Chapter 11
Abstraction, Axiomatization and Rigor: Pasch and Hilbert

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To proceed axiomatically means nothing other than to think with awareness (mit Bewußtsein denken)

Hilbert (1922), 201

Abstract In the late nineteenth century, Pasch made a well known statement concerning the conditions of attaining rigor in geometrical proof. The criterion he offered called not only for the elimination of appeals to geometrical figures, but of appeals to meanings of geometrical terms more generally. Not long after Pasch, Hilbert (and others) proposed an alternative standard of rigor. My aim in this paper is to clarify the relationship between Pasch's and Hilbert's standards of rigor. There are, I believe, fundamental differences between them.

Keywords Rigor · Proof · Pasch · Hilbert · Lambert · Freudenthal, premisory surreption · Abstraction from meaning · Semantic abstraction · Abstraction condition · Axiomatic method · Axiomatizaton · Formalization

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11.1 Introduction

In his 1882 lectures on geometry, Moritz Pasch described and endorsed a standard of rigor for geometrical proof.

> [I]f geometry is to be genuinely deductive, the process of inferring (Process des Folgerns) must be everywhere independent of (unabhängig sein vom) the *sense* (*Sinn*) of geometrical concepts just as it must be independent of figures. It is only *relations* between geometrical concepts that should be taken into account in the propositions and definitions that are dealt with. In the course of the deduction, it is certainly legitimate (statthaft) and useful (nützlich),

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> though *by no means necessary* (*keineswegs nöthig*), to think of the meaning (Bedeutung) of the geometrical concepts involved. In fact, if it is necessary so to think, the gappiness (Lückenhaftigkeit) of the deduction and the insufficiency (Unzulänglichkeit) of the means of proof is thereby revealed, unless it is possible to remove the gaps (Lücke) by modifying the reasoning used.
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> Pasch (1882), 98 (emphases in text)

The part of this statement that will most concern me here is that in which Pasch states that the "process of inferring" in proper geometrical proof, whatever that might most reasonably be taken to be, must be "independent of" the meanings of geometrical terms. I will offer a view of what the notion of independence referred to here comes to. I will further consider what I take to be most significantly at stake in the enforcement of such a condition of independence.

The understanding of Pasch's standard on which I will focus sees it as a restriction on the justification of judgments of deductive inferential validity in geometrical proofs. By implication, it therefore also sees it as a constraint on proper judgements of deductive validity in mathematical proofs more generally.

If we let $\mathcal{C}$ be a sentence and $\mathcal{P}$ a set of sentences in a given mathematical language, we may roughly state this constraint as follows:

> Abstraction Condition: Justification of a judgment that an inference from $\mathcal{P}$ to $\mathcal{C}$ is deductively valid ought not to be based on any judgment whose contents concern the meanings or contents of non-logical expressions that occur in $\mathcal{P}$ or $\mathcal{C}$.[1]

Pasch presented this standard as a standard for rigor, where he seems to have seen this as centering on the attainment of proper justification for our judgments of validity. The featured element of propriety, moreover, was the avoidance of surreptious fillings of deductive "gaps" in inferences judged to be deductively valid. Given that such fillings are paradigmatic cases of failure of rigor, it seems appropriate to refer to Pasch's standard as a standard of rigor.

It also seems right to call it an "abstractionist" standard in as much as the justificative prescission it calls for amounts to a type of abstraction. Pasch's condition requires that the justification of a judgment of deductive validity for a mathematical proof (or for an inference in a mathematical proof) should *abstract away from*—more clearly, perhaps, should prescind from—all justificative appeal to the senses or meanings of non-logical expressions.

Pasch suggested that failure to observe this condition (or another condition to like effect) incurs a non-negligible and avoidable risk of misjudgment of validity—particularly, misjudgment owing to misidentification of the constitutive elements of the inference(s) judged to be valid.[2]

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[1] Roughly speaking, an expression E may be said to *occur* in a class of expressions $\mathcal{K}$ if (i) E is an element of $\mathcal{K}$ or (ii) E is an expression upon whose meaning the meaning of an element of $\mathcal{K}$ depends.

[2] There have also been important alternative conceptions of rigor concerning which failure of rigor is not conceived as it is conceived here. One such conception is what I have elsewhere referred to as *probative* rigor. This is rigor which, roughly speaking, concerns the extent to which everything

Failure of rigor of this type, though it concerns validity and judgements of validity, concerns more as well. Pasch in fact focused most tightly not on misjudgments of validity per se, but on certain reasons for making such misjudgments—namely, those based on misidentifications of the elements (e.g., the premises and/or conclusion) of an inference or proof judged to be valid. To say more exactly what such misidentifications consist in, and what Pasch took them to consist in, are central tasks of this paper.

In this connection, let me begin by noting that Pasch's special concern seems to have been misjudgment of validity due to misidentification of premises. More particularly still, he focused on misjudgments of validity based on failure to recognize the use of illicitly imported premises.[3]

Following tradition, I'll refer to this type of misidentification of premises as *premisory surreption*. For a given inference Inf_{Id} and a given inferring agent R,[4] I take its key elements to be as follows:

(i) Judgment by R that Inf_{Id} is valid.
(ii) Failure of R to recognize that her judgment that Inf_{Id} is valid is based on her taking it to include a premise(s) which, properly speaking,[5] it does not (or ought not to be taken to) include.[6]

Pasch's proposed antidote for the failure mentioned in (ii) was application of the Abstraction Condition. In what follows, this proposal will be my main preoccupation. I will consider, in particular, some similarities and differences between it and another proposal of roughly the same period—namely, the so-called *axiomatic method* of Hilbert and others. I will argue that despite their having important similarities, these two proposals, and their underlying conceptions of rigor, are also importantly different.

Before turning to these matters, I'll present some points of historical background that I hope serve to clarify Pasch's and Hilbert's proposals by putting them into

adverted to in a proof that is in some sense capable of being proved is in fact proved. Both Bolzano and Dedekind, as I read them, advocated probative conceptions of rigor, although the particulars of their conceptions were different. For more on this and related matters, see Detlefsen (2010, 2011).

