Maximum Angle Condition for n-Dimensional Simplicial Elements

Antti Hannukainen, Sergey Korotov, and Michal Křížek

Abstract In this paper the Synge maximum angle condition for planar triangulations is generalized for higher-dimensional simplicial partitions. In addition, optimal interpolation properties are presented for linear simplicial elements which can degenerate in certain ways.

1 Introduction

Consider a family $\mathcal{F} = \{\mathcal{T}_h\}_{h\to 0}$ of face-to-face triangulations $\mathcal{T}_h$ of a bounded polygonal domain. In 1957, J. Synge proved that linear triangular finite elements yield the optimal interpolation order in the C-norm provided the *maximum angle condition* is satisfied, i.e., there exists a constant $\gamma_0 < \pi$ such that for any triangulation $\mathcal{T}_h \in \mathcal{F}$ and any triangle $T \in \mathcal{T}_h$ one has (see [27])

$$\gamma_T \leq \gamma_0, \tag{1}$$

where γ_T is the maximum angle of T. In 1975/1976, Babuška and Aziz [3], Barnhill and Gregory [4], and Jamet [14] independently derived the optimal interpolation order in the energy norm of finite element approximations under the condition (1).

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Later the maximum angle condition was investigated in various norms in [1, 2, 15–18, 22, 24, 25].

In 1992, the condition (1) was generalized by Křížek [19] to tetrahedral elements as follows:

There exists a constant $\gamma_0 < \pi$ such that for any face-to-face tetrahedralization $\mathcal{T}_h \in \mathcal{F}$ and any tetrahedron $T \in \mathcal{T}_h$ one has

$$\gamma_{\mathrm{D}} \leq \gamma_0 \quad \text{and} \quad \gamma_{\mathrm{F}} \leq \gamma_0, \tag{2}$$

where γ_{D} is the maximum dihedral angles between faces of T and γ_{F} is the maximum angle in all four triangular faces of T. According to [20], the associated finite element approximations preserve the optimal interpolation order in the H^1-norm under the condition (1).

Note that degenerated tetrahedral elements have a lot of real-life technical applications. For example, in calculation of physical fields in eletrical rotary machines, see [20]. Flat tetrahedral elements are also used to approximate thin slots, layers, or gaps. Moreover, they are suitable when the true solution of some problem changes more rapidly in one direction than in another direction (e.g. in anisotropic materials) [1].

From Fig. 1 we observe that the condition (2) is satisfied for a needle, splinter, and wedge tetrahedron. For the other degenerated tetrahedra from Fig. 1, the interpolation error may diverge in the H^1-norm. On the other hand, the finite

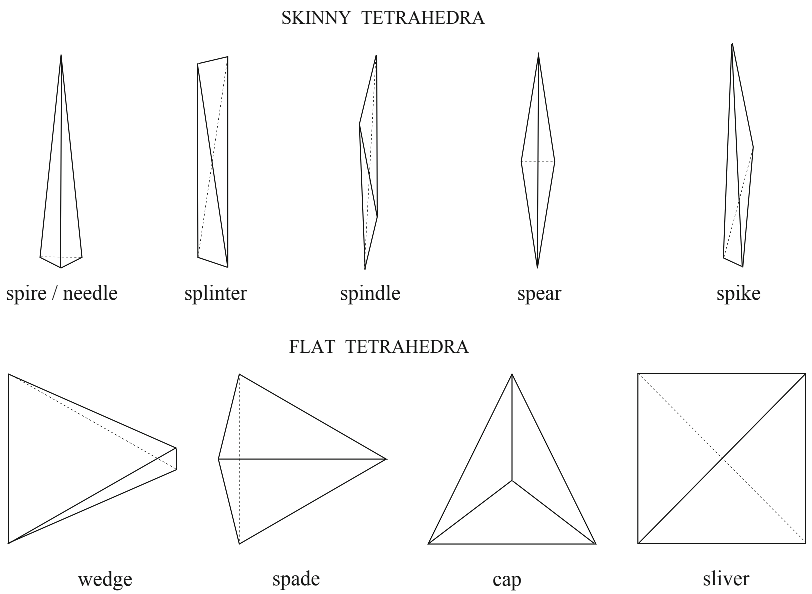

Fig. 1 Classification of degenerated tetrahedra according to [7, 9]

element method may converge in the H^1-norm, see [12]. Hence, (2) represents only a sufficient (and not necessary) condition for the convergence of the finite element method, see [21–23, 25].

