Crisp Fuzzy Implications

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Abstract. Implications play an important role in fuzzy logics as they can be used both in practical and theoretical works. There exist many works in the literature where fuzzy implications behave in a crisp manner, i.e., implications that map to either zero or one. In this sense, we call those implications as crisp fuzzy implications and our goal is to study some their main features.

1 Introduction

A great deal of studies involving fuzzy implications can be found in the literature over the last years [1,2,4–6,12]. Fuzzy implications are interesting from the theoretical point of view to its use on a variety of applications. For instance, they can be used to perform any fuzzy "if-then" rule in fuzzy systems and inference processes, which basically combine membership functions with the control rules to derive the fuzzy output. Regarding the theoretical aspect, many works have also been done aiming to generalize the traditional implication into fuzzy logic, explaining why there exists so many classes of fuzzy implications. The existence of those classes of fuzzy implications is justified by the fact that depending on the context or/and on the rules and their behavior, different implications with different properties can be adequate.

In the literature, it is possible to find examples of fuzzy implications with a crisp behavior, i.e., fuzzy implications that always map to either 0 or 1. For instance, in [13], it was defined two crisp-valued operators, named standard sharp and standard strict, as follows:

1. Standard sharp

$$I_s(x, y) = \begin{cases} 1, & \text{if } x < 1 \text{ or } y = 1 \\ 0, & \text{otherwise} \end{cases}$$

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2. Standard strict (also called Rescher-Gaines implication [8])

$$I_G(x,y) = \begin{cases} 1, & \text{if } x \leq y \\ 0, & \text{otherwise} \end{cases}$$

Those implications were used in various applications, for instance, in the domains of approximate reasoning [8], relational databases [9], fuzzy control [10,16], face recognition [17].

Thus, in this paper we intend to study the class of crisp fuzzy implications, i.e. fuzzy implications which always map to 0 or 1.

The paper is organized as follows. Section 2 summarizes some of the basic concepts demanded to understand the proposal in this work, including the concept of fuzzy implication and related properties. The study of crisp fuzzy implications is done in Sect. 3, including the most important results. At last, we finish in Sect. 4 with our final conclusions.

2 Preliminaries

Definition 1. *A function* $T : [0,1]^2 \rightarrow [0,1]$ *is said to be a* ***triangular norm*** *(**t-norm**, for short) if it satisfies the following conditions, for all* $x, y, z \in [0,1]$*:*

(T1) Symmetry: $T(x,y) = T(y,x)$*;*
(T2) Associativity: $T(x, T(y,z)) = T(T(x,y), z)$*;*
(T3) Monotonicity: If $x_1 \leq x_2$ *and* $y_1 \leq y_2$ *then* $T(x_1,y_1) \leq T(x_2,y_2)$*;*
(T4) 1-identity: $T(x,1) = x$. *(boundary condition)*

In fuzzy logic, the conjunction is often represented by a t-norm. The standard fuzzy conjunction $T_M : [0,1]^2 \rightarrow [0,1]$ given by $T_M(x,y) = min\{x,y\}$ is the only idempotent t-norm (see [11] - Theorem 3.9).

Proposition 1 *[3]. Let* T *be a t-norm. Then* $T(0,y) = 0$*, for each* $y \in [0,1]$.

Definition 2. *A t-norm* T *is called* ***positive*** *if satisfies the following condition:* $T(x,y) = 0$ *if and only if* $x = 0$ *or* $y = 0$.

Definition 3. *A function* $S : [0,1]^2 \rightarrow [0,1]$ *is said to be a* ***triangular conorm*** *(**t-conorm**, for short) if it satisfies the following conditions, for all* $x, y, z \in [0,1]$*:*

(S1) Symmetry: $S(x,y) = S(y,x)$*;*
(S2) Associativity: $S(x, S(y,z)) = S(S(x,y), z)$*;*
(S3) Monotonicity: If $x_1 \leq x_2$ *and* $y_1 \leq y_2$ *then* $S(x_1,y_1) \leq S(x_2,y_2)$*;*
(S4) 0-identity: $S(x,0) = x$. *(boundary condition)*

From an axiomatic point of view, the difference between t-norms and t-conorms is just their boundary conditions.

Definition 4. *A function $N : [0,1] \to [0,1]$ is called a **fuzzy negation** if*

(N1) N is antitonic, i.e. $N(x) \leq N(y)$ whenever $y \leq x$;
(N2) $N(0) = 1$ and $N(1) = 0$.
*A fuzzy negation N is **strict** if*
(N3) N is continuous and
(N4) $N(x) < N(y)$ whenever $y < x$.
*A fuzzy negation N is **strong** if*
(N5) $N(N(x)) = x$, for each $x \in [0,1]$.
*A fuzzy negation N is **crisp** if*
(N6) $N(x) \in \{0,1\}$, for all $x \in [0,1]$ (see [7]).
*A fuzzy negation N is **frontier** if it satisfies the following property:*
(N7) $N(x) \in \{0,1\}$ if and only if $x = 0$ or $x = 1$.

