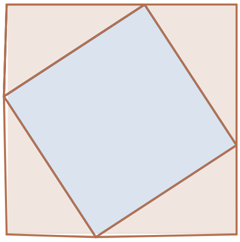




Task 2 Explain how the following figure demonstrates the Pythagorean theorem $a^2 + b^2 = c^2$.







Towards a Dialogue of the Traditional and the Dynamic Geometry

Information technology and especially the dynamic geometry software, such as GeoGebra, have provided practical instruction of geometry with countless new opportunities, and it is our belief that this development is merely the beginning. Tablet computers and the new calculator technology allow the use of dynamic geometry in a regular class space even without computers. Dynamic geometry software constructs figures on a screen based on their defining properties. The method of construction itself is often similar to drawing with a compass and a ruler; knowledge of compass-ruler constructions is an obvious benefit when learning the functions of dynamic geometry. The dynamicity of a drawing means that once created, the constructed image can be altered with the grab and drag method, by grabbing onto a free point in the figure, such as the vertex in a polygon, and moving the point around by dragging it. The shape of the figure will adjust within the limits that the definitions provided for the figure will allow. This enables studying the figure's invariances and variable properties experimentally, in other words, discovering and testing geometrical statements experimentally. The user does not have to settle for just ocular studying of the properties, since the more developed geometry software also includes a variety of tools for measuring and calculations, even if studies in nature will be approximations either way. The worldwide GeoGebra institutes network maintains a collection of softwares https://www.geogebra.org/institutes, which the reader can freely use to learn more about both dynamic geometry software GeoGebra as well as the finished applications developed with it.

When working with traditional methods, e.g. the pen and the paper, the starting point is that the student is aware of how each figure and configuration can be altered based on the known properties of the concepts. By the term *visual variation* we refer to the individual's ability to intentionally create visualizations of different examples related to the scope of the concept in a way that allows for controlling the variation of the form or other geometrical properties of a figure via visualization (Silfverberg, 1999, 92). The reader can test out the role of visual variation in their own thinking by, for instance, considering how an isosceles triangle can be rectangular, or how a triangle can be considered a trapezoid. Dynamic geometry is expected to provide the learner with support in developing their skill in visual variation, which will then complement concept learning generally, as a mental function. Many other learning processes typical of geometry, such as testing out prototypical concepts, developing skills in definition and training deductive skills, are emphasized in this kind of learning environment. At its best, activity in dynamic learning environments improves skills in testing out generalizations and developing hypotheses by mental images even without computer environment.

From the viewpoint of the development of geometrical thinking and conceptual geometrical knowledge, working with dynamic geometry software offers an excellent learning environment for learning the meaning behind geometrical concepts and studying the relationships between concepts. Working in an environment of dynamic geometry can naturally be linked with student-centred and investigative approaches to learning geometry (Erbas, & Yenmez, 2011; Hannafin & Scott, 2001). Studies have proven that dynamic geometry, when skillfully used, is also an effective way of learning geometry. As an example, Chan and Leung's (2014) meta-analysis revealed that the overall effect size of DGS-based instruction on achievement scores was distinctive compared to traditional instruction. Subgroup analysis found some groups to have better effectiveness, for example, short-term instruction with dynamic geometry software significantly improved the mathematical achievement of elementary school students. Traditional geometry with its example figures drawn on paper and dynamic geometry are not contradictory or exclusive; instead, they should be considered mutually complementary tools for developing geometrical understanding (see, e.g. Patsiomitou & Emvalotis, 2010).

Geometry and Learning Difficulties

Learning difficulties related to geometry have been studied considerably less than those apparent in the numerical and algebraic domains of mathematics. Regardless, many factors that contribute to learning difficulties are shared between both domains of mathematics.

One characteristic property of school geometry is the abundance of concepts in a hierarchical structure. There is plenty to learn and remember, especially if the learner cannot understand the logical relationships between the concepts. Working memory is under pressure as well, especially when the task requires the individual to simultaneously keep in mind many kinds of information (Bobis, Sweller, & Cooper, 1993). One part of the required information can be in text format, one part symbolical and one part marked on the illustration provided with the assignment. For example, a study by Silfverberg (1999) demonstrated that the van Hiele levels used to categorize a student's level of geometrical understanding had a clear correlation with the student's working memory capacity.

Another characteristic of school geometry is its pronounced visuality. In geometry, visual information is utilized both directly as a tool for perception and indirectly, when the required information is not provided outright, but must be deduced based on the other information available in the picture. A picture itself can be either static or a dynamically configurable image on a computer screen. The learning process can be hindered by similar visual processing disorders as the ones that hinder numerical tasks (see, e.g. Gal & Linchevski, 2010), for example, when visualizing fractions with different graphical representations or when learning to understand the part-whole relation with the help of visual models. Visual processing disorders can manifest as a difficulty of recognizing spatial relations, a reduced ability to sort out visual information, or a difficulty of recognizing known objects in images or to perceive pictures' parts and wholes (NCLD, 1999).

When a pupil has difficulties in learning geometry, it is good to remember that these challenges can be caused by many factors. On the one hand, learning difficulties can be caused by a pupil's general, non-verbal learning difficulties, such as the limitations of their visuospatial working memory (Mammarella, Giofrè, Ferrara & Cornoldi, 2013), other visual processing disorders (Gal & Linchevski, 2010) or disorders of verbal development. On the other hand, geometry itself can occasionally prove challenging in terms of content, especially when moving on to deductive geometry. Traditional Euclidean geometry with its definitions, theorems and proof requires thinking at van Hiele level 3 or higher, as well as necessary skills in logical reasoning. It is best not to forget that teaching geometry can be challenging both intellectually and didactically for the teacher, as well. Kuzniak and Rauscher (2011) demonstrate illustratively with examples how different didactic approaches to issues studied in geometry are differently vulnerable to the development of learning difficulties.

Summary

It happens all too often that school geometry is regarded solely as a means of adding new content to a geometrical reservoir of information. According to Houdement and Kuzniak (2003) as well as the van Hiele theory (1957), geometry appears as a very diverse field to learners at different ages. The gradual but comprehensive change in geometrical thinking and the accumulation of geometrical knowledge last from early childhood to adulthood, guiding learners step by step from one geometrical approach to another, as the abovementioned interpretations G1–G3 to geometry demonstrate.

It is obvious that teachers teaching geometry must know the main characteristics of the development of geometrical thinking at all school levels and adjust their teaching to match the current stage in development. Even so, the task remains pedagogically challenging. The geometrical thinking of a pupil should be guided with tasks and activities that are both understandable at the level their geometrical thinking is at, but still provide the opportunity and challenge of a higher-level analysis. With the previous examples, I strove to illustrate how the basic elements of geometrical thinking, such as how concepts grow comprehensible either through their visual images or definitions and how different kinds of arguments can make claims presented as facts appear plausible, are fundamentally different in the various geometrical interpretations.

Even though the kind of geometry that is taught in school is based on centuries-old knowledge, it is important to note that didactics of geometry is currently facing a major turning point. These days, geometry is increasingly studied in the learning environments of dynamic geometry, which are characterized by the students' explorative and experimental work. For now, not much is known about whether this modern style of geometrical education based on socio-constructivist-oriented co-operation and explorative approach will provide learners with better skills in spatial and geometrical thinking than the traditional methods of teaching geometry. The tools for determining this should also be improved, both on the theoretical and the practical levels. Like Mammarella, Giofrè and Caviola (2017) rightfully noted, the theories describing the progress of geometrical thinking are over 30 years old. In reality, this is when the theories reached the level of common knowledge; however, for example, the most well-known, the van Hiele theory, is dated back to the 1950s, a time during which the status of geometrical education within the curricula as well as the method of teaching geometry were completely different from what they are today.

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Part V

Mathematical Learning Difficulties: Approaches to Recognition and Intervention

Chapter 37
Assessing Mathematical Competence and Performance: Quality Characteristics, Approaches, and Research Trends

Jan Lonnemann and Marcus Hasselhorn

Introduction

Educational and psychological assessment involves the identification of individual preconditions and developmental constraints of children's learning. Thus gathered, the assessment information is, for example, used to distinguish between children with and without mathematics learning difficulties. Based on the identification of individual difficulties and resources, training programs to enhance learning can be adaptively developed and applied. The evaluation of the effectiveness of these programs by identifying relevant changes at the individual level also constitutes an important function of educational psychological assessment. In order to assess children's individual level of age-appropriate mathematical competencies, as well as their developmental trajectories, different diagnostic approaches have been

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developed. In the following sections, the quality characteristics of such approaches are described and different categories are presented for the classification of such approaches. In this context, different principles of task selection and their consequences for the interpretation of test results are discussed, and some promising research trends are finally outlined.

Quality Characteristics

To ensure the quality of diagnostic test results and related diagnostic decisions, specific criteria of evaluation must be adhered to. Within the scope of the classical test theory, based on which most of the currently used diagnostic tools were developed, reliability is the central criterion of evaluation. Reliability refers to the consistency or stability of a measure. The more consistency and stability is achieved for a specific diagnostic tool, the lesser the extent of measurement errors of the test is and the more reliable the test is. A standard procedure to estimate test reliability is the replication of a test with the same participants within a short period of time. Given that both test applications lead to comparable results (so-called test–retest reliability), the test under scrutiny is classified as being reliable.

Objectivity and validity are additional criteria of evaluation which are considered to be indispensable. A measurement is objective if the test result is not influenced by the person who is conducting the test. Objectivity should be ensured with regard to collection, analysis, and interpretation of data. To meet this requirement, detailed instructions for test administration, scoring, and interpretation are typically provided in a test manual. Validity refers to the accuracy of a measure. A measure is valid if it actually measures what it purports to measure. For example, testing a specific group of participants with two different tests should yield similar results if both tests are designed to measure the same competence. In the context of measuring and analyzing performance, scaling (provision of norms) can be regarded as another important criterion of evaluation. Testing norms should be representative and up to date.

Categories of Classification

Existing diagnostic approaches to assessing mathematical competence and performance can be classified into different categories. The categories *norm-referenced* versus *not-norm-referenced tests*, *individual testing* versus *group testing*, *paper-and-pencil tests* versus *interviews* versus *computer-based tests*, *chronological* versus *educational age–oriented tests*, *speed tests* versus *power tests*, and the *principles of task selection* are addressed in the following sections.

Norm-Referenced Versus Not-Norm-Referenced Tests

Norm-referenced tests allow for an evaluation of individual performance based on the performance of an appropriate reference population. In this way, it is possible to determine whether the test performance of a child is within the range of performance levels that most children of this age or grade level achieve, or whether the child's performance is above or below the average performance level of the reference population. Objective, reliable, and valid assessment of a child's below-average performance level is necessary for diagnosing developmental dyscalculia. Tests that are not norm referenced typically involve qualitative assessments, which can help gain insights into children's individual ways of thinking and their strategies. In this regard, it should be noted that two children can show identical performances (e.g., correctly solving the task $12 - 4 = ?$ in the same period of time) but may differ with regard to the strategies they use to solve the problem. While one child may solve the problem by counting backward, the other may use a decomposition strategy (e.g., $12 - 2 = 10$ and $10 - 2 = 8$). These two ways of solving the problem point to the understanding of different concepts and may perhaps represent different levels of mathematical competence. Based on more qualitative interpretations of children's math problem–solving behavior, hypotheses about individual competencies and deficits can thus be derived and individualized training programs can be developed. Individual reference parameters are suitable to evaluate the effectiveness of such programs, and they allow the inspection of individual performances over time and thereby the detection of changes at the individual level.

Individual Versus Group Testing

Mathematical competence and performance can be assessed either individually or in a group setting. A number of tests can be conducted either individually or in a group setting. The norm-reference data for such tests are typically assessed in a group setting and used to evaluate both the results of individual tests and the results of group tests. This, however, requires that similar test results can be expected under the conditions of group testing and under the conditions of individual testing.

Paper-and-Pencil Tests Versus Interviews Versus Computer-Based Tests

Most contemporary approaches to assessing mathematical competence and performance rely on paper-and-pencil tests. Interviews are especially used for children at preschool age and rather seldom in school contexts. In addition, a number of computer-based math tests are available today. Computer-based tests can have the advantage of objective and economical data analysis. Besides, they can offer the possibility to individually adapt task difficulty without great effort and to provide a playful atmosphere in the testing situation, which can have a motivating effect.

Chronological Versus Educational Age–Oriented Tests

While some approaches to assessing mathematical competence and performance are designated for specific chronological age ranges, other approaches consider the educational age, i.e., the individual level attained in the educational system. Approaches considering the educational age are, for example, designated for children in their last year of kindergarten or for children attending specific grade levels.

Speed Versus Power Tests

A further category for the classification of approaches to assessing mathematical competence and performance refers to the question of whether the assigned tasks have to be solved under time pressure or not. So-called speed tests involve problem solving under time pressure and are typically based on the assumption that a higher level of competence is expressed by more efficient solution strategies. Indeed, automatized and thus relatively fast procedures for solving simple arithmetic problems are considered to be fundamental to the development of arithmetic skills in elementary school (e.g., Geary, 2000). The interpretation of speed test measurements is, however, ambiguous because children's performance in speed tests might reflect not only their mathematical competence but also their level of motivation and/or their general processing speed. If a child can solve only a few problems under time pressure, this does not necessarily mean that the child has not understood the underlying mathematical concepts. To gain more detailed information about children's understanding of mathematical concepts, so-called power tests involving problem solving without obvious time pressure are thus more suitable. However, some approaches to assessing mathematical competence and performance consider both perspectives: speed and power.

Principles of Task Selection

Aside from the aforementioned categories of classification, the approaches to assessing mathematical competence and performance differ with regard to the principles of task selection. Curriculum-based measures including tasks that represent a specific curriculum do exist, as do measures for which task selection is based on neuropsychological or developmental psychology theories. In the following sections, these different principles of task selection and their consequences for the interpretation of test results are discussed by reference to selected approaches to assessing mathematical competence and performance.

Outline of Different Approaches

Curriculum-Based Measures

Curriculum-based measures include tasks representing a specific curriculum (see e.g., the German Mathematics Test for First Grade (DEMAT 1+)) (Krajewski, Küspert, Schneider, & Visé, 2002; Krajewski & Schneider, 2009). Provided that norm-reference data are available and that the school a child is attending adheres to the curriculum, the evaluation of the child's performance can serve to assess whether a child has achieved the curriculum goals in the same way as most of the peers, or whether the child's performance is above or below the average performance level. The concept of *curriculum-based measurement* (CBM) (see e.g., Deno, 1985) comprises repeated measures of students' academic performance in order to monitor individual progress *regarding the respective curriculum outcomes*, and to evaluate and improve instruction. Most of the CBM research in mathematics has initially focused on computational fluency in elementary school children. The *Monitoring Basic Skills Progress* (MBSP) math computation measures (Fuchs, Hamlett, & Fuchs, 1998), for example, have been widely utilized and validated through a series of studies (see e.g., Foegen, Jiban, & Deno, 2007 for an overview). Typically, CBM consists of a group-administered paper-and-pencil test containing samples of problems that represent the respective year's curriculum scope and sequence. The problems that are used to target a specific grade level comprise problems that children should be able to solve at the end of the respective school year. In order to monitor individual progress, the children have already been confronted with these problems at the beginning of the respective school year. Repeated observations of individuals' performance should be structured so that children respond to samples of problems that are always different yet of comparable difficulty. In CBM research, the targeted equivalence of difficulty across different samples of problems is a major concern (see e.g., Montague, Penfield, Enders, & Huang, 2010). If the different samples of problems are of comparable difficulty, it is possible to determine whether a child's performance level has improved, deteriorated, or remained as is. Computer-adaptive testing (CAT) has recently emerged as a method for progress monitoring. CAT involves the presentation of tasks via computer with subsequent items adjusted based on the accuracy of the child's response and the difficulty of the respective task (see e.g., STAR-Math (Renaissance Learning, 2012)). Owing to the adaptive administration, different children can receive different sets of test items. *Item response theory* (IRT) is a psychometric model that allows equitable scores to be computed across different sets of items. CAT has the capability of increasing diagnostic efficiency by reducing test administration and evaluation time without affecting test precision. Indeed, adaptive tests can provide uniformly precise scores for most children, while nonadaptive tests usually provide the best precision for children in a specific ability group (e.g., children with developmental dyscalculia). The development of computer-adaptive tests can, however, be far more expensive than development of paper-and-pencil fixed-form tests.

To summarize, it can be noted that curriculum-based measures for assessing mathematical competence and performance exist that are designated for repeatedly evaluating a specific mathematical skill to get information about the competence gains of a child, and others that are designated for single usage. The test results of curriculum-based measures allow evaluation of individual skills in terms of given curriculum goals.

Approaches Based on Neuropsychological Theories

The triple-code model of number processing (e.g., Dehaene, 1992; Dehaene & Cohen, 1995) provides a basis for different approaches to assessing mathematical competence and performance. The model proposes that numbers can be mentally represented in three different codes, which are linked to different brain regions: a visual Arabic number form, an auditory–verbal word frame, and an analogue magnitude representation. The visual Arabic code encodes numbers as strings of Arabic numerals. The auditory–verbal word frame represents numerals lexically, phonologically, and syntactically. It is assumed to mediate retrieval processes for simple addition and multiplication facts. These visual and verbal codes are thought to be nonsemantic and more related to the surface format of numerical input and output processes. The analogue magnitude code, conversely, provides a semantic representation of the size and distance relations between numbers.

Based on the triple-code model of number processing, von Aster and Shalev (2007) conceptualized a four-step developmental model of number acquisition. The four proposed steps are assumed to be associated with the development of different brain regions. Accordingly, children possess an innate or *very early acquired* core system for representing numerical magnitude information (step 1). This is assumed to be a necessary precondition for children to learn to associate a perceived number of objects or events with spoken (step 2) and written Arabic symbols (step 3). Nonsymbolic numerical magnitude representations (step 1), verbal representations (step 2), and symbolic, visual Arabic representations (step 3) are assumed to be merged into a mental number line representation (step 4), which provides the basis for the development of arithmetic skills.

Based on the four-step developmental model of number acquisition, the *Neuropsychological Test Battery for Number Processing and Calculation in Children* (ZAREKI-R) (von Aster, Weinhold Zulauf, & Horn, 2006) was developed (see Table 37.1 for an overview of selected diagnostic approaches). This test battery

Table 37.1 Categorization of selected diagnostic approaches for assessing mathematical competence and performance

Test	Authors (publication year)	Content	Reference population	Individual versus group testing	Paper and pencil vs. interview vs. computer-based	Chronological vs. educational age	Speed vs. power	Principles of task selection
ASER Math	ASER Centre (2017)	Number recognition, subtraction, division, varying additional tasks (e.g., word problems with currency or telling time)	About 700,000 children annually assessed in India	Individual testing	Paper and pencil	5–16 years	Power	Curriculum-based
Dyscalculia Screener	Butterworth (2003)	Simple reaction time, dot enumeration, number comparison, addition, multiplication	$N = 549$, UK	Individual testing	Computer-based	6–14 years	Speed	Based on neuropsychology and developmental psychology theories
MARKO-D	Ricken et al. (2013), Henning et al. (in press)	Count numbers, ordinal number line, cardinality and decomposability, class inclusion and embeddedness, relationality, bundling and unbundling	$N = 1095$, Germany; $N = 1186$, South Africa (Afrikaans, Sesotho, Zulu, English)	Individual testing	Paper and pencil	4–6 years; first grade	Power	Based on developmental psychology theories
NSS	Jordan and Glutting (2012)	Counting, number recognition, numerical magnitude comparisons, nonverbal calculations, arithmetic story problems, arithmetic number combinations	$N = 300$, USA	Individual testing	Paper and pencil	Kindergarten to first grade	Power	Based on developmental psychology theories

(continued)

Table 37.1 (continued)

Test	Authors (publication year)	Content	Reference population	Individual versus group testing	Paper and pencil vs. interview vs. computer-based	Chronological vs. educational age	Speed vs. power	Principles of task selection
ENT	van Luit, van de Rijt, and Pennings (1994)/van Luit, Van de Rijt, and Hasemann (2001)/van Luit, Van de Rijt, and Aunio (2006); Aunio (2006)	Concepts of comparison, classification, one-to-one correspondence, seriation, use of number words, structured and resultative counting, general understanding of numbers	*N* = 823, the Netherlands; *N* = 330, Germany; *N* = 1029, Finland	Individual testing	Interview	4–7 years	Power	Based on developmental psychology theories
Uwezo Numeracy	Uwezo (2014)	Counting, number recognition, place value understanding, addition, subtraction, multiplication, division, varying additional tasks (e.g., shapes or time)	About 350,000 children annually assessed in Kenya, Tanzania, and Uganda	Individual testing	Paper and pencil	6–16 years	Power	Curriculum-based

REMA	Clements, Sarama, and Liu (2008)	Verbal and object counting, subitizing, number comparison and sequencing, connecting numerals to quantities, number shape composition/decomposition, adding and subtracting, place value, recognition, shape composition/decomposition, congruence and construction of shapes, spatial imagery, measurement, patterning	–	Individual testing	Interview	Prekindergarten to second grade	Power	Based on curriculum and developmental psychology theories
Tedi-Math	van Nieuwenhoven et al. (2001), Grégoire et al. (2004), Kaufmann et al. (2008)	Counting, number recognition, reading numbers, writing numbers, seriation, classification, estimation, comparison, conservation, inclusion, additive decomposition of numbers, addition, subtraction, multiplication, arithmetic operations, visual and verbal formats, arithmetic concepts	$N = 583$, Belgium and France (French language); $N = 550$, Belgium (Dutch language); $N = 873$, Austria and Germany	Individual testing	Paper and pencil	Kindergarten to third grade	Speed and power	Based on neuropsychology theories

(continued)

Table 37.1 (continued)

Test	Authors (publication year)	Content	Reference population	Individual versus group testing	Paper and pencil vs. interview vs. computer-based	Chronological vs. educational age	Speed vs. power	Principles of task selection
TEMA-3	Ginsburg and Baroody (2003)	Numbering (e.g., verbal counting), number comparison (e.g., determining which of two spoken number words is larger), numeral literacy (e.g., reading numbers), mastery of number facts (e.g., retrieving multiplication facts), calculation skills (e.g., written addition and subtraction), number concepts (e.g., answering how many tens are in one hundred)	$N = 1219$, USA	Individual testing	Paper and pencil, interview	3–8 years	Power	Based on developmental psychology theories
WIAT-III	Wechsler (2009, 2010, 2016)	Numerical operations, math problem solving, math fluency (addition, subtraction, multiplication)	$N = 3000$, USA; $N = 822$, Canada; $N = 1132$, Australia and New Zealand	Individual testing	Paper and pencil	Kindergarten to 12th grade; 4–50 years	Speed and power	Based on curriculum and developmental psychology theories
WJ IV	Schrank, Mather, and McGrew (2014)	Calculation, math problem solving, math facts fluency	$N = 7416$, USA	Individual testing	Paper and pencil, interview	Prekindergarten to 12th grade; 2–90 years	Speed and power	Based on curriculum and developmental psychology theories

ZAREKI-R	von Aster et al. (2006), von Aster and Dellatolas (2006), Santos et al. (2012)	Counting dots, counting backward, reading numbers, writing numbers, number comparison (oral and written), mental calculation (addition, subtraction, and multiplication presented aurally), number line activities, forward and backward repetition of digit sequences, perceptive/contextual estimation, arithmetic story problems	$N = 764$, Germany and Switzerland; $N = 249$, France; $N = 172$, Brazil	Individual and group testing	Paper and pencil	First to fourth grades	Power	Based on neuropsychology theories
ZAREKI-K	von Aster et al. (2009)	Counting forward, counting backward, counting in steps of two, identifying preceding and succeeding numbers, counting dots, word problems, repeating numbers, addition/subtraction based on quantities, mental addition/subtraction, number line activities, subitizing, perceiving and comparing differently arranged quantities, reading numbers, writing numbers, linking quantities and numbers, contextual estimation, oral and written numerical magnitude comparison	$N = 429$, Switzerland	Individual testing	Paper and pencil	Last year of kindergarten	Power	Based on neuropsychology theories

ASER Annual Status of Education Report, *ENT* Early Numeracy Test, *MARKO* Mathematics and Arithmetic Test for Assessing Concepts, *NSS* Number Sense Screener, *REMA* Research-Based Early Maths Assessment, *Tedi-Math* Diagnostic Test for Basic Competencies in Mathematics, *TEMA* Diagnostic Test for Basic Competencies in Mathematics, *WIAT-III* Wechsler Individual Achievement Test (3rd Edition), *WJ IV* Woodcock–Johnson IV Tests of Early Cognitive and Academic Development, *ZAREKI* Neuropsychological Test Battery for Number Processing and Calculation in Children is designated for identifying elementary school children with developmental dyscalculia, and it has been used with children from different countries—for instance, from Switzerland, Germany, France, Greece, and Brazil (Dellatolas, von Aster, Willardino-Braga, Meier, & Deloche, 2000; Koumoula et al., 2004; von Aster & Dellatolas, 2006).

A kindergarten version of the test battery (ZAREKI-K) (von Aster, Bzufka, & Horn, 2009) is also available. To illustrate the principle of such tests derived from neuropsychological theories, ZAREKI-K is now described in more detail. The declared aim of this test battery is to provide a risk estimation for the development of mathematical learning difficulties. ZAREKI-K is a paper-and-pencil test. It is conducted in an individual setting without time pressure and consists of 18 subtests (counting forward, counting backward, counting in steps of two, identifying preceding and succeeding numbers, counting dots, word problems, repeating numbers, addition/subtraction based on quantities, mental addition/subtraction, number line activities, subitizing, perceiving and comparing differently arranged quantities, reading numbers, writing numbers, linking quantities and numbers, contextual estimation, oral numerical magnitude comparison, and written numerical magnitude comparison). These subtests are assigned to three index scales: (1) counting and number knowledge; (2) semantic numerical knowledge and arithmetic; and (3) working memory. The norm-reference data are based on 429 children from Switzerland who were examined in their last year of kindergarten before entering elementary school. Norms are available for overall performance, for performance on each of the three index scales, and for performance in each of the 18 subtests. The identification of children at risk for the development of mathematical learning difficulties is either based on overall performance or based on performance on one of the two index scales 1 and 2. The third index scale is not used for risk estimation. At the end of second grade, 378 children from the norming sample of ZAREKI-K were re-examined using ZAREKI-R. Thereby, the prognostic validity of ZAREKI-K should be evaluated. Adhering to the test manual, 26 out of the 378 children were identified as having developmental dyscalculia based on their performance in ZAREKI-R. Among these 26 children, 16 were correctly identified as children at risk for the development of mathematical learning difficulties. Another 16 children were identified as having a risk for the development of mathematical learning difficulties but based on their performance in ZAREKI-R, they were not identified as children with developmental dyscalculia. According to the authors, analyses of individual test results allow not only the identification of children at risk for the development of mathematical learning difficulties but also the selection of appropriate training materials. The existing norms for performance in each of the 18 subtests should thus serve to identify individual profiles of strengths and weaknesses. With regard to reliability, due to the small number of tasks in the different subtests (a maximum of 11 tasks), it is advisable to perform data interpretation at the level of the index scales, not at the level of the subtests. It should also be noted that the norming sample is relatively small and that the norms apply to the complete year of kindergarten before school entry, which does not appear to be optimal, because of the assumable learning progress during this period.

Approaches based on neuropsychological theories, like ZAREKI, allow for the inspection of different mathematical skills, which are associated with the functionality of different brain regions and might be impaired separately. Another example of this category is the *Diagnostic Test for Basic Competencies in Mathematics* (Tedi-Math) (van Nieuwenhoven, Grégoire, & Noël, 2001). Tedi-Math is also designated for identifying children with developmental dyscalculia. The original French version has been adapted for use with Dutch- and German-speaking children (see Grégoire, Noël, & van Nieuwenhoven, 2004; Kaufmann et al., 2008). Like ZAREKI-K, Tedi-Math is a paper-and-pencil test, and it is conducted in an individual setting without time pressure. In some of the subtests of Tedi-Math, however, qualitative assessments as well as assessments of children's processing speed are scheduled. The German version of Tedi-Math consists of 28 subtests (e.g., counting, logical operations on numbers, and arithmetical operations) addressing the different forms of numerical representations proposed in the triple-code model of number processing. Tedi-Math can be used with children from 4 to 8 years of age (kindergarten to third grade) and norms are provided for 6-month intervals. Depending on the individual level attained in the educational system, a selection of subtests is specified. Norms are provided for overall performance but also for performance in each of the different subtests, serving to identify individual profiles of strengths and weaknesses. Moreover, according to the authors, Tedi-Math has acceptable psychometric properties for progress monitoring purposes. The estimation of test reliability based on norm-reference data for the German version ($N = 873$), however, revealed a comparatively low retest reliability in kindergarten ($r = 0.23$).

Approaches Based on Developmental Psychology Theories

Different approaches to assessing mathematical competence and performance have been conceptualized on the basis of developmental theories of number understanding postulated by Resnick (1983) and Fuson (1988). For example, Fritz and Ricken (2008) (see also Fritz, Ehlert, & Balzer, 2013) proposed a model for the development of numerical concepts from ages 4 to 8; a similar model was presented by Krajewski (2008) (see also Krajewski & Schneider, 2009). In accordance with this model, a theory-based grouping of items was performed, which is used in the *Mathematics and Arithmetic Test for Assessing Concepts at Preschool Age* (MARKO-D) (Ricken, Fritz, & Balzer, 2013). An adapted version for first graders is also available (MARKO-D1+) (Fritz, Ehlert, Ricken, & Balzer, 2017). Both tests intend to provide a quantitative and qualitative description of children's individual stages of development. The different developmental levels are defined by the following concepts and tasks:

- Level I—count numbers: The ability to distinguish small sets and to count and enumerate them. Children know the number word sequence and can count out small collections of objects by allocating one object to one number word. This level is assessed by tasks such as "Give me 5 disks."

- Level II—mental number line: Children construct a mental number line representation, i.e., an ordinal representation of the number word sequence, on which numbers are aligned as gradually increasing quantities. This enables them to identify preceding and succeeding numbers, as well as understanding that adding means getting more and counting forward (moving to numbers that appear later on the mental number line). At this level, children can solve verbally presented number word problems such as "Peter puts 3 books in the cupboard. Alice puts another three books in the cupboard. How many books are in the cupboard?" by counting out first the partial quantities and then the total quantity individually, using their fingers.
- Level III—cardinality and decomposability: Children understand that number words are linked to quantities and that a number word represents a quantity with a specific number of elements. Once children conceive that a number is a composite unit that consists of individual elements, they also begin to understand that numbers can be decomposed again. At this level, children can solve tasks that require them to count from a partial to a total quantity, e.g., "I want 10 disks; I already have 4. How many are missing?"
- Level IV—class inclusion and embeddedness: Children understand that quantities can be decomposed and composed in different ways. They conceive that addition and subtraction problems can be considered as being composed of subset–subset–whole set. This level is assessed by tasks such as "Bring me 5 flowers; three of them should be red."
- Level V—relationality: Children understand that the number word sequence is a sequence of cardinal units, and they conceive that intervals between successive number words are of the same size (+ 1). This enables them to compare quantities and to determine differences between quantities precisely. This level is assessed by tasks involving the recognition of differences between sets. Beyond that, children realize that numbers represent not only concrete quantities but also counting acts, which themselves can be counted. Therefore, tasks like the following become solvable: "What number is 3 smaller than 7?"
- Level VI—units in numbers: Based on the concept of relationality, children realize that the distances between the numbers on the number word line are always the same. Therefore, segments of the same size, or bundles, can be formed on the number line (e.g., 2 × 4; 4 × 4). Conversely, a number can also be decomposed into partial quantities of the same size. With this knowledge, children are able to find different decompositions of the same magnitude (bundles) for a number. The level is assessed by tasks such as "How can you decompose the number 12 into different bundles of the same size?"

The construction of MARKO-D1 follows the construction of MARKO-D and expands the range of mathematical competencies to be tested. MARKO-D consists of 55 items, assessing levels I–V; MARKO-D1+ has 48 items, assessing levels I/II–VI. For each level, anchor items are defined, which are included in both tests. It was checked whether the items selected for both tests were assigned to the same theoretically postulated levels.

A probabilistic IRT model was used, which allowed the representation of both item difficulties and personal abilities of the children on the same scale. By means of the obtained point values, the current competence level of a child can be qualitatively determined. Quantitative data analyses can be performed on the basis of norm-reference data from 1095 German children aged 48 to 87 months (MARKO-D) and 1672 German children aged 71 to 109 months (MARKO-D1+). Norms are provided for 6-month intervals, whereby the database sample in MARKO-D is relatively small for the oldest group (e.g., 6 years 6 months and older, $N = 26$). Both tests are paper-and-pencil tests, to be conducted in an individual setting without time pressure. Recently, MARKO-D was translated into four South African languages: English, Afrikaans, isiZulu, and Sesotho (see Henning et al. (in press)).

The tests MARKO-D and MARKO-D1+ complement the quantitative evaluation of individual performance based on the scaling data from an appropriate reference population sample by a qualitative description of individually available and used mathematical concepts. For children whose developmental stage is not age appropriate, the training program MARKO-T can be used (Gerlach, Fritz, & Leutner, 2013), which draws on children's individual developmental stage identified by MARKO-D.

Approaches based on developmental psychology theories like MARKO-D and MARKO-D1+ thus allow us to assess children's individual stages of development and to identify children whose developmental stage is not age appropriate. Another example of this category is the *Early Numeracy Test* (ENT) (van Luit, van de Rijt, & Pennings, 1994). In addition, screening measures like the *Number Sense Screener* (NSS) (Jordan & Glutting, 2012) have been developed, targeting risk estimation for the development of mathematical learning difficulties (see Gersten et al. (2012) for an overview). The NSS assesses children's skills related to counting, number recognition, numerical magnitude comparisons, nonverbal calculations, arithmetic story problems, and arithmetic number combinations. Similar skills are also assessed as part of broader test batteries covering children's academic development; see for example the *Woodcock–Johnson IV Tests of Early Cognitive and Academic Development* (WJ IV ECAD) (Schrank, McGrew, & Mather, 2015) and the *Wechsler Individual Achievement Test (3rd Edition)* (WIAT-III) (Wechsler, 2009).

Research Trends

In the course of internationally comparative assessments of student achievement like *Trends in International Mathematics and Science Study* (TIMSS) or the *Programme for International Student Assessment* (PISA), the concept of competence became more important in educational psychology. The identification of individual performance levels and their evaluation based on the performance of an appropriate reference population remains an important function of educational psychological assessment. Beyond the mere identification of individual performance levels, attempts to assess children's understanding of mathematical concepts have increasingly been pursued, as well as attempts to identify individually available mathematical concepts and to assign them to a specific developmental stage. Developmental theories of mathematical understanding are thus required. Regarding the development of numerical concepts, such theories are at hand (see e.g., Fritz & Ricken, 2008; Krajewski, 2008). On the basis of these theories, competence levels can be defined that correspond to circumscribable developmental milestones and can be used as criteria for the evaluation of individual developmental trajectories. While there has been an emphasis on the development of numerical concepts in the past, the relevance of other mathematical domains, like geometry, is increasingly being highlighted (see e.g., Spelke, Lee, & Izard, 2010).

Developmental theories have guided the conception of diagnostic approaches that are based on probabilistic test theory (e.g., MARKO-D) (Ricken et al., 2013) rather than on classical test theory, which was formerly widely used and thus is the basis of most tests hitherto applied. Given a probabilistic theory model, it is intended to represent individual aptitudes of children and item difficulties on the same scale, in order to be able to determine individual competence levels. The probabilistic test theory also provides the basis for approaches to monitoring individual progress. Recently, internet-based realizations of such approaches have been developed (see e.g., Goo, Watt, Park, & Hosp, 2012 for an overview), which can have the advantage of providing comprehensive and up-to-date norms, as well as allowing the user to be independent of specific programs and their maintenance (e.g., updates etc.).

Monitoring individual progress is an essential component of the r*esponse to intervention* (RTI) approach (see e.g., Jimerson, Burns, & VanDerHeyden, 2016), which has attracted attention in discussions surrounding the handling of children with specific educational needs. The RTI approach can be regarded as a method to identify (and remediate) students with learning disabilities. It is an alternative to the ability–achievement discrepancy model, based on which diagnosing a learning disability requires a significant discrepancy between the intellectual abilities and the academic achievement of a child. By repeated assessments, children who do not make adequate progress despite high-quality instruction are identified and then provided with increasingly intensive, multitiered interventions. If these interventions are ineffective, special education placement may eventually be called for. Accordingly, the RTI approach makes use of progress monitoring or frequent assessment of children's achievement level in order to allow for appropriate instructional decisions. Empirical findings suggest that monitoring of individual progress may have beneficial effects on children's competence development (see e.g., Foegen et al. (2007) for an overview regarding progress monitoring measures in mathematics). Especially in the case of marked heterogeneity of individual learning processes in classrooms, which will presumably increase with the recent political endeavors to change schools into inclusive education institutions in several countries, measures of individual progress may become more and more prominent.

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Chapter 38
Diagnostics of Dyscalculia

Johannes E. H. van Luit

Roughly a quarter of the population worldwide experience difficulties with mathematics (Dowker, 2017). This can have major consequences for their further educational career and for their ability to live independently in society. Math problems that are very serious and persistent in nature may indicate developmental dyscalculia. Although there is inconsistent use of terminology in the literature, researchers agree that dyscalculia refers to the existence of a severe disability in learning mathematics. Ruijssenaars, Van Luit, and Van Lieshout (2016, p. 28) defined dyscalculia as a disorder characterized by persistent problems with learning and fluency and/or accurate recall and/or application of mathematical knowledge (facts and understanding). The prevalence of dyscalculia is estimated to be between 2% and 3% in students in the Netherlands (Ruijssenaars et al., 2016). Percentages are higher in international research (3–8%), depending on how researchers define a mathematical disorder or dyscalculia (Desoete, Roeyers, & De Clercq, 2004; Dowker, 2005; Shalev, Manor, & Gross-Tsur, 2005). The disability can be highly selective, affecting learners with normal intelligence (e.g., Landerl, Bevan, & Butterworth, 2004), although it also co-occurs with other developmental disorders, including reading disorders (Ackerman & Dykman, 1995; Gross-Tsur, Manor, & Shalev, 1996) and attention-deficit hyperactivity disorder (ADHD; Monuteaux, Faraone, Herzig, Navsaria, & Biederman, 2005).

The World Health Organization (WHO) initiated the “International Classification of Diseases” (ICD). ICD is the foundation for the identification of health trends and statistics globally and the international standard for reporting diseases and health conditions. It is the diagnostic classification standard for all clinical and research purposes. In the ICD-10 (version 2016), dyscalculia is mentioned as “specific disorder of arithmetical skills” (code: F 81.2). This classification involves a specific impairment in arithmetical skills that is not solely explicable on the basis of general

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mental retardation or of inadequate schooling. The deficit concerns mastery of basic computational skills of addition, subtraction, multiplication, and division rather than of the more abstract mathematical skills involved in algebra, trigonometry, geometry, or calculus.

Another very common used classification system worldwide is the *Diagnostic and Statistical Manual of Mental Disorders* (DSM-IV-TR; American Psychiatric Association, 2000). The now-obsolete diagnostic criteria for mathematics disorder (code: 315.1) were:

A. Mathematical ability, as measured by individually administered standardized tests, is substantially below than expected given the person's chronological age, measured intelligence, and age-appropriate education.
B. The disturbance in Criterion A significantly interferes with academic achievement or activities of daily living that require mathematical ability.
C. If a sensory deficit is present, the difficulties in mathematical ability are in excess of those usually associated with it.

The fifth edition of the *Diagnostic and Statistical Manual of Mental Disorders* (DSM-5; American Psychiatric Association, 2013) takes a different approach to learning disorders than previous editions by broadening the category, in order to increase diagnostic accuracy and effectively target care. Specific learning disorder is now a single, overall diagnosis, incorporating deficits that impact academic achievement. The criteria describe shortcomings in general academic skills and provide detailed specifiers for the areas of reading, mathematics, and written expression. Diagnosis requires persistent difficulties in reading, writing, arithmetic, or mathematical reasoning skills during the formal years of schooling. Symptoms may include inaccurate or slow and effortful reading, poor written expression that lacks clarity, difficulties remembering number facts, or inaccurate mathematical reasoning. Current academic skills must be well below the average range of scores in culturally and linguistically appropriate tests of reading, writing, or mathematics. The individual's difficulties must not be better explained by developmental, neurological, sensory (vision or hearing), or motor disorders and must significantly interfere with academic achievement, occupational performance, or activities of daily living.

Despite the changes from DSM-IV-TR to DSM-5, it remains necessary to perform extensive diagnostic testing to establish whether dyscalculia is present. Since recent research has increasingly recognized the heterogeneity of dyscalculia by differentiating between underlying cognitive deficits (Kaufmann et al. 2013; Rubinsten & Henik, 2009; Skagerlund & Träff, 2016), identification of dyscalculia does not on its own provide enough information about the educational needs of an individual student with math problems. The Dutch protocol "Dyscalculia: Diagnostics for Behavioural Professionals" (DDBP protocol; Van Luit, Bloemert, Ganzinga, & Mönch, 2014) describes how behavioral experts can examine whether a student has dyscalculia or a severe difficulty in math.

The DDBP protocol deals with three criteria that must be met in order to diagnose dyscalculia (Van Luit, 2012; Van Luit et al., 2014):

Criterion 1: To determine the presence and severity of the math problem
Criterion 2: To determine the math problem related to the personal abilities
Criterion 3: To determine obstinacy of the mathematical problem

In the protocol is also mentioned that in many research a fourth criterion is included: the difficulties already exist before the age of 7 years. For most children this is true, but (high) gifted children are mostly recognized for dyscalculia at a later age.

Differential Diagnosis of Dyscalculia

Kucian and Von Aster (2015) mention that dyscalculia is assumed to be a very heterogeneous disorder putting special challenges to define homogeneous diagnostic criteria. Dyscalculia is a disorder that can be characterized through perseverant problems in the process of learning and easy and/or accurate applying of math knowledge (facts/agreements). The DDBP protocol (Van Luit et al., 2014) contains guidelines and suggestions about the variables that can be investigated, and the methods used, during a diagnostic examination of dyscalculia. Due to its structured and comprehensive nature, the DDBP protocol has now been systematically implemented in most social care settings in education in the Netherlands and Flanders (Belgium). The DDBP protocol deals with three criteria that must be met in order to diagnose dyscalculia (Van Luit et al., 2014).

Criterion 1: To Determine the Presence and Severity of the Math Problem

The criterion of severity is determined by deficiencies in both automated and substantive math skills of the diverse domains.

Criterion of severity:

1a. There is a significant delay in automated math skills as compared to peers and/or fellow children.
2a. There is a significant delay in mastering the substantive math skills of the domains.

To meet criteria 1a and 1b, there has to be at the end of primary school (sixth grade) a delay of at least 2 years on a standard (national) math test. For such a test, this would mean that a student at the end of sixth grade should perform not higher than moderate at a test that is adequate for children at the end of fourth grade. Dyscalculia is rarely diagnosed before the end of third grade. The delay has to be at least 1 year at that moment to meet this criterion. For the diagnosis it is important that math education was given at an adequate level.

Dyscalculia can also be diagnosed in children in secondary school, when there is a delay of at least 2 years in their level of math skills compared to children at the end of primary school. In secondary school, students with dyscalculia also perform weak in, for example, economics, geography, and physics.

To determine criterion 1a, an automation/memorization test is necessary. To determine criterion 1b and clarify the extent of the math delay, the therapist has to conduct a process research on the math level of the student. Process research produces more detailed information than standard tests. The therapist can see what the child can and cannot do, furthermore determining the extent of the delay. The therapist can determine the quality, stability, flexibility, and agility of the knowledge and skills.

Process research can be conducted with problems adopted from a regular math test that is adequate for the skill level of the student. To conduct process research, it is important to start at an appropriate level, not too difficult nor too easy. To determine on which specific level can be started, the therapist needs information from the teacher about the exact math level of the student.

