Stability of Higher-Order ALE-STDGM for Nonlinear Problems in Time-Dependent Domains

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Abstract In this paper we investigate the stability of the space-time discontinuous Galerkin method for the solution of nonstationary, nonlinear convection-diffusion problem in time-dependent domains. At first we define the continuous problem and reformulate it using the Arbitrary Lagrangian-Eulerian (ALE) method, which replaces the classical partial time derivative by the so called ALE-derivative and an additional convective term. Then the problem is discretized with the aid of the ALE space-time discontinuous Galerkin method (ALE-STDGM). The discretization uses piecewise polynomial functions of degree $p \geq 1$ in space and $q > 1$ in time. Finally in the last part of the paper we present our results concerning the unconditional stability of the method. An important step is the generalization of a discrete characteristic function associated with the approximate solution and the derivation of its properties, namely its continuity in the $\|\cdot\|_{L^2}$-norm and in special $\|\cdot\|_{DG}$-norm.

1 Introduction

Problems in time-dependent domains are very important in many areas of science and technology, for example, fluid-structure interaction problems.

In this paper we deal with the stability analysis of the ALE-STDGM with arbitrary polynomial degree in space as well as in time, applied to a nonstationary, nonlinear convection-diffusion problem equipped with initial and Dirichlet boundary condition. The ALE-STDGM analyzed here corresponds to the technique used in [3] and [4] for the numerical simulation of airfoil vibrations induced by compressible flow, which means that the ALE mapping is not prescribed globally in the whole time interval, but separately for each time slab.

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We present here new technique of theoretical analysis in contrast to [1] and [2], where we proved the unconditional stability of the ALE-STDGM with arbitrary polynomial degree in space, but only linear approximation in time. The new technique is based on generalization of the discrete characteristic function in time-dependent domains.

2 Formulation of the Continuous Problem

We consider an initial-boundary value nonstationary, nonlinear convection-diffusion problem in a time-dependent bounded domain $\Omega_t,\ t \in (0,T)$:

Find a function $u = u(x,t)$ with $x \in \Omega_t,\ t \in (0,T)$ such that

$$\frac{\partial u}{\partial t} + \sum_{s=1}^{d} \frac{\partial f_s(u)}{\partial x_s} - \operatorname{div}(\beta(u)\nabla u) = g \quad \text{in } \Omega_t,\ t \in (0,T), \tag{1}$$

$$u = u_D \quad \text{on } \partial\Omega_t,\ t \in (0,T), \tag{2}$$

$$u(x,0) = u^0(x), \quad x \in \Omega_0. \tag{3}$$

We assume that f_s, β, g, u_D, u_0 are given functions, $|f_s'| \le L_f,\ s = 1,\dots,d$, and function β is Lipschitz-continuous and bounded: $\beta : \mathbb{R} \to [\beta_0, \beta_1]$ where $0 < \beta_0 < \beta_1 < \infty$.

Problem (1)–(3) can be reformulated using the Arbitrary Lagrangian-Eulerian (ALE) method. First we consider a standard ALE formulation prescribed globally in the whole time interval, used in a number of works (cf., e.g., …). It is based on a regular one-to-one ALE mapping of the reference domain Ω_0 onto the current configuration Ω_t:

$$\mathcal{A}_t : \overline{\Omega_0} \to \overline{\Omega_t}, \quad X \in \overline{\Omega_0} \to x = x(X,t) = \mathcal{A}_t(X) \in \overline{\Omega_t}, \quad t \in [0,T]. \tag{4}$$

Usually it is supposed that the ALE mapping is sufficiently regular, e.g., $\mathcal{A} \in W^{1,\infty}(0,T; W^{1,\infty}(\Omega_t))$. Now we introduce the domain velocity

$$\tilde{z}(X,t) = \frac{\partial}{\partial t}\mathcal{A}_t(X),\ z(x,t) = \tilde{z}(\mathcal{A}_t^{-1}(x),t),\ t \in [0,T],\ X \in \Omega_0,\ x \in \Omega_t, \tag{5}$$

and define the ALE derivative of a function $f = f(x,t)$ for $x \in \Omega_t$ and $t \in [0,T]$ using the chain rule as

$$\frac{Df}{Dt} = \frac{\partial f}{\partial t} + z \cdot \nabla f, \tag{6}$$

which allows us to reformulate problem (1)–(3) in the ALE form:

