(T, N)-Implications and Some Functional Equations

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Abstract. Fuzzy implications has drawn attention of many authors along the years, as their theoretical features seem to be a useful tool in a fair amount of applications. Meanwhile, functional equations are those in which the unknowns are functions instead of a traditional variable, and within the fuzzy logic, they can be considered generalizations of some tautologies of the classical logic. In this paper we investigate the validity of five functional equations for the class of (T, N)-implications, namely, we have selected the law of importation and four distributivity properties and have studied them in the context of the aforementioned operator.

1 Introduction

Fuzzy implications [1,4,18] are one of the most relevant operators in fuzzy logics. Many applications have been constructed making use of them as we can see in [2,3,18] and they are also applied in different areas such as approximate reasoning, control and decision-making theories, expert systems, fuzzy mathematical morphology, image processing, among others [8,9,11,15,20,21,24,27].

In [22,23], a new class of fuzzy implication named (T, N)-implication (firstly presented by [5]) was studied. Such implications were given by the composition of a fuzzy negation and a t-norm. The conditions under which such functions preserved the principal properties of fuzzy implications were also investigated and it was proved the necessary and sufficient conditions for a function $I : [0, 1]^2 \to [0, 1]$ to be a (T, N)-implication.

It is important to recall that the classical implication is found in various tautologies in classical logic. It is clear that not all generalizations of these tautologies hold for all fuzzy operators. That is the reason why one should have a deep

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and careful study on such tautologies in order to convert them into functional equations that would include fuzzy operations. In [1] it is stated that considering the generalized equivalences, some properties of implications had received more attention due to their value on the many applications, namely the properties of contrapositive symmetry, the law of importation, the distributivity properties of fuzzy implications over t-norms and t-conorms, and also the T-conditionality property. In [11], a fuzzy generalization for $I(x, I(y, z)) = I(I(x, y), I(x, z))$ law was made, providing the requirements for that Boolean-like law to be valid within some classes of fuzzy implications, among which the (T, N)-implications. In this sense, the aim of this work is to study the law of importation and four distributivity properties regarding the class of (T, N)-implications over t-norms and t-conorms.

The paper is organized as follows. Sect. 2 recalls some of the basic concepts demanded to comprehend the developments in this work, including the concept of fuzzy implication and related properties. The study of (T, N)-implications and functional equations is done in Sect. 3, including the most important results. Al last, we conclude in Sect. 4 with our final remarks and discuss some ideas for future works.

2 Preliminares

Definition 1. *A function $T : [0, 1]^2 \to [0, 1]$ is called a* ***triangular norm (t-norm****, for short) if it satisfies the following conditions:*

(T1) $T(x, y) = T(y, x)$ *for all* $x, y \in [0, 1]$*;*
(T2) $T(x, T(y, z)) = T(T(x, y), z)$ *for all* $x, y, z \in [0, 1]$*;*
(T3) If $x_1 \leq x_2$ *and* $y_1 \leq y_2$ *then* $T(x_1, y_1) \leq T(x_2, y_2)$*, for all* $x_1, x_2, y_1, y_2 \in [0, 1]$*;*
(T4) $T(x, 1) = x$*, for all* $x \in [0, 1]$. *(boundary condition)*

Proposition 1. *Let T be a t-norm. Then $T(0, y) = 0$ for each $y \in [0, 1]$.*

In fuzzy logic, the conjunction is often represented by a t-norm. The standard fuzzy conjunction $T_M : [0, 1]^2 \to [0, 1]$ given by $T_M(x, y) = min\{x, y\}$ is the only idempotent t-norm (see [17] - Theorem 3.9).

