Interactive Expansion of Achiral Polyhedra

Vera Viana[1(✉)], João Pedro Xavier[2], Ana Paula Aires[3], and Helena Campos[3]

[1] CEAU—Centre of Studies of Architecture and Urbanism, Porto, Portugal
vlopes@arq.up.pt

[2] FAUP—Faculdade de Arquitectura Da, Universidade Do Porto, Porto, Portugal
jxavier@arq.up.pt

[3] UTAD—Universidade de Trás Os Montes E Alto Douro, Vila Real, Portugal
{aaires,hcampos}@utad.pt

Abstract. The representation of geometric concepts of three-dimensional space is a well-identified predicament that can undermine the understanding of geometry in particular, and mathematics, in general. Branco Grünbaum mentioned that no method of presentation is satisfactory for much more than the simplest situations and expressed the hope "that computer-based modes of presentation will alleviate this difficulty in the near future" [1]. Aiming to address specific concepts of polyhedral geometry, we've been exploring a 3D modelling software and its graphical algorithm editor as digital tools to illustrate certain concepts through accurate graphical descriptions that are dynamic and interactive and imply the knowledge of several geometric operations. Among the several possibilities that could illustrate the software potential, we have chosen the concepts of expansion and contraction of polytopes conceived by the Irish mathematician Alicia Boole Stott in 1910 [2], in our opinion, one of the most visually interesting for a dynamic description. For the sake of concision, we will restrict our presentation to two- and three-dimensional polytopes and illustrate the possibilities of dynamically interact with virtual models to visualize, in real-time, the expansion and contraction of regular polygons and uniform convex polyhedra. The purpose of this research is thus to graphically clarify Stott's methods through a dynamic approach made possible with contemporary digital tools and demonstrate how this kind of analysis may simplify further researches on the subject and enhance the didactics of these concepts in particular and, more generally, of polyhedral geometry.

Keywords: Expansion · Contraction · Achiral polyhedra · Algorithmic modelling

L. Cocchiarella (Ed.), *ICGG 2018—Proceedings of the 18th International Conference on Geometry and Graphics*, Advances in Intelligent Systems and Computing 809, pp. 1116–1128, 2019. https://doi.org/10.1007/978-3-319-95588-9_96

1 Digital Tools to Work in Space

To achieve an interactive representation of the concepts devised by Alicia Boole Stott, we have chosen a specific three-dimensional modelling software[1] in articulation with its graphic algorithm editor.[2] The key-features that, in our belief, consistently contribute to the results accomplished, are highlighted in the following:

- the quality of visualization and its accuracy, through which the models are depicted in linear perspective, axonometric and/or orthogonal projection, that can be shown in single or multiple viewports in the graphic area, combined with the possibility to navigate around the models and zoom in to a high power of magnification (and even analyze the models as if cut by a moving clipping plane with any orientation);
- the level of precision modelling and the myriad of geometric operations made possible with the software's 2D and 3D editing tools, that allows the user to accurately work with geometric concepts and transformations as if "directly in space" [3];
- the possibility to visualize and graphically manipulate the parameters and operations involved in the generation algorithm created for each construction,[3] with no need of any knowledge on programming, scripting or source code editing [4];
- the possibility of editing predefined parameters at any stage of the modelling process, without affecting the hierarchy and connections between the elements involved in the construction. Any changes performed in "the chains of scripted graphical transformation result in an immediate visual update" [5] of the active viewport(s).

The last two are characteristic features of the software's algorithmic editor, that perform very differently from, for instance, the "Construction Protocol" of the dynamic geometry software GeoGebra, that might me navigated to add steps to any construction but does not allow editing previous steps. With Stella4D, a powerful software with many interesting features through which three- and four-dimensional polytopes can be visualized, created and manipulated, a movable dynamic expansion of the faces of a polyhedron and the visualization of the resulting polyhedron would not have been possible, given that the necessary geometric operations are not included in its tools [6].

2 Expansion and Contraction of Polyhedra

Drawing inspiration from the "stereographic photographs of semiregular polyhedra sent to her by Schoute" [7], Alicia Boole Stott published in 1910 a research [2] on the expansion and contraction of 2-, 3- and 4-dimensional polytopes (Fig. 1), in which their limits are equally moved away from the center (or contracted inwards), until, for each case, a new uniform polytope is outlined. Coxeter denotes that Stott's expansion e_k is equivalent to the insertion of a ring between the initial and new polytopes, while contraction matches "the removal of a ring" [8]. The n-dimensional version of expansion is known as

[1] Rhinoceros 3D Homepage, https://www.rhino3d.com, last accessed 2018/04/12.