Let us point out that the two conditions in (2) are independent. For instance, for a sliver (and also cap) tetrahedron the condition $\gamma_D \le \gamma_0$ does not hold while $\gamma_F \le \gamma_0$ holds. On the other hand, for a spike tetrahedron $\gamma_D \le \gamma_0$ holds and $\gamma_F \le \gamma_0$ is violated.

Since there are six dihedral angles of each tetrahedron and twelve angles between its adjacent edges, a direct generalization of (2) into higher dimensions would be technically quite complicated. Therefore, in the next section we introduce another concept which is based on the d-dimensional sine for $d > 1$. We will survey the main results from our previous paper [13], see also [5, 6].

2 The Maximum Angle Condition in Higher Dimensions

Recall that a d-simplex S in $\mathbf{R}^d$, $d \in \{1, 2, 3, \dots\}$, is the convex hull of $d+1$ vertices $A_0, A_1, \dots, A_d$ that do not belong to the same $(d-1)$-dimensional hyperplane, i.e.,

$$S = \operatorname{conv}\{A_0, A_1, \dots, A_d\}.$$

Let

$$F_i = \operatorname{conv}\{A_0, \dots, A_{i-1}, A_{i+1}, \dots, A_d\}$$

be the facet of S opposite to the vertex A_i for $i \in \{0, \dots, d\}$.

In 1978, Eriksson has introduced a generalization of the sine function to an arbitrary d-dimensional spatial angle, see [10, p. 74].

Definition 1 Let $\hat{A}_i$ be the angle at the vertex A_i of the simplex S. Then d-sine of the angle $\hat{A}_i$ for $d > 1$ is given by

$$\sin_d(\hat{A}_i | A_0 A_1 \dots A_d) = \frac{d^{d-1} (\operatorname{meas}_d S)^{d-1}}{(d-1)! \, \mathbf{\Pi}_{j=0, j\neq i}^{d} \operatorname{meas}_{d-1} F_j}. \tag{3}$$

Remark 2 Let us show that d-sine is really a generalization of the classical sine function. Set $d = 2$ and consider an arbitrary triangle $A_0A_1A_2$. Denote by $\hat{A}_0$ its angle at the vertex A_0. Then, obviously,

$$\operatorname{meas}_2(A_0A_1A_2) = \frac{1}{2}|A_0A_1||A_0A_2| \sin \hat{A}_0. \tag{4}$$

Comparing this relation with (3), we find that

$$\sin \hat{A}_0 = \sin_2(\hat{A}_0, A_0A_1A_2). \tag{5}$$

Definition 3 A family $\mathcal{F} = \{\mathcal{T}_h\}_{h\to 0}$ of partitions of a polytope into d-simplices is said to satisfy the *generalized minimum angle condition* if there exists $C > 0$ such that for any $\mathcal{T}_h \in \mathcal{F}$ and any $S = \text{conv}\{A_0, \dots, A_d\} \in \mathcal{T}_h$ one has

$$\forall\, i \in \{0, 1, \dots, d\} \qquad \sin_d(\hat{A}_i | A_0A_1 \dots A_d) \geq C > 0. \tag{6}$$

This condition is investigated in the paper [6]. It generalizes the well-known Zlámal minimum angle condition for triangles (see [8, 28, 29]), which is stronger than (1).