[3]There are distinctions between different types of inferential failures that should be borne in mind here. Among these is failure to recognize *that* premisory importation has occurred when it has occurred. Related to, though also distinct from, this type of failure is failure properly and correctly to identify the premise(s) imported. These seem to be distinct types of failures though their differences will not feature in what follows.

[4]Conceived as I conceive them, inferring agents include not only those who may devise a given piece of reasoning, but those who, though they may not devise it, nonetheless judge it to be valid.

[5]Proper, that is, for purposes of judging the validity of Inf_{Id}.

[6]What is fundamentally wrong, then, with judging a premisorily surreptious argument to be valid is not that, taken to include its surreptious premises, it is not valid. Rather, it is that premises sufficient to warrant a judgement of validity have not been properly identified or registered as premises.

clearer historical and logical perspective. These include, though they do not center on, a challenge to an influential claim(s) concerning Pasch's priority as an advocate of an abstractionist standard of rigor.

11.2 Background

The Abstraction Condition represents a significant departure from older standards of rigor that importantly influenced the thinking of eighteenth and nineteenth century mathematicians. Chief among these was a standard I have elsewhere referred to as the *presentist* standard.[7]

Like the standard of rigor based on the Abstraction Condition, the presentist standard rested on a conception of rigor which sees it as attainment of a type of gaplessness in reasoning. The gaplessness of concern to the presentist, though, was different in character from that pursued by supporters of semantic abstraction as a standard of rigor.

On the presentist conception, mathematical reasoning—particularly, proof—was regarded as having a *subject* of some type (e.g. a geometrical figure). A proof (or, perhaps better, a proving) was judged to be rigorous to the extent that its subject was gaplessly retained before a prover's mind throughout the course of a proof (or proving) as the *subject of* the various judgments whose deductive arrangement makes up the proof.[8]

Poncelet expressed the core idea of such a view as follows:

> In ordinary geometry, which one often calls synthetic …the figure is described, one never loses sight of it (jamais on ne la perd de vue), one always reasons with quantities and forms that are real (réelles) and existing (existantes), and one never draws consequences which

[7]Cf. Detlefsen (2005), 237, 264–66 and Detlefsen (2010), 176.

[8]There are at least two different ways to understand gapless retention of subject. One is to emphasize a notion of awareness, and to take gapless retention of subject to consist in some type of continuity of the objects of *awareness* of a prover throughout the course of a proof.

Gapless retention of subject might also be conceived along more logical lines. On such a view, proofs would be seen as characteristically having parts—in particular, constituent judgments and inferences. Each of these parts would itself have a subject, and gapless retention of subject throughout the course of a proof would consist in the subjects of the relevant parts of a proof standing in a certain relationship to each other (e.g. being *identical to* or in a relevant sense *continuous with* each other) and to the overall subject of the proof.

Of these two broad understandings of gapless retention of subject, the latter might seem the more appealing. On the surface, at least, it would appear to allow that gapless retention of subject be an objective matter. This may be deceiving, though, in that it is possible that any satisfactory understanding of the central notion of a proof's having a *subject* would have to make use of a subjective element, perhaps in the form of an appeal to a prover's awareness. There may ultimately be no other way to make sense of the idea of a proof's being *about* something that a prover, in order properly to be a prover, must associate with it as its subject.

> cannot be depicted in the imagination (à l'imagination) or before one's eyes (à la vue) by sensible objects (objets sensibles).
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> Poncelet (1822), xxj[9, 10]

There seems to be a tension between abstractionism and presentism. Presentism sees proofs as characteristically having contentual subjects (e.g. geometrical figures), and it takes rigor to consist in some type of *constancy* or *continuity* concerning the subject-bearing parts of a proof throughout its course.[11, 12]

The abstractionist reasoner, on the other hand, seeks detachment from rather than continuous contact with or immersion in the contentual subjects of proofs. More particularly, she requires that no judgment concerning the validity of an inference or proof should depend for its justification on judgments whose contents are even partially constituted by contents of non-logical expressions. Indeed, Pasch and other abstractionists sometimes went farther and advocated practical measures whose intent seems to have been to reduce, at least in particular contexts, the role of geometrical contents in geometrical reasoning. Only in this way, they believed, could the dangers posed to rigor by contentual associations be reasonably managed.

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[9] Despite what this passage may suggest, Poncelet's endorsement of traditional synthetic procedure was qualified. He seems particularly to have had reservations concerning its laboriousness, which he saw as being primarily due to a perceived need for the prover to take things back to rudimentary constructions—or, as he put it, "to reproduce the entire series of primitive arguments from the moment where a line and a point have passed from the right to the left of one another, etc." (*ibid.*).

[10] Presentist standards of rigor seem to have been familiar to writers well before Poncelet's time. My reason for mentioning him is to indicate the influence that such ideas still had on nineteenth century mathematicians.

An older description of presentism, and (some of) its supposed virtues, can be found in Berkeley.

> It hath been an old remark that Geometry is an excellent Logic. And it must be owned, that …when from the distinct Contemplation and Comparison of Figures, their Properties are derived, *by a perpetual well-connected chain of Consequences, the Objects being still kept in view, and the attention ever fixed upon them*; there is acquired a habit of reasoning, close and exact and methodical: which habit strengthens and sharpens the Mind …
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> Berkeley (1734), sec. 2, emphasis added

It should be noted that though Berkeley described a "presentist" conception of rigor in this remark, he did not generally subscribe to such a conception.

[11] By *constancy* of the subject-bearing parts of a proof, I mean constancy or identity of subjects throughout the subject-bearing parts of a proof (or, more exactly, throughout the series of judgments and inferences which together make up a proof).

In speaking of subjectivally *continuous* proof, I mean roughly proof in which the subjects of the subject-bearing parts of a proof are in some sense *continuous with* each other and with the overall subject of the proof, even though they may not be constant. Roughly speaking, continuity in this sense assumes that though the subjects of the subject-bearing parts of a proof may be distinct, the transitions from one to another are in some important way(s) conservative. No clearer formulation of these ideas is necessary for my purposes here.

[12] In mentioning the "course" of a proof here, I am assuming that proofs are characteristically divided, or at least divisible, into stages or steps. Nothing I propose here, though, depends on a particular working out of this idea.