Process research has a twofold goal: examining the way in which a student creates an answer and determining in detail the achievement level on the different subdomains. In process research, often prototypical tasks are used to test hypotheses regarding the nature of the problem of the student. In process research the focus is determining the math delay (or deviance of the norm) in terms of math goals/end terms and the underlying shortcomings in procedures and strategies (including declarative knowledge). The four steps the therapist follows to adjudicate the severity of the math problem are (see for a concrete example of step d the Appendix):

(a) Observing open actions (use of blocks), hidden actions (secretly counting on fingers or with the use of looking to a big corral necklace in the classroom), and task approach during math exams (orientating or immediately starting)
(b) Questioning the strategy chosen by the student to resolve the math task (thinking out loud regarding the problem solution, questioning the strategy chosen)
(c) Variation in different math tasks which are in terms of questioning and level near the most difficult correctly and the easiest wrongly dissolved tasks
(d) Providing help examining the degree in which the student needs which type of help, by means of continuing "the five phases of math help" (see for an example the Appendix). The help is provided at the most difficult tasks (step 3) that students just couldn't solve themselves in each domain. The five phases of help are:

    1. Offering more structure
    2. Decrease of complexity
    3. Giving verbal help
    4. Giving material help
    5. Modelling of step d (demonstrate, associate, mimic)

During process research different levels of tasks will be varied, and no tests will be fully completed. As a consequence, it is not always possible to make precise pronunciations on the exact extent of the student's math delay. This means that the degree of severity is an estimation. For example, a student who cannot count with steps of 25 (matching the math skill level of third or fourth grade), but can count with steps of 5 (matching the math skill level of second grade), therefore a pronunciation of the level of the student regarding this math aspect is possible. The inconsistency in what the student can and cannot do produces important information regarding the ultimate support and treatment needed. It is advisable that the focus of help provided after the diagnostic research is on the specific subjects the student does not master sufficiently. The information provided by school produces supporting information regarding the student's level. Examining the way the student failed or succeeded to solve tasks is possible using the provided test sheets, including scratch paper.

Criterion 2: To Determine the Math Problem Related to the Personal Abilities

This means: There is a significant delay with respect to what can be expected from a child, based on his/her individual development. The cognitive level is mostly determined by an intelligence test. Children with dyscalculia can have an under- or above-average intelligence level. It is not possible to determine dyscalculia when the student has an intelligence score of 70 or below, because in that case, the mathematical skills are expected related to the personal abilities. In case the total IQ score is between 71 and 85, diagnosing dyscalculia has to be done with caution. Mathematics requires a complex skillset, which relies on higher cognitive functions. Therefore it is not realistic to expect from children with an IQ score between 71 and 85 to develop and achieve the same math level as their peers with an average IQ score. This means that, to determine dyscalculia in these children, the math problems should not only be explicable by a lower intelligence score. There should, for example, also be a lag of speed in problem-solving and a deficiency in insight. Note that there is no scientific evidence for ascripting a declaration of dyscalculia to a person with an IQ score between 71 and 85. In that case the lag in mathematical skills needs to be larger (at the end of grade six at least 3 years) than the lag of mathematical skills of a person with an average intelligence score (at the end of grade six at least 2 years).

From an analysis of the student's file, there should appear a specific failure in mathematics. In case the performance in other learning areas is also low, this indicates a general learning problem or a broad learning disorder. Low performances on tests in reading and spelling can also indicate dyslexia as a comorbid disorder. In the latter case, a protocol of diagnostics of dyslexia must be followed to determine whether the student (also) has dyslexia. Dyscalculia can also be diagnosed in children with dyslexia when their mathematical performances are not significantly lower than their (limited) performances on tests for reading and spelling. In that case dyscalculia is diagnosed when there is a discrepancy between the mathematical performances and the intelligence, and it is clear that the mathematical problems are not a consequence of reading problems. Whether the mathematical problems are a consequence of reading problems can be investigated during process research.

Mathematical problems of gifted children often get noticed later than usual. In the beginning of their school career, these children often profit from their good memory which enables them to conceal their lags of mathematical skills by applying fast solving strategies. In the course of primary school, this strategy is not enough anymore and their lag in mathematical skills reveals itself. It is important not to withhold these children a declaration of dyscalculia, because of their sufficient performances in the first math tests at the start of secondary school.

In case a student has a comorbid disorder (e.g., ADHD), his or her math performances have to be considerably lower than usually expected of children with the disorder (see Kuhn, 2015). For other comorbid disorders like dyslexia, autism, and DCD, such lower scores are not necessary.

Criterion 3: To Determine Obstinacy of the Mathematical Problem

This means: There is an obstinate mathematical problem, which is resistant to specialized help. To determine the obstinacy, the third and latter criteria for determining dyscalculia, the structural and specialized help a student received in mathematics is investigated. Leading are the reports of offered help. According to the model of "response-to-instruction," didactic resistance can only be determined with great certainty, when the conditions on all three levels have been complied (Fuchs & Vaughn, 2012). Dyscalculia cannot be diagnosed if criterion 3 has not been complied. This also applies for children in secondary school. To determine if the criterion of didactic resistance has been met, the available school records (student file) need to be evaluated. In addition a process research is necessary to determine whether a student is able to learn new strategies and reproduce them independently.

As mentioned earlier, adequate instruction and practice can be described on three different levels, according to the model of "response-to-instruction." In the description is disclosed whether the instruction and practice given by the teacher were adequate. In the description is also disclosed what was concluded based on the evaluation of the individual education program, and in which way during 6 months, at least 1 h per week (individual or in a little group), help of good quality with mathematics has been given by a qualified therapist. The help with mathematics needs to be based on a start measurement, specific and measurable goals have to be set for the mathematical behavior, and the content needs to match the most basic failure. The effectiveness of specialized extra help with mathematics is evaluated, examining the progress during the period of help. The help has been effective when the student's performance after 6 months on the same test as the start measurement reveals a progress comparable to 4 months of education (see Fig. 38.1). The help in

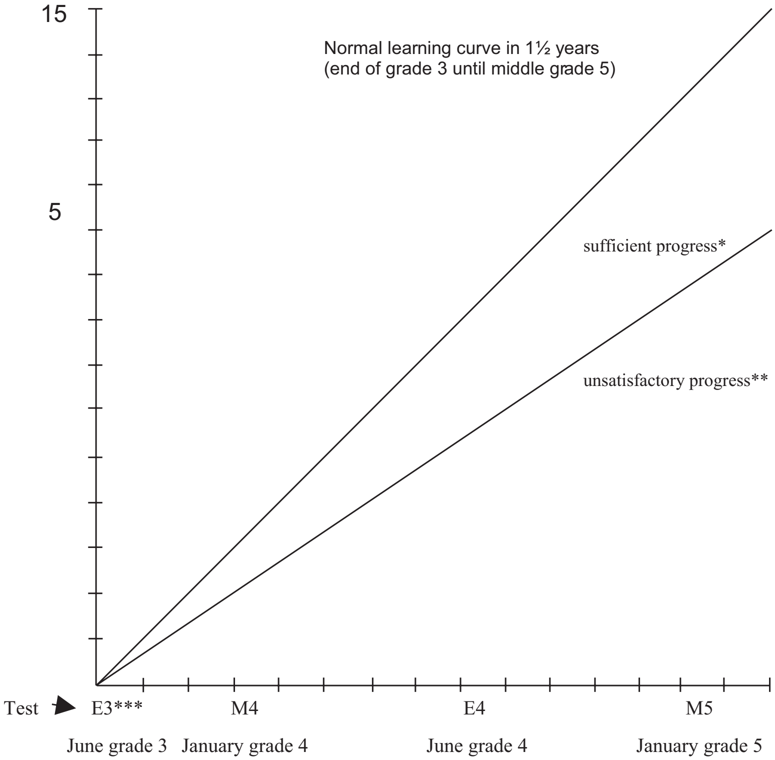

Fig. 38.1 Performance graph for progress in math (example) (Van Luit et al., 2014). * A score in this area or above: the progress is sufficient; ** A score in this area: the progress is not sufficient; ***E = end, M = middle

mathematics has not been effective when the progress after 6 months is less than 4 months. This is about determining whether a student barely benefited from help of good quality. In that case didactic resistance (third criterion) is determined. The above raised questions like "How do we determine the start measurement? What content do we choose, and especially how broad should the offer of content be? Will the student receive mathematics instruction together with his class? After half a year, do we administer the same test (which test)?" Logically it is not possible to determine one direction for every individual case. Per student an individual strategy has to be determined.

For example, a student at the start of third grade starts with half a year specialized help in mathematics. The regular content is too complex for the student. The therapist establishes a baseline at the level of halfway second grade. This means that the student does not get the same content as the rest of the class, but preferably takes math lessons in second grade. The student also could practice mathematics with materials given by the therapist, in its own classroom. What test does the therapist choose to evaluate the effect of the help? Of course the same test suitable for halfway second grade could be administered. This is helpful when determining possible development. If the result is better than expected, then it is also possible to administer a test suitable for children at the end of second grade. However, it is often impossible to tackle every component of mathematics in half a year. In that case the effectiveness of the help can also be determined administering a selective, qualitative test with the components that have been practiced during the period of help.

In case the student has an IQ score between 71 and 85, the didactic resistance is determined when the progress after help of good quality after 6 months is no more than 3 months (which is more or less half of the progress that is expected in children without mathematical problems). Note that to determine didactic resistance, the learning efficiency with regular mathematics instruction is less than 67% (Fig. 38.1). This is also the criterion with adequate math help.

Literature is missing clinically oriented research involving an adequate number of students diagnosed with persistent mathematical learning disabilities (i.e., dyscalculia). The limited amount of research into dyscalculia is largely due to issues of feasibility and generalisability. The difference between mathematics disorder and dyscalculia seems to be the gravity. About the lowest 25% of the children with a mathematics disorder (this is the case for 1 out of 10 children) does have dyscalculia. The other 75% has also severe difficulties, but doesn’t meet all three criteria for dyscalculia fully. What is called as dyscalculia in one study may be conceptualized as a form of mathematical disorder in another study (Kaufmann et al., 2013).

Process Research

In process research help is enhanced gradually to determine whether a student is learnable concerning mathematics. In case of systematic errors (classification), using a number of selected tasks in process research is investigated what the reasons for those errors are and what the starting points are to offer help (indication analysis).

In process research the “five steps of help in mathematics” have been mentioned earlier. Some children find it difficult to deduce the relevant information from the task context. For those children offering more structure can be helpful. In case children have difficulty when it comes to big numbers, the complexity can be reduced to see if that enables a student to solve the task. There are also children who benefit from verbal support while solving the task. This can be helpful to more or less guiding the student from one step to another. Other children benefit from material support, using, for example, cubes or beads (corral necklace), but this is also called visualizing the task. In particular when a student has a weak working memory, in this manner he always has the necessary information at his disposal. When a student has no idea how to solve a task, modelling the strategy could be helpful. In that way a student can learn the most efficient strategy for a specific type of task (Van Luit et al., 2014). In the Appendix several tasks are elaborated (per (sub)component) and the different levels of help are demonstrated. This way insight is given in how the diagnosticians can variate in levels of offering help. The target is, on one hand, to determine how much the student benefits from the offered help and, on the other hand, to determine explanations for the difficulties the student has with mathematics and what his or her strengths are. It is important to get an insight in what level of help is adequate for the student. By variating the levels of help, it is possible to investigate what is and what is not helpful for the student (Van Luit et al., 2014).

Learnability

The process research is ideal to investigate the learnability of a student. Preferably a process research takes place two times in a diagnostic investigation with 1 or 2 weeks in between. To clarify the learnability, it's advisable to look for the zone of proximal development. This concept concerns investigating the level of mathematics at which an individual student can solve tasks independently and how much and of what intensity a student needs help when the complexity of tasks is slightly enhanced. By offering help (using the five phases of help) a few proceedings can be practiced with the student. At the second research moment could be checked in which quantity the student has remembered the taught proceeding and is able to reproduce it. The ease with which the student can reproduce the strategies after 1 or 2 weeks with similar tasks is an indication for learnability. We realize that it is hard to enforce during practice, but we strongly advice to proceed. For some children the taught strategies appear to be unfamiliar, even when the strategies have been practiced intensively. In that case their learnability appears to be small. Other children are capable of reproducing the taught and practiced strategies with similar tasks, which indicates at least some learnability.

Math Problems in Early Education

Children learn many mathematical skills already before formal mathematics instruction starts at primary school. These early numeracy skills, especially counting skills, have been found good predictors for later mathematics performance at first, second, and third grades but are even predictors for math knowledge in secondary school (Siegler, 2009). Therefore in this chapter, there will be also attention for this part of (difficulties in) math education. In addition, in younger children, the tendency to spontaneously focus on numerosities was found a good predictor for counting skills in preschool age. The main goal to highlight early math here is because prevention is more promising than remediation.

As a result far out, most 6-year-old children (at this age most children in Western Europe are going from kindergarten into grade 1 in primary school) have quite well developed early numeracy (primary understanding of amounts and acoustic, asynchronic, synchronic, resultative, and shortened counting; see Aunio, Heiskari, Van

Luit, & Vuorio, 2015), including the ability to make relational statements about numerical and nonnumerical quantity situations and to operate with number word sequence for whole numbers. In the first grade of primary school, children are expected to learn the basic skills of addition and subtraction. The first graders usually operate with numbers between 0 and 100, although the emphasis in addition and subtraction is with numbers 0–20.

In talks with parents and teachers from children who have mathematical difficulties or even dyscalculia diagnosed on a later age (from 9 years on), they indicate that the child with a math difficulty had already problems, especially with resultative and shortened counting during kindergarten (Dowker, 2017). Mostly when children are 9 or 10 years old, they still have difficulties with these stages of counting and they often still count on their fingers. For example, when they have to solve 5 + 3, they count their fingers starting with a dump: 1 and then 2, 3, 4, 5, and further on with the other hand 6, 7, 8. Mostly, they are not able to see five as a whole to count further on. Finger counting and not using 5 and 10 as wholes is characteristically for children who will meet difficulties in math during their school career. Using a lot of time through counting is a sign for difficulties in math (Tobia, Rinaldi, & Marzocchi, 2018). It is not possible to prevent dyscalculia; nevertheless it is possible to observe weak math performance in an early stage of the school career of a child. It is very important to help these low-performing children from an early start point (Toll & Van Luit, 2014a).

Most kindergartners develop early numeracy almost automatically, while for a minority of the children (around 20%), this development is less naturally. Research shows the importance of mastering such skills before children move toward formal math in first grade of primary school (Jordan, Glutting, & Ramineni, 2010). Especially for children who find these skills difficult, it is of great importance to support them adequately. Therefore, specific instruction and exercise in preparatory math skills are necessary in young age.

Dowker (2005) points out that more and more indications are being found that, apart from the possibility of early signaling (Van Luit & Van de Rijt, 2009), treating early mathematical learning problems improves further mathematical education (Gersten, Jordan, & Flojo, 2005; Van Luit & Schopman, 2000). Dowker (2005) states that the prevention of these problems during kindergarten forms a main challenge in research for the following decades. Morgan, Farkas, and Wu (2009) highlight that it is very important to trace problems in early numeracy as early as possible to be able to provide the best possible support at that stage, also when they later turn out to have dyscalculia.

Our own research shows that kindergartners greatly benefit from early detection. It’s not just about determining a score, but more importantly identifying specific deficits. Hereby it is possible to help children specifically at areas where they experience problems (Van Luit, 2011). For this purpose effective programs have been developed (Aunio, Hautamäki, & Van Luit, 2005; Toll & Van Luit, 2014b; Van Luit & Schopman, 2000).

Dyscalculia can’t be prevented with it, but potentially weak math children can learn a lot from it, and the necessary help for children with dyscalculia should be started at a young age for the highest chance on school success.

From Problems at a Young Age to Dyscalculia

Incorrect number sense at the preparatory level and problems with elementary arithmetic in first grade increasingly lead to more limitations in the abilities of children to adequately solve mathematical tasks. Therefore, these difficulties manifest in children with dyscalculia at a young age in gaining early numeracy during the first half of primary school. These basic difficulties become observable when kindergartners, for example, have problems with fluently naming small quantities (using structure), counting, and automatizing number symbols (Van Luit, 2011).

One of the most striking characteristics Dowker (2005) identifies for dyscalculia is a weakness in recollecting numeric knowledge from memory (e.g., in a young age – 6 to 8 years – they do not know that five is between four and six or that adding four to three equals seven). This problem can persist through older age. Furthermore, they keep using (sometimes into adulthood) number lines to solve simple math problems (12 + 6 = 12, 13, 14, 15, 16, 17, 18, while they keep track of additional units using their fingers under the table). These two characteristics, although on a more basic level, are also observable in kindergartners. It is not yet possible to diagnose dyscalculia in kindergartners. However, children who turn out to have dyscalculia from third grade and upward were also among the weakest kindergartners concerning early numeracy.

Literature provides multiple explanations for the cause of dyscalculia (Van Luit, 2015): poor planning skills, a strongly limited capability for using and learning to use correct strategies, having no control over their math actions, lacking good short-term memory, an inadequate knowledge of automatized numeracy, limited knowledge in math, little or no self-confidence, having no confidence in self-improvement, lacking faith in personal growth, and not being open to help from others. On top of that, it appears that memorization or even automatization problems (not being able to remember that 7×8 equals 56 quickly), discrimination problems (not being able to understand that the number 3 in 13 is worth less than the number 1), and thinking problems (not using association to quickly solve 19 minus 7 via 9 minus 7) play an important role in having no or strongly challenged math learning abilities (Tobia et al., 2016).

Therefore, dyscalculia is especially concerned with the failure of declarative knowledge: numeric facts and naming numbers, such as a deficiency in fluently naming numeric information like numbers and quantities (Busch, Schmidt, & Grube, 2015; Fuchs et al., 2005; Landerl et al., 2004; Willburger, Fussenegger, Moll, Wood, & Landerl, 2008). This implies that automatization problems are always present with dyscalculia (see Criterion 1). However, some children practice so intensively that they in the end of primary school are able to remember, for example, multiplication tables or all summations up to 10. Unfortunately, this knowledge is not adaptable and remains fragmented. For example, when multiplying they will know that "4×8 = 32", but they don't know how to solve "14×8" (not seeing by themselves that "14×8" consist of "10×8" and "4×8"). For children with dyscalculia, this will be true already at a young age. They would not see, for instance, that when four toy cars and one toy car are added up, they can continue counting from five toy cars when they add another one. These children will invariably start counting from the first toy car. A deficiency in declarative knowledge almost never stands alone. It will in turn complicate the establishment of procedural knowledge: understanding solution procedures. For example, using the solution procedure for solving “9×6” by subtracting “1×6” from “10×6” assumes that the facts “10×6 = 60” and “60–6 = 54” are known. It also assumes that the student has insight into the act of multiplication, is able to visualize it, has insight into the connection between different multiplication problems, multiplying and adding, and as in this example, even between multiplying and subtracting. When this insight is present, a child will understand how mathematical facts are related and would not need as much automatized knowledge. Factual and procedural knowledge are therefore strongly related.

To sum up the most important characteristics of children with dyscalculia (see also Dowker, 2005):

(a) Regarding declarative knowledge: automation/memorization deficits, namely, problems remembering the basic combinations and easy and/or accurate recalling of math facts from memory.
(b) Regarding procedural knowledge: problems implementing procedures, namely, progressive schemes, applying terms and concepts needed for applying these plans, and the sequence of these steps in complex algorithms.
(c) Regarding visual-spatial processing: problems with notion and conception of space. Problems placing numbers on a number line, mixing numbers in big digits, and problems with geometry and reading time.
(d) Regarding number knowledge: lack of notion of the number system and insufficient knowledge regarding the position value of numbers (not knowing the value of a unit, a dozen, or the value of numbers in fractions above and under the line).

Conclusion

There is need for accurate diagnostics of dyscalculia (Rubinsten & Henik, 2009). To diagnose dyscalculia, three criteria have to be determined: severity, delay, and didactic resistance. To be able to judge whether these criteria are met, different test instruments, process research, and an evaluation of the school records should be used. In addition the clinical therapist is expected to be able to observe during the research moments. It might appear that a student does not benefit from instruction and practice of new strategies on the first research moment and is not able to independently reproduce the strategies. This is an indication for obstinacy. Furthermore it is important to start in kindergarten when children lag behind in match prerequisites. Prevention and early help show better results over time than remediation on a later moment in their school career. Adequate early math education will not prevent dyscalculia, but might help for a better basis.

Appendix

The Five Steps of Math Help

Example

Tim saves for a new drum set. He has already saved € 623.
The drum set will cost € 1017. How much money does he need to” have to save more?

1. **Offering more structure**
   Reorganization of the written task

- Tim has to pay € 1017 for a new drum set.
- Tim already has € 623.
- How much does he need?

2. **Decrease of complexity**.
   Using less complex numbers

- Tim has to pay € 500 for a new drum set.
- Tim already has € 350.
- How much does he needs?

3. **Giving verbal help**.
   Asking questions about the content of the task

- How much money does Tim have?
- What is the price for the drum set?
- Does he have enough money to buy the drum set?
- How much more money does he need?

4. **Giving material help**.
   Presenting the task with help of a picture (see Fig. 38.2).

- As you can see, Tim has € 350.
- The price of the drum set is € 500.
- What is the difference between 500 and 350?
- Which sum is connected with this problem?

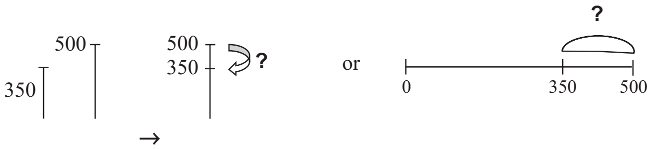

Fig. 38.2 Presenting the task with the help of a picture

- How much more money does Tim need?
- What is the answer for this sum?

5. **Modelling of step 4.**
   Demonstration, doing together, and imitation

- Tim has already € 350, but the drum set is € 500.
- The drum set is more expensive and therefore Tim needs more money.
- The question is how much money does he need.
- This is about the difference between 500 and 350.
- I make, for example, a picture of the number line. I write down 350 and 500.
- The problem is solvable by counting from 350 to 500. First I am doing plus 100 that makes 450 and after that another 50 makes 500.
- The answer is 500 minus 350 makes € 150.
- Tim needs € 150 more.

Take care

If a more easy math task has been done with a good result (in one of the phases of help), more comparable and a little bit more complex tasks can be given after that. For example:

- *The drum set costs € 800. Tim has already € 363 saved. How much does Tim needs more?*

or

- *The drum set costs € 1000. Tim has already € 360 saved. How much does Tim needs more?*

When this was going well, then you can return to the original task.

Take care

If a more easy math task has been done without a good result (in one of the phases of help), more comparable and a little bit less complex tasks can be given.
For example only using hundreds:

- *The drum set costs € 300 and Tim has already € 200 saved. How much does Tim needs more?*

With this more easy task, the same levels of help will be followed.

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Chapter 39
Three Frameworks for Assessing Responsiveness to Instruction as a Means of Identifying Mathematical Learning Disabilities

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The conventional model of learning disability (LD) identification involves calculating a discrepancy between two assessment scores, one measuring intelligence and the other measuring achievement, as an index of the degree to which a student's academic learning is commensurate with his/her academic learning potential. The intelligence-achievement discrepancy has come under attack for technical and conceptual problems (Fletcher et al., 1994), including the delay of identification and intervention until the intermediate grades (Vaughn & Fuchs, 2003). As a result, the 2004 reauthorization of the US Individuals with Disabilities Education Improvement Act (P.L. 108-446) encouraged as a supplementary framework for LD identification.

In the USA, the term *RTI* denotes a school-based prevention model, with an interconnected system of assessments and increasingly intensive levels of intervention. The first level of this RTI prevention framework is primary prevention (commonly known as Tier 1), which comprises the instructional practices general educators conduct with all students: (a) the core instructional program along with (b) classroom routines that provide opportunities for instructional differentiation, (c) accommodations that permit access to the primary prevention program for all students, and (d) problem-solving strategies for addressing students' motivational

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problems that interfere with them performing the academic skills they possess. The major function of RTI assessment within Tier 1 is to identify students who are at risk of not responding to the primary prevention program. The goal is to allocate intervention with greater intensity to these students, whose learning would suffer without it, and provided in a timely manner to circumvent severe, long-term academic difficulty. Primary prevention assessment to identify risk for poor outcomes is typically accomplished using a brief screening test administered to all students (i.e., universally). A cut-point is established based on research for the specific test used (i.e., conducted by the test developer or by the district or school) to classify students in terms of success versus failure on future important outcomes (e.g., success in later grades or passing a future high-stakes test).

When screening reveals the need for more intensive intervention, secondary prevention (Tier 2) services are made available. Tier 2 intervention typically involves small-group instruction that relies on a validated intervention program (i.e., *validation* denotes that studies with high-quality experimental designs support the program's efficacy). The validated program specifies instructional procedures, duration (typically 10–20 weeks of 20- to 45-min sessions), and frequency (3 or 4 times per week). The intensity of Tier 2 intervention differs from primary prevention in two ways. First, Tier 2 intervention is scientifically validated whereas primary prevention is based on research principles. Second, Tier 2 intervention relies on adult-led small-group tutoring whereas primary prevention relies largely on whole-class instruction. Schools can design their RTI prevention systems so students receive just one or a series of Tier 2 intervention programs. In secondary prevention, the major purpose of assessment is to formulate sound decisions about whether students have responded to the intervention and whether students should return to primary prevention without additional support or instead require more intensive intervention. This is typically based on assessment that is conducted at the end of tutoring.

More intensive intervention is at the heart of tertiary (or Tier 3) intervention. Here intervention typically begins with a validated intervention platform (which may also have been used in Tier 2). However, the Tier 3 interventionist collects weekly progress monitoring and uses the resulting data on a regular basis to determine whether progress is adequate for goal attainment and, if not, to make an individualized adjustment to the intensive intervention platform.

A fundamental assumption is that RTI can be used as an LD identification method, in which inadequate learning – in the face of high-quality instruction – is assumed due to an LD, which requires a special form of instruction. LD identification is most frequently assessed at the end of Tier 2, with intensive intervention reserved for students with identified LD. The methods for operationalizing RTI in the 2004 reauthorization were not, however, prescribed.

The purpose of this chapter is to consider three frameworks for operationalizing RTI as a means of identifying mathematical LDs. We begin with the most complex framework, *Systemic RTI Reform*, and then address two more efficient versions: *Embedded RTI* and *Dynamic Assessment*. We describe how each framework is conceptualized and operationalized and explain how each attempts to assure quality instruction and to assess responsiveness to that instruction. We conclude by comparing the strengths and weaknesses of the three frameworks. Although we contextualize our discussion within RTI, readers should note that LD identification, framed as responsiveness to high-quality instruction, can be conducted within other models of RTI or entirely outside a school-based RTI system.

Systemic RTI Reform

In this chapter, we refer to the three-tier RTI framework just summarized as *Systemic RTI Reform*, which involves at least four components. The first is universal screening, with different measures and benchmarks specified for each grade level. These are used to identify the subset of students who are likely to experience poor academic outcomes if their school program is limited to the Tier 1 general education program.

These students receive the Systemic RTI Reform's second major component: Tier 2 intervention. This intervention tier typically relies on a time-limited (i.e., 12- to 20-week) validated small-group program that has been shown, in randomized control trials or in strong quasi-experiments, to accelerate the learning of at-risk learners. It is usually conducted as a supplement to but sometimes replaces parts of the Tier 1 program. As with screening, schools must select a suite of Tier 2 intervention programs to address different grade levels and academic domains. The goal is to boost the student's academic performance to a level that supports his/her return to the Tier 1 program as a full-time participant.

During or at the end of the Tier 2 intervention, the school conducts assessment to inform a decision about whether this has been accomplished, that is, to determine whether the student has responded adequately to Tier 2 intervention. If responsiveness is adequate, the student is deemed ready for full-time Tier 1 participation. Inadequate response indicates the student requires Tier 3 intervention, which differs from or is more intensive than what is provided with a standard (packaged and validated) Tier 2 intervention. As already noted, in many schools, inadequate response to validated Tier 2 intervention also is a major determinant of the LD identification process, which in turn provides the basis for special education.

So, a student with inadequate response proceeds to the fourth major component of the Systemic RTI Reform: Tier 3 intervention. Such intensive intervention involves a specialized teacher (often a special educator), who begins with a validated Tier 2 program as the Tier 3 intervention platform. The teacher, however, does not assume the standard intervention platform will produce adequate response for this student. Instead, the teacher uses ongoing, systematic progress-monitoring data to continuously monitor the student's response to the platform and to adjust the platform over time to ensure a strong match with the student's needs. Thus, an important dimension of Tier 3 intervention is individualization.

These four Systemic RTI Reform components represent a sophisticated, complex design task for school personnel and present schools with demanding implementation challenges (Fuchs, Fuchs, & Compton, 2012b). These implementation challenges are reflected in two recent reports. In an evaluation of 68 schools in one large city in the USA, Ruffini, Lindsay, McInerney, Waite, and Miskell (2016) raised questions about the integrity with which many schools implement Systemic RTI Reform, even following 2 years of adoption. Also, the recent US National RTI Evaluation (Balu et al., 2015) raised questions about the effects of Systemic RTI Reform on student learning, at least in the domain of reading for students in the primary grades (ages 6–10).

In terms of this chapter's focus on assessing mathematical LDs, however, our central questions pertain to the second tier of intervention within Systemic RTI Reform: How is quality instruction assured? And how is response to that quality instruction assessed? Within Systemic RTI Reform, assuring quality instruction requires provision of a *validated* Tier 2 program implemented with fidelity. On the basis of faithful implementation of a validated program, the vast majority of at-risk students are expected to respond adequately. Yet, the availability of validated mathematics interventions to address the full range of curricular targets across all grade levels is limited (Fuchs, Fuchs, & Compton, 2012a), with most validated programs focused on numeration, calculations, word problems, or fractions at ages 6–12. Moreover, fidelity of implementation is not often monitored in school-based implementation, as done in research.

In terms of measuring response to quality instruction at Tier 2, additional research is also still required. Studies do, however, indicate widely varying mathematical LD prevalence and LD severity as a function of the measure used and the method by which response is gauged. The most prominent methods within the Systemic RTI Reform research base are low final performance at the end of Tier 2 intervention or low rate of improvement across the Tier 2 intervention or some combination thereof. In Fuchs et al. (2005), a large-scale randomized control trial examining the effects of a first-grade mathematics intervention focused on numeration, calculations, and word problems, we contrasted these options for operationalizing responsiveness to intervention. Results indicated the following.

For methods based on *low final performance*, the groups of students labeled unresponsive (i.e., LD) manifested uniformly more severe difficulty than responders. However, the prevalence of unresponsiveness was unrealistically high. Also, a problem with relying exclusively on low final achievement to denote unresponsiveness is that it identifies some students as unresponsive even when they make strong improvement over the course of intervention. Therefore, as an alternative, we considered *improvement* as the basis for denoting responsiveness, which provided a more representative but still unrealistically high rate of unresponsiveness. Moreover, relying exclusively on improvement presents its own problem, because it permits students who complete intervention with respectable performance level to be identified as unresponsive. For this reason, we also considered a dual discrepancy, which permits classification as unresponsive only when students experience unacceptable improvement over the course of intervention as well as unacceptable final performance level at the end of intervention. This approach reduced unresponsiveness to a realistic 4% of the general population, while retaining an acceptable degree of severity in mathematics difficulty.

The main point for the present discussion is that findings suggest different methods and measures for designating response to Tier 2 mathematics intervention yield different sets of students designated as having mathematical LDs, with varying prevalence rates and fluctuations in the severity of mathematics difficulty. A lack of standard procedure for using RTI to identify LD creates heterogeneity in the LD population. Such heterogeneity may plague Systemic RTI Reform methods of identification, as occurred with the IQ-achievement discrepancy approach to identification. Before Systemic RTI Reform can represent a tenable method for identifying mathematical LDs, research is needed to achieve concurrence on technically strong methods for designating inadequate response to intervention in ways that forecast the long-term severe difficulty students with mathematical LDs are expected to experience. Alternatively, each validated Tier 2 intervention might be required to provide a research-based cut-point for responsiveness, which corresponds with long-term mathematics success. This issue is further complicated by the fact that many Systemic RTI Reform schools appear to designate responsiveness without any formal measurement of the construct, relying instead on informal judgment. Research is needed to describe what methods schools use and how well informal judgments correspond to students' trajectories of mathematical development.

Embedded RTI

An alternative framework for operationalizing students' responsiveness to quality instruction situates the entire process within the general education classroom. In this chapter, we refer to this framework as *Embedded RTI*. This framework was introduced as an alternative to IQ-achievement discrepancy classification at the 1995 National Research Council Board on Testing and Assessment of the National Academy of Sciences in the USA, and a version of that white paper (Fuchs, 1995) was subsequently published (Fuchs & Fuchs, 1998). To our knowledge, it represents the initial conceptualization of RTI.

An important advantage of the Embedded RTI framework is that because it is situated entirely in the classroom, it does not require "systemic" reform, in which schools design, find resources to support, and monitor the fidelity of an additional Tier 2 intervention capacity not ordinarily found in schools. For schools that struggle to implement a tenable Tier 2 intervention system, available resources might be better allocated toward enhancing the quality of the general education classroom program and thereby creating a context in which responsiveness to quality instruction can be viably assessed.

At the same time, the Embedded RTI framework does instead require implementation of a validated progress-monitoring system school-wide, to produce at least twice monthly estimates of all students' slopes of improvement and performance levels (see Fuchs & Fuchs, 1998 for a full discussion of such progress-monitoring systems, their design, and the technical basis). The Embedded RTI framework requires that LD classification occurs under two conditions that reflect objective documentation of inadequate response to quality instruction. First, the ongoing progress-monitoring data must demonstrate the student is receiving instruction in a classroom with generally adequate progress. This is operationalized as a mean rate of improvement commensurate with grade-level norms for that progress-monitoring tool for the vast majority of students in the class. This is the indicator of quality instruction.

Second, the Embedded RTI framework requires that LD classification be considered only when a student's performance reveals a dual discrepancy from classroom norms for two consecutive half-month intervals: The student performs at least two standard deviations below the level demonstrated by classroom peers *and* demonstrates a learning rate at least two standard deviations below that of classmates. This is the indicator of inadequate response to the quality instruction (to which the vast majority of students in the classroom are making adequate growth).

To illustrate the rationale for this focus on dual discrepancies, we borrow an example from pediatric medicine. The endocrinologist monitoring a child's physical growth is interested in height not only in terms of level but also in terms of growth velocity over time (Rosenfeld, 1982). Given a child whose current height falls below the third percentile, the endocrinologist considers the possibility of underlying pathology and the need to intervene only if, in response to an adequately nurturing environment, the individual's growth trajectory is flatter than that of appropriate comparison groups. For 7-year-olds, for example, large-scale normative data (Tanner & Davies, 1985) operationalize this as an annual growth rate of less than 4 cm. Consequently, the physician judges a 7-year-old who manifests a large discrepancy in height status but who is nonetheless growing at least 4 cm annually in response to a nurturing environment to be deriving the available benefits from that environment to not be an appropriate candidate for special intervention.

The endocrinologist's decision logic reflects three assumptions. The first is that genetic variations underlie normal development, producing a range of heights across the population. The second assumption is that in response to a nurturing environment, a short but growing child does not present a pathological profile indicative of a need for special treatment to produce growth. Instead, this profile suggests an individual who may legitimately represent the lower end of the normal distribution on height, an individual whose development is commensurate with his or her capacity to grow. The third assumption is that under these circumstances, special intervention is unlikely to increase adult height sufficiently to warrant the risks associated with that intervention. Of course, when questions about the quality of the environment arise, the first-level response is to remove those uncertainties by enhancing nurturance, even with hospitalization (Wolraich, 1996), so that growth can be tested under adequate environmental conditions. We return to this point soon.

Applied to LD identification, this decision logic translates into three related propositions. First, because student capacity varies, educational outcomes will differ across the population of learners, and a low-performing child who is nonetheless learning may ultimately perform less well than peers. For example, we do not expect all children to achieve the same degree of mathematics competence.

Second, if a low-performing child is learning at a rate similar to the growth rates of other children in the same sufficiently strong classroom learning environment, he/she is demonstrating the capacity to profit from that educational environment. Additional intervention, therefore, is not warranted even though a discrepancy in performance level may exist. That is, given the benefits being derived from the classroom instructional environment, the student probably does not require a unique form of instruction and probably is achieving commensurate with his/her capacity to learn. Moreover, the risks and costs associated with entering the remedial or special education system are deemed inappropriate and unnecessary in this case because it is unlikely, in light of the growth already occurring, that a different long-term educational outcome could be achieved as a function of that intervention. Of course, the converse is also assumed. When a low-performing child is not demonstrating growth in a situation where others are thriving, LD identification and special intervention are warranted.

The third assumption, however, is that when the vast majority of students in a classroom are not achieving adequate growth rates (compared with local or national norms), one must question the adequacy of that educational environment before formulating decisions about individual student responsiveness and the presence of an LD. Growth under more nurturing environmental conditions must be indexed before any child's need for special intervention can be assessed. In classrooms with poor growth rates, intervention aimed at enhancing the overall quality of the *classroom* instructional program must occur. The school must bring to bear resources to help these teachers reconfigure and expand their instructional environment.

In this way, Embedded RTI is not business as usual. It instead represents its own reform, situated in the general education classroom. It also requires ongoing, objective evidence that a high-quality general education instruction is in effect, tolerating that no more than ~10% (2 of 25 students) experience inadequate growth, before students in that classroom may be identified with an LD. In the context of a classroom in which the vast majority of students are growing, school personnel may be confident that students who are identified for intervention (because their performance level and growth rate are substantially below those of classmates) are not casualties of an inadequate general education classroom environment. Embedded RTI therefore meets both fundamental assumptions of an RTI LD identification system by ensuring identification is based on inadequate response to high-quality classroom instruction.

This does, however, prompt the question: Is it possible to expect classrooms to produce generally adequate growth? After all, in the USA, Systemic RTI Reform expects approximately 20% of students to experience poor learning in the general education classroom (e.g., Vaughn & Fletcher, 2012). In a classroom of 25 students, it thus tolerates five who will require Tier 2. In some locales, rates of Tier 2 intervention in the USA are much higher. For example, in the National RTI Evaluation (Balu et al., 2015), 41% of students received Tier 2 or 3 intervention. This floods the Tier 2 intervention system and compromises its potential to provide meaningful supports.

So it is important to note that research demonstrates instructional routines situated in the general education classroom can substantially expand the range of students who experience success (see http://www.bestevidence.org/ and the forthcoming Evidence for ESSA, http://www.huffingtonpost.com/robert-e-slavin/evidence-and-the-essa_b_8750480.html). For example, in the mathematics Peer-Assisted Learning Strategy (PALS) series of randomized control trials (e.g., Fuchs et al., 1997; Fuchs, Fuchs, & Karns, 2001; Fuchs, Fuchs, Yazdian, & Powell, 2002), control condition (non-PALS) classrooms experienced inadequate growth rates averaging 25%. In PALS classrooms, that percentage was cut in half. In the general education classroom, resources must be dedicated to ensure high-quality instruction with strong instructional core programs and routines for differentiating instruction in a feasible manner (such as PALS), with accommodations that ensure instruction is accessible to all students, and with behavior management systems that guarantee students produce the work of which they are capable.

It is also important to note that some research indicates Tier 2 intervention is not necessary for identifying LDs. Compton et al. (2012), for example, showed that assessment conducted in the fall of first grade can be used productively to move students directly to the more intensive and perhaps longer-duration intervention they require, instead of first requiring students to endure 10–20 weeks or more of failure in Tier 2 intervention. In this sense, for students who eventually will emerge as having an LD, Systemic RTI Reform represents an "RTI wait-to-fail" model that delays the provision of the more intensive intervention these students require and increases RTI costs.

Dynamic Assessment

Dynamic Assessment (DA), which typically requires a 50–60-min testing session, is the most efficient of the three frameworks for assessing a student's response to quality instruction. DA provides instruction in a structured way, and the ease of the student's learning is quantified in order to predict the student's responsiveness to instruction in classrooms. In this way, DA can be used outside an RTI system to formulate LD identification decisions.

As the focus of discussion and research for more than eight decades (Kern, 1930; Penrose, 1934; as cited in Grigorenko & Sternberg, 1998), DA is designed to address the major concern that "static" estimates of learning reveal only two states: unaided success or failure. Yet, as Vygotsky (1962) proposed, children may function somewhere between these states: unable to perform a task independently but able to succeed with assistance.

This has implications for formulating distinctions among students at the lower end of the distribution. For example, when two children earn the same low score on a calculations test, they may not have the same potential to develop word-problem skill. One may succeed in solving word problems with only minimal assistance. This would suggest that the initially low score on the static assessment stems from inadequate learning opportunity in the child's present environment but indicates good learning potential with strong instruction in the future. The other child, by contrast, may struggle to learn word problems even when provided high-quality instruction, revealing a mathematical LD and the need for special intervention.

As noted, DA involves structuring a learning task, providing instruction to help the student learn the task, and indexing responsiveness to the assisted learning phase as a measure of learning potential – in the context of this chapter, as a measure of responsiveness to quality instruction. The DA's instruction is designed with graduated levels of scaffolding. The first level is minimal (for students who require only a simple set of directions explaining what the task requires). For students who require additional scaffolding to learn the unfamiliar task, instruction incorporates principles reflective of high-quality general education, and higher levels of scaffolding mirror more explicit, structured instruction that can be incorporated with differentiated routines (such as PALS). The DA's resulting score indexes the level of scaffolding needed to effect learning.

Research on DA varies as a function of the DA's structure and design and the methodological study features. In the past 15 years, most work has examined the contribution of DA in explaining future academic performance while accounting for competing predictors of outcome. This seems most applicable to the assessment of LDs. For example, Fuchs, Compton et al. (2008) examined the value of a DA at the start of third grade in predicting year-end word-problem skill. The DA's focus, learning three simple algebra skills, was selected for five reasons. First, the algebra skills are of sufficient difficulty that few third graders would solve problems without assistance, yet most beginning third graders should have mastered the simple calculation skills needed to solve the problems. Second, rules underlying the algebra skills could be delineated to construct clear explanations within a graduated sequence of prompts. Third, the sequence of prompts for the three skills could be constructed in an analogous hierarchy, to promote equal interval scaling of the DA scoring system. Fourth, the three skills are increasingly difficult (as established in pilot work), and later skills appear to build on earlier skills, such that transfer across the skills might facilitate better DA scores. Last, algebra requires understanding of the relations among quantities, as is the case for word problems.

The study controlled for other variables potentially important in predicting the word-problem outcome: children's initial word-problem and calculations skill; their language ability, attentive behavior, and nonverbal reasoning; and the quality of instruction students received over the course of third grade (whether students received conventional or validated mathematical problem-solving instruction, which was determined via random assignment). Structural equation measurement models showed that DA measured a distinct dimension of pretreatment ability. It also showed that whereas instructional quality was sufficient to account for word-problem outcome proximal to instruction, the DA, along with language and pretreatment math skills, was needed to forecast learning on word-problem outcomes more distal to instruction.

These findings, along with those of other studies (e.g., Caffrey, Fuchs, & Fuchs, 2008; Seethaler, Fuchs, Fuchs, & Compton, 2016; Swanson & Howard, 2005), suggest DA might serve an important function in predicting responsiveness to intervention, by forecasting distal aspects of learning that are not highly relevant to the present grade level's curricular focus. For example, within RTI models at first grade, supplemental intervention focuses dominantly on the subdomains of numeration and calculations. Yet, some studies (e.g., Fuchs, Fuchs, et al., 2008) show that ~25% of the population experience later word-problem solving difficulty even when calculation skill is adequate. A first-grade DA (e.g., Seethaler, Fuchs, Fuchs, & Compton, 2012) may be used to forecast development of later-emerging forms of mathematical LDs, which would help teachers design more appropriate early intervention, focused on word-problem solving, for such students.

In a different way, DA might be used productively to help schools immediately identify students, after only a single DA session, to receive the level of intensity required for students with mathematical LDs. By contrast, within a three-tier RTI framework, these students would wait 10–30 weeks to ultimately prove unresponsive to the less intensive form of supplemental intervention delivered at Tier 2. With Embedded RTI, demonstrating a dual discrepancy from classroom performance takes a minimum of 4 weeks but can run longer.

Even so, despite promising DA research conducted over the past 15 years, work is required to address the full complement of initial grade levels for the DA administration, designed to predict the full range of mathematical outcomes. These studies also must address the predictive validity of these tools, as assessed in the Fuchs, Fuchs, et al. (2008) study discussed above. Developing a system of technically strong DAs to address the many forms of mathematical learning represents an ambitious undertaking before DA may be used routinely in the assessment of mathematical LDs.

Comparisons across the Three Frameworks

We close this chapter by comparing the three frameworks for assessing responsiveness to high-quality instruction, as a means of identifying mathematical LDs, along four dimensions: ensuring that assessment provides quality instruction, technical strength of responsiveness decisions, feasibility and affordability of the assessment process, and early access to special education for students with LDs.