Remark 1. By [7], a fuzzy negation $N : [0,1] \to [0,1]$ is crisp if and only if there exists $\alpha \in [0,1)$ such that $N = N_\alpha$ or there exists $\alpha \in (0,1]$ such that $N = N^\alpha$, where

$$N_\alpha(x) = \begin{cases} 0, & \text{if } x > \alpha \\ 1, & \text{if } x \leq \alpha \end{cases} \qquad N^\alpha(x) = \begin{cases} 0, & \text{if } x \geq \alpha \\ 1, & \text{if } x < \alpha \end{cases}$$

Theorem 1 *[1]. If a function $N : [0,1] \to [0,1]$ satisfies $(N1)$ and $(N5)$, then it also satisfies $(N2)$ and $(N3)$. Moreover, N is a bijection, i.e., it satisfies $(N4)$.*

Corollary 1 *[1]. Every strong negation is strict.*

Definition 5. *A function $I : [0,1]^2 \to [0,1]$ is a **fuzzy implication** if the following properties are satisfied, for all $x, y, z \in [0,1]$:*

(I1) If $x \leq z$ then $I(x,y) \geq I(z,y)$;
(I2) If $y \leq z$ then $I(x,y) \leq I(x,z)$;
(I3) $I(0,y) = 1$;
(I4) $I(x,1) = 1$;
(I5) $I(1,0) = 0$.

The set of all fuzzy implications will be denoted by $\mathcal{FI}$.

Definition 6. *Let $I \in \mathcal{FI}$. The function $N_I : [0,1] \to [0,1]$ defined by $N_I(x) = I(x,0)$, $x \in [0,1]$ is called the **natural negation of** I or the negation induced by I.*

Definition 7. *Let T be a t-norm, S a t-conorm and N a fuzzy negation, then:*

– *A function* $I : [0,1]^2 \to [0,1]$ *is called a* ***(S, N)-implication*** *(denoted by* $I_{S,N}$ *) if* $I(x,y) = S(N(x),y)$.
– *A function* $I : [0,1]^2 \to [0,1]$ *is called an* ***R-implication*** *(denoted by* I_T *) if* $I(x,y) = sup\{t \in [0,1] \mid T(x,t) \leq y\}$.
– *A function* $I : [0,1]^2 \to [0,1]$ *is called a* ***QL-implication*** *(denoted by* $I_{S,N,T}$ *) if* $I(x,y) = S(N(x),T(x,y))$.
– *A function* $I : [0,1]^2 \to [0,1]$ *is called a* ***D-implication*** *(denoted by* $I_{S,T,N}$*) if* $I(x,y) = S(T(N(x),N(y)),y)$.

Definition 8. *A fuzzy implication I is said to satisfy:*

(i) *the* ***exchange principle*** *if, for all* $x,y,z \in [0,1]$
$I(x,I(y,z)) = I(y,I(x,z))$; *(EP)*
(ii) *the* ***left neutrality property****, if*
$I(1,y) = y, \quad y \in [0,1]$; *(NP)*
(iii) *the* ***identity principle****, if*
$I(x,x) = 1, \quad x \in [0,1]$; *(IP)*
(iv) *the* ***left-ordering property*** *if, for all* $x,y \in [0,1]$
$I(x,y) = 1$ *whenever* $x \leq y$; *(LOP)*
(v) *the* ***right-ordering property*** *if for all* $x,y \in [0,1]$
$I(x,y) \neq 1$ *whenever* $x > y$. *(ROP)*

Definition 9. *Let* $I \in \mathcal{FI}$ *and let N be a fuzzy negation. I is said to satisfy the:*

(i) ***contraposition law*** *(or in other words, the contrapositive symmetry) with respect to N, if*
$I(x,y) = I(N(y),N(x)), \ x,y \in [0,1]$; *(CP)*
(ii) ***left contraposition law*** *with respect to N, if*
$I(N(x),y) = I(N(y),x), \ x,y \in [0,1]$; *(L-CP)*
(iii) ***right contraposition law*** *with respect to N, if*
$I(x,N(y)) = I(y,N(x)), \ x,y \in [0,1]$. *(R-CP)*

If I satisfies the (left, right) contrapositive symmetry with respect to a specific N, then we will denote this by $L\!-\!CP(N)$, $R\!-\!CP(N)$ and $CP(N)$, respectively.

In [14,15], Pinheiro et al. introduced a new class of implication, named (T,N)-implications which was defined by means of fuzzy negations and a t-norm.