It has been suggested that Pasch had some type of priority as a defender of such a view and standard of rigor. Hans Freudenthal, for example, referred to him as "the father of rigor in geometry" (cf. Freudenthal 1962, 619). And some fifty years before Freudenthal's statement, J.W. Young remarked Pasch's abstractionist emphasis. "The abstract formulation of mathematics", he wrote, "seems to date back to the German mathematician Moritz Pasch." (cf. Young et al. 1911, 51). Later in the same essay he noted the link Pasch saw between abstraction and rigor: namely, that "to be rigorous …an argument must be abstract" (op. cit., 218).

It seems clear, however, that there were clear expressions of such ideas well before Pasch and his writings. An example is J. H. Lambert, who wrote:

> [It] can and must be required that one nowhere in a proof call on the thing itself (auf die Sache selbst berufe) but that the proof should be carried forward symbolically throughout (durchaus symbolisch vortrage)—if this is possible. In this aspect Euclid's postulates are the same as so many algebraic equations which one has before oneself, and from which x, y, z &c will be brought out (herausgebracht) without one's looking back at the thing itself (ohne daß man auf die Sache selbst zurücke sehe).
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> Lambert (1786), 149–150

Lambert made this remark in the context of discussing the question of the derivability of the parallel postulate from the other Euclidean axioms and postulates. He took this to be the question whether the parallel postulate can be "properly derived" (in richtige Folge hergeleitet werden könne) from the other Euclidean postulates, taken in conjunction with what might be other commonly recognized basic propositions (übrigen Grundsätze) Lambert (1786), 149 of Euclidean geometry.[13,14]

Lambert claimed that proper derivation of the parallel postulate from the other basic Euclidean propositions would require derivation which "abstracts" (abstrahiert)

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[13]This suggests that Lambert may have seen properly rigorous proof as allowing not only inclusion of axioms among the legitimate ultimate premises of a proof, but inclusion of other propositions as well—specifically, propositions which were commonly recognized as having a basicness appropriate for use in proofs of the propositions being proved. This suggests a view of proof in which the basic qualification for premises is that they be appropriately more basic than the theorems they're used to prove.

Lambert didn't say in a precise way what he took the salience of such relative basicness to be. It seems sensible enough, though, to allow for the possibility that there be propositions which are axiom-like in certain respects (e.g. their relative evidentness, or their relative evidensory primitivity), but not in others (e.g. their deductive power, or their simplicity). It is also sensible enough to hold that the basic aim of proof is to justify the seemingly less basic by the seemingly more basic to the fullest extent feasible or practicable. On such a view of the aim of proof, a proof which used relatively more basic propositions to justify relatively less basic propositions could be seen as making progress even if the progress made were not that of justificative reduction to the most basic propositions.

[14]Lambert raised a related question as well, namely, whether, supposing the parallel postulate to not be so derivable, it might nonetheless become derivable by adding to the basic Euclidean propositions other propositions which have "the same evidentness" (die gleicher Evidenz hätten) (loc. cit.) as them (i.e. the basic Euclidean propositions). This, however, seems to have been more a comment concerning how to think about the independence of the parallel postulate and its significance than a comment concerning premisory rigor per se.

(loc. cit.) from all "representation and conceivability of the things talked about" (von der Vorstellung und der Gedenkbarkeit der Sache die Rede ist) (Lambert 1786, 155) and which thus proceeds by the application of what are essentially symbolical rules.[15]

In this way, and perhaps only in this way, Lambert suggested, can one adequately guard against surreptitious importation of information (ein *Vitium subreptionis*, Lambert 1786, 156) into—hence failure of rigor of—geometrical proof.

If this is right, Pasch was not the first to propose semantic abstraction as an effective, perhaps even a necessary means of securing rigor in geometrical proof. My purpose in noting this, however, is not to diminish Pasch's importance as an advocate of abstractionist approaches to rigor.

He was neither the first[16] nor the last,[17] even of his day, to express concerns regarding the rigor of Euclid's proofs. This notwithstanding, his discovery of what has come to be known as Pasch's axiom[18] added materially to the perceived urgency of these concerns. In addition, his insistence that rigor requires the avoidance not only of appeals to diagrams, or diagrammatically conveyed contents, but to geometrical contents however conveyed, both strengthened and clarified the place of the Abstraction Condition as a constraint on geometrical proof.

These points having been noted, let me turn now to the questions identified earlier—namely, how, if at all, application of the Abstraction Condition might reasonably be taken to advance rigor, and how such application compares to and differs from application of the so-called *axiomatic* method of Hilbert and others.

11.3 Semantic Abstraction and Premisory Surreption

How is it, exactly, that application of the Abstraction Condition should provide protection against premisory surreption? An historically sensible answer would be: "By mitigating the effects of unrecognized semantically borne psychological association in inference."

According to associationist views, successional form of thinking such as proof (and reasoning more generally) are subject to influences of psychological association. Generally speaking, repeated association of one idea with another, or one proposition with another, increases the likelihood of their co-application (e.g. their being "thought" together, their being affirmed together, etc.), independently of whether such co-application is logically warranted or whether the reasoner is aware of it.

Experiences, thinkings, etc. have contents, and patterns of succession among such mental events not uncommonly induce corresponding associations among their

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[15]There are indications that Lambert took Euclid to have been trying to develop a means of arguing which left no room for thought or judgment concerning things-in-themselves in geometrical reasoning. He saw the axioms as functioning symbolically, not semantically.

[16]Cf. Todhunter (1869).

[17]Cf. Smith and Bryant (1901) and Russell (1902).

[18]On one variation, Pasch's axiom states that, in a plane, if a line that does not pass through a vertex of a triangle intersects one side of it internally (i.e., at a point between vertices of the triangle), it then internally intersects another side and externally intersects the third.

contents. These associations may in turn give rise to affirmations, hypothetizations and such other propositional attitude-takings as may generally be suited for use as premises in proofs.

