Chapter 21
Asymptotic Translation Uniform Integrability and Multivalued Dynamics of Solutions for Non-autonomous Reaction-Diffusion Equations

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Abstract In this note we introduce asymptotic translation uniform integrability condition for a function acting from a positive semi-axes of time-line to a Banach space. We prove that this condition is equivalent to uniform integrability condition. As a result, we obtain the corollaries for the multivalued dynamics (as time $t \to +\infty$) of solutions for non-autonomous reaction-diffusion equations.

21.1 Introduction

Let $\mathbb{R} = [0, +\infty)$, $\gamma \geq 1$, and $\mathscr{E}$ be a real separable Banach space. As $L_{\gamma}^{\text{loc}}(\mathbb{R}_{+}; \mathscr{E})$ we consider the Fréchet space of all locally integrable functions with values in $\mathscr{E}$, that is, $\varphi \in L_{\gamma}^{\text{loc}}(\mathbb{R}_{+}; \mathscr{E})$ if and only if for any finite interval $[\tau, T] \subset \mathbb{R}_{+}$ the restriction of φ on $[\tau, T]$ belongs to the space $L_{\gamma}(\tau, T; \mathscr{E})$. If $\mathscr{E} \subseteq L_{1}(\Omega)$, then any function φ from $L_{\gamma}^{\text{loc}}(\mathbb{R}_{+}; \mathscr{E})$ can be considered as a measurable mapping that acts from $\Omega \times \mathbb{R}_{+}$ into $\mathbb{R}$. Further, we write $\varphi(x, t)$, when we consider this mapping as a function from $\Omega \times \mathbb{R}_{+}$ into $\mathbb{R}$, and $\varphi(t)$, if this mapping is considered as an element from $L_{\gamma}^{\text{loc}}(\mathbb{R}_{+}; \mathscr{E})$; cf. Gajewski et al. [5, Chapter III]; Temam [10]; Babin and Vishik [1]; Chepyzhov and Vishik [3]; Zgurovsky et at. [12] and references therein.

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A function $\varphi \in L_\gamma^{\mathrm{loc}}(\mathbb{R}_+; \mathscr{E})$ is called *translation bounded* in $L_\gamma^{\mathrm{loc}}(\mathbb{R}_+; \mathscr{E})$, if

$$\sup_{t\geq 0} \int_t^{t+1} \|\varphi(s)\|_{\mathscr{E}}^{\gamma} ds < +\infty; \tag{21.1}$$

Chepyzhov and Vishik [4, p. 105].

Let $N = 1, 2, \ldots$ and $\Omega \subset \mathbb{R}^N$ be a *bounded domain*. A function $\varphi \in L_1^{\mathrm{loc}}(\mathbb{R}_+; L_1(\Omega))$ is called *translation uniform integrable one (t.u.i.)* in $L_1^{\mathrm{loc}}(\mathbb{R}_+; L_1(\Omega))$, if

$$\lim_{K\to+\infty} \sup_{t\geq 0} \int_t^{t+1} \int_\Omega |\varphi(x,s)| \chi_{\{|\varphi(x,s)|\geq K\}} dx ds = 0; \tag{21.2}$$

Gorban et al. [6–9]. Dunford-Pettis compactness criterion provides that a function $\varphi \in L_1^{\mathrm{loc}}(\mathbb{R}_+; L_1(\Omega))$ is t.u.i. in $L_1^{\mathrm{loc}}(\mathbb{R}_+; L_1(\Omega))$ if and only if for every sequence of elements $\{\tau_n\}_{n\geq 1} \subset \mathbb{R}_+$ the sequence $\{\varphi(\cdot + \tau_n)\}_{n\geq 1}$ contains a subsequence which converges weakly in $L_1^{\mathrm{loc}}(\mathbb{R}_+; L_1(\Omega))$. We note that for any $\gamma > 1$ Hölder's and Chebyshev's inequalities imply that every translation bounded in $L_\gamma^{\mathrm{loc}}(\mathbb{R}_+; L_\gamma(\Omega))$ function is t.u.i. in $L_1^{\mathrm{loc}}(\mathbb{R}_+; L_1(\Omega))$, because

$$\int_t^{t+1} \int_\Omega |\varphi(x,s)| \chi_{\{|\varphi(x,s)|\geq K\}} dx ds \leq \frac{1}{K^{\gamma-1}} \sup_{t\geq 0} \int_t^{t+1} \int_\Omega |\varphi(x,s)|^\gamma dx ds \to 0 \text{ as } K \to +\infty.$$

Let us introduce the definition of asymptotic translation uniform integrable function.

