Chapter 2
The Universe in Expansion

Oras ubicumque locaris extremas, quaeram: quid telo denique fiet?
(wherever you shall set the boundaries, I will ask: what will then happen to the arrow?)

Lucretius, De Rerum Natura

We introduce in this chapter the geometric basis of cosmology and the expansion of the universe. A part from the technical treatment, historical, theological and mythological introductions to cosmology can be found in Ryden (2003) and Bonometto (2008).

2.1 Newtonian Cosmology

In order to do cosmology we need a theory of gravity, because gravity is a long-range interaction and the universe is pretty big. Electromagnetism is also a long-range interaction, but considering the lack of evidence that the universe is charged or made up of charges here and there, it seems reasonable that gravity is what we need in order to describe the universe on large scales.

Which theory of gravity do we use for describing the universe? It turns out that Newtonian physics works surprisingly well! It is also surprising that attempts of doing cosmology with Newtonian gravity are well posterior to relativistic cosmology itself.

In particular, the first work on Newtonian cosmology can be dated back to Milne and McCrea in the 1930s (McCrea and Milne 1934; Milne 1934). These were models of pure dust, while pressure was introduced later by McCrea (1951) and Harrison (1965). More recently the issue of pressure corrections in Newtonian cosmology has been tackled again in Lima et al. (1997), Fabris and Velten (2012), Hwang and Noh (2013) and Baqui et al. (2016).

O. Piattella, *Lecture Notes in Cosmology*, UNITEXT for Physics,
https://doi.org/10.1007/978-3-319-95570-4_2

Newtonian cosmology works as follows. Imagine a sphere of dust of radius r. This radius is time-dependent because the configuration is not stable since there is no pressure, thus $r = r(t)$. We assume homogeneity of the sphere during the evolution, i.e. its density depends only on the time:

$$\rho(t) = \frac{3M}{4\pi r(t)^3} , \tag{2.1}$$

where M is the mass of the dust sphere and is constant. Now, imagine a small test particle of mass m on the surface of the sphere. By Newton's gravitation law and Gauss's theorem one has:

$$F = -\frac{GMm}{r(t)^2} \quad \Rightarrow \quad \ddot{r} = -\frac{4\pi G}{3}\rho r , \tag{2.2}$$

where we have used Eq. (2.1) and the dot denotes derivation with respect to the time t. This is the same acceleration equation that we shall find later using GR, cf. Eq. (2.51).

Exercise 2.1 Integrate Eq. (2.2) and show that:

$$\frac{\dot{r}^2}{r^2} = \frac{8\pi G}{3}\rho - \frac{K}{r^2} , \tag{2.3}$$

where K is an integration constant.

We shall also see that Eq. (2.3) is the same as Friedmann equation in GR, cf. (2.50). The integration constant K can be interpreted as the total energy of the particle. Indeed, we can rewrite Eq. (2.3) as follows:

$$E \equiv -\frac{mK}{2} = \frac{m}{2}\dot{r}^2 - \frac{GMm}{r} , \tag{2.4}$$

which is the expression of the total energy of a particle of mass m in the gravitational field of the mass M.

2.2 Relativistic Cosmology

In GR we have geometry and matter related by Einstein equations:

$$G_{\mu\nu} = \frac{8\pi G}{c^4}T_{\mu\nu} , \tag{2.5}$$

where $G_{\mu\nu}$ is the Einstein tensor, computed from the metric, and $T_{\mu\nu}$ is the energy-momentum or stress-energy tensor, and describes the matter content.

In cosmology, which is the metric which describes the universe and what is the matter content? It turns out that both questions are very difficult to answer and, indeed, there are no still clear answers, as we stressed in Chap. 1.

2.2.1 Friedmann–Lemaître–Robertson–Walker Metric

The metric used to describe the universe on large scales is the Friedmann–Lemaître–Robertson–Walker (FLRW) metric. This is based on the assumption of very high symmetry for the universe, called the **cosmological principle**, which is minimally stated as follows: the universe is isotropic and homogeneous, i.e. there is no preferred direction or preferred position.

A more formal definition can be found in Weinberg (1972, p. 412) and is based on the following two requirements:

1. The hypersurfaces with constant cosmic standard time are maximally symmetric subspaces of the whole of the spacetime;
2. The global metric and all the cosmic tensors such as the stress-energy one $T_{\mu\nu}$ are form-invariant with respect to the isometries of those subspaces.

We shall come back in a moment to maximally symmetric spaces. Roughly speaking, the second requirement above means that the matter quantities can depend only on the time.

The cosmological principle seems to be compatible with observations at very large scales. According to Wu et al. (1999): *on a scale of about 100* h^{-1} *Mpc the rms density fluctuations are at the level of* $\sim$10% *and on scales larger than 300* h^{-1} *Mpc the distribution of both mass and luminous sources safely satisfies the cosmological principle of isotropy and homogeneity*.

In a recent work Sarkar and Pandey (2016) find that the quasar distribution is homogeneous on scales larger than 250 h^{-1} Mpc. Moreover, numerical relativity seems to indicate that the average evolution of a generic metric on large scale is compatible with that of FLRW metric (Giblin et al. 2016).

According to the cosmological principle, the constant-time spatial hypersurfaces are maximally symmetric.[1] A maximally symmetric space is completely characterised by one number only, i.e. its scalar curvature, which is also a constant. See Weinberg (1972, Chap. 13).

Let R be this constant scalar curvature. The Riemann tensor of a maximally symmetric D-dimensional space is written as:

$$R_{\mu\nu\rho\sigma} = \frac{R}{D(D-1)}(g_{\mu\rho}g_{\nu\sigma} - g_{\mu\sigma}g_{\nu\rho}) \; . \tag{2.6}$$

Contracting with $g^{\mu\rho}$ we get for the Ricci tensor:

[1]This means that they possess 6 Killing vectors, i.e. there are six transformations which leave the spatial metric invariant (Weinberg 1972).

$$R_{\nu\sigma} = \frac{R}{D} g_{\nu\sigma} \ , \tag{2.7}$$

and then R is the scalar curvature, as we stated, since $g^{\nu\sigma} g_{\nu\sigma} = D$. Since any given number can be negative, positive or zero, we have three possible maximally symmetric spaces. Now, focusing on the 3-dimensional spatial case:

1. $ds_3^2 = |d\mathbf{x}|^2 \equiv \delta_{ij} dx^i dx^j$, i.e. the Euclidean space. The scalar curvature is zero, i.e. the space is flat. This metric is invariant under 3-translations and 3-rotations.
2. $ds_3^2 = |d\mathbf{x}|^2 + dz^2$, with the constraint $z^2 + |\mathbf{x}|^2 = a^2$. This is a 3-sphere of radius a embedded in a 4-dimensional Euclidean space. It is invariant under the six 4-dimensional rotations.
3. $ds_3^2 = |d\mathbf{x}|^2 - dz^2$, with the constraint $z^2 - |\mathbf{x}|^2 = a^2$. This is a 3-hypersphere, or a hyperboloid, in a 4-dimensional pseudo-Euclidean space. It is invariant under the six 4-dimensional pseudo-rotations (i.e. Lorentz transformations).

Exercise 2.2 Why are there six independent 4-dimensional rotations in the 4-dimensional Euclidean space? How many are there in a D-dimensional Euclidean space?

Let us write in a compact form the above metrics as follows:

$$ds_3^2 = |d\mathbf{x}|^2 \pm dz^2 \ , \qquad z^2 \pm |\mathbf{x}|^2 = a^2 \ . \tag{2.8}$$

Differentiating $z^2 \pm |\mathbf{x}|^2 = a^2$, one gets:

$$z dz = \mp \mathbf{x} \cdot d\mathbf{x} \ . \tag{2.9}$$

Now put this back into ds_3^2:

$$ds_3^2 = |d\mathbf{x}|^2 \pm \frac{(\mathbf{x} \cdot d\mathbf{x})^2}{a^2 \mp |\mathbf{x}|^2} \ . \tag{2.10}$$

In a more compact form:

$$ds_3^2 = |d\mathbf{x}|^2 + K \frac{(\mathbf{x} \cdot d\mathbf{x})^2}{a^2 - K|\mathbf{x}|^2} \ , \tag{2.11}$$

with $K = 0$ for the Euclidean case, $K = 1$ for the spherical case and $K = -1$ for the hyperbolic case. The components of the spatial metric in Eq. (2.11) can be immediately read off and are:

$$g_{ij}^{(3)} = \delta_{ij} + K \frac{x_i x_j}{a^2 - K|\mathbf{x}|^2} \ . \tag{2.12}$$

Exercise 2.3 Write down metric (2.11) in spherical coordinates. Use the fact that $|d\mathbf{x}|^2 = dr^2 + r^2 d\Omega^2$, where

$$d\Omega^2 \equiv d\theta^2 + \sin^2\theta d\phi^2 \ , \tag{2.13}$$

and use:

$$\mathbf{x} \cdot d\mathbf{x} = \frac{1}{2} d|\mathbf{x}|^2 = \frac{1}{2} d(r^2) = r dr \ . \tag{2.14}$$

Show that the result is:

$$\boxed{ds_3^2 = \frac{a^2 dr^2}{a^2 - Kr^2} + r^2 d\Omega^2} \tag{2.15}$$

Calculate the scalar curvature $R^{(3)}$ for metric (2.15). Show that $R_{ij}^{(3)} = 2K g_{ij}^{(3)}/a^2$ and thus $R^{(3)} = 6K/a^2$.

If we normalise $r \to r/a$ in metric (2.15), we can write:

$$ds_3^2 = a^2 \left(\frac{dr^2}{1 - Kr^2} + r^2 d\Omega^2 \right) \ , \tag{2.16}$$

and letting a to be a function of time, we finally get the FLRW metric:

$$\boxed{ds^2 = -c^2 dt^2 + a^2(t) \left(\frac{dr^2}{1 - Kr^2} + r^2 d\Omega^2 \right)} \tag{2.17}$$

The time coordinate used here is called **cosmic time**, whereas the spatial coordinates are called **comoving coordinates**. For each t the spatial slices are maximally symmetric; $a(t)$ is called **scale factor**, since it tells us how the distance between two points scales with time.