Tendencies to associate contents, however, generally expose reasoners to premisory surreption in proof by dint of provers' unrecognized co-application of associated premises with recognized premises. The seriousness of such exposure was widely recognized as regards the use of geometrical figures in geometrical reasoning.[19]

Pasch seems to have been concerned with threats to rigor that are posed by the forces of psychological association. He saw these forces as posing a threat to non-diagrammatically presented as well as diagrammatically presented contentual appeals. Accordingly, he proposed a standard of rigor which called for judgments of inferential validity to be "independent" not only of uses of diagrams, but of uses of all appeals to semantical contents of geometrical terms.

My reading of the remark by Pasch quoted in the introductory section thus sees it as supporting not only a broadly logical but a psycho-criteriological understanding of this "independence."[20] On this understanding, in order to certify a putative inference as valid it should not only be logically unnecessary for a reasoner to know or even to be aware of the senses or referents of the non-logical terms that occur in the inference, it should be psychologically unnecessary as well. This at any rate is what I take Pasch's statement of the desired independence of geometrical inference from the meanings of geometrical terms to suggest.

Pasch didn't give specific directions for the practical achievement of such independence, but he seems to have believed that it *is* practically achievable. Somehow and in some sense, he suggested, we schematize inferences in axiomatic reasoning by treating their non-logical terms as "variables" rather than as constants. That is, we treat them as terms which range over or admit of different contents and not as terms that have fixed (or relatively fixed) particular contents.

At the same time, Pasch believed, we come to realize that (i) deductive validity depends only on *relationships between* non-logical terms (and not on the contents of those terms themselves), and that (ii) judgments concerning deductive validity ought only to appeal to such relationships.[21] Therefore, by whatever practical means we may achieve abstraction from the meanings of non-logical terms in our judgments of validity, our doing so is key, in Pasch's view, to minimizing the threat of premisory surreption.

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[19]Here by the "use" of a geometrical figure I mean a justificative appeal to a judgment(s) concerning the properties of said diagram or of the figure(s) it may be taken to represent. The justification of such a judgment is presumably based on some type of "diagrammatic" grasp or examination of the figure involved.

[20]By a broadly logical understanding of this independence, I mean a view according to which to know that a proposition follows deductively from other propositions, it is not (broadly) logically necessary to know or to be in any way aware of senses, referents or images commonly associated with non-logical terms these propositions contain.

[21]This is a way of affirming the traditional idea that, properly speaking, deductive validity ought only to depend on the (logical) *forms* of the premises and conclusion of an inference.

As Pasch saw it, then, practical achievement of rigor requires a kind of psychological discipline in geometrical proof—a discipline aimed at psychological separation of the inferences in geometrical proofs from considerations of the contents of geometrical terms. Pasch seems to have seen the exercise of such discipline as a practically effective means of mitigating the rigor-compromising risks of contentual association.

In light of this, it is perhaps the more remarkable that Pasch did not propose the elimination of *all* appeals to contents in judgments of validity, including, specifically, appeals to the meanings of *logical* terms.

To my mind, that he did not represents an asymmetry in his views concerning the relationship between (attainment of) rigor and appeals to contents in the justification of validity judgements. On the one hand, he took the threat of premisory surreption posed by appeals to the contents of geometrical terms as substantial. On the other hand, he seems to have treated the threat of premisory surreption arising from contentual uses of logical terms as (at least relatively) insubstantial.

This requires explanation, and it suggests that Pasch may have held some such view as the following

> Asymmetry: Contentual use of an expression or figure that is peculiar to or distinctive of reasoning in a given topic- or subject-area τ[22] poses a greater risk of premisory surreption in a proof belonging to τ than does contentual use of an expression or figure that is not peculiar to or distinctive of reasoning in τ.

Supposing that Pasch did hold Asymmetry, or something like it, certain additional premises would also be necessary for justification of his views on rigor and contentual discipline in reasoning. Prominent among these would be a second type of asymmetry claim intended to help articulate what is meant by saying that an expression is "peculiar to" or "distinctive of" reasoning in a given subject-area. Part of the thinking here would presumably be that appeal to logical terms is necessary for reasoning generally and that it does not therefore apply in any peculiar or asymmetic way to any particular area reasoning such as geometry.

My purpose here, however, is not to evaluate or even to analyze Asymmetry. Rather, it is to call attention to a way not taken in the further modern development of the rigor concept and of standards for its attainment. The remarks just made concerning Asymmetry might lead one to expect that post-Paschian development of standards of rigor would follow increasingly fine-grained analyses of surreptive contentual association and means of avoiding it.

This does not seem to describe the post-Paschian development, though, nor even the development from Pasch to Hilbert. Instead of finer analysis of contentual association and of possible means of managing it, there seems rather to have been a basic

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[22]Here, by contentual use of an expression or figure $\mathcal{E}$, I mean, roughly, use of a judgment $\mathcal{J}$ to justify belief in the validity of an inference or proof where the (propositional) content of $\mathcal{J}$ is in part determined by the content of $\mathcal{E}$.

I am also supposing that, to count as being *proper* to a theory or subject-area τ, a use of $\mathcal{E}$ must be thought to apply in some special or distinctive—some asymmetric—way, or to some asymmetric extent, to reasoning belonging to τ.

if also largely unremarked change in the conception of rigor and even the underlying conception of reasoning to which it has been attached. This at any rate is how I propose we approach understanding of Hilbert's mature post-Paschian writings concerning mathematical proof and rigor.

11.4 Axiomatic Reasoning and Rigor in Hilbert

Pasch emphasized abstraction away from non-logical contents together with reliance on judgments of logical form as means of achieving rigor in mathematical proof. Somewhat more accurately, he saw mathematics as having two parts. One was a more "rigid", properly mathematical part exclusively concerned with deduction. The other was a more "pliable" not-properly-mathematical part concerned with provision of material (i.e. basic starting propositions) for deductions.[23]

In Pasch's view, the proofs of properly mathematical geometry were exclusively concerned with deductive relationships between geometrical propositions and not with their truth or evidentness. Rigor seems similarly to have been conceived as avoidance of surreption in judgments of logical or deductive connection.