Definition 2. *A function $S : [0, 1]^2 \to [0, 1]$ is called a* ***triangular conorm (t-conorm****, for short) if it satisfies the following conditions, for all $x, y, z \in [0, 1]$:*

(S1) $S(x, y) = S(y, x)$ *for all* $x, y \in [0, 1]$*;*
(S2) $S(x, S(y, z)) = S(S(x, y), z)$ *for all* $x, y, z \in [0, 1]$*;*
(S3) If $x_1 \leq x_2$ *and* $y_1 \leq y_2$ *then* $S(x_1, y_1) \leq S(x_2, y_2)$*, for all* $x_1, x_2, y_1, y_2 \in [0, 1]$*;*
(S4) $S(x, 0) = x$ *for all* $x \in [0, 1]$. *(boundary condition)*

The standard fuzzy disjunction $S_M : [0, 1]^2 \to [0, 1]$ given by $S_M(x, y) = max\{x, y\}$ is the only idempotent t-conorm (see [17] - Theorem 3.14).

Definition 3. *A function $N : [0,1] \rightarrow [0,1]$ is a* ***fuzzy negation*** *if*

(N1) N is antitonic, i.e. $N(x) \leq N(y)$ whenever $y \leq x$;
(N2) $N(0) = 1$ and $N(1) = 0$.
A fuzzy negation N is said to be ***strict*** *if*
(N3) N is continuous and
(N4) $N(x) < N(y)$ whenever $y < x$.
A fuzzy negation N is said to be ***strong*** *if*
(N5) $N(N(x)) = x$, for each $x \in [0,1]$.
A fuzzy negation N is said to be ***crisp*** *if*
(N6) $N(x) \in \{0,1\}$, for all $x \in [0,1]$.

By [14], a fuzzy negation $N : [0,1] \rightarrow [0,1]$ is crisp if and only if there exists $\alpha \in [0,1)$ such that $N = N_\alpha$ or there exists $\alpha \in (0,1]$ such that $N = N^\alpha$, where

$$N_\alpha(x) = \begin{cases} 0, & \text{if } x > \alpha \\ 1, & \text{if } x \leq \alpha \end{cases} \tag{1}$$

and

$$N^\alpha(x) = \begin{cases} 0, & \text{if } x \geq \alpha \\ 1, & \text{if } x < \alpha \end{cases}. \tag{2}$$

Definition 4. *Let T be a t-norm, S be a t-conorm and N be a strict fuzzy negation. Then S is said to be* **N-*dual to* T** *if, for all $x, y \in [0,1]$,*

$$N(S(x,y)) = T(N(x), N(y)) \tag{3}$$

and T is said to be **N-*dual to* S** *if, for all $x, y \in [0,1]$,*

$$N(T(x,y)) = S(N(x), N(y)). \tag{4}$$

Definition 5. *A function $I : [0,1]^2 \rightarrow [0,1]$ is a* ***fuzzy implication*** *if the following properties are satisfied, for all $x, y, z \in [0,1]$:*

(*I1*) *If $x \leq z$ then $I(x,y) \geq I(z,y)$;* (*left antitonicity*)
(*I2*) *If $y \leq z$ then $I(x,y) \leq I(x,z)$;* (*right isotonicity*)
(*I3*) $I(0,y) = 1$; (*left boundary condition*)
(*I4*) $I(x,1) = 1$; (*right boundary condition*)
(*I5*) $I(1,0) = 0$. (*boundary condition*)

Definition 6. *[1] Let I be a fuzzy implication and T be a t-norm. We say that I satisfies the* ***Law of importation (LI)*** *with respect to a t-norm T if*

$$I(T(x,y),z) = I(x, I(y,z)), \tag{5}$$

for all $x, y, z \in [0,1]$.

3 Functional Equations and (T, N)-Implications

As already mentioned, functional equations are the ones in which the unknowns are functions instead of being a traditional variable. In this section we investigate the validity of some functional equations by the function I_T^N, introduced in [22]. In [1], Baczyński states that functional equations come up as generalizations of the corresponding tautologies in classical logic involving boolean implications. The results presented in the sequel consider the law of importation (LI), Eq. 5, and four basic distributive equations involving an implication, which will be discussed later.

Proposition 2. *Let T be a t-norm and let N be a fuzzy negation. Then the function $I_T^N : [0,1]^2 \to [0,1]$ defined by*

$$I_T^N(x, y) = N(T(x, N(y))) \tag{6}$$

is a fuzzy implication, for all $x, y \in [0,1]$.

