Chapter 5
A Mathematical Model of Peer Instruction and Its Applications

Hideo Nitta

Abstract In this chapter, we review mathematical models of learning. The focus is on the mathematical model of peer instruction (PI) based on the master equation that describes the dynamic change of students' response in PI. In this model, for evaluating the effectiveness of each question for PI, the "peer instruction efficiency (PIE)" is introduced in analogy with the Hake gain. It is shown that, in the simplest approximation, PIE becomes proportional to the relative number of students answering correctly before discussion. The mathematical model is applied to introductory physics courses at a university and a high school. It is found that overall practical data of PIE moderately agree with theory. Application of theoretical results to practical data, such as identifying effective PI questions, is also discussed.

5.1 Introduction

Physics describes the properties of physical substances and interaction among them. Then, human beings as "physical substances" composed of atoms and molecules should, in principle, be described by physics. Of course, at the present stage of physics, it is absolutely a desperate attempt to construct a mathematical theory of dynamics, including learning, for a student. However, it may become possible to describe a learning process for *many students* if we express it with only a few "macroscopic variables." This describing is analogous to statistic-mechanical explanation of macroscopic properties of a gas. Although it is eventually impossible to predict the motion of molecules of the gas, the thermodynamic variables, such as pressure, specific heat, etc., are very well described by theory.

In this chapter, we first introduce a few mathematical models of learning that developed previously. Then we present our mathematical model that describes dynamics of the response of students in peer instruction (PI). In this model, for evaluating the effectiveness of each question for PI, the "Peer-Instruction Efficiency

H. Nitta (✉)
Department of Physics, Tokyo Gakugei University, Tokyo, Japan
e-mail: nitta@u-gakugei.ac.jp

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(PIE)" is introduced in analogy with the Hake gain. It is shown that, in the simplest approximation, PIE becomes proportional to the relative number of students answering correctly before discussion. The mathematical model is applied to introductory physics courses at a university and physics classes at a high school. It is found that overall practical data of PIE moderately agrees with theory. Application of PIE to data analysis is also discussed.

5.2 The Hake Gain

The recent trend in physics education research is to evaluate the effect of teaching in a quantitative manner. This trend originated from the celebrated work by Hake (1998), in which the normalized learning gain (the Hake gain) was introduced for evaluating the students' achievement in the class. The Hake gain is defined by

$$\langle g \rangle = \frac{\langle S_{\mathrm{post}} \rangle - \langle S_{\mathrm{pre}} \rangle}{100 - \langle S_{\mathrm{pre}} \rangle}, \tag{5.1}$$

where $\langle S_{\mathrm{pre}} \rangle$ and $\langle S_{\mathrm{post}} \rangle$ are the class-average scores of the pretest and posttest, respectively. The Hake gain makes numerical comparison of the effectiveness of various teaching methods possible. By using Eq. (5.1), Hake showed that the average value of $\langle g \rangle$ for the interactive engagement classes is more than twice of $\langle g \rangle$ for the traditional lectures. This surprising result not only triggered the worldwide spread of the active learning method in physics classes but also impressed researchers into the importance of quantitative evaluation of teaching methods.

5.3 A Generalized Ising Model of Teaching-Learning Process

Although there have been a lot of interests in evaluating teaching methods quantitatively, only a few mathematical theories of teaching-learning process have been developed. An interesting theory was developed by Bordogna and Albano (2001, 2003) to simulate teaching-learning process using the Monte-Carlo method. Their model, which we call the "BA model", is based on a generalized Ising model that has been used for spin systems, neural networks, and social systems. The BA model assumes that the knowledge of the jth student at time t is given by $\sigma_j(t)$ which satisfies $-1 \leq \sigma_j(t) \leq 1$, where $\sigma_j(t) = 1$ represents perfect knowledge of the target subject considered. The knowledge $\sigma_j(t)$ develops by the effects of "cognitive impact (CI)" acting on the student. The BA model considers three types of CI: $CI^{\mathrm{TS}}(j, t)$, $CI^{\mathrm{SS}}(j, t)$, and $CI^{\mathrm{BS}}(j, t)$. $CI^{\mathrm{TS}}(j, t)$ represents the CI of the teacher on the jth student at time t, $CI^{\mathrm{BS}}(j, t)$ the student-student interaction, and $CI^{\mathrm{BS}}(j, t)$ the bibliography and other sources of information. By performing the Monte-Carlo simulation, Bordogna and Albano showed that the learning achievements become higher for students engaging in collaborating work than for those only attending lectures. This result is consistent with finding by Hake mentioned above. They also showed that the structure of students' groups may influence the achievements. The result of simulation indicated that lower-knowledge students may learn at the expense of their higher-knowledge peers.