To provide *quality instruction within the RTI assessment*, Systemic RTI Reform relies on validated Tier 2 intervention programs. This offers the advantage of small-group instructional programs demonstrated to boost the mathematics trajectories of at-risk students but suffers at least three important problems: the paucity of available programs to address the range of mathematics curricular topics across the grade levels, inadequate monitoring of the extent to which Tier 2 interventions are implemented in schools according to prescribed methods used in validation studies, and school schedules that complicate or undermine student availability for supplemental intervention.

Embedded RTI takes a very different tack, which restricts the focus of the RTI assessment to the general education classroom. Here, quality is operationalized not in terms of an analysis of the programs delivered but rather in terms of the adequacy of student learning for the students served in those classrooms. This is achieved via the school-wide ongoing progress-monitoring system. Implementing such a progress-monitoring system represents an ambitious undertaking but pales in comparison to the complexity involved in introducing a system of validated supplemental interventions into school cultures not accustomed to, resourced for, or skilled in an additional layer of services. A disadvantage of Embedded RTI echoes one of Systemic RTI Reform's problems: inadequate coverage of the mathematics curriculum in existing progress-monitoring systems.

Dynamic Assessment avoids the complexity of a Tier 2 intervention system and the challenges of school-wide progress monitoring by encapsulating the quality instruction experience with a self-contained, hour-long, standardized assessment. A tester need only follow the prescribed testing procedures to ensure the delivery of the graduated series of instructional scaffolds that constitute the quality instructional experience. Even so, as with Systemic RTI Reform and Embedded RTI, the breadth of DA's coverage – for the various mathematics domains and across the grade levels – is lacking.

With respect to the *technical strength of responsiveness decisions*, Systemic RTI Reform suffers the most tenuous basis for classifying adequate versus inadequate response to Tier 2 intervention. As discussed earlier, research suggests excessive variability in the groups of students designated as unresponsive as a function of the measures and methods used. Moreover, it is unclear what methods schools use, beyond informal judgments, for distinguishing adequate from inadequate response. At the present time, there is no basis for estimating whether school-based responsiveness decisions represent a reliable or valid basis for sorting students back into Tier 1 classroom participation versus more intensive intervention. By contrast, Embedded RTI and Dynamic Assessment provide clear rules for making such designations, with technical data that support such decision-making. Even so, all three frameworks suffer from few decision utilization studies examining how LD versus non-LD classifications based on RTI data predict or classify students according to their long-term mathematics performance and success.

On the *feasibility and affordability* criterion, DA is the easy winner among these three frameworks for operationalizing RTI as a means of mathematics LD assessment. It can take up to one full hour. This represents a long test. However, it involves one single session and can be conducted by a well-trained school psychologist. Embedded RTI offers the next most feasible and affordable option, by containing resources within the general education classroom and encouraging the reform of classroom instruction to expand the range of its effectiveness. Systemic RTI Reform is clearly the most resource intensive framework for schools to implement, requiring a new level of validated intervention without school personnel who are trained in or supervised and without schools organized for successful implementation.

Finally, in terms of *early access to special education* for students with LDs – a major impetus for the 2004 Reauthorization's focus on RTI – Systemic RTI Reform requires students with mathematical LDs to pass through at least one level of supplemental (less intensive) intervention before gaining access to the most intensive and individualized services available within the school building. Students must demonstrate risk for inadequate response to classroom instruction before becoming eligible for Tier 2 intervention and then must demonstrate inadequate response to Tier 2 intervention before entering the formal LD assessment process and eventually special education services.

Yet, some research indicates that LD students who ultimately fail to respond to Tier 2 intervention can be identified *before* they enter Tier 2 intervention. This avoids (a) an extended period of waiting-to-fail before gaining access to services with the appropriate level of intensity and (b) the costs associated with providing ineffective Tier 2 intervention. Embedded RTI shortens this process, with its exclusive focus on response within the classroom instructional program. It does, however, require schools to implement a systematic data-based process of ensuring a classroom is providing quality instruction, as evidenced via generally strong student learning, before LD identification may occur. DA provides the most direct and fastest route to the appropriate level of intensity for students with mathematical LDs.

This analysis of strengths and weaknesses provides the basis for a systematic program of research designed to describe relevant school practices, assess the accuracy and feasibility of methods for assessing responsiveness to quality instruction, examine the long-term decision utility of responsiveness decisions, systematically consider the feasibility and affordability of contrasting assessment processes, and determine how alternative methods affect early access to special education services for students with mathematical LDs.

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Chapter 40
Technology-Based Diagnostic Assessments for Identifying Early Mathematical Learning Difficulties

Gyöngyvér Molnár and Benő Csapó

Introduction

The work presented in this chapter is located in the overlapping area of four research and development domains of education which have recently received growing attention. (1) Information communication technologies (ICT) have proliferated in all areas of life, including school learning. The ubiquitous ICT has made it more realistic to transfer all assessment to computerised platforms; therefore, *technology-based assessment* can be widely utilised to support everyday educational processes (Csapó, Ainley, Bennett, Latour, & Law, 2012). (2) Adapting education to the individual needs of students, and thus giving special support to those who really need it, has always been an intention of educators, but such a goal required assessment instruments that could diagnose students' difficulties early enough and monitor their progress. Therefore, assessment *for* learning, i.e. formative and *diagnostic assessment*, has recently become a dominant field within the research on educational assessment (Black & Wiliam, 1998), especially technology-based assessment. (3) Research has shown that preschool development and the first school years determine later success (Aunola, Leskinen, Lerkkanen, & Nurmi, 2004; Watts, Duncan, Siegler, & Davis-Kean, 2014); therefore *early childhood education* is one of the most rapidly growing areas in educational research, which development is strongly supported by the means of technology-based formative and diagnostic assessment. (4) Finally, mathematics is one of the most important school subjects;

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success in learning it has a strong impact on a number of other areas of education, including science. Due to its importance in modern societies in everyday life as well as in science and technology-related professions, *mathematics education* has become one of the focal areas in improving educational systems. Besides reading and science, it is one of the three most frequently tested domains, both in international and national assessment programmes, thus attracting broad public attention.

There are many initiatives in progress, and a number of computer-based tests are available in the field of mathematics, but they are mainly developed for summative assessment, as well as the large-scale international (OECD PISA, IEA TIMSS) and national assessment programmes (e.g. MAP, Missouri Assessment Program, Missouri; SOL, Standards of Learning, Virginia; OAKS, Oregon Assessment of Knowledge and Skills, Oregon; SBAC, Smarter Balanced Assessment Consortium; PARCC, The Partnership for Assessment of Readiness for College and Careers). There are much fewer formative or diagnostic tests available, especially for measuring younger students' mathematics knowledge and skills. In general, there is a lack of research-based online diagnostic mathematics tests available for everyday classroom applications. Although there are several initiatives for online assessments (see, e.g. Pearson's MyMathLab (n.d.); Let's Go Learn (n.d.); The Diagnostic assessment part of PARCC (n.d.); Math Garden from the University of Amsterdam (n.d.); PAT: Mathematics in New Zealand (n.d.)), but these are all commercial products not completely and freely available for students and teachers.

As an exception, Panamath is available for free, but it is measuring only one part of mathematics' knowledge, students' approximate number system (ANS) aptitude. In the Panamath tasks, students are presented two sets of dots (blue and yellow), and they have to decide in a brief flash whether the number of blue or yellow dots is greater. The result tells about the accuracy of the test takers' basic sense for numbers. It can be used across the entire lifespan from 2-year-olds to old adults (DeWind & Brannon, 2016).

As there are large differences between students in a number of dimensions, successful mathematics education, especially in the first school years, requires differentiated and personalised teaching. This includes early identification of learning difficulties, frequent feedback, individualised well-targeted interventions, and continuous monitoring of development. An assessment system which can diagnose learning difficulties and can be used frequently enough must be built on a deeper understanding of students' developmental processes, the impacts of mathematics education on it, and the organisation of students' knowledge in general.

The first part of this chapter presents the advantages and possibilities of technology-based assessment. It describes how technology and its advantages initiated to rethink the purpose of assessment focusing more on diagnostic instead of summative assessment and realising efficient testing for personalised learning. The second part of the chapter summarises the scientific foundations for the diagnostic assessments. The theoretical foundations of framework development have resulted in a three-dimensional framework that outlines mathematics learning and the development of mathematical abilities and skills in three dimensions. These dimensions cover students' psychological development, the applicability of their knowledge, and the curricular content of teaching. To implement the diagnostic assessment, a complex online platform called *eDia* has been constructed to support the entire assessment process from item writing through item banking, test delivery, and storing and analysing the data to providing feedback to students and their teachers. The third part of the chapter shows how the mathematics framework has been mapped into an item bank containing over a thousand items by dimension for the first six grades of primary school. The diagnostic assessment system has been offered to schools for application in everyday practice. The fourth part shows the implementation process, some early results from field testing, scaling issues, and framework validation. Finally, the last part discusses how the system can be further developed and how it can be integrated into everyday educational processes to support personalised education and provide customised support for atypical learners of mathematics.

Advantages and Possibilities of Technology-Based Assessment: The Move from Summative to Diagnostic Assessment to Realise Efficient Testing for Personalised Learning

The most prominent educational developments of the past few decades have been aimed at establishing the feedback mechanisms of different levels of educational systems. Therefore, both the theory and the practice of educational assessment have seen considerable advances. Large-scale international assessments have become regularly administered by collaborative teams of experts of the leading test centres of the world. As a result, a huge improvement of data transfer technology and data analysis methods could be witnessed. Systems of assessment and evaluation in national contexts taking into account both the international trends and the local characteristics have been gradually set up. Due to the rapid development, the means of paper-based assessments most widespread and accepted at the millennium imposed serious constraints on their usability. To facilitate potential improvement and meet the twenty-first century needs of the new kinds of assessment and evaluation, an essential qualitative change had to be made (Scheuermann & Pereira, 2008). The direction of the change was mainly determined by technology. The fact that technology has developed, spread, and become accessible offers extraordinary opportunities for the improvement of the practice of educational assessment. Applying technology allows more exact and more varied testing procedures of significantly more complex skills and abilities by devising tasks in more realistic, application-oriented, and authentic testing environments than those of the earlier, paper-based assessments (Beller, 2013; Bennett, 2002; Breiter, Groß, & Stauke, 2013; Bridgeman, 2010; Christakoudis, Androulakis, & Zagouras, 2011; Csapó, Ainley, et al., 2012; Farcot & Latour, 2009; Kikis, 2010; Martin, 2010; Martin & Binkley, 2009; Moe, 2010; Ripley, 2010; van Lent, 2010). Its effectiveness and the increase of effectiveness under certain conditions could be detected on every level of assessment and evaluation.

- The economy of testing (Bennett, 2003; Choi & Tinkler, 2002; Farcot & Latour, 2008; Peak, 2005).
- The diversity of test editing and development (Csapó, Ainley, et al., 2012) and the speed of test administration and data flow (Csapó, Lőrincz, & Molnár, 2012).
- The opportunity to provide instant, objective, and standardised feedback (Becker, 2004; Dikli, 2006; Mitchell, Russel, Broomhead, & Aldridge, 2002; Valenti, Neri, & Cucchiarelli, 2003).
- The motivation of the students for testing changes (Meijer, 2010; Sim & Horton, 2005).
- Innovative item development opportunities, multimedia, dynamic, and interactive items, applying second- and third-generation tests (Pachler, Daly, Mor, & Mellar, 2010; Strain-Seymour, Way, & Dolan, 2009), which were impracticable in a paper-based form (Molnár, Greiff, Wüstenberg, & Fischer, 2017).
- An adaptive test algorithm has become available, which allows a more exact assessment of levels of knowledge and skills and abilities (Frey, 2007; Jodoin, Zenisky, & Hambleton, 2006).
- The circle of test takers could be extended (e.g. audio version of tasks and instructions could be played, which makes testing of children who cannot read possible) (Csapó, Molnár, & Nagy, 2014).
- Technology serves as an effective means of logging and analysing contextual data (e.g. the time needed for the execution of a task could be measured; besides the number of attempts made by the student to modify their solutions, the number and location of a student’s clicks during a test could also be mapped) (Csapó et al., 2014). Consequently, instead of the only indicator used in paper-based testing, which is the test result, a rich and well-structured database is available, which makes a more thorough following and analysis of the student’s movements and behaviour possible during the test (Molnár & Lőrincz, 2012).
- Indicators of test goodness criteria could increase (Csapó et al., 2014; Jurecka & Hartig, 2007; Ridgway & McCusker, 2003).

Although approaching the problem from different perspectives, major relevant research and development projects in an international context (e.g. *Assessment and Teaching of 21st Century Skills* – ATC21S, Class of 2020 Action Plan; Griffin, McGaw, & Care, 2012; SETDA, 2008) have all agreed that the direction for improvement could be computer-based testing exclusively (Csapó, Ainley, et al., 2012; Pearson, 2012; Scheuermann & Björnsson, 2009). Today computer-based assessment permits more effective assessments than traditional face-to-face or paper-based testing. Therefore, within a reasonably foreseeable time, all important assessment will probably be put on a technological basis. International summative tests have already shown such a tendency. Furthermore, given the opportunity to provide instant feedback on assessments, besides the predominantly summative approach, recently, there has been an emphasis on individualised diagnostic testing in order to enhance fast and effective learning by means of exploiting the learning supporting function of diagnostic testing (Kettler, 2011; Redecker & Johannessen, 2013; Van der Kleij, Eggen, Timmers, & Veldkamp, 2012). Traditional paper-based tests are not suitable for diagnostic assessment, which bottom line is sufficiently frequent student assessment. The development of technology together with that of assessment and evaluation in the past 15 years has created numerous new opportunities in early childhood assessment, which so far have mainly been based on individual data collection (Csapó et al., 2014).

Theoretical Foundations of Framework Development: A Three-Dimensional Model of Mathematical Knowledge

In the history of mathematics education, three perennial goals have remained clear from the very beginning of the history of schooling up to present-day approaches. To create a diagnostic assessment system which can precisely identify students' weaknesses and strengths, a framework must be created which clearly distinguishes these three directions, three types of goals.

Cultivating general cognitive abilities has always been one of the main declared goals of learning mathematics. Adjusting learning to students' mental development is a precondition of successful teaching, while obtaining feedback on how maths teaching stimulates the developing mind requires regular testing. To create assessment instruments to meet this goal, psychological processes must be studied.

Another obvious goal is that mathematics education should provide learners with practical skills applicable outside the school context. Seneca's often cited aphorism, "Non scholae sed vitae discimus", expresses the expectations of modern societies as well, and this aim, making mathematics education more relevant for the average learner, is embodied in national and international assessment projects.

Finally, mathematics is one of the oldest and best organised bodies of human knowledge. As Banach has formulated his admiration, "Mathematics is the most beautiful and most powerful creation of the human spirit". To comprehend the organisation of this branch of knowledge, students must study mathematics as a discipline, including its specific terminology, its axioms, theorems, definitions, proofs, etc. Another set of goals can be deduced from this need which can be further shaped taking into account the educational requirements of those students who prepare to be professional users of mathematics, becoming research mathematicians or dealing with high-level applications in a number of other areas of research and development.

A Three-Dimensional Model of Students' Knowledge for Diagnostic Assessment in Early Education

The arguments for assessing students' progress in three dimensions may be further elaborated by analysing some national and international assessment frameworks. Large-scale international assessment programmes publish their frameworks well before the actual assessment. The first international assessment programmes in mathematics were conducted by the IEA (International Association for the Assessment of Educational Progress) in the early 1970s and 1980s, and the assessments became regular since 1995 under the acronym TIMSS (Trends in International Mathematics and Science Studies). The early IEA assessments focused on the curricular content of mathematics teaching and were closer to the disciplinary view of mathematics. Although the curricula in the participating countries remained the primary source of content for recent TIMSS assessments, they distinguish the content domains (covering the main domains of mathematics as a discipline) and the cognitive domains which are knowing, applying, and reasoning (Mullis & Martin, 2013).

The other large-scale international programme launched in 2000 under the aegis of the OECD, PISA (Programme for International Student Assessment), aims to assess the knowledge and skills that students are expected to possess at the age of 15 to be prepared for the challenges they will face in modern societies. To characterise the type of broadly applicable knowledge, PISA extended the conception of *literacy* and termed the assessment domains reading literacy, mathematical literacy, and scientific literacy. For mathematical literacy, a novel definition was developed:

> Mathematical literacy is defined in PISA as: the capacity to identify, to understand, and to engage in mathematics and make well-founded judgments about the role that mathematics plays, as needed for an individual's current and future private life, occupational life, social life with peers and relatives, and life as a constructive, concerned, and reflective citizen. (OECD, 2000, p. 50)

Based on this definition, the framework was elaborated in three dimensions, dealing with *mathematical processes*, *mathematical content*, and *situations and contexts* of applying mathematical knowledge. Both the definition of mathematical literacy and the detailed framework proceeding from it as well as item development placed much stronger emphasis on the application of knowledge as the disciplinary content and the mathematical processes were embedded in contexts and situations relevant for young students living in developed societies. Over the assessment cycles, the conception of mathematical literacy further evolved, and its core idea remained very similar to the original:

> Mathematical literacy is an individual's capacity to formulate, employ and interpret mathematics in a variety of contexts. It includes reasoning mathematically and using mathematical concepts, procedures, facts and tools to describe, explain and predict phenomena. It assists individuals to recognize the role that mathematics plays in the world and to make the well-founded judgments and decisions needed by constructive, engaged and reflective citizens. (OECD, 2016, p. 65)

Following the traditions of framework development in international assessment projects and taking into account several further theoretical considerations (see Csapó, 2004, 2010) and empirical results (Csapó, 2007), a three-dimensional model of teaching and learning goals was proposed. This approach (outlined in Fig. 40.1) assumes that these three aspects of teaching should be present at the same time in school education, to develop the intellect and to cultivate thinking and general cognitive abilities. These goals must not exclude each other, and they should not compete for teaching time. Focusing on one of these goals, e.g. teaching the disciplinary

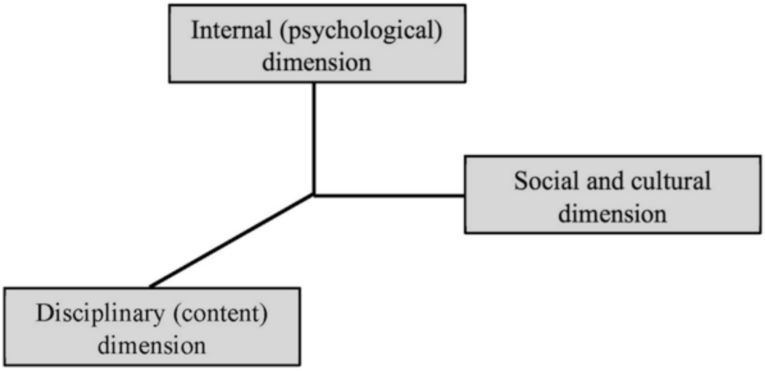


Fig. 40.1 A three-dimensional model for developing a framework of diagnostic assessments. (Csapó, 2007)

content (which traditionally happens in many education systems), is not satisfying in modern societies; students are expected to apply their mathematical knowledge in a broad variety of contexts (as PISA assesses it), and they should be able to solve problems in unknown, novel situations (as was assessed, e.g. in PISA 2012 in the domain of problem-solving, see OECD, 2014). These goals (teaching disciplinary content knowledge in mathematics, preparing students to apply it in a broad range of contexts, and developing thinking skills; see Csapó & Szendrei, 2011) have been competing with each other for teaching time over the past few decades. One or another became from time to time dominant in the curricula; however, they should receive equal attention for interacting and reinforcing each other.

The model in Fig. 40.1 has been elaborated for each assessment domain taking into account the specific characteristics of the particular domain and has been published in three parallel volumes (see Csapó & Csépe, 2012 for reading; Csapó & Szendrei, 2011 for mathematics; and Csapó & Szabó, 2012 for science). The similarities and differences of these frameworks highlighted the specific roles each domain plays in education. Reading is the basis for all further learning, including mathematics, while mathematics provides foundations for learning certain sciences. Further developmental work (creating items and carrying out assessments with them) based on this three-dimensional framework indicates the validity of the approach in educational practice.

As for mathematics, each dimension has been separately considered and elaborated in detail in the light of literature from the particular field of research. It is of great use to separate these different dimensions in diagnostic assessments because a precise identification of areas of delayed differences is a precondition of personalised interventions. The scope of studying these dimensions is also different. The roots of the psychological development of mathematical reasoning may be universal as far as early neurocognitive development in children is alike across cultures and societies. Studies related to the application dimension can mostly be shared with researchers dealing with the contexts and expectations of developed countries, while the curricular content is related to the national educational system.

The *psychological dimension* has been conceptualised as the interaction between students' cognitive development and learning mathematics at school (Nunes & Csapó, 2011). The questions in this dimension are how well mathematics education is adjusted to students' psychological development, on the one hand, and how learning mathematics can contribute to the development of specific reasoning skills and how effectively it stimulates students' general cognitive development, on the other. Research in this field provides rich resources ranging from the classical works of Piaget (see, e.g. Inhelder & Piaget, 1958) to the most recent neurocognitive studies. A long list of skills can be taken into account in this field that are strongly embedded in psychological development, such as counting skills, additive and multiplicative reasoning as well as spatial, probabilistic, combinatorial, and proportional reasoning, and so on. Assessments of a number of such skills are especially crucial at the beginning of schooling and in the first school years, as their developmental level determines later success (see Nguyen et al., 2016).

The *application dimension* of the goals of learning mathematics is interpreted as mastering mathematical literacy, the type of skills that make mathematics useful in areas other than the immediate school context. Mathematics is applied in a number of areas, ranging from other school subjects to a broad cross section of everyday life (Csíkos & Verschaffel, 2011). The key questions in this field are how students can construct mathematical models of problems they face and how well they can mobilise mathematical knowledge to solve those problems. Transfer of knowledge to new contexts is not automatic, and children must learn and practise applying their knowledge. Research on realistic mathematical modelling is the most useful source for elaborating the assessment framework of this dimension (see, e.g. Verschaffel, De Corte, & Lasure, 1994). The tasks that can be taken into account for the measurement of this dimension range from pseudo-real-world to real-world problems which embed mathematical knowledge in a number of relevant contexts and real-life situations.

The disciplinary dimension can be defined as the mathematics content knowledge described in the national core curriculum. This is the prescribed content on which textbooks, local curricula, and teachers' actual work are based (Szendrei & Szendrei, 2011). A precise translation of the core curriculum into an assessment framework and later on into test tasks makes it possible to monitor how students progress with their daily mathematics studies. Previous research has indicated that mastering and reproducing the immediate teaching material does not necessarily have a long-term impact on students' cognitive development (see, e.g. Csapó, 2007), but for a precise diagnosis, it is necessary to know if students actually learn what they are expected to in mathematics lessons.

Teaching students disciplinary content knowledge in mathematics, preparing them to apply it, and developing their thinking skills are not considered as exclusive alternatives but processes that reinforce and interact with each other. That is, education must achieve these objectives in an integrated way, but for diagnostic purposes the tests must be able to show if there is insufficient progress in one or another of these dimensions, thus they should be treated as distinct dimensions in diagnostic assessments.

Taking this principle into account and considering the specific aspects of early education and the diagnostic orientation of the assessment, the former tridimensional

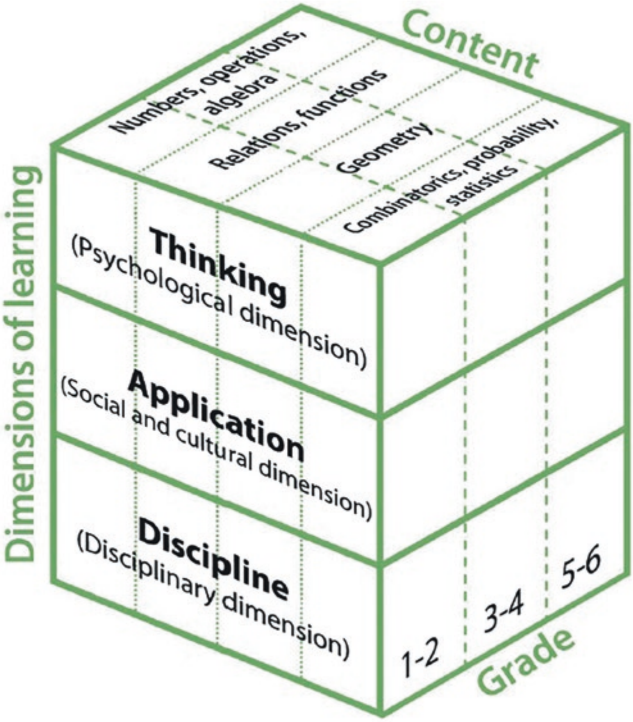


Fig. 40.2 The model of mathematical knowledge to develop the framework for diagnostic assessment in Grades 1–6. (Csíkos & Csapó, 2011)

model was further developed and used as a foundation for item development. The continuum of the first six grades has been divided into 2-year sections, and the test items have been prepared to cover these periods (Csíkos & Csapó, 2011, see Fig. 40.2).

The three-dimensional approach indicates that these aspects of learning are not independent of each other. Disciplinary content is the means of developing students' reasoning skills, and this is what students are then expected to apply in other contexts. The following sections show how items were developed for these dimensions, how students' knowledge is measured in these dimensions, and how their disciplinary knowledge, reasoning skills, and applicable knowledge are related.

Creating an Assessment System: Online Platform Building and Innovative Item Writing

Based on the model of mathematical knowledge described in the previous section, an item bank was constructed for diagnostic assessments. This item bank contains 6182 tasks (each task consists of several items) to measure disciplinary content knowledge in mathematics (MD; $n = 2119$), mathematical reasoning (MR; $n = 1965$), and mathematical literacy (MA; $n = 2098$) in first to sixth grades (age 6–12). The content of the assessment as a function of the three dimensions of learning and target population is shown in Fig. 40.2.

The tasks were grouped into clusters (4–5 tasks per cluster), meaning 15–20 items per cluster for the lower grades and 20–25 items for the higher grades. One 45-min test contains at least three clusters (at least 45–50 items).

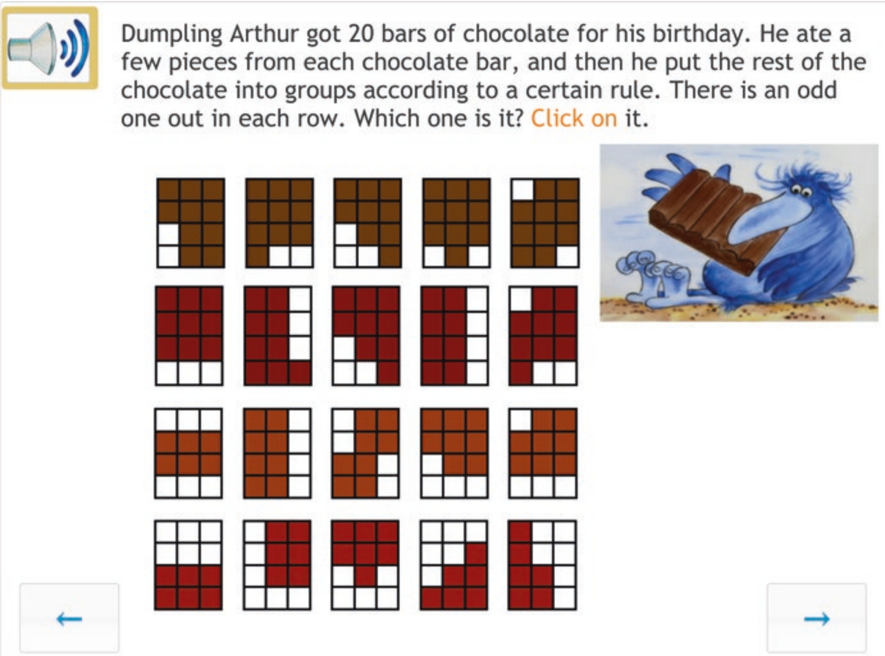
In the first to third grades, instructions are provided both in written form and online by a prerecorded voice to prevent reading difficulties and ensure the validity of the results. Thus, students must use headphones during the administration of the tests. After listening to the instructions, they must indicate their answer using the mouse or keyboard (in the case of desktop computers, which is the most common infrastructure in the Hungarian educational system) or directly tapping, typing, or dragging the elements of the tasks with their fingers on tablets. It takes no more than 45 min (one school lesson) to complete the test.

At the beginning of the tests, participants are provided with instructions, including a trial (warm-up) task with immediate feedback, in which they can learn how to use the programme: (1) at the top of the screen, a yellow bar indicates how far the have advanced in the test; (2) they must click on the speaker to be able to listen to the task instruction; (3) to move on to the next task, they must click on the "next" button; and, finally, (4) after completing the last task, they receive game-based immediate feedback with one to ten balloons depending on their achievement. The better their results are, the more balloons they will see over Piglet's head. The immediate feedback also contains their achievement in each dimension of knowledge.

The feedback system, which is available for the teacher, is more elaborated. As the tasks in the item bank have been scaled by means of IRT, students' achievement can be objectively compared. Teachers receive feedback on students' achievements both in percentage and in ability scores, which are comparable to each other and also contain a point of reference to the national standards. In each of the grades and fields, the national-level average achievement was transferred to 500 points with a 100-point standard deviation, which constructs the point of reference to the students' achievement.

Mathematical Reasoning Items

Based on the framework for the diagnostic assessment of mathematics (Csapó & Szendrei, 2011), reasoning items encompass the measurement of inductive reasoning, deductive reasoning, combinative reasoning, systematisation skills, and proportional reasoning. The task presented in Fig. 40.3 combines the mathematical concept of whole numbers with the assessment of students' inductive reasoning skills within the context of a famous Hungarian cartoon. In the task, students must



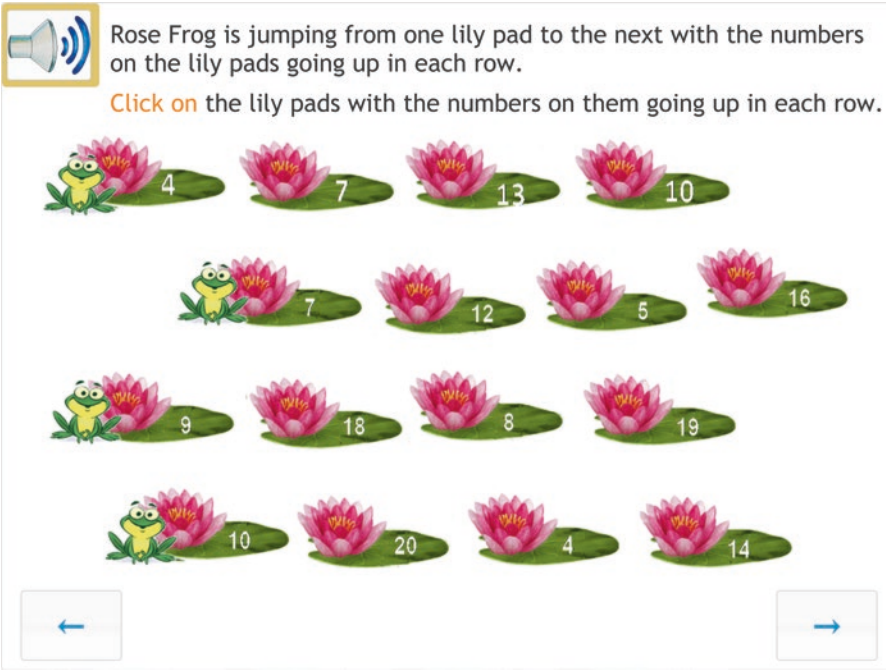
Dumpling Arthur got 20 bars of chocolate for his birthday. He ate a few pieces from each chocolate bar, and then he put the rest of the chocolate into groups according to a certain rule. There is an odd one out in each row. Which one is it? Click on it.

Fig. 40.3 Mathematical reasoning task: combining the mathematical concept of whole numbers with the assessment of students' inductive reasoning skills in a familiar Hungarian cartoon

discover regularities by detecting dissimilarities with respect to attributes of different objects. In this operation, they must use their knowledge of quantities and their understanding of the relations of greater than, less than and the same. According to Klauer's definition (1993) of inductive reasoning, students must use the operation of discrimination in this item. In the present case, students can provide their answers by clicking on the "odd-one-out" element, scoring a maximum of 4 points, one in each group of chocolate bars. As demonstrated, inductive reasoning tasks are often connected to other areas of mathematics, in this case to whole numbers and computation.

In the task presented in Fig. 40.4, students' systematisation skills and their level of understanding of the number concept are assessed. The formulation and development of the number concept must be supported from three directions: number symbols, the name of the numbers, and the quantities indicated by the numbers.

The tasks, which support the connections between these representations, are suitable for diagnostic purposes for the reasoning dimension of counting. The present task provides an example of the combination of number symbols and quantities in a reasoning context. Students need to recognise number symbols and then connect them to quantities and place them in increasing order by clicking on the numbers. In short, the order of the clicking was evaluated.



Rose Frog is jumping from one lily pad to the next with the numbers on the lily pads going up in each row.

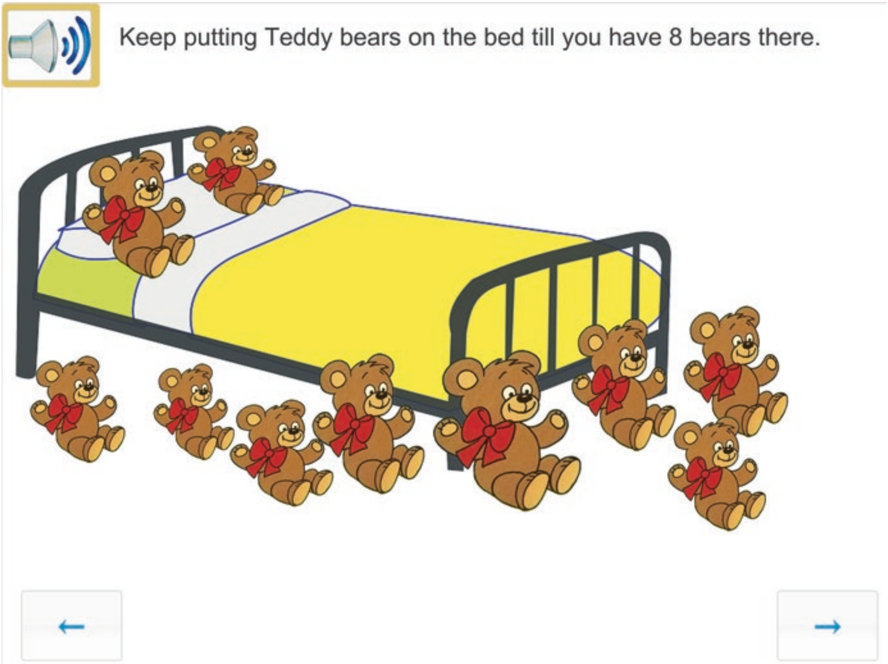
Click on the lily pads with the numbers on them going up in each row.

Fig. 40.4 Mathematical reasoning task: recognising and combining number symbols and quantities

Mathematical Literacy Items

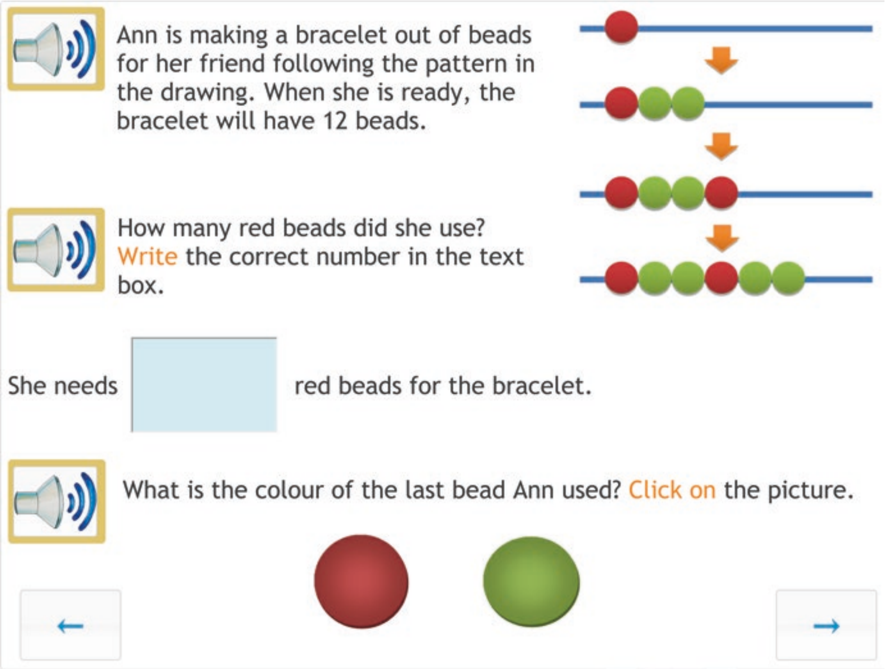
In the lower grades, mathematical problems become realistic when everyday experiences and observations come to play an active role in the problem-solving process. It is easier to interpret the problem if it is supported by a relevant picture or situation. The word problems can be made realistic if they can be solved with the accompanying picture or by manipulating the pictures given. The task presented in Fig. 40.5 using online technology encompasses an important feature of an authentic problem beyond the real-life-like context; namely, several solutions are possible, and the students can interact with the problem environment. With the scoring procedure, it is all the same which of the teddy bears are placed – dragged and dropped – on the bed; only the number of teddy bears counts. All of the combinations are accepted. The task measures skill level addition up to 10 in a realistic application context.

The task presented in Fig. 40.6 illustrates that it is impossible to split the tasks of the three dimensions from each other. It is a mathematical literacy type task, which measures number concept and relations and functions in a realistic application context. The aim is to measure students' ability to follow, recognise, and continue periodically repeating rhythms and movements by detecting similarities in relations among objects in an application context. Task scoring is automatised for all of the



Keep putting Teddy bears on the bed till you have 8 bears there.

Fig. 40.5 Mathematical literacy task: adding up to ten in a realistic application context



Ann is making a bracelet out of beads for her friend following the pattern in the drawing. When she is ready, the bracelet will have 12 beads.

How many red beads did she use? Write the correct number in the text box.

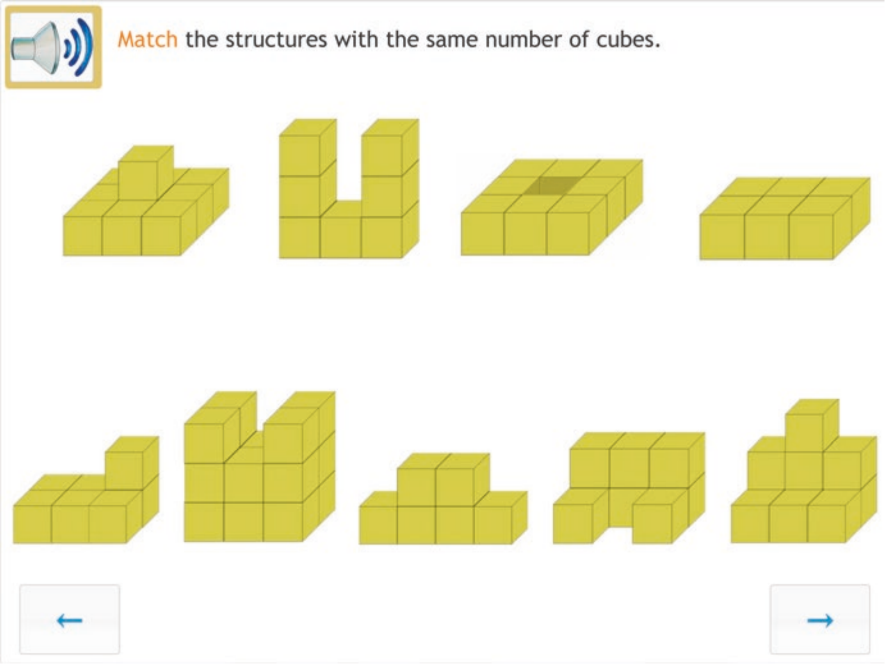
She needs ☐ red beads for the bracelet.

What is the colour of the last bead Ann used? Click on the picture.

Fig. 40.6 Mathematical literacy task: following, recognising, and continuing periodically repeating rhythms and movements in a realistic application context tasks in the item bank, even those with several correct answers. In the present case (Fig. 40.6), students had the option of typing their answer for the number of beads used in the bracelet in several ways, e.g. using number symbols or letters, using small letters or capitals or a mixture of them, or using spaces. All of these possibilities were accepted by the scoring system.

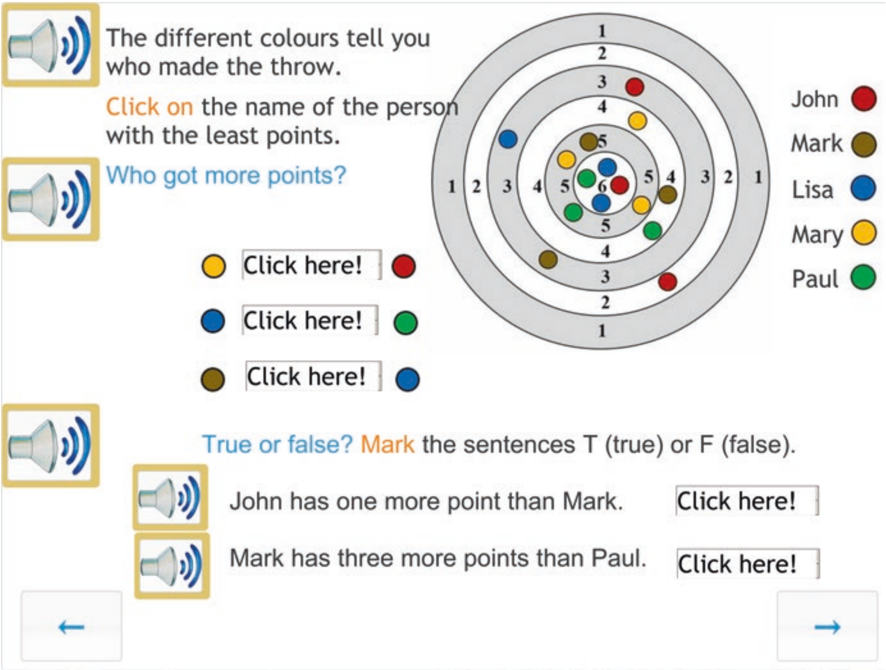
Items that Assess Disciplinary Mathematics Knowledge

In early mathematics education, among the most effective teaching methods are learning-by-doing activities. This is also the case for geometry, where students need to discover three-dimensional forms through different activities. The experience gained during these activities provides the foundations and in many cases determines the conceptual building work in lower and higher grades. In an online environment, the possibilities of manipulation play an important role. The task presented in Fig. 40.7 illustrates this. Students need to connect three-dimensional forms built out of cubes with other three-dimensional forms consisting of the same number of cubes by clicking on them to draw the connections. As GeoGebra elements and tasks uploaded from GeoGebraTube can be used in the *eDia* system, students can even rotate and engage in a manipulative interaction with these three-dimensional geometric forms.



Match the structures with the same number of cubes.

Fig. 40.7 Mathematical disciplinary task from geometry involving learning-by-doing with the possibility of manipulative interaction



The different colours tell you who made the throw.

Click on the name of the person with the least points.

Who got more points?

True or false? Mark the sentences T (true) or F (false).

John has one more point than Mark.

Mark has three more points than Paul.

Fig. 40.8 Mathematical disciplinary task: integrating the understanding of number symbols, the operation of addition, the comparison of quantities and numerosities, and the knowledge of relation symbols

In the first few years of schooling, operations with whole numbers, which build the foundations for additive reasoning, form an essential part of mathematics education. They include not only the operation of addition but all the knowledge elements for comparing quantities and numerosities. By reading the different numbers, sums, and differences, students are prepared for the mathematical concepts of addition and subtraction. In the process of understanding and interpreting addition and subtraction, the number line plays an important role. The task presented in Fig. 40.8 integrates the understanding of number symbols, the operation of addition, the comparison of quantities and numerosities, and the knowledge of relation symbols. During the solution process, students had to click on the name of the child who scored the least points and then choose the right relation symbol from the drop-down menus.