[2] Grasshopper Community Homepage, http://www.grasshopper3d.com, last accessed 2018/04/12.

[3] A "Grasshopper Definition", hereafter referred to as "Definition".

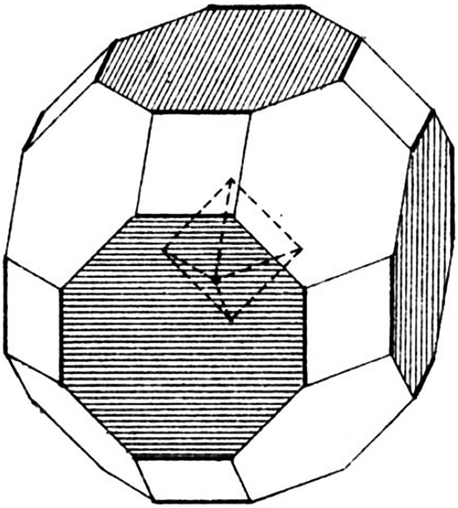

Fig. 1. The rhombitruncated cuboctahedron, derived from a cube, reduced to an octahedron (drawn by A. B. Stott [2] to illustrate an example of combination of inverse operations)

cantellation [9]. For dimension 3, Stott described a method to obtain the achiral Archimedeans that Ball & Coxeter considered as "far more elegant" [10] than Kepler's. The limits (in this case, the faces) of a specific convex uniform polyhedron are uniformly displaced from the centroid, each face in the direction of its normal, until the distance between the new location of consecutive vertices equals the edge's length of the original polyhedron. The expanded faces remain parallel to the former, retaining their original size. The gaps between the new edges that are coplanar are filled by regular faces and, as such, the new polyhedron remains semiregular. No transformation is necessary (as in Kepler's method to construct the rhombic and rhombitruncated Archimedeans[4]) and the eleven achiral Archimedean are constructible from a specific regular, quasirregular or semiregular polyhedron. The two remaining Archimedeans, both enantiomers, are, at first, obtainable from the expansion of the faces of the cube and the regular dodecahedron, respectively, but a slight rotation [12] of the expanded faces is necessary to fill the gaps between the faces with equilateral triangles.

Through contraction, the whole procedure is inverted, as the limits of a given polyhedron are moved "uniformly toward the centre until they meet" [13].

3 Expansion e_1

In dimension 2, the e_1 expansion stands for the edges of a regular polygon being equally moved away from the centre in the direction perpendicular to each side, so that, as Stott denotes, "any regular polygon changes into a regular polygon having the same length of

[4] All the designations for polyhedra follow [11].

edge and twice as many sides" [2]. Figure 2 illustrates the expansion of the edges of any regular n-gon made possible by a Definition (of which a section in shown in the right side) considering $n > 3$. The point (shown as a small sphere, in the left) moves along the line segment that connects the nearest polygon vertex to the intersection between the planes depicted. The left image portrays the situation in which this point is halfway along the path that transforms a regular heptagon into a regular tetradecagon. The number of segments of the base-polygon are selectable through a slider and every resulting polygon confirms Stott's assertion that any $\{n\}$ expands into a $\{2n\}$.

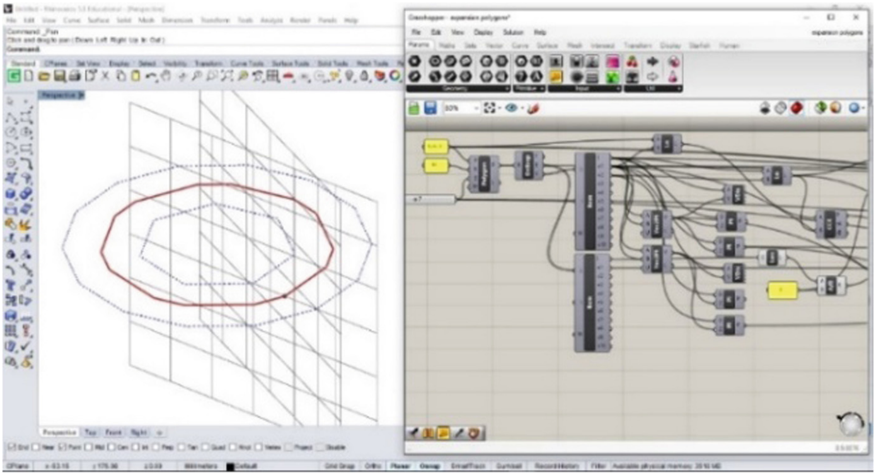

Fig. 2. A regular n-gon expands into a regular $2n$-gon

An adaptation of this Definition allowed us to verify how the edges' expansion work for regular n-gons of density $1 < d < n/2$ in which n and d are coprime [14], in other words, regular star-gons, identified with the Schläfli symbol $\{n/d\}$. As Fig. 3 demonstrates for a $\{5/2\}$ and a $\{7/2\}$, regular pentagrams and heptagrams respectively expand into regular decagons and tetradecagons, confirming that any $\{n/d\}$ expands into a regular polygon of density 1, $2n$ sides and edge length equal to the side of $\{n/d\}$. Figure 3 shows the initial and final situations and the stage in which the movable point is half-way through its path.