Definition 21.1 A function $\varphi \in L_1^{\mathrm{loc}}(\mathbb{R}_+; L_1(\Omega))$ is called *asymptotic translation uniform integrable one (a.t.u.i.)* in $L_1^{\mathrm{loc}}(\mathbb{R}_+; L_1(\Omega))$, if

$$\lim_{K\to+\infty} \overline{\lim_{t\to+\infty}} \int_t^{t+1} \int_\Omega |\varphi(x,s)| \chi_{\{|\varphi(x,s)|\geq K\}} dx ds = 0. \tag{21.3}$$

Remark 21.1 The limit (as $K \to +\infty$) in (21.2) ((21.3)) exists because the function

$$K \mapsto \sup_{t\geq 0} \Big(\overline{\lim_{t\to+\infty}} \Big) \int_t^{t+1} \int_\Omega |\varphi(x,s)| \chi_{\{|\varphi(x,s)|\geq K\}} dx ds \tag{21.4}$$

is nonincreasing in $K > 0$.

The main result of this note has the following formulation.

Theorem 21.1 *Let* $\varphi \in L_1^{\mathrm{loc}}(\mathbb{R}_+; L_1(\Omega))$. *Then there exists* $\tilde{T} \geq 0$ *such that* $\varphi(\,\cdot\, + \tilde{T})$ *is t.u.i. in* $L_1^{\mathrm{loc}}(\mathbb{R}_+; L_1(\Omega))$ *iff* φ *is a.t.u.i. in* $L_1^{\mathrm{loc}}(\mathbb{R}_+; L_1(\Omega))$.

In Sect. 21.3 we apply Theorem 21.1 to non-autonomous nonlinear reaction-diffusion system.

21.2 Proof of Theorem 21.1

Let us prove Theorem 21.1. The t.u.i. of $\varphi(\,\cdot\, + \tilde{T})$ for some $\tilde{T} \geq 0$ implies a.u.t.i. of $\varphi(\,\cdot\,)$ because for each sequence $\{a_n\}_{n=1,2,\ldots} \subset \overline{\mathbb{R}}$ its limit superior is no greater than its supremum, that is, (21.2) implies (21.3). Let us prove the converse statement: if $\varphi(\,\cdot\,)$ is a.t.u.i., then $\varphi(\,\cdot\, + \tilde{T})$ is t.u.i. for some $\tilde{T} \geq 0$. We provide the proof in several steps.

Step 1 The following equalities hold:

$$
\begin{aligned}
0 &= \lim_{K\to+\infty} \overline{\lim_{t\to+\infty}} \int_t^{t+1}\int_\Omega |\varphi(x,s)|\chi_{\{|\varphi(x,s)|\geq K\}}dxds \\
&= \inf_{K>0}\inf_{T\geq 0}\sup_{t\geq T} \int_t^{t+1}\int_\Omega |\varphi(x,s)|\chi_{\{|\varphi(x,s)|\geq K\}}dxds \\
&= \inf_{T\geq 0}\inf_{K>0}\sup_{t\geq T} \int_t^{t+1}\int_\Omega |\varphi(x,s)|\chi_{\{|\varphi(x,s)|\geq K\}}dxds.
\end{aligned}
\tag{21.5}
$$

Indeed, the first equality follows from a.t.u.i. of $\varphi(\,\cdot\,)$, the second equality holds because the mapping

$$
K \mapsto \overline{\lim_{t\to+\infty}} \int_t^{t+1}\int_\Omega |\varphi(x,s)|\chi_{\{|\varphi(x,s)|\geq K\}}dxds
$$

is nonincreasing and for each $a : [0, +\infty) \mapsto \overline{\mathbb{R}}$ the equality

$$
\overline{\lim_{t\to+\infty}}\, a(t) = \inf_{T\geq 0}\sup_{t\geq T} a(t)
$$

holds, and the last equality follows from the basic properties of infimum.