Hilbert's views of proof and rigor were different. Weyl called attention to what he saw as a chief such difference in his comments on Hilbert's 1927 address to the Hamburg Mathematical Seminar.[24]

He particularly stressed what he took to be a pivotal difference between Hilbert's and Brouwer's views as regards adherence to the traditional contentual conception of proof.[25]

> Before Hilbert constructed his proof theory everyone thought of mathematics as a system of contentual (inhaltliche), meaningful (sinnerfüllte), and evident (einsichtige) truths; this point of view was the common platform of all discussions. …Brouwer, like everyone else, required of mathematics that its theorems be (in Hilbert's terminology) "real propositions", meaningful truths.
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> Weyl (1928), 22[26]

[23] Cf. "Mathematics is a system with two parts that should be distinguished. The first, properly mathematical, part, is focused exclusively on deduction. The second makes deduction possible by introducing and elucidating a series of insights that are to serve as material for deduction."

Pasch (1918), 228

For a useful discussion of this and related ideas of Pasch's see Pollard (2010).

[24] The text of this address was published as Hilbert (1928).

[25] According to this view, a proof is a finite sequence of judgments whose propositional contents are judged to stand in an appropriate deductive relationship to one another. This traditional view of proof, however, was something that Brouwer shared with many a non-intuitionist. Thus, though Hilbert directed his criticism towards Brouwer, it might just as justifiably have been aimed at Frege (cf. Frege 1906, 387), or any of a number of other thinkers of the late nineteenth and early twentieth centuries.

[26] Cf. Weyl (1944), 640, Brouwer (1923), 336 and Brouwer (1928), 490–492 for related statements concerning the traditional view of proof.

Hilbert rejected what Weyl here described as the traditional view as representing a distortion of traditional mathematical practice. Its chief inaccuracy, he believed, was its under-estimation of the importance of non-contentual reasoning to traditional mathematics.

Hilbert did not deny the importance of contentual judgement and proof to traditional mathematics. He insisted only that non-contentual methods also figured importantly in making traditional mathematics the successful science it evidently was.

For present purposes, the salient difference between contentual and non-contentual proof is that the latter, unlike the former, does not require the logical or deductive connection of (the propositional contents of) conclusory judgements with (the propositional contents of) premisory judgements. Rather, at least generally speaking, it requires only the formal or symbolic connection of formulae.[27] Hilbert thus offered the following general description of a new view of mathematical reasoning.

> [I]n mathematics the objects of our thinking are concrete signs (konkreten Zeichen) themselves, whose shapes (Gestalt), according to the conception adopted, are immediately clear and re-cognizable (unmittelbar deutlich und wiedererkennbar). …The propositions (Aussagen) which constitute mathematics are replaced (umgesetzed) by formulae, so that, mathematics proper (die eigentliche Mathematik), becomes a stock of formulae (Bestande an Formeln). …A proof becomes an array of formulas given as such to our perceptual intuition.
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> [I]n my theory [of proof, MD] contentual inference (das inhaltliche Schließen) is replaced by outwardly manifest manipulation of signs according to rules (äußeres Handeln nach Regeln). In this way the axiomatic method attains that reliability (Sicherheit) and perfection that it can and must reach if it is to become the basic instrument of all theoretical research.
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> Hilbert (1928), 2, 4[28,29]

Unlike real or contentual proof, then, with its deductive connection of genuine (i.e., contentual) propositions, Hilbert's ideal proofs featured symbolic expressions connected by applications of rules stated in terms of their (i.e., the expressions') outward appearances.

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[27] Hilbert referred to such processes of reasoning as "formaler Denkprozesse" in later writings (cf. Hilbert 1930, 380).

[28] To be more exact, the view described here was taken to apply to what Hilbert and Bernays later referred to as formal (formale) axiomatic reasoning, a type of axiomatic reasoning they distinguished (cf. Hilbert and Bernays 1934, §1) from contentual (inhaltliche) axiomatic reasoning. "[I]n contentual axiomatics (inhaltlichen Axiomatik)", they said, "the basic relations are taken to be something found in experience or in intuitive conception (anschaulicher Vorstellung), and thus something contentually determined, about which the sentences of the theory make assertions (Behauptungen)." (Hilbert and Bernays 1934, 6.)

In formal axiomatization (formale Axiomatik), on the other hand, "the basic relations are not taken as having already been determined contentually. Rather, they are determined implicitly by the axioms from the very start. And in all thinking with an axiomatic theory only those basic relations are used that are expressly formulated in the axioms." (op. cit., 7).

In his proof-theoretic writings Hilbert sometimes wrote 'axiomatic' where, more strictly speaking, he meant 'formal axiomatic'.

[29] Cf. Hilbert (1926), 177 for one of a number a similar statements by Hilbert.

In Hilbert's view, the history of mathematics had amply illustrated the benefits of using such non-contentual (or ideal) methods in mathematics. He considered these benefits to be, broadly speaking, benefits of simplicity or, perhaps better, efficiency, and he considered them to be considerable enough to warrant development of a general plan for their systematic justification.

Allowing the use of ideal methods in mathematical proof presents a problem concerning rigor, though, and a problem that seems to go quite deep. Rigor, as Pasch conceived of it, was a property taken to apply to contentual inference. More accurately, it was a property taken to apply in the first instance to judgments of validity concerning contentual inferences. A contentual inference was to qualify as rigorous just in case the justification of its validity avoided all premisory (and other relevant types of) surreption.

Such a conception of rigor does not apply even in principle to ideal proofs. The "premises" and "conclusions" of inferences in ideal proofs, generally speaking, are not and do not express propositions. Nor are they intended to.[30]

As a consequence, they do not admit of genuine logical connection or failure of genuine logical connection. Rather, they are formulae whose use in our reasoning consists in their being manipulated according to rules stated in terms of the outward appearances of the expressions to which they are intended to apply.

What becomes of rigor when contentual inference and proof is replaced by formal manipulation of the type just described? Is there a meaningful and important conception of rigor that remains and is capable of serving as an ideal of formal reasoning in something like the way that avoidance of premisory surreption (or, more generally, of logical gaps of all types) serves as an ideal of contentual deductive inference?