Find $u = u(x, t)$ with $x \in \Omega_t,\ t \in (0, T)$ such that

$$\frac{Du}{Dt} + \sum_{s=1}^{d} \frac{\partial f_s(u)}{\partial x_s} - z \cdot \nabla u - \mathrm{div}(\beta(u)\nabla u) = g \quad \text{in } \Omega_t,\ t \in (0, T), \tag{7}$$

$$u = u_D \quad \text{on } \partial\Omega_t,\ t \in (0, T), \tag{8}$$

$$u(x, 0) = u^0(x), \quad x \in \Omega_0. \tag{9}$$

Moreover we assume the following properties of the domain velocity: There exists a constant $c_z > 0$ such that

$$|z(x, t)|,\ |\mathrm{div} z(x, t)| \le c_z \quad \text{for } x \in \Omega_t,\ t \in (0, T). \tag{10}$$

3 ALE–Space Time Discretization

We consider a time partition $0 = t_0 < t_1 < \cdots < t_M = T$ and set $\tau_m = t_m - t_{m-1}$, $I_m = (t_{m-1}, t_m)$ for $m = 1, \ldots, M$. The space-time discontinuous Galerkin method (STDGM) has an advantage that on every time interval $\overline{I}_m = [t_{m-1}, t_m]$ it is possible to consider a different space partition. Here we also use this property of the STDGM in the ALE framework: we consider an ALE mapping separately on each time interval $[t_{m-1}, t_m)$ for $m = 1, \ldots, M$. The resulting ALE mapping in $[0, T]$ may be discontinuous at time instants t_m, which means that $\mathcal{A}(t_m-) \neq \mathcal{A}(t_m+)$ in general. Such situation appears in the numerical solution of fluid-structure interaction problems, when both the ALE mapping and the approximate flow solution are constructed successively on time intervals I_m by the STDGM (see [6]).

3.1 ALE Mappings and Triangulations

For every $m = 1, \ldots, M$ we consider a standard conforming triangulation $\hat{\mathcal{T}}_{h,t_{m-1}}$ in $\Omega_{t_{m-1}}$, where $h \in (0, \overline{h})$, $\overline{h} > 0$ and introduce a one-to-one ALE mapping

$$\mathcal{A}_{h,t}^{m-1} : \overline{\Omega}_{t_{m-1}} \xrightarrow{\text{onto}} \overline{\Omega}_t \quad \text{for } t \in [t_{m-1}, t_m),\ h \in (0, \overline{h}). \tag{11}$$

We assume that $\mathcal{A}_{h,t}^{m-1}$ is in space a piecewise affine mapping, continuous in space variable $X \in \Omega_{t_{m-1}}$ as well as in time $t \in [t_{m-1}, t_m)$ and $\mathcal{A}_{h,t_{m-1}}^{m-1} = \mathrm{Id}$ (identical mapping). For every $t \in [t_{m-1}, t_m)$ we define the conforming triangulation

$$\mathcal{T}_{h,t} = \left\{ K = \mathcal{A}_{h,t}^{m-1}(\hat{K});\ \hat{K} \in \hat{\mathcal{T}}_{h,t_{m-1}} \right\} \text{ in } \Omega_t. \tag{12}$$

At $t = t_m$ we define the one-sided limit $\mathcal{A}^{m-1}_{h,t_m-}$, and introduce the corresponding triangulation. As we see, for every $t \in [0, T]$ we have a family $\{\mathcal{T}_{h,t}\}_{h\in(0,\overline{h})}$ of triangulations of the domain Ω_t.