Field Trial and Empirical Validation of the Theoretical Model

We launched a field trial study to ascertain the applicability of computer-based tests in regular educational practice for assessing students at the beginning of schooling and for the empirical validation of the theoretical model of mathematical knowledge introduced above. The objectives of the study were threefold. First, we examined the applicability of an online diagnostic assessment system in the field of mathematics for students at the beginning of schooling. We then empirically validated the three-dimensional model of mathematical knowledge based on research results collected with first graders using *eDia*, the Hungarian online diagnostic assessment system. Finally, we examined the relationship between disciplinary content knowledge (MD), mathematical reasoning (MR), and mathematical literacy (ML) and answered the research question: how are the three different dimensions of mathematical knowledge related?

The sample was drawn from first-grade students in Hungarian primary schools. School classes formed the sampling units. 5115 first graders were involved in the study. The proportion of girls and boys was about the same.

The instrument was only a part of the whole test battery; it consisted of 48 items, which measured MD, ML, and MR in that order. To prevent reading difficulties, instructions were provided online using a prerecorded voice. Children had to indicate their answer by using the mouse or keyboard. Testing took place in the computer labs at the participating schools. Test completion lasted no more than 45 min (one school lesson). The tests were automatically scored, and students received immediate feedback at the end of the testing.

Reliability, time-on-task, and missing and achievement data were analysed to test the applicability of the online assessment system by first graders. The Rasch model was used to scale the data and draw the three-dimensional item-person map of mathematics. We conducted confirmatory factor analyses (CFA) within structural equation modelling (SEM; Bollen, 1989) to test the underlying measurement model of mathematical knowledge with the three different dimensions: disciplinary knowledge, literacy, and reasoning. Bivariate correlations, partial correlations, and SEM analyses were employed to test construct validity, that is, the relations between the three dimensions of mathematical knowledge.

Why have we conducted confirmatory factor analyses and what is it good for? Confirmatory factor analysis is a special form of factor analyses. In the present case, it is used to test whether the model based on the empirical data is consistent with our understanding of the nature and of the three-dimensional model of mathematical knowledge. That is, the objective of confirmatory factor analysis is to test whether the data fit a hypothesised measurement model, which is based on the three-dimensional theory of knowledge.

Bivariate correlation indicates the numerical relationship, the strength of the association between two measured variables, while partial correlation measures the degree of this association with the effect of controlling variables removed. Bivariate correlations can give misleading results if there is another variable that is related to both of the examined variables. This misleading information can be avoided by computing the partial correlation coefficient. Both of the coefficients take on a value in the range from -1 to 1. The value 0 conveys that there is no relationship, the value -1 means a perfect negative correlation, and the value 1 conveys a perfect positive association.

Construct validity describes the degree to which a test measures what it claims, indicating how well it really covers the targeted content; whether the scale behaves like the theory predicts a measure of that construct should behave. It describes the degree to which empirical evidence and theoretical rationales support the adequacy and appropriateness of inferences and actions based on test scores (Messick, 1995).

Applicability of the Diagnostic System in Everyday School Practice

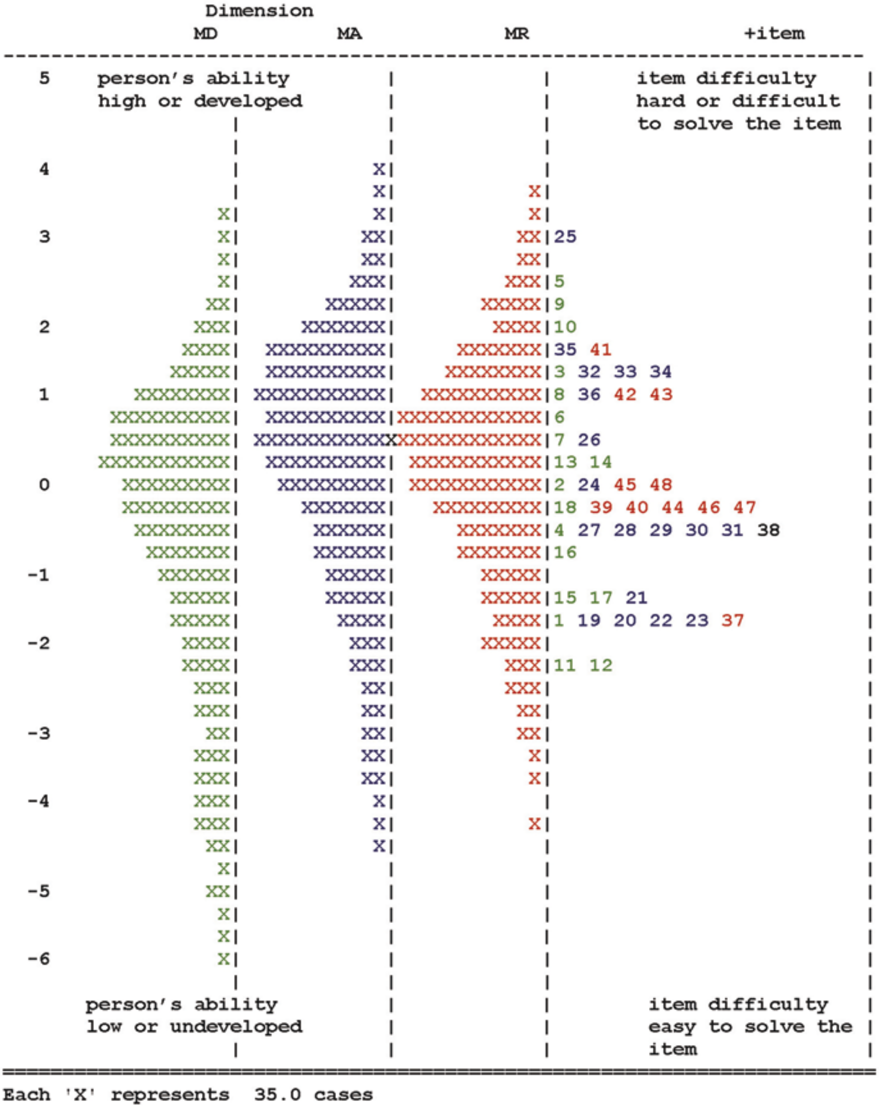
The results confirmed our hypotheses. The internal consistency of the mathematics test proved to be high both on the test ($\alpha = 0.942$) and subtest levels (α_MD = 0.89; α_MR = 0.83; α_ML = 0.89), so the results are reliable and generalisable. Less than 0.4%, that is, 18 students out of 5115, were not able to finish the test on time (within 45 min). As none of them completed more than 70% of the test and reached the third subtest, all of their data were deleted from the databases that form the data for the 5097 students involved in the analyses. Generally, the students managed to finish the test within the given timeframe, 1690 seconds on average (sd = 673).

Scaling and Item Difficulty

Participants' score distribution on the mathematics test also confirmed the applicability of the online assessment system. The mean achievement was about 50% (49.39%, sd = 23.87). The subtest level achievement distribution changed (M_md = 42.29, $s = 26.66$; M_ml = 53.96, $s = 26.67$; M_mr = 53.18, $s = 28.14$) and was significantly different (t_md_ml = −40.96, $p < 0.01$; M_md_mr = −33.45, $p < 0.01$; M_ml_mr = 2.35, $p < 0.05$). The level of standard deviations indicated that the test could be used to test the variability of the sample even on a subtest level.

The three-dimensional item-person map (Fig. 40.9) shows the match between the item difficulty distribution and the distribution of students' Rasch-scaled achievement estimates for MD, MA, and MR. For any person engaged with an item located at that person's level, the Rasch model routinely sets the probability of success on the item at 50% on an item-person logit scale.

The probability of success increases to 75% for an item that is 1 logit easier or decreases to 25% for an item that is 1 logit more difficult. The MD (green signs) and MA (blue signs) items were well matched to the sample ("x" and number are parallel), and with MR some hard and easy items were missing from the test. The achievement distribution in MD was the highest; there were more low-developed students than in the two other dimensions. Generally, the test was suitable for measuring and discriminating student achievement based on the three-dimensional model of mathematical knowledge in first grade in an online environment.



```
                 Dimension
                MD          MA            MR                  +item
------------------------------------------------------------------------
   5    person's ability        |            |       item difficulty    |
        high or developed       |            |       hard or difficult  |
                   |            |            |       to solve the item  |
                   |            |            |                          |
   4               |           X|            |                          |
                   |           X|           X|                          |
                  X|           X|           X|                          |
   3              X|          XX|          XX|25                        |
                  X|          XX|          XX|                          |
                  X|         XXX|         XXX|5                         |
                 XX|       XXXXX|       XXXXX|9                         |
   2            XXX|     XXXXXXX|        XXXX|10                        |
               XXXX|  XXXXXXXXXX|     XXXXXXX|35 41                     |
              XXXXX|  XXXXXXXXXX|    XXXXXXXX|3 32 33 34                |
   1       XXXXXXXX| XXXXXXXXXXX|  XXXXXXXXXX|8 36 42 43                |
         XXXXXXXXXX|  XXXXXXXXXX|XXXXXXXXXXXX|6                         |
         XXXXXXXXXX| XXXXXXXXXXXXXXXXXXXXXXXX|7 26                      |
        XXXXXXXXXXX|  XXXXXXXXXX| XXXXXXXXXXX|13 14                     |
   0      XXXXXXXXX|   XXXXXXXXX| XXXXXXXXXXX|2 24 45 48                |
          XXXXXXXXX|     XXXXXXX|   XXXXXXXXX|18 39 40 44 46 47         |
           XXXXXXXX|      XXXXXX|     XXXXXXX|4 27 28 29 30 31 38       |
            XXXXXXX|      XXXXXX|     XXXXXXX|16                        |
  -1         XXXXXX|       XXXXX|       XXXXX|                          |
              XXXXX|       XXXXX|       XXXXX|15 17 21                  |
              XXXXX|        XXXX|        XXXX|1 19 20 22 23 37          |
  -2           XXXX|         XXX|       XXXXX|                          |
               XXXX|         XXX|         XXX|11 12                     |
                XXX|          XX|         XXX|                          |
                XXX|          XX|          XX|                          |
  -3             XX|          XX|          XX|                          |
                XXX|          XX|           X|                          |
                XXX|          XX|           X|                          |
  -4            XXX|           X|            |                          |
                XXX|           X|           X|                          |
                 XX|           X|            |                          |
                  X|            |            |                          |
  -5             XX|            |            |                          |
                  X|            |            |                          |
                  X|            |            |                          |
  -6              X|            |            |                          |
                   |            |            |                          |
       person's ability         |            |        item difficulty   |
       low or undeveloped       |            |        easy to solve the |
                   |            |            |        item              |
=========================================================================
Each 'X' represents  35.0 cases
```

Fig. 40.9 The three-dimensional item-person map of first graders' mathematical knowledge

Gender-level achievement differences changed between the different dimensions. Girls' achievement proved to be significantly higher on the test level (M_girl = 50.36, s_girl = 23.46, M_boy = 49.01, s_boy = 23.92, $t = -2.011$, $p = 0.044$); however, the level of significance might only have been caused by the large sample size. On the subtest level, there were no gender-level achievement differences on the MD and MA subtests, while significant differences could be detected on the reasoning part of the test ($t = -2.923$, $p < 0.01$), thus causing the gender-level differences on the test level.

Based on this result, we can conclude that, first, computer-based assessment can be carried out even at the very beginning of schooling without any modern touch screen technology on normal desktop computers using a general browser and the school infrastructure, and, second, the online diagnostic system can be used to test students' mathematics knowledge at the beginning of schooling in a school context.

Dimensionality and Structural Validity

In validating the three-dimensional model of mathematical knowledge, SEM analyses were outperformed. The three-dimensional measurement model for mathematics showed a good model fit (Table 40.1), based on Hu and Bentler's (1999) recommended cut-off values. The comparative fit index (CFI) and the Tucker-Lewis index (TLI) value above 0.95 and the root mean square error of approximation (RMSEA) below 0.06 indicate a good global model fit. As significant and high correlations were found between the pairs of dimensions ($r_{MD\_MR} = 0.685$, $r_{MD\_ML} = 0.749$, $r_{ML\_MR} = 0.634$, $p < 0.001$) on a latent level – latent variables are not directly observed but are inferred from other variables that are observed (directly measured) – within the three-dimensional model, we also tested the one-dimensional model with the three dimensions combined under one general factor. With the one-dimensional model, the fit indices decreased considerably.

In order to test which model fitted the data better, a special χ^2-difference test was carried out in Mplus, which showed that the three-dimensional model fitted significantly better than the one-dimensional model ($\chi^2 = 3389.111$; $df = 6$; $p < 0.001$). In summary, the three-dimensional model fitted well and better than the one-dimensional model. Thus, the disciplinary, literacy, and reasoning dimensions of mathematical knowledge were empirically distinguished, supporting our hypothesis.

The bivariate correlations between MD, ML, and MR were high, ranging from 0.63 to 0.71 (Fig. 40.10). The relationships proved to be similar between MR and either ML or MD ($r = 0.63$ and 0.64, $p < 0.001$, respectively), and they were significantly weaker than the correlation between ML and MD ($r = 0.71$, $p < 0.001$).

Partial correlations were significantly lower as all bivariate relationships were influenced by the third construct ($r_{MR\_ML} = 0.32$; $r_{MR\_MD} = 0.35$; $r_{ML\_MD} = 0.51$, $p < 0.001$). Like the bivariate correlations, the partial correlation coefficients between MR and either ML or MD were of the same strength ($p < 0.001$), while the partial

Table 40.1 Goodness of fit indices for testing dimensionality of mathematics

Model	χ^2	*Df*	*p*	CFI	TLI	RMSEA (90% CI)	*n*
Three-dimensional	16955.213	1067	0.001	0.965	0.963	0.054 (0.053–0.055)	5097
One-dimensional	31445.929	1073	0.001	0.931	0.928	0.075 (0.075–0.076)	5097

Note: *df* degrees of freedom, *CFI* comparative fit index, *TLI* Tucker-Lewis index, *RMSEA* root mean square error of approximation, χ^2 and *df* are estimated by WLSMV

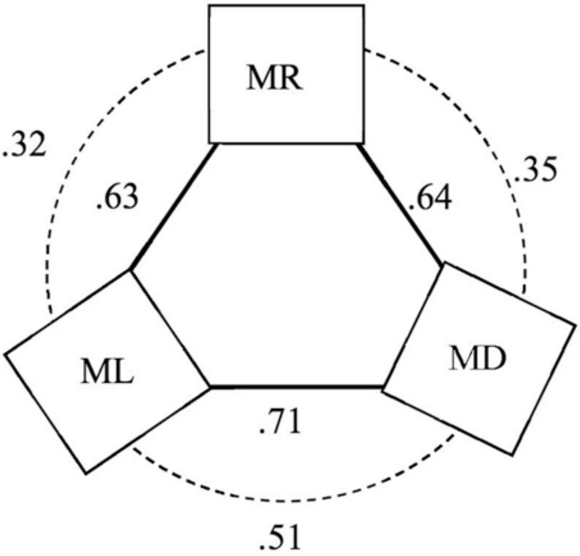


Fig. 40.10 Relations between MR, ML, and MD (Solid lines depict bivariate correlations; dotted lines represent partial correlations. All coefficients are significant at the $p < 0.001$ level)

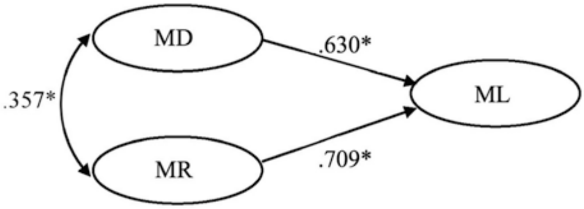


Fig. 40.11 A structural model of mathematical knowledge: disciplinary knowledge and mathematical reasoning as predictors of mathematical literacy (*$p < 0.01$)

correlation between ML and MD proved to be the highest. This is supported and was indicated by the correlation coefficients on a latent level as well (see above).

We assumed that disciplinary knowledge and mathematical reasoning predict performance in literacy, the application dimension of mathematics, since we need that dimension of mathematics most in everyday life. Thus, we regressed MD and MR on ML and estimated the proportion of variance explained. The results showed that MD and MR explained performance in ML on a high level (90%) but with a different effect (see Fig. 40.11). The residuals of measures of MD and MR were still correlated on a moderate level ($r = 0.35$), indicating common aspects of MD and MR that are separable from ML. The model fit well (CFI = 1.000, TLI = 1.000, RMSEA = 0.000).

To sum up, our results showed that MD, MR, and ML are highly correlated constructs, though not identical. Students' levels of disciplinary knowledge and mathematical reasoning strongly influence and predict achievement in the context of mathematical application. That is, if we enhance disciplinary knowledge in mathematics and students' thinking skills, we can expect a stronger transfer from the disciplinary to the application contexts. This suggests that beyond factual knowledge, thinking skills should become an integral part of school agendas (de Koning, 2000) and should be incorporated into a broad range of school-related mathematical learning activities.

Conclusions and Further Research and Development

In this chapter, we have presented the theoretical foundations and technological realisation of an online diagnostic assessment system in the domain of mathematics. The applicability of this system in educational practice was demonstrated in an ecologically valid context, when the online tests were administered to a large sample in real school settings. The assumption that computer-based assessment is applicable even in the early school grades was confirmed. We validated the three-dimensional model of mathematical knowledge empirically, having addressed the psychological, application, and disciplinary dimensions of knowledge. These results strengthen the foundations for a complex online diagnostic assessment platform called *eDia*, which contains about 2000 tasks (8000 items) per dimension for the first six grades of primary school.

According to the empirical results, the three-dimensional approach is valid; the disciplinary, application, and reasoning aspects of learning are neither independent of nor identical to each other. Consequently, each of these three aspects of knowledge must be enhanced at the same level and at the same time at school, and all of them must be incorporated into a broad range of mathematical learning activities and must not be mutually exclusive. In modern societies, it is neither sufficient nor satisfying to focus on only one of these goals, a common tendency in many education systems in which the teaching of disciplinary content is favoured.

The system can be used to identify students with atypical development, that is, children whose achievement is significantly lower in one of the three dimensions. Teachers receive prompt feedback about their students' development in each of the dimensions separately in a comparable way. At this moment, in the phase of system development, the system administers the tests having different difficulty levels to the students in a random way; it is not enough to provide only percentage-based feedback to the teachers, as they are strictly taken not objectively comparable to each other. The feedback is based on students IRT-based ability levels in ability points, which can be referred to the national mean ability values that is transferred to 500 (with 100-point standard deviation) in each grade, which constructs the point of reference to the students' achievement. Beyond the student-level results and national standards, teachers receive feedback about their class-level and school-level achievement with comparison to the other class-level, school-level, regional-level, and strata-level achievements. Our future plan is to put the test administration on an adaptive level.

Training programmes adjusted to their specific deficiencies can then be implemented to help them catch up. The efficacy of such a training can also be monitored with the assessment system. Further research can be carried out with the diagnostic assessment system to explore the reason for an atypical mathematical development and the ways in which the different dimensions of mathematical knowledge can be effectively enhanced.

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Chapter 41
Small Group Interventions for Children Aged 5–9 Years Old with Mathematical Learning Difficulties

Pirjo Aunio

Introduction

This chapter begins with a description of some important concepts – who are the children having problems in mathematics, what do we mean with the concept of intervention, what does Responsiveness to Intervention mean, and which intervention features have been found effective for children aged 5–9 years with learning difficulties in mathematics. Then, I describe the research and developmental work that has been done in Finland on designing web services for educators related to mathematical learning difficulties, assessments, and interventions. The two web services (LukiMat and ThinkMath) have been developed by two different, but related, research teams at the Niilo Mäki Institute (University of Jyväskylä) and the University of Helsinki.

Learning Difficulties in Mathematics

In literature, there are several different terms used in relation to learning difficulties in mathematics, such as low performance in mathematics, mathematical learning disability, dyscalculia, mathematics disorder, and many more. These various terms refer to different definitions (e.g., in terms of various cutoff scores) and different origins of the problems ranging from neurological dysfunctions to inappropriate opportunities to learn and practice mathematical skills (e.g., low socioeconomic status of the child's family) (Ansari, 2015; Mazzocco, 2009). Geary (2013) suggests that children who score at or below the tenth percentile on standardized mathematics achievement tests for at least two consecutive academic years are categorized as

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having an MLD (mathematical learning disability). He further suggests that all children scoring between the 11th and 25th percentiles, inclusive, across 2 consecutive years are classed as LA (low achievers). The various terms are quite confusing, but when we talk about young children just starting their school career, it seems to be appropriate to use the terms “low performing” or “mathematical learning difficulties,” thus avoiding the “stronger terms” like “mathematical learning disability” and “dyscalculia,” which clearly indicate to the possible neurological dysfunctions in the background of severe learning problems in basic arithmetic learning which is mostly visible in educational context only after a couple of years of math learning. For teachers it is also important to understand that mathematics performance is a continuum; there is no definite point where the problem starts.

Intervention

At the moment the concept “intervention” is a popular term and used with various meanings in education. Intervention can refer to the intervention programs which are used for children who have learning difficulties to change the originally bad learning prognosis (i.e., extra educational support). Intervention can also refer to the research design that is used to study children’s development, and aims to investigate what factors affect learning. This approach is often used by developmental psychologists. In addition, intervention research design can be used to investigate the effects of a particular intervention program, which can then be published and used by educators. This approach is common among special education and educational psychology research.

The most important way to measure the effectiveness of the educational intervention programs is to study the increase in learning (i.e., achievement) of the children as a result of extra practice, hence intervention (Jimerson, Burns, & VanDerHeyden, 2007). The recommended and often used intervention research design includes a pretest (i.e., baseline measurement) and immediate and delayed post-test measurements with control groups. The intervention and control group design allows researchers and teachers to investigate whether the children receiving intervention develop faster than their peers who are not getting extra attention, for instance, in mathematics learning. Researchers use approaches that are a bit different to judge whether the intervention program is effective. In general, it is possible to say that an intervention program is effective if the children with low performance or learning difficulties progress better than their performance control peers. Secondly, an intervention program shows better results if the children with low performances are able to maintain their head start compared with the control group even after the intervention phase has ended. Thirdly, the results would be best, in addition to the aforementioned effects, if the low-performing children closed the gap to their average performing peers. It is the researchers’ task to explain these possibilities of effectiveness measuring to educators who need to make decisions about how to support children with learning difficulties (Jimerson et al., 2007).

However, to determine which intervention program is best for particular children is more complex than only deciding how effects need to be detected. When we need to make a decision on which intervention program to use, we need to compare programs and studies with different features. This task needs to be carried out carefully as intervention programs and studies can differ in various aspects, which can make comparison difficult (Fischer, Moeller, Cress, & Nuerk, 2013; Mononen, Aunio, Koponen, & Aro, 2014). The interventions can vary in terms of target children, comparison group, aims, setting, duration, mathematical content, conductor and professional developmental support, and instructional design features, which all can have an impact, individually or combined, on the intervention's effectiveness (Fischer et al., 2013; Mononen et al., 2014).

The aims of an intervention program can be remedial or preventive. Remedial intervention is needed when children have already been identified as having a severe mathematical learning difficulty (i.e., mathematical learning disability, dyscalculia) (Kucian et al., 2011). Preventive intervention programs aim to avoid later learning problems. Preventive interventions are often used with younger children in preschool and primary grades and aim to assure that children form a basis of fundamental skills needed in later learning (Toll & Van Luit, 2014). The focus groups can differ in intervention studies; they can be children who are diagnosed with severe problems in learning mathematics (Kucian et al., 2011), or children who have low achievement (i.e., performance) in mathematics (Toll & Van Luit, 2012), or children who are at risk of developing learning difficulties based on their low socioeconomic family background (Dyson, Jordan, & Glutting, 2011). Target groups can also differ in their age, at the moment most research is done with younger children (preschool and primary grades) (Bryant, Bryant, Gersten, Scammacca, & Chavez, 2008; Clarke, Doabler, Smolkowski, Baker et al., 2016; Clarke, Doabler, Smolkowski, Kurtz-Nelson et al. 2016), but there also is good progress in interventions for older students (Moser Opitz et al., 2016; Xin et al., 2017). There can also be differences in intervention settings: interventions can be conducted individually (Fuchs, Fuchs, & Compton, 2012; Hunt, Tzur, & Westenskow, 2016), in pairs (i.e., dyads) (Barnes et al., 2016), in small groups of 3–8 children (Bryant et al., 2008; Mononen & Aunio, 2014; Moran, Swanson, Gerber, & Fung, 2014), or with a whole classroom (Clarke et al., 2011). In terms of the setting, the intervention can be core instruction, thus taking place during regular mathematics lessons and replacing the math curriculum previously used in that classroom (Clarke et al., 2011). Intervention can be supplementary, during which children follow the average mathematics lessons and on top of that get extra educational support in skills they have a higher need of support in (Powell et al., 2015). Time practiced is also an important feature (i.e., exposure time for treatment); intervention programs can be short, e.g. a couple of hours, or more extensive with a duration of more than 60 hours; also the duration of one session can vary a lot, for instance, from 10 to 60 min, and on top of that the number of sessions can differ. For instance, Salminen and her colleagues (2015) investigated the differences in time used in computer-assisted instruction research in the field of mathematical learning difficulties and found them to vary between 2 weeks and a whole semester and sessions lasting from 1 to 60 min, and there also appeared to be great variance in number of sessions, from 7 sessions to 50 sessions. Dennis and his colleagues (2016) reported the intervention length in minutes to vary between 400 and 5400 min in mathematics learning small group interventions for kindergarteners. Mathematical content can also vary. There are intervention programs that practice only some quite narrow skill, like numerical magnitude comparison and number line estimation in a study of Siegler and Ramani (2009), and then there are intervention programs that practice several mathematical skills (Aunio, Hautamäki, & Van Luit, 2005; Barnes et al., 2016). The skills practiced can be very basic skills by nature (Siegler & Ramani, 2009), or the focus can be on complex mathematical problem-solving (Pfannenstiel, Bryant, Bryant, & Porterfield, 2015; Sharp & Dennis, 2017).

Interventions can be led by researchers (Dyson et al., 2011) or educators (Mononen & Aunio, 2014, 2016) (i.e., agents of intervention). If the intervention is conducted by the teacher, there is a need for good professional development support so that she/he understands the principles and way of conducting the intervention the same way as has been the developers' idea; this way the ecological validity is secured (Cary et al., 2017). Interventions can include various instructional features such as explicit and systematic instruction (Toll & Van Luit, 2014), use of visual representations in the introduction of mathematics ideas at concrete-representational-abstract (CRA) levels (Mononen & Aunio, 2014, 2016), or use of computer-assisted instruction (CAI) (Salminen et al., 2015). When the effectiveness of interventions is studied, it is important to measure the impact related to comparative groups of children, so children on similar performance levels are compared with each other in similar learning environments; ideally the intervention is the only difference between participating children.

In summary, finding the best intervention program to support children is a complex issue. We need more results comparing similar intervention programs applied in a similar way, to be able to be sure about the best ways to support children in their learning. Maybe good a guideline for educators is to think about what kind of mathematical learning problems children have (what skills are the ones the child lacks) and then to look at the literature to find out what kind of intervention programs have been developed to meet those learning needs. Then it might be sensible for the educator to check whether the situation (children, learning needs, learning environment) is similar to that in which the particular intervention program has been found efficient.

The Features of Effective Instruction for Children with Mathematical Learning Difficulties

There has been fast progress in the development in intervention study methodology. At first there were individual intervention studies with quite small samples with convenient sampling but with quite many control variable measures. Currently there seems to be a high demand of randomized control trials (RCT), large-scale interventions, and replication studies (e.g., Gersten et al., 2015), to produce reliable evidence about effectiveness of interventions. The alternative ways to understand the effectiveness of interventions in children's learning are the meta-analyses, reviews, and systematic reviews which aim to summarize the previous intervention research results. They provide a broader picture of the field of interventions than individual studies do. Research reviews have produced some results with interventions for students with learning difficulties in mathematics (Chodura, Kuhn, & Holling, 2015; Codding, Burns, & Lukito, 2011; Gersten et al., 2009; Jitendra et al., 2018; Kroesbergen & Van Luit, 2003; Maccini, Mulcahy, & Wilson, 2007; Zhang & Xin, 2012), but only few have concentrated on young children (Dennis et al., 2016; Mononen et al., 2014).

In the review of Mononen et al. (2014), the interventions show small to average effect sizes in improvement of the early numeracy skills of children aged 4–7. Results indicate that different types of instructional design features, including explicit instruction, computer-assisted instruction (CAI), game playing, or the use of concrete-representational-abstract levels in representations of math concepts, lead to improvements in mathematics performance. Therefore, rather than waiting to provide effective mathematics interventions at school (e.g., Baker, Gersten, & Lee, 2002; Slavin & Lake, 2008), evidence-based programs before the onset of school could be used to promote early numeracy skills, especially for low-performing children and to children from low socioeconomic environments.

In a recent meta-analysis that included younger children, Dennis et al. (2016) found that studies conducted at kindergarten level yielded significantly weaker effects than studies conducted at elementary level. Their results also showed that the interventions provided for students who had low math performance (at or below 35th percentile) at the time of identification yielded strong intervention effects compared to children performing above the 35th percentile. In addition, interventions were more effective when they were provided by the researchers and researcher-trained graduate assistants; those provided by teachers and paraprofessionals produced weaker effects. Dennis et al. (2016) found effective instructional variables to be peer-assisted learning and explicit teacher-led instruction (i.e., sequencing tasks from easy to difficult, task analysis), but interventions including the use of technology were least effective in improving the mathematics performance of students with mathematical learning difficulties. In addition, they found that intervention delivered in form of small group instruction was more effective for students with mathematical learning difficulties.

Dennis et al. (2016) replicated the results, at least partly, in a previous meta-analysis concerning group-based interventions for children with mathematical learning difficulties (Baker et al., 2002; Swanson, Hoskyn, & Lee, 1999). These studies show that intervention studies that used explicit and strategic instructional procedures with students with learning difficulties have been found to have larger effect sizes compared to other instructional approaches (Baker et al., 2002; Chodura et al., 2015; Gersten et al., 2009; Kroesbergen & Van Luit, 2003; Miller, Butler, & Lee, 1998; Mononen et al., 2014; Swanson et al., 1999). Explicit interventions included, for instance, sequencing of instruction into logical sequences; providing a clear presentation of subject matter, guided practice, and independent practice; and evaluating student learning on a regular basis. Explicit instruction often includes using a concrete-representational-abstract (CRA) sequence which has been found to be an effective instructional feature (Miller et al., 1998; Mononen et al., 2014; Xin & Jitendra, 1999). Peer-assisted instruction has been found to be an effective instructional feature with younger students (Baker et al., 2002; Kunsch, Jitendra, & Sood, 2007). The effects of CAI in interventions for children with learning difficulties in mathematics are controversial, some found support (Kroesbergen & Van Luit, 2003; Miller et al., 1998; Mononen et al., 2014), and other ones were rather criticized (Dennis et al., 2016; Räsänen, Salminen, Wilson, Aunio, & Dehaene, 2009).

Previous meta-analysis (Chodura et al., 2015; Dennis et al., 2016; Jitendra et al., 2018; Mononen & Aunio 2012a; 2012b; 2012c) has pointed out some weaknesses in intervention studies in the field of mathematical learning difficulties. These are, for instance, longitudinal effects of an intervention which are hard to study as there is no delayed measurement used; there is also not enough information to know how children with mathematical learning difficulties are identified (challenges with selection and outcome measurements and cutoff criteria).

To know whether the intervention studies published after the latest meta-analysis (Dennis et al., 2016) have faced the pointed weaknesses, the author made a small review with intervention studies published after 2014 in peer-reviewed English journals, conducted in small groups of children, applied with an at least quasi-experimental design with a control group, and focused on early numeracy (grade K-2) and children with learning difficulties in mathematics (Table 41.1). I found seven intervention studies published in peer-reviewed English journals, all of them had been made in the United States; in half of the studies, some way to randomize students in the intervention and control group was used, and they were used as supplementary, not replacing the core mathematics instruction. The children showed low performance in early numeracy in six studies and possibly also in Clarke, Doabler, Smolkowski, Baker et al., (2016); Clarke, Doabler, Smolkowski, Kurtz-Nelson et al. (2016) in which the teacher identified those children who most likely benefit from small group instruction. In three studies the children also had a low-income family background (Barnes et al., 2016; Dyson, Jordan, Beliakoff, & Hassinger-Das, 2015; Hassinger-Das, Jordan, & Dyson, 2015). Three (Clarke, Doabler, et al., 2016; Clarke, Dobler, et al., 2016; Doabler et al., 2016) out of seven studies used the ROOTS intervention program developed by Clarke's research group in University of Oregon. All of the intervention studies focus on several mathematical skills. Cutoff criteria for low early performance varied between below 10 and below 35 percentile in standardized mathematics measurement, resulting in a quite big variation in skills in the target group of children. In all of the studies, a variety of standardized measurements was used (such as the Number Sense Brief Screener; SAT; TEMA-3: WJ-III), but also measurements designed by the research group were used (EN-CBM, ASPENS) as outcome measurements. All seven studies reported significant intervention effects on children's early mathematics performance. But only three studies reported the delayed measurements results, confirming the lasting effects of interventions (Clarke, Doabler, et al., 2016; Dyson et al.,

Table 41.1 Small group intervention studies for young children with mathematical learning difficulties

Authors	Title	Age	At-risk status	Number of participants	Duration	Leader	Instructional design feature	Math measure	Pre-post effect size	Follow-up (Yes/No)
Barnes et al. (2016)	Effects of tutorial interventions in mathematics and attention for low-performing preschool children	4.50 years	Low performance (below 25th percentile in TEMA-3) and low-income family	N=541 (M+Att n=181; M only n=180; business-as-usual (BaU) n=180)	24 weeks (1920 min)	Researcher	Explicit + systematic instruction, cumulative review; teaching to mastery, scaffolding, progress monitoring. Pre-K-Mathematical Tutorial (PKMT)	Child Math Assessment (CMA) (Starkey, Klein, & Wakeley, 2004); Test of Early Mathematics Ability – Third Edition (TEMA-3) Ginsburg & Baroody, 2003)	In independent contracts with BaU group, both the M+ATT group (ES=.43) and the M only group (ES=.60) had greater math knowledge at post-test CMA	No
Clarke, Doabler, Smolkowski, Baker et al. (2016)	Examining the efficacy of a tier 2 kindergarten mathematics intervention	5.54 years	Teacher identified five lowest performing children who would benefit from small group math instruction. Of the 122 children 91% scored at or below 10th percentile in TEMA	N=140 (intervention n=67, control n=73)	20 weeks (1200 minutes)	Researcher	ROOTS (whole number understanding), explicit and systematic instruction: modeling and demonstrating, guided practice, visual representations, feedback. Math verbalization. Systematic instruction: prioritizing instruction around critical content, connecting new content with students' background knowledge, selecting and sequencing instructional examples and scaffolding instruction	Test of Early Mathematics Ability (TEMA-3, Ginsburg & Baroody in table 41.1. see above), Early Numeracy Curriculum-Based Measurement (EN-CBM, Clarke & Shinn, 2004)	Found statistically significant gains among the intervention students over those in control classrooms on TEMA standard scores (t= 2.19, df 27, p=.0371) but not in the EN-CBM total score (t=1.35, df 27, p=.1870). The corresponding to Hedge's g effect sizes of .38 for the TEMA standard score and .30 for the EN-CBM	No

(continued)

Table 41.1 (continued)

Authors	Title	Age	At-risk status	Number of participants	Duration	Leader	Instructional design feature	Math measure	Pre-post effect size	Follow-up (Yes/No)
Clarke, Doabler, Smolkowski, Kurtz-Nelson et al. (2016)	Testing the immediate and long-term efficacy of tier 2 kindergarten mathematics intervention	5.2 years	Low performance. Children qualified for the intervention if they scored 20 or less on the NSB (Jordan, Glutting & Ramineni 2008) and had composite score on ASPENS that placed in the strategic or intensive range (Clarke, Gersten et al., 2011)	*N*=290 (two-student ROOTS condition (*n*=58); five-student ROOTS condition (*n*=145), no-treatment control condition (*n*=87)	10 weeks (1000 min)	Researcher	ROOTS (whole number understanding), explicit and systematic. Explicit instruction: (a) teacher modeling, (b) deliberate practice (including scaffolding), (c) visual representations of mathematics, (d) academic feedback	ROOTS Assessment of EARLY Numeracy Skills (RAENS) (Doabler, Clarke & Fien (2012); Assessing Student Proficiency in Early Number Sense (ASPENS) (Clarke, Gersten et al., 2011); Number Sense Brief (NSB) Screener (Jordan, Glutting & Ramineni 2008)); Test of Early Mathematics Ability – Third Edition (Ginsburg & Baroody, 2003); The Stanford Achievement Test – Tenth Edition (SAT)	Statistically significant differences by condition in gains from fall to strong for four dependent variables. Students in the ROOTS condition made greater gains than control students on the ASPENS (t=5.20, df=136, p<.0001), oral counting (t02.14, df=132, p=.0333), TEMA standard scores (t=3.35, df=142, p=.0010), and RAENS (t=6.84, df=162, p<.0001). The time X condition model estimated differences in gains between conditions of 0.75 for the NSB (Hedge's g= .16), 19.7 for ASPENS (g=.58), 6.5 for oral counting (g=.28), 2.45 for the TEMA standard score (g=.32), and 4.7 for the RAENS (g=.75)	Yes

Doabler et al. (2016)	Testing the efficacy of a tier 2 mathematics intervention: A conceptual replication study	5.2 years	Low performance. Children with both an NSB score of 20 or less and an ASPENS composite score in the “strategic” or intensive” ranges were considered at risk for MD and eligible for intervention	$N=319$ ROOTS-small condition ($n=67$), ROOTS-large condition ($n=162$), and control condition ($n=90$)	10 weeks (1000 min)	Researcher	ROOTS (whole number understanding), explicit and systematic. Explicit instruction: (a) teacher modeling, (b) deliberate practice (including scaffolding) (C) visual representations of mathematics (d) academic feedback	ROOTS Assessment of EARLY Numeracy Skills (RAENS) (Doabler, Clarke & Fien (2012); Assessing Student Proficiency in Early Number Sense (ASPENS) (Clarke, Gersten et al., 2011); Number Sense Brief (NSB) Screener (Jordan, Glutting & Ramineni 2008)); Test of Early Mathematics Ability – Third Edition (Ginsburg & Baroody, 2003); The Stanford Achievement Test – Tenth Edition (SAT)	Students in the ROOTS condition made greater gains than control students on the NSB ($t=3.15$, df=94, $p=.0022$), ASPENS ($t=5.60$, df=118, $p<.0001$). The Time x Condition model estimated differences in gains between conditions of 1.94 for the NSB ($g=0.40$), 21.78 for the ASPENS ($g=0.64$), 2.43 for the TEMA-3 standard score ($g=0.31$), and 6.50 for the RAENS ($g=1.08$)	No

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Table 41.1 (continued)

Authors	Title	Age	At-risk status	Number of participants	Duration	Leader	Instructional design feature	Math measure	Pre-post effect size	Follow-up (Yes/No)
Dyson et al. (2015)	A kindergarten number-sense intervention with contrasting conditions for low-achieving children	5.5 years	Low income and low performance. Below 25th percentile (number sense screener, Jordan et al., 2010)	*N*=126 (number list practice (*n*=40), number-fact practice (*n*=44), control (*n*=42))	8 weeks (720 min)	Researcher	Researchers designed intervention material practicing number, number relations, number fact practice, and number list practice. Practice conditions. Children in both conditions received the same 25-minute number-sense intervention. Lessons differed for the two intervention groups only in the last 5 minutes of the 30-minute lesson. Number-list practice, each child played a version of the Great Race game (Ramani & Siegler, 2008). Number-fact practice: children participated in an activity that engaged quick answers to addition and subtraction combinations that had been taught in the lesson that day or in previous lesson	Number competencies – Number Sense Screener (Jordan, Glutting, Ramineni & Watkins, 2010); Arithmetic Fluency (Jordan & Montani, 1997), Mathematics Calculation Achievement (Woodcock-Johnson III Tests of Achievement (WJ-III) Standard Test Book Form A: Calculation Subtest (Woodcock, McGrew & Mathe, 2007)). Background variables : nonverbal reasoning (WPPSI, Wechsler, 2002), spatial ability (The Children's Mental Transformation Task (CMTT) (Levine, Huttenlocher, Taylor & Langrock (1999)), inattentive behavior (SWAN Rating Scale, Swanson et al., 2006); reading achievement (WJ-III Standard Test Book Form A: Letter-Word Identification Subtest (Woodcock et al., 2007))	There was a significant main effect for group at each time point and for each measure. For number sense, the number-list condition outperformed the control group (although not always significantly) at both post-test and delayed post-test with effect sizes greater than .25 (ES=.32 and ES= .26, respectively). The effect sizes for the number-fact practice versus control were more than twice those of the number-list practice at both time points (ES=.82 and ES=.56, respectively). The effect of the number-fact condition over the number-list condition produced an effect size even greater than the effect of the number list over the control (ES=.42)	Yes

Gersten et al. (2015)	Intervention for first graders with limited number knowledge: large-scale replication of a randomized controlled trial	First graders	Lowest 35th of students screened in six math subtests (timed computation, Fuchs, Hamlett & Fuchs, 1990; concepts/ applications, Fuchs et al., 1990; story problems, Jordan, Kaplan Locuniak & Ramineni, 2007; the number knowledge test, Okamoto & Case, 1996; quantity discrimination, Clarke, Baker, Cahrd & Otterstedt, 2006; digit-span backward, Geary, 1993)	*N*= 994 (intervention *n*=615, control *n*=379)	17 weeks (1920 min)	Teacher	Number Rockets program applied the concrete–representational–abstract model, which relies on concrete objects to promote conceptual learning. The sequence of topics was identifying and writing numbers to 99; identifying more, less, and equal with objects; sequencing numbers; skip counting by 10s, 5s, and 2s; writing number sentences; place value, addition, and subtraction	Screening measures: timed computation (Fuchs, Hamlett & Fuchs, 1990); concepts/ applications (Fuchs et al., 1990); story problems (Jordan, Kaplan, Locuniak & Ramineni, 2007); the number knowledge test (Okamoto & Case, 1996); quantity discrimination (Clarke, Baker, Chard & Otterstedt, 2006); digit-span backward (Geary, 1993). Outcome measures: Test of Early Mathematics Ability – Third Edition (TEMA-3) and assessment to explore any unintended negative consequences – The Woodcock-Johnson III Letter/Word Subtest	Significant effect size of .34 standard deviation (SD) units is relatively large for large-scale research study. The effect size is virtually identical to the mean effect size found in the original efficacy study of .337 SD units. Thus, one can clearly infer that the original findings were replicated in the large-scale study	No

(continued)

Table 41.1 (continued)

Hassinger-Das et al. (2015)	Reading stories to learn math. Mathematics vocabulary instruction for children with early numeracy difficulties	Kindergarten	Low income; low scores (<=22 out of 44 = below 25th percentile) Number Sense Brief (NSB, Jordan et al., 2010)	*N*=124 (a storybook number competencies (SNC) intervention, a number sense intervention, business-as-usual control)	8 weeks (720 min)	Researcher	Storybook number competencies (SNC) intervention targeting mathematics vocabulary knowledge (e.g., equal, more, less) and number concepts	Measures: Mathematics vocabulary (The Bracken Basic Concept Scale – Third edition). Receptive: Quantity Subtest (BBCS-3R; Bracken, 2006); Number Sense (Jordan et al., 2010); Mathematics achievement (Woodcock-Johnson III Test of Achievement Normative Update Brief Battery/Form C (WJ-III)); Applied Problems and Calculation Subtest (Woodcock, McGrew & Mather, 2007)	Findings demonstrated that the SNC intervention group outperformed the other groups on measures of mathematics vocabulary, both in terms of words that were closely aligned to the intervention and those that were not	Yes

2015; Hassinger-Das et al., 2015). This small review confirms the pervious findings that explicit and systematic small group interventions have effects on early numeracy learning of low-performing students. From the European point of view, it would be good to validate the findings also with samples outside the United States. One challenge that science face here is that we have to develop ways on how to describe our measurements, criteria, and outcome, so that it becomes possible to relate them to measurements designed in other countries as well. In some countries we still lack good quality standardized measurements to identify mathematical learning difficulties and to follow the development in core skills. We still need more intervention studies to report the results from delayed measurements.