5.4 Mathematical Learning Models by Pritchard Et al.

Pritchard et al. developed mathematical models that describe learning processes of tabula rasa, constructivism, and tutoring (Pritchard et al. 2008). Their aim of developing the theory is to provide a quantitative tool that allows the parametrization of measured learning data through the Hake gain, Eq. (5.1). They assumed that a concept inventory (CI) covers the knowledge domain T composed of the known knowledge domain $K_T(t)$ and the unknown knowledge domain $U_T(t)$. The knowledge domain is normalized to satisfy the relation $K_T(t)+U_T(t) = 1$, where t is the amount of teaching or instruction.

In the pure memory model, Pritchard et al. assumed the following simple differential equation:

$$\frac{dU_T(t)}{dt} = -\alpha_m U_T(t), \tag{5.2}$$

where α_m represents the sticking probability of taught things in student's mind for the pure memory model.

In the simple connected model, based on the idea of constructivism that learning occurs by constructing an association between the new knowledge and prior knowledge, Pritchard et al. assumed the logistic differential equation

$$\frac{dU_T(t)}{dt} = -\alpha_c U_T(t) K_T(t), \tag{5.3}$$

which can be solved analytically. They also proposed the combined model of pure memory and constructivism called the connectedness model.

In the tutoring model, it is assumed that knowledge grows in a uniform rate of learning:

$$K_T(t) = \alpha_{tu} t + K_T(0), \tag{5.4}$$

where $K_T(0)$ represents the initial knowledge before tutoring.

Based on the analytic solutions of these differential equations, Pritchard et al. obtained an analytic expression of the Hake gain for each learning model. They compared the analytic expressions with data and found reasonable agreement (Pritchard et al. 2008).

5.5 Mathematical Theory of Peer Instruction

Peer instruction (PI) is one of the simplest methods of instruction for increasing students' engagement in lectures (Mazur 1997). In a PI-based lecture, the teacher presents a concept-oriented multiple-choice question (MCQ) about the material just covered by his/her lecture. Then students are encouraged to discuss the question with their neighbor. Due to its simplicity as well as having no limitations on the class size, PI is suitable for introducing essence of active learning to an old-fashioned lecture without changing the curriculum.

For the success of a PI-based lecture, it is crucial to present an effective MCQ for discussion. A good MCQ should induce intensive discussion among students as well as effectively support students to understand the target concept. For the purpose to evaluate MCQs, it is important to develop a measure of their effectiveness. To obtain such a measure, we try to express the dynamics of peer instruction by a mathematical model.

We consider the following process. First, students in a class are posed a MCQ. Each student considers the MCQ without discussion to give her/his own answer. Then students discuss the MCQ with their neighbor, exchanging ideas about their answers.

Let us first consider the dynamics describing the change of the number of students who answer correctly before and after peer discussion for a MCQ. We define that $\rho_{\mathrm{b}}(q, a)$ and $\rho_{\mathrm{a}}(q, a)$ are the normalized numbers of students choosing the answer a for the MCQ (denoted by q) before and after discussion, respectively, and that $T_{a'a}(q)$ is the transition rate from an answer a to the other one a' by peer discussion. Then $\rho_i(q, a)$ (i = a,b) satisfy the following master equation (Nitta 2010):

$$\rho_a(q, c) - \rho_{\mathrm{b}}(q, c) = -\sum_{d(\neq c)} T_{dc}(q)\rho_{\mathrm{b}}(q, c) + \sum_{d(\neq c)} T_{cd}(q)\rho_{\mathrm{b}}(q, d) \tag{5.5}$$