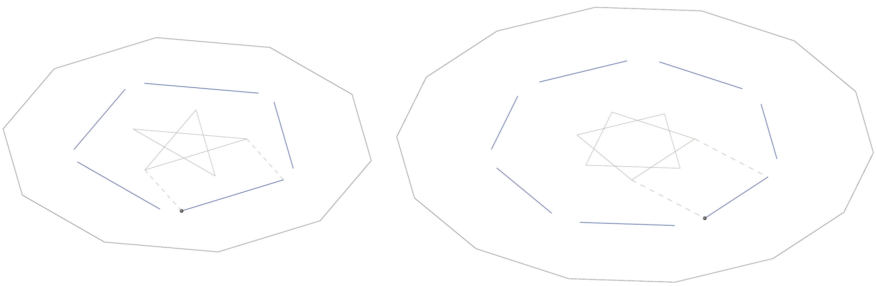

Fig. 3. A regular star-gon expands into a regular convex polygon

In the following, we analyse the results of the expansion e_1 in dimension 3 with convex regular polyhedra, whose edges expand to outline faces with twice as many sides, as we have seen. Every vertex will thus transform into a face with the configuration of the corresponding vertex-figure and, consequently, the base-polyhedron into its semiregular truncated version.

Much the same way as Stott, we begin with the expansion of the edges of a cube, whose faces transform into regular octagons (Fig. 4, left). Through translation of these to a suitable position (centre), a semiregular truncated version of the cube is obtained (right) and the vertices of {4,3} transform into triangular faces. Through our Definition, it is not only possible to dynamically see the transformation of the cube, but to conclude that the distance between the edges of the cube and the closest parallel edges of the truncated cube equals the edge length. This distance is illustrated (right), with the dashed lines parallel to one of the 2-fold symmetry axis.

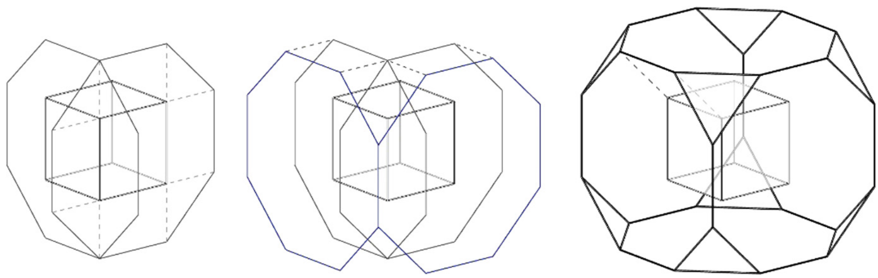

Fig. 4. A truncated cube results from the expansion of the edges of a cube

Figure 5 shows three situations in which the edges of a regular tetrahedron are expanded in the direction of the corresponding 2-fold symmetry axes. This expansion has a semiregular truncated tetrahedron (right image) as result if, and only if, the distance between the closest parallel edges equals the tetrahedron's bimedian, equivalent to the diameter of the mid-sphere of the {3,3}.

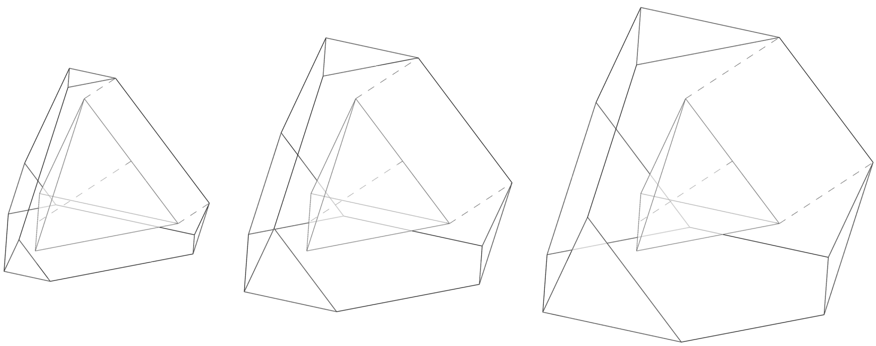

Fig. 5. A truncated tetrahedron results from the expansion of the edges of a tetrahedron

When the edges of a regular octahedron are expanded in the direction of its 2-fold axes, the octahedron transforms into a semiregular truncated octahedron, if the distance between the closest parallel edges equals the edge length. Figure 6 (left) highlights this congruence through the edges of a triangular cupola (Johnson Solid *J*3) whose hexagonal base coincides with a face of the truncated octahedron. The octahedron's faces transformed into regular hexagons and its vertices into squares (represented by its edges in the right image), the corresponding vertex figures of the {3,4}.