Definition 10 *[15]. Let N and N′ be fuzzy negations and T be a t-norm. The function* $I_{T,N}^{N'}$ *defined by* $I_{T,N}^{N'}(x,y) = N'(T(x,N(y)))$ *is called a* $(\mathbf{N}', \mathbf{T}, \mathbf{N})$***-implication****.*

Actually, in [15] for (T,N)-implications we had $N' = N$ which is different from the previous definition where we have different negations. In order to avoid misunderstanding between definitions, from here forth, implications defined according to Definition 10 we be called (N',T,N)-implications.

3 Crisp Fuzzy Implications

In classical logic there is only one bivalent implication, however in the fuzzy setting the notion of bivalence gives rise to an uncountable family of such implications. They are called here Crisp fuzzy implication.

Definition 11. *Let $I : [0,1]^2 \to [0,1]$ be a fuzzy implication. We say that I is a* ***crisp fuzzy implication*** *if $I(x,y) \in \{0,1\}$ for all $x, y \in [0,1]$.*

Proposition 2. *Let $I : [0,1]^2 \to [0,1]$ be a fuzzy implication. Then I is crisp if and only if one of the following conditions are satisfied, for all $x, y \in [0,1]$:*

(C1) If there exists $\alpha \in (0,1]$ and $\beta \in [0,1)$ such that $I(x,y) = I_{\alpha,\beta}(x,y)$, where

$$I_{\alpha,\beta}(x,y) = \begin{cases} 0, & if\ x \geq \alpha\ and\ y \leq \beta \\ 1, & otherwise \end{cases};$$

(C2) If there exists $\alpha \in [0,1)$ and $\beta \in (0,1]$ such that $I(x,y) = I^{\alpha,\beta}(x,y)$, where

$$I^{\alpha,\beta}(x,y) = \begin{cases} 0, & if\ x > \alpha\ and\ y < \beta \\ 1, & otherwise \end{cases};$$

(C3) If there exists $\alpha, \beta \in (0,1]$ such that $I(x,y) = {I_{\alpha}}^{\beta}(x,y)$, where

$${I_{\alpha}}^{\beta}(x,y) = \begin{cases} 0, & if\ x \geq \alpha\ and\ y < \beta \\ 1, & otherwise \end{cases};$$

(C4) If there exists $\alpha, \beta \in [0,1)$ such that $I(x,y) = {I^{\alpha}}_{\beta}(x,y)$, where

$${I^{\alpha}}_{\beta}(x,y) = \begin{cases} 0, & if\ x > \alpha\ \ and\ \ y \leq \beta \\ 1, & otherwise \end{cases}.$$

Proof. First, suppose I is crisp, then as $I(0,0) = 1$ and $I(1,0) = 0$, we have by (I1) that there exists: (1) $\alpha \in (0,1]$ such that $I(x,0) = \begin{cases} 0, & \text{if } x \geq \alpha \\ 1, & \text{otherwise} \end{cases}$ or (2) $\alpha \in [0,1)$ such that $I(x,0) = \begin{cases} 0, & \text{if } x > \alpha \\ 1, & \text{otherwise} \end{cases}$. By (I2), we have for case (1) that there exist: $(i)_1$ $\beta \in [0,1)$ such that $I(x,y) = \begin{cases} 0, & \text{if } x \geq \alpha \text{ and } y \leq \beta \\ 1, & \text{otherwise} \end{cases}$ or $(ii)_1$ $\beta \in (0,1]$ such that $I(x,y) = \begin{cases} 0, & \text{if } x \geq \alpha \text{ and } y < \beta \\ 1, & \text{otherwise} \end{cases}$. Hence, $I(x,y) = I_{\alpha,\beta}(x,y)$ or $I(x,y) = {I_{\alpha}}^{\beta}(x,y)$, for all $x, y \in [0,1]$, respectively.

Similarly, still by (I2), for case (2) there exist: $(i)_2$ $\beta \in [0,1)$ such that $I(x,y) = \begin{cases} 0, & \text{if } x > \alpha \text{ and } y \leq \beta \\ 1, & \text{otherwise} \end{cases}$ or $(ii)_2$ $\beta \in (0,1]$ such that $I(x,y) = \begin{cases} 0, & \text{if } x > \alpha \text{ and } y < \beta \\ 1, & \text{otherwise} \end{cases}$. Hence, $I(x,y) = I^{\alpha}{}_{\beta}(x,y)$ or $I(x,y) = I^{\alpha,\beta}(x,y)$, for all $x, y \in [0,1]$, respectively.

The reciprocal case follows straightforward.

Definition 12. *Let I be a crisp fuzzy implication. Independently from I being of type C1, C2, C3 or C4, the pair (α, β) is called the* ***threshold pair of*** *I.*

In the following proposition, we can observe that we can obtain a crisp fuzzy implications from any fuzzy implication I.