Definition 4 A family $\mathcal{F} = \{\mathcal{T}_h\}_{h\to 0}$ of partitions of a polytope into d-simplices is said to satisfy the *generalized maximum angle condition* if there exists $C > 0$ such that for any $\mathcal{T}_h \in \mathcal{F}$ and any $S = \text{conv}\{A_0, \dots, A_d\} \in \mathcal{T}_h$ one can always choose d edges of S, which, when considered as vectors, constitute a (higher-dimensional) angle whose d-sine is bounded from below by the constant C.

Remark 5 From (4) and (5) we observe that the condition stated in Definition 4 for $d = 2$ is equivalent to the maximum angle condition (1).

Remark 6 Let us show that in case of tetrahedra the validity of the maximum angle condition (2) implies the desired property in Definition 4, i.e. it is really a generalization. So, let (2) be valid for a given tetrahedron T. Then one can always find, see the proof of Theorem 7 in [19, pp. 517–518], three unit vectors t_1, t_2, and t_3 parallel to three edges of T, so that the volume of the parallelepiped $\mathcal{P}(t_1, t_2, t_3)$ generated by t_1, t_2, t_3 is bounded from below by some constant $c > 0$. Now we use formula (3) to estimate the 3-dimensional sine of the angle formed by the vectors t_1, t_2, t_3 as follows

$$\sin_3(t_1, t_2, t_3) = \frac{3^2\,(\text{meas}\,_3 S(t_1, t_2, t_3))^2}{2!\,\mathbf{\Pi}^3_{j=0, j\neq i}\text{meas}\,_2 F_j} \geq \frac{3^2\,(\frac{1}{6}\text{meas}\,_3 \mathcal{P}(t_1, t_2, t_3))^2}{2!(\frac{1}{2})^3} \geq c^2, \tag{7}$$

where $S(t_1, t_2, t_3)$ is the tetrahedron made by t_1, t_2, t_3 originating at 0, and the area of each of the three faces F_j involved is bounded from above by $1/2$ due to the fact that t_1, t_2, t_3 are unit vectors. The constant c can be, in fact, estimated from below by

$$m := \min\Big(\sin\frac{\pi - \gamma_0}{2}, \sin\gamma_0\Big),$$

see [19, p. 518].

Now we present the main interpolation theorem of this paper using the standard Sobolev space notation.

Theorem 7 *Let $\mathcal{F}$ be a family of partitions of a polytope into d-simplices satisfying the generalized maximum angle condition. Then there exists a constant $C > 0$ such that for any $\mathcal{T}_h \in \mathcal{F}$ and any $S \in \mathcal{T}_h$ we have*

$$\|v - \pi_S v\|_{1,\infty} \le C h_S |v|_{2,\infty} \qquad \forall v \in \mathcal{C}^2(S), \tag{8}$$

where π_S is the standard Lagrange linear interpolant and $h_S = \operatorname{diam} S$.

For the proof see [13].

3 Examples

Example 1 Denoting by A_0, A_1, A_2, A_3 the vertices of spindle, spear, spike, spade, cap, or sliver tetrahedron from Fig. 1 in an arbitrary way, we find by (3) that

$$\sin_3(\hat{A}_0, A_0A_1A_2A_3) \to 0$$

as the discretization parameter tends to zero. The same is true for the splinter tetrahedron. However, by Definition 4 we may choose three edges which constitute a spatial angle whose 3-sine is bounded from below by a fixed constant $C > 0$. Choosing the two short edges of the splinter tetrahedron and one long edge, we find that the generalized maximum angle condition holds.