Definition 7. *Let T be a t-norm and let N be a fuzzy negation. The function I_T^N defined by Eq. (6) is called* ***(T, N)-implication****.*

The principle of exchange is one of the crucial properties of fuzzy implications. Due to the commutativity property of the t-norm T, one of the conditions for an implication to satisfy it is that (LI) is also satisfied. The well-known fuzzy implications called (S, N), R, QL and D-implications satisfy (LI) under some conditions (see [15,19]). In addition, some possible applications were pointed out in [15]. As follows, we show under which conditions (T, N)-implications satisfy (LI).

Proposition 3. *Let I_T^N be a (T, N)-implication. Then:*

(i) If N is strong then I_T^N satisfies (LI) with respect to the t-norm T;
(ii) If N is continuous and I_T^N satisfies (LI) with respect to the t-norm T, then N is strong.

Proof. (i) Indeed, for all $x, y, z \in [0,1]$

$$\begin{aligned} I_T^N(x, I_T^N(y,z)) &= N(T(x, N(N(T(y, N(z)))))) \\ &= N(T(x, T(y, N(z)))) \\ &\overset{(T2)}{=} N(T(T(x,y), N(z))) \\ &= I_T^N(T(x,y), z). \end{aligned}$$

(ii) As I_T^N satisfies (LI) with respect to the t-norm T, then, for $x = y = 1$, $I_T^N(1, I_T^N(1,z)) = I_T^N(T(1,1), z) \overset{(T4)}{\Rightarrow} N(T(1, N(N(T(1, N(z)))))) = N(T(1, N(z)))$ for all $z \in [0,1]$, still by (T4),

$$N(N(N(N(z)))) = N(N(z)). \tag{7}$$

Given that N is continuous, for all $y \in [0,1]$ there exists $x' \in [0,1]$ such that $N(x') = y$. Only for this x' there exists $x \in [0,1]$ such that $N(x) = x'$. Thus, for all $y \in [0,1]$ there exists $x \in [0,1]$ such that $N(N(x)) = y$. Therefore, by Eq. (7), $N(N(N(N(x)))) = N(N(x)) \Rightarrow N(N(y)) = y$, for all $y \in [0,1]$.

Note that if N is continuous and non-strong then I_T^N does not satisfy (LI). However, there are non-continuous negations N such that I_T^N satisfies (LI) for some t-norm T. See the following example:

Example 1. Take a crisp negation N given by $N = N_\alpha$ and the minimum t-norm T, so

$$
\begin{aligned}
I_T^{N_\alpha}(x, I_T^{N_\alpha}(y,z)) &= N_\alpha(T(x, N_\alpha(N_\alpha(T(y, N_\alpha(z)))))) \\
&= \begin{cases} N_\alpha(T(x, N_\alpha(N_\alpha(y)))), & \text{if } z \leq \alpha \\ 1, & \text{if } z > \alpha \end{cases} \\
&= \begin{cases} N_\alpha(x), & \text{if } z \leq \alpha \text{ and } y > \alpha \\ 1, & \text{if } z > \alpha \text{ or } y \leq \alpha \end{cases} \\
&= \begin{cases} 0, & \text{if } z \leq \alpha \text{ and } y > \alpha \text{ and } x > \alpha \\ 1, & \text{otherwise} \end{cases}
\end{aligned}
$$

and

$$
\begin{aligned}
I_T^{N_\alpha}(T(x,y), z) &= N_\alpha(T(T(x,y), N_\alpha(z))) \\
&= \begin{cases} N_\alpha(T(T(x,y),1)), & \text{if } z \leq \alpha \\ 1, & \text{if } z > \alpha \end{cases} \\
&= \begin{cases} N_\alpha(T(x,y)), & \text{if } z \leq \alpha \\ 1, & \text{if } z > \alpha \end{cases} \\
&= \begin{cases} 0, & \text{if } z \leq \alpha \text{ and } T(x,y) > \alpha \\ 1, & \text{if } z > \alpha \text{ or } T(x,y) \leq \alpha \end{cases} \\
&= \begin{cases} 0, & \text{if } z \leq \alpha \text{ and } x > \alpha \text{ and } y > \alpha \\ 1, & \text{otherwise} \end{cases},
\end{aligned}
$$

thus, $I_T^{N_\alpha}$ satisfies (LI).