where c and d represent the correct answer and the incorrect answers, respectively. The left-hand-side of Eq. (5.5) represents the difference of the number of students answering correctly before and after discussion. The first term on the right-hand side (r.h.s.) of Eq. (5.5) represents the normalized number of students who at first choose the correct answer and then, after discussion, change it into one of the incorrect answers. This process is usually called the "outgoing process" in the theory of irreversible statistical mechanics. Similarly, the second term on the r.h.s. of Eq. (5.5) represents the normalized number of students who give incorrect answers before discussion and the correct answer after discussion (i.e., the "incoming process"). Of course, $\rho_a(q, c)$, $T_{dc}(q)$, etc. are extremely simplified functions. We have neglected enormous number of parameters such as the quality of MCQ; students' character, knowledge, and reasoning ability; influence of teachers; literature; quality of peer discussion; and many others. It should be noted that $T_{dc}(q) \neq T_{cd}(q)$, i.e., the detailed balance which is normally assumed in physics problems does not hold here. Indeed, if PI is an effective education method at all, it should result in $T_{dc}(q) \ll T_{cd}(q)$.

One may feel that Eq. (5.5) is rather hypothetical. However, this is not so. We emphasize that Eq. (5.5) is, though phenomenological, an exact equation in that all variables and functions in Eq. (5.5) can be determined by PI data for a MCQ. Indeed, by using an audience response system, called "clicker system," the teacher can collect all responses from students for the MCQ, q, before and after the discussion session. Then, from the response data, one can determine the transition probability, $T_{a'a}(q)$, as well as the normalized number of students, $\rho_{\rm b}(q,a)$ and $\rho_{\rm a}(q,a)$.

Although Eq. (5.5) is exact in the above sense, it cannot predict anything without providing a certain analytical expression for $T_{a'a}(q)$. Naturally, as mentioned before, it is not realistic to derive such an expression from the first principle because it should depend on enormous number of student parameters. Here we take a phenomenological approach for obtaining an analytical expression.

Let us neglect

1. The transition from the correct answer to an incorrect answer, i.e., the first term of the r.h.s. of Eq. (5.5)
2. The dependence of $T_{cd}(q)$ on incorrect answers
3. The MCQ dependence, q

Then we obtain,

$$\rho_{\rm a}(c) - \rho_{\rm b}(c) = T \sum_{d(\neq c)} \rho_{\rm b}(d), \tag{5.6}$$

where T stands for the "average" transition rate from incorrect answers to the correct answer. Using the identity $\sum_{d(\neq c)}\rho_{\rm b}(d) = 1 - \rho_{\rm b}(c)$, which means that "the (*normalized*) number of students giving the incorrect answers" is equal to "the number of all students" minus "the number of students giving the correct answer," we obtain

$$\rho_{\rm a} - \rho_{\rm b} = T(1 - \rho_{\rm b}), \tag{5.7}$$

where now $\rho_{\rm b}$ and $\rho_{\rm a}$ represent simply the normalized number of students giving the correct answer before and after discussion, respectively. At this stage, only one parameter, T, i.e., the transition rate from incorrect answers to the correct answer, is left. Here we further assume that T is the function of $\rho_{\rm b}$: $T = T(\rho_{\rm b})$. Then we may expand T into the power series:

$$T = k_0 + k_1\rho_{\rm b} + k_2\rho_b^2 + \cdots, \tag{5.8}$$

where k_i $(i = 0, 1, 2, \cdots)$ are constants. Under the condition that $\rho_{\rm b} = 1 \rightarrow T = 1$ and $\rho_{\rm b} = 0 \rightarrow \rho_{\rm a} = 0$, we have $k_1 = 1$ and $k_0 = 0$; hence $T = \rho_{\rm b}$. This simple result suggests that the more the number of students answering correctly before discussion increases, the more transition from incorrect answers to the correct answer happens. Using $T = \rho_b$ with Eq. (5.7), we obtain

$$\rho_a - \rho_b = \rho_b(1 - \rho_b), \tag{5.9}$$

For evaluating the effectiveness of PI, let us introduce the "Peer Instruction Efficiency" (PIE) for a MCQ q by

$$\eta(q) = \frac{\rho_a(q,c) - \rho_b(q,c)}{1 - \rho_b(q,c)}. \tag{5.10}$$

Although the definition of PIE looks the same as the Hake gain, they are different in character. The Hake gain represents the overall learning gain for a series of lectures as a whole, whereas PIE represents the efficiency of students' discussion for each MCQ.

Substituting Eq. (5.9) into Eq. (5.10), we obtain the very simple expression for PIE:

$$\eta = \rho_b. \tag{5.11}$$

We will discuss the use of PIE in the next section.