Proposition 3. *Let $I \in \mathcal{FI}$. Then, for any $\gamma \in (0,1]$, $I_\gamma(x,y) = \begin{cases} 1, & if\ I(x,y) \geq \gamma \\ 0, & if\ I(x,y) < \gamma \end{cases}$ is a crisp fuzzy implication.*

Proof. We will first prove that I_γ satisfies the conditions demanded in Definition 5. Indeed,

(I_γ1) For all $x, y, z \in [0,1]$, such that $x \leq y$, we have by (I1) that $I(y,z) \leq I(x,z)$. We will analyze the following cases: (1) If $I(x,z) \geq I(y,z) \geq \gamma$, then $I_\gamma(x,z) = I_\gamma(y,z) = 1$; (2) If $I(x,z) \geq \gamma > I(y,z)$, then $I_\gamma(x,z) = 1 > 0 = I_\gamma(y,z)$ and (3) if $\gamma > I(x,z) \geq I(y,z)$, then $I_\gamma(x,z) = I_\gamma(y,z) = 0$. Therefore, I_γ satisfies (I1).

(I_γ2) For all $x, y, z \in [0,1]$, such that $y \leq z$, we have by (I2) that $I(x,y) \leq I(x,z)$. We will analyze the following cases: (1) If $\gamma \leq I(x,y) \leq I(x,z)$, then $I_\gamma(x,y) = I_\gamma(x,z) = 1$; (2) If $I(x,y) < \gamma \leq I(x,z)$, then $I_\gamma(x,y) = 0 < 1 = I_\gamma(x,z)$ and (3) if $I(x,y) \leq I(x,z) < \gamma$, then $I_\gamma(x,y) = I_\gamma(x,z) = 0$. Therefore, I_γ satisfies (I2).

(I_γ3) For all $y \in [0,1]$, we have by (I3) that $I(0,y) = 1$. So, $I(0,y) \geq \gamma$ and thereby $I_\gamma(0,y) = 1$. Therefore, I_γ satisfies (I3).

(I_γ4) For all $x \in [0,1]$, we have by (I4) that $I(x,1) = 1$. So, $I(x,1) \geq \gamma$ and thereby $I_\gamma(x,1) = 1$. Therefore, I_γ satisfies (I4).

(I_γ5) By (I5), $I(1,0) = 0$. So, $I(1,0) < \gamma$ and thereby $I_\gamma(1,0) = 0$. Therefore, I_γ satisfies (I5).

We conclude that I_γ is a fuzzy implication. As, $I_\gamma(x,y) \in \{0,1\}$ for all $x, y \in [0,1]$, then I_γ is a crisp fuzzy implication.

Notice that if we take $\gamma \in [0,1)$, and $I^\gamma(x,y) = \begin{cases} 1, & \text{if } I(x,y) > \gamma \\ 0, & \text{if } I(x,y) \leq \gamma \end{cases}$, I^γ is also a crisp fuzzy implication.

Proposition 4. *Let $I^{N'}_{T,N}$ be a (N', T, N)-implication. Then $I^{N'}_{T,N}$ is crisp if and only if N' is a crisp fuzzy negation.*

Proof. Suppose N' is not crisp, then there is $z \in (0,1)$ such that $N'(z) \notin \{0,1\}$. So, for any t-norm T and any fuzzy negation N we have that $T(z, N(0)) = T(z,1) = z$, thus $I^{N'}_{T,N}(z,0) = N'(T(z,N(0))) = N'(z) \notin \{0,1\}$. Therefore $I^{N'}_{T,N}$ is not crisp. Conversely, if N' is crisp then, for any t-norm T and any fuzzy negation N, $I^{N'}_{T,N}(x,y) = N'(T(x,N(y))) \in \{0,1\}$.

Corollary 2. *Let I^N_T be a (T,N)-implication. Then I^N_T is crisp if and only if N is a crisp fuzzy negation.*

Proposition 5. *Let I be a (T, N)-implication for any crisp negation N and any t-norm T. Then:*

(i) I satisfies (LOP);
(ii) I does not satisfy (ROP).

Proof. Since N is crisp we have, by Corollary 2, that $I = I^N_T$ is crisp. Note that:

$$I(x,y) = N(T(x,N(y))) = \begin{cases} 1, & \text{if } N(y) = 0 \\ N(x), & \text{if } N(y) = 1 \end{cases}.$$

Then,

(i) For all $x, y \in [0,1]$ such that $x \leq y$, since N is crisp, we will analyze two cases:
(1) if there exists $\alpha \in [0,1)$ such that $N = N_\alpha$, then

$$I(x,y) = \begin{cases} 1, & \text{if } y > \alpha \\ N(x), & \text{if } y \leq \alpha \end{cases}. \tag{1}$$

For $y \leq \alpha$, as $x \leq y$, then $x \leq \alpha$. So $N_\alpha(x) = 1$ and hence $I(x,y) = 1$.
(2) If there exists $\alpha \in (0,1]$ such that $N = N^\alpha$, then

$$I(x,y) = \begin{cases} 1, & \text{if } y \geq \alpha \\ N(x), & \text{if } y < \alpha \end{cases}. \tag{2}$$

For $y < \alpha$, as $x \leq y$, then $x < \alpha$. So $N_\alpha(x) = 1$ and therefore, $I(x,y) = 1$. Thus, I satisfies (LOP).