The FLRW metric was first worked out by Friedmann (1922, 1924) and then derived on the basis of isotropy and homogeneity by Robertson (1935, 1936) and Walker (1937). Lemaître's work (Lemaitre 1931) had been also essential to develop it.[2]

A further comment concerning FLRW metric (2.17) is in order here. The dimension of distance is being carried by the scale factor a itself, since we rescaled the radius $r \to r/a$. Indeed, as we computed earlier, the spatial curvature is $R^{(3)} = 6K/a(t)^2$, also time-varying, and it is a real, dimensional number as it should be.

[2] See also Lemaître (1997) for a recent republication and translation of Lemaître's 1933 paper.

2.2.2 The Conformal Time

A very useful form of rewriting FLRW metric (2.17) is via the **conformal time** η:

$$a d\eta = dt \qquad \Rightarrow \qquad \eta - \eta_i = \int_{t_i}^{t} \frac{dt'}{a(t')} \ . \tag{2.18}$$

As we shall see later, but as we already can guess from the above integration, $c(\eta - \eta_i)$ represents the comoving distance travelled by a photon between the times η_i and η, or t_i and t. The conformal time allows to rewrite FLRW metric (2.17) as follows:

$$\boxed{ds^2 = a(\eta)^2 \left(-c^2 d\eta^2 + \frac{dr^2}{1 - Kr^2} + r^2 d\Omega^2 \right)} \tag{2.19}$$

i.e. the scale factor has become a conformal factor (hence the name for η). Recalling the earlier discussion about dimensionality, if a has dimensions then $c\eta$ is dimensionless. On the other hand, if a is dimensionless, then η is indeed a time.

Note also that metric (2.19) for $K = 0$ is Minkowski metric multiplied by a conformal factor.

2.2.3 FLRW Metric Written with Proper Radius

A third useful way to write FLRW metric (2.17) is using the **proper radius**, which is defined as follows:

$$\mathcal{D}(t) \equiv a(t) r \ . \tag{2.20}$$

We shall discuss in more detail the proper radius, or proper distance, in Sect. 2.5.

Exercise 2.4 Using $\mathcal{D}$ instead of r, show that the FLRW metric (2.17) becomes:

$$ds^2 = -c^2 dt^2 \left(1 - H^2 \frac{\mathcal{D}^2/c^2}{1 - K\mathcal{D}^2/a^2} \right) - \frac{2H\mathcal{D} dt d\mathcal{D}}{1 - K\mathcal{D}^2/a^2} + \frac{d\mathcal{D}^2}{1 - K\mathcal{D}^2/a^2} + \mathcal{D}^2 d\Omega^2 \ , \tag{2.21}$$

where

$$\boxed{H \equiv \frac{\dot{a}}{a}} \tag{2.22}$$

is the **Hubble parameter**. The dot denotes derivation with respect to the cosmic time.

2.2.4 Light-Cone Structure of the FLRW Space

Let us consider the $K = 0$ case, for simplicity. Moreover, consider also $d\Omega = 0$. In this case, the radial coordinate is also the distance. Then, putting $ds^2 = 0$ in the FLRW metric gives the following light-cone structures.

Cosmic Time-Comoving Distance

From the FLRW metric (2.17), the condition $ds^2 = 0$ gives us:

$$\frac{cdt}{dr} = \pm a(t) \; . \tag{2.23}$$

We put our observer at $r = 0$ and $t = t_0$. The plus sign in the above equation then describes an outgoing photon, i.e. the future light-cone, whereas the negative sign describes an incoming photon, i.e. the past-light cone, which is much more interesting to us. So, let us keep the negative sign and discuss the shape of the light-cone.

Assume that $a(0) = 0$. Therefore, the slope of the past light-cone starts as $-a(t_0)$, which we can normalise as -1, i.e. locally the past light-cone is identical to the one in Minkowski space. However, a goes to zero, so the light-cone becomes flat, encompassing more radii than it would for Minkowski space. See Fig. 2.1. We can show this analytically by taking the second derivative of Eq. (2.23) with the minus sign:

$$\frac{c^2 d^2 t}{dr^2} = -\dot{a}\frac{cdt}{dr} = a\dot{a} \; . \tag{2.24}$$

Being $a > 0$ and $\dot{a} > 0$ (we consider just the case of an expanding universe), the function $t(r)$ is convex (i.e. it is "bent upwards").

Conformal Time-Comoving Distance

For the FLRW metric (2.19), the condition ds^2 gives:

$$\frac{cd\eta}{dr} = \pm 1 \; . \tag{2.25}$$

The latter is exactly the same light-cone structure of Minkowski space. Indeed, Friedmann metric written in conformal time and for $K = 0$ is Minkowski metric multiplied by a conformal factor $a(\eta)$. See Fig. 2.2.

Cosmic Time-Proper Distance

In order to find the light-cone structure for the FLRW metric (2.21) with $K = 0$, we need to solve the following equation:

$$-\frac{c^2 dt^2}{d\mathcal{D}^2}\left(1 - \frac{H^2\mathcal{D}^2}{c^2}\right) - \frac{2H\mathcal{D}}{c}\frac{cdt}{d\mathcal{D}} + 1 = 0 \; . \tag{2.26}$$

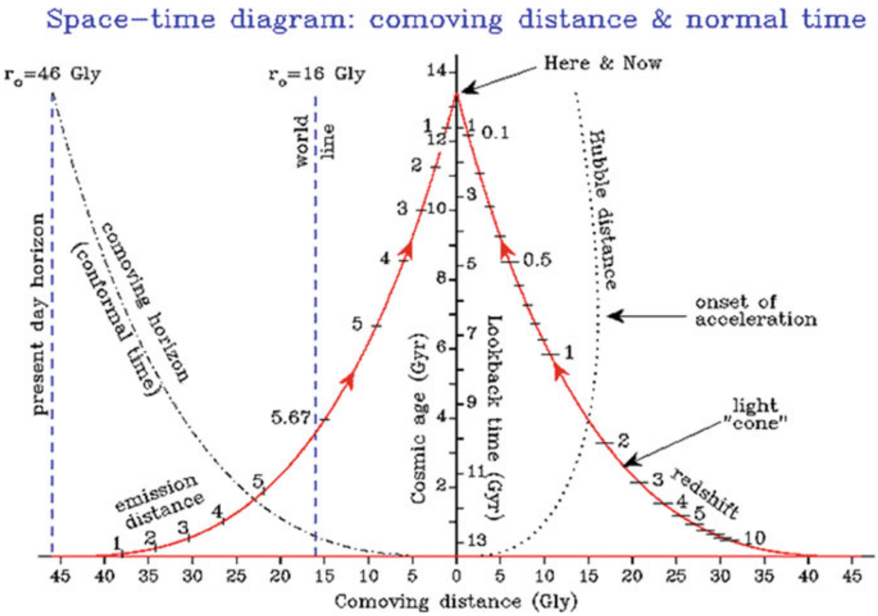

Fig. 2.1 Space-time diagram and light-cone structure for the FLRW metric (2.17). Credit: Prof. Mark Whittle, University of Virginia

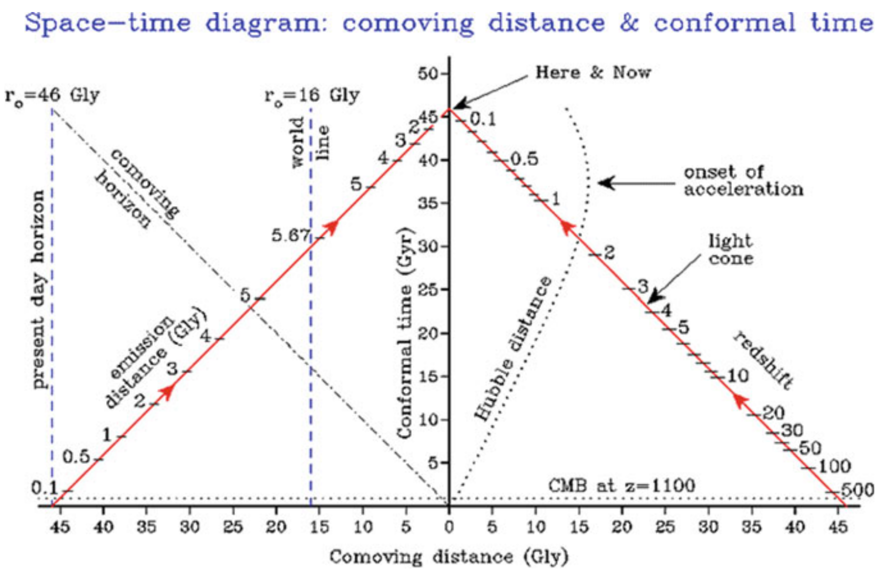

Fig. 2.2 Space-time diagram and light-cone structure for the FLRW metric (2.19). Credit: Prof. Mark Whittle, University of Virginia

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Exercise 2.5 Solve Eq. (2.26) algebraically for $cdt/d\mathcal{D}$ and show that:

$$\frac{cdt}{d\mathcal{D}} = \left(\frac{H\mathcal{D}}{c} \pm 1\right)^{-1} . \tag{2.27}$$

For $t = t_0$ we have $H(t_0) > 0$ and $\mathcal{D} = 0$. Therefore, from Eq. (2.27) we have that $(cdt/d\mathcal{D})(t_0) = \pm 1$ and thus we must choose the minus sign in order to describe the past light-cone. Going back in time, $H\mathcal{D}$ grows, until

$$\frac{H\mathcal{D}}{c} = 1 , \tag{2.28}$$

for which $cdt/d\mathcal{D}$ diverges. This means that no signal can come from beyond this distance $\mathcal{D} = c/H$, which is the **Hubble radius** that we met in Chap. 1. See Fig. 2.3. The lower part of this figure is explained as follows. First of all $H\mathcal{D}$ becomes larger than 1, and this explains the change of sign of the slope of the light cone. Then, $H \to \infty$ for $a \to 0$ (if we assume a model with Big Bang) and therefore $cdt/d\mathcal{D} \to 0$. This is why the light-cone flattens close to $t = 0$ in Fig. 2.3.