Step 2 We set

$$\delta(T) := \inf_{K>0} \sup_{t\geq T} \int_t^{t+1} \int_\Omega |\varphi(x,s)| \chi_{\{|\varphi(x,s)|\geq K\}} dxds, \tag{21.6}$$

$T \geq 0$, and notice that (21.5) directly implies the existence of $\tilde{T} \geq 0$ such that

$$\delta(T) < +\infty \text{ for each } T \geq \tilde{T} \text{ and } \delta(T) \searrow 0 \text{ as } T \to \infty. \tag{21.7}$$

Step 3 According to (21.6) and (21.7), for each $T \geq \tilde{T}$ there exists $K_T > 0$ such that

$$\sup_{t\geq T} \int_t^{t+1} \int_\Omega |\varphi(x,s)| \chi_{\{|\varphi(x,s)|\geq K\}} dxds < \delta(T) + \frac{1}{T} < +\infty, \tag{21.8}$$

for each $K \geq K_T$.

Step 4 Since for each $n = 0, 1, \ldots$

$$\begin{aligned} \int_{\tilde{T}+n}^{\tilde{T}+n+1} \int_\Omega |\varphi(x,s)| dxds &= \int_{\tilde{T}+n}^{\tilde{T}+n+1} \int_\Omega |\varphi(x,s)| \chi_{\{|\varphi(x,s)|\leq K_{\tilde{T}}\}} dxds \\ &+ \int_{\tilde{T}+n}^{\tilde{T}+n+1} \int_\Omega |\varphi(x,s)| \chi_{\{|\varphi(x,s)|\geq K_{\tilde{T}}\}} dxds \\ &\leq K_{\tilde{T}} \text{meas}(\Omega) + \delta(\tilde{T}) + \frac{1}{\tilde{T}} < +\infty, \end{aligned}$$

where the first inequality follows from (21.8), and the second inequality holds because $\text{meas}(\Omega) < +\infty$, then absolute continuity of the Lebesgue integral implies that for each $T > \tilde{T}$ and $t \in [\tilde{T}, T]$ there exists $K(\tilde{T}, T) > 0$ such that

$$\int_t^{t+1} \int_\Omega |\varphi(x,s)| \chi_{\{|\varphi(x,s)|\geq K\}} dxds \leq \int_{\tilde{T}}^{T+1} \int_\Omega |\varphi(x,s)| \chi_{\{|\varphi(x,s)|\geq K\}} dxds < \frac{1}{T}$$

for each $K \geq K(\tilde{T}, T)$, that is,

$$\sup_{t \in [\tilde{T}, T]} \int_t^{t+1} \int_\Omega |\varphi(x, s)| \chi_{\{|\varphi(x,s)| \geq K\}} dx ds \leq \frac{1}{N}, \tag{21.9}$$

for each $T > \tilde{T}$ and $K \geq \tilde{K}_T^{\tilde{T}} := \sup_{t \in [\tilde{T}, T]} \{K_T; K(\tilde{T}, T)\}$.

Step 5 Inequalities (21.8) and (21.9) imply that

$$\sup_{t \geq \tilde{T}} \int_t^{t+1} \int_\Omega |\varphi(x, s)| \chi_{\{|\varphi(x,s)| \geq K\}} dx ds < \delta(T) + \frac{1}{T},$$

for each $T > \tilde{T}$ and $K \geq \tilde{K}_T^{\tilde{T}}$. Thus, according to (21.6),

$$\delta(\tilde{T}) = \inf_{K > 0} \sup_{t \geq \tilde{T}} \int_t^{t+1} \int_\Omega |\varphi(x, s)| \chi_{\{|\varphi(x,s)| \geq K\}} dx ds < \delta(T) + \frac{1}{T}, \tag{21.10}$$

for each $T > \tilde{T}$.