Responsiveness to Intervention Practice in Supporting Children with Learning Difficulties

At the beginning of the twenty-first century in the United States and Europe, the way to approach individuals with learning difficulties started to change. The focus shifted from diagnosing the individual in clinical settings to viewing individuals' learning as part of his or her learning context and emphasizing the early identification of learning difficulties to provide early interventions (i.e., Responsiveness to Intervention, RtI) (Hallahan, Pullen, & Ward, 2013). Responsiveness to Intervention can be seen as a pedagogical problem-solving model, whose most important goal is to provide all children with the most efficient instruction and intervention according to their needs (Jimerson et al., 2007). The instruction and intervention are mostly divided into three levels of support: Tier 1, Tier 2, and Tier 3 (Riley-Tillman & Burns, 2009), but other tier systems also exist (Fuchs, Fuchs, & Schumacher, 2011). Increasing levels mean that the focus becomes more individualized, the support becomes more intense, and the support is provided over a longer period of time (Riccomini & Smith, 2011). Bryant et al. (2008) describe the relations between tiers so that Tier 1 consists of evidence-based core instruction for all children, Tier 2 includes supplementary intervention and ongoing progress monitoring for children who struggle with learning, and Tier 3 is designed for children who are struggling so much that they require intensive intervention. Previous research shows that research-based intervention programs that are provided with care and whose effectiveness has been investigated produce better learning results in the classroom than non-research-based interventions (e.g., Jacob & Parkinson, 2015; Slavin & Lake, 2008).

In general, intervention programs can be used on classroom level, small group, and individual level. The most important difference between them is their focus. The classroom interventions mostly try to raise the level of a whole group of learners; these can be called Tier 1 interventions if RtI is applied. The need for such interventions comes from the information about, for instance, the differences between schools and school districts. Small group interventions are designed to meet the specific learning needs of children who have learning difficulties. Individual interventions focus on learning difficulties of an individual student. Individual and small group interventions are often used in Tiers 2 and 3 if RtI is applied. Small group interventions offer good possibilities for children to work together and practice skills that they have problems with, utilizing tasks designed to their level of knowledge and needs. When there are only 4–8 children in a group, it is easier for the teacher to focus on children's learning; she is able to guide and coach their learning. In small groups, there is also a possibility for the teacher to teach the target skill or topic and then let the children practice together and individually. The main challenge with individual interventions is the demand of resources; at the moment schools do not have enough resources to offer individual interventions for children who need support.

Finnish Web Services for Educators

In Finland there has been a positive tendency over the last 10 years to boost teachers' levels of knowledge concerning individual learning differences in early reading and mathematical skills. The emphasis has been mainly on the early identification of learning difficulties and early intervention, with the aim of moving toward the Responsiveness to Intervention model and general (Tier 1), intensified (Tier 2), and special educational support (Tier 3) in the national education system (National core curriculum for basic education (2014/2016)). The nationwide attempts in the field of early mathematics funded by the National Ministry of Education and Culture have focused on producing evidence-based knowledge for educators and providing them with assessment tools and intervention programs to be used with children struggling with learning. The author have been part of two teams that have designed two Web services for educators, namely, LukiMat (www.lukimat.fi) and ThinkMath (http://blogs.helsinki.fi/thinkmath/in-english/). From these Web services, ThinkMath focuses on small group intervention programs; thus it is in focus of this chapter; LukiMat has been described in another paper (Aunio, 2016).

ThinkMath Web service development started at the University of Helsinki in 2011. It provides educators with hands-on intervention materials to be used with children, aged 5–8 years, who have problems with learning early mathematical skills. The main idea behind ThinkMath was that educators needed evidence-based materials for offline use, as there was a significant lack of computer devices for young children to use in early childhood settings or in early primary school grades. ThinkMath delivers intervention materials and knowledge to educators. There is a knowledge base with evidence-based information concerning (1) mathematical skills development and learning difficulties, (2) thinking skills development, (3) motivational issues related to learning, (4) executive functions relevance for learning, and (5) (special) educational interventions. In the knowledge base, we have provided short videos to explain the main ideas to educators as clear and fast as possible. The Material section offers group-based intervention materials for practicing, for instance, mathematical skills with children in small groups.

The base for the development of mathematics knowledge base and materials was the core factor model of the mathematical skills in children aged 5–8 years (Aunio & Räsänen, 2015), which we originally developed for LukiMat. The model aimed to be a working model for the educators by presenting them with an overview of the most important skills that develop in early childhood and, secondly, aimed to make educators aware of the individual differences in early mathematical skill development. This model was based on a systematic literature review of longitudinal studies investigating mathematical development in this age group. We also analyzed the assessment batteries designed for identifying children with potential learning difficulties in mathematics. We were able to categorize skills into four main groups of numerical factors that are most crucial for the development of mathematical skills: symbolic and nonsymbolic number sense, understanding mathematical relations, counting skills, and basic skills in arithmetic (Aunio & Räsänen, 2015). In the ThinkMath materials, we focused on practicing these skills with children performing low.

The design related to pedagogical characteristics followed the findings in the research literature (Mononen et al., 2014). In the ThinkMath mathematical skill intervention programs, explicit teaching was one of the main guidelines along with several ways to practice the skills in focus (e.g., Gersten et al., 2008, 2009). In line with these recommendations, each lesson consists of a teacher-guided activity to model a new mathematical learning concept or strategy as well as guided and peer activities (e.g., hands-on activities with manipulatives or card and board games based on the current topic). At the end of the lesson, there is a short, paper-and-pencil individual activity. Another general feature is that mathematical ideas are represented following the concrete, representational, and abstract levels, thus giving meaning to abstract concepts by using visual representations (e.g., cubes, bundles of sticks, dot cards structured in tens and hundreds) (e.g., Mononen, 2014). The teacher manual includes 12–15 lesson plans of 35–45 min each. The lesson plans include specific instructions for teachers to follow in each activity. The manipulatives are made of low-cost, everyday materials found in every classroom, combined with printable materials (e.g., dot and place value cards) included in the manual. During the development of the intervention materials, we worked closely with educators and investigated the effects of these intervention programs on low-performing children through a quasi-experimental, pre-post measurement with intervention and control groups in different age groups (Mononen, Aunio, & Leijo, in revision; Mononen & Aunio, 2014; Mononen & Aunio, 2016).

Studies with ThinkMath Intervention Programs

The second-grade intervention study (Mononen & Aunio, 2014) was done with 88 children (M age 8 years and 2 months) from 4 classes in schools located in 2 southern Finnish cities. The intervention program used in this study was Improving Mathematics Skills in the Second Grade (Mononen & Aunio, 2012a). It aims to practice number-word sequence skills, counting, and conceptual place value knowledge in the 1–1000 range, following the guidelines of explicit instruction. Children's mathematical skills were measured with the Assessment of Mathematics Skill in the Second Grade (AMS-2) (Aunio & Mononen, 2012a). It is a paper-and-pencil test and measures (1) the number of forward and backward word sequence skills; (2) numerical relational skills associated with base 10 and place value knowledge; (3) addition and subtraction word problems; (4) multi-digit addition and subtraction calculations with number symbols, all within a 1–1000 range; and (5) addition and subtraction facts in the 1–20 range (40 items, 2 min' time). Children's thinking skills were assessed using the Assessment of Thinking Skills in the Second Grade (Hotulainen, Mononen, & Aunio, 2012a). Reading comprehension and fluency skills were measured using a standardized reading test for primary grades (Lindeman, 2005). Mathematical skills were measured three times: shortly before the intervention, immediately following the intervention, and 3 months after the intervention. The thinking and reading skills were assessed at the first of the three time points. Children were divided in the low-performing intervention group (n = 11), the low-performing control group (n = 13), and the typically performing control group (n = 64 children). The intervention program lasted 6 weeks, and there were two 45-min intervention sessions per week. The results demonstrated that the low-performing intervention group made significant improvements in mathematics whole scale and addition and subtraction facts but did not show significantly better scores compared to the low-performing control group. In addition, neither the intervention children nor the control children were able to perform at the same level of their peers following the intervention. There was no difference between low performance children in the control and intervention groups in terms of their thinking and reading skills. Although there were not many scientifically significant results, there was a trend to be seen that when children with low mathematical skills received extra support, their skills developed, but when the intensified instruction ended, so did the development of their skills. This was especially true of arithmetical fluency skills.

Mononen and Aunio (2016) investigated the impact of ThinkMath intervention for Finnish first graders (N = 151, M age = 7 years and 2 months) with low numerical performance. The children were from nine classrooms located in three cities of Southern Finland. This program focused on increasing the counting skills knowledge, including the number sequence and enumeration skills in number range 1–20 (Mononen & Aunio, 2012b). The study applied a quasi-experimental design using control groups. The effects of intervention were examined using the Assessment of Mathematics Skills in the First Grade (Aunio & Mononen, 2012b). This group-based paper-and-pencil test includes mathematical tasks in the range from 1 to 100: (1) mathematical relational skills (i.e., number comparison), (2) counting skills (i.e., verbal and object counting), and (3) word problems (i.e., verbal addition and subtraction problems). Single-digit addition and subtraction facts in the range from 1 to 20 are also assessed (within a 2-min time limit). A sum score for the subscales 1–3 (i.e., a combined scale) was used to identify the low-performing children. In addition, the relations between inductive reasoning (Assessment of Thinking Skills in the First Grade by Hotulainen, Mononen, & Aunio, 2012b), language (reading fluency Allu by Lindeman, 2005; listening comprehension Ytte test by Kajamies, Poskiparta, Annevirta, Dufva, & Vauras, 2003), and mathematical skills were examined. The intervention program was provided in small groups 12 times during 8 weeks; 1 session lasted about 45 min. The development of intervention children (*n* = 11) was compared to the development of low-performing (*n* = 26) and typically performing (*n* = 114) children. The results showed significant effects of intervention, as the children in the intervention group made significantly greater gains in their mathematical performance from Time 1 to Time 2, compared with the low-performing control and typically performing children. One important finding was that the children with low performance in mathematical skills showed lower performance also in their inductive reasoning and reading fluency skills than did children with typical performance. This means that when supporting these children, we also need to think about how to support children's thinking skills early on. The main conclusion is that a relatively short counting skill intervention that applied explicit teaching showed promising effects in improving low-performing children's mathematical performance as a supplemental instruction.

In our kindergarten intervention study (Mononen, Aunio, & Leijo, in revision), the children in the low-performing group were studied in more detail. The children in this group were children whose mathematics performances were below the 10th percentile (i.e., very low performing, *n* = 20) and children whose mathematical performances lay between the 11th and 25th percentiles (i.e., low performing, *n* = 18). The results were collected with a scale (Aunio & Mononen, 2012c) measuring mathematical relational skills, number-word sequence skills, and counting skills which showed that the number of children who reached an average level of performance at the posttest stage was higher among the group of children with low performance (67%) versus those with very low performance (35%) (Mononen, Tapola, & Aunio, 2015).

Westerholm and Aunio (submitted) investigated the effects of ThinkMath intervention for Finnish as second language kindergarteners. There were nine children (M, age 6 years 8 months) participating in the study from one metropolitan area school.

In this study we used ThinkMath: mathematical relational and counting skill intervention program (Mononen & Aunio, 2012b) as was used in the study of Mononen and Aunio (2016). Children practiced making comparisons on quantities and numbers using related concepts such as more and less and counting number sequences orally. In addition, counting objects and matching them with number words and number symbols was practiced. Based on the research literature concerning the learning challenges of children with the instruction language as a second language (Arnold, Fisher, Doctoroff, & Dobbs, 2002; Clements & Sarama, 2008; Codding, Chan-Iannetta, Palmer, & Lukito, 2009; Fuchs et al., 2008; Fuchs, Fuchs ym., 2008; Klein, Starkey, Clements, Sarama, & Iyer, 2008; Mercer & Sams, 2006; Starkey, Klein, & Wakeley, 2004), we added motivation (i.e., prize), explaining mathematical talk in the intervention program. Children's mathematical skills were measured with Early Numeracy Test (Van Luit, Van de Rijt, & Aunio, 2006) before, immediately after, and five weeks after the intervention ended. The intervention program lasted 8 weeks having 35–75 min sessions twice a week. The results showed that the ThinkMath intervention program with added motivating features and explicit mathematical talk was a useful way to support the early mathematics learning of children that had Finnish as a second language and low mathematical performance in kindergarten.

ThinkMath service has risen interest outside Finland as well. We now have two international research projects, one in South Africa (e.g., Aunio, Mononen, Ragpot, & Törmänen, 2016) and one in Norway (https://thinkmathglobal.com). The most important scientific question still is, if it is possible to enhance the early mathematics learning of children with low performance with ThinkMath small groups interventions. In South Africa the learning context is quite different to Finland; it will be interesting to see what kind of challenges the differences in teacher education, classroom organization, and children's background offers us. Our project in Norway will increase our knowledge about how well our originally Finnish programs work in system where school starting age is different and teacher education is less research based than in Finland.

Conclusion

The existing intervention studies, meta-analysis, and reviews have shown that it is beneficial to use explicit and structured small group mathematics interventions with low-performing young children. The work on Finnish LukiMat and ThinkMath projects have not only shown that it is possible to develop evidence-based materials and that educators appreciate them but also that Web services are a very efficient way to deliver such information and materials. Even though it is challenging to obtain significant and lasting learning effects with intervention studies in natural educational settings, these studies are essential in providing educators with evidence-based materials. In the future we will need more research reviews and large-scale intervention studies to be able to understand how to support children with various age and needs in mathematical learning.

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Chapter 42

Perspectives to Technology-Enhanced Learning and Teaching in Mathematical Learning Difficulties

Pekka Räsänen, Diana Laurillard, Tanja Käser, and Michael von Aster

Today, technology is a part of almost every aspect of life of those living in a developed country. People are constantly "online" and have an easy access to information and services. The speed of change has been high. Therefore, predicting how our digitalized life will change within the next 5 to 10 years could only go wrong. One new innovation in battery or processor chip technology will change totally how the future will look. Innovations appear nearly every day. Education is one of the areas where this rapid development of technologies has opened up a lot of new possibilities, but it has also raised fears in the same way as when schoolbooks were introduced 100 years ago. There were fears that introducing study books at school would destroy children's abilities to memorize (Wakefield, 1998). But despite the fears, now that it is cheaper to buy tablets for children instead of printing books for them, the discussion is changing more toward questions about the key elements of the pedagogies inside the technologies.

It would be difficult to cover all possible ways in which technology-enhanced learning (TEL) is and could be used at schools or homes for learning, collaboratively or individually, and therefore we only take the reader on a short global trip to

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consider the barriers and possibilities there are at the moment in using TEL in mathematics education and learning. Our focus is especially on using TEL tools to support children with low performance in mathematics. Technology is not only used for educational applications; there are plenty of computer-assisted tools being used to organize and plan the education system, as well as tools for assessment, and in research on mathematical skills and disabilities. In addition, there are more and more computer-assisted tests, tools for brain imaging, and technologies affecting brain activations that have increased our understanding about the mathematical brain. However, these methodological and technical advancements are out of the scope of this chapter. Hence, we concentrate on technologies directed toward advancing learning and the pedagogies of mathematics in the classrooms (Fig. 42.1).

The majority of the chapters in this book tell us that the mathematical learning disabilities (MLD) are by definition something special: standard classroom education is not enough for these children (see Landerl, Chap. 2, Santos Carvalho & Vitor Geraldi Haase, Chap. 22, this volume), and alternative pedagogies should be used. However, teachers have a limited amount of knowledge and information about effective and research-informed pedagogies on MLD. Additional knowledge is needed to understand the phenomena, how to identify these children in the classroom, what kinds of alternative ways there exist, and how to use them to support the children in this special group. Technology can offer globally accessible media to inform teachers and other professionals about MLD on a level never seen before. Therefore, we will raise the issue of teacher professional education as one of the key global issues where TEL tools will play a significant role.

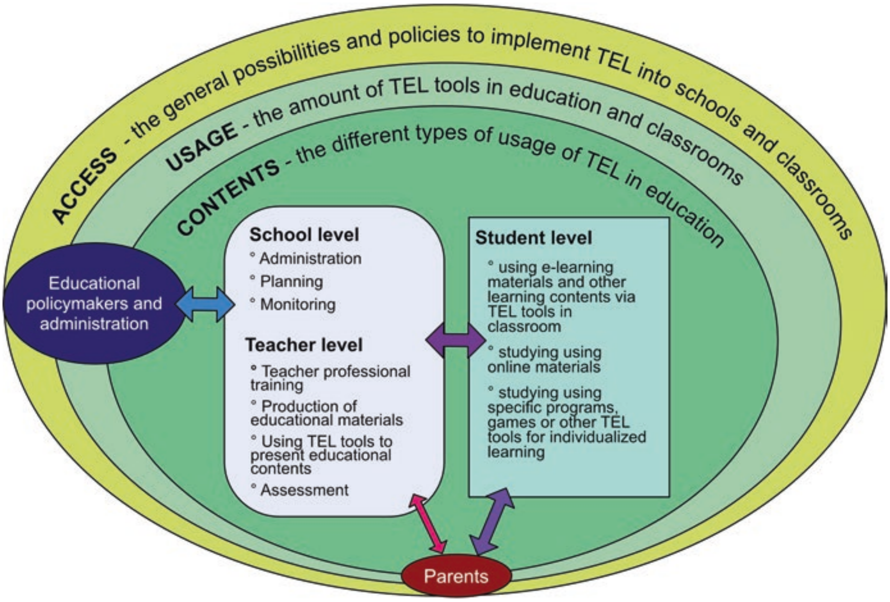

Fig. 42.1 Current questions in usage of TEL at schools

The second story in this book is that there is a huge variety in mathematical skills in general and also within the group of children with MLD. There is a variance in both domain-general and domain-specific skills, both affecting learning and teaching and requiring very individual pathways for development. Partly, this variation in individual skills is connected to differences in brain functioning (see, De Smedt, Peters, & Ghesquière, Chap. 23, this volume) and partly to learning environments and pedagogies (see Gaidoschik, Chap. 6, this volume). The brain development of the frontoparietal numerical network of the children with MLD seems to function differently than in typically developing children (McCaskey et al., 2017). However, two recent studies with computer-assisted training showed that abnormally functioning connectivity in MLD will be normalized on the neuronal level by rather short, intensive training (Iuculano et al., 2015; Michels, O'Gorman, & Kucian, 2017). This indicates that at least under the age of 10, children with MLD do not develop compensatory mechanisms to reach the same level of proficiency, but the training offers them the means to build representations that are similar to those of their typically performing peers.

Technology offers many possibilities for this kind of very individualized intensive training that has not been possible to conduct in large classrooms. However, before we discuss the different types of interventions, we need to look at the global questions in ICT in classrooms. One of them is access to technologies.

Global Inequalities in Access to Learning Technologies

An access to electricity is one of the key issues for educational equality today. The United Nations Department of Economic and Social Affairs reported that even though the access to electricity has more than tripled from 1990s still "about 90 percent of children in Sub-Saharan Africa go to primary schools that lack electricity, 27 percent of village schools in India lack electricity access, and fewer than half of Peruvian schools are electrified. Collectively, 188 million children attend schools not connected to any type of electricity supply" (UNDESA, 2014).

According to UNDESA the educational benefits of electrification are clear. Lighting extends the studying hours by enabling longer school days, more reading time, and possibilities to do homework. It also allows teachers to prepare learning materials after the school days. Electricity enables both students and teachers to use modern mass media tools such as radio, television, computers, and the Internet. Likewise, it improves the quality of the basic circumstances, such as sanitation and health. Electricity in schools is needed for many of the basic tools used daily in developed and developing countries: audiotapes, projectors and slide projectors, printers and copy machines, digital cameras, radios and television, phones, and the ICT technologies – computers, tablets, mobile phones, and the Internet. Therefore, it is not a surprise that the electrified schools outperform non-electrified schools on crucial educational indicators and that electricity enables broader social and economic development of the communities (see, e.g., UNESCO, 2014a, 2014b; Zhang, Postlehwaite, & Grisay, 2008).

As always when there is development in access to technologies, inside the big positive picture, the data from individual cases show contradictory results. While some studies on access to electricity do find a positive effect, some find no effect, and some even negative effects. For example, when we look at the effects of bringing electricity to the area, some studies do find an effect in an increase in time spent studying (Barron & Torero, 2014; Khandker, Samad, Ali, & Barnes, 2012), but some do not (Bensch, Kluve, & Peters, 2011). Squires (2015) actually found that access to electricity in rural areas in Honduras increased the school dropout and produced less attendance to the school due to increased "need for child labour" at homes. Whatever the question with technology, just having it does not guarantee positive outcomes.

From 2015 in developing countries, more households had a mobile phone than they had electricity or running water (World Bank, 2016). Mobile devices, with their increasing affordability and storage, can contain a vast amount of educational content, including reading and learning materials and games targeted to on a range of ages. In addition, unlike computers, handheld mobile devices require substantially less electricity or infrastructure. Due to the advantages in solar power, mobile devices are capable of reaching even the most marginalized communities, and research has shown mobile learning devices have the potential to widen access and supplement education in remote and underserved areas of the world (Kim et al., 2012; Ling, 2004). In their meta-analyses Sung, Chang, and Liu (2016) showed that the effect size of implementing mobile devices into classroom education was significantly more effective than teaching methods that only use pen and paper or desktop computers. For mathematics the effect size was 0.34 including different types of approaches from cooperative learning to games. That is about the same level as the other meta-analyses have given to using TEL in mathematics education (summarized in Räsänen, 2015).

Still only one in seven persons in the world has access to a high-speed Internet connection. High-speed connections are needed for the rich educational contents already available on the Internet. The situation in access to TEL tools via the Internet is changing rapidly. However, there is only a limited amount of up-to-date worldwide information about the current situation of ICT at schools. UNESCO has recently started a project to collect such information (http://uis.unesco.org/).

Online Learning, Virtual Worlds, and Social Learning Environments

All teachers, most parents, and some students know that poor mathematical skills will affect chances in life. In fact, poor mathematical skills are more of a handicap in life than poor reading (Bynner & Parsons, 1997; Parsons & Bynner, 2005). The problem for teachers and parents is to identify the causes of the MLD. It is often assumed that children who cannot learn even simple arithmetic must be low intelligence (just as it used to be assumed that children who were unable to learn to read must be stupid). In fact, intelligence has little to do with the ability to learn arithmetic. Almost any child can. However, there is a small proportion of children who are unable to learn arithmetic in the normal way. These children are dyscalculic. That is, they show quite early and specific cognitive deficits, just as we now know, as well as dyslexics with different specific deficits making it more difficult to acquire reading skills (Vanbinst, Ansari, Ghesquière, & De Smedt, 2016). We can now easily distinguish children with dyslexia from children with other difficulties that prevent the normal acquisition of reading, and we can now distinguish dyscalculia from other causes of poor math attainment. However, very few teachers or parents or educational psychologists, not to mention education authorities, have heard about dyscalculia and therefore have never been trained to identify it or to provide the specialist help needed for the people concerned.

One way of spreading the word about the MLD and dyscalculia is to use a technology that is accessible to most of the key professionals in many developing countries, namely, the Internet. Technology works best by responding to the most challenging problems, and education has plenty to offer. By 2025, the global demand for higher education will double to ~200 m per year, mostly from emerging economies (NAFSA, 2010). It has been estimated that there is a need for millions of new teaching posts for universal primary education (UNESCO, 2014b), the largest growth being in sub-Saharan Africa.

One of the big challenges is how to reach the children who need good primary education and, especially, how to train teachers to spot MLD as early as possible. There are straightforward tests that have already been standardized in many countries for identifying the children at risk and models of how to build personalized learning plans, including the use of interactive games that could be applied. Many of these games are digital, adaptive, and available widely at low cost. This is an example of how teachers – and other professionals – globally could make use of an educational technology resource that has been created and developed in one place. A different technology-enhanced method, for developing teachers on the large scale, is to create massive open online courses (MOOC), and a start has been made on this, aimed at primary teachers (Laurillard, 2016a, b).

An international course team working in partnership with UNESCO developed a Coursera MOOC on "ICT in Primary Education," which reached ~10,000 teachers around the world, over 1200 of whom were located in low-income countries (see Laurillard, 2016a, b), showing that such an approach is not confined to reaching only the rich countries. A more niche course on Primary Education, dyscalculia and other mathematics disabilities, with targeted marketing, could certainly provide collaborative professional development for teachers and leaders in most of the countries of the world.

To generate the network of development in the most underdeveloped areas, one possibility could be that each of these MOOC-participating teachers could work locally to engage 25 teachers in collaborating on using the course resources to develop improved localized classroom methods at regional level. To reach the children in need, each of those teachers could then set up support groups of eight adults in villages, townships, and communities, working together to train them to become more able teachers. This multiplies up to hundreds of thousands of teachers. The large-scale technological capability is only needed at the first stage. After that the local systems can be used, making use of the cascaded digital resources and innovative ideas. However, for the collaborative approach to be preserved, it is important for the teachers experimenting with their localized solutions to pass their ideas and experiences back up the chain.

Increasingly, some of the most challenging contexts – remote rural areas, urban slums, and border cities – are beginning to have access to mobile devices and connectivity. It is not the technology that makes it difficult, but the organization and support for the human systems in the network. In the urban slum areas, for example, adults set up their own private schools where there are too few government schools, but they are unofficial, so they have no support or access to professional development (Oketch, Mutisya, Ngware, & Ezeh, 2010). Providing this kind of support could now be affordable but would still have to overcome the political barriers.

This is where digital technology could make the critical difference by offering the means for collaborative professional development. The two-way communication and sharing of designs, products, and localized solutions is a way of building professional knowledge of effective practice. This is not the typical trajectory of pilot – rollout – fade. For example, Khan Academy, an educational website with thousands of free video lessons on various topics, especially illustrating ways to present mathematical content from basics to upper classes, has more than ten million users monthly. And it is not only students using it, but also teachers, to improve their pedagogical skills.

It is worth asking, for any big challenge, “how can technology help?” because digital tools and environments operate on the very large scale and vastly increase efficiency and scope. MOOCs are an opportunity for the academic community to think through how such technologies could serve our moral imperative to achieve a wider reach and greater contribution to society. A new model of collaborative professional development is one way to do that.

In the long run, we can expect that the Internet access will harmonize the pedagogies used in mathematics teaching and interventions on MLD globally. Some of the models of effective, research-informed methods used to support children with MLD in top performing schools can be localized and mimicked in rural schools and in less advanced schools. Most of the pedagogical methods do not require additional resources; the critical issue is how concepts are presented and what kinds of elements do the interactions between the teachers and students contain. As important as it is to use TEL to support the students, the effects can be multiplied via supporting teachers in their work.

Availability: The Surge of Learning Games

Shuler, Levine, and Ree (2012) calculated 6 years ago that there were approximately half a million apps available on the Appstore. In the education category of the apps, general early learning was the most popular subject (47%) and mathematics the second (13%). From that the total amount of apps has raised to over two million with educational apps reaching soon 200,000 individual apps or games, and the comparable store for Android machines shows even larger figures. However, there is no clear criteria what an educational app means in these distribution channels. In their discussion of mathematics apps, Pelton and Pelton (2012) noted that "while some are commendable, almost all of the rest are simple flashcards, numeric procedures, or mobile textbooks. Very few currently available apps have engaged best practices by integrating visual models to support sense-making." The ease of access and the fact that the majority of these apps and games are low cost, often totally free, mean that they are readily available to the general population, but the question remains, what quality they have and what is being learned by using these apps. There are relatively few applications that are built on research or have an evidence-based background (Doabler, Fien, Nelson-Walker, & Baker, 2012; Young et al., 2012). The What Works Clearinghouse (WWC, see https://ies.ed.gov/ncee/wwc/FWW/) collects information about promising intervention programs including TEL applications, but only a few of them have any research that could be used to evaluate the program efficacy.

Usage: Does Using TEL Tools Help to Produce Better Learning?

With increased use of TEL at schools, the question of effectiveness has been in focus of discussions during the last years. The answer to the question depends on the data and the design used. The often-found result from individual controlled studies (e.g., Carter, Greenberg, & Walker, 2017) to the international comparison datasets (OECD, 2015) has been that increasing computer usage in studying at schools does not per se produce better learning. This discussion is actually old. Already Clark (1983; Kozma, 1994) stressed that the content is more important than the media that is used to deliver it. They argued that separating media from educational method is an unnecessary schism which does not produce real new insights in education.

Nevertheless, the question is still important to educational policy-makers. Technologies require large investments but age very quickly. Are these investments worth doing, or could it be more beneficial to invest in something else? In this question, we often turn to the international datasets. However, those have not shown promising results on these investments, but the complexity of interpreting these results is clear. If we look at the results from international datasets, like PISA (illustrated in Fig. 42.2), and compare the country averages in mathematics and the number of students that have used computers in the classroom to do mathematics during the last month before the PISA assessment, we find a negative correlation ($r = -0.36$). The more students there were who had used computers in a math class, the lower the respective country's average score. The negative correlation seems to have two roots. Firstly the majority of the high-performing countries (e.g., Korea, Japan, or Finland) have taken a slow start in moving to digitalized education, and secondly, there are some below average-performing countries which have invested a

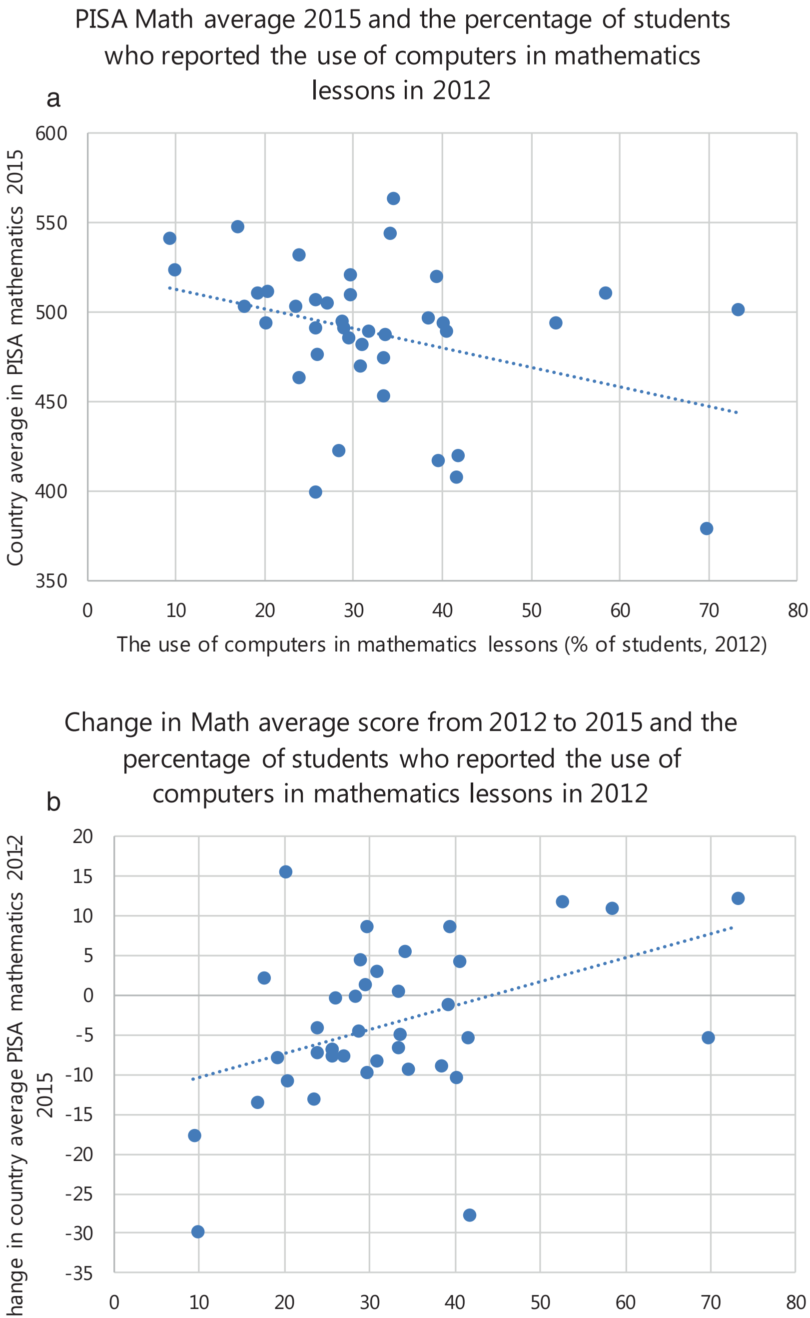

Fig. 42.2 The percentage of students who had used computers in mathematics lessons during the month prior to the PISA test in 2012 contrasted against (**a**) the country average in PISA 2015 (trend line $r = -0.36$, $p < 0.03$) and (**b**) change in the average score from 2012 to 2015 (trend line $r = 0.41$, $p < 0.01$). (Source of the data: OECD, 2015)

lot to using TEL in education. If we remove these extremes from the equation, the negative effect disappears. Also, the within-country data shows a nonlinear effect: students not using or using a lot of computers in classrooms do not perform as well as those who use computers moderately. Moderate usage seems to indicate better pedagogical considerations when and where the TEL is used. An alternative look on exactly the same data leads to another result. If we change the level of performance to improvement in average performance during the last 3 years, the correlation turns into positive ($r = +0.41$). It means that more usage of computers in a math class turns to better results 3 years later. Why is this? When we look at the gain score from the latest PISA studies (i.e., here the change from 2012 to 2015), we notice that many countries that had invested a lot in using ICT in education (e.g., Denmark, Sweden, Uruguay) have shown low average scores in 2012 but rapid improvement and at the same time many high-performing countries (e.g., Korea, Taipei, Finland) showed the largest declines. How much these changes are connected to ICT or whether they are connected to other changes in educational cultures and to other investments in education require more detailed analyses from various national and international datasets.

The effectiveness of computer-assisted interventions has been studied since the 1960s. This offers us another set of data to look at the value of investment to TEL tools in mathematics. There are several meta-analytic studies (summarized in, e.g., Räsänen, 2015) showing that there have been three most-gaining subgroups across the years of using TEL: (1) younger children show a larger gain than older children, (2) children with special needs seem to show more benefit than children in studies with more heterogeneous samples, and (3) studies where TEL has been used as supplementary education instead of replacing the teacher have shown better results (e.g. Lavin & Sanders, 1983; Li & Ma, 2010; Niemiec & Walberg, 1987; Slavin & Lake, 2008). In addition, the studies conducted in developing countries tend to show higher effectiveness than those done in developed countries. These results indicate that it seems to be easier to produce better results when the starting level is lower, especially if the reasons for a lower starting level have been poor access to education or low SES and not cognitive factors. Children with cognitive deficits or mathematical learning disabilities have only recently become a focus of research. Chodura, Kuhn, and Holling (2015) looked specifically at interventions for children with MLD. They found a similar level of effectiveness for computer-assisted and face-to-face interventions. These two are often contrasted: While some stress the importance of direct contact and social interaction in the learning process, one of the most commonly presented reasoning on using TEL tools has been that the gamification brings engagement and motivation into the learning that the standard classroom or special education lacks.

Affective and Motivational Factors

Educational TEL programs are typically designed in a format of a game. The serious games, as educational games are often called, are hypothesized to address both the cognitive and the affective dimensions of learning (O'Neil, Wainess, & Baker, 2005), to enable learners to adapt learning to their cognitive needs and interests and to provide motivation for learning (Malone, 1981). However, the assumption underlying the motivational appeal of serious games is based on the addictive nature the commercial computer games have. However, the results of a meta-analysis show that serious games are not more motivating than other instructional methods (ES = 0.26, a nonsignificant difference; Wouters, Van Nimwegen, Van Oostendorp, & Van Der Spek, 2013).

An essential difference between leisure computer games and serious games is that playing for entertainment is chosen by the players and played whenever and for as long as the player wants, whereas with the serious games, the playing and playing time are defined by someone else (e.g., teacher, game developer, researcher). In addition, logic of effort needed and cognitive load in entertainment and educational games are different. Educational games typically aim to increase the difficulty and cognitive load systematically to match the players current skill level to boost performance (about the adaptive logic inside math games, see, e.g., Räsänen et al., 2015), while in entertainment games, the cognitive load varies more freely and does not aim to maximize the performance level. Therefore, there is no strong evidence that gamification of TEL in mathematics education would produce by itself stronger and long-lasting internal motivation and that it would produce better learning that way.

Contents: What Is Inside the Intervention Games for MLD?

There have been two content areas that have dominated the studies with TEL games targeted to children with MLD or low performance. The first has been the key symptom area in MLD, the lack of arithmetic fluency. This approach dominated the first decades of research with flashcard-type of training and its variants to improve memory retrieval of arithmetic facts. The rise of ideas of number sense as a core difficulty behind the difficulties to learn the basic arithmetic skills (e.g., Piazza et al., 2010) has led to focus the interventions to either symbolic or nonsymbolic number sense and to games combining arithmetic to number line representation to illustrate the distances and relations between numbers. The latter approach has dominated the research during the last 10 years, and we will concentrate on these findings. A third raising hypothesis has been that training domain-general skills strongly connected to numerical cognition might likewise boost learning mathematics. Especially working memory (Passolungi & Costa, Chap. 25, this volume) and increasingly also, spatial skills (Resnick et al., Chap. 26, this volume) have recently grasped the attention of the researchers.

Training Number Sense

There is an ongoing debate about the roles of symbolic and nonsymbolic number sense in the development of MLD. Dehaene (1997) suggested that an evolutionarily grounded analogue magnitude representation, also called an approximate number system (ANS) or "number sense," underlies the numerical understanding. After this suggestion many studies have aimed to train the ANS with the intention of transferring improvements to symbolic arithmetic. There are some grounds for this idea. The ANS, typically measured by requiring participants to choose which of two dot arrays contains more dots, correlates with measures of symbolic math in both adults and children (e.g., DeWind & Brannon, 2012; Halberda, Ly, Wilmer, Naiman, & Germine, 2012; Halberda, Mazzocco, & Feigenson, 2008; Lyons & Beilock, 2011) and also predicts from preschool to math achievement tests at school age (Gilmore, McCarthy, & Spelke, 2010; Mazzocco, Feigenson, & Halberda, 2011a). Likewise, children with MLD perform less well in ANS tasks compared with typically performing children (Mazzocco, Feigenson, & Halberda, 2011b).

For example, Park and Brannon (2013, 2014) showed that adults after a very short training with nonsymbolic addition and subtraction tasks improved the performance in symbolic addition tasks. In their training two clouds of dots were presented to the participant, who needed to estimate without counting, which one of the two new clouds of dots would match with the answer of the calculation presented. According to Park and Brannon this illustrates a causal link between these two representations. Wang, Odic, Halberda, and Feigenson (2016) got similar results with 5-year-old children (see however Merkley, Matejko, & Ansari, 2017 about the critics). Park and colleagues (2016) gave a similar tablet game with nonsymbolic approximate addition and subtraction of large arrays of items to 3–5-year-old children finding selective improvements in math skills after multiple days of playing compared with children who played a memory game. Khanum and others (Khanum, Hanif, Spelke, Berteletti, & Hyde, 2016) have replicated these results using similar training tasks with Pakistani children of school-age, demonstrating that there is a cultural invariance in these results.

In addition to these experimental games, Wilson and colleagues (Wilson, Revkin, Cohen, Cohen, & Dehaene, 2006) have developed a free adaptive computer learning game for children with MLD (http://www.thenumberrace.com/nr/). The Number Race game is reminiscent of traditional board games in which one throws a dice and advances by that number of steps. The dice throwing has been replaced with a comparison task (nonsymbolic and symbolic). The game is constantly adapting based on an artificial intelligence algorithm. This algorithm represents the learner's current skill level ("knowledge space") in three dimensions, and it is programmed to ensure an average accuracy of 75%. The three dimensions of the model are the ratio of the quantities presented (the distance effect), the time allowed to respond, and the conceptual complexity of the format in which the quantities are presented (from dot patterns to arithmetic). Several studies have used this game module to test if this kind of training would produce benefits to learning (Obersteiner, Reiss, & Ufer, 2013; Räsänen, Salminen, Wilson, Aunio, & Dehaene, 2009; Sella, Tressoldi, Lucangeli, & Zorzi, 2016; Wilson, Dehaene, Dubois, & Fayol, 2009). Szűcs and Myers (2017) critically analyzed these studies and concluded that there is no conclusive evidence that specific ANS training improves symbolic arithmetic. They found many problems in these studies, not limited to the fact that it was unclear whether the game directly focused on ANS or on some other numerical processes more important for learning arithmetic. In their study, Sella et al. (2016) divided 4–6-year-old children into two groups, one playing Number Race and a control group playing with a drawing program. There were clear effects of the Number Race game compared to drawing activities to boost numerical skills of typically performing young children. However, the result does not directly point to the benefits of computer-assisted number sense training in early development, because it was a comparison between math and non-math training. In the study of Obersteiner et al. (2013), exact numerical representations were contrasted against approximate training, and he found no difference in learning between these two trainings. Räsänen et al. (2009) used the Number Race training with 6-year-old children with a risk of MLD and contrasted this against a training with a game with explicit training of number symbols, where the latter, according to a reanalysis of Szűcs and Myers (2017), seemed to produce slightly better results. In a similar fashion, Honoré and Noël (2016) contrasted symbolic and nonsymbolic training. Both trainings produced significant learning effects compared to control conditions, but symbolic training led to a significantly larger improvement in arithmetic than did nonsymbolic training.

Maertens and his colleagues (2016) used another type of approach to train the relations between numbers. They contrasted the above-described comparison tasks to number line training. Performance on tasks where the child is asked to estimate the position of numbers in the number line has been shown to be related to children's mathematical achievement (e.g., Booth & Siegler, 2008; Friso-van den Bos et al., 2015; Muldoon, Towse, Simms, Perra, & Menzies, 2013; Siegler & Booth, 2004). Moreover, interventions that have focused on improving numerical representations through game-based number line tasks have shown transfer to arithmetic learning and mathematical performance (Fischer, Moeller, Bientzle, Cress, & Nuerk, 2011; Link, Moeller, Huber, Fischer, & Nuerk, 2013; Siegler & Ramani, 2008). Maertens et al. (2016) found that both comparison and number line estimation trainings had a positive effect on arithmetic. However, there were no transfer effects from one task to another. This suggests that comparison and number line estimation rely on different mechanisms and probably influence arithmetic through different mechanisms.

Another game that uses number line as a way to present numbers and calculations is Calcularis (Käser, Baschera, et al., 2013). Because it is one of the few research-informed games developed for children with MLD, we look at it in more detail. The model of the game is based on the theory of a hierarchical development of mental number representations (von Aster & Shalev, 2007): The game builds up on early available concrete number representations (number as a set of objects) and the verbal symbolization (spoken number) that develops during preschool age followed by the development of the Arabic symbolization taught in school. At the last level, the mental number line is gradually built over the first years of elementary school. Children with MLD often exhibit problems in constructing and accessing this mental number line representation (Kaufmann et al., 2009; Kucian, Loenneker, Dietrich, Martin, & von Aster, 2006; Mussolin et al., 2010). The scientific evaluation of the precursor version of Calcularis (called Rescue Calcularis) demonstrated that children with and without MLD benefit from a number line training. Kucian and others (2011) showed that the neuronal changes observed after playing the game indicated a refined mental number representation as well as more efficient number processing.

Calcularis turns these findings on number processing and numerical cognition into the design of different instructional games, which are hierarchically structured according to number ranges and can be further divided into three areas (a content model: numerical understanding and representations, addition and subtraction, multiplication and division). The first area focuses on different number representations as well as number understanding in general. Transcoding between alternative representations is trained, and children learn the three principles of number understanding: cardinality, ordinality, and relativity. The first area is exemplified by the LANDING game illustrated in Fig. 42.3a. In this game, children need to indicate the position of a given number on a number line. To do so, a falling cone has to be steered using a joystick or the right and left arrow key. The second and third areas cover cognitive operations and procedures with numbers. In this area, children train the concepts and automation of arithmetic operations. In the PLUS-MINUS game (see Fig. 42.3b), children solve addition and subtraction tasks using blocks of tens and ones to model them.

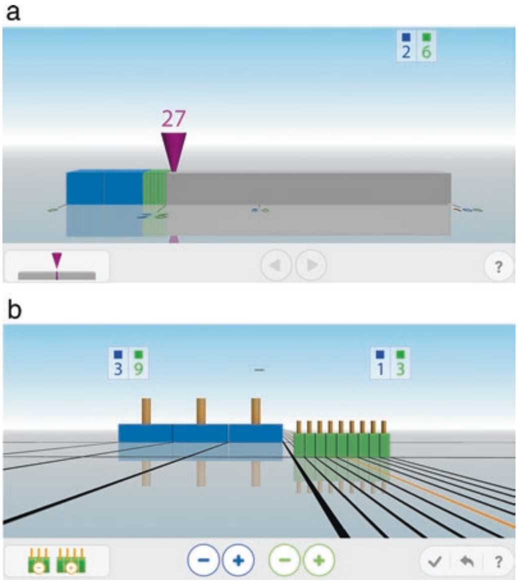

Fig. 42.3 In the LANDING game (**a**), the position of the displayed number (16) needs to be indicated on the number line. In the PLUS-MINUS game (**b**), the task displayed needs to be modeled with the blocks of tens and ones

To offer optimal learning conditions, the training program adapts to the knowledge state of a specific child (Käser et al., 2012; Käser, Busetto, et al., 2013). All children start the training with the same game. After each item, the program estimates the knowledge state of the child and displays a new task adjusted to this state.