2.2.5 *Christoffel Symbols and Geodesics*

Exercise 2.6 Assume $K = 0$ in metric (2.17), rewrite it in Cartesian coordinates and calculate the Christoffel symbols. Show that:

$$\Gamma^0_{00} = 0 , \qquad \Gamma^0_{0i} = 0 , \qquad \Gamma^0_{ij} = \frac{a\dot{a}}{c}\delta_{ij} , \qquad \Gamma^i_{0j} = \frac{H}{c}\delta^i{}_j . \tag{2.29}$$

We now use these in the geodesic equation:

$$\frac{dP^\mu}{d\lambda} + \Gamma^\mu_{\nu\rho} P^\nu P^\rho = 0 , \tag{2.30}$$

where $P^\mu \equiv dx^\mu/d\lambda$ is the four-momentum and λ is an affine parameter. For a particle of mass m, one has $\lambda = \tau/m$, where τ is the proper time. The norm of the four-momentum is:

$$\mathcal{P}^2 \equiv g_{\mu\nu} P^\mu P^\nu = -\frac{E^2}{c^2} + p^2 = -m^2c^2 , \tag{2.31}$$

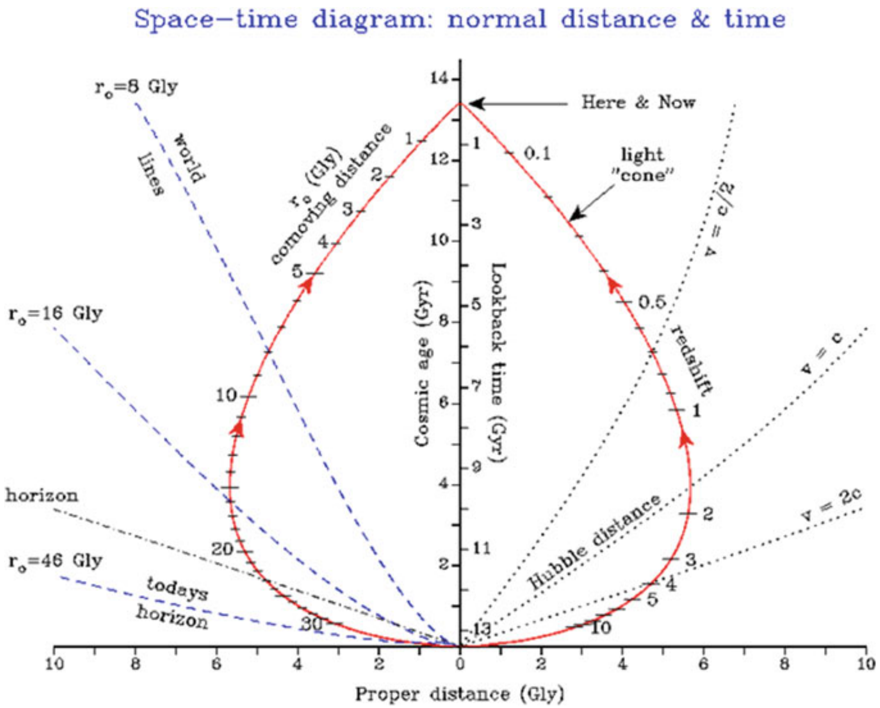


Fig. 2.3 Space-time diagram and light-cone structure for the FLRW metric (2.21). Credit: Prof. Mark Whittle, University of Virginia

where we have defined the energy and the **physical momentum** (or **proper momentum**):

$$\frac{E^2}{c^2} \equiv -g_{00}(P^0)^2 \ , \qquad p^2 \equiv g_{ij} P^i P^j \ , \tag{2.32}$$

and the last equality of Eq. (2.31), which applies only to massive particles, comes from:

$$\frac{ds^2}{d\lambda^2} = \frac{m^2 ds^2}{d\tau^2} = -m^2c^2 \ , \tag{2.33}$$

since, by definition, $ds^2 = -c^2 d\tau^2$. We have recovered above the well-known dispersion relation of special relativity. The metric $g_{\mu\nu}$ used above is, in principle, general. But, of course, we now specialise it to the FLRW one.

For a photon, $m = 0$ and $E = pc$. The time-component of the geodesic equation is the following:

$$\frac{dP^0}{d\lambda} + \frac{a\dot{a}}{c}\delta_{ij} P^i P^j = 0 \ . \tag{2.34}$$

Introducing the proper momentum as defined in Eq. (2.32), one gets:

$$c\frac{dp}{d\lambda} + Hp^2 = 0 \;. \tag{2.35}$$

Exercise 2.7 Solve Eq. (2.35) and show that $p = E/c \propto 1/a$, i.e. the energy of the photon is proportional to the inverse scale factor.

Therefore, we can write:

$$\frac{E_{\rm obs}}{E_{\rm em}} = \frac{a_{\rm em}}{a_{\rm obs}} \;. \tag{2.36}$$

On the other hand the photon energy is $E = hf$, with f its frequency. Therefore:

$$\frac{a_{\rm em}}{a_{\rm obs}} = \frac{E_{\rm obs}}{E_{\rm em}} = \frac{f_{\rm obs}}{f_{\rm em}} = \frac{\lambda_{\rm em}}{\lambda_{\rm obs}} = \frac{1}{1+z} \;. \tag{2.37}$$

This is the relation between the redshift and the scale factor. We have connected observation with theory. Usually, $a_{\rm obs} = 1$ and the above relation is simply written as $1 + z = 1/a$.

What does happen, on the other hand, to the energy of a massive particle? The time-geodesic equation for massive particles is identical to the one for photons, but the dispersion relation is different, i.e. $E^2 = m^2c^4 + p^2c^2$. Therefore:

$$E = \sqrt{m^2c^4 + \frac{p_i^2 a_i^2 c^2}{a^2}} \;, \tag{2.38}$$

where p_i is some initial proper momentum, at the time t_i and $a_i = a(t_i)$. For $m = 0$ we recover the result already obtained for photons. For massive particles the above relation can be approximated as follows:

$$E = mc^2\left(1 + \frac{p_i^2 a_i^2}{2a^2m^2c^2} + \cdots\right) \;, \qquad (mc \gg p) \;, \tag{2.39}$$

i.e. performing the expansion for small momenta which is usually done in special relativity. The second contribution between parenthesis is the classical kinetic energy of the particle, whose average is proportional to $k_{\rm B}T$. Therefore:

$$T \propto a^{-1} \;, \text{ for relativistic particles,} \tag{2.40}$$

$$T \propto a^{-2} \;, \text{ for non-relativistic particles.} \tag{2.41}$$

We shall recover the above result also using the Boltzmann equation.

Exercise 2.8 Show that $p \propto 1/a$ by using not the time-component geodesic equation but the spatial one:

$$\frac{dP^i}{d\lambda} + 2\Gamma^i_{0j} P^0 P^j = 0 \; . \tag{2.42}$$

Why is there a factor two in this equation?

2.3 Friedmann Equations

Given FLRW metric, Friedmann equations can be straightforwardly computed from the Einstein equations:

$$G_{\mu\nu} + \Lambda g_{\mu\nu} = R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R + \Lambda g_{\mu\nu} = \frac{8\pi G}{c^4} T_{\mu\nu} \; , \tag{2.43}$$

where Λ is the cosmological constant.

Exercise 2.9 Calculate from FLRW metric (2.17) the components of the Ricci tensor. Show that:

$$R_{00} = -\frac{3}{c^2}\frac{\ddot{a}}{a} \; , \qquad R_{0i} = 0 \; , \qquad R_{ij} = \frac{1}{c^2} g_{ij} \left(2H^2 + \frac{\ddot{a}}{a} + 2\frac{Kc^2}{a^2} \right) \; , \tag{2.44}$$

and show that the scalar curvature is:

$$R = \frac{6}{c^2} \left(\frac{\ddot{a}}{a} + H^2 + \frac{Kc^2}{a^2} \right) \; . \tag{2.45}$$

Finally, compute the Einstein equations:

$$\boxed{H^2 + \frac{Kc^2}{a^2} = \frac{8\pi G}{3c^2} T_{00} + \frac{\Lambda c^2}{3}} \tag{2.46}$$

$$\boxed{g_{ij} \left(H^2 + 2\frac{\ddot{a}}{a} + \frac{Kc^2}{a^2} - \Lambda c^2 \right) = -\frac{8\pi G}{c^2} T_{ij}} \tag{2.47}$$

These are called **Friedmann equations** or **Friedmann equation** and **acceleration equation** or **Friedmann equation** and **Raychaudhuri equation**.

Which stress-energy tensor $T_{\mu\nu}$ do we use in Eqs. (2.46) and (2.47)? Having fixed the metric to be the FLRW one, we have some strong constraints:

- First of all: $G_{0i} = 0$ implies that $T_{0i} = 0$, i.e. there cannot be a flux of energy in any direction because it would violate isotropy;
- Second, since $G_{ij} \propto g_{ij}$, then $T_{ij} \propto g_{ij}$.
- Finally, since $G_{\mu\nu}$ depends only on t, then it must be so also for $T_{\mu\nu}$.

Therefore, let us stipulate that

$$T_{00} = \rho(t)c^2 = \varepsilon(t) \ , \qquad T_{0i} = 0 \ , \qquad T_{ij} = g_{ij} P(t) \ , \tag{2.48}$$

where $\rho(t)$ is the rest mass density, $\varepsilon(t)$ is the energy density and $P(t)$ is the pressure. In tensorial notation we can write the following general form for the stress-energy tensor:

$$\boxed{T_{\mu\nu} = \left(\rho + \frac{P}{c^2}\right) u_\mu u_\nu + P g_{\mu\nu}} \tag{2.49}$$

where u_μ is the four-velocity of the fluid element. In this form of Eq. (2.49), the stress-energy tensor does not contain either viscosity or energy transport terms. Matter described by (2.49) is known as **perfect fluid**. For more detail about the latter see Schutz (1985) whereas for more detail about viscosity, heat fluxes and the imperfect fluids see e.g. Weinberg (1972) and Maartens (1996).

