Steady-State Behaviour of the Rigid Jeffcott Rotor Comparing Various Analytical Approaches to the Solution of the Reynolds Equation for Plain Journal Bearing

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Abstract A planar 2 DOF model of an unbalanced rigid disc on a massless rigid shaft (rigid Jeffcott rotor) is extended considering nonlinear forces in plain journal bearings. To express the fluid-film forces in the journal bearings, several approximate analytical solutions of the Reynolds equation are used, including widely used approximations for infinitely long and infinitely short journal bearing and a method using correction polynomial functions to extend the area of aspect ratios. The differences in steady-state response of such a rotor are studied. The influence of the approximate solution type, eccentricity ratio and aspect ratio is analysed. The aim is to find out the more effective approach to journal bearing description which could be further used in detailed dynamical analyses of both stable and unstable dynamic behaviour along with nonlinear phenomena like bifurcations and transitions to chaotic motions.

Keywords Rigid Jeffcott rotor · Steady-state response · Reynolds equation
Analytical solution · Plain journal bearings

1 Introduction

The *Jeffcott rotor* [1], also known as the *Laval rotor* [2], is one of the most-simplified mathematical models of rotor. It was used as the first-approximation in simulations of rotor behaviour. Although the 2 DOF model could be perceived as an oversimplification, its simplicity enables to easily include various complex phenomena and to

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make a fundamental analyses of their influence on the rotor behaviour. The modifications of Jeffcott rotor are widely used to show various rotordynamics phenomena such as influence of rotor clearances and rotor-stator contacts [3], effects of seals [4], parametric vibration caused by cracked shaft [5] and a number of other phenomena.

One of the most studied subjects is the influence of a hydrodynamic lubrication in journal bearings on the dynamics of the rotor. A partial differential equation which describes the pressure distribution of thin viscous films in journal bearings was derived at the end of 19th century by Reynolds [6]. An analytical solution of the Reynolds equation for the finite length journal bearing has been found only lately [7]. However, the approximate analytical solutions could be highly useful as well, considering their relative simplicity and closed-form expression that can be easily used in wide class of engineering applications. There are two well-known and widely used approximations for the solution of pressure distribution in journal bearing: infinitely long journal bearing approximation (ILJB) for aspect ratios[1] $L/D \gg 1$ and infinitely short journal bearing approximation (ISJB) for aspect ratios $L/D \ll 1$. In case of ILJB [8], the pressure gradient in the circumferential direction is much larger than in the axial direction and in case of ISJB [9, 10] the pressure gradient in the circumferential direction is much smaller than in the axial direction. In both cases, the Reynolds partial equation is reduced to an ordinary differential equation, whose solution in closed form can be found. However, the solution is limited by the appropriate assumptions and it does not hold for aspect ratios $L/D \rightarrow 1$.

There are some approaches such as perturbation method that extends the interval of aspect ratios where the solution holds. The method and the solution are discussed in [11] for the Reynolds equation without squeeze-term and it has been subsequently extended for full Reynolds equation [12]. However, even if using these methods, the analytical solution for aspect ratios $L/D \in \langle 0.5; 2\rangle$ does not hold as the assumptions are not satisfied. One of the possible ways to obtain an analytical closed-form expression for the solution of Reynolds equation with $L/D \rightarrow 1$ is a usage of correction functions [13]. The multiplicative polynomial functions are used to fit the approximate solution (ISJB, ILJB) to the referential (numerically obtained) pressure distribution in the bearing.

There are several another approaches for solving the Reynolds equation. Hydrodynamic lubrication in systems with Hertzian contacts is often computed employing Grubin's approximation [14]. Another analytical methods are focused for the special cases of finite journal bearings such as a porous bearing [15], which employs analytical solution for the infinitely long porous bearing and Warner's correction factors [16], a journal bearing in a turbulent flow regime [17], or a tilting pad bearing [18].

Relatively small number of authors employ analytical methods for a stability analysis of finite length journal bearings. Various perturbation methods [19, 20] and spectral element methods [21] are the most widely used techniques for such analysis.