Step 6 Since the function

$$K \mapsto \sup_{t \geq \tilde{T}} \int_t^{t+1} \int_\Omega |\varphi(x, s)| \chi_{\{|\varphi(x,s)| \geq K\}} dx ds$$

is nonincreasing, we have that

$$\lim_{K \to +\infty} \sup_{t \geq \tilde{T}} \int_t^{t+1} \int_\Omega |\varphi(x, s)| \chi_{\{|\varphi(x,s)| \geq K\}} dx ds = \delta(\tilde{T}) < \delta(T) + \frac{1}{T}. \tag{21.11}$$

for each $T > \tilde{T}$, where the inequality follows from (21.10). According to (21.7), $\delta(T) + \frac{1}{T} \searrow 0$ as $T \to +\infty$. Therefore, (21.11) implies that

$$\lim_{K \to +\infty} \sup_{t \geq \tilde{T}} \int_t^{t+1} \int_\Omega |\varphi(x, s)| \chi_{\{|\varphi(x,s)| \geq K\}} dx ds = 0,$$

that is, $\varphi(\cdot)$ is t.u.i.

21.3 Examples of Applications

Let $N, M = 1, 2, \ldots, \Omega \subset \mathbb{R}^N$ be a bounded domain with sufficiently smooth boundary $\partial\Omega$. We consider a problem of long-time behavior of all globally defined weak solutions for the non-autonomous parabolic problem (named RD-system)

$$\begin{cases} y_t = a\Delta y - f(x,t,y), & x \in \Omega,\ t > 0, \\ y|_{\partial\Omega} = 0, \end{cases} \tag{21.12}$$

as $t \to +\infty$, where $y = y(x,t) = (y^{(1)}(x,t), \ldots, y^{(M)}(x,t))$ is unknown vector-function, $f = f(x,t,y) = (f^{(1)}(x,t,y), \ldots, f^{(M)}(x,t,y))$ is given function, a is real $M \times M$ matrix with positive symmetric part.

We suppose that the listed below assumptions hold.

Assumption I Let $p_i \geq 2$ and $q_i > 1$ are such that $\frac{1}{p_i} + \frac{1}{q_i} = 1$, for any $i = 1, 2, \ldots, M$. Moreover, there exists a positive constant d such that $\frac{1}{2}(a + a^*) \geq dI$, where I is unit $M \times M$ matrix, a^* is a transposed matrix for a.

Assumption II The interaction function $f = (f^{(1)}, \ldots, f^{(M)}) : \Omega \times \mathbb{R}_+ \times \mathbb{R}^M \to \mathbb{R}^M$ satisfies the standard Carathéodory's conditions, i.e. the mapping $(x,t,u) \to f(x,t,u)$ is continuous in $u \in \mathbb{R}^M$ for a.e. $(x,t) \in \Omega \times \mathbb{R}_+$, and it is measurable in $(x,t) \in \Omega \times \mathbb{R}_+$ for any $u \in \mathbb{R}^M$.

Assumption III (Growth Condition) There exist an a.t.u.i. in $L_1^{\text{loc}}(\mathbb{R}_+; L_1(\Omega))$ function $c_1 : \Omega \times \mathbb{R}_+ \to \mathbb{R}_+$ and a constant $c_2 > 0$ such that

$$\sum_{i=1}^{M} \left| f^{(i)}(x,t,u) \right|^{q_i} \leq c_1(x,t) + c_2 \sum_{i=1}^{M} \left| u^{(i)} \right|^{p_i}$$

for any $u = (u^{(1)}, \ldots, u^{(M)}) \in \mathbb{R}^M$, and a.e. $(x,t) \in \Omega \times \mathbb{R}_+$.

Assumption IV (Sign Condition) There exists a constant $\alpha > 0$ and an a.t.u.i. in $L_1^{\text{loc}}(\mathbb{R}_+; L_1(\Omega))$ function $\beta : \Omega \times \mathbb{R}_+ \to \mathbb{R}_+$ such that

$$\sum_{i=1}^{M} f^{(i)}(x,t,u)u^{(i)} \geq \alpha \sum_{i=1}^{M} \left| u^{(i)} \right|^{p_i} - \beta(x,t)$$

for any $u = (u^{(1)}, \ldots, u^{(M)}) \in \mathbb{R}^M$, and a.e. $(x,t) \in \Omega \times \mathbb{R}_+$.