Combine Eqs. (2.46), (2.47) and (2.48). The Friedmann equation becomes:

$$\boxed{H^2 = \frac{8\pi G}{3}\rho + \frac{\Lambda c^2}{3} - \frac{K c^2}{a^2}} \tag{2.50}$$

while the acceleration equation is the following:

$$\boxed{\frac{\ddot{a}}{a} = -\frac{4\pi G}{3}\left(\rho + \frac{3P}{c^2}\right) + \frac{\Lambda c^2}{3}} \tag{2.51}$$

Exercise 2.10 Write Eqs. (2.50) and (2.51) using the conformal time introduced in Eq. (2.18). Show that the Friedmann equation becomes:

$$\boxed{\mathcal{H}^2 = \frac{8\pi G}{3}\rho a^2 + \frac{\Lambda c^2 a^2}{3} - K c^2} \tag{2.52}$$

and that the acceleration equation becomes:

$$\frac{a''}{a} = \frac{4\pi G}{3}\left(\rho - \frac{3P}{c^2}\right) a^2 + \frac{2\Lambda c^2 a^2}{3} - K c^2 \ , \tag{2.53}$$

where the prime denotes derivation with respect to the conformal time η and

$$\boxed{\mathcal{H} \equiv \frac{a'}{a}} \tag{2.54}$$

is the **conformal Hubble factor**.

In the Friedmann and acceleration equations, ρ and P are the total density and pressure. Hence, they can be written as sums of the contributions of the individual components:

$$\rho \equiv \sum_x \rho_x \;, \qquad P \equiv \sum_x P_x \;. \tag{2.55}$$

The contribution from the cosmological constant can be considered either geometrically or as a matter component with the following density and pressure:

$$\rho_\Lambda \equiv \frac{\Lambda c^2}{8\pi G} \;, \qquad P_\Lambda \equiv -\rho_\Lambda c^2 \;. \tag{2.56}$$

The scale factor a is, by definition, positive, but its derivative can be negative. This would represent a contracting universe. Note that the left hand side of the Friedmann equation (2.50) is non-negative. Therefore, $\dot{a}$ can vanish only if $K > 0$, i.e. for a spatially closed universe. This implies that, if $K \leqslant 0$ and if there exists an instant for which $\dot{a} > 0$, then the universe will expand forever.

2.3.1 The Hubble Constant and the Deceleration Parameter

When the Hubble parameter H is evaluated at the present time t_0, it becomes a number: the Hubble constant H_0 which we already met in Chap. 1 in the Hubble's law (1.1). Its value is

$$H_0 = 67.74 \pm 0.46 \text{ km s}^{-1} \text{ Mpc}^{-1} \;, \tag{2.57}$$

at the 68% confidence level, as reported by the Planck group (Ade et al. 2016). Usually H_0 is conveniently written as

$$H_0 = 100\, h \text{ km s}^{-1} \text{ Mpc}^{-1} \;. \tag{2.58}$$

The unit of measure of the Hubble constant is an inverse time:

$$H_0 = 3.24\, h \times 10^{-18} \text{ s}^{-1} \;, \tag{2.59}$$

whose inverse gives the order of magnitude of the age of the universe:

$$\frac{1}{H_0} = 3.09 \text{ h}^{-1} \times 10^{17} \text{ s} = 9.78 \text{ h}^{-1} \text{ Gyr} , \tag{2.60}$$

and multiplied by c gives the order of magnitude of the size of the visible universe, i.e. the Hubble radius that we have already seen in Eq. (2.28) but evaluated at the present time $t = t_0$:

$$\frac{c}{H_0} = 9.27 \text{ h}^{-1} \times 10^{25} \text{ m} = 3.00 \text{ h}^{-1} \text{ Gpc} . \tag{2.61}$$

But what does "present time" t_0 mean? Time flows, therefore t_0 cannot be a constant! That is true, but if we compare a time span of 100 years (the span of some human lives) to the age of the universe (about 14 billion years), we see that the ratio is about 10^{-8}. Since this is pretty small, we can consider t_0 to be a constant, also referred to as the age of the universe.[3] We can calculate it as follows:

$$t_0 = \int_0^{t_0} dt = \int_0^1 \frac{da}{\dot{a}} = \int_0^1 \frac{da}{H(a)a} = \int_0^\infty \frac{dz}{H(z)(1+z)} . \tag{2.62}$$

Exercise 2.11 Prove the last equality of Eq. (2.62).

The integration limits of Eq. (2.62) deserve some explanation. We assumed that $a(t = 0) = 0$, i.e. the Big Bang. This condition is not always true, since there are models of the universe, e.g. the de Sitter universe, for which a vanishes only when $t \to -\infty$. The other assumption is that $a(t_0) = 1$. This is a pure normalisation, done for convenience, which is allowed by the fact that the dynamics is invariant if we multiply the scale factor by a constant.

Recall that, in cosmology, when a quantity has subscript 0, it usually means that it is evaluated at $t = t_0$.

The Deceleration Parameter

Let us focus now on Eq. (2.51). It contains $\ddot{a}$, so it describes how the expansion of the universe is accelerating. The key-point is that if the right hand side of Eq. (2.51) is positive, i.e. $\rho + 3P/c^2 < 0$, then $\ddot{a} > 0$. There exists a parameter, named **deceleration parameter**, with which to measure the entity of the acceleration. It is defined as follows:

$$\boxed{q \equiv -\frac{\ddot{a}a}{\dot{a}^2}} \tag{2.63}$$

[3] Pretty much the same happens with the redshift. A certain source has redshift z which, actually, is not a constant but varies slowly. This is called **redshift drift** and it was first considered by Sandage (1962) and McVittie (1962). Applications of the redshift drift phenomenon to gravitational lensing are proposed in Piattella and Giani (2017).

In Riess et al. (1998) and Perlmutter et al. (1999) analysis based on type Ia supernovae observation have shown that $q_0 < 0$, i.e. the deceleration parameter is negative and therefore the universe is in a state of accelerated expansion. We perform a similar but simplified analysis in Sect. 11.1 in order to illustrate how data in cosmology are analysed.

2.3.2 Critical Density and Density Parameters

Let us now rewrite Eq. (2.50) incorporating Λ in the total density ρ:

$$H^2 = \frac{8\pi G\rho}{3} - \frac{Kc^2}{a^2} \,. \tag{2.64}$$

The value of the total ρ such that $K = 0$ is called **critical energy density** and has the following form:

$$\boxed{\rho_{\rm cr} \equiv \frac{3H^2}{8\pi G}} \tag{2.65}$$

Its present value (Ade et al. 2016) is:

$$\boxed{\rho_{\rm cr,0} = 1.878\, h^2 \times 10^{-29}\ {\rm g\ cm^{-3}}} \tag{2.66}$$

It turns out that ρ_0 is very close to $\rho_{\rm cr,0}$, so that our universe is spatially flat. Such an extreme fine-tuning in K is a really surprising coincidence, known as the **flatness problem**. A possible solution is provided by the inflationary theory which we shall see in detail in Chap. 8.

Instead of densities, it is very common and useful to employ the density parameter Ω, which is defined as

$$\boxed{\Omega \equiv \frac{\rho}{\rho_{\rm cr}} = \frac{8\pi G\rho}{3H^2}} \tag{2.67}$$

i.e. the energy density normalised to the critical one. We can then rewrite Friedmann equation (2.50) as follows:

$$1 = \Omega - \frac{Kc^2}{H^2a^2} \,. \tag{2.68}$$

Defining

$$\Omega_K \equiv -\frac{Kc^2}{H^2a^2} \,, \tag{2.69}$$

i.e. associating the energy density

$$\rho_K \equiv -\frac{3Kc^2}{8\pi G a^2} \ , \tag{2.70}$$

to the spatial curvature, we can recast Eq. (2.68) in the following simple form:

$$1 = \Omega + \Omega_K \ . \tag{2.71}$$

Therefore, the sum of all the density parameters, *the curvature one included*, is always equal to unity. In particular, if it turns out that $\Omega \simeq 1$, this implies that $\Omega_K \simeq 0$, i.e. the universe is spatially flat. From the latest Planck data (Ade et al. 2016) we know that:

$$\boxed{\Omega_{K0} = 0.0008^{+0.0040}_{-0.0039}} \tag{2.72}$$

at the 95% confidence level.

It is more widespread in the literature the normalisation of ρ to the *present-time* critical density, i.e.

$$\boxed{\Omega \equiv \frac{\rho}{\rho_{\rm cr,0}} = \frac{8\pi G \rho}{3H_0^2}} \tag{2.73}$$

because it leaves more evident the dependence on a of each material component. With this definition of Ω, Friedmann equation (2.50) is written as:

$$\frac{H^2}{H_0^2} = \sum_x \Omega_{x0} f_x(a) + \frac{\Omega_{K0}}{a^2} \ , \tag{2.74}$$

where $f_x(a)$ is a function which gives the a-dependence of the material component x and $f_x(a_0 = 1) = 1$. Consistently:

$$\boxed{\sum_x \Omega_{x0} + \Omega_{K0} = 1} \tag{2.75}$$

also known as **closure relation**. We shall use the definition $\Omega_x \equiv \rho_x/\rho_{\rm cr,0}$ throughout these notes.

2.3.3 The Energy Conservation Equation

The energy conservation equation

$$\boxed{\nabla_\nu T^{\mu\nu} = 0} \tag{2.76}$$

is encapsulated in GR through the Bianchi identities. Therefore, it is not independent from the Friedmann equations (2.50) and (2.51). For the FLRW metric and a perfect fluid, it has a particularly simple form:

$$\boxed{\dot{\rho} + 3H\left(\rho + \frac{P}{c^2}\right) = 0} \tag{2.77}$$

This is the $\mu = 0$ component of $\nabla_\nu T^{\mu\nu} = 0$ and it is also known from fluid dynamics as **continuity equation**.

Exercise 2.12 Derive the continuity equation (2.77) by combining Friedmann and acceleration equations (2.50) and (2.51). Derive it in a second way by explicitly calculating the four-divergence of the energy-momentum tensor.

The continuity equation can be analytically solved if we assume an equation of state of the form $P = w\rho c^2$, with w constant. The general solution is:

$$\boxed{\rho = \rho_0 a^{-3(1+w)}} \qquad (w = \text{constant}) , \tag{2.78}$$

where $\rho_0 \equiv \rho(a_0 = 1)$.

Exercise 2.13 Prove the above result of Eq. (2.78).