(ii) We will analyze two cases again:
(1) if $N = N_\alpha$ for some $\alpha \in [0,1)$, then there exists $x, y \in [0,1]$ such that $x > y > \alpha$. So, by Eq. (1), $I(x,y) = 1$.
(2) If $N = N^\alpha$ for some $\alpha \in (0,1]$, then there exists $x, y \in [0,1]$ such that $y < x < \alpha$. So, by Eq. (2) $I(x,y) = N^\alpha(x) = 1$. In any case, there exists $x > y$, but $I(x,y) = 1$, therefore I does not satisfy (ROP).

Proposition 6. *Let I be a crisp fuzzy implication. Then:*

(i) I satisfies (EP);
(ii) I satisfies $R\text{–}CP(N_I)$, where N_I is the natural negation of I;
(iii) I does not satisfy (NP);

Proof. (i) Since I is crisp, then by Proposition 2, I satisfies one of the conditions (C1), (C2), (C3) or (C4). If I satisfies (C1), then there exists $\alpha \in (0,1]$ and $\beta \in [0,1)$ such that $I(x,y) = I_{\alpha,\beta}(x,y)$. So, (1) for $z \leq \beta$, we have $I(y,z) = \begin{cases} 0, & \text{if } y \geq \alpha \\ 1, & \text{if } y < \alpha \end{cases}$. Thus,

$$I(x, I(y,z)) = \begin{cases} I(x,0), & \text{if } y \geq \alpha \\ I(x,1), & \text{if } y < \alpha \end{cases} = \begin{cases} 0, & \text{if } x \geq \alpha \text{ and } y \geq \alpha \\ 1, & \text{otherwise} \end{cases},$$

and we also have $I(x,z) = \begin{cases} 0, & \text{if } x \geq \alpha \\ 1, & \text{if } x < \alpha \end{cases}$. Thus,

$$I(y, I(x,z)) = \begin{cases} I(y,0), & \text{if } x \geq \alpha \\ I(y,1), & \text{if } x < \alpha \end{cases} = \begin{cases} 0, & \text{if } x \geq \alpha \text{ and } y \geq \alpha \\ 1, & \text{otherwise} \end{cases}.$$

Then, for $z \leq \beta$, (EP) is satisfied. Now, (2) for $z > \beta$, we have $I(y,z) = I(x,z) = 1$. So $I(x, I(y,z)) = I(x,1) = 1 = I(y,1) = I(y, I(x,z))$.

Cases (C2), (C3) and (C4) are similar to the previous one.Therefore, I satisfy (EP).

(ii) Indeed, for all $x, y \in [0,1]$,

$$I(x, N_I(y)) = I(x, I(y,0)) \overset{(EP)}{=} I(y, I(x,0)) = I(y, N_I(x)).$$

(iii) As $I(x,y) \in \{0,1\}$, since I is crisp then, for all $y \in (0,1)$, $I(x,y) \neq y$.

Proposition 7. *Let I be a crisp fuzzy implication with (α,β) as its threshold. If $\beta < \alpha$ then I satisfies (IP).*

Proof. Indeed, if I is of type (C1) then there is no $x \in [0,1]$ such that $x \geq \alpha$ and $x \leq \beta$ simultaneously, since $\beta < \alpha$. So $I(x,x) = 1$, for all $x \in [0,1]$. Similarly, if I is of type (C2), (C3) or (C4) we prove that $I(x,x) = 1$. Therefore, in any case, I satisfies (IP).

Proposition 8. *Let I be a crisp fuzzy implication with (α,β) as its threshold. If $\alpha < \beta$ then I does not satisfy (IP).*

Proof. Indeed, because there exists x between α and β such that $I(x,x) = 0$, therefore I does not satisfy (IP).

Proposition 9. *Let I be a crisp fuzzy implication with (α,β) as its threshold. If $\alpha = \beta$ then:*

(i) If I is of type (C1) then I does not satisfy (IP);
(ii) If I is of type (C2), (C3) or (C4) then I satisfies (IP).

Proof. In fact,

(i) There is $x = \alpha$ such that $I(x, x) = 0 \neq 1$, so I does not satisfy (IP).
(ii) If I is of type (C2) then there is no $x \in [0, 1]$ such that $x > \alpha$ and $x < \beta$ simultaneously, since $\beta = \alpha$. So $I(x, x) = 1$, for all $x \in [0, 1]$. Similarly, if I is of type (C3) or (C4) we prove that $I(x, x) = 1$. Therefore, in any case, I satisfies (IP).