[1]The aspect ratio is formulated for axial length L of the bearing and journal diameter D.

2 Jeffcott Rotor with Hydrodynamic Journal Bearing Forces

Based on the scheme depicted in the Fig. 1, the mathematical model of Jeffcott rotor can be formulated. In the original Jeffcott model, the elastic forces of flexible shaft or elastic forces in the bearing were considered. However, the phenomena occuring in the journal bearings are more complex and the bearing forces can be represented more precisely using an approximate solution of Reynolds equation which describes a pressure distribution in the bearing and which can be transformed to the force acting to the journal.

As shown in the Fig. 1, a position of the geometric centre C of journal is described in the non-rotating space by the horizontal displacement x and vertical displacement y. The angular speed of the rotor is ω, mass of the rotor is m and its static unbalance is ΔmE. In central position, the radial clearance between the journal and a bearing is c.

The mathematical model of the Jeffcott rotor in journal bearings can be formulated in the matrix form

$$\begin{bmatrix} m & 0 \\ 0 & m \end{bmatrix} \begin{bmatrix} \ddot{x} \\ \ddot{y} \end{bmatrix} = \begin{bmatrix} 0 \\ -mg \end{bmatrix} + \Delta mE\omega^2 \begin{bmatrix} \cos\omega t \\ \sin\omega t \end{bmatrix} + \\ + \begin{bmatrix} F_{rad}^{(X)}(e,\dot{e},\dot{\Phi},\omega)\cos\Phi - F_{tan}^{(X)}(e,\dot{e},\dot{\Phi},\omega)\sin\Phi \\ F_{rad}^{(X)}(e,\dot{e},\dot{\Phi},\omega)\sin\Phi + F_{tan}^{(X)}(e,\dot{e},\dot{\Phi},\omega)\cos\Phi \end{bmatrix}, \tag{1}$$

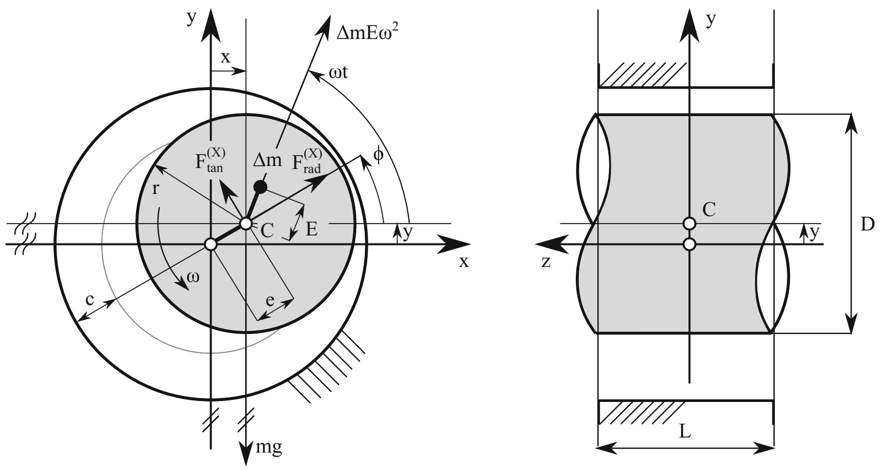

Fig. 1 The schematic of the considered Jeffcott rotor

where indices $X = IL, IS, IL_{cor}, IS_{cor}$ correspond to the infinitely long, infinitely short, corrected infinitely long and corrected infinitely short journal bearing, respectively.

The eccentricity e and relative eccentricity ε are defined in the form

$$e(t) = \sqrt{x(t)^2 + y(t)^2}, \qquad \varepsilon(t) = \frac{e(t)}{c}. \tag{2}$$

The derivatives of excentricity and relative excentricity are

$$\dot{e}(t) = \frac{de(t)}{dt} = \frac{x(t)\dot{x}(t) + y(t)\dot{y}(t)}{e(t)}, \qquad \dot{\varepsilon}(t) = \frac{d\varepsilon(t)}{dt} = \frac{1}{c}\frac{de(t)}{dt} = \frac{\dot{e}(t)}{c}. \tag{3}$$