There are three particular values of w which play a major role in cosmology:

Cold matter: $w = 0$, i.e. $P = 0$, for which $\rho = \rho_0 a^{-3}$. As we have discussed in Chap. 1, the adjective cold refers to the fact that particles making up this kind of matter have a kinetic energy much smaller than the mass energy, i.e. they are non-relativistic. If they are thermally produced, i.e. if they were in thermal equilibrium with the primordial plasma, they have a mass much larger than the temperature of the thermal bath. We shall see this characteristic in more detail in Chap. 3.

Cold matter is also called **dust** and it encompasses all the non-relativistic known elementary particles, which are overall dubbed **baryons** in the jargon of cosmology. If they exist, unknown non-relativistic particles are called **cold dark matter** (CDM).

Hot matter: $w = 1/3$, i.e. $P = \rho/3$, for which $\rho = \rho_0 a^{-4}$. The adjective hot refers to the fact that particles making up this kind of matter are relativistic.

For this reason they are known, in the jargon of cosmology, as **radiation** and they encompass not only the relativistic known elementary particles, but possibly the unknown ones (i.e. **hot dark matter**). The primordial neutrino background belonged to this class, but since neutrino seems to have a mass of approximately 0.1 eV, it is now cold. We shall see why in Chap. 3.

Vacuum energy: $w = -1$, i.e. $P = -\rho c^2$ and ρ is a constant. It behaves as the cosmological constant and provides the best (and the simplest) description that we have for dark energy, though plagued by the serious issues that we have presented in Chap. 1.

2.3.4 The ΛCDM Model

The most successful cosmological model is called ΛCDM and is made up of Λ, CDM, baryons and radiation (photons and massless neutrinos). The Friedmann equation for the ΛCDM model is the following:

$$\frac{H^2}{H_0^2} = \Omega_\Lambda + \frac{\Omega_{c0}}{a^3} + \frac{\Omega_{b0}}{a^3} + \frac{\Omega_{r0}}{a^4} + \frac{\Omega_{K0}}{a^2} \, . \tag{2.79}$$

We already saw in Eq. (2.72) the value of the spatial curvature contribution. From Ade et al. (2016) here are the other ones:

$$\boxed{\Omega_\Lambda = 0.6911 \pm 0.0062 \, , \qquad \Omega_{m0} = 0.3089 \pm 0.0062} \tag{2.80}$$

at 68% confidence level, where $\Omega_{m0} = \Omega_{c0} + \Omega_{b0}$, i.e. it includes the contributions from both CDM and baryons, since they have the same dynamics (i.e. they are both cold). It is however possible to disentangle them and one observes:

$$\boxed{\Omega_{b0}h^2 = 0.02230 \pm 0.00014 \, , \qquad \Omega_{c0}h^2 = 0.1188 \pm 0.0010} \tag{2.81}$$

also at 68% confidence level. The radiation content, i.e. photons plus neutrinos, can be easily calculated from the temperature of the CMB, as we shall see in Chap. 3. It turns out that:

$$\boxed{\Omega_{\gamma 0}h^2 \approx 2.47 \times 10^{-5} \, , \qquad \Omega_{\nu 0}h^2 \approx 1.68 \times 10^{-5}} \tag{2.82}$$

Since $h = 0.68$, and recalling the closure relation of Eq. (2.75), we can conclude that today 69% of our universe is made of cosmological constant, 26% of CDM and 5% of baryons. Radiation and spatial curvature are negligible. That is, the situation is pretty obscure, in all senses.

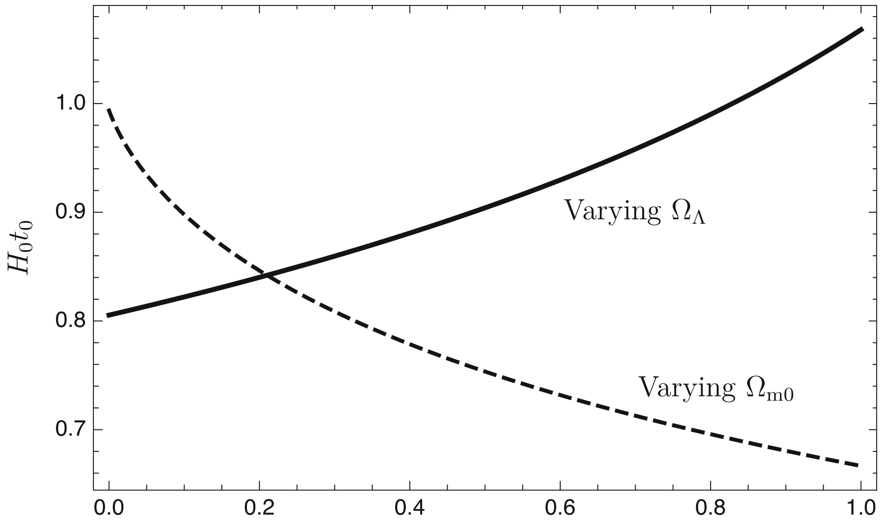


Fig. 2.4 Dimensionless age of the universe $H_0 t_0$ as function of Ω_Λ (keeping fixed the matter and radiation content) and as function of Ω_{m0} (keeping fixed the radiation content and with no Λ)

Let us now calculate the age of the universe for the ΛCDM model. Using Eq. (2.62), we get:

$$t_0 = \frac{1}{H_0} \int_0^1 da \frac{a}{\sqrt{\Omega_\Lambda a^4 + \Omega_{m0} a + \Omega_{r0} + \Omega_{K0} a^2}} \,. \tag{2.83}$$

Using the numbers shown insofar, we get upon numerical integration:

$$\boxed{t_0 = \frac{0.95}{H_0} = 13.73 \text{ Gyr}} \tag{2.84}$$

The value reported by Ade et al. (2016) is 13.799 ± 0.021 at 68% confidence level. Note how $H_0 t_0 \approx 1$. This fact has been dubbed **synchronicity problem** by Avelino and Kirshner (2016). In Fig. 2.4 we plot the dimensionless age of the universe $H_0 t_0$ for models with or without Λ and in Fig. 2.5 we plot the evolution of $H_0 t$ as function of a in order to show indeed how $H_0 t_0 \approx 1$ is quite a peculiar instant of the history of the universe.

As one can see, in presence of Λ the dimensionless age of the universe reaches values larger than unity. This, mathematically, is due to the a^4 factor multiplying Ω_Λ in Eq. (2.83). Note that we can obtain the observed value $H_0 t_0 \approx 0.95$ also in absence of a cosmological constant and for a curvature-dominated universe, i.e. $\Omega_{K0} \approx 0.97$.

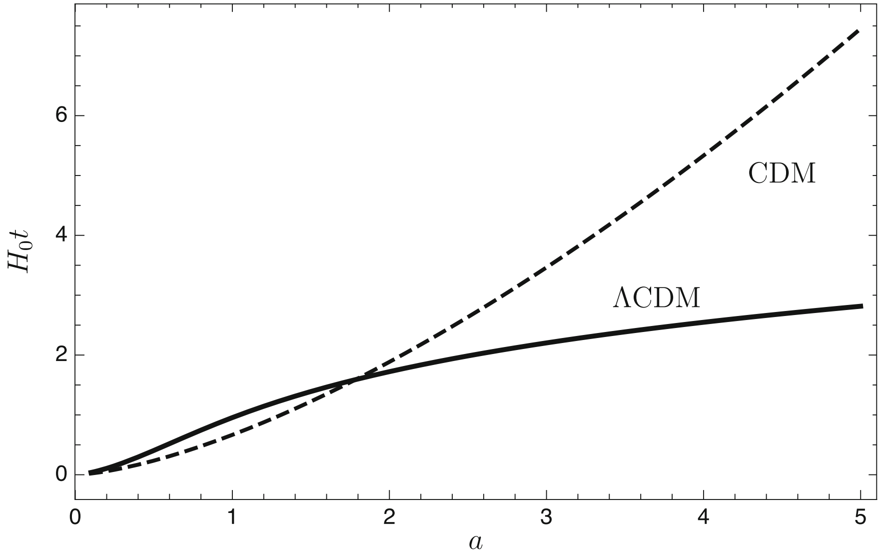


Fig. 2.5 Dimensionless age of the universe H_0t as function of a for the ΛCDM model (solid line) and in a model made only of CDM (dashed line)

2.4 Solutions of the Friedmann Equations

The Friedmann equations can be solved exactly for many cases of interest.

2.4.1 The Einstein Static Universe

As the first application of his theory to cosmology, Einstein was looking for a static universe, since at his time there was not yet compelling evidence of the contrary. Therefore, we must set $\dot{a} = \ddot{a} = 0$. Since ρ is positive, we must have $K = 1$, therefore the Einstein Static Universe (ESU) is a closed universe. Its radius is:

$$\frac{8\pi G}{3}\rho = \frac{c^2}{a^2} \qquad \Rightarrow \qquad a = \sqrt{\frac{3c^2}{8\pi G\rho}} \, . \tag{2.85}$$

From the acceleration equation we get that

$$\rho + 3P/c^2 = 0 \, , \tag{2.86}$$

therefore we cannot have simply ordinary matter because we need a negative pressure. Here enters the cosmological constant Λ. We assume that $\rho = \rho_{\rm m} + \rho_\Lambda$, so that

$$\rho + 3P/c^2 = 0 \,, \qquad \Rightarrow \qquad \rho_m + \rho_\Lambda - 3\rho_\Lambda = 0 \,, \tag{2.87}$$

and therefore $\rho_m = 2\rho_\Lambda$. The radius can thus be written as

$$a = \frac{c}{\sqrt{4\pi G \rho_m}} = \frac{1}{\sqrt{\Lambda}} \,. \tag{2.88}$$

Until here all seems to be fine. But it is not. The problem is indeed the condition $\rho_m = 2\rho_\Lambda$, which makes the ESU unstable. In fact, if this condition is broken, say $\rho_m/\rho_\Lambda = 2 + \epsilon$, then the universe expands or collapses, depending on the sign of ϵ.

Exercise 2.14 Prove that the ESU is unstable. Hint: use $\rho_m/\rho_\Lambda = 2 + \epsilon$ in the Friedmann and acceleration equations.