Let us generalize the idea of using the master equation for learning processes. Instead of $\rho_{a,b}(q,c)$, we consider the fraction of students in the correct stage of knowledge c about a concept q at time t, $p(q;c;t)$. Following the idea of Eq. (5.5), we assume that the development of knowledge during the time Δt is described by the master equation

$$\Delta p(q;c;t) = -\sum_{d(\neq c)} T_{dc}(q;t)p(q;c;t) + \sum_{d(\neq c)} T_{cd}(q;t)p(q;d;t), \tag{5.12}$$

where $\Delta p(q;c;t) = p(q;c;t{+}\Delta t) - p(q;c;t)$ and d represent incorrect or blank state of knowledge about the concept q and $T_{cd}(q;t)$ the transition rate of the state of knowledge from d to c during the time interval Δt. If we neglect the "outgoing process" of the knowledge construction, i.e., the first term of the r.h.s. of Eq. (5.12), and make other similar approximations as before, we obtain

$$\Delta p(t) = (\alpha \Delta t)(1 - p(t)) \tag{5.13}$$

where we have omitted the parameters q and c and denoted the transition rate to the correct state of knowledge per unit time as α. Further, we neglect the explicit time dependence of α but assume that α is the function of $p(t)$. Then, by expanding α into the power series and taking the lowest-order terms like Eq. (5.8), we obtain

$$\alpha = k_0 + k_1 p(t), \tag{5.14}$$

where k_0 and k_1 are constants. Substituting Eq. (5.14) into Eq. (5.13) and taking the limit $\Delta t \rightarrow 0$, we obtain the differential equation

$$\frac{dp(t)}{dt} = (k_0 + k_1 p(t))(1 - p(t)), \tag{5.15}$$

which is essentially the same differential equation given by Pritchard et al. for their "connectedness model" (Pritchard et al. 2008). It is worthwhile to note that if we set $k_0 = \alpha_m$, $k_1 = 0$, and $1 - p(t) = U_T(t)$, Eq. (5.15) is reduced to the pure memory model of Eq. (5.2). On the other hand, by setting $k_0 = 0$, $k_1 = \alpha_c$, and $p(t) = K_T(t)$, we recover the simple connected model of Eq. (5.3).

5.6 Applications

Now we compare our theoretical result with class data. In Fig. 5.1, we compare the theoretical curve given by Eq. (5.9) and data of the fraction of correct answer before and after peer discussion. The data was taken from an introductory physics course for the first year students in TGU during the academic years 2009–2011 (Nitta et al.

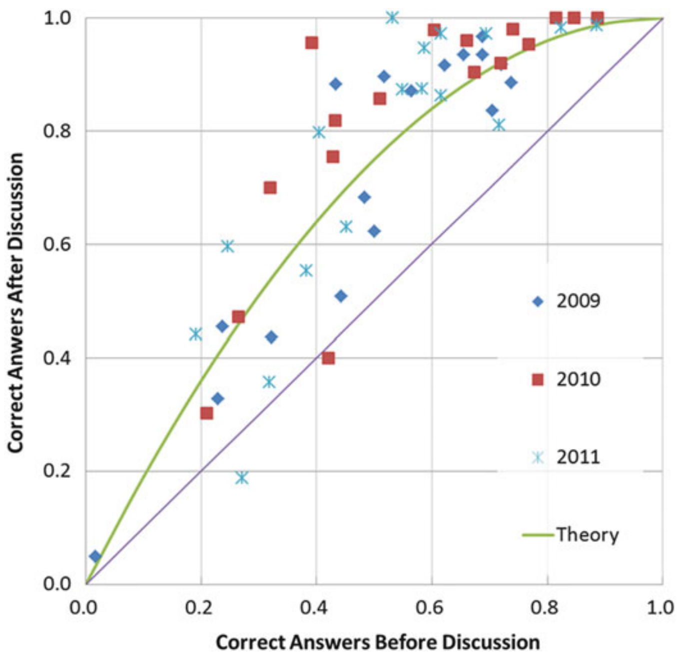

Fig. 5.1 Comparison of theory with data of the fraction of correct answer before and after peer discussion (Nitta et al. 2014)