3.2 Discrete Function Spaces

Let $p \geq 1$ be an integer and $P^p(\hat{K})$ the space of all polynomials on $\hat{K}$ of degree $\leq p$. Then for every $m = 1, \ldots, M$ we consider the space

$$S_h^{p,m-1} = \left\{\varphi \in L^2(\Omega_{t_{m-1}});\ \varphi|_{\hat{K}} \in P^p(\hat{K})\ \forall\, \hat{K} \in \hat{\mathcal{T}}_{h,t_{m-1}}\right\}. \tag{13}$$

Further, for $q \geq 1$ by $P^q(I_m; S_h^{p,m-1})$ we denote the space of mappings of the time interval I_m into the space $S_h^{p,m-1}$ which are polynomials of degree $\leq q$ in time. We set

$$S_{h,\tau}^{p,q} = \left\{\varphi;\ \varphi(t) \circ \mathcal{A}^{m-1}_{h,t}|_{I_m} \in P^q(I_m; S_h^{p,m-1}),\ m = 1, \ldots, M\right\}. \tag{14}$$

This means that if $\varphi \in S_{h,\tau}^{p,q}$, then

$$\varphi\left(\mathcal{A}^{m-1}_{h,t}(X), t\right) = \sum_{i=0}^{q} \vartheta_i(X)\, t^i, \quad \vartheta_i \in S_h^{p,m-1},\ X \in \Omega_{t_{m-1}},\ t \in \overline{I}_m. \tag{15}$$

3.3 Some Notation and Important Concepts

Over a triangulation $\mathcal{T}_{h,t}$, for each positive integer k, we define the broken Sobolev space $H^k(\Omega_t, \mathcal{T}_{h,t}) = \{v;\ v|_K \in H^k(K) \quad \forall K \in \mathcal{T}_{h,t}\}$.

By $\mathcal{F}_{h,t}$ we denote the system of all faces of all elements $K \in \mathcal{T}_{h,t}$. It consists of the set of all inner faces $\mathcal{F}^I_{h,t}$ and the set of all boundary faces $\mathcal{F}^B_{h,t}$. Each $\Gamma \in \mathcal{F}_{h,t}$ will be associated with a unit normal vector $\mathbf{n}_\Gamma$. By $K_\Gamma^{(L)}$ and $K_\Gamma^{(R)} \in \mathcal{T}_{h,t}$ we denote the elements adjacent to the face $\Gamma \in \mathcal{F}^I_{h,t}$. Moreover, for $\Gamma \in \mathcal{F}^B_{h,t}$ the element adjacent to this face will be denoted by $K_\Gamma^{(L)}$. We shall use the convention, that $\mathbf{n}_\Gamma$ is the outer normal to $\partial K_\Gamma^{(L)}$.

If $v \in H^1(\Omega_t, \mathcal{T}_{h,t})$ and $\Gamma \in \mathcal{F}_{h,t}$, then $v_\Gamma^{(L)}$ and $v_\Gamma^{(R)}$ will denote the traces of v on Γ from the side of elements $K_\Gamma^{(L)}$ and $K_\Gamma^{(R)}$, respectively. We set $h_K = \operatorname{diam} K$ for $K \in \mathcal{T}_{h,t}$, $h(\Gamma) = \operatorname{diam} \Gamma$ for $\Gamma \in \mathcal{F}_{h,t}$ and $\langle v\rangle_\Gamma = \frac{1}{2}\left(v_\Gamma^{(L)} + v_\Gamma^{(R)}\right)$, $[v]_\Gamma = v_\Gamma^{(L)} - v_\Gamma^{(R)}$, for $\Gamma \in \mathcal{F}^I_{h,t}$.