2.4.2 The de Sitter Universe

For $\rho = 0$, the Friedmann equation (2.50) becomes:

$$H^2 = \frac{\Lambda c^2}{3} - \frac{Kc^2}{a^2} \,. \tag{2.89}$$

When spatial curvature is taken into account, it is more convenient to solve the acceleration equation (2.51) rather than Friedmann equation. Indeed:

$$\ddot{a} = \frac{\Lambda c^2}{3} a \,, \tag{2.90}$$

is straightforwardly integrated:

$$a(t) = C_1 \exp\left(\sqrt{\frac{\Lambda}{3}} ct\right) + C_2 \exp\left(-\sqrt{\frac{\Lambda}{3}} ct\right) \,, \tag{2.91}$$

where C_1 and C_2 are two integration constants. One of these is constrained by Friedmann equation (2.89). Calculating $\dot{a}^2$ and a^2 from Eq. (2.91) we get:

$$\dot{a}^2 = \frac{\Lambda c^2}{3} \left[C_1^2 \exp\left(2\sqrt{\frac{\Lambda}{3}} ct\right) + C_2^2 \exp\left(-2\sqrt{\frac{\Lambda}{3}} ct\right) - 2C_1C_2 \right] \,, \tag{2.92}$$

$$a^2 = C_1^2 \exp\left(2\sqrt{\frac{\Lambda}{3}} ct\right) + C_2^2 \exp\left(-2\sqrt{\frac{\Lambda}{3}} ct\right) + 2C_1C_2 \,. \tag{2.93}$$

Combining them, one finds:

$$\dot{a}^2 = \frac{\Lambda c^2}{3}(a^2 - 4C_1C_2) \ . \tag{2.94}$$

Using Eq. (2.89), we can put a constraint on the product of the two integration constants:

$$\frac{4\Lambda}{3}C_1C_2 = K \ , \tag{2.95}$$

so that we have freedom to fix just one of them. Assuming that $C_1 \neq 0$, we can write the general solution as:

$$a(t) = C_1 \exp\left(\sqrt{\frac{\Lambda}{3}}ct\right) + \frac{3K}{4\Lambda C_1}\exp\left(-\sqrt{\frac{\Lambda}{3}}ct\right) \ . \tag{2.96}$$

When $K = \pm 1$ we can set C_1 such that we can write the solution as follows:

$$a(t) = \begin{cases} \sqrt{3/\Lambda}\sinh\left(\sqrt{\Lambda/3}ct\right) \ , & \text{for } K = -1 \ , \\ a_0 \exp\left(\sqrt{\Lambda/3}ct\right) \ , & \text{for } K = 0 \ , \\ \sqrt{3/\Lambda}\cosh\left(\sqrt{\Lambda/3}ct\right) \ , & \text{for } K = 1 \ . \end{cases} \tag{2.97}$$

where a_0 is some initial $t = 0$ scale factor. Note that there is no a_0 for the solutions with spatial curvature because we fixed the integration constant in order to have the hyperbolic sine and cosine.

Exercise 2.15 From the Einstein equations (2.43) show that $R = 4\Lambda$ for the de Sitter universe. Verify that the above solutions (2.96) and (2.97) satisfy this relation by substituting them into the expression in Eq. (2.45) for the Ricci scalar.

The de Sitter universe (de Sitter 1917, 1918a, b, c) is eternal with no Big-Bang (i.e. when $a = 0$) for $K = 1$. Here we have rather a bounce at the minimum value a_0 for the scale factor. For $K = 0$ there is a Big-Bang at $t = -\infty$. For $K = -1$ we might have negative scale factors, which we however neglect and consider the evolution as starting only at $t = 0$, for which there is another Big-Bang. Note that the Big-Bang's we are mentioning here are not singularities. There is no physical singularity in the de Sitter space, since it is maximally symmetric.

The deceleration parameter is the following

$$q = -\frac{\ddot{a}a}{\dot{a}^2} = -\frac{\Lambda c^2}{3}\frac{a^2}{\dot{a}^2} = -\left(1 - \frac{3K}{\Lambda a^2}\right)^{-1} \ , \tag{2.98}$$

i.e. always negative.

2.4.3 Radiation-Dominated Universe

For $\rho = \rho_0 a^{-4}$, $K = 0$ and $\Lambda = 0$, the solution of Eq. (2.50) is:

$$a = \sqrt{\frac{t}{t_0}} \; , \tag{2.99}$$

The deceleration parameter is $q_0 = 1$ and the age of the universe is:

$$t_0 = \frac{1}{2H_0} \; . \tag{2.100}$$

Exercise 2.16 Prove the results of Eqs. (2.99) and (2.100).

It is quite complicated to analytically solve Friedmann equation (2.50) for a radiation-dominated universe when $K \neq 0$. On the other hand, solving the acceleration equation (2.53) is much easier. For $\rho c^2 = 3P$, Eq. (2.53) becomes:

$$a'' + Kc^2 a = 0 \; , \tag{2.101}$$

whose general solution is:

$$a(\eta) = \begin{cases} C_1 \exp(c\eta) + C_2 \exp(-c\eta) \; , & \text{for } K = -1 \; , \\ C_3 + C_4\eta \; , & \text{for } K = 0 \; , \\ C_5 \sin(c\eta) + C_6 \cos(c\eta) \; , & \text{for } K = 1 \; . \end{cases} \tag{2.102}$$

First of all, we can choose $a(0) = 0$. Thus, the general solution (2.102) becomes:

$$a(\eta) = \begin{cases} 2C_1 \sinh(c\eta) \; , & \text{for } K = -1 \; , \\ C_4\eta \; , & \text{for } K = 0 \; , \\ C_5 \sin(c\eta) \; , & \text{for } K = 1 \; . \end{cases} \tag{2.103}$$

Second, these solutions are subject to the constraint of Friedmann equation (2.52), written of course in the radiation-dominated case:

$$a'^2 = \frac{8\pi G}{3}\rho a^4 - Kc^2 a^2 \; , \tag{2.104}$$

where notice that $\rho a^4 = \rho_0$, i.e. a constant. When $a = 0$, i.e. $\eta = 0$, then

$$a'^2(\eta = 0) = \frac{8\pi G}{3}\rho_0 \equiv a_m^2 c^2 \; , \tag{2.105}$$

and the solutions (2.103) become:

$$a(\eta) = a_m \begin{cases} \sinh(c\eta) , & \text{for } K = -1 , \\ c\eta , & \text{for } K = 0 , \\ \sin(c\eta) , & \text{for } K = 1 . \end{cases} \tag{2.106}$$

If we want to recover the cosmic time from the above solutions, we need to solve the following integration:

$$\int_0^{\eta} a(\eta')d\eta' = t . \tag{2.107}$$

Using Eq. (2.106), one obtains:

$$ct = a_m \begin{cases} \cosh(c\eta) - 1 , & \text{for } K = -1 , \\ (c\eta)^2/2 , & \text{for } K = 0 , \\ 1 - \cos(c\eta) , & \text{for } K = 1 . \end{cases} \tag{2.108}$$

Inverting these relations allows you to find $\eta = \eta(t)$, which once substituted in Eq. (2.106) allows to find $a = a(t)$.

Exercise 2.17 Using the solutions (2.108), find the explicit form of $a(t)$. Show that $ct = a_m \eta^2/2$ leads to Eq. (2.99).

2.4.4 Cold Matter-Dominated Universe

For $\rho = \rho_0 a^{-3}$, $K = 0$ and $\Lambda = 0$, the solution of Friedmann equation (2.50) is straightforwardly obtained:

$$a = \left(\frac{t}{t_0}\right)^{2/3} . \tag{2.109}$$

The deceleration parameter is $q_0 = 1/2$ and the age of the universe is:

$$t_0 = \frac{2}{3H_0} = 6.52\ h^{-1}\ \text{Gyr} . \tag{2.110}$$

This model of universe is also known as the **Einstein-de Sitter universe**. A part the fact that it does not predict any accelerated expansion, there are also problems with the age of the universe given in Eq. (2.110): it is smaller than the one of some globular clusters (Velten et al. 2014).

Exercise 2.18 Prove the results of Eqs. (2.109) and (2.110).

Worse than the radiation-dominated case, it is impossible to analytically solve Friedmann equation (2.50) for a dust-dominated universe when $K \neq 0$. But, as in the radiation-dominated case, it is possible to find an exact solution for $a(\eta)$. Let's write Eq. (2.53) for the dust-dominated case:

$$a'' = \frac{4\pi G}{3}\rho a^3 - Kc^2 a \ . \tag{2.111}$$

Note that $\rho a^3 = \rho_0 =$ constant. The general solution is therefore the general solution of Eq. (2.101) plus a particular solution of Eq. (2.111), that is:

$$a(\eta) = \begin{cases} C_1 \sinh(c\eta) + C_2 \cosh(c\eta) - \frac{4\pi G}{3c^2}\rho_0 \ , & \text{for } K = -1 \ , \\ C_3 + C_4\eta + \frac{2\pi G}{3}\rho_0\eta^2 \ , & \text{for } K = 0 \ , \\ C_5 \sin(c\eta) + C_6 \cos(c\eta) + \frac{4\pi G}{3c^2}\rho_0 \ , & \text{for } K = 1 \ . \end{cases} \tag{2.112}$$

Exercise 2.19 Using the condition $a(0) = 0$ and employing Friedmann equation for a dust-dominated universe, i.e.

$$a'^2 + Kc^2a^2 = \frac{8\pi G}{3}\rho a^4 \ , \tag{2.113}$$

as constraint, show that Eq. (2.112) can be cast as:

$$a(\eta) = \frac{4\pi G}{3c^2}\rho_0 \begin{cases} \cosh(c\eta) - 1 \ , & \text{for } K = -1 \ , \\ (c\eta)^2/2 \ , & \text{for } K = 0 \ , \\ 1 - \cos(c\eta) \ , & \text{for } K = 1 \ . \end{cases} \tag{2.114}$$

Recovering the cosmic time from Eq. (2.114) one has:

$$ct = \frac{4\pi G}{3c^2}\rho_0 \begin{cases} \sinh(c\eta) - c\eta \ , & \text{for } K = -1 \ , \\ (c\eta)^3/6 \ , & \text{for } K = 0 \ , \\ c\eta - \sin(c\eta) \ , & \text{for } K = 1 \ . \end{cases} \tag{2.115}$$

Unfortunately, the above relations for $K = \pm 1$ cannot be explicitly inverted in order to give $\eta(t)$ and then $a(t)$.