In [4], the conditions under which the Boolean-like law holds for some classes of fuzzy implications were given. Here, we prove that it is valid for a crisp fuzzy implication whenever (IP) is satisfied.

Proposition 10. *Let I be a crisp fuzzy implication. Then:*

(i) If I satisfies (IP) then $I(x, I(y, x)) = 1$, for all $x, y \in [0, 1]$;
(ii) If I does not satisfy (IP) then there are $x, y \in [0, 1]$ such that $I(x, I(y, x)) \neq 1$.

Proof. Indeed,

(i) If I satisfies (IP) then $I(x, x) = 1$, for all $x \in [0, 1]$, so by Proposition 6: $I(x, I(y, x)) \overset{(EP)}{=} I(y, I(x, x)) = I(y, 1) = 1$.
(ii) If I does not satisfy (IP) then there exist $x \in (0, 1)$ such that $I(x, x) = 0$. So, for $y = 1$, $I(x, I(1, x)) = I(1, I(x, x)) = I(1, 0) = 0 \neq 1$.

Corollary 3. *Let I be a crisp fuzzy implication with (α, β) as its threshold.*

(i) If $\beta < \alpha$ then $I(x, I(y, x)) = 1$, for all $x, y \in [0, 1]$;
(ii) If $\alpha < \beta$ then there exists $x, y \in [0, 1]$ such that $I(x, I(y, x)) \neq 1$;
(iii) If $\alpha = \beta$ then there exists $x, y \in [0, 1]$ such that $I(x, I(y, x)) \neq 1$ whenever I is of type (C1). And $I(x, I(y, x)) = 1$, for all $x, y \in [0, 1]$ whenever I is of type (C2), (C3) or (C4).

Proof. It follows straight from Propositions 7, 8 and 9.

Definition 13. *Let I be a crisp fuzzy implication with (α, β) as its threshold and let N be a fuzzy negation. We say that IN is a **dual crisp fuzzy implication of I with respect to** N, or dual NCrisp, if it satisfies one of the following types, for all $x, y \in [0, 1]$:*

(NC1) $IN(x, y) = I_{N(\beta), N(\alpha)}(x, y)$, whenever I satisfies (C1);
(NC2) $IN(x, y) = I^{N(\beta), N(\alpha)}(x, y)$, whenever I satisfies (C2);
(NC3) $IN(x, y) = {I^{N(\beta)}}_{N(\alpha)}(x, y)$, whenever I satisfies (C3);
(NC4) $IN(x, y) = {I_{N(\beta)}}^{N(\alpha)}(x, y)$, whenever I satisfies (C4).

Definition 14. *Let I be a crisp fuzzy implication and N be a fuzzy negation. I is said to be a*

*(i) **Crisp-CP** with respect to N, if*
$$I(x, y) = IN(N(y), N(x)), \quad \text{(C-CP)}$$

(ii) ***Crisp Left-CP*** *with respect to* N*, if*
$I(N(x), y) = IN(N(y), x),$ *(C-LCP)*
(iii) ***Crisp Right-CP*** *with respect to* N*, if*
$I(x, N(y)) = IN(y, N(x)),$ *(C-RCP)*

where IN *is its dual NCrisp.*

Proposition 11. *Let* I *be a crisp fuzzy implication and* N *be a fuzzy negation. If* N *is strict, then* I *is C-CP with respect to* N*.*

Proof. (1) If I satisfies ($C1$), then, by Proposition 2, there exist $\alpha \in (0, 1]$ and $\beta \in [0, 1)$ such that $I(x, y) = I_{\alpha,\beta}(x, y)$. So, as N is strict, $x \geq \alpha$ if and only if $N(x) \leq N(\alpha)$ and $y \leq \beta$ if and only if $N(y) \geq N(\beta)$. Therefore, by (NC1)

$$IN(N(y), N(x)) = I_{N(\beta),N(\alpha)}(N(y), N(x))$$
$$= \begin{cases} 0, & \text{if } N(y) \geq N(\beta) \text{ and } N(x) \leq N(\alpha) \\ 1, & \text{otherwise} \end{cases}$$
$$= \begin{cases} 0, & \text{if } x \geq \alpha \text{ and } y \leq \beta \\ 1, & \text{otherwise} \end{cases} = I_{\alpha,\beta}(x, y) = I(x, y).$$