3.4 Discretization

Let $t \in (0, T)$ be an arbitrary but fixed time instant. For $u, \varphi \in H^2(\Omega_t, \mathcal{T}_{h,t}), \theta \in I\!R$ and $c_W > 0$ we introduce the following forms

$$a_h(u, \varphi, t) := \sum_{K \in \mathcal{T}_{h,t}} \int_K \beta(u) \nabla u \cdot \nabla \varphi \, \mathrm{d}x$$

$$- \sum_{\Gamma \in \mathcal{F}^I_{h,t}} \int_\Gamma \Big(\langle \beta(u) \nabla u \rangle \cdot \mathbf{n}_\Gamma \, [\varphi] + \theta \, \langle \beta(u) \nabla \varphi \rangle \cdot \mathbf{n}_\Gamma \, [u] \Big) \, \mathrm{d}S$$

$$- \sum_{\Gamma \in \mathcal{F}^B_{h,t}} \int_\Gamma \big(\beta(u) \nabla u \cdot \mathbf{n}_\Gamma \, \varphi + \theta \beta(u) \nabla \varphi \cdot \mathbf{n}_\Gamma \, u - \theta \beta(u) \nabla \varphi \cdot \mathbf{n}_\Gamma \, u_D \big) \, \mathrm{d}S,$$

$$J_h(u, \varphi, t) := c_W \sum_{\Gamma \in \mathcal{F}^I_{h,t}} h(\Gamma)^{-1} \int_\Gamma [u] \, [\varphi] \, \mathrm{d}S + c_W \sum_{\Gamma \in \mathcal{F}^B_{h,t}} h(\Gamma)^{-1} \int_\Gamma u \, \varphi \, \mathrm{d}S,$$

$$A_h(u, \varphi, t) = a_h(u, \varphi, t) + \beta_0 \, J_h(u, \varphi, t),$$

$$b_h(u, \varphi, t) := - \sum_{K \in \mathcal{T}_{h,t}} \int_K \sum_{s=1}^d f_s(u) \frac{\partial \varphi}{\partial x_s} \, \mathrm{d}x$$

$$+ \sum_{\Gamma \in \mathcal{F}^I_{h,t}} \int_\Gamma H(u_\Gamma^{(L)}, u_\Gamma^{(R)}, \mathbf{n}_\Gamma) \, [\varphi] \, \mathrm{d}S + \sum_{\Gamma \in \mathcal{F}^B_{h,t}} \int_\Gamma H(u_\Gamma^{(L)}, u_\Gamma^{(L)}, \mathbf{n}_\Gamma) \, \varphi \, \mathrm{d}S,$$

$$d_h(u, \varphi, t) := - \sum_{K \in \mathcal{T}_{h,t}} \int_K \sum_{s=1}^d z_s \frac{\partial u}{\partial x_s} \varphi \, \mathrm{d}x = - \sum_{K \in \mathcal{T}_{h,t}} \int_K (z \cdot \nabla u) \varphi \, \mathrm{d}x,$$

$$l_h(\varphi, t) := \sum_{K \in \mathcal{T}_{h,t}} \int_K g \varphi \, \mathrm{d}x + \beta_0 \, c_W \sum_{\Gamma \in \mathcal{F}^B_{h,t}} h(\Gamma)^{-1} \int_\Gamma u_D \, \varphi \, \mathrm{d}S.$$

Let us note that in integrals over faces we omit the subscript Γ. We consider $\theta = 1$, $\theta = 0$ and $\theta = -1$ and get the symmetric (SIPG), incomplete (IIPG) and nonsymmetric (NIPG) variants of the approximation of the diffusion terms, respectively. In $b_h(u, \varphi, t)$, H is a numerical flux which is Lipschitz-continuous, consistent and conservative.

For a function φ defined in $\bigcup_{m=1}^M I_m$ we denote

$$\varphi_m^\pm = \varphi(t_m \pm) = \lim_{t \to t_m \pm} \varphi(t) \quad \text{and} \quad \{\varphi\}_m = \varphi(t_m+) - \varphi(t_m-). \tag{16}$$

Now we define an ALE-STDG approximate solution of our problem.