The goniometric functions of the angle Φ are obviously

$$\cos\Phi = \frac{x}{e}, \qquad \sin\Phi = \frac{y}{e}, \tag{4}$$

and the angle Φ can be directly expressed in the form

$$\Phi = \mathrm{arctg2}\left(\frac{y}{x}\right) = \begin{cases} \mathrm{arctg}\left(\frac{y}{x}\right) & x > 0 \wedge y > 0, \\ \mathrm{arctg}\left(\frac{y}{x}\right) + \pi & x < 0, \\ \mathrm{arctg}\left(\frac{y}{x}\right) + 2\pi & x > 0 \wedge y < 0. \end{cases} \tag{5}$$

The derivative $\dot{\Phi}$ of angle Φ is

$$\dot{\Phi} = \frac{x(t)\dot{y}(t) - y(t)\dot{x}(t)}{x(t)^2 + y(t)^2}. \tag{6}$$

Formulas for forces in HD journal bearings introduced in [8] (IL) and [9, 10] (IS) have been summarized in [13]. Corresponding to the directions indicated in the Fig. 1 (the influence of oil film on the shaft), the forces can be expressed as

$$F_{rad}^{(IL)} = -6\mu RL\left(\frac{R}{c}\right)^2\left[|\omega - 2\dot{\Phi}|\frac{2\varepsilon^2}{(2+\varepsilon^2)(1-\varepsilon^2)} + \frac{\pi\dot{\varepsilon}}{(1-\varepsilon^2)^{3/2}}\right], \tag{7}$$

$$F_{tan}^{(IL)} = 6\mu RL\left(\frac{R}{c}\right)^2\left[(\omega - 2\dot{\Phi})\frac{\pi\varepsilon}{(2+\varepsilon^2)(1-\varepsilon^2)^{1/2}} + \frac{4\dot{\varepsilon}}{(1+\varepsilon)(1-\varepsilon^2)}\right], \tag{8}$$

$$F_{rad}^{(IS)} = -\mu RL\left(\frac{L}{c}\right)^2\left[|\omega - 2\dot{\Phi}|\frac{\varepsilon^2}{(1-\varepsilon^2)^2} + \frac{\pi(1+2\varepsilon^2)\dot{\varepsilon}}{2(1-\varepsilon^2)^{5/2}}\right], \tag{9}$$

$$F_{tan}^{(IS)} = \mu RL\left(\frac{L}{c}\right)^2\left[(\omega - 2\dot{\Phi})\frac{\pi\varepsilon}{4(1-\varepsilon^2)^{3/2}} + \frac{2\varepsilon\dot{\varepsilon}}{(1-\varepsilon^2)^2}\right]. \tag{10}$$

The IL-based and IS-based forces corrected for the finite journal bearing have been published in [13]. They are considered in the form

$$F_{rad}^{(IL_{cor})} = C_{rad}^{(IL_{cor})}(L/D, \varepsilon)\, F_{rad}^{(IL)}, \qquad F_{tan}^{(IL_{cor})} = C_{tan}^{(IL_{cor})}(L/D, \varepsilon)\, F_{tan}^{(IL)}, \tag{11}$$

$$F_{rad}^{(IS_{cor})} = C_{rad}^{(IS_{cor})}(L/D, \varepsilon)\, F_{rad}^{(IS)}, \qquad F_{tan}^{(IS_{cor})} = C_{tan}^{(IS_{cor})}(L/D, \varepsilon)\, F_{tan}^{(IS)}, \tag{12}$$

where $C_Y^{(X)}(L/D, \varepsilon)$ are correction polynomials defined for $X = IL, IS$ and $Y = rad, tan$.