In order to adapt the difficulty level and the task selection to the needs of a specific child, the training program needs to represent and estimate the mathematical knowledge of the child. This knowledge is modeled with a dynamic Bayesian network representing different mathematical skills and their dependencies as a directed acyclic graph. The model used for Calcularis consists of more than 100 different skills. A small excerpt of the network is displayed in Fig. 42.4. The skills are sorted into different number ranges. Within a number range, they are ordered according to their difficulties. The difficulty of a task depends on the magnitude of the numbers involved in the task, the complexity of the task, and the means allowed to solve the task. Modeling "46 + 33 = 79" with one, ten, and hundred blocks (*Support Addition 2,2*) is easier than calculating it mentally (*Addition 2,2*). Furthermore, tasks including a carry such as "46 + 37 = 83" (*Addition 2,2 with bridging to ten*) are more complex to solve than tasks not requiring carrying. In order to also be able to adapt to specific problems of a child, the program contains a bug library storing typical error patterns. If a child commits a typical error several times, the controller systematically selects actions for remediation.

The effects of the training program have been assessed in a pilot study with 41 children conducted in Switzerland (Käser, Baschera, et al., 2013) and a following comprehensive study with 138 children in Germany, where children were randomly assigned to 1 of 3 conditions (Calcularis training group, waiting control group, spelling control training group) with 6 and 12 weeks of training time (Rauscher et al., 2016, 2017). The results largely confirmed those of the pilot study: Compared to the two control conditions, children of the Calcularis training group demonstrated significant improvements with regard to arithmetic performance and spatial number processing abilities. These effects were already present after 6 weeks of training and became even larger after 12 weeks. In addition, the positive effects in math performance were accompanied by a significant decrease of math anxiety, which is

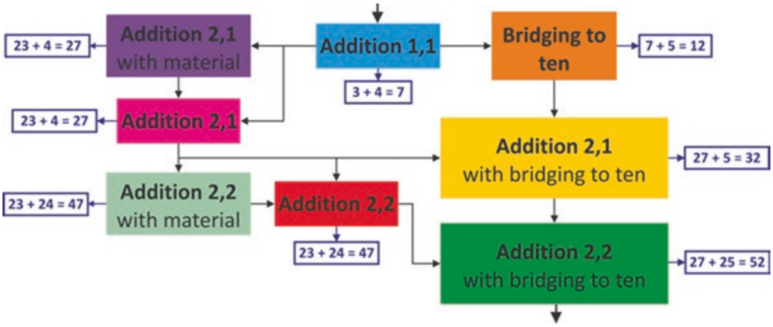

Fig. 42.4 Addition skill net in the number range 0–100 with example tasks

known to substantially contribute to developmental dyscalculia. Due to its adaptive nature, Calcularis is widely used in inclusive classroom settings to achieve intra-class differentiation. It is also suitable for intervention, in which children practice at home without direct supervision. The supervisors can monitor their students' work with the coaching application, which, as Calcularis itself, is browser-based.

Another game that has used the number line representation to illustrate the numbers and calculations is a free tablet game Vektor (http://cognitionmatters.org). However, it differs from the other new research-informed games in a critical way that it combines numerical and cognitive training. In the Vektor game, the numbers and calculations are presented both as five- and ten-pals (see also "number bonds to ten" in Butterworth, Varma, & Laurillard, 2011) and in symbolic calculation tasks with a number line representation. The cognitive training in Vektor is based on WM training with predominantly visuospatial tasks that have previously been shown to be effective in increasing WM efficiency (Bergman-Nutley & Klingberg, 2014; Melby-Lervåg & Hulme, 2013). The newest version of the game also contains visuospatial and visuospatial reasoning tasks, because it has been shown that visuospatial WM predicts later mathematical skills and that especially the number line representation is tied to visuospatial skills (Simms, Clayton, Cragg, Gilmore, & Johnson, 2016). However, even though there is a lot of evidence on the connections between visuospatial and numerical skills (Resnick et al., Chap. 26, this volume), we still lack studies about direct transfer effects from spatial training to arithmetic skills (however, see Lowrie, Logan, & Ramful, 2017).

Passolunghi and Costa (2016) have shown that working memory (WM) training significantly enhances children's numeracy abilities involving concepts of comparison, classification, correspondence, seriation, counting, and general knowledge of numbers (see also Holmes, Gathercole, & Dunning, 2009; Kroesbergen, van't Noordende, & Kolkman, 2014; Kuhn & Holling, 2014; St Clair-Thompson, Stevens, Hunt, & Bolder, 2010; Witt, 2011). In their study with the Vektor game on combining working memory and arithmetic training with 6-year-old children, Nemmi and his colleagues (2016) found that a combined training of cognitive skills and arithmetic was more effective than either WM or arithmetic training alone. However, they also found that when going beyond these group effects to a more individual level, there is a whole new world for researchers to tackle.

Typically, the effectiveness of a training is analyzed at a group level. Effectiveness of an intervention is considered to be good when children in the experimental group improve significantly more than children in the control groups. Theoretically, the education is the same in all subjects in the experimental group of the TEL intervention study. In a similar fashion as the education is the same to all children in the classroom. However, in the classroom the teacher should focus on individual performances. In reality, some children learn more, and some less, irrespective of the method used. This is a challenge to the teacher on how to raise the level of learning of the children who learned less. The same method and pedagogy is not beneficial to all. And every teacher knows this.

Likewise, this is a challenge to researchers. Instead of concentrating on groups, more research is needed about the factors behind the individual gains than about the effects at the group level. In the study of Nemmi and others (2016), they divided the children into subgroups based on the baseline level of WM and mathematical skills measured before the intervention. The main finding of the effectiveness of the combined WM and numerical training over numerical training alone was nonexistent in children with low WM and in children with below average skills in mathematics. The impact of an intervention varied by a factor of 3 between the subjects, depending on their baseline performance. Therefore, while one intervention can be extremely beneficial to some, another child with different profile of numerical and cognitive skills may not benefit from that specific training at all. Focusing on this question would bring researchers closer to the educational practices within the classroom. What do we need to know about the child to learn what kind of intervention is beneficial? As soon as we have more understanding about this question, we can build individualized, adaptive, and effective interventions with TEL tools.

From the Classrooms to the Lab

The majority of the noncomputerized interventions for children with MLD and programs for learning the basic number concepts recommend using manipulatives: small collections of objects to be ordered, categorized, compared, and counted (Clements & Sarama, 2011; Samara & Clements, 2009). This is a common knowledge for well-trained special needs teachers, who use a wide variety of activities with manipulatives in their classes (Dowker, 2004; Emerson & Babtie, 2014). They help children with MLD to learn the meaning of numbers by using concrete materials as well as by articulating their practice in multiple representations of diagrams and number lines and then building up to symbols and equations (Emerson & Babtie, 2014). These kinds of activities are also offered as computerized tasks in some TEL programs, such as the Building Blocks early educational program (Sarama & Clements, 2004), and in the NumberBeads game targeted to children and adults with severe MLD (Laurillard, 2016b).

Investigations using computerized manipulatives for geometry and fractions show that these can lead to statistically significant gains in learning new concepts (Reimer & Moyer, 2005). Olson (1988) found that students who used both physical and software manipulatives demonstrated a greater sophistication in classification and logical thinking than did a control group that used physical manipulatives alone. A computer environment offers students greater control and flexibility over the manipulatives, allowing them to, for example, duplicate and modify the computer bean sticks (Char, 1989; Moyer, Niezgoda, & Stanley, 2005).

Digital entertainment games are more and more combining the realities of virtual and real worlds. Games happen in 3D worlds, and the players can more and more realistically manipulate objects in these worlds. The educational applications are slowly moving to this direction, and most probably the next wave of intervention research on supporting children with MLD using TEL will concentrate on bringing in the effective traditions used by the experienced and well-informed special need teachers. Studies like Iuculano et al. (2015) show that an intensive face-to-face intervention is very effective in helping children with MLD to built up numerical skills and, thus, will provide a good starting point for researchers to think about the key features of the effective interactions and pedagogies needed in TEL tools. Do we have virtual manipulatives in virtual classrooms only, or do we see virtual teachers as well? Most probably yes, but first, there is still a lot to learn from the best teachers.

Final Word

Technologies are spreading fast, and almost all children in OECD countries have computers at home, and cheaper mobile technologies are reaching even the most underdeveloped areas. Even though technologies have a promise of advancing the education, the OECD report on technology usage at schools gives a serious warning: “perhaps the most disappointing finding of the report is that technology is of little help in bridging the skills divide between advantaged and disadvantaged students. Put simply, ensuring that every child attains a baseline level of proficiency in reading and mathematics seems to do more to create equal opportunities in a digital world than can be achieved by expanding or subsidising access to high-tech devices and services” (OECD, 2015).

The question of access to technology is easier to solve than the question of effective contents for learning, especially when we aim to improve the skills of those children with learning disabilities. In this chapter we have tried to introduce some new ideas from research during the last decade on how the question of content has been approached. Advances in basic neuroscientific research will uncover more about the mechanisms of learning, raising new contents to be implemented in serious games and even to electronic school books.

Technology is making education a joint global issue. It offers teachers new sources for collaboration as well as for professional development and training. It also gives access everywhere to the same TEL tools. Innovations connecting online systems with adaptive learning systems can easily create extensive datasets from tens if not hundreds of thousands of children to uncover individual learning pathways and mechanisms. Usage of big data is opening up new possibilities for using technology in educational research.

In the end, the successful solutions for TEL on MLD depend on building bridges between the best educational practices and basic research on the mathematical brain. These are at best combined on intervention studies that can inform us both about the mechanisms of numerical learning and about the effective methods of using TEL tools in education and remediation. While the paper books are more and more changing to net-connected e-books, the researchers will have a totally new possibility to build such studies in collaboration with teachers as part of children’s daily school and homework.

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Chapter 43
Executive Function and Early Mathematical Learning Difficulties

Douglas H. Clements and Julie Sarama

Executive Function and Early Math Learning Difficulties

Young children who struggle for any reason in learning mathematics need support and personal resources, both cognitive and emotional. For most children, *executive function* (EF) processes develop most quickly in the early childhood years (i.e., birth to third grade) and provide resources that allow children to control their own thinking and emotions. Another category of resources that children need includes *content* or *mathematical* knowledge, skills, and dispositions. What role does each of these categories play in young children's learning of mathematics? How are they related? How might we provide support for all young children, especially those with special needs, so that their struggles become productive challenges?

The Role of Cognitive Executive Function

Children need to plan ahead, focus attention, and remember past experiences in all subject-matter areas, but these abilities may be particularly important to mathematics. Cognitive EF processes include *attention shifting*, *cognitive flexibility*, *inhibitory control*, and *updating working memory*, all of which may affect approaches to learning (Vitiello, Greenfield, Munis, & George, 2011). *Attention shifting* is switching a

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"mental set" while simultaneously ignoring distractions. Counting by different units (e.g., tens and ones) and keeping them straight is an example of attention shifting. *Cognitive flexibility* is similarly involved in avoiding *functional fixedness*. For example, children who solve every word problem in a set by adding the numbers may continue to do so, even if the situation changes to multiplication. *Inhibitory control* involves restraining an initial response (e.g., the first answer that occurs to you) to think about better strategies or ideas. Consider the following problem: "There were six birds in a tree. Three birds already flew away. How many birds were there from the start?" Children have to inhibit the immediate desire to subtract prompted by the words "flew away" and perform addition instead. Over the last 100 years, the demand for the application of EF processes such as inhibitory control has greatly increased in math education (Baker et al., 2010). Finally, *updating working memory* involves maintaining and manipulating relevant information and keeping it in active memory often while engaging in another cognitively demanding task. Children solving multistep mathematics problems are helped by keeping the problem situation and their solution strategy in mind while performing necessary computations and then use the result without forgetting where they were in the process.

The Role of Emotional Executive Function

Emotional EF allows children to work on difficult problems without quitting in frustration or anger. Children with low emotional EF lack social skills and may act with aggression (Broidy et al., 2003), leading to outbursts, inattention, and feeling overwhelmed (Saarni, Mumme, & Campos, 1998). These children are at higher risk for disciplinary problems and are less likely to make successful transitions through their early years (Huffman, Mehlinger, & Kerivan, 2000). Because they have difficulty cooperating or resolving conflicts successfully, children who lack emotional EF also do not participate in classroom *learning* activities in a productive way (Ladd, Birch, & Buhs, 1999). This limits positive teacher-child interactions, perpetuating the negative behavioral cycle (Hamre & Pianta, 2001; Neuenschwander, Röthlisberger, Cimeli, & Roebers, 2012). Lack of emotional control further impacts engagement, as evidenced by children's rejection or angry withdrawal from tasks that become difficult (Bassett, Denham, Wyatt, & Warren-Khot, 2012).

Finally, emotional EF processes are also frequently inhibitory. That is, they include the ability to suppress one response, so you can respond in a better way. For example, a child may need to suppress the immediate impulse to grab another child's math manipulatives, instead of asking to share manipulatives or work together. This process may involve affective decision-making or persistence in the face of difficulties (Zelazo et al., 2003). Together, emotional and cognitive EF allow children attend to and engage with tasks even when facing difficulties in problem-solving or learning, fatigue, distraction, or decreased motivation (Blair & Razza, 2007; Neuenschwander et al., 2012). Considering this, it is unsurprising that early childhood teachers argue that such EF abilities are equally as important as academic subject matter (Bassok, Latham, & Rorem, 2016).

The Executive Function of Children with Special Needs

EF abilities are particularly important to certain groups of children, especially those identified as having special needs such as developmental delays or experiential gaps. EF tasks are used to screen or diagnose people with learning disabilities (Ikeda, Okuzumi, Kokubun, & Haishi, 2011; Toll, van der Ven, Kroesbergen, & Van Luit, 2010); however, categorizations are often not assigned in the earliest years, specifically when *response to intervention* strategies have not yet been attempted (Methe & VanDerHeyden, 2013). Deficits in EF processes, such as updating working memory, are prevalent in children with difficulties learning mathematics and literacy (Gathercole et al., 2016; Mammarella, Hill, Devine, Caviola, & Szűcs, 2015). Deficits in EF may also underlie ADHD (Barkley, 1997) and reading difficulties (Biscaldi, Gezeck, & Stuhr, 1998; Moll, Snowling, Göbel, & Hulme, 2015). Additionally, working memory (but not attention) mediated the relation between groups to mathematics learning for older students with spina bifida myelomeningocele (Raghubar et al., 2015).

Research has demonstrated that children from low-resource communities who experience gaps in opportunities for learning may also have lower EF (e.g., Blair, Protzko, & Ursache, 2011; Blair & Razza, 2007; Bull & Scerif, 2001; McLean & Hitch, 1999; Raver, 2013), and this risk is exacerbated for children who are second-language learners (Wanless, McClelland, Tominey, & Acock, 2011). Children identified as gifted and talented may also have exceptional needs in this domain (Mooji, 2010). Differences in EF between groups raise important *equity* issues that we must address to fairly serve all children and thus the entire community of learners. As such, children with special needs likely require special interventions to develop EF competencies (Harris, Friedlander, Saddler, Frizzelle, & Graham, 2005; Lyon & Krasnegor, 1996; Mazzocco & Hanich, 2010; Raches & Mazzocco, 2012; Toll et al., 2010).

The Role of Subject-Matter Knowledge

Content knowledge, or knowledge of math concepts and skills, is notably important as well (Passolunghi & Lanfranchi, 2012)—early mathematical knowledge in particular. For example, researchers found that early math knowledge is the *best predictor* of later knowledge of math (Koponen, Salmi, Eklund, & Aro, 2013). The math that children know when they enter kindergarten and first grade has predicted their math achievement for years to come, throughout their school career. Moreover, what children know in math has also predicted their reading achievement as well as early literacy skills (Duncan et al., 2007; Duncan & Magnuson, 2011; Koponen et al., 2013). Mathematics then, including logical and mathematical reasoning, appears to be a core component of cognition (Clements & Sarama, 2011). Therefore, the combination of both EF and subject-matter competencies is critical in learning the important subject of early mathematics (cf. Blair, 2002), especially for children who need extra support in either or both of these.

Teaching Executive Function

Given that EF develops quickly in the early years for most, but not all, children, recent work has sought to promote its development. Studies have identified successful methods of teaching EF, including three types of interventions: computer games, direct training of specific EF tasks, and particular curricula or educational programs. Findings about the utility of computer games are mixed (Otero, Barker, & Naglieri, 2014; Razza & Raymond, 2015). As an example, 4- and 6-year-old children showed increases in attention after 5 days of computer game-based training, involving tasks such as helping direct a cartoon cat on the computer screen to move through a maze (Rueda, Rothbart, McCandliss, Saccomanno, & Posner, 2008). In another study, preschool children received training of either visuospatial working memory or inhibition for 5 weeks (Thorell, Lindqvist, Nutley, Bohlin, & Klingberg, 2009). For training in working memory, children had to remember the location and order of objects. To train inhibition, children worked on go/no-go tasks, in which the child was told to respond ("go") when a certain stimulus (e.g., a fruit) was presented, but to make no response ("no-go") when another stimulus (e.g., a fish) was presented. Children trained on visuospatial working memory improved on spatial and verbal working memory, as well as attention; however, training on inhibition did not transfer to working memory or attention tasks (Thorell et al., 2009). Goldin et al. (2014) also demonstrated that computer games lead to an increase in some, but not all, EF processes for 6-year-olds. Finally, 20 or more days of training on computer activities increased working memory and response inhibition of elementary-aged children diagnosed with ADHD and resulted in a reduction of the parent-rated inattentive behaviors (Klingberg et al., 2005; Klingberg, Forssberg, & Westerberg, 2002).

Direct training of specific EF tasks is the second type of intervention that has garnered some success. For example, providing 4-year-olds with feedback and reflection training helped children change the attribute for sorting. Training occurred after every mistake and consisted of the following: the child was asked to name the attribute, such as color or size, given an example of a correct sort, and then asked to resort with assistance (Espinet, Anderson, & Zelazo, 2012). No information about the transferability of this training to other situations exists.

The third category of EF interventions includes enhancement of EF capabilities through the utilization of particular curricula or early childhood programs (e.g., Bierman, Nix, Greenberg, Blair, & Domitrovich, 2008; Diamond, Barnett, Thomas, & Munro, 2007; Diamond & Lee, 2011; Lillard & Else-Quest, 2007; Raver et al., 2011; Weiland, Ulvestad, Sachs, & Yoshikawa, 2013; Weiland & Yoshikawa, 2013). While specific teaching approaches have proved successful, such as guiding impulsive children to self-monitor their behavior by talking to themselves (Reid, Trout, & Schartz, 2005), much remains to be understood about how to teach EF capabilities. The *Tools of the Mind* program is an example of an intervention that was designed to develop EF (Barnett et al., 2008; Diamond et al., 2007). This program was based on the Vygotskian theory that mature, intentional dramatic play represents the primary social context where children practice EF behaviors. Unfortunately, research has demonstrated limited support for the utility of this program. Four separate randomized cluster trial evaluations of the *Tools* program showed no effects on EF, even with good fidelity (Clements et al., 2017; Farran, Lipsey, & Wilson, 2011; Lonigan & Phillips, 2012; Morris, Mattera, & Maier, 2016). Consequently, evidence demonstrated that *Tools* was effective only with intensive and extensive supports for implementation, or as one reviewer concluded, pretend play was *not* crucial to building EF or other competencies {Lillard et al., 2013 #5779}. Furthermore, according to Elliott and colleagues (2010), neither training teachers to provide educational environments sensitive to children with working memory difficulties nor direct attempts to train children's working memory were effective in enhancing EF. Despite these discouraging results, children who begin school with very low EF competencies may still benefit from the utilization of curricula designed to enhance EF (Tominey & McClelland, 2011).

Relationships Between EF and Math

If we do develop successful training for children who need EF competencies, will that help them learn mathematics? Most studies on this topic are correlational—that is, they examine the relationship between achievement and EF but cannot tell us if one *causes* the other. Interestingly, although many researchers and other educators believe that EF will support later mathematics learning, the correlational research suggests a "two-way" relationship—that each may help support the development of the other throughout life.

Relationships Between EF and Math Learning

Several studies have shown positive correlations between EF and achievement in various subjects in young children (e.g., Best, Miller, & Naglieri, 2011; Bierman et al., 2008; Blair et al., 2011; Blair & Razza, 2007; Cameron et al., 2012; Clements, Sarama, & Germeroth, 2016; Viterbori, Usai, Traverso, & De Franchis, 2015; Welsh, Nix, Blair, Bierman, & Nelson, 2010), albeit with some exceptions (Edens & Potter, 2013). For example, inhibitory control and attention-shifting EF processes in preschoolers were related to measures of math and literacy ability in kindergarten (Blair & Razza, 2007). Children with higher behavioral EF, including attention, working memory, and inhibitory control, also achieved at higher levels in literacy, language, and math (McClelland et al., 2007).

Evidence showed that EF is more associated with math than literacy or language (Blair et al., 2011; Blair, Ursache, Greenberg, Vernon-Feagans, & The Family Life Project Investigators, 2015; Bock et al., 2015; Gathercole, Pickering, Knight, & Stegmann, 2004; Ponitz, McClelland, Matthews, & Morrison, 2009). Notably, though, deficits in EF evident in children with special needs also played a crucial role in math learning. For instance, Alloway (2007) identified low working memory as a substantial barrier to learning for children with developmental coordination disorder. EF has also been established as a predictor of children's involvement during learning opportunities, which in turn was related to their learning of literacy and mathematics (Nesbitt, Farran, & Fuhs, 2015).

Relationship Between Specific EF Skills and Math Learning A growing number of studies have indicated that inhibitory control and updating working memory may have a particularly close relationship to math learning and achievement (Bull, Espy, & Wiebe, 2008; Geary, 2011; Harvey & Miller, 2016; Miller, Rittle-Johnson, Loehr, & Fyfe, 2016; Neuenschwander et al., 2012; van der Ven, Kroesbergen, Boom, & Leseman, 2012), although some studies showed strong relationships for all three EF processes (Purpura, Schmitt, & Ganley, 2016). Inhibition and working memory tasks predicted success in math, and working memory tasks predicted math learning disabilities, above and beyond the predictive value of earlier mathematical abilities (Toll et al., 2010). Further, a lack of inhibition or working memory for children with lower mathematical ability represented a specific deficit that results in difficulty shifting and evaluating new strategies for dealing with math tasks (Bull & Scerif, 2001). Working memory may also be uniquely important for children with learning difficulties or disabilities (Toll et al., 2010). Thus, one can argue that these two components of EF play particularly important roles in the learning of math for young children, especially for those with low initial mathematical ability.

Although most studies have focused on cognitive EF, emotional EF should not be ignored. In one ironical example, first and second graders with the highest working memory also had the highest capacity to use advanced problem-solving strategies; yet, these children avoided utilizing advanced strategies when they were also high in math anxiety (Ramirez, Chang, Maloney, Levine, & Beilock, 2016). As a result, these children underperformed in math compared with their lower working memory peers. Environments that are designed to reduce stress, foster emotional well-being, and promote emotional EF help prepare children socially and cognitively for successful learning and problem-solving in preschool, the primary grades, and beyond (Blair, 2002).

Although EF processes are important for math, we must remember that EF processes are not the only factors that influence mathematical achievement. Content-specific competencies such as numerical competence also contribute to subsequent math achievement (Passolunghi & Lanfranchi, 2012). Even fine motor or spatial skills predict math and literacy achievement beyond measures of EF (Cameron et al., 2012). Similarly, a combination of EF and spatial skills of preschoolers predicted 70% of the variance in later math performance, with spatial skills uniquely predicting 27% of the variance in math competence (Verdine, Irwin, Golinkoff, & Hirsh-Pasek, 2014). Thus, EF and mathematical competence work together to influence mathematical achievement—EF does not determine success in math on its own.

Exploring Causality in the Relationship Between EF and Math Learning

Correlational studies cannot identify a causal connection in which EF competencies support learning of math (e.g., Best et al., 2011); however, some types of correlational studies are designed to make stronger suggestions of causation. For example, if early EF predicts mathematics achievement years *later*, the suggestion that EF contributes to math achievement can be made (Best et al., 2011; Blair & McKinnon, 2016; Clark, Pritchard, & Woodward, 2010; LeFevre et al., 2013). In one study, EF of entering preschoolers predicted end-of-the-pre-K-year literacy skills (Bierman et al., 2008). In another, executive control at age 3 years predicted math performance in kindergarten (Clark et al., 2010). While studies such as these are only correlational, they link EF *early* in life to academic achievement later in life, ultimately suggesting a causal interpretation. It is important to note, however, that these studies *did not always test the relationship in both directions*. For example, EF measures were found to predict math and reading achievement in the primary grades (Bull et al., 2008), but the researchers did *not* test for a relationship between *early* math skills and *later* EF competencies (see also Clark et al., 2010; LeFevre et al., 2013). In a similar vein, McClelland and others (2007) focused exclusively on the role of behavioral EF in supporting achievement in literacy and math. They suggested that strengthening these skills “prior to kindergarten may be an effective way to ensure that children also have a foundation of early academic skills” (McClelland et al., 2007, p. 956); however, they did not mention that in their own data, fall academic achievement predicted spring EF just as well as the reverse.

Further supporting the hypothesis of a two-way relationship, a longitudinal study showed that EF and math achievement influence each other (van der Ven et al., 2012). Interestingly, this may not be true of literacy competencies (Fuhs, Nesbitt, Farran, & Dong, 2014; Ponitz et al., 2009; Weiland, Barata, & Yoshikawa, 2014; Welsh et al., 2010), furthering support for the notion that the relationship between EF and math competencies may be particularly strong and significant. Finally, contrary to researchers’ expectation, none of the EF measures in one study predicted later math curriculum; however, early math achievement predicted all measures of EF (Watts et al., 2015), raising the possibility that mathematics may influence EF as much, or even more, than EF influences mathematics.

Overall, the results of such correlational, longitudinal analyses are mixed, with some suggesting that EF contributes to academic achievement rather than the reverse and many others finding that early math competencies or experiences can predict later-developing EF competencies. Remembering that these findings are *correlational*, *not causal*, is of the utmost importance, as the above set of studies delineated a two-way relationship between EF and math achievement, with each supporting the development of the other.

Causation: Experimental Studies of EF and Math Interventions

The gold standard for research on *causation* is the randomized control experiment, in which teachers and children are randomly assigned to treatment groups and only one group receives an intervention. What do experiments show us about the relationship between EF and math?

Checking Whether Teaching EF Causes Math Achievement

As previously illustrated, evidence in support of existing methods of teaching EF is limited. A review of experiments evaluating programs teaching EF, such as *Tools of the Mind*, found no reliable evidence pointing to the positive impact of increasing EF and correspondingly raising achievement (Jacob & Parkinson, 2015). For example, Barnett et al.'s (2008) experiment suggested that the *Tools* curriculum improved classroom quality yet actually served as a pertinent example of limited evidence in support of this method of instruction. Within the context of this study, lower scores on a measure of problem behaviors were interpreted as indicative of improvement in children's EF by the original authors; however, Jacob and Parkinson {2015 #6137} claimed that there were only minimum effects on achievement measure, the largest on language, and that these effects were not statistically significant when adjusted for hierarchical structure (classroom grouping of children). They also noted that there were no significant effects on math achievement, even though the curriculum included activities designed to promote math skills as well as EF skills. Thus, the study did not portray evidence that any cognitive EF skills increased after program participation, nor did it demonstrate that the program facilitated math learning (Jacob & Parkinson, 2015).

Further, computer training for 7-year-olds low in updating working memory and math achievement produced only small improvements in working memory immediately post-training; however, these small improvements were sustained 6 months later. Neither this computer training nor a comparison (Cogmed) training resulted in better performance in mathematics or generalized to other working memory tasks that differed from those included (Ang, Lee, Cheam, Poon, & Koh, 2015).

Some studies suggested that the relationship may run in the opposite direction—rather than EF competencies leading to achievement in school, it may be the case that we can improve children's EF competencies by teaching math. In one study, first graders were provided either computerized attention training or computerized academic training. Both forms of training positively affected *attention* according to a teacher rating scale, and this impact was greater for *academic* training. Further, only the academic training affected measures of academic achievement (Rabiner, Murray, Skinner, & Malone, 2010). As the above research suggests that teaching EF may not be a reliable way to support academic achievement, what else might we do?

Alternative Approaches, Especially for Children with Learning Difficulties

One possible explanation for the lack of evidence documenting a connection between teaching EF and higher levels of academic achievement is that teaching EF in a *decontextualized* way may simply be ineffective. Indeed, some studies have shown that teaching children to better use EF processes does in fact increase learning when taught *within a subject-matter context* (Naglieri & Gottling, 1995). Children with learning disabilities and mild mental impairments have benefited from verbalizing and reflecting on their strategies on arithmetic computation worksheets (Naglieri & Johnson, 2000), and effects were stronger for those with low planning skills (Naglieri & Gottling, 1997). Similarly, teaching planning strategies has supported children (again, especially those with poor planning processes) learning reading comprehension (Haddad et al., 2003). Children also benefited from instruction on EF strategies to read math word problems with comprehension (Capraro, Capraro, & Rupley, 2011; Fuchs, Fuchs, & Prentice, 2004).

Teaching Math Can Cause Both Math Learning and EF Development

Recall that some predictive research suggested a bidirectional relationship between the development of EF and academic competencies. Even when effects on EF are not planned, teaching math has had significant effects not just on math but on EF as well. For example, the combination of the *Building Blocks* math curriculum (Clements & Sarama, 2013) and the *OWL* literacy curriculum resulted in unplanned but positive, albeit small, statistically significant impacts on EF for children at risk for later difficulties in learning (Weiland & Yoshikawa, 2013). This "spill-over" phenomenon supported the hypothesis that cognitively demanding curricula improve other cognitive developmental domains such as EF, even without specifically targeting EF processes.

As discussed, two evaluations of the *Tools of the Mind* did not establish that this program had a strong effect on EF; however, these evaluations did produce intriguing results regarding the benefits of math activities. The first large-scale evaluation (Farran et al., 2011) found that the *Tools* program had little effect on EF; however, it also demonstrated that a positive association existed between the classroom and teacher's level of focus on math and children's gains in *both* math and EF (Farran et al., 2011). The second large-scale evaluation compared three treatment groups. The researchers hypothesized that the *Tools* + *Building Blocks* group would perform better than a *Building Blocks-only (BB)* group on EF, and perhaps even on math, given the facilitative effect of the (presumed) gains in EF (Clements et al., 2017). It was further hypothesized that both of these groups would outperform the control group in math. The results of this evaluation were surprising as the *BB* group not only had higher math scores but also outperformed the control group on one of the EF measures (HTKS) and the *BB + Tools* group on another working memory task (backward digit span).

These results are surprising and important. If confirmed, they may indicate that educators might replace (or complement) efforts to develop EF separately to build capacity for learning mathematics and other subjects and instead focus on implementing effective mathematics activities as a way of developing both mathematics and EF competencies.

Math Activities that May Develop EF

A *Building Blocks* activity from the experiments included above asks children to find all pairs of positive whole numbers that sum to six. This may require children to suppress initial responses (e.g., only stating "it's 3 plus 3"), manipulate abstract structures (add 1 to one of the numbers and subtract 1 from the other to find a new pair), and remain cognitively flexible. The stronger relationship between EF and math activities, as opposed to literacy activities (e.g., Fuhs et al., 2014), may reflect that math makes greater demands on working memory and attention control. The ability to hold relevant information in mind, to operate on it while shifting attention appropriately among problem elements, and to inhibit automatic responding to only one aspect of a given problem represent prime examples of these increased demands. Indeed, it may be that 100 years of rising population mean IQ in the United States is due to the increasing cognitive demands of mathematical curricula (Blair, Gamson, Thorne, & Baker, 2005)!

Importantly, EF may be developed by *learning* the math *in the context* of challenging activities, not by simply "exercising" the math once learned. When children learn arithmetic facts, they use the frontal areas of their brains that support EF and working memory; however, once they know facts fluently, they use regions that store verbal memories and process symbols (Butterworth, Varma, & Laurillard, 2011). Thus, EF may develop in the early stages of learning when children are first exposed to mathematical material. Support for this idea was established by a study in which a computer game that challenged children to improve their arithmetic performance enhancement may have simultaneously improved their working memory capacity (Núñez Castellar, All, de Marez, & Van Looy, 2015). In summary, the research on the relationship between EF competencies and math learning suggests that *high-quality math education may have the dual benefit of teaching an important content area and developing at least some EF competencies.* A group of researchers funded by the Heising-Simons Foundation (https://dreme.stanford.edu) are currently exploring the impact of intentionally developed math curricula based on recent EF research that allows us to facilitate both benefits more effectively.

Moreover, research has identified mathematical environments (Fisher, Godwin, & Seltman, 2014) and teaching practices that can help children pay attention and grow in their ability to do so, as well as to develop general EF competencies (see

Clements & Sarama, 2014, for numerous activities so designed). Carefully guiding children to attend to specific mathematical features, such as the number in a collection or the corners of a polygon, is likely to improve their learning. The predisposition to spontaneously recognize numbers, for example, is a skill but also a habit of mind and includes the ability to direct attention to numbers (Lehtinen & Hannula, 2006). These habits of mind generate further development of specific mathematical knowledge and the ability to direct attention to math in situations in which it is relevant, that is, to generalize and transfer knowledge to new situations, as well as to develop both emotional and cognitive EF.

Conclusions

Learning and doing mathematics requires both affective and cognitive resources. This is especially true for young children with learning difficulties. Executive function (EF)—the ability to control and supervise one's own emotions and thinking—may be one of the most important resources children need to succeed in math. Both emotional and cognitive EF contribute to social-emotional development and academic learning that are especially important in math curricula that increasingly require higher-order skills such as those provided by EF (Baker et al., 2010).

As EF develops most rapidly in the early childhood years, educators need to use research to provide environments, curricula, and experiences that develop these processes, especially for children at risk due to developmental delays or low entering competencies. Several approaches to teaching EF have showed promise, but few have been consistently successful at a *practically* significant level. The research discussed in this chapter reveals the lack of causal evidence that interventions to develop EF will increase achievement. Although relationships between early EF and literacy are often weak, high-quality *mathematics* education may have the dual benefit of teaching an important content area and developing at least some EF competencies. Given that early math knowledge predicts later mathematics achievement, later literacy achievement, and later EF (Watts et al., 2015) but that early EF does not predict these things, this approach stands as a potentially promising path. Ultimately, the evidence discussed in this chapter underlines the importance of increasing the intentional development of math curricula based on recent research on EF that may contribute to more effective facilitation of both math and literacy achievement.

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Chapter 44
Children's Mathematical Learning Difficulties: Some Contributory Factors and Interventions

Ann Dowker

Difficulty with arithmetic is a common problem (Butterworth, Sashank, & Laurillard, 2011). For example, about 22% of the adult population in the UK have severe numeracy difficulties that have a serious practical and social impact on their daily lives, whereas only about 5% have similar levels of difficulty in literacy assessed by the same criteria (Bynner & Parsons, 1997; Parsons & Bynner, 2005).

This chapter will first discuss some of the factors that contribute to arithmetical difficulties. There are genetic and other brain-based developmental factors that contribute to mathematical difficulties, which are discussed elsewhere in this book. This chapter will focus on some environmental and motivational factors. It will then proceed to discuss some interventions that have been used for mathematical difficulties, especially at primary school level.

National and Cultural Factors: What Do We Learn from International Comparisons?

International comparisons, such as those of TIMSS (Mullis, Martin, Foy, & Hooper, 2016; Mullis, Martin, & Loveless, 2016) and PISA (OECD, 2013), report considerably better performance in mathematics by children in some countries than in others. In particular children from countries in the Far East, such as Japan, Korea, Singapore and China, tend to perform better in arithmetic than do children in most parts of Europe and America.

It is important to be cautious in interpreting the results of such international comparisons (Jerrim, 2011; Sturman, 2015). For example, there may be issues with

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sampling. Subtle differences in the ways in which pupils or schools are selected for testing may significantly affect the results. For example, the implications of selecting a class of pupils of a particular age may differ according to whether promotion by age is automatic or is dependent on school achievement and on whether the schools divide pupils into ability groups by streaming or setting. Moreover, differences are typically studied at a whole-country level and may fail to take into account differences between regions or school types within a country. For example, in the TIMSS 2011 study, Massachusetts obtained mathematics scores similar to those in the highest-achieving countries, while Alabama obtained mathematics scores below the international median. China usually comes close to the top in international comparisons; but it must be remembered that this is an extremely large and diverse country and that it can be difficult to be sure that the sampling is representative. Children in the larger, more readily accessible cities are more likely to be sampled than those in remote rural schools. For example, the PISA 2012 study included only schools in Shanghai and Hong Kong (OECD, 2014). There is a lot of evidence that rural Chinese children are relatively economically and educationally disadvantaged compared to urban children and their omission from most international comparisons may lead to somewhat misleading results.

Nevertheless, despite such cautions, there are some fairly consistent findings of relatively high performance by the Pacific Rim countries and by Finland (Mullis, Martin, & Loveless, 2016). Educators, researchers and policymakers have made several attempts to examine the reasons.

Might International Differences in Teaching Methods Affect Performance?

For example, people have attempted to investigate whether there are particular teaching methods or school characteristics in higher-achieving countries that might be transferable to other countries and lead to improvement.

There has, for instance, been considerable emphasis recently on the ‘mastery’ approaches of East Asian countries, resulting in some attempts to introduce methods based on this approach into schools in the UK and the USA (see discussion of whole-class interventions at the beginning of this chapter). One difficulty in doing so is deciding which of the numerous aspects of the ‘mastery’ approaches are particularly important. For example, such approaches involve (1) breaking down different parts of the curriculum into units and (2) clearly defining the goals, and the aim is to ensure that all pupils have mastered each unit before going on to another one. While the approach is sometimes misinterpreted as involving whole-class teaching without any adaptations to weaker pupils, in fact teachers are expected to look at the children’s work, and to intervene immediately with individuals’ misconceptions before moving on. Thus, at least according to the ideal, the same person combines, within a short space of time, the role of class teacher and deliverer of interventions. This of course places high demands on the teacher.

Finland, another high-achieving country, differs from the Pacific Rim countries in some key ways. Whereas children in the Pacific Rim countries start school quite early and experience repeated high-stake testing, Finnish children do not begin formal instruction until 7 and experience little high-stake testing.

One thing that Finland and the Pacific Rim countries appear to have in common is the high status of the teaching profession, with high levels of selection for education courses, extended courses, and opportunities for continuous professional development. It may be that this is at least as important as any details of the curriculum.

Socio-economic Differences

Parental social class is an important predictor of children's academic performance in all subjects, including mathematics (Perry & Francis, 2010). British adults with severe persisting numeracy difficulties are far more likely than those without such difficulties to have come from 'working-class' backgrounds and to have been poor (Bynner & Parsons, 1997, 2006). In countries with greater social inequalities, such as Brazil, social class effects on academic performance are even greater (Nunes, Schliemann, & Carraher, 1993). For example, Davie, Butler, and Goldstein (1972) found that even at the age of 4, there was a year's difference in British children's performance on intellectual tasks, including numeracy tasks, between working-class children living in a deprived area and middle-class children living in a comfortable area.

There are many possible ways in which social class could influence academic performance, including mathematical performance. Better-off parents can afford more books and other academic materials for their children. On average, they may have more time to talk to and interact with their children (though this is not always the case). The children are likely to attend schools with better resources. Moreover, better-off parents have usually had more formal education themselves and are therefore likely to be more able to help their children to learn academic subjects. Also, the culture of the school is likely to be less alien to a child from a family with a high level of formal education than one from a family with lower formal education (Biddle, 2001; Brooker, 2002).

Evidence suggests that early influences of social class are much greater on verbal than nonverbal aspects of mathematics. For instance, Jordan, Huttenlocher, and Levine (1992, 1994) found no social class differences in pre-schoolers' ability to do nonverbal addition and subtraction problems; but middle-class children were better than working-class children at verbal arithmetic. Presumably, middle-class children have more experience than working-class children with conventional mathematical language, which gives them an advantage in verbal arithmetic, but not necessarily in nonverbal arithmetic.

There are several preschool intervention programs for children at risk, usually children living in poverty. They appear to be commonest in the USA and include, for example, the mathematical components of the Head Start Program

(Arnold, Fisher, Doctoroff, & Dobbs, 2002), the Pre-K Mathematics curriculum (Klein, Starkey, Clements, Sarama, & Iyer, 2008), the Big Math for Little Kids program of Ginsburg and his colleagues (Greenes, Ginsburg, & Balfanz, 2004) and the Building Blocks program of Clements and Sarama (2011).

Ramani and Siegler (2008, 2011) proposed that SES differences in early numeracy may reflect differing prior experience with informal numerical activities, such as numerical board games. They found that the numerical magnitude knowledge of pre-schoolers from low-income families lagged behind that of peers from better-off backgrounds. But playing a simple numerical board game for four 15-min sessions eliminated the differences in numerical estimation proficiency. Playing games that substituted colours for numbers did not have this effect. These findings have been replicated with groups of children in the UK (Whyte & Bull, 2008) and Sweden (Elofsson, Gustafson, Samuelsson, & Traff, 2016).

The Role of Attitudes and Emotions

Mathematical development and performance depend not only on our learning and intellectual abilities, and the teaching that we have received, but also on emotions and attitudes. There is a wide spectrum of attitudes that people have with regard to mathematics, ranging from the extremely positive to the extremely negative. Unfortunately, the latter are all too common, and while they sometimes involve mere dislike or boredom with mathematics, many people suffer from severe anxiety, even fear, with regard to mathematics (Ashcraft, 2002; Maloney & Beilock, 2012).

Mathematics anxiety has been defined as ‘a feeling of tension and anxiety that interferes with the manipulation of numbers and the solving of mathematical problems in[…]ordinary life and academic situations’ (Richardson & Suinn, 1972). People who fear mathematics are seriously restricted in their choice of occupation (Brown, Brown, & Bibby, 2008; Chipman, Krantz, & Silver, 1992). They may experience difficulty and anxiety in managing their finances, and, if the fear is severe, even in such activities as reading train and bus timetables.

Estimates as to the frequency of mathematics anxiety are varied and depend both on the ways in which mathematics anxiety is defined and assessed and on the nature of the sample. Given that people with extreme mathematics anxiety are probably less likely to attend university, university student samples may be biased toward comparatively low levels of such anxiety: nevertheless, mathematics anxiety is common even in such samples. Richardson and Suinn (1972) estimated that 11% of university students show high enough levels of mathematics anxiety to be in need of counselling. Johnston-Wilder, Brindley, and Dent (2014) found a higher figure in a group of apprentices, with about 30% showing high mathematics anxiety and a further 18% affected to a lesser degree. Ashcraft and Moore (2009) estimated that 17% of the population have high levels of mathematics anxiety.

Mathematics anxiety is an important problem, not only because it is an unpleasant and stressful emotion but because many studies in different countries show that attitudes to mathematics are correlated with actual mathematical performance and in particular that mathematics anxiety is negatively correlated with performance (Baloğlu & Koçak, 2006; Dulaney et al., 2017; Hembree, 1990; Ho et al., 2000; Ma & Kishor, 1997; Miller & Bichsel, 2004). What is less clear is the direction of causation. On the one hand, mathematics anxiety may cause poorer performance by reducing motivation and leading to reduced practice, or to active avoidance (Chinn, 2009), or by overloading working memory (Ashcraft, 2002). On the other hand, mathematical difficulties may lead to mathematics anxiety, by causing experiences of failure, confusion and embarrassment associated with mathematics.

Maloney and Beilock (2012) proposed that mathematics anxiety is likely to be due both to pre-existing difficulties in mathematical cognition and to social factors, e.g. exposure to teachers who themselves suffer from mathematics anxiety. They suggest that children with initial mathematical difficulties are also likely to be more vulnerable to the negative social influences and that this may create a vicious circle.