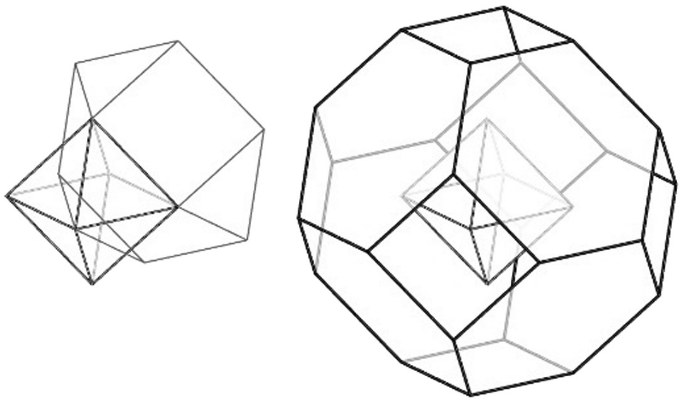

Fig. 6. A truncated octahedron results from expansion of the edges of an octahedron

Figure 7 demonstrates that the expansion of the edges of the regular dodecahedron (left) and the regular icosahedron (right) in the direction of the corresponding 2-fold symmetry axis transforms them into their semiregular truncated versions. The dodecahedron's faces and vertices transform into decagons and triangles; and the icosahedron's, into hexagons and pentagons. Each Definition allowed us to conclude that the dashed line representing the distance between the edges of the {5,3} and the closest parallel edges of its truncated version equals the edge length multiplied by the golden ratio. As such, the distance between the initial and expanded edges is in the golden ratio to the diameter of the dodecahedron's mid-sphere. The corresponding distance between the {3,5} and the truncated icosahedron's edges matches the diameter of the icosahedron's mid-sphere, that is to say, the distance between the midpoints of opposed edges.

Through Table 1, that summarizes the cases that we have seen, we conclude that every expanded $\{p,q\}$ is a $t\{p,q\}$ or, in Cundy & Rollett's symbols (that slightly modify Schläfli's) [15], every expanded p^q is a $q.(2p)^2$ (with vertex configuration $q.2p.2p$).

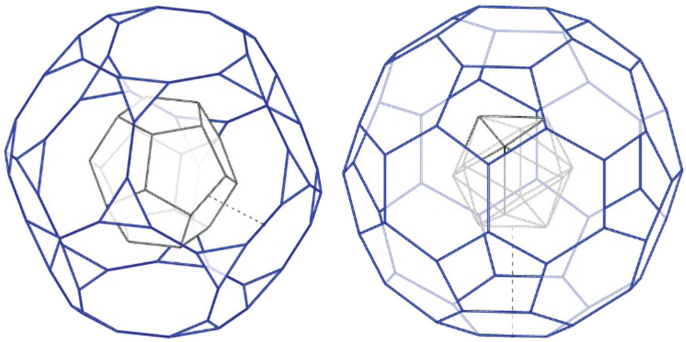

Fig. 7. A truncated dodecahedron and a truncated icosahedron result, respectively, from the expansion of the edges of a dodecahedron (left) and an icosahedron (right)

Table 1. Uniform expansion of the edges of convex regular polyhedra (with unit edge length)

Regular polyhedron		Polyhedron resulting from the edges' expansion		Distance between the closest parallel edges
Tetrahedron	3^3	Truncated Tetrahedron	3.6^2	Diameter of the mid-sphere
Cube	4^3	Truncated Cube	3.8^2	1
Octahedron	3^4	Truncated Octahedron	4.6^2	1
Dodecahedron	5^3	Truncated Dodecahedron	3.10^2	$\frac{1+\sqrt{5}}{2}$ (golden ratio)
Icosahedron	3^5	Truncated Icosahedron	5.6^2	Diameter of the mid-sphere