In further arguments we will use standard functional Hilbert spaces $H = (L_2(\Omega))^M$, $V = (H_0^1(\Omega))^M$, and $V^* = (H^{-1}(\Omega))^M$ with standard respective inner products and norms $(\cdot,\cdot)_H$ and $\|\cdot\|_H$, $(\cdot,\cdot)_V$ and $\|\cdot\|_V$, and $(\cdot,\cdot)_{V^*}$ and $\|\cdot\|_{V^*}$, vector notations $\mathbf{p} = (p_1, p_2, \ldots, p_M)$ and $\mathbf{q} = (q_1, q_2, \ldots, q_M)$, and the spaces

$$\mathbf{L_p}(\Omega) := L_{p_1}(\Omega) \times \ldots \times L_{p_M}(\Omega), \quad \mathbf{L_q}(\Omega) := L_{q_1}(\Omega) \times \ldots \times L_{q_M}(\Omega),$$
$$\mathbf{L_p}(\tau, T; \mathbf{L_p}(\Omega)) := L_{p_1}(\tau, T; L_{p_1}(\Omega)) \times \ldots \times L_{p_M}(\tau, T; L_{p_M}(\Omega)),$$
$$\mathbf{L_q}(\tau, T; \mathbf{L_q}(\Omega)) := L_{q_1}(\tau, T; L_{q_1}(\Omega)) \times \ldots \times L_{q_M}(\tau, T; L_{q_M}(\Omega)), \ 0 \le \tau < T < +\infty.$$

Let $0 \le \tau < T < +\infty$. A function $y = y(x,t) \in \mathbf{L}_2(\tau, T; V) \cap \mathbf{L_p}(\tau, T; \mathbf{L_p}(\Omega))$ is called a *weak solution* of Problem (21.12) on $[\tau, T]$, if for any function $\varphi = \varphi(x) \in (C_0^\infty(\Omega))^M$, the following identity holds

$$\frac{d}{dt}\int_\Omega y(x,t)\cdot\varphi(x)dx + \int_\Omega \{a\nabla y(x,t)\cdot\nabla\varphi(x) + f(x,t,y(x,t))\cdot\varphi(x)\}dx = 0 \tag{21.13}$$

in the sense of scalar distributions on (τ, T).

In the general case Problem (21.12) on $[\tau, T]$ with initial condition $y(x,\tau) = y_\tau(x)$ in Ω has more than one weak solution with $y_\tau \in H$ (cf. Balibrea et al. [2] and references therein).

Assumptions I–IV and Chepyzhov and Vishik [4, pp. 283–284] (see also Zgurovsky et al. [11, Chapter 2] and references therein) provide the existence of a weak solution of Cauchy problem (21.12) with initial data $y(\tau) = y^{(\tau)}$ on the interval $[\tau, T]$, for any $y^{(\tau)} \in H$. The proof is provided by standard Faedo–Galerkin approximations and using local existence Carathéodory's theorem instead of classical Peano results. A priori estimates are similar. Formula (21.13) and definition of the derivative for an element from $\mathscr{D}([\tau, T]; V^* + \mathbf{L_q}(\Omega))$ yield that each weak solution $y \in X_{\tau,T}$ of Problem (21.12) on $[\tau, T]$ belongs to the space $W_{\tau,T}$. Moreover, each weak solution of Problem (21.12) on $[\tau, T]$ satisfies the equality:

$$\int_\tau^T\int_\Omega\left[\frac{\partial y(x,t)}{\partial t}\cdot\psi(x,t) + a\nabla y(x,t)\cdot\nabla\psi(x,t) + f(x,t,y(x,t))\cdot\psi(x,t)\right]dxdt = 0, \tag{21.14}$$

for any $\psi \in X_{\tau,T}$. For fixed τ and T, such that $0 \le \tau < T < +\infty$, we denote

$$\mathscr{D}_{\tau,T}(y^{(\tau)}) = \{y(\cdot) \mid y \text{ is a weak solution of (21.12) on } [\tau, T],\ y(\tau) = y^{(\tau)}\}, \quad y^{(\tau)} \in H.$$