Most studies of attitudes of mathematics anxiety have dealt with the problem of negative attitudes. But positive attitudes such as enjoyment of mathematics and self-confidence in mathematics are also important topics to study, and cannot be reduced to the simple absence of anxiety. For example, Villavicencio and Bernardo (2016) found that positive emotions toward mathematics predicted achievement in a Filipino adolescent group, even after controlling for anxiety. It is possible that positive attitudes to mathematics could act as a protective factor in pupils with risk factors for low mathematical attainment.

The significance of attitudes to mathematics makes it important to find ways of intervening to improve attitudes and in particular to treat or prevent mathematics anxiety. Treatments include desensitization and cognitive behavior therapy and related treatments that are used for many forms of anxiety (e.g. Hembree, 1990). Beilock and colleagues have found that 'writing out' one's anxieties can reduce anxiety and improve performance, both in mathematics anxiety and other forms of academic performance anxiety (Park, Ramirez, & Beilock, 2014; Ramirez & Beilock, 2011). Such treatments are of course only feasible with those who are old enough to have reasonable facility with writing.

There is evidence that interventions that improve mathematical performance may also improve attitudes and reduce anxiety. Levitt and Hutton (1983) found that training in basic arithmetical skills and in relevant study skills such as note taking can reduce mathematics anxiety. Supekar, Iuculano, Chen, and Menon (2015) used an intensive 8-week programme to improve the mathematical skills of children in Grade 3 with high and low mathematics anxiety. The program was based on MathWise (Fuchs et al., 2008, 2013) and included both games and more formal activities involving practice with addition and subtraction, training in efficient counting strategies for arithmetic and exposure to concepts such as the commutativity of addition and that adding zero to a number does not change it. Children in general improved in mathematical problem-solving, and those who started with high mathematics anxiety showed a significant reduction in mathematics anxiety on a questionnaire measure. Moreover, fMRI scanning showed that before training, children with high mathematics anxiety showed brain activation differences from those with low mathematics anxiety, and in particular in the higher amygdala; but after the training these group differences disappeared.

Interventions for Mathematical Difficulties

As indicated in the above paragraph, interventions can be effective in ameliorating mathematical difficulties and possibly improve attitudes as well. How can we intervene best?

There have been some intervention programs for children with mathematical difficulties since at least the first half of the twentieth century, especially in the USA (Brownell, 1929; Tilton, 1947; Williams & Whitaker, 1937). However, there are relatively few numeracy interventions available until recently; and mostly such interventions have not been used on a large scale.

However, in the twenty-first century, there has been increasing interest in developing interventions for children with numeracy difficulties (Butterworth et al., 2011; Chodura, Kuhn, & Holling, 2015; Clements & Sarama, 2011; Cohen Kadosh, Dowker, Heine, Kaufmann, & Kucian, 2013; Dowker, 2017; Dowker & Sigley, 2010; Gersten et al., 2009; Kucian et al., 2011; Rasanen, Salminen, Wilson, Aunio, & Dehaene, 2009). Only a few of these interventions will be discussed in detail here: for a more comprehensive account, see Dowker (2017).

Interventions in numeracy (as well as literacy) have sometimes been classified into three categories of varying degrees of intensiveness termed ‘waves’ in the UK and ‘tiers’ in the USA. Wave 1 involves whole-class teaching designed to be suitable for children of a variety of attainment levels. Wave 2 involves lighter-touch, less-intensive interventions in small groups (or sometimes limited-time one-to-one interventions) with children who are experiencing mild or moderate difficulties in the subject. Wave 3 involves more intensive, usually individualized interventions for children with more significant problems.

Whole-Class Approaches

While there has been interest in developing and improving mathematics curricula and educational techniques for quite a long time, there has been increasing recent interest in investigating the possible role of certain new whole-class programs in improving overall performance and reducing the incidence of numeracy difficulties. One programs which is attracting current interest from this point of view, especially in the UK, is Mathematics Mastery, a program inspired by some aspects of Singapore mathematics education. This is supported by NCETM (https://www.ncetm.org.uk). Compared to traditional curricula, fewer topics are covered in more depth, and greater emphasis is placed on problem-solving and on encouraging mathematical thinking. A current evaluation by the Education Endowment Foundation has so far indicated that the use of the program in Year 1 results in an average increased gain in mathematics age of 2 months in the first year, and the use of the program in Year 7 results in an average increased gain in mathematics age of 1 month. Further investigation is desirable to see whether these gains are maintained or extended over time.

Adapting classroom instruction to take account of individual needs may also involve allowing for independent individualized or small-group work within a class (El-Naggar, 1996). This may involve progressing through a textbook at one's own pace, the use of individualized worksheets and/or the individualized use of educational computer software.

Such approaches are potentially more flexible and have more potential for taking account of the componential nature of arithmetical ability, than giving all children the same instruction, or streaming or setting. Unlike streaming and setting, which have usually been found to have negative effects on low attainers' performance, Lou et al. (1996) indicated that within-class grouping had a positive effect on the performance of low achievers, but only if it was accompanied by provision of appropriate materials and activities. The potential disadvantages of such approaches include the risks that even within one class, some pupils may be labelled as 'low attainers' and live down to expectations as well as that work will become so individualized that pupils will not benefit from mutual discussion and exploration of ideas.

Light-Touch Individualized and Small-Group Interventions

There are clearly many children who have significant need of more targeted intervention, but do not need extremely intensive intervention. Relatively light-touch interventions are needed for such individuals. Such interventions are usually delivered by teachers or teaching assistants within the school. They are often delivered in small groups, or sometimes individually but on a relatively infrequent basis.

For example, in the UK, Askew, Bibby, and Brown (2001) developed a small-group intervention technique that involved the use of derived fact strategies. Teachers worked with small groups (four per group) of 7- to 8-year-old children, who had performed below average in school achievement at age 7. The children underwent intervention once a week for 20 weeks. These children improved significantly more than the controls, both in accuracy and in their use of known and derived facts rather than needing to resort to counting strategies.

In an American study, Bryant, Bryant, Gersten, Scammacca, and Chavez (2008) delivered 15-min small-group interventions 3–4 days a week for 18 weeks. The interventions dealt with counting, quantity representation, basic facts and place value concepts. The 26 pupils in the intervention did not differ significantly from controls on a standardized test at the end of the intervention period. However, intervention programs lasting longer than 15 min at a time and/or continuing over a longer period of time have given positive results. Bryant et al. (2008) used similar interventions to the above, but lasting 20 min at a time, 4 days a week, for 23 weeks. The 42 children in this study did perform significantly better than controls on a standardized test.

Thus, relatively small-group interventions can have a significant impact on the progress of children with mathematical difficulties. The ways in which such interventions are delivered seem to affect the level and nature of their effectiveness.

There are also programs which, though administered individually, involve relatively small amounts of time and may often be delivered by teaching assistants rather than specialist teachers.

An example is Catch Up® Numeracy, which was developed through a collaboration between myself and Catch Up®, a not-for-profit charity (Dowker, 2017; Dowker & Sigley, 2010; Holmes & Dowker, 2013). The main target pupils have been pupils aged from 6 to 11 years, who have moderate difficulties with mathematics. It has recently been extended for use with 11 to 14 year olds. It consists of two 15-min sessions per week for approximately 30 weeks.

Catch Up Numeracy focuses on assessing and targeting specific strengths and weaknesses. The intervention begins by assessing the children on the ten components of numeracy. Each child is assessed individually by a trained teacher or more usually a teaching assistant. This assessment is used to construct a 'Catch Up Numeracy' learner profile, which determines the entry level for each of the ten Catch Up Numeracy components. Children are provided with mathematical games and activities targeted to their specific levels in specific activities.

The ten components include (1) counting orally; (2) counting objects; (3) reading and writing numbers; (4) comparing, adding and subtracting tens and units; (5) ordinal numbers; (6) word problems; (7) translation between different formats (numerals, number words and sets of objects); (8) derived fact strategies (the use of known facts, combined with arithmetical principles such as commutativity, to derive new facts; e.g. if $8 + 6 = 14$, then $6 + 8$ must also be 14); (9) estimation of quantities and of answers to arithmetic problems; and (10) remembered number facts.

Studies where children were pretested and post-tested on the Hodder Basic Number Screening Test (Gillham & Hesse, 2012) have shown that children make about twice as much progress as would be expected from the passage of time alone and that they make significantly more progress than business-as-usual controls (Dowker, 2017; Holmes & Dowker, 2013). A randomized controlled study is currently underway to investigate whether the gains are significantly greater than those of children receiving equivalent-time mathematics intervention.

Highly Intensive Interventions

There are some children whose difficulties are so severe and/or resistant to intervention that light-touch interventions will not prove sufficient. Much more intensive interventions, perhaps involving daily individualized sessions with a teacher highly trained in intervention techniques, may be necessary. A well-known example of such an intensive intervention is Mathematics Recovery. The Mathematics Recovery program was designed in Australia by Wright and his colleagues (Wright & Ellemor-Collins, 2018; Wright, Martland, & Stafford, 2006). In this program, teachers provide intensive individualized intervention to low-attaining 6- and 7-year-olds. Children in the program undergo 30 min of individualized instruction per day over a period of 12–14 weeks.

The choice of topics within the program is based on the Learning Framework in Number, originally devised by Steffe (1992). This divides the learning of arithmetic into five broad stages. These stages are (1) emergent (some simple counting, but few numerical skills), (2) perceptual (can count objects and sometimes add small sets of objects that are present), (3) figurative (can count well and use 'counting all' strategies to add), (4) counting on (can add by 'counting on from the larger number' and subtract by counting down, can read numerals up to 100 but have little understanding of place value) and (5) facile (know some number facts, are able to use some derived fact strategies, can multiply and divide by strategies based on repeated addition, may have difficulty with carrying and borrowing). Children are assessed, before and after intervention, in a number of key topics. They undergo interventions based on their initial performance in each of the key topics. The key topics that are selected vary with the child's overall stage. For example, the key topics at the emergent stage are (i) number word sequences from 1 to 20, (ii) numerals from 1 to 10, (iii) counting visible items (objects), (iv) spatial patterns (e.g. counting and recognizing dots arranged in domino patterns and in random arrays), (v) finger patterns (recognizing and demonstrating quantities up to 5 shown by number of fingers) and (vi) temporal patterns (counting sounds or movements that take place in a sequence). The key topics at the next perceptual stage are (i) number word sequences from 1 to 30, (ii) numerals from 1 to 20, (iii) figurative counting (counting on and counting back, where some objects are visible but others are screened), (iv) spatial patterns (more sophisticated use of domino patterns; grouping sets of dots into 'lots of 2', 'lots of 4', etc.), (v) finger patterns (recognizing, demonstrating and manipulating patterns up to 10 shown by numbers of fingers) and (vi) equal groups and sharing (identifying equal groups and partitioning sets into equal groups). The key topics at later stages place greater emphasis on arithmetic and less on counting. Despite the overall division into stages, the program acknowledges and adapts to the fact that some children can be at a later stage for some topics than for others.

Smith, Cobb, Farran, Cordray, and Munter (2013) found significant improvement in both standardized tests and researcher-derived tests in a randomized field trial of first-grade pupils assigned to Mathematics Recovery versus a waiting list control group. Effect sizes ranged from 0.21 to 0.28 for standardized tests and were higher for the researcher-derived tests. Future studies should assess the longer-term impact.

Numbers Count

In order to set up an intensive intervention for children with serious numeracy difficulties in the UK, and to test its effectiveness, Every Child Counts was set up as a partnership initiative between the Every Child a Chance charity (a coalition of business partners and charitable trusts) and government. The aim was to enable the lowest-attaining children to make greater progress toward expected levels of attainment in mathematics, catching up with their peers and performing at least at average levels on school assessment tests, wherever possible, by the end of the second year of primary school. The original intention was to provide intensive support in mathematics to 30,000 Year 2 children annually, though this has been significantly reduced due to the financial crisis of 2008 and subsequent government spending cuts.

Dunn, Matthews, and Dowrick (2010) developed the Numbers Count program, which draws on aspects of three existing interventions: Multi-Sensory Mathematics (developed in Leeds using Numicon materials), Numeracy Recovery (developed in Hackney) and Mathematics Recovery (Wright et al., 2006). This program involved careful assessment of individual children's strengths and weaknesses, followed by intervention targeted to addressing specific weaknesses, and emphasizes the development of number concepts through multisensory teaching. It included a wide variety of components of arithmetic but places particular emphasis on methods of counting and number representation. Children received a half an hour of individualized or sometimes very small-group (two or three children to a teacher) intervention per day. It was delivered by teachers who have received Masters level training. In the initial stages of the project, 2621 Year 2 children, across 27 English local authorities, took part in Numbers Count. They received an average of 40 half-hour individualized Numbers Count lessons in a term, delivered by teachers who had received Masters level training. The participating children were given the Sandwell Test, a standardized arithmetic test, before and after entering the program, and were retested 3 months and 6 months later.

Torgerson, Wiggins, Torgerson et al. (2011) carried out an independent evaluation of the program. About 12 children within each of 44 schools were randomly allocated to either an intervention group or a waiting-list control group. Children in the intervention group received an average of 40 half-hour individualized Numbers Count lessons in a term, delivered by teachers who had received Masters level training. The participating children were given the Sandwell Test, a standardized arithmetic test, as a pretest and were post-tested on the Sandwell Test after 3 months and 6 months and also the Progress in Maths 6 (PIM 6) test after 3 months. Findings showed that the intervention group performed significantly better than the controls on the PIM 6 test (effect size 0.33).

The changes in the Sandwell scores were greater. Before entering the program, the children's Number Age was on average 11 months below their Chronological Age. On average, they gained 14 months in Number Age in one term, a ratio gain of over 4 (months gained in Mathematical Age divided by mean duration of intervention in months), and were scoring at chronological age level by the time they exited the program. However, it must be noted that, while the PiM scores were marked by people blind to the children's group assignment, the Sandwell scores were not, so that there could have been unconscious bias with regard to the latter. As always, the question arises of whether the gains will be maintained over the long term. A long-term evaluation is currently being carried out to investigate whether the effects of the intervention persist to the end of primary school and into secondary school.

What Makes Interventions Effective?

As can be seen, programs vary considerably, both according to the theories and interests of the researchers and according to the target group, e.g. age, nature and extent of mathematical difficulties, etc. However, there are several features that effective programs share. These include effective assessment of pupils' initial performance level, including, in the case of pupils with mathematical difficulties or low attainment, diagnosis of individual strengths and weaknesses. They also involve taking a developmental approach and applying knowledge of how performance and knowledge typically develop in the age group being studied. They also involve careful planning, taking availability of resources into account, and appropriate use of school staff: the best-designed program will not work if teaching staff are unavailable, excessively overburdened or not adequately trained to deliver the program. Another key feature of an effective program is its ability to motivate pupils, and to prevent or counteract the association of mathematics with boredom, or worse, fear and anxiety. The use of games, for example, is a recurring feature of promising programs, especially with preschool and primary school children.

Primary school age is where interventions have been most focussed. There are several reasons for this. Mathematical difficulties become noticeable by this stage, whereas they are harder to detect at an earlier stage. Moreover, primary and early secondary pupils are the ones who are most universally doing mathematics at school. Pre-schoolers are by definition not attending formal school and until recently were unlikely to be in an educational setting at all. Until relatively recently secondary pupils who were struggling with mathematics could partially or completely abandon the subject at an early stage. But all primary pupils are spending a significant amount of time on mathematics.

Moreover, it is desirable to intervene at the primary school stage in case of difficulties with numeracy, in order to prevent these problems from having a negative impact on pupils reaching later stages in the curriculum or leading to their developing serious mathematics anxiety.

It is noteworthy that nearly half of the British adult working population are performing only at primary school level in mathematics (BIS, 2011). While part of this is undoubtedly because many pupils struggle with mathematical concepts taught in secondary school, it is also likely that pupils are failing to progress because they have not fully come to grips with primary school material. This makes interventions at the primary school level important: ranging from whole-school interventions to improve overall performance to highly individualized interventions for pupils who are seriously struggling.

There is still much work to be done in developing and evaluating programs. Ideally, there needs to be long-term follow-up of programs, to see whether the programs have an impact on pupils' long-term mathematical performance and on other outcomes. There should also be more comparisons between different programs, with a particular focus on investigating whether different programs are more effective with different groups of pupils. Furthermore, there should be more study of whether the size and nature of the group, in which programs are delivered, affects effectiveness: e.g. whether and under what circumstances individualized interventions are more effective than group interventions, or small-group than large-group interventions.

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Chapter 45
Beyond the "Third Method" for the Assessment of Developmental Dyscalculia: Implications for Research and Practice

Vivian Reigosa-Crespo

Three approaches have been established related to identification of learning disabilities, including those associated to the domain of mathematics: (1) ability-achievement discrepancy (Mastropieri & Scruggs, 2002), (2) response to intervention (RTI) (Fuchs & Fuchs, 2006), and (3) cognitive patterns of strengths and weaknesses (PSW) (Panel, 2014). In some way, the main ideas stated by the authors in this section were based on these approaches, and a solid consensus concerning to limitations of the discrepancy method can be noticed.

The most important criticism about the discrepancy model is concerning the poor sensitivity of the model to developmental differences in cognition and achievement, the overidentification of children coming from diverse environments, the promotion of the "wait-to-fail" model because early identification is unlikely, the poor decision-making based on measurement problems, and the difficulty for distinction between children with specific learning disabilities (SLDs) and low achievers; also it is unclear which IQ score should be used to determine "ability" for discrepancy calculation.

On the other hand, although most authors recognize the role of RTI for preventing learning problems and promoting early intervention services, there is scarce evidence supporting the use of RTI alone in identifying all children with SLD and in addressing their intervention needs. Moreover, there are numerous reasons for which children are not responsive to intervention beyond the SLD. Other practical problems with the RTI approach are related to a lack of agreement about using standard protocols or problem-solving strategies and on a measurement model for defining responsiveness. Additionally, RTI research has largely focused on word reading at the early elementary grades, with methods across grades and content areas not empirically established. Lynn Fuchs and Douglas Fuchs attempt to solve those practical issues in their chapter as part of this section.

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However, other issues still need further empirical evidence. Consequently, there is no agreement concerning the teacher training standards and supervision methods, and the RTI approach does not have effective algorithms for differential diagnosis of SLD and other disorders.

In an attempt to sort out the limitations of those approaches, a group of 58 experts wrote a white paper regarding SLD identification and intervention. Based on survey response and empirical evidence, they arrived at five important conclusions:

> "(1) Maintain the SLD definition and strengthen statutory requirements in SLD identification procedures; (2) neither ability-achievement discrepancy analyses nor failure to respond to intervention (RTI) alone is sufficient for SLD identification; (3) to meet SLD statutory and regulatory requirements, a "third method" approach that identifies a pattern of psychological processing strengths and deficits, and achievement deficits consistent with this pattern of processing deficits, makes the most empirical and clinical sense; (4) an empirically validated RTI model could be used to prevent learning problems in children, but comprehensive evaluations should occur whenever necessary for SLD identification purposes, and children with SLD need individualized interventions based on specific learning needs, not merely more intense interventions designed for children in general education; and (5) assessment of cognitive and neuropsychological processes should be used not only for identification, but for intervention purposes as well, and these assessment-intervention relationships need further empirical investigation." (Panel, 2014) p. 62.

Although diverse models of PSW exist, they follow these general principles: (i) the full-scale IQ is irrelevant except for intellectual disability diagnoses; (ii) children classified as having a SLD have a pattern in which most academic skills and cognitive abilities are within the average range. However, they have isolated weaknesses in academic and cognitive functioning, (iii) each model demands that we "match" deficits in specific cognitive processes to the specific area of academic concern without testing children with numerous measures in an attempt to find a deficit, and (iv) most cognitive abilities that do not relate to the area of academic concern are average or above.

According to Zirkel (2013), the posture declared in a previous version of the white paper published in 2010 is legally flawed in terms of its reliance on the "pattern of strengths and weaknesses" provision, its failure to consider state special education laws, and its overemphasis on the processing component. As a result, the position is justifiable only as advocating revisions, rather than in finding support, in the law¨ (Zirkel, 2013, p.93). This critique was not thoroughly addressed in the last version of the white paper.

In spite of this limitation, the nature of the five conclusions stated in the referred white paper reveals the importance of a clear definition about specific math disability or dyscalculia in order to differentiate from low achievement in math. This issue is relevant because dyscalculia is qualitatively and functionally different from low achievement only (Mazzocco, 2007). In consequence, low achievement alone is not a suitable diagnostic indicator for dyscalculia. This conclusion does not imply that only children with dyscalculia should receive intervention for their learning difficulties or that those with low achievement should not receive instructional support. Rather, it argues that assuming the differentiation of dyscalculia from low achievement allow those with low achievement to receive special education services, which has occurred in the past with poor implementation of discrepancy approaches for SLD identification, is not appropriate (Panel, 2014). On the contrary, evidence from research suggests that children with low achievement would likely benefit from an RTI approach, where greater intensity of instruction should likely lead to response for a significant percentage of struggling students (Stoiber & Gettinger, 2016). Actually, this approach depends on curriculum-based definitions of typical arithmetical development. It focussed on each child’s conceptual gaps in understanding and giving individual support with activities designed to eliminate each gap. However, nonresponsive children subsequently identified with a pattern of cognitive strengths and weaknesses that underlie dyscalculia need individualized instruction to meet their academic requirements (Panel, 2014).

Evidence provided by genetic, brain, and cognitive research strongly supports that dyscalculia is a neurocognitive developmental disorder which exhibits high rates of co-occurrence with other SLD as dyslexia (Gross-Tsur, Manor, & Shalev, 1996) and attention-deficit/hyperactivity disorder (Monuteaux, Faraone, Herzig, Navsaria, & Biederman, 2005). Many studies have sought to explain dyscalculia and its co-occurrence in terms of domain-general cognitive capacities such as those related to general intelligence, memory, language, and space (Geary & Hoard, 2005).

However, much research has been motivated by the postulation of core cognitive deficits that can give rise to the observed behavior. Core deficits themselves can have many causes and variable behavioral manifestations. Children with dyscalculia show a core deficit in processing numerosities, which is revealed in slower and less accurate enumeration of small sets of objects and in comparing the numerosities of sets of objects or the magnitude of digits (B. Butterworth & Kovas, 2013). However, good language abilities appear to be needed for the typical development of counting, calculation, and arithmetical principles (Dolan, 2007). It is not yet known whether this impairment interacts with other cognitive impairments to create identifiable symptom pictures or subtypes of dyscalculia; and it does not exclude the possibility that there are other causes of learning difficulties in mathematics, even selective learning difficulties (B. Butterworth & Reigosa-Crespo, 2007). Empirical evidence that neurocognitive processes affect math achievement suggests that the assessment of these processes is critical not only for dyscalculia or low achievement identification and service delivery but for intervention purposes as well.

From the author’s perspective, in the case of dyscalculia, the inclusion of the concept of the core neurocognitive deficit may address two additional points of criticisms concerning the PSW approach. The first criticism is related to poor stability of the profiles of cognitive ability. In 2012, Reeve and colleagues published the results of a 6-year longitudinal study (all primary grades), and they found that dot enumeration and number comparison ability measures of core number competence were broadly stable across the study (Reeve, Reynolds, Humberstone, & Butterworth, 2012). They also demonstrated that the neurocognitive profiles based on core numerical abilities were not related to general cognitive abilities but were related to computation abilities. Several reports based on longitudinal studies support this conclusion (Halberda, Mazzocco, & Feigenson, 2008; Reigosa-Crespo et al., 2013).

Some authors argued, as second limitation of the PSW approach, that cognitive profiles do not provide information about appropriate interventions because of the literature showing that association among different aspects of cognitive ability and academic achievement is inconsistent (Fletcher, Denton, & Francis, 2005). However, supportive evidence for a relationship between core numerical capacities and math competence has been revealed in studies of typical children and those with known mathematical disabilities (Reigosa-Crespo et al., 2012). Those authors also argued that there is little evidence that instruction addressing strengths and weaknesses in cognitive skills is related to intervention outcomes (Fletcher et al., 2005). Moreover, two recent meta-analyses on specific cognitive skills training have indicated that it does not appear to result in improved academic achievement for most students (Kearns & Fuchs, 2013).

Numerous attempts have been made to design educational interventions to foster the development of core numerical processing. In the majority, the results have showed improvement using nonsymbolic (Booth & Siegler, 2008; Hyde, Khanum, & Spelke, 2014) and symbolic quantities (Obersteiner, Reiss, & Ufer, 2013; Siegler & Ramani, 2008) with transfer to improvements in math competence. Nonetheless, further research is needed on the consequences of intervention based on core neurocognitive deficits in children with dyscalculia.

The inclusion of the concept of the core neurocognitive deficit in the PSW approach has several practical implications. The diagnosis of dyscalculia can consist of simple tests of the basic capacities to estimate and compare numerosities; for example, a screener for DD based on these principles is widely used in the UK (B. Butterworth, 2003) and forms the basis of the Minimat test that was used to carry out a very-large-scale cohort study of dyscalculia in Cuba (Reigosa-Crespo et al., 2012). In the assessment of individual cognitive capacities, set enumeration and comparison can supplement performance on curriculum-based standardized tests of arithmetic to differentiate dyscalculia from other causes of low math achievement (Landerl, Bevan, & Butterworth, 2004).

On the other hand, dyscalculia and other neurocognitive disorders are rarely identified until relatively late in childhood because specialized assessments are difficult to access and teachers and parents are often poorly informed about them (Goswami, 2008). Moreover, because of the high rates of co-occurrence, it is likely that an unassessed SLD will be treated as the consequence of the assessed SLD. This may occur when one condition is more noticeable than the other or when one SLD has been studied more than another (B. Butterworth & Kovas, 2013).

Implications for intervention are clear. If the basic capacities for understanding numerosities are weak, these should form the focus of a training strategy rather than learning number bonds and other arithmetical facts by rote (B. Butterworth, Varma, & Laurillard, 2011). If it turns out that there are subtypes of DD due to interaction with other cognitive deficits or due to domain-specific information processing impairments yet to be identified, new and appropriate interventions will need to be devised. Moreover, neuroscientific research suggests that rather than address isolated curricular gaps, remediation should build the foundational number concepts first. It offers a precise cognitive target for assessment and intervention that is largely independent of the learners' social and educational contexts.

According to several authors, for solving these challenges, further research into the atypical developmental trajectories of neurocognitive processes leading to dyscalculia are needed (B. Butterworth & Kovas, 2013). It is also important to study the comorbidities among SLDs, as well as the causes and effects of comorbidities and their educational consequences. This knowledge also helps with individualizing education for all learners. Additionally, and determined by the first, educators, school psychologists, and clinicians need to be trained to understand dyscalculia, to differentiate dyscalculia from low math achievement, and, also, to design specific learning pathways for each child who struggles with math.

For concluding this chapter, an implementation based on this approach will be presented briefly. The educational neuroscience lab in the Cuban Centre for Neuroscience (CCN) developed several tools that may serve as scaffolding for implementing a school-based program focused on neurocognitive development (henceforth, SBND program). These tools are questionnaires for the identification of signs of core neurocognitive deficits associated with poor achievement on reading and mathematics and also test for profiling an individual's neurocognitive status. This profile may facilitate interventions focusing on individual differences into the classroom. A SBND program has five main features.

A Closed Cycle Approach: Red Flags → Neurocognitive Profile → Intervention → Monitoring A closed cycle approach means that a SBND program involves (i) the identification of signs of atypical neurocognitive development in the learners, (ii) profiling of individual differences relative to strength and weakness in neurocognitive capacities related to reading and mathematics, (iii) personalized intervention in the classroom based on neurocognitive profiles, and (iv) monitoring the student's progress by "reuse" of the tools for detection and profiling.

An Ecological Approach: The School Is the Best Place The SBND program is designed to run in schools and to avoid clinical practices mainly focusing on diagnosis and treatment of disorders. Indiscriminate use of these practices can lead to stigmatization and segregation of those with special needs. Under the SBND program, teachers who receive training in science of learning and ICT can use the screening tools in mobile devices like smartphones or tablets for identifying signs of warning concerning the neurocognitice development of individual students ("red flags"). Based on these early signs and also on the information resulting from individual neurocognitive profiles, teachers can elaborate multiple strategies for attending to individual differences in the classroom. This approach supports an "ecological" perspective since SBND programs benefit from the natural conditions of the school environment and everyday teaching-learning interactions. At the same time, these everyday educational processes and outcomes may be positively impacted as consequence of these programs.

Taking Advantage of ICT Facilities As is known, several global organizations are promoting the integration of ICT into curriculum, teaching, learning, and assessment as a main goal for education until 2030 (UNESCO-IBE, 2016). In line with this endeavor, the SBND programs take advantage of current ICT availability (Santos et al., 2015). Tools for detecting "red flags" are based on mobile solutions, whereas tools for profiling neurocognitive development are computerized tests that facilitate precision and accuracy in the assessment. Both have been developed as client-server applications. The teacher training in educational neuroscience is designed as an e-learning environment (Fig. 45.1). The intervention includes strategies for attending to individual differences in the classroom and, also, neurocognitive training using theoretically based video games.

Teaching the Teachers A SBND program may drive teacher training and professional development in two ways. On the one hand, teachers gain knowledge about the neurobiology of learning, the neurocognitive development of learners and its relationship with literacy and numeracy, and also how this knowledge can impact on educational practices. On the other hand, teachers acquire skills to use ICT as part of this educational process.

Inclusive Education and Then Inclusive Intervention Identifying "red flags" in neurocognitive development may be a powerful way to produce early preschool-based and school-based neurocognitive interventions. However, educators must understand the relationship between the brain, cognition, and learning in order to manage individual differences in neurocognitive development in educational settings. The most effective strategies could be those in which individual differences are seen as opportunities rather than problems that need to be addressed. In this sense, differences can provide opportunities to experiment with strategies that involve all learners in meaningful activities. Cooperative learning is one of them, for example.

At present, a SBND program is carried out under the direction of the Educational Ministry of Ecuador in collaboration with the educational neuroscience lab of the CCN. This study has recruited 20,030 children as well as 1598 teachers and other

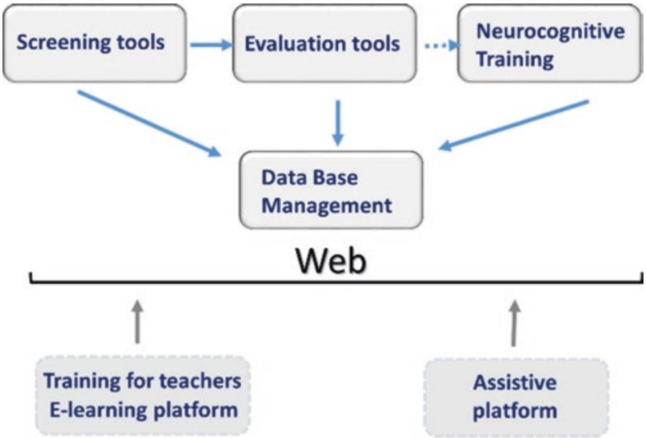

Fig. 45.1 Tools of the school-based programs for improving neurocognitive development are based on ICT facilities

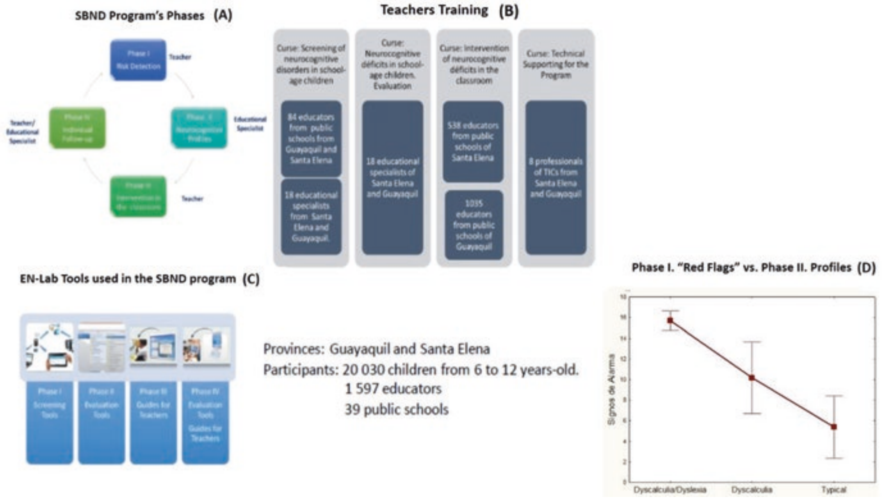

Fig. 45.2 Implementation of a school-based program focused on neurocognitive development in Ecuador

educational practitioners. The main goal is to evaluate and validate the principles of a SBND program. At present, the study is in progress, but actions related with training teachers and detection of "red flags" have been concluded. Figure 45.2 shows the SBND program's phases (A), the training courses for teachers (B), the tools used in each phase of the SBND program (C), and the relationship between "red flags" identified by teachers and the neurocognitive profiles (D). In this case, notice that more "red flags" indicate a more atypical profile.

Challenges for Educational Policy and Practice

Researchers, practitioners, policy-makers, and teachers should take in mind that several conditions are relevant for a successful SBND program:

Barriers to translation. There is a lack of an integrated knowledge base that limits the effectiveness of disseminating findings from the laboratory into the classroom. A common platform and a common language become necessary for helping to identify and to address misunderstandings as they arise and to develop concepts and messages that are both scientifically valid and educationally informative (Howard-Jones et al., 2016). A critical component of this endeavor is that tangible financial resources must materialize for progress to be made, from the local level all the way up to the governments. Notice that, as a worldwide practice, the government education budget spent on research is significantly smaller compared, for example, with the government health budget spent on research.

The teaching profession. The majority of in-service teachers have no background knowledge about science of learning, and pre-service teachers do not receive information about that. Removing the barriers to translation is a necessary condition for effectively teaching teachers about the brain, cognition, and learning. In line with this, a thoughtful way for training teachers is to create courses in partnership between teachers and researchers on science of learning (Pickering & Howard-Jones, 2007).

On the other hand, introducing science of learning into initial teacher education requires faculty cooperation across department and college lines (Dubinsky, 2010). In this sense, effective mechanisms should be established in order to diminish administrative barriers relating to the development of tuition-sharing arrangements, calculating faculty time assignments, etc. and, also, for coordinating the participation of the faculties which have different sets of pressures and priorities. At the level of individual faculties, communication and cooperation among people with expertise in each area should be required. For example, concepts such as synapsis, neural plasticity, sensitive periods, memory recovery, and cognitive process must be explained by researchers to educators; and concepts such as curriculum, assessment, and learning trajectories must be explained by educators to researchers. A final challenge for introducing science of learning content into initial teacher education is that university-level teacher educators need to be convinced that doing so will result in preparing better classroom teachers. Finally, policy-makers need to keep in mind that, introducing science of learning concepts as a background for initial teacher education, new entry requirements and new qualifications for future teachers have to be taken into consideration.

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Chapter 46
Challenges and Future Perspectives

Pekka Räsänen, Vitor Geraldi Haase, and Annemarie Fritz

This book started with a note about the situation in our schools: globally, six out of ten children and adolescents are not able to read or handle mathematics with proficiency by the time they are of age to complete primary education. That makes over 600 million children and teenagers (56%) whom we fail to teach the basic skills required for an independent adult life (UNESCO Institute for Statistics 2017). This number exceeds with a multitude of those considered to have learning disabilities in mathematics. Definitely, the first and most urgent question in mathematics education globally is how to improve the overall quality of education and how to offer proper learning opportunities for all.

Studies from different countries using different criteria have estimated that the prevalence of persistent difficulties in learning mathematics is about 5–7% (Zhang et al., 2018; Landerl, Chap. 2, this volume). That is approximately the same number of teenagers as there is in the highest-performing countries below the lowest level of performance in the international comparison studies like the Programme for International Student Assessment (PISA) (see Haase and Krinzinger, Chap. 20, this

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A. Fritz et al. (eds.), *International Handbook of Mathematical Learning Difficulties*, https://doi.org/10.1007/978-3-319-97148-3_46

volume). Therefore, even if all countries could raise the quality of mathematics education to the level of the top Nordic or Asian countries and were able to offer educational opportunities for all, still more than 70 million school-aged children and adolescents would show extensive difficulties in learning basic mathematical skills. This number is similar to the total population of France, or that of countries like Argentina, Chile, Uruguay, and Paraguay put together. We are talking about a large challenge to work on for education and research.

Judging from the chapters presented in the section “Mathematics learning and its difficulties around the world” and from the considerations above, the most pressing problems are related to environmental (i.e., policy) questions. How can we provide good mathematics education for all? The answer to this question is more political than scientific, except for those 1–2 children in every classroom who have learning disabilities, i.e., developmental dyscalculia, and who continue to struggle with learning basic numeracy despite the best available education.

We Need Research from Genes to Behavior to Build Bridges Between Them

In the search for a solution for math education of individuals with inherent difficulties, attention has been turned to neuropsychology and cognitive neurosciences. A new field of research—educational neuroscience—has emerged (Della Salla & Anderson, 2012; Mareschal, Butterworth, & Tolmie, 2013). The most enthusiastic supporters of this new field of research have believed that educational neuroscience will subsidize the education of individuals with developmental dyscalculia. It is also hoped that educational neuroscience will contribute insights into the education of the majority of children who, despite not experiencing more severe impairments, still find learning mathematics difficult or overly demanding.

Therefore, we need to discuss the burgeoning field of cognitive neuroscience. However, a word of caution is necessary. Understanding the neurocognitive underpinnings of a behavior such as math learning is a complex enterprise. A way to reduce complexity is thinking in terms of levels of analysis, as proposed by Frith (1992). She distinguished four levels of analysis: etiological (environmental and genetic interactions); neural (neuromodulation, patterns of brain connectivity and activation); cognitive (cognitive processes and architectures); and behavioral (relations with environmental contingencies). It is important to understand that there is no qualitative hierarchy between these levels of description. Understanding the neural level is nothing deeper than understanding observable behavior. A detailed analysis and understanding of both is needed before we can understand their relations.

Bridges across these levels are still fragile and the waters underneath are turbulent (Ansari & Coch, 2006). Researchers are working hard to build and strengthen these bridges. The concept of the endophenotype is a possible way to build these bridges (see Haase and Carvalho, Chaps. 22 and 23, this volume). Endophenotypes are intermediate phenotypes that may help to simplify the analysis and to establish links between the etiological and behavioral levels.

Molecular genetic research discloses multiplexed relations between the etiological and the phenotypical levels (see Carvalho and Haase, Chaps. 22 and 23, this volume). The hope of the endophenotype research program is to reduce this complexity by characterizing the intermediate steps. These intermediate steps may be characterized at the neural level as patterns of expression of neuromodulators or patterns of task-related brain activation. At the cognitive level, the goal is to identify the cognitive mechanisms underlying typical and atypical math learning. The goals of the research program in contemporary numerical cognition may be described as uncovering the endophenotypes underlying math learning (Henik & Fias, 2018). If it is still risky to cross these bridges, it is worth trying—or, at least, the levels of analyses should consolidate or be made compatible.

Educational Neuroscience: Where Are We?

Technological advances in recent decades, such as various functional neuroimaging and molecular genetic investigation techniques, have raised enormous interest in psychology and pedagogy. It is increasingly possible to investigate the biological foundations of psychological processes at relatively low cost. It makes less and less sense to investigate psychological phenomena from a purely functional or cognitive perspective, disregarding biological evidence. It seems that, finally, psychology is fulfilling the Darwinian ideal and is being integrated into biology. As stated by Dehaene (2007, p. 527):

> An ultimate goal of psychology is to provide lawful explanations of mental mechanisms in terms of a small set of rules, preferably framed in the language of mathematics, which capture the regularities present in human and animal behavior. Furthermore, those psychological laws should not remain stated solely at a descriptive level (although obtaining valid descriptive rules of behavior is usually an indispensable step on that road). Rather, they should be ultimately grounded in a neurobiological level of explanation, through a series of additional bridging laws linking the molecular, synaptic, cellular, and circuit levels with psychological representation and computations.

It is clear that neuroscience alone cannot solve the puzzle of learning. Neuroscience needs a detailed analysis of behavior, otherwise it is useless: "behavioral work provides understanding, whereas neural interventions test causality" (Krakauer, Ghazanfar, Gomez-Marin, MacIver, & Poeppel, 2017). Likewise, cognitive skills that we describe with concepts like spatial skills (see Resnick et al., Chap. 26, this volume), working memory (see Passolungi and Costa, Chap. 25, this volume) or executive functioning (see Sarama and Clements, Chap. 43, this volume) all contribute strongly to mathematics learning and learning disabilities, not forgetting the emotional and motivational aspects in learning (see Baten et al., Chap. 28, this volume; Haase et al., Chap. 29, this volume). However, we are still far from fully understanding the mechanisms and structures of these concepts we continuously use in psychology and cognitive sciences. The reductive ideal of a neural model can only be as detailed as the concepts we use to describe the functions in it.

There is also a risk of overexplanation in educational neurosciences. This habit is common among neuroscientists and among teachers, who gladly take ideas from neuroscience as guidelines for educational practices. Many of those complexity-simplifying ideas, often called neuromyths, are produced by non-neuroscientists, but they are well-marketed as such. Typically, they are only loosely grounded in neuroscience, building incorrect and ineffective ideas of learning and teaching (Howard-Jones, 2014). Unfortunately, teachers who are interested in learning more about the brain tend to believe more of these false simplifications than teachers who are not interested in it (Dekker, Lee, Howard-Jones, & Jolles, 2012; Gleichgerrcht, Lira Luttges, Salvarezza, & Campos, 2015). What is most unfortunate is that these misconceptions seem hard to change (Im, Cho, Dubinsky, & Varma, 2018). Therefore, researchers need to be careful in their interpretations, as well as working to correcting the oversimplifications of these complex issues.

Another source of overexplanation are neuroscientific discoveries themselves. We have had a habit of explaining a cross-sectional group-level difference in brain activation patterns between typical and learning disabled as a cause, not a consequence, of more complex development (see similar discussion on dyslexia research, e.g., Ramus & Szenkovits, 2008). Luckily, now there are more and more interventions and longitudinal studies bringing insights into the causal mechanisms of the learning process itself. We also easily make oversimplifications from the brain to education, resulting in recommendations without empirical grounds (Alferink & Farmer-Dougan, 2010). Therefore—always—a warning must be added when we make interpretations about mechanisms between different levels of explanations (genetic, neural, cognitive, behavioral) and especially when we aim to make practical suggestions from cognitive neuroscientific studies for practices in the classroom (see Ansari, Chap. 7, this volume).

Will this building of knowledge on the neurocognitive and neurogenetic foundations of psychological processes provide an evidence-base for promoting all sorts of learnings required to function effectively in the current and future world? In the long run, the answer seems to be definitely positive, but this is just a hunch. There still is no imaging machine that allows us to predict the future. Caution dictates that we should focus on the current state of knowledge and its implications.

At the outset, it is important to recognize that neuroscientific advances have remained largely restricted to examining the validity of hypotheses and models previously proposed in psychology (Bowers, 2016). Neuropsychology and functional neuroimaging are powerful tools to settle disputes between rival cognitive models of some psychological processes. In showing, for example, that two cognitive processes are implemented by distinct neural systems, cognitive neuroscience helps to overcome the intrinsic limitations of purely psychological methods, supporting the ontological reality of psychological constructs (e.g., Bechara, Damasio, Tranel, & Anderson, 1998). New structural–functional correlations have also been discovered (e.g., Koenigs et al., 2007). So far, no neuroscientific breakthrough has occurred with radical epistemological or pedagogical implications.

Other examples relate to interventions for remediating learning difficulties. Functional and structural neuroimaging studies have consistently shown that interventions modify patterns of connectivity among regions implicated in specific learning impairments (Michels, O'Gorman, & Kucian, 2017). This finding stresses the important role of synaptic plasticity in learning and in learning impairments, providing a solid basis for interventions. However, success criteria are still formulated in behavioral terms.

Considerable interest has also been elicited by noninvasive biologically based therapies such as transcranial microcurrent stimulation or neurofeedback (Cortese et al., 2016; Kadosh, Dowker, Heine, Kaufmann, & Kucan, 2013; Wang & Sourina, 2013). Research on biological therapies is still incipient but growing. It is to be expected that in the near future, a breakthrough may occur. Perhaps, on a longer time horizon, neurochips or genetic therapies will be available that overcome the limitations and variability of human learning potential.

In the following subsection, we comment on two important advances of neurocognitive and neurogenetic research on mathematics learning with potential implications for pedagogy: the modeling of arithmetic learning and the role of acquiring arithmetic fact knowledge.

What Is Learning Arithmetic from a Neuroscientific Perspective?