3 Application and Results

To find a numerical solution for the system (1), the set of second order ODEs can be rewritten in state-space. Mathematical model (1) can be formally rewritten as

$$\boldsymbol{M}\ddot{\boldsymbol{q}} = \underbrace{\boldsymbol{f}_G + \boldsymbol{f}_e(t) + \boldsymbol{f}_o^{(X)}(\boldsymbol{q}, \dot{\boldsymbol{q}}, \omega, t)}_{\boldsymbol{f}^{(X)}(\boldsymbol{q}, \dot{\boldsymbol{q}}, \omega, t)}, \qquad X = IS, IL, IS_{cor}, IL_{cor}, \tag{13}$$

where $\boldsymbol{q} = [x, y]^T$ is vector of generalized coordinates and $\boldsymbol{M}$ is mass matrix. At the right hand side, a vector of gravitation forces $\boldsymbol{f}_G$, vector of unbalance forces $\boldsymbol{f}_e(t)$ and vector of oil-film forces $\boldsymbol{f}_o^{(X)}(\boldsymbol{q}, \dot{\boldsymbol{q}}, \omega, t)$ are defined. The model can be written in the state-space with state vector $\boldsymbol{u} = [\boldsymbol{q}^T, \dot{\boldsymbol{q}}^T]^T$ as

$$\dot{\boldsymbol{u}} = \begin{bmatrix} \dot{\boldsymbol{q}} \\ \ddot{\boldsymbol{q}} \end{bmatrix} = \begin{bmatrix} \dot{\boldsymbol{q}} \\ \boldsymbol{M}^{-1}\boldsymbol{f}^{(X)}(\boldsymbol{q}, \dot{\boldsymbol{q}}, \omega, t) \end{bmatrix}, \qquad X = IS, IL, IS_{cor}, IL_{cor}. \tag{14}$$

The set of first order Eq. (14) is solved using the Runge-Kutta method with adaptive time step. The simulation is performed for $t \in \langle 0, 1.5 \rangle$ [s] and the time interval $t_{ss} \in \langle 1, 1.5 \rangle$ [s] of steady-state behaviour is subjected to the subsequent analyses. Particular parameters of the Jeffcott rotor used in the analyses are shown in the Table 1. For the comparison of results, numerical simulations in *AVL Excite* software have been performed. Herein, the model is created using a multibody approach and the Reynolds equation is solved using finite element method.

Two different types of analyses were performed to investigate the dynamical behaviour of the system: *unbalance response analysis* of the Jeffcott rotor and the *analysis of whirl instability* caused by the fluid-film of HD bearings. In the Figs. 2, 3 and 4, an unbalance response of the Jeffcott rotor is shown via orbits of the centre of journal for chosen rotational speeds. The aspect ratio $\eta = 1$ is considered for finite length bearing (Fig. 3) and for limit values of the approximate solutions $\eta = 0.5$, $\eta = 2$ for IS and IL journal bearing (Figs. 2 and 4, respectively). Qualitative and quantitative change of the orbits occurs with increasing speed of the rotor. However, orbits of the rotor also differ for the different HD bearings models at the same

Table 1 Parameters of the Jeffcott rotor in HD journal bearings [13]

Parameter	Value
Radial clearance	$c = 0.9 \times 10^{-3}$ m
Unbalance	$\Delta m = 0.01$ kg
Unbalance eccentricity	$E = 0.01$ m
Rotor mass	$m = 3$ kg
Rotor RPM	$n = \langle 500; 2900 \rangle$ RPM
Dynamic viscosity	$\mu = 0.07$ Pa·s
Bearing diameter	$D = 47.37 \times 10^{-3}$ m
Bearing axial length	$L = \eta D, \eta \in \langle 0, 5; 2 \rangle$

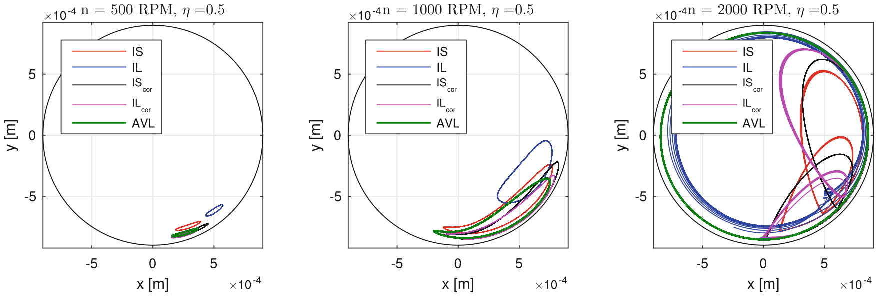

Fig. 2 Orbits of the Jeffcott rotor with all the considered hydrodynamic forces for aspect ratio $\eta = \frac{L}{D} = 0.5$