(2) If I satisfies ($C2$), then, by Proposition 2, there exist $\alpha \in [0, 1)$ and $\beta \in (0, 1]$ such that $I(x, y) = I^{\alpha,\beta}(x, y)$. So, as N is strict, $x > \alpha$ if and only if $N(x) < N(\alpha)$ and $y < \beta$ if and only if $N(y) > N(\beta)$. Therefore, by (NC2)

$$IN(N(y), N(x)) = I^{N(\beta),N(\alpha)}(N(y), N(x))$$
$$= \begin{cases} 0, & \text{if } N(y) > N(\beta) \text{ and } N(x) < N(\alpha) \\ 1, & \text{otherwise} \end{cases}$$
$$= \begin{cases} 0, & \text{if } x > \alpha \text{ and } y < \beta \\ 1, & \text{otherwise} \end{cases} = I^{\alpha,\beta}(x, y) = I(x, y).$$

(3) If I satisfies ($C3$), then, by Proposition 2, there exist $\alpha, \beta \in (0, 1]$ such that $I(x, y) = I_{\alpha}{}^{\beta}(x, y)$. So, as N is strict, $x \geq \alpha$ if and only if $N(x) \leq N(\alpha)$ and $y < \beta$ if and only if $N(y) > N(\beta)$. Therefore, by (NC3)

$$IN(N(y), N(x)) = I^{N(\beta)}{}_{N(\alpha)}(N(y), N(x))$$
$$= \begin{cases} 0, & \text{if } N(y) > N(\beta) \text{ and } N(x) \leq N(\alpha) \\ 1, & \text{otherwise} \end{cases}$$
$$= \begin{cases} 0, & \text{if } x \geq \alpha \text{ and } y < \beta \\ 1, & \text{otherwise} \end{cases} = I_{\alpha}{}^{\beta}(x, y) = I(x, y).$$

(4) If I satisfies ($C4$), then, by Proposition 2, there exist $\alpha, \beta \in [0, 1)$ such that $I(x, y) = I^{\alpha}{}_{\beta}(x, y)$. So as N is strict, $x > \alpha$ if and only if $N(x) < N(\alpha)$ and

$y \leq \beta$ if and only if $N(y) \geq N(\beta)$. Therefore, by (NC4)

$$\begin{aligned} IN(N(y), N(x)) &= {I_{N(\beta)}}^{N(\alpha)}(N(y), N(x)) \\ &= \begin{cases} 0, & \text{if } N(y) \geq N(\beta) \text{ and } N(x) < N(\alpha) \\ 1, & \text{otherwise} \end{cases} \\ &= \begin{cases} 0, & \text{if } x > \alpha \text{ and } y \leq \beta \\ 1, & \text{otherwise} \end{cases} = {I^{\alpha}}_{\beta}(x, y) = I(x, y). \end{aligned}$$

Therefore, in any case, I is C-CP.

Notice that for C-LCP and C-RCP, the requirements are different from Proposition 11. The proof for both is analogous as we can see in the following proposition.

Proposition 12. *Let I be a crisp fuzzy implication and N be a fuzzy negation. If N is strong, then I is C-LCP and C-RCP with respect to N.*

Proof. Straightforward.

Finally, we also studied that is impossible for some implications to be crisp fuzzy implications.

Proposition 13. *None of the following classes of fuzzy implications (S, N)-, R-, QL- and D-implication is a crisp fuzzy implication.*

Proof. In fact,

(1) if I is an (S, N)-implication, then there exist a t-conorm S and a fuzzy negation N such that $I(x, y) = S(N(x), y)$ for all $x, y \in [0, 1]$. In particular, for $x = 1$, $I(1, y) = S(N(1), y) = S(0, y) \overset{(S4)}{=} y$. So, for all $y \in (0, 1)$, $I(1, y) \notin \{0, 1\}$. Therefore, I is not crisp.
(2) If I is an R-implication, then there exists a t-norm T such that $I(x, y) = sup\{t \in [0, 1] \mid T(x, t) \leq y\}$ for all $x, y \in [0, 1]$. In particular, for $x = 1$, $I(1, y) = sup\{t \in [0, 1] \mid T(1, t) \leq y\} \overset{(T4)}{=} sup\{t \in [0, 1] \mid t \leq y\} = y$. So, for all $y \in (0, 1)$, $I(1, y) \notin \{0, 1\}$. Therefore, I is not crisp.
(3) If I is a QL-implication, then there exist a t-norm T, a t-conorm S and a fuzzy negation N such that $I(x, y) = S(N(x), T(x, y))$ for all $x, y \in [0, 1]$. In particular, for $x = 1$, $I(1, y) = S(N(1), T(1, y)) = S(0, T(1, y)) \overset{(S4)}{=} T(1, y) \overset{(T4)}{=} y$. So, for all $y \in (0, 1)$, $I(1, y) \notin \{0, 1\}$. Therefore, I is not crisp.
(4) If I is a D-implication, then there exist a t-norm T, a t-conorm S and a fuzzy negation N such that $I(x, y) = S(T(N(x), N(y)), y)$ for all $x, y \in [0, 1]$. In particular, for $x = 1$, $I(1, y) = S(T(N(1), N(y)), y) = S(T(0, N(y)), y) = S(0, y) \overset{(S4)}{=} y$. So, for all $y \in (0, 1)$, $I(1, y) \notin \{0, 1\}$. Therefore, I is not crisp.