4 Expansion e_2

Through the e_2 expansion in dimension 3, the faces of a given polyhedron expand until a new uniform polyhedra is obtained. The example we begin with is the regular tetrahedron (Fig. 8). Every face is expanded in the direction of the corresponding 3-fold symmetry axis (coincident with the face's normal), so that every vertex transforms into a triangle (similar to the corresponding vertex figure) and every edge, into a square. Figure 8 (left) shows that the location of one of the expanded vertices is geometrically determined by the intersection of the normal to a given face of the 3^3 with the plane perpendicular to the edge opposed to that vertex. With the determination of this point and the translation of the corresponding face, we outlined the remaining faces through a combination of geometric transformations involving rotational symmetry and reflection. The right image shows the quasiregular cuboctahedron resulting from the expansion. The distance between the triangular faces matches the radius of the tetrahedron's circum-sphere which, for a tetrahedron with unit edge length, is equivalent to $\sqrt{\frac{3}{8}}$.

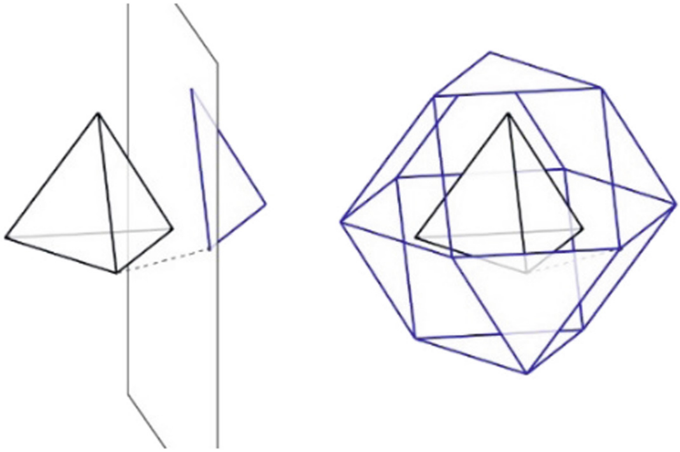

Fig. 8. A cuboctahedron results from the expansion of the faces of a tetrahedron

The uniform expansion of the faces of a cube leads to the semiregular rhombicuboctahedron, provided that the distance between the faces of the 4^3 and the expanded faces of the 3.4^3 equals half of the face's diagonal, specifically, $\frac{1}{2}\sqrt{2}$ (for a cube with unit edge length), as the dashed segments and arc in Fig. 9 (left) denote.

The expansion of the faces of the cube's dual leads us to a semiregular rhombicuboctahedron as well (Fig. 9, right), as "the number of vertices lying in a face of one being equal to the number of faces meeting in a vertex of the other" [2:7]. The distance between the triangular faces of both polyhedra corresponds to the altitude of the triangular faces, which, for an equilateral triangle of n edge length, is $\frac{n}{2}\sqrt{3}$. Needless to say, the dual pair illustrated in Fig. 9 (and Fig. 11) do not share the same reciprocating sphere.

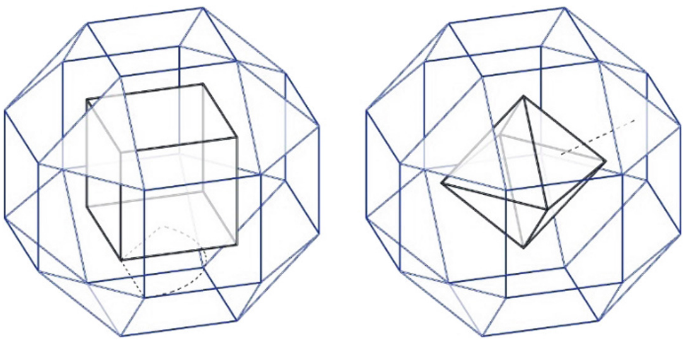

Fig. 9. A rhombicuboctahedron results from the expansion of the faces of a cube or an octahedron. 3.4^3 is therefore recognizable as an expanded cube or an expanded octahedron

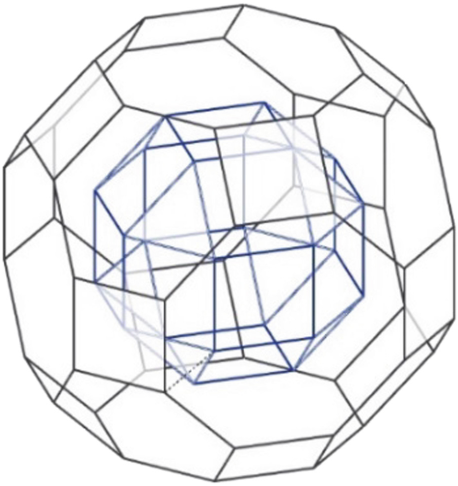

Fig. 10. A rhombitruncated cuboctahedron results from expanding the square faces of a rhombicuboctahedron

The expansion of the square faces of the rhombicuboctahedron that are not perpendicular to the 4-fold symmetry axes produces the semiregular rhombitruncated cuboctahedron, if the distance between the square faces of both equals the edge length (Fig. 10). As such, twelve cubes fit between the parallel square faces of both polyhedra.