I believe there is. In saying this, though, I do not intend to deny that the differences between genuinely logical reasoning and symbolic reasoning are considerable and that they dictate a change in the very conception of rigor. On the new conception, the aim of rigor will no longer be avoidance of premisory surreption and other validity-nullifying gaps in reasoning. Rather, it will be avoidance of deficiencies of explicitness or transparency in formal reasoning.

In Hilbert's view, axiomatic reasoning[31] was intended to avoid just such deficiencies. Formal axiomatic proofs were taken to be concrete objects that are distinguished from each other, and from non-proofs, by outwardly manifest characteristics.

The axioms of a formal axiomatic system, in particular, were supposed to be syntactically rather than semantically specified. In the end, this meant that they were to be elements of reasoning whose use involves (indeed, substantially consists in) their being exhibited. That is, they are elements of reasoning whose use is to be made

[30] I put 'premises' and 'conclusions' in scare-quotes because, in the present case, they are formulae, and not what premises and conclusions have traditionally been taken to be, namely, propositions or propositional attitude-takings (e.g., judgement or hypothesis).

[31] More specifically, what he and Bernays called *formal* axiomatic reasoning.

manifest by their being displayed or exhibited, and whose contributions to reasoning are a function of their use in various formal-manipulatory procedures or activities.[32]

They are not elements of reasoning that are to be identified by giving a formula that *expresses* them, and whose contributions to reasoning are essentially a function of contents (e.g., propositions, or propositional-functions) they express.

In Hilbert's view, since it is formulae rather than propositions that are capable of being exhibited, it is formal rather than contentual proofs that are capable of being fully explicit and, so, fully rigorous, according to a conception of rigor in which rigor is taken to consist in transparency or explicitness of usage. In my view, it is this or something like this explicitness of formal axiomatic reasoning that Hilbert intended to emphasize when, as in the epigraph at the top of this paper, he described axiomatic thinking as thinking with awareness or consciousness.

Proper rigor in our reasoning should guarantee avoidance of logical surreption when our reasoning is of such a type as to include genuine logical inference. Not all our reasoning is reasoning of this type, however. That it is not suggests the need for an adjustment in our understanding of rigor—one which sees it as applying not only to contentual reasoning but to formal or symbolic reasoning as well.

Extended in this way, rigor consists in a type of explicitness—explicitness in which every element of a piece of reasoning, as well as its use within that reasoning, is outwardly manifest. Hilbert believed formalization to be the key to attaining such rigor, regardless of whether the reasoning in question was what Hilbert and Bernays referred to as *contentual* (inhaltlich) axiomatic reasoning or what they generally termed *formal* (formale) axiomatic reasoning. In each case, it was formalization that was supposed to provide for that explicitness or transparency of use on which rigor was taken to fundamentally depend.[33]

[32]Roughly speaking, exhibition in the current sense consists in the presentation (whatever, exactly, that might mean) of a particular concrete expression as an exemplar for other concrete expressions—specifically, expressions whose external features are sufficiently similar to those of the exemplar to qualify them as tokens of the same type as it.

[33]In Hilbert's view, formal axiomatic thinking was not only the "basic instrument of all theoretical research" (Hilbert 1928, 4), it was also a general and pervasive form of human thought.

> In our theoretical sciences we are accustomed to the use of formal thought processes (formaler Denkprozesse) and abstract methods …[But] already in everyday life (täglichen Leben) one uses methods and concept-constructions (Begriffsbildungen) which require a high degree of abstraction and which only become plain through unconscious application of the axiomatic method (nur durch unbewußte Anwendung der axiomatischen Methoden verständlich sind). Examples include the general process of negation and, especially, the concept of infinity.
>
> Hilbert (1930), 380

The last sentence of this remark raises questions concerning how Hilbert might have understood "unconscious" applications of the formal axiomatic method. Would it be possible to unconsciously apply a method of reasoning whose essence is consciousness of its own elements? As a matter of strict logical possibility, the answer would seem to be 'yes.' Whether this represents some other type of incoherence, though, is more difficult to say and something I lack space to consider further here.

11.5 Conclusion

Pasch believed that achievement of rigor in mathematical inference required practical separation of judgments of inferential validity from the mathematical subject-matters with which the premises and conclusions of inferences might be concerned. To achieve this separation, he suggested, every judgment of inferential validity should be justified in complete abstraction from the meanings of the mathematical terms that occur in it. Only by applying such abstraction, he believed, could a reasoner be properly assured that there is no deductive gap between the premises and conclusion of a given piece of reasoning.

Hilbert too adopted a view according to which attainment of rigor in mathematical reasoning requires a type of separation of that reasoning from the contents of mathematical terms that occur in it. If I am not mistaken, though, the separation he envisioned was quite different both in character and in intended purpose from that which Pasch had in mind.

Pasch generally conceived of mathematical theories as what, since the late nineteenth century, have been called *abstract* sciences. On this conception, the axioms of mathematical theories are not taken to be propositions that are intended to characterize determinate pre-axiomatically given classes of objects (e.g., traditionally conceived points or lines) and relations between them. Rather, they are regarded as propositional-schemata (or perhaps propositional functions)[34] which, though perhaps applying to pre-axiomatically foreseen domains, nonetheless characteristically apply to unforeseen domains as well.

Conceived in this schematic way, the axioms of abstract sciences were taken to have only their schematic forms to contribute to proofs in which they occurred. Specifically, they had no propositional contents to contribute to them. If, then, as Pasch suggested, the justification of a judgment that a proof or an inference in a proof is valid were to appear to appeal to the content of a mathematical term, there would be reason to view it (i.e., the justification) with suspicion.

Pasch's commitment to the Abstraction Condition, and to the type of separation from contents that it brought to geometrical reasoning, reflected his view that axiomatic geometries are generally best seen as abstract sciences whose axioms are propositional schemata rather than propositions. This meant in turn that geometrical proofs were characteristically to be seen as finite sequences of items (viz. propositional schemata) whose contributions to the proofs in which they occurred were their schematic forms.

Hilbert, too, particularly in his writings around the turn of the twentieth century (cf. Hilbert 1899, 1900), stressed a conception of axiomatic reasoning according to which the axioms of an axiomatic system are not intended to describe or capture some pre-axiomatically given content, but, rather, to give "an exact (genaue) and, for mathematical purposes, complete (vollständige) specification (Beschreibung)" (Hilbert 1899), ch. 1, §1; (Hilbert 1900, 181) of those elements which may rightly be used without proof in an axiomatic proof.