Example 2 We show that if $\sin_3$ of two trihedral angles (cf. [11]) are the same numbers, then the magnitude of these solid angles in steradians need not be the same. Let $A_0A_1A_2A_3$ be the regular tetrahedron whose edges have length 1. Then the altitude of its faces is $\sqrt{3}/2$ and the spatial altitude is $\sqrt{6}/3$. Consequently, from (3) for the trihedral angle $\hat{A}_0$ we get

$$\sin_3(\hat{A}_0, A_0 \dots A_3) = \frac{3^2(\frac{1}{3}\frac{\sqrt{3}}{4}\frac{\sqrt{6}}{3})^2}{2!(\frac{\sqrt{3}}{4})^3} = \frac{4\sqrt{3}}{9}.$$

Consider now the tetrahedron with vertices $B_0 = (0,0,0)$, $B_1 = (1,0,0)$, $B_2 = (\frac{\sqrt{33}}{9}, \frac{4\sqrt{3}}{9}, 0)$, and $B_3 = (0,0,1)$. Then by (3) we also find that

$$\sin_3(\hat{B}_0, B_0 \dots B_3) = \frac{3^2(\frac{1}{3}\operatorname{meas}_2 B_0B_1B_2)^2}{2!\frac{1}{4}\operatorname{meas}_2 B_0B_1B_2} = \frac{4\sqrt{3}}{9}.$$

Now by the Girard Theorem for the spherical excess we have (see [26, p. 83])

$$\hat{B}_0 = \frac{\pi}{2} + \frac{\pi}{2} + \arcsin\frac{4\sqrt{3}}{9} - \pi = 0.8785\dots \text{ steradians},$$

whereas

$$\hat{A}_0 = 3 \arccos \frac{1}{3} - \pi = 0.5512\ldots \text{ steradians},$$

where all dihedral angles α of $A_0A_1A_2A_3$ are $\approx 70.52°$, since $\cos\alpha = \frac{1}{3}$.

Example 3 The sliver element from Fig. 1 can be narrowed as follows. For $h \to 0$ consider the tetrahedron with vertices

$$(0, 0, 0), \quad (h^2, 0, 0), \quad (0, h, 0), \quad (h^2, h, h^3).$$

Example 4 A higher-dimensional example can be constructed in the following way. Consider positive numbers $r_1, r_2, \ldots, r_d$ and a simplex with vertices $A_0, A_1, \ldots, A_d$. We fix some number k so that $0 \le k \le d$. The first $k + 1$ vertices of the simplex are defined as follows. Let $A_0 = (0, 0, \ldots, 0, \ldots, 0)$. Further, let

$$A_1 = (r_1, 0, \ldots, 0, \ldots, 0),\ A_2 = (0, r_2, \ldots, 0, \ldots, 0),\ \ldots,$$

$$A_k = (0, \ldots, 0, r_k, 0, \ldots, 0).$$

The remaining vertices are:

$$A_{k+1} = (0, \ldots, 0, r_{k+1}, 0, 0, \ldots, 0),\ A_{k+2} = (0, \ldots, 0, r_{k+1}, r_{k+2}, 0, \ldots, 0),$$

$$A_{k+3} = (0, \ldots, 0, r_{k+1}, r_{k+2}, r_{k+3}, 0, \ldots, 0),\ \ldots,$$

$$A_d = (0, \ldots, 0, r_{k+1}, r_{k+2}, r_{k+3}, \ldots, r_d).$$

Therefore, for $k = 0$, we get the path-simplex, and for $k = d$ the hypercube-corner simplex. Allowing some of the r_k's to approach zero with different rates, in general, we get various degenerated simplices still satisfying the generalized maximum angle conditions.

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References

1. T. Apel, *Anisotropic Finite Elements: Local Estimates and Applications*. Advances in Applied Mathematics (B.G. Teubner, Stuttgart, 1999)
2. T. Apel, M. Dobrowolski, Anisotropic interpolation with applications to the finite element method. Computing **47**, 277–293 (1992)