We remark that $\mathscr{D}_{\tau,T}(y^{(\tau)}) \ne \emptyset$ and $\mathscr{D}_{\tau,T}(y^{(\tau)}) \subset W_{\tau,T}$, if $0 \le \tau < T < +\infty$ and $y^{(\tau)} \in H$. Moreover, the concatenation of Problem (21.12) weak solutions is a weak solutions too, i.e. if $0 \le \tau < t < T$, $y^{(\tau)} \in H$, $y(\cdot) \in \mathscr{D}_{\tau,t}(y^{(\tau)})$, and $v(\cdot) \in \mathscr{D}_{t,T}(y(t))$, then

$$z(s) = \begin{cases} y(s), & s \in [\tau, t], \\ v(s), & s \in [t, T], \end{cases}$$

belongs to $\mathscr{D}_{\tau,T}(y^{(\tau)})$; cf. Zgurovsky et al. [12, pp. 55–56].

Each weak solution y of Problem (21.12) on a finite time interval $[\tau, T] \subset \mathbb{R}_+$ can be extended to a global one, defined on $[\tau, +\infty)$. For arbitrary $\tau \geq 0$ and $y^{(\tau)} \in H$ let $\mathscr{D}_\tau(y^{(\tau)})$ be the set of all weak solutions (defined on $[\tau, +\infty)$) of Problem (21.12) with initial data $y(\tau) = y^{(\tau)}$. Let us consider the family $\mathscr{K}_\tau^+ = \cup_{y^{(\tau)} \in H} \mathscr{D}_\tau(y^{(\tau)})$ of all weak solutions of Problem (21.12) defined on the semi-infinite time interval $[\tau, +\infty)$.

Consider the Fréchet space

$$C^{\mathrm{loc}}(\mathbb{R}_+; H) := \{y : \mathbb{R}_+ \to H \ : \ \Pi_{t_1,t_2} y \in C([t_1, t_2]; H) \text{ for any } [t_1, t_2] \subset \mathbb{R}_+\},$$

where Π_{t_1,t_2} is the restriction operator to the interval $[t_1, t_2]$; Chepyzhov and Vishik [3, p. 918]. We remark that the sequence $\{f_n\}_{n\geq 1}$ converges (converges weakly respectively) in $C^{\mathrm{loc}}(\mathbb{R}_+; H)$ towards $f \in C^{\mathrm{loc}}(\mathbb{R}_+; H)$ as $n \to +\infty$ if and only if the sequence $\{\Pi_{t_1,t_2} f_n\}_{n\geq 1}$ converges (converges weakly respectively) in $C([t_1, t_2]; H)$ towards $\Pi_{t_1,t_2} f$ as $n \to +\infty$ for any finite interval $[t_1, t_2] \subset \mathbb{R}_+$.

We denote $T(h)y(\cdot) = y_h(\cdot)$, where $y_h(t) = y(t + h)$ for any $y \in C^{\mathrm{loc}}(\mathbb{R}_+; H)$ and $t, h \geq 0$.

In the non-autonomous case we notice that $T(h)\mathscr{K}_0^+ \nsubseteq \mathscr{K}_0^+$. Therefore (see Gorban et al. [8]), we need to consider *united trajectory space* that includes all globally defined on any $[\tau, +\infty) \subseteq \mathbb{R}_+$ weak solutions of Problem (21.12) shifted to $\tau = 0$:

$$\mathscr{K}_\cup^+ := \bigcup_{\tau \geq 0} \left\{ y(\,\cdot\, + \tau) \in W^{\mathrm{loc}}(\mathbb{R}_+) \ : \ y(\,\cdot\,) \in \mathscr{K}_\tau^+ \right\}. \tag{21.15}$$