Definition 1 A function U is an approximate solution of problem (7)–(9), if $U \in S_{h,\tau}^{p,q}$ and

$$\int_{I_m} \Big(\big(D_t U, \varphi\big)_{\Omega_t} + A_h(U, \varphi, t) + b_h(U, \varphi, t) + d_h(U, \varphi, t) \Big) \, \mathrm{d}t \tag{17}$$

$$+(\{U\}_{m-1}, \varphi_{m-1}^{+})_{\Omega_{t_{m-1}}} = \int_{I_m} l_h(\varphi, t) \, \mathrm{d}t \quad \forall \varphi \in S_{h,\tau}^{p,q}, \quad m = 1, \ldots, M,$$

$$U_0^- \in S_h^{p,0}, \quad (U_0^- - u^0, v_h) = 0 \quad \forall v_h \in S_h^{p,0}. \tag{18}$$

4 Analysis of the Stability

In the space $H^1(\Omega_t, \mathcal{T}_{h,t})$ we define the norm $\| \cdot \|_{DG,t}$ by the relation $\|\varphi\|_{DG,t}^2 = \sum_{K \in \mathcal{T}_{h,t}} |\varphi|_{H^1(K)}^2 + J_h(\varphi, \varphi, t)$. Moreover, over $\partial\Omega_t$ we define the norm of the Dirichlet boundary condition by $\|u_D\|_{DGB,t}^2 = c_W \sum_{\Gamma \in \mathcal{F}_{h,t}^B} h(\Gamma)^{-1} \int_\Gamma |u_D|^2 \, \mathrm{d}S$.

If we use $\varphi := U$ as a test function in (17), we get the basic identity

$$\int_{I_m} \big((D_t U, U)_{\Omega_t} + A_h(U, U, t) + b_h(U, U, t) + d_h(U, U, t) \big) \, \mathrm{d}t \tag{19}$$

$$+(\{U\}_{m-1}, U_{m-1}^{+})_{\Omega_{t_{m-1}}} = \int_{I_m} l_h(U, t) \, \mathrm{d}t.$$

4.1 Important Estimates

Here we estimate forms from (19) individually. The proofs can be carried out similarly as in [1]. For a sufficiently large constant c_W we obtain the coercivity of the diffusion and penalty terms.

Lemma 2 *Let*

$$c_W \geq \frac{\beta_1^2}{\beta_0^2} c_M (c_I + 1) \quad \textit{for} \quad \theta = -1 \ (\textit{NIPG}),$$

$$c_W \geq \frac{\beta_1^2}{\beta_0^2} c_M (c_I + 1) \quad \textit{for} \quad \theta = 0 \ (\textit{IIPG}),$$

$$c_W \geq \frac{16\beta_1^2}{\beta_0^2} c_M (c_I + 1) \quad \textit{for} \quad \theta = 1 \ (\textit{SIPG}),$$

where constants c_M and c_I are from the multiplicative trace inequality and the inverse inequality, respectively. Then

$$\int_{I_m} A_h(U, U, t)\,\mathrm{d}t \geq \frac{\beta_0}{2}\int_{I_m} \|U\|^2_{DG,t}\,\mathrm{d}t - \frac{\beta_0}{2}\int_{I_m} \|u_D\|^2_{DGB,t}\,\mathrm{d}t. \tag{20}$$

Further, we estimate the convection terms and the right-hand side term:

Lemma 3 *For each $k_1, k_2, k_3 > 0$ there exist constants $c_b, c_d > 0$ such that we have*

$$\int_{I_m} |b_h(U, U, t)|\mathrm{d}t \leq \frac{\beta_0}{2k_1}\int_{I_m} \|U\|^2_{DG,t}\mathrm{d}t + c_b\int_{I_m} \|U\|^2_{\Omega_t}\mathrm{d}t. \tag{21}$$

$$\int_{I_m} |d_h(U, U, t)|\,\mathrm{d}t \leq \frac{\beta_0}{2k_2}\int_{I_m} \|U\|^2_{DG,t}\,\mathrm{d}t + \frac{c_d}{2\beta_0}\int_{I_m} \|U\|^2_{\Omega_t}\,\mathrm{d}t. \tag{22}$$

$$\begin{aligned}\int_{I_m} |l_h(U, t)|\,\mathrm{d}t \leq{}& \frac{1}{2}\int_{I_m} \left(\|g\|^2_{\Omega_t} + \|U\|^2_{\Omega_t}\right)\mathrm{d}t \\ &+\frac{\beta_0 k_3}{2}\int_{I_m} \|u_D\|^2_{DGB,t}\,\mathrm{d}t + \frac{\beta_0}{2k_3}\int_{I_m} \|U\|^2_{DG,t}\,\mathrm{d}t.\end{aligned} \tag{23}$$