2.4.5 Radiation Plus Dust Universe

The mixture of radiation plus matter is a cosmological model closer to reality and with which we can describe the evolution of our universe on a larger timespan than the single component-dominated cases. Consider the total density:

$$\rho = \rho_{\rm m} + \rho_{\rm r} = \frac{\rho_{\rm eq}}{2}\frac{a_{\rm eq}^3}{a^3} + \frac{\rho_{\rm eq}}{2}\frac{a_{\rm eq}^4}{a^4}\ , \tag{2.116}$$

where $a_{\rm eq}$ is the equivalence scale factor, i.e. the scale factor evaluated at the time at which dust and radiation densities were equal. At this time, we dub the total density as $\rho_{\rm eq}$. Now write down the acceleration equation (2.53) for the dust plus radiation model:

$$a'' = \frac{4\pi G}{3}\rho_m a^3 - Kc^2 a\ . \tag{2.117}$$

It is identical to the dust-dominated case, viz. Eq. (2.111)! Indeed, the fact that radiation is also present will enter when we set the constraint from Friedmann equation, which is the following:

$$a'^2 + Kc^2a^2 = \frac{4\pi G\rho_{\rm eq}}{3}\left(a_{\rm eq}^3 a + a_{\rm eq}^4\right)\ . \tag{2.118}$$

Solving Eq. (2.117) with the condition $a(0) = 0$ leads to the following solutions:

$$a(\eta) = \frac{2\pi G\rho_{\rm eq}a_{\rm eq}^3}{3c^2}\begin{cases} C_1\sinh(c\eta) + \cosh(c\eta) - 1\ , & \text{for } K = -1\ , \\ C_2\eta + (c\eta)^2/2\ , & \text{for } K = 0\ , \\ C_3\sin(c\eta) + 1 - \cos(c\eta)\ , & \text{for } K = 1\ . \end{cases} \tag{2.119}$$

Now use the constraint from Friedmann equation, i.e.

$$a'^2(\eta = 0) = \frac{4\pi G\rho_{\rm eq}}{3}a_{\rm eq}^4\ , \tag{2.120}$$

and find that:

$$C_1 = C_3 = c\sqrt{\frac{3}{\pi G\rho_{\rm eq}a_{\rm eq}^2}} \equiv c\tilde{\eta}\ , \qquad C_2 = c^2\tilde{\eta}\ , \tag{2.121}$$

so that:

$$a(\eta) = \frac{2a_{\rm eq}}{c^2\tilde{\eta}^2}\begin{cases} c\tilde{\eta}\sinh(c\eta) + \cosh(c\eta) - 1\ , & \text{for } K = -1\ , \\ c^2\tilde{\eta}\eta + (c\eta)^2/2\ , & \text{for } K = 0\ , \\ c\tilde{\eta}\sin(c\eta) + 1 - \cos(c\eta)\ , & \text{for } K = 1\ . \end{cases} \tag{2.122}$$

In particular, the solution for $K = 0$ is:

$$a(\eta) = a_{eq}\left(2\frac{\eta}{\tilde{\eta}} + \frac{\eta^2}{\tilde{\eta}^2}\right) . \tag{2.123}$$

Exercise 2.20 Show that the conformal time at equivalence η_{eq} and $\tilde{\eta}$ are related by:

$$\eta_{eq} = (\sqrt{2} - 1)\tilde{\eta} . \tag{2.124}$$

Exercise 2.21 Solve Friedmann equation for the ΛCDM model, neglecting radiation and spatial curvature:

$$\frac{H^2}{H_0^2} = \frac{\Omega_{m0}}{a^3} + \Omega_\Lambda . \tag{2.125}$$

Show that:

$$a(t) = \left[\frac{\Omega_{m0}}{\Omega_\Lambda}\sinh^2\left(\frac{3}{2}\sqrt{\Omega_\Lambda}H_0 t\right)\right]^{1/3} . \tag{2.126}$$

Exercise 2.22 Solve the Friedmann equation for the curvature-dominated universe:

$$H^2 = -\frac{Kc^2}{a^2} . \tag{2.127}$$

This is the Milne model (Milne 1935). Clearly, only $K = -1$ is allowed.

Show then that $a = ct$. Substitute this solution into the expression for the Ricci scalar (2.45). Show that $R = 0$.

Write down explicitly the FLRW metric with $a = ct$ and show that it is Minkowski metric written in a coordinate systems different from the usual.

The above last result is not completely surprising, since Milne model has no matter (empty universe) and no cosmological constant. The spatial hypersurfaces are already maximally symmetric because of the cosmological principle and the absence of matter add even more symmetry to the spacetime.

2.5 Distances in Cosmology

We present and discuss in this section the various notions of distance that are employed in cosmology. See e.g. Hogg (1999) for a reference on the subject.

2.5.1 Comoving Distance and Proper Distance

We have already encountered comoving coordinates in the FLRW metric (2.17) and the proper radius $\mathcal{D}(t) \equiv a(t)r$ in the FLRW metric (2.21). We must be clearer about the difference between the radial coordinate and the distance. They are equal only when $d\Omega = 0$. The comoving square infinitesimal distance is indeed, from FLRW metric (2.17) the following:

$$d\chi^2 = \frac{dr^2}{1 - Kr^2} + r^2 d\Omega^2 , \tag{2.128}$$

i.e. it has indeed a radial part, but also has a transversal part. So, if χ is the comoving distance between two points, the proper distance at a certain time t is $d(\chi, t) = a(t)\chi$.

The comoving distance is a notion of distance which does not include the expansion of the universe and thus does not depend on time.

The proper distance is the distance that would be measured instantaneously by rulers. For example, imagine to extend a ruler between GN-z11 (the farthest known galaxy, $z = 11.09$) and us. Our reading at the time t would be the proper distance at that time.

Suppose that $d\Omega = 0$. Then the comoving distance to an object with radial coordinate r is the following:

$$\chi = \int_0^r \frac{dr'}{\sqrt{1 - Kr'^2}} = \begin{cases} \text{arcsin}\, r \ , \ \text{for } K = 1 \ , \\ r \ , \ \text{for } K = 0 \ , \\ \text{arcsinh}\, r \ , \ \text{for } K = -1 \ . \end{cases} \tag{2.129}$$

Deriving d with respect to the time one gets:

$$\dot{d} = \dot{a}\chi = \frac{\dot{a}}{a} d = Hd \ , \tag{2.130}$$

which recovers the Hubble's law for $t = t_0$.

2.5.2 The Lookback Time

Imagine a photon emitted by a galaxy at a time t_{em} and detected at the time t_0 on Earth. A very basic notion of distance is $c(t_0 - t_{\text{em}})$, i.e. it is the light-travel distance, based on the fact that light always travels with speed c. The quantity $t_0 - t_{\text{em}}$ is called **lookback time** and suggestively reminds the fact that when we observe some source in the sky we are actually looking into the past, because of the finiteness of c.

From the FLRW metric, by putting $ds^2 = 0$, we can relate the lookback time with the comoving distance as follows:

$$cdt = a(t)d\chi \, . \tag{2.131}$$

This seems quite similar to the proper distance, but careful: the proper distance is defined as $a\chi$ and evidently $ad\chi \neq d(a\chi)$. The lookback time is the photon time of flight and thus it includes cumulatively the expansion of the universe. On the other hand, the proper distance is the distance considered between two simultaneous events and therefore the expansion of the universe is not taken into account cumulatively.

Since we observe redshifts, is there a way to calculate the lookback time from z? In principle yes: one solves Friedmann equation, finds $a(t)$, inverts this function in order to find $t = t(a)$, uses $1 + z = 1/a$ and finally gets a relation $t = t(z)$. For example, for the flat Einstein-de Sitter universe, using Eqs. (2.109) and (2.110) one gets:

$$1 + z = \left(\frac{2}{3H_0 t}\right)^{2/3} \qquad \Rightarrow \qquad t = \frac{2}{3H_0(1+z)^{3/2}} \, . \tag{2.132}$$

This approach is model-dependent because in order to solve the Friedmann equation we must know it and this is possible only if we know, or *model*, the energy content of the universe. Hence the model-dependence.

A model-independent way of relating lookback time and redshift is **cosmography**, a word which means “measuring the universe”. In practice, cosmography consists in a Taylor expansion of the scale factor about its today value:

$$a(t) = a(t_0) + \left.\frac{da}{dt}\right|_{t_0} (t - t_0) + \frac{1}{2}\left.\frac{d^2a}{dt^2}\right|_{t_0} (t - t_0)^2 + \cdots \tag{2.133}$$

where we stop at the second order, for simplicity. This can be written as

$$a(t) = a(t_0)\left[1 + H_0(t - t_0) - \frac{1}{2}q_0 H_0^2 (t - t_0)^2 + \cdots\right] , \tag{2.134}$$

i.e. the first coefficient of the expansion is the Hubble constant, whereas the second one is proportional to the deceleration parameter. The third is usually called *jerk* and the fourth *snap*. All these parameters are evaluated at t_0 in the above expansion.

Exercise 2.23 For $a(t_0) = 1$ and introducing the redshift show that:

$$z \sim H_0(t_0 - t) + \frac{1}{2}(q_0 + 2)H_0^2(t_0 - t)^2 \, . \tag{2.135}$$

Here is a direct, model-independent relation between the redshift and the lookback time $(t_0 - t)$.

2.5.3 *Distances and Horizons*

For a photon, not unexpectedly,

$$d\chi = \frac{cdt}{a(t)} = cd\eta \ , \tag{2.136}$$

i.e. the comoving distance is equal to the conformal time, which we introduced in Eq. (2.18). We might say that the comoving distance is a *lookback conformal time*.

By integrating $cdt/a(t)$ from t_{em} to t_0 we get the comoving distance from the source to us, or the conformal time spent by the photon travelling from the source to us:

$$\chi = \int_{t_{em}}^{t_0} \frac{cdt'}{a(t')} = \int_a^1 \frac{cda'}{H(a')a'^2} \ . \tag{2.137}$$

For the dust-dominated case one has $H = H_0/a^{3/2}$ and the comoving distance as a function of the scale factor and of the redshift is:

$$\chi(a) = \frac{c}{H_0}\int_a^1 \frac{da'}{\sqrt{a'}} = \frac{2c}{H_0}\left(1-\sqrt{a}\right) \ , \quad \chi(z) = \frac{2c}{H_0}\left(1-\frac{1}{\sqrt{1+z}}\right) \ . \tag{2.138}$$

When $z \to 0$, $\chi \sim cz/H_0$. Comparing with Eq. (2.135) one sees that, at the first order in the redshift, the lookback time distance is equivalent to the comoving one.