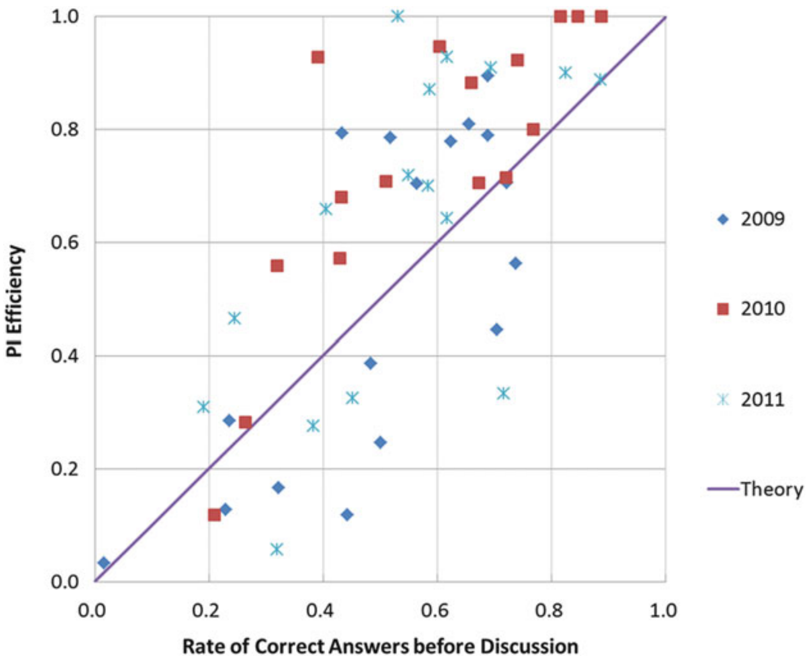

Fig. 5.2 The peer instruction efficiency

2014). The numbers of enrolled students were 60 (2009), 55 (2010), and 81 (2011). Teacher A taught 2009 class while teacher B taught 2010 and 2011 classes. The contents of the courses and MCQs were almost the same between teacher A and teacher B. In Fig. 5.1, data from 17 MCQs in each year were plotted, i.e., 51 plots altogether are shown.

In Fig. 5.2, the same data of Fig. 5.1 are represented in the form of PIE. The straight line corresponds to the simple theoretical expression of PIE, $\eta = \rho_b$. Although the data are dispersed, the theoretical line roughly agrees with the data. This agreement indicates that the approximation $T = \rho_b$ is basically valid. In other words, the transition rate from the incorrect answers to the correct answer by peer discussion tends to increase as the number of students answering correctly before discussion increases.

In Figs. 5.3 and 5.4, we show the data of PI taken from an introductory physics course in TGU High School. Only the data about mechanics are shown. The number of plots is 161, which have been gathered from 2008 to 2016. Each plot represents responses of about 40–120 students for 1–3 classrooms. The best-fit curves in Figs. 5.3 and 5.4 are given by, respectively, $y = -1.2x^2 + 2.3x - 0.069$ and $y = 1.0x - 0.012$. In Fig. 5.4, although the best-fit line agrees well with the theoretical line, data seem to deviate from theory at the region where the rate of correct answers before discussion is low, typically ρ_b is less than 0.2. In this region, almost every point is lower than theoretical line. This indicates that discussion is

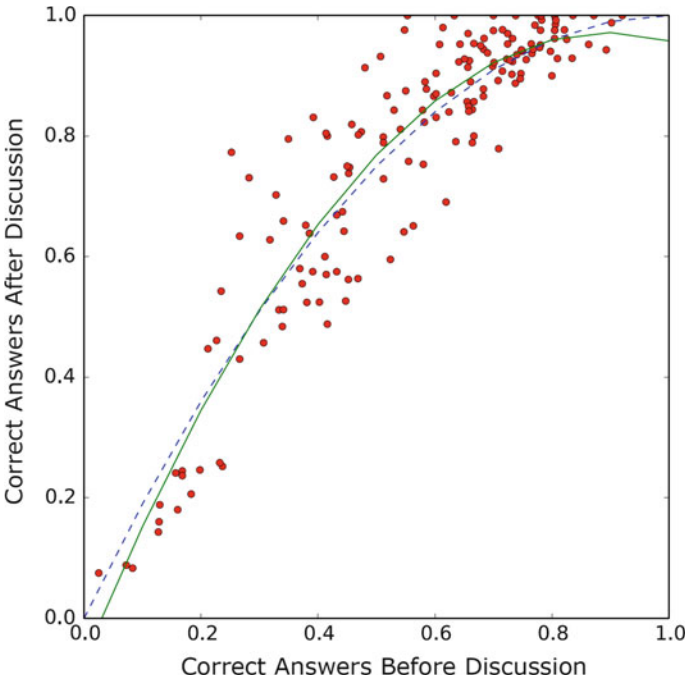

Fig. 5.3 Comparison of theory with data of the fraction of correct answer before and after PI for high school classes. The dashed line and the solid line represent the theoretical curve and the best-fit curve in the quadratic function, respectively

ineffective when there is not enough fraction of students who can lead the discussion into the right direction.