3. I. Babuška, A.K. Aziz, On the angle condition in the finite element method. SIAM J. Numer. Anal. **13**, 214–226 (1976)
4. R.E. Barnhill, J.A. Gregory, Sard kernel theorems on triangular domains with applications to finite element error bounds. Numer. Math. **25**, 215–229 (1976)
5. J. Brandts, S. Korotov, M. Křížek, On the equivalence of ball conditions for simplicial finite elements in $\mathbf{R}^d$. Appl. Math. Lett. **22**, 1210–1212 (2009)
6. J. Brandts, S. Korotov, M. Křížek, Generalization of the Zlámal condition for simplicial finite elements in $\mathbf{R}^d$. Appl. Math. **56**, 417–424 (2011)
7. S.W. Cheng, T.K. Dey, H. Edelsbrunner, M.A. Facello, S.H. Teng, Sliver exudation, in *Proceedings of 15-th ACM Symposium on Computational Geometry* (1999), pp. 1–13
8. P.G. Ciarlet, *The Finite Element Method for Elliptic Problems* (North-Holland, Amsterdam, 1978)
9. H. Edelsbrunner, Triangulations and meshes in computational geometry. Acta Numer. **9**, 133–213 (2000)
10. F. Eriksson, The law of sines for tetrahedra and n-simplices. Geom. Dedicata **7**, 71–80 (1978)
11. J.W. Gaddum, The sums of dihedral and trihedral angles in a tetrahedron. Am. Math. Mon. **59**, 370–371 (1952)
12. A. Hannukainen, S. Korotov, M. Křížek, The maximum angle condition is not necessary for convergence of the finite element method. Numer. Math. **120**, 79–88 (2012)
13. A. Hannukainen, S. Korotov, M. Křížek, Generalizations of the Synge-type condition in the finite element method. Appl. Math. **62**, 1–13 (2017)
14. P. Jamet, Estimation de l'erreur pour des éléments finis droits presque dégénérés. RAIRO Anal. Numér. **10**, 43–60 (1976)
15. K. Kobayashi, T. Tsuchiya, On the circumradius condition for piecewise linear trian-gular elements. Jpn. J. Ind. Appl. Math. **32**, 65–76 (2015)
16. K. Kobayashi, T. Tsuchiya, A priori error estimates for Lagrange intrepolation on triangles. Appl. Math. **60**, 485–499 (2015)
17. K. Kobayashi, T. Tsuchiya, Extending Babuška-Aziz theorem to higher-order Largange interpolation. Appl. Math. **61**, 121–133 (2016)
18. M. Křížek, On semiregular families of triangulations and linear interpolation. Appl. Math. **36**, 223–232 (1991)
19. M. Křížek, On the maximum angle condition for linear tetrahedral elements. SIAM J. Numer. Anal. **29**, 513–520 (1992)
20. M. Křížek, P. Neittaanmäki, *Mathematical and Numerical Modelling in Electrical Engineering: Theory and Applications* (Kluwer Academic Publishers, Dordrecht, 1996)
21. V. Kučera, A note on necessary and sufficient condition for convergence of the finite element method, in *Proceedings of Conference on Applied Mathematics 2015*, ed. by J. Brandts et al. (Institute of Mathematical, Prague, 2015), pp. 132–139
22. V. Kučera, Several notes on the circumradius condition. Appl. Math. **61**, 287–298 (2016)
23. V. Kučera, On necessary and sufficient conditions for finite element convergence. Numer. Math (submitted). Arxiv 1601.02942
24. S. Mao, Z. Shi, Error estimates of triangular finite elements under a weak angle condition. J. Comput. Appl. Math. **230**, 329–331 (2009)
25. P. Oswald, Divergence of FEM: Babuška-Aziz triangulations revisited. Appl. Math. **60**, 473–484 (2015)
26. K. Rektorys, *Survey of Applicable Mathematics I* (Kluwer Academic Publishers, Dordrecht, 1994)
27. J.L. Synge, *The Hypercircle in Mathematical Physics* (Cambridge University Press, Cambridge, 1957)
28. A. Ženíšek, The convergence of the finite element method for boundary value problems of a system of elliptic equations (in Czech). Appl. Math. **14**, 355–377 (1969)
29. M. Zlámal, On the finite element method. Numer. Math. **12**, 394–409 (1968)