Exercise 2.24 Calculate the comoving distance as a function of the scale factor and of the redshift for a radiation-dominated universe and for the de Sitter universe.

When the lower integration limit in Eq. (2.137) is $a = 0$, i.e. the Big Bang, one defines the **comoving horizon** χ_p (also known as **particle horizon** or **cosmological horizon**). This is the conformal time spent from the Big Bang until the cosmic time t or scale factor a. It is also the maximum comoving distance travelled by a photon (hence the name particle horizon) since the Big Bang and so it is the comoving size of the visible universe.

In the dust-dominated case, using Eq. (2.138), with $a = 0$ or $z = \infty$ one obtains:

$$\chi_p = c\eta_0 = \frac{2c}{H_0} \ . \tag{2.139}$$

Note that this is not the age of the universe given in Eq. (2.110), but three times its value.

When the upper integration limit of Eq. (2.137) is infinite, one defines the **event horizon**:

$$\chi_e(t) \equiv c\int_t^\infty \frac{dt'}{a(t')} = c\int_a^\infty \frac{da'}{H(a')a'^2} \ , \tag{2.140}$$

which of course makes sense only if the universe does not collapse. This represents the maximum distance travelled by a photon from a time t. If it diverges, then no event horizon exists and therefore eventually all the events in the universe will be causally connected. This happens, for example, in the dust-dominated case:

$$\chi_e = \frac{c}{H_0} \int_a^\infty \frac{da'}{\sqrt{a'}} = \infty \, . \tag{2.141}$$

But, in the de Sitter universe we have

$$\chi_e = \frac{c}{H_0} \int_a^\infty \frac{da'}{a'^2} = \frac{c}{H_0 a} \, . \tag{2.142}$$

The proper event horizon for the de Sitter universe is a constant:

$$a\chi_e = \frac{c}{H_0} \, . \tag{2.143}$$

2.5.4 The Luminosity Distance

The luminosity distance is a very important notion of distance for observation. It is based on the knowledge of the intrinsic luminosity L of a source, which is therefore called **standard candle**. Type Ia supernovae are standard candles, for example. Then, measuring the flux F of that source and dividing L by F, one obtains the square luminosity distance:

$$d_L^2 \propto \frac{L}{F} \, . \tag{2.144}$$

Now, imagine a source at a certain redshift z with intrinsic luminosity $L = dE/dt$. The observed flux is given by the following formula:

$$F = \frac{dE_0}{dt_0 A_0} \, , \tag{2.145}$$

where A_0 is the area of the surface on which the radiation is spread:

$$A_0 = 4\pi a_0^2 \chi^2 \, , \tag{2.146}$$

i.e. over a sphere with the proper distance as the radius. We must use the proper distance, because this is the instantaneous distance between source and observer at the time of detection. Note that χ is the comoving distance between the source and us.

We do not observe the same photon energy as the one emitted, because photons suffer from the cosmological redshift, thus:

$$\frac{dE}{dE_0} = \frac{a_0}{a} \,. \tag{2.147}$$

Finally, the time interval used at the source is also different from the one used at the observer location:

$$\frac{dt}{dt_0} = \frac{a}{a_0} \,. \tag{2.148}$$

We can easily show this by using FLRW metric with $ds^2 = 0$, i.e. $cdt = a(t)d\chi$. Consider the same $d\chi$ at the source and at the observer's location. Thus, $cdt = a(t)d\chi$ and $cdt_0 = a(t_0)d\chi$ and the above result follows.

Putting all the contributions together, we get

$$F = \frac{dE_0}{dt_0 A_0} = \frac{a^2 dE}{a_0^2 dt 4\pi a_0^2 \chi^2} = \frac{dE}{dt 4\pi a_0^2 \chi^2 (1+z)^2} \,. \tag{2.149}$$

Hence, the luminosity distance is defined as:

$$\boxed{d_{\rm L} \equiv a_0 (1+z)\chi} \tag{2.150}$$

From this formula and the observed redshifts of type Ia supernovae we can determine if the universe is in an accelerated expansion, in a model-independent way. In order to do this, we first need to know how to expand χ in series of powers of the redshift.

Using the definition (2.137) and the expansion (2.135), we get:

$$\chi = \int_t^{t_0} \frac{cdt'}{a(t')} = \frac{c(t_0 - t)}{a_0} + \frac{cH_0}{2a_0}(t_0 - t)^2 + \cdots \tag{2.151}$$

where we stop at the second order only. This is the expansion of the comoving distance with respect to the lookback time. We must invert the power series of Eq. (2.135) in order to find the expansion of the lookback time with respect to the redshift. This can be done, for example, by assuming the following ansatz:

$$H_0(t_0 - t) = \alpha + \beta z + \gamma z^2 + \cdots \tag{2.152}$$

and substitute it into Eq. (2.135), keeping at most terms $\mathcal{O}(z^2)$.

Exercise 2.25 Show that:

$$\alpha = 0 \,, \qquad \beta = 1 \,, \qquad \gamma = -\frac{1}{2}(q_0 + 2) \,, \tag{2.153}$$

and thus

$$H_0(t_0 - t) = z - \frac{1}{2}(q_0 + 2)z^2 + \cdots \tag{2.154}$$

Substituting the expansion of Eq. (2.154) back into Eq. (2.150), one gets

$$d_{\mathrm{L}} = \frac{c}{H_0}\left[z + \frac{1}{2}(1 - q_0)z^2 + \cdots\right] , \tag{2.155}$$

where note again that at the lowest order the luminosity distance is cz/H_0, identical to the comoving distance and to the lookback time distance.

Since d_{L} and z are measured, one can fit the data with this quadratic function and determine q_0, thereby establishing if the universe expansion is accelerated or not. Note that H_0 is an overall multiplicative factor, thus does not determine the shape of the function $d_L(z)$.

In the case of a dust-dominated universe, using Eq. (2.138), the luminosity distance has the following expression:

$$d_{\mathrm{L}} = \frac{2c}{H_0}\left(1 + z - \sqrt{1+z}\right) . \tag{2.156}$$

For small z, this distance can be expanded in powers of the redshift as:

$$d_{\mathrm{L}} = \frac{c}{H_0}\left(z + \frac{1}{4}z^2 + \cdots\right) , \tag{2.157}$$

which, when compared with Eq. (2.150), provides $q_0 = 1/2$, as expected.

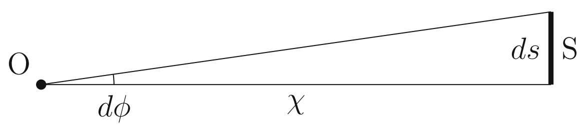
2.5.5 Angular Diameter Distance

The angular diameter distance is based on the knowledge of proper sizes. Objects with a known proper size are called **standard rulers**. Suppose a standard ruler of transversal proper size ds (small) to be at a redshift z and comoving distance χ. Moreover, this object has an angular dimension $d\phi$, also small. See Fig. 2.6 for reference.

At a fixed time t, we can write the FLRW metric as:

$$ds^2 = a(t)^2 d\chi^2 . \tag{2.158}$$

Fig. 2.6 Defining the angular diameter distance



Since the object is small and we are at the origin of the reference frame, the comoving distance χ is also the radial distance. Therefore, the transversal distance is:

$$ds = a(t)\chi d\phi \ . \tag{2.159}$$

Dividing the proper dimension of the object by its angular size provides us with the angular diameter distance:

$$d_{\rm A} = a(t)\chi \ . \tag{2.160}$$

For the case of a dust-dominated universe, one has:

$$d_{\rm A} = \frac{2c}{H_0}\left[\frac{1}{1+z} - \frac{1}{(1+z)^{3/2}}\right] . \tag{2.161}$$

In the limit of small z, we find $d_{\rm A} \sim cz/H_0$. All the distances that we defined insofar coincide at the first order expansion in z.

Note the relation:

$$d_{\rm L} = (1+z)^2 d_{\rm A} \ , \tag{2.162}$$

known as **Etherington's distance duality** (Etherington 1933).

In gravitational lensing applications it is often necessary to know the angular-diameter distance between two sources at different redshifts (i.e. the angular-diameter distance between the lens and the background source). In order to compute this, let us refer to Fig. 2.7.

The problem is to determine the angular-diameter distance between L and S, say $d_{\rm A}({\rm LS})$. Is this the difference between the angular-diameter distances $d_{\rm A}({\rm S}) - {\rm d}_{\rm A}({\rm L})$? We now show that this is not the case. Simple trigonometry is sufficient to establish that:

$$ds = a(t_{\rm S})\chi_{\rm S} d\phi_{\rm S} = a(t_{\rm S})\chi_{\rm LS} d\phi_{\rm L} \ , \tag{2.163}$$

And for comoving distances we do have that $\chi_{\rm LS} = \chi_{\rm S} - \chi_{\rm L}$. Therefore, we have

$$\boxed{d_{\rm A}({\rm LS}) = a(t_{\rm S})\chi_{\rm LS} = a(t_{\rm S})(\chi_{\rm S} - \chi_{\rm L})} \tag{2.164}$$

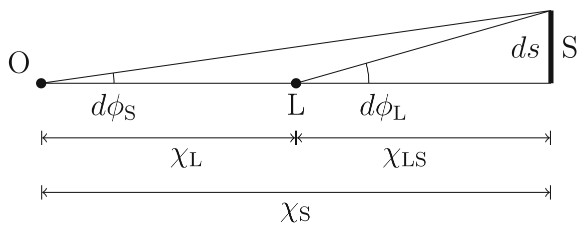


Fig. 2.7 The angular diameter distance between two different redshifts

which is the relation we were looking for, and it is different from the difference between the angular diameter distances:

$$d_A(S) - d_A(L) = a(t_S)\chi_S - a(t_L)\chi_L \ . \tag{2.165}$$

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