Finally we need to estimate the term with the ALE derivative. The proof is based on the Reynolds transport theorem and on (10).

Lemma 4 *It holds that*

$$\int_{I_m} (D_t U, U)_{\Omega_t}\,\mathrm{d}t \geq \frac{1}{2}\left(\|U^-_m\|^2_{\Omega_{t_m}} - \|U^+_{m-1}\|^2_{\Omega_{t_{m-1}}} - c_z\int_{I_m} \|U\|^2_{\Omega_t}\,\mathrm{d}t\right), \tag{24}$$

$$\begin{aligned}&\left(\{U\}_{m-1}, U^+_{m-1}\right)_{\Omega_{t_{m-1}}} \\ &= \frac{1}{2}\left(\|U^+_{m-1}\|^2_{\Omega_{t_{m-1}}} + \|\{U\}_{m-1}\|^2_{\Omega_{t_{m-1}}} - \|U^-_{m-1}\|^2_{\Omega_{t_{m-1}}}\right),\end{aligned} \tag{25}$$

Theorem 5 *There exists a constant $C_T > 0$ such that*

$$\begin{aligned}&\|U^-_m\|^2_{\Omega_{t_m}} - \|U^-_{m-1}\|^2_{\Omega_{t_{m-1}}} + \|\{U\}_{m-1}\|^2_{\Omega_{t_{m-1}}} + \frac{\beta_0}{2}\int_{I_m} \|U\|^2_{DG,t}\mathrm{d}t \\ &\leq C_T\left(\int_{I_m} \|g\|^2_{\Omega_t}\mathrm{d}t + \int_{I_m} \|u_D\|^2_{DGB,t}\mathrm{d}t + \int_{I_m} \|U\|^2_{\Omega_t}\mathrm{d}t\right).\end{aligned} \tag{26}$$

Proof From (19), by virtue of (24), (20), (21), (22), (25) and (23), after some manipulation and choosing $k_1 = k_2 = k_3 = 6$, we get (26) with $C_T = \max\{1, 7\beta_0, c_z + 1 + c_d/\beta_0 + 2c_b\}$. □

4.2 Discrete Characteristic Function

In our further considerations, the concept of a discrete characteristic function will play an important role. Here it is generalized to time-dependent domains.

For $m = 1, \dots, M$ we use the following notation: $U = U(x,t),\ x \in \Omega_t,\ t \in I_m$, will denote the approximate solution in Ω_t, and $\tilde{U} = \tilde{U}(X,t) = U(\mathcal{A}_t(X),t),\ X \in \Omega_{t_{m-1}},\ t \in I_m$, denotes the approximate solution transformed to the reference domain $\Omega_{t_{m-1}}$.

Definition 6 The discrete characteristic function to $\tilde{U}$ at a point $s \in I_m$ is defined as $\tilde{\mathcal{U}}_s = \tilde{\mathcal{U}}_s(X,t) \in P^q(I_m; S_h^{p,m-1})$ such that

$$\int_{I_m} (\tilde{\mathcal{U}}_s, \varphi)_{\Omega_{t_{m-1}}}\,\mathrm{d}t = \int_{t_{m-1}}^{s} (\tilde{U}, \varphi)_{\Omega_{t_{m-1}}}\,\mathrm{d}t \quad \forall \varphi \in P^{q-1}(I_m; S_h^{p,m-1}), \tag{27}$$

$$\tilde{\mathcal{U}}_s(X, t_{m-1}^+) = \tilde{U}(X, t_{m-1}^+),\ X \in \Omega_{t_{m-1}}. \tag{28}$$