Difficulty in learning arithmetic skills is one of the central behavioral features of developmental dyscalculia. Very often it is used as the main criterion in research and diagnostics. However, arithmetic and written language are biologically secondary cognitive abilities (Geary, 2007). These abilities are relatively recently acquired and not universally present cultural artifacts. The relatively short time span since the invention of arithmetic and written language was probably not sufficient to evolve specific genetic mechanisms for intuitively learning these abilities. There also does not seem to be an intrinsic motivational system selected to acquire these abilities. Learning to read and write words and numbers is an arduous process consuming 3–4 years of hard work from a child and his or her teachers (Dehaene, 2009; Moura et al., 2015).

Research has proposed models based on the concept of exaptation, such as the cultural recycling (Dehaene & Cohen, 2007) and redeployment (M. L. Anderson, 2010) models, that explain the difficulties inherent in the processes of learning to read and write words and numbers, as well as doing math at the neural and computational levels. Acquisition of numerical symbols, for example, requires the establishment of synaptic connections between neural systems evolved for different adaptative symptoms, i.e., it constitutes a form of exaptation. Symbolic numerals seem to acquire their quantitative meaning through de novo establishment of synaptic connections between ancient, biological primary neurocognitive systems devoted to approximate number magnitude representation (the intraparietal sulcus) and neural systems implementing complex linguistic forms (the left perisylvian language areas) and visual forms (the inferolateral occipitotemporal transition) (Dehaene & Cohen, 2007).

According to one model, severe and persistent math learning difficulties could originate from representational inaccuracy of numerical magnitude in the intraparietal sulcus and/or difficulties in establishing synaptic connections among areas originally evolved with different finalities (Noël & Rousselle, 2011). Studies of developmental dyscalculia lag behind, but considerable genetic evidence supports the exaptation hypothesis of developmental dyslexia (Kere, 2014; Paracchini, Diaz, & Stein, 2016; see also Carvalho and Haase 2018, Chap. 22). The best-replicated genes associated with developmental dyslexia impair processes of neuronal cell migration and the establishment and maintenance of synaptic connectivity. This suggests that specific learning impairments could be considered to be disconnection disorders (Mitchell, 2011); that is, specific learning impairments could be caused by variability in the ability to establish new synaptic connections under epigenetic control. The same holds for genetic syndromes such as fragile X, Turner, velocardiofacial, and Williams syndromes, in which severe math learning difficulties are an important phenotypical trait (Haase & Carvalho, 2018; Carvalho & Haase, chaps. 22 and 23). The physiopathology of most syndromes consists of impairments in the synaptic plasticity mechanisms required for learning (Ismail, Fatemi, & Johnston, 2017; Johnston, 2004).

The pedagogical implications of the exaptation models are clear. Learning to read and write symbolic numerals, as well as using these symbols in exact calculation, heavily relies on experience-dependent synaptic plasticity (Hebb, 1949). The lack of intuitive cognitive mechanisms or intrinsic motivation promoting the acquisition of these abilities and the experience-dependent nature of the process indicate the need for considerable engagement and training on the part of pupils and teachers, not only to understand but also to automatize procedures and facts. We next turn our focus to the acquisition of arithmetic facts—an important foundational ability for the development of more complex arithmetic skills.

Difficulty in automatically retrieving arithmetic facts is the cardinal symptom of developmental dyscalculia (Butterworth, Sashank, & Laurillard, 2011). Evidence suggests the numerical/arithmetic system is both hierarchically and compositionally organized (Dowker, 2015). Rehearsal of arithmetic facts is a foundational ability for future acquisitions such as multidigit calculation and word problem solving (Raghubar et al., 2009; Verschaffel, Depaepe, & van Dooren, 2015). At the same time, some rare cases of developmental dyscalculia present very specific impairments in arithmetic facts (De Visscher & Noël, 2013; Temple, 1991).

The process of acquiring arithmetic facts has been experimentally modeled in young adults learning multiplication facts across several sessions (Zamarian, Ischebeck, & Delazer, 2009). At the beginning of the process, functional magnetic resonance imaging (fMRI) records higher activation levels in the prefrontal regions associated with controlled processing. As the individual acquires proficiency in the task, the activation focus moves to the posterior regions, especially the left angular gyrus (see also Grabner et al., 2009).

Another study has compared the effects of two learning strategies (Delazer et al., 2005). Learning by problem solving was more efficient and activated the medial areas of the parietal lobes (probably related to visuospatial imagery) in comparison with learning by drill, which mainly activated the left perisylvian language areas. Although less efficient, the learning-by-drill strategy seems to activate different neural systems compared to learning by problem solving. The two strategies could thus play a complementary role in the acquisition and rehearsal of arithmetic facts.

Neuroimaging methods have also been used to investigate the acquisition of arithmetic facts by children across distinct ages and stages of the learning process. In general, the results have corroborated the progression from controlled processing in the prefrontal areas to automatic processing in the posterior cortical areas (Rivera, Reiss, Eckert, & Menon, 2005). However, studies with children have called attention to an important difference in comparison with research with adults. Research with children in different phases of the acquisition of arithmetic facts indicates that during this process a temporary pattern of hippocampal activation is observed (Cho et al., 2012; Qin et al., 2014). This finding contrasts with adult studies, in which no hippocampal activation has been detected (De Smedt, 2016; Menon, 2015). Hippocampal activation during arithmetic fact learning fits well with the role this structure plays in the consolidation of information in long-term memory. The transitory nature of hippocampal activation in children and its absence in adults suggest there may be a kind of developmental window opportunity of hippocampal memory-related functions important for the acquisition of arithmetic facts.

Results from cognitive neuroscience are, once again, crystal clear: acquisition of arithmetic facts is not only a foundational ability for further math developments; it is an experience-dependent complex process mediated by distinct neural systems at the cost of considerable effort.

However modest, the results of neuroscientific investigations such as the exaptation models and the acquisition of arithmetic facts may be difficult to integrate into current pedagogical theory and practice (e.g., Brazil, 2016; Marope, Griffin, & Gallagher, 2017). On the one hand, the epistemological point of departure of neuroscience is clearly a “positivistic” one, as illustrated by Dehaene’s (2007) quote. Neuroscientists seem to be interested in “mathematizing” and “biologizing” psychological processes. Numerical cognitive research has focused mainly on the representations and processes underlying specific math abilities such as magnitude representation, counting, transcoding or calculation. Neuroscientific results suggest math proficiency depends not only on conceptual understanding and ingenuity on the part of the kids, but also on the gradual building of factual and procedural knowledge that requires a considerable degree of training and automatization.

On the other hand, math pedagogy has moved away from the traditional drill and practice to a constructivist approach (Klein, 2003). Understanding, reasoning, and creativity are valued over calculation fluency. Pedagogical goals are not formulated anymore in terms of the acquisition of skills, but as contextualized competencies that should prepare pupils to effectively function as critical citizens, displaying more sophisticated, flexible, adaptive, and critical forms of quantitative reasoning (Brazil, 2016; Marope et al., 2017). Additionally, the epistemology of pedagogy is largely hermeneutic–qualitative. More than just a science in strict terms, pedagogy is an art engaged with ethical and not only instrumental goals.

It seems then that we have a long way to go before math pedagogy and neuroscience can cooperate more effectively. We have reviewed neuroscientific results that challenge pedagogy in several forms: Is it possible to reach the goal of adaptive expertise presumably required for the world of the future without a considerable degree of routine expertise? Is it possible to operationalize current competence goals regarding compatibility with the investigation of their neurocognitive underpinnings? At the same time, neuroeducational research requires massive doses of humility on the part of neuroscientists, and an openness of mind to understand and learn the complexity of the tasks, goals, and ethics underlying pedagogical efforts to prepare future citizens. Fortunately, both neuroscientists and pedagogues are increasingly interested in this dialogue. We hope this book helps these two fields to approach each other.

Focus on Early Development

There is no doubt that we are born with some level of quantitative understanding. We share the same ability to estimate relative differences between quantities as has been found in numerous studies with different animal species. Numerical competencies have been reported from both social and nonsocial animals, from fish to our closest relatives, chimpanzees (e.g., ants: Reznikova & Ryabko, 2011; bears: Vonk & Beran, 2012; fish: Agrillo, Piffer, & Bisazza, 2011). The chimpanzee has been shown even to be able to learn to match quantities to human-invented numerical symbols in laboratory conditions (Biro & Matsuzawa, 2001).

However, the symbolic system of numbers and mathematics has been one of the most important discoveries made by human beings. Whether this discovery is an invention of the human mind or a discovery of the existing laws of the universe is an essential philosophical discussion but is out of the scope of this book (regarding this discussion, see Butterworth, Gallistel, & Vallortigara, 2018). In all cases, mathematics is an innovation that allows us to communicate about exact amounts with each other, to share, to calculate, to measure, and to build bridges that do not collapse (for a historical view, see, for example, Menninger, 2013).

Learning even the basics of this cultural tool takes a long time. For example, just creating an understanding that the word “six” always refers to a specific number of things, and that the word “seven” comes after that in the counting sequence and means one more, typically takes from 4 to 7 years from birth, even though children as young as 18 months old already show signs of picking up the correct order of the number sequence (Ip, Imuta, & Slaughter, 2018). The variance in the age of learning this skill is large, partly because of individual differences but also to a large extent because of differences in learning environments.

The mother’s (and father’s) activity in numerical interactions (Casey et al., 2018; Sorariutta & Silvén, 2018) and/or access to mathematically aware high-quality early education (Ulferts, Anders, Leseman, & Melhuish, 2016) have been continuously shown to be the strongest predictors for successful learning of the number symbol system and arithmetic skills.

In the UK, Melhuish et al. (2008) used a large sample to study the unique predicting power of different factors for mathematical skills at the age of 10 years. Several variables showed a significant impact ($d > 0.30$): the mother's education, home learning environment, school effectiveness, socioeconomic status, and income of the family. Importantly, the quality of early education measured more than 5 years earlier still contributed significantly to the skills even after controlling for the effects of all other variables.

The mother's education has been consistently shown to have one of the largest effect sizes to later school success. Multiple things are connected to this maternal and home effect. There are biological (genetic, nutrition) and especially social factors that contribute to this: informal learning (see Lehtinen et al., Chap. 3, this volume) and learning opportunities, math talk, and other linguistic input at home, without forgetting the parents' interest, support and encouragement at preschool, and school work (see, for example, Boonk, Gijselaers, Ritzen, & Brand-Gruwel, 2018). Many studies have shown that the socioeconomic effects are mostly mediated by the home environment (Duncan & Brooks-Gunn, 2000), and the key factor within the effects of the mother's education seems to be sensitive parenting (Collins, Maccoby, Steinberg, Hetherington, & Bornstein, 2000).

In addition to the home environment, a growing body of research in recent years—in particular from neuroscience, sociology, and psychology—has proved that early childhood education and care (ECEC) provides a crucial foundation for future learning by providing children with basic functions of learning, communication, and cognitive and emotional skills, on which learning in schools can build (OECD, 2018).

Implementation of early education in the education system has been adopted in the educational policies of most countries. Given the potential benefits, there is growing awareness of the role ECEC can play in compensating for the adverse effects of childhood poverty and disadvantages with long-lasting effects, both in developed countries (Lehrl, Kluczniok, Rossbach, & Anders, 2017) and in developing countries (Rao, 2014). Children from the poorest families seem to benefit most from high-quality early education (Christian, Morrison, & Bryant, 1998), especially when it is combined with holistic support for the children and their family in terms of nutritional, health, social, and psychological services. Interestingly, Chor (2018) showed a multigenerational effect of participation in early education. When mothers had been in high-quality early education, this supportive effect carried over to school success in mathematics in the next generation.

However, it has to be said clearly: merely offering early education and giving children access to it is not a guarantee of the positive effects ECEC can produce. The beneficial effects of ECEC—in this instance, on mathematics learning—depend on the quality of the education and care provided.

The policies of ECEC should stand on many legs, as described in the following sections.

Development of Scientifically Based Content Acknowledging the importance of early education and our knowledge about the basic prerequisites, the educational content has to be chosen carefully. Possible predictors of mathematics attainment are an important issue of scientific research. It has already been focused on for several decades, resulting in a wide range of evidence-based empirical findings. Math training (support) in early education for children at preschool age should follow this substantial knowledge and be structured on the basis of these findings. At the same time, the developmental circumstances and prerequisites of the children have to be taken into account.

Development of Organizational Conditions In order to be effective, preschool intervention has to be institutionalized and binding to make it accessible for all children. Suitable institutions are kindergartens and schools, though country-specific conditions and structures have to be taken into account. In some countries, kindergarten attendance is not compulsory or early education cannot be implemented in kindergartens. Establishing early education in school, however, requires structural changes in the school system.

Development of Personnel Conditions The success of early education strongly depends on personnel resources—that is to say, the qualifications of teachers who carry out early education. As a result, implementing early education goes hand in hand with considering qualification structures of early education teachers. It is not sufficient to provide children with mathematical games, relying on autonomous acquisition of concepts in the course of playing (Chien et al., 2010; Clements, Fuson, & Sarama, 2017). The understanding of mathematical concepts depends on the instructions and on the provision of different examples. Thus, without profound qualifications, successes in supporting preschool-aged children through early education cannot be expected.

Globally, we are far from having equal access for all children to attend quality early education, and even in cases of access to ECEC, not all early education programs consider mathematics an important topic. The majority of early education teachers are not skilled in teaching mathematics (OECD, 2018). There is an urgent need for improving both the access and the quality of pedagogies for early age. And when we talk about pedagogies for early age, academically oriented programs within early education are not as effective as holistic approaches that take into account support for the whole family (Britto et al., 2017; Richter et al., 2017).

Mathematical skills at preschool age have been shown to be the strongest single predictor of later success at school (Duncan et al., 2007). Evidence suggests that severe difficulties in math learning are associated with cognitive and basic numerical processing impairments and that these impairments may be toned down when properly recognized at an early age (Siegler & Braithwaite, 2017). But there is still a lot to be done at school. It all starts from early recognition of learning difficulties. There, the research has not provided good guidelines for classroom teachers and policy makers with its varying criteria and definitions of mathematical learning difficulties (MLD) from one study to another.

Lack of Tools for Screening and Monitoring Learning

The demands for early recognition and for modifying teaching according to individual students' current conceptual knowledge, learning capabilities, and learning pace require a close connection of diagnostics and pedagogical action (cf. Heinzel & Prengel, 2012; Müller, Ehlert, & Fritz, 2017). In that respect, pedagogical and psychological diagnostics is a continuing process of generating hypotheses about current knowledge, which are then transformed into pedagogical action. Then again this is monitored and evaluated in order to be able to conduct further modifications if necessary. This cyclic process has been the guiding line of discussions for several decades.

Diagnostics and didactics, whether in lessons or in additional interventions, are closely connected and intertwined in this respect. They are to be carried out as a mutual process, composed of different parts: capturing data about learning prerequisites ↔ planning and conducting remedial support aligned to the students' needs ↔ monitoring and evaluation of support according to the targets set (Müller et al., 2017).

With this approach, diagnostic processes gain a different significance, which goes hand in hand with the need to establish new procedures, actions, and methods in pedagogy and instruction. Apart from approaches that allow comparison of individual attainment with that of the age reference group or the achievement of learning objectives (summative assessment), approaches to optimize teaching and learning processes (formative assessment) are needed. Methods of formative assessment of attainment can be arranged on a continuum of informal to formal steps (Bell & Cowie, 2001). Every interaction between students and teachers taking place inside the classroom can serve as informal information. This includes all sorts of assessment of performance, as well as observations during lessons, provided that the obtained information is utilized for optimizing teaching practice (McMillan, 2000).

The emergence of e-learning materials and e-books for schools has opened up the possibility of new methods of assessment. E-learning offers the possibility to quantify in databases all interactions of the child with the learning materials, quickly producing masses of data about the student's performances. Two overlapping groups of researchers have been interested in the analysis of this data: those interested in educational data mining and those interested in learning analytics (Baker & Inventado, 2014). Even though it is still in its infancy, learning analytics gives researchers and teachers new tools to monitor the progress of children, and it increases the probability of recognizing those who are at risk of dropping out from learning (Kurvinen et al., 2015; Minaei-Bidgoli, Kashy, Kortemeyer, & Punch, 2003). Unlike classical (summative) diagnostics, which usually only displays a state of knowledge at a selected point in time, the continual capturing of students' learning process depicts development of competence over a span of time. In order to do this, suitable tests of constant difficulty that assess the level of the very same competence at multiple times are necessary (cf. Klauer, 2014; Strathmann & Klauer, 2010).

Improvement can be quantified with regard to different reference systems: A preferred reference system is the *curricular reference system* (however, see below about the problems with mathematics syllabi), oriented toward the attainment of *learning objectives*. The students' learning progress is determined with regard to the curriculum (Leuders, 2014; Ufer, Reiss, & Heinze, 2009). However, the curriculum focuses on different mathematical ideas during the course of a school year. Thus, the assessment has to be parallelized for this frame of time. Comparable assessments require sticking to a certain subarea of mathematical competence (e.g., numbers and operations), which has to be assessed repeatedly, using different tasks. The challenge lies in aligning and matching the difficulty of the tests during the course of a school year or even over several years to achieve comparability (Strathmann & Klauer, 2010).

The construction of curricular tests making continuing assessment of students' learning process possible has proved to be challenging with regard to content and empirical assessment. Special obstacles arise due to the particular requirements and conditions of assessing changes. There are strict requirements concerning the test instruments, as well as the measurement theory in question. As the classical testing theory does not allow any reliable assessments in this context, there have been calls to abandon it (Klauer, 2014).

Recently, competence or developmentally oriented assessments based on the probabilistic testing theory have been preferred (Fritz, Ehlert, & Leutner, 2018). *Development-oriented reference systems* follow developmental concepts of capturing a learning domain step by step. Usually these are based on empirical evidence and theoretically modeled (based on a model of the development of the knowledge in question) capturing the entire complexity of the learning domain. Therewith, the developmental level and the further learning progress of a student can be depicted systematically. This new focus goes along with a change of perspective. It is no longer about checking which content to teach (method of input control) but about assessing what students' capabilities are at the end of each grade (method of output control).

The meaning of the competence or development-oriented reference standards becomes even plainer when deriving interventional actions from diagnostic results. Namely, if the following learning progress is to be predicted prescriptively by referring to a diagnosed current state, then the test—as well as the attainment to be predicted—has to relate to a common competence, supported by empirical evidence about their development. In other words, the testing procedure has to be based on a theory or model that represents the development of the according knowledge domain. A theoretical explanation of diagnostic action is strongly demanded (Hellmich, 2007; Müller et al., 2017).

Numerous empirical studies have provided proof that continuing capture of students' learning process and gains in knowledge positively affects their educational development (for an overview, see Stecker, Fuchs, & Fuchs, 2005). If teachers acknowledge diagnostics as the basis for their actions instead of using them for giving marks, and if they make use of diagnostics to provide their students with productive feedback on their learning progress, then formative assessments will foster students' learning success (Hattie & Timperley, 2007; Shute, 2008).

To rely solely on continually capturing students' learning process will not be sufficient for learning success, though. The adaptation of lessons and teaching to the documented learning progress has a crucial impact on learning success as well (Stecker et al., 2005), instead of just following the curricula and syllabus.

Monitoring-Based Framework for Interventions in Schools

The learning objectives formulated in curricula and syllabi provide a framework for the content that has to be learned during the course of a school year. The standard is defined by a social reference. But having to stick to these normative guidelines, instruction is addressed to the entire learning group expected to attain these learning objectives, even though we know that is not appropriate for every single child in the classroom. In fact, there is a match between the syllabus and skill level only for a small proportion of students (Hellmich, 2007).

Besides questioning the adequacy of learning objectives for all students, the ratification of the UN Convention on the Rights of Persons with Disabilities in 2009 brought up a further increase in the heterogeneity of a classroom. Hence, if teaching is aimed at enabling all students to make progress, the learning objectives stated in the curricula cannot apply to all students in a same way. Low-achieving students or students with learning difficulties need educational standards that focus on teaching sustainable basic knowledge that provides students with a fundamental basis for further acquisition of knowledge in various contexts (school, occupation). Such an individual reference standard focuses on the individual's learning progress. It thus prevents a lack of coherence between previous knowledge and current teaching. In some countries, the discourse about normative guidelines and empirical findings about low-achieving students' capabilities has led to a submission of minimum standards. This was meant to relieve teachers from getting all students to meet the curricular learning obstacles and, furthermore, to provide teachers with a guideline when developing expectations for low-achieving students (Klieme et al., 2003).

We do not see the minimum standards as a solution. The minimum standards easily become the highest requirements for some children, especially for children from a disadvantaged or immigrant background. When the aim is to get every child to develop to their highest potential with a strong feeling of competence at their own skill level, the standards for those who will not meet the curricular aims need to be individualized. This calls for more trained professionals in special needs education for schools, who can work together with teachers to build possibilities for more individualized standards. These professionals should have strong background knowledge of cognitive and psychological development and disabilities. There is plenty of research on teachers' attitudes to and effects of inclusion, what we lack is research about effective management and organizational models in inclusive schools. The discussion about inclusion has also raised questions about teaching that is appropriate and adaptive for all children. This means aligning teaching to students' individual prerequisites and levels of learning capabilities. Referring to an individual reference standard, students' acquisition of knowledge consequently has to be fostered on an individual level.

The US response-to-intervention (RTI) approach offers a possible guideline for realizing these targets. The research that led to adopting the RTI approach was the discovery that learning difficulties, especially in the case of dyslexia, were often connected to inadequate teaching and instruction methods rather than individual deficits (Vellutino, Scanlon, & Reid Lyon, 2000). The critique was especially targeted to the discrepancy criteria (i.e., the difference between IQ and reading skill). One of the key findings was that general intelligence did not play a significant role in learning technical reading. In the case of dyscalculia—persistent difficulties in learning the basic number skill—similar views have been presented that the core deficit in dyscalculia might not be related to general skills (Ehlert, Schroeders, & Fritz, 2012; Landerl, Bevan, & Butterworth, 2004; Reeve, Reynolds, Humberstone, & Butterworth, 2012), even though mathematical reasoning is known to correlate strongly with general reasoning skills.

According to the PISA studies, in Organization for Economic Co-operation and Development (OECD) countries there are four times more children who fail in acquiring the basic skills than the studies on dyscalculia would predict. In developing countries that figure is often more than tenfold. This indicates that the majority of the children who fail in learning mathematics do not meet the criteria for learning disabilities. Therefore, there are both diagnostic and practical reasons to adopt the preventive philosophy of RTI as a guideline for schools and classrooms. Every child who fails to learn should have a right to additional support without delay, and if he or she does not respond to this, then special educational and or remedial support is needed. As a legal statement, this kind of approach would force policy makers and school directors to revise educational practices in schools. In some countries these approaches have already been put into action (see, for example, https://www.european-agency.org/country-information/finland/legislation-and-policy).

The RTI approach intends to apply a multilevel diagnostic and interventional process; the central characteristics of this approach are early identification, prevention, and a closely following intervention in the case of learning difficulties. This approach can be depicted as a data-based multilevel remedial support system for students (Reschley & Bergstrom, 2009). As a whole, constant diagnostics and closely following support interlock, thereby offering different solutions to situations in pedagogical practice.

In the USA and in scientific research, different modifications of the RTI approach exist. Typically, three levels of support can be distinguished, bringing about structural changes in schools (see Fuchs and Fuchs, Chap. 39, this volume).

- Level I: High-quality lessons based on empirical evidence. A continuing learning progress evaluation (monitoring) serves the purpose of identifying problematic developments in educational progress.
- Level II: Those children who do not make enough progress receive additional support, taking place in small groups three to five times a week. Progress is monitored.

- Level III: If the second level of support does not suffice either, further diagnosis is carried out in order to check whether there is a need for support in developmental areas that have not been targeted yet, making optimized and even more efficient support possible.

The Challenges of the Response-to-Intervention Approach

The approach of considering difficulties in math learning not as a deficit of the child but as a problem of the school system, which is not offering the appropriate teaching and training methods (see Fuchs & Fuchs, Chap. 39, this volume), is undoubtedly a big step forward, but it involves the risk of overlooking the individual needs of specific children. Difficulties in dealing with arithmetic are caused by domain-general cognitive and noncognitive factors and domain-specific factors that underlie the development of mathematical abilities. For remediation, it is important to understand the impact of the different factors in order to arrange appropriate support.

Whatever type of mathematical learning difficulties children are suffering from, on the behavioral level the main problems are a lack of conceptual understanding and development of factual knowledge. Due to their problems, they have acquired insufficient basic knowledge, on which no further knowledge can build. But even as far as their procedural knowledge is concerned, they often stick to simple strategies, preferably using counting strategies (see Gaidoschik, Chap. 6, this volume). This strategy overloads the working memory and impedes the acquisition of more efficient arithmetic strategies.

Beyond that, mathematical learning difficulties are highly heterogeneous and are often associated with comorbid disorders such as developmental dyslexia or attention deficit hyperactivity disorder (ADHD) (see Krinzinger, Chap. 24, this volume). And as language and mathematics are linked together very closely, in a time when multilingual classrooms have become the norm rather than the exception, the linguistic competence of the children has to be taken into account (see Prediger et al., Chap. 27, this volume). Likewise, several studies pinpoint the importance of cognitive variables in the development of mathematical disabilities (see section titled "Approaches to recognition and intervention," this volume; Szucs, Devine, Soltesz, Nobes, & Gabriel, 2013; Zhang et al., 2018). This small and only rough listing of factors affecting numerical learning sheds light on what we expect from teachers and what knowledge is needed to individually tailor remedial support.

Another critique of the RTI model that has been presented is that it requires all children to go through the different levels before they are offered individualized support. Many of these "hard-to-remediate" children could be recognized already at the first level and be offered the support they need without repeated failure to respond. However, there is still a lot to do in developing tools for early recognition. How can we reliably recognize those children with dyscalculia for whom the best solution would be a very individualized approach from the beginning of the school or even much earlier? Even though there have been many attempts to build reliable indicators and assessment batteries to recognize these children as early as possible (e.g., Gersten, Jordan, & Flojo, 2005; Reeve et al., 2012), we are still far from a consensus about what are the most reliable core markers of persistent dyscalculia. Secondly, we lack cross-cultural validation of the reliability of these measures. How greatly do the markers of dyscalculia depend on differences in educational cultures? We assume the answer is less than we would expect, but that requires empirical confirmation.

One important way to support early recognition is to improve the general level of mathematics education. This question typically implies a need for more resources—more investment in education. Here, we raise two issues that do not necessarily imply a need for more recourses: firstly, the teacher who is already there in the classroom and applies different pedagogies, and how he or she is equipped to respond to the requirements to adopt new teaching methods; secondly, we focus on the syllabus, i.e., the question of what is taught in mathematics in classrooms, and when. The RTI model works or fails on these two key elements of education.

Professional Development for Teachers

This paradigm shift from classroom learning for all children to more individual learning progress means a challenge for teachers in many ways. It demands that they be proficient in math-specific content knowledge and its structure. The more profound and elaborate the content knowledge of the teachers, the more appropriate and understandable their instructions, leading to an enhanced level of teaching standard (Ball & Bass, 2000). Beyond that they need knowledge about methods to instruct children effectively (math-specific pedagogical knowledge). As teaching is a complex activity with an enormous number of variables, it cannot be expected to identify a few core practices or methods that help improve students' performance in every grade, school form, and country.

The main factors that apply to all high-performing countries are high levels of professional education for teachers, high status of the teaching profession, and expanded opportunities for continuous professional development. Technology has come to help in offering masses of teachers an inexpensive, globally accessible means of professional development (see Räsänen et al., Chap. 8). Universities and researchers should take this as an opportunity to provide open lectures for the masses, and not only open lectures but also structured learning and teaching materials that can be applied in classrooms. They should build open lectures and MOOCs for the masses in different languages, and as well, structured learning and teaching materials that can be applied in classrooms.

Children with mathematical learning disabilities need more practice and time to reach the same level of proficiency as that of typically performing children. The learning time (time on task, time for exercises) has been proved to be one of the most significant variables in learning; a fair relationship of time allocated and time needed, meeting the needs of the students, has a large impact on students' performance (Artelt et al., 2003; Carroll, 1989; Marzano, 2003; OECD, 2018). However, just adding time for mathematics learning is not enough. A recent meta-analysis on learning time clearly showed that what works for reading does not work for math (Kidron & Lindsay, 2014). Added learning time had only small effects on math learning and only when using direct instruction by a qualified teacher, and this effect was not even significant in groups of children at risk of failure in learning.

In general, the existing meta-analyses on the efficiency of teaching and remedial instruction for children with learning difficulties show that instructed learning processes, including well-structured explanations and exercises, adapted to the learners' previous knowledge, outperform socioconstructivist or experimental approaches. These approaches, in contrast, tend to overload the working memory and do not help the child to build up cumulative knowledge (Grünke, 2006; Hattie, 2009; Kroesbergen & van Luit, 2003).

Therefore, we urgently need a broad range of evidence-based intervention programs. Although we are looking at a wealth of evidence-based programs, there are some bottlenecks. These relate primarily to older students; most of the trainings focus on students in elementary school. But more important are programs that are easy to implement in schools (Fuchs, Fuchs, & Compton, 2012) without extensive further professional training needed for teachers. Programs carried out in schools by teachers usually have only small effect sizes (if any) compared to the effects in experimental control studies (Müller & Fritz, 2017; Philipp & Souvignier, 2016). Philipp and Souvignier (2016) are talking about a research to practice gap, meaning there is a lack of scientific research about the conditions of how to implement evidence-based training in schools by ordinary teachers without specific training on the program used so that it would increase the students' performance in at least the majority of the classes conducted by the majority of teachers. The ecological validity of the training programs should be a guideline for future research on interventions.

The Scaffold of Teaching Math Content at School

The RTI/intervention model relies heavily on the curriculum and syllabus. However, when we talk about learning difficulties, we rarely pay attention to the syllabus of mathematics education itself. How do the curricula contribute to better or worse performance of children and youth?

What students should learn and teachers should teach in the mathematics classroom is settled in national curricula. National curricula always represent policy statements. They give a detailed description of students' learning objectives for each grade separately. National curricula are therefore to be understood as normative guidelines for teaching. The curriculum and syllabus provide the framework for the content of the schoolbooks used, for the evaluations done, for what children have mastered, and to define who has not reached the level of the standards. Many countries use national assessments to evaluate pupils' attainment in curricular content.

Each country has had its own processes and comparisons with other countries while they have built and reformed their curricula. Typically, there have been slow modifications done in time for teachers and schoolbook creators to adjust to the changes. Therefore, even though mathematics is typically considered a hierarchically learned subject by nature, for historical reasons the ages at which children are taught and expected to master different types of content vary a lot. Some curricula are very ambitious in their aims, while some curricula follow a philosophy of "less is more" (Sahlberg, 2011), proceeding slowly and giving children more time to learn the basics. We illustrate these differences with two examples. In Table 46.1 we present examples from three different continents of how the timing of presenting new arithmetic content in the sample countries is organized.

Clearly, the ambitiousness of the curriculum and syllabus does not automatically lead to higher average performance nor to a higher percentage of top-performing adolescents. Too-high demands at a very early age increase the risk of early failure and dropout from future learning. Naturally, the curricula and syllabus of mathematics are only one variable in the equation, and therefore between-countries comparisons may not be the best way to compare educational systems or to start to reform them. Differences in geography; the numbers of students and schools; the training of teachers; educational cultures; the student makeup (in terms of language, culture, and socioeconomic profile); and how the school system is structured, resourced, and managed all have significant impacts on the results.

But if the aim is to reduce the number of children with poor performance, then what is taught—and especially when and in which phase new content is presented—are key variables. This should be the guiding question of the future planning of curricula and syllabi: Is there a match between the wishes of the educational policy making and the actual performance levels of the children and the reality in the classrooms? Therefore, there is a need for a philosophical analysis of mathematics curricula and syllabi, and how well the normative (*how things should be*) and empirical (*how things actually are*) dimensions do match. Definitely, more empirical studies on the aims of mathematics education and the results (i.e., the performance levels of children) are needed and, especially, studies on how these are connected to levels of poor performance and difficulties in learning mathematics. There is a need for dialogue between researchers and policy makers about the reality in classrooms and about what the slow steps for better education in each country should be. Slowness in reforms is important because teachers, even more than students, need time to learn the new know-how.

As a second example, we present in more detail the defined expectations of learning outcomes after the second grade in two different countries with very different performance levels, namely Germany and South Africa. In the chapter by Kotzé and van der Berg in Chap. 5 of this volume, eight Latin American and sub-Saharan countries were compared to each other. It became evident that "South Africa has the highest proportion of students who are functioning at below-acceptable levels of numeracy (Pre-numeracy and Emergent Numeracy). Kenya and Tanzania both have a much higher proportion of students living in poverty, but also have a much higher proportion of these students performing at acceptable (Basic Numeracy) to above-average levels (Beginning Numeracy, Competent Numeracy and Mathematically Skilled)" (p. 12).

Table 46.1 Syllabi of arithmetic described in national-level or regional-level curriculum frameworks in three examples from different continents

Age (years)	Southern African	Latin American	Northern European
6	Counting forward and backward between 0 and 100 (write 1–20) Addition and subtraction 0–20 Addition of the same number repeatedly to 20	Reading, writing, and ordering numbers 1–999 Addition and subtraction 1–99	Playing with quantities Learning the symbols of numbers 0–9
7	Counting forward and backward and writing between 0 and 200 Addition and subtraction 0–99 Multiplication of numbers 1 to 10 by 2, 5, 3, and 4 to a total of 50 Using and naming unitary fractions in familiar contexts, including halves, quarters, thirds, and fifths	Numbers 1–10,000 Basics of multiplication and division Structure of fractions	Numbers 1–20 Addition and subtraction with a focus on 1–10
8	Counting forward and backward and writing between 0 and 1000 Addition and subtraction 0–999 Multiplication and division by 2, 3, 4, 5, and 10 to a total of 100 Using and naming unitary and nonunitary fractions in familiar contexts, including halves, quarters, eighths, thirds, sixths, fifths, and tenths	Numbers to eternity Multiples, divisors, and prime factors Addition and subtraction of fractions with the same denominator	Numbers 1–100 Multiplication by 1–5, 10 The concept of groups of equal size (division)
9	Addition and subtraction of whole numbers (1–9999) Multiplication of 2-digit by 2-digit whole numbers Division of 3-digit by 1-digit whole numbers Estimation and rounding Comparing and ordering common fractions with different denominators; addition with the same denominators	Addition and subtraction of fractions, decimals, and percentages	Base-ten system (1–1000) Multiplication 1–10 The concept of division and fractions
10	Addition and subtraction of whole numbers (1–99,999) Multiplication and division of 3-digit by 2-digit whole numbers Factors of 2-digit whole numbers to at least 100 Counting forward and backward in fractions Addition and subtraction of common fractions with the same denominators, and mixed numbers	Multiplication and division of fractions and decimals Calculation of powers of a number	Multiplication and division of whole numbers Addition and subtraction of fractions
11	Addition and subtraction of whole numbers of at least 6 digits Multiplication and division of 4-digit by 3-digit whole numbers Factors of 2-digit and 3-digit whole numbers Prime factors of numbers (1–100) Addition and subtraction of common fractions in which one denominator is a multiple of another, and mixed numbers	Distinguishing between rational and irrational numbers, and giving examples of both Understanding the concept of root extraction and its relation to potentiation	Relations between fractions, decimals, and percentages Estimation and rounding

These findings contradict the monocausal assumption that poor achievement of students is solely due to economic conditions in the respective country.

Construction of Curricula in a Tension Between the Two Poles of Individual Prerequisites and Normative Guidelines

Hardly any other subject is based on a system of knowledge organized as hierarchically as that of mathematics (Stern, 2003, 2005). This system of knowledge has to be taught precisely and systematically, paying special attention to its cumulative structure.

If the system of mathematical knowledge is organized in the curriculum in a sensible way, if the learning objectives are coherent, and if the students can take enough time to practice, then, step by step, students can gain more and more expertise. They show their increasing competence by successfully applying their knowledge to tasks they could not carry out before or by solving tasks with more effective strategies. This newly acquired expertise in turn forms the basis for the acquisition of further expertise or knowledge in that area.

If, however, knowledge is not acquired systematically and thus cannot be interconnected in a meaningful way with already existing knowledge, there is no basis for further expertise. A lack of knowledge, especially fragmentary basic knowledge, impedes any further acquisition. Students who start school with poor previous knowledge and/or do not acquire viable basic knowledge during the first years of school face a widening gap between their abilities and those of their peers or the learning objectives of their grade. The same applies to higher grades if new knowledge cannot be connected properly. This results in attainment differences equivalent to 3 years or more compared to the current grade's objectives (Spaull & Kotze, 2015). Moreover, the knowledge is fragmentary and not sufficiently expandable.

It is therefore quite evident that the learner's previous knowledge, successively acquired through learning processes, becomes more and more important for math academic achievement throughout the course of the school career, becoming even more significant than intelligence (Ausubel, 1968; Sternberg, 2005). Therefore, knowledge of a content area is the most important prerequisite for future learning in the same knowledge domain (Hattie, 2009).

Based on this conclusion, curricula have to be evaluated in terms of composition and structure. It is important to check whether they support systematic, cumulative acquisition of knowledge and allow enough time to practice and develop a stable basis of knowledge. If, however, curricula are too complex and lack a cumulative structure, if they neither provide possibilities to connect new knowledge to existing knowledge nor allow for taking the time necessary for the new knowledge to sink in, curricula in themselves might present learning obstacles, preventing students from gaining mathematical expertise and competence.

The curricula of the two countries illustrated in Table 46.2 not only are comparatively demanding but also lack a cumulative structure and thus go against the fundamental principle of new mathematical learning content having to be based on

Table 46.2 Comparison of expectations at the end of the "Schuleingangsphase" (the school entrance phase) in Germany and learning objectives at the end of grade 2 in the South African curriculum for the subject of mathematics, in the field of numbers and operations

Germany	South Africa
Numbers up to 100	Numbers up to 200
Counting forward and backward in steps Simple multiplication (tables up to 10) up to 100	Counting forward and backward in steps of 1s, 10s, 5s, 2s, 3s, and 4s from any multiple of that number up to 200 Multiplication of numbers 1 to 10 by 2, 5, 3, and 4 to a total of 50
Describing, comparing, and ordering numbers up to 100	Describing, comparing, and ordering objects up to 99
Structuring numbers up to 100 according to the place value system	Recognizing the place value of two-digit numbers up to 99
Switching between different levels of abstraction (enactive, iconic, symbolic, and verbal)	Using drawings or concrete apparatus to solve problems
Doubling and halving	Doubling and halving Unitary fractions
Describing own solution to problems verbally and in written form	Explaining own solution to problems
Addition (to 100), subtraction (to 100), multiplication (to n^2, $n \leq 10$), and division in context Using arithmetic laws, characteristics of operations, and calculation strategies	Addition (to 99), subtraction (from 99), multiplication (to 50), and grouping and sharing leading to division (up to 50) (including remainders) in context Using calculation strategies
Using units for money (Euros and cents)	Recognizing and identifying coins and banknotes Solving money problems involving totals and change
Using appropriate technical terms (plus, minus, times, divided by)	Using appropriate symbols (+, −, ×, ÷, =)
Rapidly mentally calculating up to 100	Rapidly mentally recalling addition and subtraction facts up to 20 Rapidly mentally adding or subtracting multiples of 10 from 0 to 100

previous knowledge. Furthermore, these curricula rely on previous knowledge that most of the students in these countries do not possess.

In conclusion, the concept of curricula, presenting policy statements and normative guidelines for teaching, is of great importance and thus has to be reflected on the basis of scientific insights and data. In order to facilitate cumulative learning, the curricula should be prepared according to the hierarchically constructed knowledge system and ensure that children are allowed as much time as they need to understand the content.

Reforming Math Education in the Twenty-First Century

Mathematics education has always been in crisis. As long as there has been public schooling for the masses, the discussion about policies in mathematics education has been a constant demand for reform. The results of these reforms have been less than flattering. Davison and Mitchell (2008) state that "The history of education reform in the twentieth century documents one failure followed by another." They even ask if these reform policies have been poorly designed or designed by individuals and organizations with little working knowledge of what really goes on in schools.

What is clear is that the polarization of the discussion about the aims of mathematics education is partly caused by different views of mathematics itself. While one end sees mathematics as a strongly abstract axiomatic system, the other end sees it as a socially developed construct for practical needs. We can see the effects of these philosophical extremes easily in reforms like "New Math" in the USA in the 1960s and the similar Bourbaki school-based reforms in European countries. In both of them, the aim was to improve children's learning via mathematizing mathematics education. It may come as a surprise to some, but neither of them had any research behind them to support the reforms (Schoenfeld, 2004). The counterattack of the "back to basics" schooling in the USA was also fast, as was the disappearance of the "set theory–based" mathematics education in many European countries. Again, they disappeared before the researchers had time to study their effectiveness and whether there were elements that should be saved. If we think about how long it takes for an average and skillful teacher to learn new ways to think about mathematics, to develop pedagogical models based on that, and especially to implement all that as effective teaching practices in a classroom of children with heterogeneous skills, we can easily see that the question Davidson and Michell raised about understanding the nature of teaching work in the classroom is extremely valid when we consider the time needed for effective reform and the time needed for research to confirm effectiveness. Schooling should not be an experiment, but experiments are needed to develop schooling.

Now, when technology and automatization are changing work life and the world rapidly, the question of educational reforms is even more topical. In the internet era, the access to information and also the amount of information production has skyrocketed. Therefore, knowledge and the ways of knowing something has dramatically changed. Education and school have to change with these societal changes. Partly for these reasons, for example, the United Nations Educational, Scientific, and Cultural Organization (UNESCO) International Bureau of Education (IBE-UNESCO) has been an active advocate of competence-based curricula—curricula about the competencies needed in the twenty-first century. A competence-based curriculum is a model that emphasizes. Most of the recent new reforms of the curricula from the least-developed countries to the wealthiest societies have adopted these ideas in their curriculum frameworks.

Table 46.3 Role of learners in competence-based curricula

Teacher-led transmission	Learner-led enquiry
From passive recipients of an accepted body of knowledge	To developing increasing responsibility for their own learning
From memorization and regurgitation	To active enquiry, interrogation, and management of a variety of competing information sources
From compliance without engagement	To co-construction and enthusiastic engagement in framing enquiries and outcomes
From answering teacher' questions	To framing and exploring learners' own questions
From competing against one another	To collaborating with one another and with the teacher
From compartmentalized learning in single subjects	To integrated, multidisciplinary connections across subjects
From more remote and formal teacher–learner relationships	To trust and rapport between teachers and learners and among learners
From "silo-based" subject learning that lacks connection with the learner background and context	To relevant learning, drawing on prior knowledge and the cultural context to clarify and refine conceptual understanding
From shallow, surface learning (and extrinsic motivation) reliant on teacher talk and demonstration to pass exams	To deep learning and intrinsic motivation: • Investigating a range of perspectives/ways of looking at issues/problems • Subjecting information and processes to critical interrogation • Examining alternatives and seeking creative solutions • Justifying conclusions/decisions/choices based on evidence/evaluation

From Marope et al. (2017)

Marope et al. (2017) describe how the role of the learner is expected to change in the curriculum (see Table 46.3). They contrast the learner-led competence-based approach to teacher-led educational practices that are still often found in countries with low levels of teacher education.

Even though everyone can basically put their signature on these high principles of good educational practices, again the lessons learned from the previous reforms should be taken with seriousness. The analyses of the PISA data and other studies show that direct teacher-led instruction has its benefits and it is especially beneficial for children with lower performance levels. We cannot just throw that evidence away when developing new pedagogies. In particular, children with mathematical learning difficulties may benefit more if we also implement the best practices of teacher-led transmission in future schools.

The UNESCO report on the global paradigm shift in education states "A real threat to instituting and sustaining pedagogical approaches that support competence-based curricula is that in many countries, especially developing countries, teaching is not yet professionalized" (p. 33). What kind of competence-based educational models best suit children with mathematical learning difficulties and children with other learning disabilities? How can these basically positive ideas of student-centered learning be turned into practice in classrooms, with sensible support for those with cognitive disabilities? Without research guiding this development in educational practices, there is again a big risk that ideological wishes will overcome reality, as has happened in many attempts to reform schools. Every reform should start from one question: How will the most vulnerable in the classroom benefit from this? The answer to that question can only be found via the methods of research. The authors in this book have written about the topics in which they are respected experts. Their message is that more research is needed, and we couldn't agree more, research that translates from labs to classrooms.

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