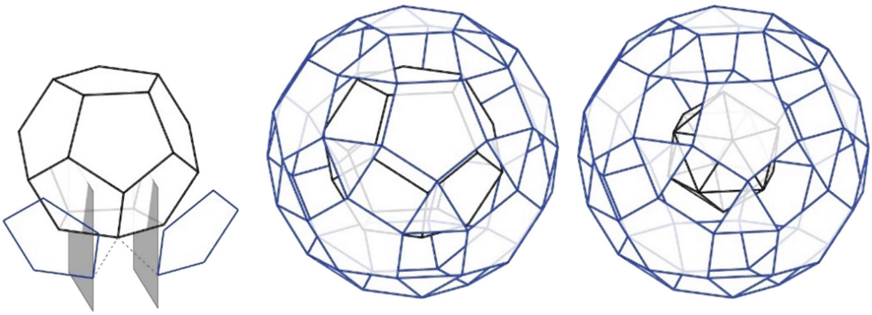

Fig. 11. A rhombicosidodecahedron results from the expansion of the faces of a dodecahedron or an icosahedron. As such, the 3.4.5.4 is an expanded dodecahedron or an expanded icosahedron

Further expansion of the rhombitruncated cuboctahedron's faces does not yield a uniform polyhedron, since its vertex-figures are cyclic irregular polygons [16: 404].

Figure 11 shows that the expansion of the faces of a regular dodecahedron and a regular icosahedron (duals of each other) have, both, a semiregular rhombicosidodecahedron as result. In the left, the procedure for the determination of the distance between the initial and expanded faces (similar to the one shown in Fig. 8) is demonstrated. The remaining faces of the rhombicosidodecahedron (in the center) are determined by a series of geometric transformations involving reflection and rotational symmetry, through which the thirty edges of the 5^3 are transformed into the thirty square faces of the 3.4.5.4, and the twenty triangular faces of the latter are the result of the expansion of the dodecahedron's vertices. In the expansion of the 3^5, however, the edges and vertices transform, respectively, into square faces and into pentagonal faces.

In our Definition, in which both polyhedra have unit edge length, the distance between the initial faces of the dodecahedron and the expanded (pentagonal) faces is 0.951056516295; and the distance between the faces of the icosahedron and the expanded (triangular) faces is 1.401258538444. Interestingly enough, the distance mentioned for the dodecahedron matches exactly the radius of the circum-sphere of the icosahedron; while the distance for the icosahedron equals the radius of the dodecahedron's circum-sphere.[5]

Expanding the square faces of the rhombicosidodecahedron has a semiregular rhombitruncated icosidodecahedron as result (Fig. 12), provided that the distance between the square faces equals the edge length multiplied by the golden ratio. As such, the distance between the square faces equals the diagonal of the pentagonal faces. Figure 12 depicts the situations in which the point that controls the distance between the closest parallel square faces is halfway through (left) before the end (right). For a better understanding of the models, only the square faces of the expanded polyhedron have been represented.

As with the 4.6.8 in Fig. 10, the faces of the 4.6.10 are not expandable onto other uniform polyhedron, not only because their vertex figures are irregular triangles, but because the remaining uniform polyhedra with convex octagonal or decagonal faces[6] [16: 434–437] are not obtainable through uniform expansion of these Archimedeans.

As Stott did [2: 7], we are now able to conclude that uniformly expanding the faces of any regular polyhedron produces the three types of faces of the expanded polyhedron:

1. the expanded faces, parallel to those of the regular polyhedron;
2. the square faces that result from the two new positions of each edge of the regular polyhedron;
3. the faces that have the configuration of the vertex figure of the regular polyhedron and correspond, so to speak, to its vertices expansion.

As Figs. 10 and 12 demonstrate, certain faces of the achiral Archimedeans are expandable to obtain other Archimedeans.[7] Expanding faces of a certain type produces a specific polyhedron and expanding faces of another type produces another

[5] These distances confirm the displacements of $\sqrt{\frac{1}{8}\left(5+\sqrt{5}\right)}$ and $\sqrt{\frac{3}{8}\left(3+\sqrt{5}\right)}$ between the initial and expanded faces mentioned in [17], as well as the formulas for the circumradius of the regular icosahedron [18] and the regular dodecahedron in [19].