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[34] Cf. Whitehead (1906), 2, Huntington (1911), §20.

Hilbert didn't expand further in Hilbert (1899, 1900) on the exactness and completeness mentioned in this last remark. In the fuller course of his work (e.g. in Hilbert 1922, 1928), though, he did. The remarks referred to earlier in which he proposed replacing contentual inference (inhaltliche Schließen) with operations on concretely exhibited (konkret ausweisbaren)[35] symbolic expressions according to explicit rules illustrates the development in his understanding of the axiomatic method.

This development is in my view most plausibly seen as the adoption of a new view concerning the nature of rigor—a view which focuses on explicitness or transparency. Judged from this vantage, failures of rigor are fundamentally failures to recognize or to identify elements that ought properly to be seen as belonging to a piece of reasoning. Such failures, in turn, are generally taken to be due to deficiencies of explicitness or transparency in our reasoning.

Axiomatic proof, as Hilbert conceived of it, was intended to protect against such deficiencies by offering the ultimate in explicitness. Axioms were to be identified by their outward shapes, and these shapes, in turn, were to be given by their being exhibited. The idea, if I am right, is that full explicitness in proof can be achieved only through such exhibition. It cannot be achieved by semantical expression. In other words, rigor, or full explicitness in proof, can only be achieved by axiomatization if the axioms of the system are themselves objects which can be exhibited or displayed, and not merely, as with Hilbert's predecessors, propositions, propositional-schemata or other contents taken to be *semantically expressed by* exhibitable objects.

Rather, it requires that axioms be given by being exhibited—that is, by presenting a concrete expression[36] that is identifiable by its outwardly manifest characteristics. An expression given in this way is to serve as an exemplar of similarly shaped concrete expressions that are taken to belong to a given syntactical category of a given formal language.[37]

Hilbert's "decontentualization" of proof—his proposed replacement of propositions and other contentual items which figure centrally in the traditional conception of proof by the formal objects of (his formal conception of) axiomatic proof was thus in his view a transformation that is necessary if the legitimate demands of rigor are generally to be met.

His view is complicated by the fact that, in addition to urging a place for the above-described conception of axiomatic proof and its accompanying conception of rigor, Hilbert continued to see a place in mathematics (and metamathematics) for contentual proof as well. How he may have conceived of rigor for such proof, to what extent he may have taken rigor so conceived to be achievable for this type of proof and how his views on these matters may have compared to and/or contrasted with Pasch's views are matters I will leave for another occasion.

[35] Cf. Hilbert (1928), 1.

[36] By 'expression' here, I mean simply a string of characters in a language. I do not mean that this string serves to express a semantical content of some type.

[37] The need to bring syntactical categories into the picture is necessary in order to distinguish between similarly shaped syntactical objects that belong to different syntactical categories (e.g., a formula considered as a line of a proof versus a similarly shaped object which is taken to constitute a one line proof).

Postscript

In 1990–1991, I coordinated one of the Notre Dame philosophy department's *Perspectives in Philosophy* lecture series. Hilary was one of the speakers I invited. His lectures primarily concerned various of Wittgenstein's ideas on proof.

This led at one point to a discussion concerning "Hilbert's Thesis" and its possible bearing on and/or presuppositions concerning matters of rigor. In his 1984 paper "Proof and experience", Hilary presented Hilbert's Thesis as the claim that "derivability in quantification theory[38] captures the intuitive mathematical notion of *deduction*, just as recursiveness captures the intuitive mathematical notion of *computability*" (cf. Putnam (1984), 32, emphases in text).

Hilary was of the view that the completeness of quantification theory provides strong evidence for Hilbert's Thesis.

> [C]ompleteness is easily explained: a sentence which cannot be derived from given axioms by means of quantification theory doesn't, in fact, *follow* from those axioms. "Doesn't follow" in the very intuitive sense that there is, in fact, a possible structure which can be used to interpret the language in such a way that the axioms come out *true* while the sentence that wasn't derivable comes out *false*. This is very strong evidence for ..."Hilbert's Thesis" ...
>
> Putnam (1984), 31–32 (emphases in text)

For such a view to be plausible, I think, one must adopt a semantical or contentual understanding of deduction or deducibility—that is, an understanding according to which a sentence ϕ is properly said to be deducible from a set of sentences Γ only if the propositions expressed by the elements of Γ logically imply the proposition expressed by ϕ.

I have tried to indicate reasons for doubting that Hilbert held such a view of deduction or deducibility. More accurately, I have tried to indicate why I think Hilbert did not take deducibility to consist in or to be constituted by logical implication. (This does not suggest, of course, that Hilbert would have denied their extensional coincidence.)

Relatedly, I have argued that Hilbert's mature view of rigor was one which took it to consist not in logical gaplessness per se, but in full explicitness as regards the constituent elements of a piece of reasoning. To put it another and, I think, not very surprising way, whether Hilbert would have accepted Hilbert's Thesis—the claim that "derivability in quantification theory captures the intuitive mathematical notion of *deduction*" (loc. cit.)—depends crucially on how one understands the notion of *capture* that figures here.

[38] Hilary glossed the term "quantification theory" as "first-order logic" on p. 31 of Putnam (1984).

References

Berkeley, G. (1734). The analyst; or, a discourse addressed to an infidel mathematician. Wherein it is examined whether the object, principles, and inferences of the modern analysis are more distinctly conceived, or more evidently deduced, than religious mysteries and points of faith, Printed for J. Jonson, London.

Bour, P.-E., Rebuschi, M., & Rollet, L. (Eds.). (2010). *Construction*. London: College Publications.

Brouwer, L. E. J. (1923). Über die Bedeutung des Satzes vom ausgescholessenen Dritten in der Mathematik, insbesondere in der Funktiontheorie. *Journal für die reine und angewandte Mathematik*, *154*, 1–7. English translation in van Heijenoort (1967), pp. 334–345. Page references are to this translation.