4 Final Remarks

One can find many examples of studies which use fuzzy implications with crisp behavior such as [8–10,17]. Our purpose in this work was to study those fuzzy implications which always map to 0 or 1, therefore named crisp fuzzy implications. We provided a characterization for those implications, presenting four possible classes (Proposition 2) and studied some properties and conditions under which fuzzy implications need in order to be considered crisp.

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References

1. Baczyński, M., Balasubramaniam, J.: Fuzzy Implications. Springer, Heidelberg (2008). https://doi.org/10.1007/978-3-540-69082-5
2. Baczyński, M., Beliakov, G., Bustince, H., Pradera, A. (eds.): Advances in Fuzzy Implication Functions, Studies in Fuzziness and Soft Computing, vol. 300. Springer, Heidelberg (2013). https://doi.org/10.1007/978-3-642-35677-3
3. Bedregal, B.C.: A normal form which preserves tautologies and contradictions in a class of fuzzy logics. J. Algorithms **62**(3–4), 135–147 (2007)
4. Cruz, A., Bedregal, B.C., Santiago, R.H.N.: On the characterizations of fuzzy implications satisfying I(x, I(y, z)) = I(I(x, y), I(x, z)). Int. J. Approx. Reason. **93**, 261–276 (2018)
5. Dimuro, G.P., Bedregal, B., Santiago, R.H.: On (G, N)-implications derived from grouping functions. Inf. Sci. **279**, 1–17 (2014)
6. Dimuro, G.P., Bedregal, B.C.: On residual implications derived from overlap functions. Inf. Sci. **312**, 78–88 (2015)
7. Dimuro, G.P., Bedregal, B., Bustince, H., Jurio, A., Baczyński, M., Mis, K.: QL-operations and QL-implication functions constructed from tuples (O, G, N) and the generation of fuzzy subsethood and entropy measures. Int. J. Approx. Reason. **82**, 170–192 (2017)
8. Dubois, D., Prade, H.M., Bezdek, J.C.: Fuzzy Sets in Approximate Reasoning and Information Systems. Kluwer Academic Publishers, Norwell (1999)
9. Dubois, D., Nakata, M., Prade, H.: Extended divisions for flexible queries in relational databases. In: Pons, O., Vila, M.A., Kacprzyk, J. (eds.) Knowledge Management in Fuzzy Databases. Studies in Fuzziness and Soft Computing, vol. 39, pp. 105–121. Physica, Heidelberg (2000). https://doi.org/10.1007/978-3-7908-1865-9_6
10. Godo, L., Sandri, S.: Dealing with imprecise inputs in a fuzzy rule-based system using an implication-based rule model. In: Bouchon-Meunier, B., Gutiérrez-Ríos, J., Magdalena, L., Yager, R.R. (eds.) Technologies for Constructing Intelligent Systems 1. Studies in Fuzziness and Soft Computing, vol. 89. Physica, Heidelberg (2002). https://doi.org/10.1007/978-3-7908-1797-3_4
11. Klir, G.J., Yuan, B.: Fuzzy Sets and Fuzzy Logic: Theory and Applications. Prentice Hall, Upper Saddle River (1995)
12. Mas, M., Monserrat, M., Torrens, J., Trillas, E.: A survey on fuzzy implication functions. IEEE Trans. Fuzzy Syst. **15**(6), 1107–1121 (2007)

13. Oh, K.W., Bandler, W.: Properties of fuzzy implication operators. Int. J. Approx. Reason. **1**(3), 273–285 (1987)
14. Pinheiro, J., Bedregal, B., Santiago, R.H., Santos, H.: (T, N)-implications. In: 2017 IEEE International Conference on Fuzzy Systems (FUZZ-IEEE), pp. 1–6 (2017)
15. Pinheiro, J., Bedregal, B., Santiago, R.H., Santos, H.: A study of (T, N)-implications and its use to construct a new class of fuzzy subsethood measure. Int. J. Approx. Reason. **97**, 1–16 (2018)
16. Sakly, A., Benrejeb, M.: On the choice of the adequate fuzzy implication operator with the center of gravity defuzzification method based on precision criterion in fuzzy control. In: Proceedings of the 2002 IEEE International Conference on Fuzzy Systems (FUZZ-IEEE 2002), Honolulu, HI, pp. 482–487 (2002)
17. Santos, A.S., Valle, M.E.: A fast and robust max-c projection fuzzy autoassociative memory with application for face recognition. In: IEEE World Congress on Computational Intelligence (2018, submitted)