Note that $T(h)\{y(\,\cdot\, + \tau) \ : \ y \in \mathscr{K}_\tau^+\} \subseteq \{y(\,\cdot\, + \tau + h) \ : \ y \in \mathscr{K}_{\tau+h}^+\}$ for any $\tau, h \geq 0$. Therefore,

$$T(h)\mathscr{K}_\cup^+ \subseteq \mathscr{K}_\cup^+$$

for any $h \geq 0$. Further we consider *extended united trajectory space* for Problem (21.12):

$$\mathscr{K}_{C^{\mathrm{loc}}(\mathbb{R}_+; H)}^+ = \mathrm{cl}_{C^{\mathrm{loc}}(\mathbb{R}_+; H)} \left[\mathscr{K}_\cup^+ \right], \tag{21.16}$$

where $\mathrm{cl}_{C^{\mathrm{loc}}(\mathbb{R}_+; H)}[\,\cdot\,]$ is the closure in $C^{\mathrm{loc}}(\mathbb{R}_+; H)$. We note that

$$T(h)\mathscr{K}_{C^{\mathrm{loc}}(\mathbb{R}_+; H)}^+ \subseteq \mathscr{K}_{C^{\mathrm{loc}}(\mathbb{R}_+; H)}^+$$

for each $h \geq 0$, because

$$\rho_{C^{\mathrm{loc}}(\mathbb{R}_+; H)}(T(h)u, T(h)v) \leq \rho_{C^{\mathrm{loc}}(\mathbb{R}_+; H)}(u, v) \text{ for any } u, v \in C^{\mathrm{loc}}(\mathbb{R}_+; H),$$

where $\rho_{C^{\mathrm{loc}}(\mathbb{R}_+; H)}$ is a standard metric on Fréchet space $C^{\mathrm{loc}}(\mathbb{R}_+; H)$.

Let us provide the result characterizing the compactness properties of shifted solutions of Problem (21.12) in the induced topology from $C^{\text{loc}}(\mathbb{R}_+; H)$.

Theorem 21.2 *Let Assumptions I–IV hold. If* $\{y_n\}_{n\ge 1} \subset \mathcal{K}^+_{C^{\text{loc}}(\mathbb{R}_+;H)}$ *is an arbitrary sequence, which is bounded in* $L_\infty(\mathbb{R}_+; H)$*, then there exist a subsequence* $\{y_{n_k}\}_{k\ge 1} \subseteq \{y_n\}_{n\ge 1}$ *and an element* $y \in \mathcal{K}^+_{C^{\text{loc}}(\mathbb{R}_+;H)}$ *such that*

$$\|\Pi_{\tau,T}\, y_{n_k} - \Pi_{\tau,T}\, y\|_{C([\tau,T];H)} \to 0, \quad k \to +\infty, \tag{21.17}$$

for any finite time interval $[\tau, T] \subset (0, +\infty)$*. Moreover, for any* $y \in \mathcal{K}^+_{C^{\text{loc}}(\mathbb{R}_+;H)}$ *the estimate holds*

$$\|y(t)\|_H^2 \le \|y(0)\|_H^2 e^{-c_3 t} + c_4, \tag{21.18}$$

for any $t \ge 0$*, where positive constants* c_3 *and* c_4 *do not depend on* $y \in \mathcal{K}^+_{C^{\text{loc}}(\mathbb{R}_+;H)}$ *and* $t \ge 0$*.*

Proof This statement directly follows from Gorban et al. [8, Theorem 4.1] and Theorem 21.1.

A set $\mathcal{P} \subset \mathcal{F}^{\text{loc}}(\mathbb{R}_+) \cap L_\infty(\mathbb{R}_+; H)$ is said to be a *uniformly attracting set* (cf. Chepyzhov and Vishik [3, p. 921]) for the extended united trajectory space $\mathcal{K}^+_{\mathcal{F}^{\text{loc}}(\mathbb{R}_+)}$ of Problem (21.12) in the topology of $\mathcal{F}^{\text{loc}}(\mathbb{R}_+)$, if for any bounded in $L_\infty(\mathbb{R}_+; H)$ set $\mathcal{B} \subseteq \mathcal{K}^+_{\mathcal{F}^{\text{loc}}(\mathbb{R}_+)}$ and any segment $[t_1, t_2] \subset \mathbb{R}_+$ the following relation holds:

$$\text{dist}_{\mathcal{F}_{t_1,t_2}}(\Pi_{t_1,t_2} T(t)\mathcal{B}, \Pi_{t_1,t_2}\mathcal{P}) \to 0, \quad t \to +\infty, \tag{21.19}$$

where $\text{dist}_{\mathcal{F}_{t_1,t_2}}$ is the Hausdorff semi-metric.