Brouwer, L. E. J. (1928). Intuitionistische Betrachtungen über den Formalismus. *Koninklijke Akademie van wetenschappen de Amsterdam, Proceedings of the Section of Sciences*, *31*, 324–329. English translation with introduction in van Heijenoort (1967), pp. 490–492. Page references are to this translation.

Cellucci, C., Grosholz, E., & Ippoliti, E. (Eds.). (2011). *Logic and knowledge*. Newcastle upon Tyne: Cambridge Scholars Publishing.

Detlefsen, M. (2005). Formalism. In Shapiro (2005), pp. 236–317.

Detlefsen, M. (2010). Rigor, Re-proof and Bolzano's Critical Program. In Bour et al. (2010), pp. 171–184.

Detlefsen, M. (2011). Dedekind against intuition: Rigor, scope and the motives of his logicism. In Cellucci et al. (2011), pp. 273–288.

Engel, F., & Stäckel, P. (1895). Die Theorie der Parallelinien von Euklid bis auf Gauss. Leipzig: Teubner.

Frege, G. (1906). Über die Grundlagen der Geometrie, *Jahresbericht der Deutschen Mathematiker-Vereinigung*, *15*, 293–309, 377–403, 423–430. English translation in Frege (1971).

Frege, G. (1971). *Gottlob Frege: On the foundations of geometry and formal theories of arithmetic*, trans. and ed., with an introduction, by Eike-Henner W. Kluge. New Haven: Yale University Press.

Freudenthal, H. (1962). The main trends in the foundations of geometry in the 19th century. In Suppes et al. (1962), pp. 613–621.

Hilbert, D. (1899). *Grundlagen der Geometrie*, Published in *Festschrift zur Feier der Enthüllung des Gauss-Weber Denkmals in Göttingen*. Leipzig: Teubner.

Hilbert, D. (1900). Über den Zahlbegriff. *Jahresbericht der Deutschen Mathematiker-Vereinigung*, *8*, 180–194.

Hilbert, D. (1922). Neubegründung der Mathematik. Erste Mitteilung. *Abhandlungen aus dem Mathematischen Seminar der Hamburgischen Universität*, *1*, 157–77. Page references are to the reprinting in Hilbert (1935), Vol. III.

Hilbert, D. (1926). Über das Unendliche. *Mathematische Annalen*, *95*, 161–190.

Hilbert, D. (1928). Die Grundlagen der Mathematik. *Abhandlungen aus dem mathematischen Seminar der Hamburgischen Universität*, *6*, 65–85. Reprinted in *Hamburger Einzelschriften*, *5*, pp. 1–21. Leipzig: Teubner. Page references are to the latter.

Hilbert, D. (1930). Naturerkennen und Logik. *Die Naturwissenschaften*, *18*, 959–963. Printed in Hilbert (1935), Vol. III. Page references are to this printing.

Hilbert, D., & Bernays, P. (1934). *Grundlagen der Mathematik* (Vol. I). Berlin: Springer. Page references are to Hilbert and Bernays (1968), the second edition of this text.

Hilbert, D. (1932–1935). *Gesammelte Abhandlungen*, three volumes. Bronx: Chelsea Pub. Co.

Hilbert, D., & Bernays, P. (1968). *Grundlagen der Mathematik* (Vol. I, 2nd ed.). Berlin: Springer.

Huntington, E. V. (1911). *The fundamental propositions of algebra*. Long Island University, Brooklyn: Galois Institute Press.

Lambert, J. H. (1786). Theory der Parallelinien I. *Leipziger Magazin für reine und angewandte Mathematik*, *1*(2), 137–164. Reprinted in Engel and Stäckel (1995), pp. 152–176.

Lambert, J. H. (1786). Theory der Parallelinien II. *Leipziger Magazin für reine und angewandte Mathematik*, *1*(3), 325–358. Reprinted in Engel and Stäckel (1995), pp. 176–207.
Pasch, M. (1882). *Vorlesungen über neuere Geometrie*. Leipzig: Teubner.
Pasch, M. (1918). Die Forderung der Entscheidbarkeit. *Jahresbericht der Deutschen Mathematiker-Vereinigung*, *27*, 228–232.
Pollard, S. (2010). 'As if' reasoning in Vaihinger and Pasch. *Erkenntnis*, *73*, 83–95.
Poncelet, J.-V. (1822). *Traité des propriétés projectives de figures*. Paris: Bachelier.
Putnam, H. (1984). Proof and experience. *Proceedings of the American Philosophical Society*, *128*, 31–34.
Russell, B. (1902). The teaching of Euclid. *The Mathematical Gazette*, *2*(33), 165–167.
Shapiro, S. (Ed.). (2005). *Oxford handbook of the philosophy of mathematics and logic*. Oxford: Oxford University Press.
Smith, C., & Bryant, S. (1901). *Euclid's elements of geometry: Books I–IV, VI and XI*. London: Macmillan.
Suppes, P., Nagel, E., & Tarski, A. (Eds.) (1962). *Logic, methodology, and philosophy of science: Proceedings of the 1960 International Congress*. Stanford: Stanford U Press.
Todhunter, I. (1869). *The elements of Euclid for the use of schools and colleges*. London: Macmillan.
van Heijenoort, J. (1967). *From Frege to Gödel: A source book in mathematical logic, 1879–1931*. Cambridge: Harvard University Press.
Weyl, H. (1928). Diskussionsbemerkungen zu dem zweiten Hilbertschen Vortrag über die Grundlagen der Mathematik. *Abhandlungen aus dem mathematischen Seminar der Hamburgischen Universität*, *6*, 86–88. Reprinted in *Hamburger Einzelschriften*, *5*, pp. 22–24, Teubner, Leipzig, 1928. English translation in van Heijenoort (1967).
Weyl, H. (1944). David Hilbert and his mathematical work. *Bulletin of the American Mathematical Society*, *50*, 612–654.
Whitehead, A. N. (1906). *The axioms of projective geometry*. Cambridge: University Press.
Young, J. W., Denton, W. W., & Mitchell, U. G. (1911). *Lectures on fundamental concepts of algebra and geometry*. New York: Macmillan Co.

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