[6] Only three concave uniform polyhedra with octagonal faces exist (models number 69, 79 and 86 in [11]) and nine with decagonal faces (72, 74, 75, 82, 84, 89, 90, 91 and 98 in [11]).

[7] Although the distance between the initial and expanded faces might be determined geometrically, further researches might find [17] of great interest, since the distances between the initial faces of the Platonic or Archimedean solids and the resulting faces are summarized and, for each case, the faces to expand (or contract) specified.

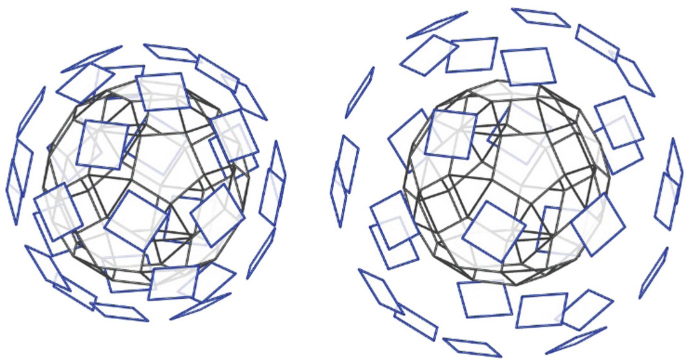

Fig. 12. A rhombitruncated icosidodecahedron results from expanding the square faces of a rhombicosidodecahedron

polyhedron. For instance, if we expand the triangular faces of a quasiregular cuboctahedron with unit edge length until the distance between the initial and expanded faces equals the triangle's height, $\frac{1}{2}\sqrt{3}$, a semiregular truncated cube is obtained. If, on the other hand, we expand the square faces as far as the distance of half the square diagonal, $\frac{1}{2}\sqrt{2}$, the resulting polyhedron is the semiregular truncated octahedron. Furthermore, if the hexagonal faces of the latter are expanded with $\frac{1}{2}\sqrt{3}$ of distance in between, the rhombitruncated cuboctahedron is obtained.

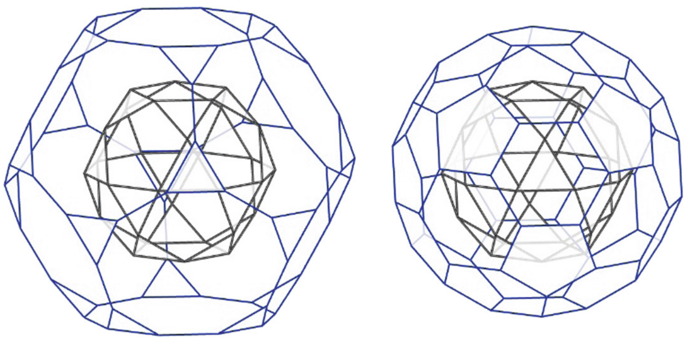

Fig. 13. A truncated dodecahedron and a truncated icosahedron result from the expansion of, respectively, the triangular faces or the pentagonal faces of an icosidodecahedron

Figure 13 demonstrates that the expansion of the triangular faces of the icosidodecahedron produces the semirregular truncated dodecahedron (left), while the expansion of the pentagonal faces produces the semirregular truncated icosahedron (right). In the first situation, the distance between parallel triangular faces corresponds to the circumradius of a dodecahedron with the same edge length as the icosidodecahedron. In the second situation, the distance between parallel pentagonal faces matches the radius of the circum-sphere of an icosahedron with similar edge length. As we have already seen, these distances are respectively equivalent, for an icosidodecahedron of unit edge length, to $\sqrt{\frac{3}{8}\left(3+\sqrt{5}\right)}$ and $\sqrt{\frac{1}{8}\left(5+\sqrt{5}\right)}$.

Last but not least, we present an example of contraction of a semiregular truncated tetrahedron and the regular octahedron obtained within (Fig. 14, left). The distance between parallel triangular faces is the same through which, in Fig. 8, we expanded a tetrahedron of unit edge length to obtain the truncated tetrahedron—specifically, the circumradius of the tetrahedron, $\sqrt{\frac{3}{8}}$, although the direction now is the opposite, that is to say, inwards. If, on the other hand, the hexagonal faces of the truncated tetrahedron are expanded outwards considering the same distance, a semirregular truncated octahedron will be obtained (Fig. 14, right).