Our aim to introduce PIE has been for evaluating the effectiveness of each PI. We shall try to demonstrate such usage of PIE for the evaluation of PI or MCQs. Figure 5.5 represents the deviation of the PIE data, shown in Fig. 5.2, from the theoretical value. The deviation δ is defined by

$$\delta = \mathrm{PIE}(\mathrm{datum}) - \rho_b, \tag{5.16}$$

The average value and the standard deviation of δ are $\overline{\delta}_{\mathrm{tot}} = 0.062$ and $\sigma_{\mathrm{tot}} = 0.22$, respectively. These values can be used for evaluating the effectiveness of MCQs grouped in specific subjects. For example, let us consider the δ for MCQs about interpretation of kinematics graphs. The frequency distribution of δ for MCQs about interpretation of kinematics graphs is shown in Fig. 5.6. By the one-tailed Welch's *t*-test, the average value of delta, $\overline{\delta}_{\mathrm{graph}} = 0.20$, turns out to be larger than $\overline{\delta}_{\mathrm{tot}}$ with the significance level of $p < 0.05$. In other words, PI for interpretation of kinematic graphs is more effective than other types of PI. We may interpret this result in the following way. The difficulties students have on understanding kinematic graphs are

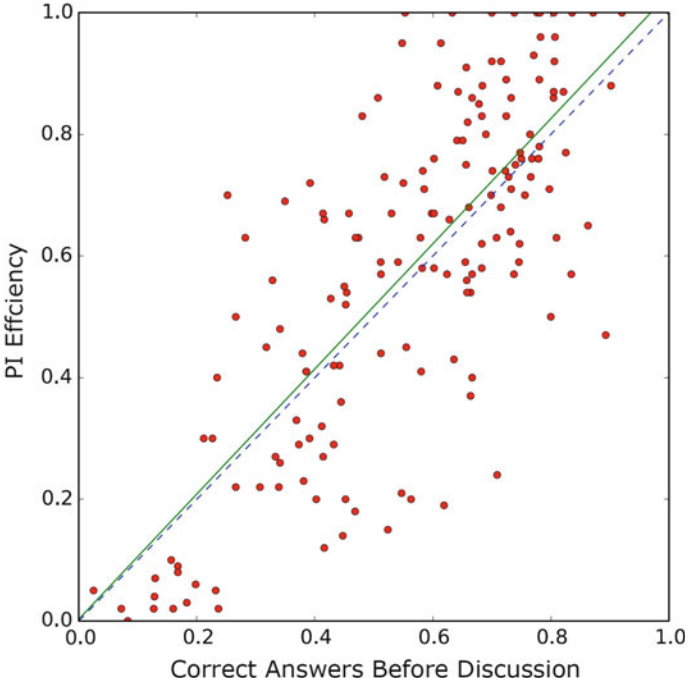

Fig. 5.4 PIE for high school classes. The dashed line and solid line represent the theoretical line and the best-fit line by the linear function, respectively

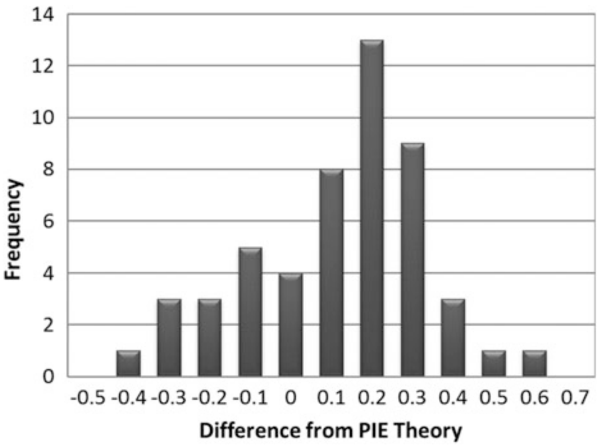

Fig. 5.5 Deviation of PIE data from theory

not conceptual difficulties but rather simple technical problems that can be easily resolved by instruction from other students who have already overcome the technical difficulties. This result indicates that interpretation of kinematic graphs is one of the best subjects for PI in that students' difficulties can be overcome very effectively by