Further, we introduce the discrete characteristic function $\mathcal{U}_s = \mathcal{U}_s(x,t), x \in \Omega_t, t \in I_m$ to $U \in S_{h,\tau}^{p,q}$ at a point $s \in I_m$:

$$\mathcal{U}_s(x,t) = \tilde{\mathcal{U}}_s(\mathcal{A}_t^{-1}(x),t),\ x \in \Omega_t,\ t \in I_m. \tag{29}$$

Hence, in view of (14), $\mathcal{U}_s \in S_{h,\tau}^{p,q}$ and for $X \in \Omega_{t_{m-1}}$ we have

$$\mathcal{U}_s(X, t_{m-1}+) = U(X, t_{m-1}+). \tag{30}$$

In what follows, we prove that the discrete characteristic function mapping $U \to \mathcal{U}_s$ is continuous with respect of the norms $\|\cdot\|_{L^2(\Omega_t)}$ and $\|\cdot\|_{DG,t}$.

Theorem 7 *There exist constants $c_{CH}^{(1)}, c_{CH}^{(2)} > 0$, such that*

$$\int_{I_m} \|\mathcal{U}_s\|_{\Omega_t}^2\,\mathrm{d}t \le c_{CH}^{(1)} \int_{I_m} \|U\|_{\Omega_t}^2\,\mathrm{d}t \tag{31}$$

$$\int_{I_m} \|\mathcal{U}_s\|_{DG,t}^2\,\mathrm{d}t \le c_{CH}^{(2)} \int_{I_m} \|U\|_{DG,t}^2\,\mathrm{d}t \tag{32}$$

for all $s \in I_m,\ m = 1, \dots, M$ and $h \in (0, \overline{h})$.

Proof The proof is very long and technical. It is based on three steps. At first, the discrete characteristic function $\mathcal{U}_s$ is transformed to the reference domain, i.e. $\tilde{\mathcal{U}}_s = \mathcal{U}_s \circ \mathcal{A}_t$. In the second step we apply continuity properties from [5] of the discrete characteristic function in the reference (fixed) domain. Finally in the last step we transfer it back to the current configuration. □

Using the definition and properties (31)–(32) of the discrete characteristic function, we can prove the following theorem. The proof is very long and technical.

Theorem 8 *There exist constants* $C, C^* > 0$ *such that*

$$\int_{I_m} \|U\|^2_{\Omega_t} \mathrm{d}t \le C\,\tau_m \left(\|U^-_{m-1}\|^2_{\Omega_{t_{m-1}}} + \int_{I_m} \left(\|g\|^2_{\Omega_t} + \|u_D\|^2_{DGB,t} \right) \mathrm{d}t \right) \tag{33}$$

provided $0 < \tau_m < C^*$.

Finally we arrive to our main result concerning the unconditional stability of the method.

Theorem 9 *Let* $0 < \tau_m \le C^*$ *for* $m = 1, \ldots, M$. *Then there exists a constant* $C_S > 0$ *such that*

$$\|U^-_m\|^2_{\Omega_{t_m}} + \sum_{j=1}^{m} \|\{U_{j-1}\}\|^2_{\Omega_{t_{j-1}}} + \frac{\beta_0}{2} \sum_{j=1}^{m} \int_{I_j} \|U\|^2_{DG,t}\,\mathrm{d}t$$
$$\le C_S \left(\|U^-_0\|^2_{\Omega_{t_0}} + \sum_{j=1}^{m} \int_{I_j} R_t\,\mathrm{d}t \right), \quad m = 1, \ldots, M,\; h \in (0, \overline{h}),$$

where $R_t = (C_T + C\,\tau_j)\,(\|g\|^2_{\Omega_t} + \|u_D\|^2_{DGB,t})$ *for* $t \in I_j$.

Proof The proof is based on (26), (33) and the use of the discrete Gronwall inequality. □

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