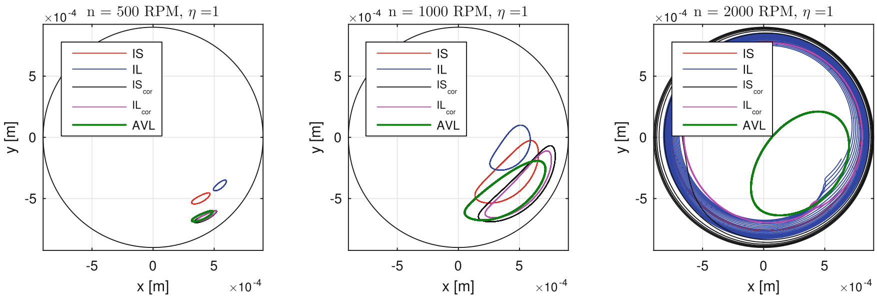

Fig. 3 Orbits of the Jeffcott rotor with all the considered hydrodynamic forces for aspect ratio $\eta = \frac{L}{D} = 1$

speeds. The IS_{cor} and IL_{cor} solutions are relatively close, particularly at lower speeds. As expected, the approximate solution IS is closer to the both corrected solutions for $\eta = 0.5$ and IL is closer to the both corrected solution for $\eta = 2$.

The analysis of the oil-film whirl instability is induced even in case of perfectly balanced rotor ($\Delta mE = 0$). The motion of Jeffcott rotor with parameters shown in the Table 1 has been simulated for all the considered HD forces in journal bearings and

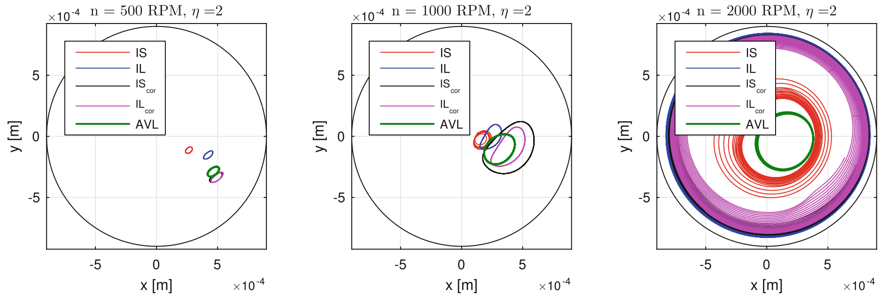

Fig. 4 Orbits of the Jeffcott rotor with all the considered hydrodynamic forces for aspect ratio $\eta = \frac{L}{D} = 2$