A set $\mathcal{U} \subset \mathcal{K}^+_{\mathcal{F}^{\text{loc}}(\mathbb{R}_+)}$ is said to be a *uniform trajectory attractor* of the translation semigroup $\{T(t)\}_{t\ge 0}$ on $\mathcal{K}^+_{\mathcal{F}^{\text{loc}}(\mathbb{R}_+)}$ in the induced topology from $C^{\text{loc}}(\mathbb{R}_+; H)$, if

1. $\mathcal{U}$ is a compact set in $C^{\text{loc}}(\mathbb{R}_+; H)$ and bounded in $L_\infty(\mathbb{R}_+; H)$;
2. $\mathcal{U}$ is strictly invariant with respect to $\{T(h)\}_{h\ge 0}$, i.e. $T(h)\mathcal{U} = \mathcal{U}$ $\forall h \ge 0$;
3. $\mathcal{U}$ is a minimal uniformly attracting set for $\mathcal{K}^+_{C^{\text{loc}}(\mathbb{R}_+;H)}$ in the topology of $C^{\text{loc}}(\mathbb{R}_+; H)$, i.e. $\mathcal{U}$ belongs to any compact uniformly attracting set $\mathcal{P}$ of $\mathcal{K}^+_{C^{\text{loc}}(\mathbb{R}_+;H)}$: $\mathcal{U} \subseteq \mathcal{P}$.

Note that uniform trajectory attractor of the translation semigroup $\{T(t)\}_{t\ge 0}$ on $\mathcal{K}^+_{C^{\text{loc}}(\mathbb{R}_+;H)}$ in the induced topology from $C^{\text{loc}}(\mathbb{R}_+; H)$ coincides with the classical global attractor for the continuous semi-group $\{T(t)\}_{t\ge 0}$ defined on $\mathcal{K}^+_{C^{\text{loc}}(\mathbb{R}_+;H)}$.

Assumptions I–IV are sufficient conditions for the existence of uniform trajectory attractor for weak solutions of Problem (21.12) in the topology of $C^{\text{loc}}(\mathbb{R}_+; H)$.

Theorem 21.3 *Let Assumptions I–IV hold. Then there exists an uniform trajectory attractor $\mathscr{U} \subset \mathscr{K}^{+}_{C^{\mathrm{loc}}(\mathbb{R}_+;H)}$ of the translation semigroup $\{T(t)\}_{t\geq 0}$ on $\mathscr{K}^{+}_{C^{\mathrm{loc}}(\mathbb{R}_+;H)}$ in the induced topology from $C^{\mathrm{loc}}(\mathbb{R}_+;H)$. Moreover, there exists a compact in $C^{\mathrm{loc}}(\mathbb{R}_+;H)$ uniformly attracting set $\mathscr{P} \subset C^{\mathrm{loc}}(\mathbb{R}_+;H) \cap L_\infty(\mathbb{R}_+;H)$ for the extended united trajectory space $\mathscr{K}^{+}_{C^{\mathrm{loc}}(\mathbb{R}_+;H)}$ of Problem (21.12) in the topology of $C^{\mathrm{loc}}(\mathbb{R}_+;H)$ such that $\mathscr{U}$ coincides with ω-limit set of $\mathscr{P}$:*

$$\mathscr{U} = \bigcap_{t\geq 0} \mathrm{cl}_{C^{\mathrm{loc}}(\mathbb{R}_+;H)} \left[\bigcup_{h\geq t} T(h)\mathscr{P} \right]. \tag{21.20}$$

Proof This statement directly follows from Gorban et al. [8, Theorem 3.1] and Theorem 21.1.

21.4 Conclusions

Asymptotic translation uniform integrability condition for a function acting from positive semi-axe of time line to a Banach space is equivalent to uniform integrability condition. As a result, we claim only asymptotic (as time $t \to +\infty$) assumptions of translation compactness for parameters of non-autonomous reaction-diffusion equations.

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