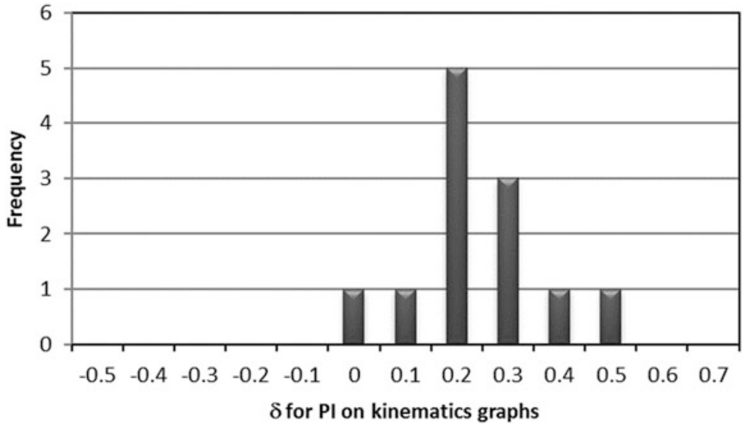

Fig. 5.6 The distribution of δ for PI on interpretation of kinematics graphs

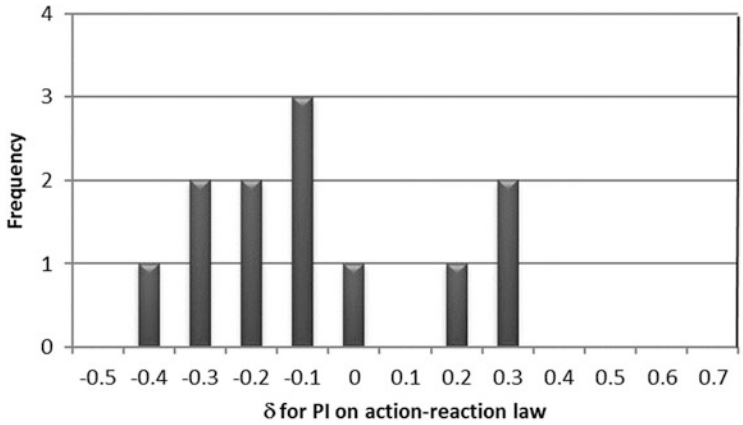

Fig. 5.7 The distribution of δ for PI on action-reaction law

instruction among students. In other words, on this specific subject, instruction by peers will be more effective than that by a teacher. It should be noted, however, that interpretation of graphs on kinematics are rather special than other subjects related to laws of physics, such as graphs on force and motion. Since the relation among position, velocity, and acceleration is mathematical, the origin of students' difficulties would not come from understanding of the law of physics but from "technical difficulties" on interpreting mathematical meaning of graphs.

Another example of the distribution of δ is shown in Fig. 5.7 for MCQs about action-reaction law. In this case, the average value of delta, $\overline{\delta}_{ar} = -0.13$, is significantly lower than $\overline{\delta}_{tot}$ ($p < 0.05$). This result implies that students have so robust naïve conceptions on action-reaction forces that they do not likely to change their beliefs by peer discussion.

5.7 Concluding Remarks

In this chapter, we have presented a mathematical theory of peer instruction (PI) that describes the dynamics of student discussion. The derived simple expression gives a kind of standard line that reasonably agrees with data. We have demonstrated that PIE is useful for rough estimations of the effectiveness of multiple-choice questions (MCQs) for students' discussion. If PIE is larger than the normalized rate of correct answers before discussion, then the MCQ used for the PI turns out to be, roughly speaking, more effective than standard; if smaller, less effective. Although this evaluation is very rough, we find it useful for improving MCQs and lectures in our practice for years.

Finally, we would like to point out that, by combining data of PIE with the results of pre/post concept inventory, such as FCI, one may obtain rich information about students' naïve conceptions as well as the effectiveness of her/his PI-based lectures (Nitta et al. 2014).

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