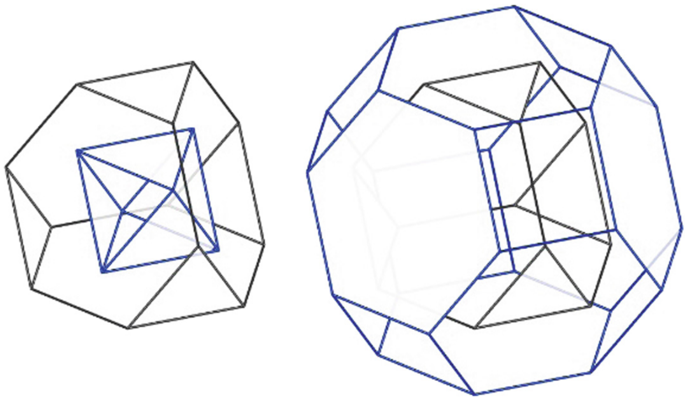

Fig. 14. An octahedron results from the contraction of the triangular faces of a truncated tetrahedron. A truncated octahedron results from the expansion of its hexagonal faces

To conclude, and as the examples presented aimed to demonstrate, the algorithmic editor of the software selected proved to be especially adequate for this research, being fully capable of handling the more complex situations without losing any of its characteristic, to which the accuracy is of the outmost importance. Further researches will allow us to systematize the results obtained and dynamically model more situations to understand the expansion of concave uniform polyhedra.

Acknowledgements. This assignment is co-financed by the European Regional Development Fund (ERDF) through the COMPETE 2020—Operational Programme Competitiveness and Internationalization (POCI) and national funds by the FCT under the POCI-01-0145-FEDER-007744 project.

References

1. Grünbaum, B.: Acoptic polyhedra. Contemp. Math. Adv. Discrete Comput. Geom. 163–199 (1999). https://doi.org/10.1090/conm/223/03137
2. Stott, A.: Geometrical deduction of semiregular from regular polytopes and space fillings. Verhandelingen der Koninklijke Akad. Wetenschappen Amsterdam **11**, 3–24 (1910)
3. Jakobsen, I., Matthiasen, J.: Descriptive geometry and/or computer technology? What mathematics is required for doing and understanding architecture? Nexus Netw. J. **16**(2), 505–516 (2014). https://doi.org/10.1007/s00004-014-0199-3
4. Rhino 6 Features Homepage. https://www.rhino3d.com/6/features. Last accessed 2018/04/19
5. Lagios, K., Niemasz, J., Reinhart, C.: Animated building performance simulation (ABPS)—linking rhinoceros/grasshopper with radiance/daysim. In: Proceedings of SimBuild 2010, International Building Performance Simulation Association, New York City (2010)
6. Webb, R.: Stella: polyhedron navigator. Symmetry: Cult. Sci. **11**(1–4), 231–268 (2003)
7. Polo-Blanco, I.: Theory and History of Geometric Models, pp. 152–158. S.l.: S.n. (2007)
8. Coxeter, H.: Regular Polytopes. Dover Publications, New York (1973)
9. Weisstein, E.: Cantellation. From mathworld—a wolfram web resource. http://mathworld.wolfram.com/Cantellation.html. Last accessed 2018/05/07
10. Ball, W., Coxeter, H., Ball, W.: Mathematical recreations and essays. Dover Publications, New York (1987)
11. Wenninger, M.: Polyhedron Models. University Press, Cambridge (1971)
12. Ostermann, A., Wanner, G.: Geometry by Its History. Springer, Berlin, Berlin (2014)
13. Polo-Blanco, I.: Alicia Boole Stott, a geometer in higher dimension. Historia Math. **35**(2), 123–139 (2008). https://doi.org/10.1016/j.hm.2007.10.008
14. Grünbaum, B.: Regular polyhedra—old and new. Aequationes Math. **15**(1), 118–120 (1977). https://doi.org/10.1007/bf01837888
15. Cundy, H., Rollett, A.: Mathematical Models. Tarquin Publications, Norfolk (1989)
16. Coxeter, H., Longuet-Higgins, M., Miller, J.: Uniform polyhedra. Phil. Trans. R. Soc. A: Math. Phys. Eng. Sci. **246**(916), 401–450 (1954). https://doi.org/10.1098/rsta.1954.0003
17. Weisstein, E.: Expansion. From mathworld—a wolfram web resource. http://mathworld.wolfram.com/Expansion.html. Last accessed 2018/05/07
18. Weisstein, E.: Regular Icosahedron. From mathworld—a wolfram web resource. http://mathworld.wolfram.com/RegularIcosahedron.htm. Last accessed 2018/05/13
19. Weisstein, E.: Regular Dodecahedron. From mathworld—a wolfram web resource. http://mathworld.wolfram.com/RegularDodecahedron.html. Last accessed 2018/05/13