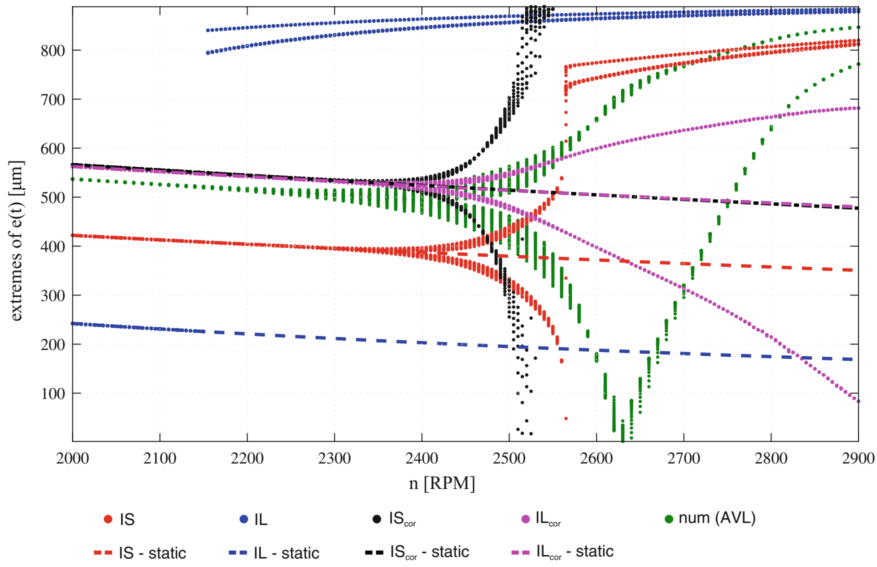

Fig. 5 Bifurcation of eccentricities of the Jeffcott rotor with all the considered hydrodynamic forces for $\eta = \frac{L}{D} = 1$. Solution obtained by the static analysis is depicted by the dashed line

for the rotational speeds $\omega = \pi n/30$, $n = \langle 2000, 2900\rangle$ RPM. To depict the whirl instability, bifurcation diagram Fig. 5 was used. The extremes of the eccentricities $e(t_{ss})$ are evaluated at all the considered rotational speeds. It enables to distinguish the areas with fixed point attractor (single point at the particular speed) and areas of the limit-cycle attractor (two or more different points at the particular speed).

The diagram in Fig. 5 shows almost identical extremes of eccentricities for both corrected models IS_{cor} and IL_{cor} in the area of fixed point attractor and the bifurcation occurs at the pretty close rotational speeds. However, the behaviour of both models differs in the area of limit cycle attractor. The amplitudes of IS based model grow more rapidly and it even does not converge to the limit cycle for larger values of rotational speeds. The location of a bifurcation point differs for IS and IL approximations.

An additional information of the bifurcation analyses can be obtained using static analysis. This follows from (13) with omitted dynamical forces: inertial forces $\boldsymbol{M}\ddot{\boldsymbol{q}}$ and centrifugal forces $\boldsymbol{f}_e(t)$. All the terms in formulas of oil-film forces (7)–(12) that are dependent on $\dot{\varepsilon}$ also equals zero in the static case. The problem can be formulated in the form of set of nonlinear algebraic equations

$$\boldsymbol{f}_G + \boldsymbol{f}_o(\boldsymbol{q}, \omega) = \boldsymbol{0} \tag{15}$$

which is solved using Trust-Region Dogleg Method. For the chosen rotational speed ω of the rotor, a static solution $\boldsymbol{q}_{static}$ is find which satisfies (15). Corresponding eccentricities [transformed using (2)] are depicted in the bifurcation diagram by the dashed line.

Obviously, for the stable area before bifurcation of the dynamical solution a static solution directly corresponds to the dynamical solution (without unbalance). However, in the area after bifurcation point, the static solution corresponds to unstable dynamical solution.

4 Conclusions

The paper focuses on the behaviour of Jeffcott rotor supported by HD journal bearings. To describe bearing forces, various approximate analytical solutions of Reynolds equation for plain journal bearings are used. The differences in the rotor behaviour are demonstrated using the unbalance response and the analysis of the whirl instability. The unbalance response analysis shows the similar behaviour of the corrected IL and IS models, particularly at lower speeds. Both of these two models come to the IL or IS approximations with the aspect ratio coming close to the corresponding limit state (IS or IL).

The analysis of whirl instability shows the different location of Hopf bifurcation for all the considered forces. Qualitatively different behaviour in the limit-cycle area is observed. The dynamical steady-state computations are supplemented by the static analysis of the system via numerical solution of set of nonlinear algebraic equations. This shows the possible unstable equilibria in the area of limit-cycle attractor of the Jeffcott rotor.

To provide a comparison, multibody simulation-based computations have been performed in AVL Excite where the Reynolds equation is solved using the finite element method. The comparison shows satisfactory agreement particularly for the corrected IL and IS based approximations in wide range of aspect ratios $\eta = \frac{L}{D} \in \{0.5; 1; 2\}$ in case of unbalance response. The agreement of the bifurcation point for corrected IS and IL models with numerical model have been provided for the case of $\eta = 1$.

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