Another example can be given by taking the crisp fuzzy negation $N = N_\alpha$ with $\alpha = 0$ and any t-norm T. In this case, by Proposition 1 we also have that $I_T^{N_\alpha}$ satisfies (LI).

In classic logic, the distributivity of binary operators over one another can somehow define the framework of the algebra imposed by these operators. In fuzzy logic, one can find a variety of studies on the distributivity of t-norms over t-conorms [6,7,10,16]. In this sense, taking into account the four basic distributive equations involving an implication, Eqs. 8, 9, 10, 11, we present in the next proposition the generalizations of them which yields to the distributivity of (T, N)-implications over t-norms and t-conorms.

$$I(T(x,y),z) = S(I(x,z),I(y,z)) \tag{8}$$
$$I(S(x,y),z) = T(I(x,z),I(y,z)) \tag{9}$$
$$I(x,S_1(y,z)) = S_2(I(x,y),I(x,z)) \tag{10}$$
$$I(x,T_1(y,z)) = T_2(I(x,y),I(x,z)) \tag{11}$$

Proposition 4. *Let I_T^N be a (T, N)-implication and S be a t-conorm. Then:*

(i) If T is N-dual of S and the range of N is a subset of the idempotent elements of T then I_T^N satisfies Eq. (8) with respect to the t-norm T and to the t-conorm S;
(ii) If I_T^N satisfies Eq. (8) with respect to the t-norm T and to the t-conorm S, then
(1) T is N-dual of S and,
(2) If N is strict then the range of N is a subset of the idempotent elements of T.

Proof. (i) As T is N-dual of S and the range of N is a subset of the idempotent elements of T, i.e., $T(N(x),N(x)) = N(x)$ for all $x \in [0,1]$, then, for all $x,y,z \in [0,1]$:

$$\begin{aligned} S(I_T^N(x,z),I_T^N(y,z)) &= S(N(T(x,N(z))),N(T(y,N(z)))) \\ &\overset{Eq.\ (4)}{=} N(T(T(x,N(z)),T(y,N(z)))) \\ &\overset{(T2)(T1)}{=} N(T(T(x,y),T(N(z),N(z)))) \\ &= N(T(T(x,y),N(z))) \\ &= I_T^N(T(x,y),z). \end{aligned}$$

(ii) (1) As I_T^N satisfies Eq. (8) with respect to the t-norm T and to the t-conorm S, then, for $z = 0$, $N(T(T(x,y),N(0))) = S(N(T(x,N(0))), N(T(y,N(0))))$, so by (T4), $N(T(x,y)) = S(N(x),N(y))$ for all $x,y \in [0,1]$ and
(2) For $x = y = 1$, $S(I_T^N(1,z),I_T^N(1,z)) = N(T(T(1,1),N(z)))$, so by (T4), $S(N(N(z)),N(N(z))) = N(N(z))$ for all $z \in [0,1]$, since T is N-dual of S we have $N(T(N(z),N(z))) = N(N(z)) \overset{N\ strict}{\Rightarrow} T(N(z),N(z)) = N(z)$, for all $z \in [0,1]$.

Corollary 1. *Let N be a strict negation and T be a t-norm. Then, I_T^N satisfies Eq. (8) if and only if $T = T_M$ and $S = S_M$.*

In the previous corollary, the continuity of N ensures that if I_T^N satisfies Eq. (8) then T is minimum. However, there are non-continuous negations such that I_T^N satisfies Eq. (8) for some t-norms. See the following example:

Example 2. Take a crisp negation N given by $N = N_\alpha$ and take T as the minimum t-norm, so

$$\begin{aligned} S(\ I_T^{N_\alpha}(x,z),\ I_T^{N_\alpha}(y,z)) = & \\ = \quad & S(N_\alpha(T(x,N_\alpha(z))), N_\alpha(T(y,N_\alpha(z)))) \\ = \quad & \begin{cases} S(N_\alpha(x),N_\alpha(y)), & \text{if } z \le \alpha \\ 1, & \text{if } z > \alpha \end{cases} \\ = \quad & \begin{cases} 0, & \text{if } z \le \alpha \text{ and } x > \alpha \text{ and } y > \alpha \\ 1, & \text{otherwise} \end{cases} \end{aligned}$$

and, by Example 1

$$I_T^{N_\alpha}(T(x,y),z) = \begin{cases} 0, & \text{if } z \le \alpha \text{ and } x > \alpha \text{ and } y > \alpha \\ 1, & \text{otherwise} \end{cases}$$

thus, $I_T^{N_\alpha}$ satisfies Eq. (8).

Another example can be given for any t-norm T. Just take the crisp fuzzy negation $N = N_\alpha$ with $\alpha = 0$. Then, by Proposition 1 we also have that $I_T^{N_0}$ satisfies Eq. (8).

Proposition 5. *Let I_T^N be a (T, N)-implication. Then,*

(i) I_T^N satisfies Eq. (9) for T_M and S_M, i.e., considering T_M as T and S_M as S in Eq. (9);
(ii) If I_T^N satisfies Eq. (9) with respect to the t-norm T and to the t-conorm S, then
(1) S is N-dual of T and
(2) If N is strict then the range of N is a subset of the idempotent elements of S.

Proof. (i) For all $x, y, z \in [0,1]$, if $x \le y$ then $S_M(x,y) = y$ and, by (T3) and (N1), $I_T^N(y,z) \le I_T^N(x,z)$, so

$$T_M(I_T^N(x,z), I_T^N(y,z)) = I_T^N(y,z) = I_T^N(S_M(x,y),z).$$

Therefore, I_T^N satisfies Eq. (9). Similarly, if $x > y$ the result follows.

(ii) (1) As I_T^N satisfies Eq. (9) with respect to the t-norm T and to the t-conorm S, then, for $z = 0$, $N(T(S(x,y),N(0))) = T(N(T(x,N(0))),$ $N(T(y,N(0))))$, so by (T4), $N(S(x,y)) = T(N(x),N(y))$ for all $x, y \in [0,1]$ and
(2) for $x = y = 1$, $T(I_T^N(1,z), I_T^N(1,z)) = I_T^N(S(1,1),z)$, so by (T4), $T(N(N(z)),N(N(z))) = N(N(z))$ for all $z \in [0,1]$, since S is N-dual of T we have $N(S(N(z),N(z))) = N(N(z)) \overset{N \text{ strict}}{\Rightarrow} S(N(z),N(z)) = N(z)$, for all $z \in [0,1]$.

Corollary 2. *Let N be a strict negation and T be a t-norm. Then, I_T^N satisfies Eq. (9) if and only if $T = T_M$ and $S = S_M$.*

Proposition 6. *Let I_T^N be a (T, N)-implication and S_1 and S_2 be t-conorms. Then:*

(i) If $S_1 = S_2 = S_M$ then, for any t-norm T and any negation N, I_T^N satisfies Eq. (10);
(ii) If I_T^N satisfies Eq. (10) with respect to t-conorms S_1 and S_2, then:
(1) The range of N is a subset of the idempotent elements of S_2 and
(2) If N is strict then $S_1 = S_2 = S_M$.

Proof. (i) For all $x, y, z \in [0,1]$, if $y \leq z$ then $S_M(y,z) = z$ and, by (N1) and (T3), $I_T^N(x,y) \leq I_T^N(x,z)$, so

$$S_M(I_T^N(x,y), I_T^N(x,z)) = I_T^N(x,z) = I_T^N(x, S_M(y,z)).$$

Therefore, I_T^N satisfies Eq. (10). Similarly, if $y > z$ the result follows.
(ii) (1) As I_T^N satisfies Eq. (10) then, in particular for $y = z = 0$,

$$N(T(x, N(S_1(0,0)))) = S_2(N(T(x, N(0))), N(T(x, N(0)))),$$

so by (T4), $N(x) = S_2(N(x), N(x))$, for all $x \in [0,1]$. (2) Since N is strict and $S_2(N(x), N(x)) = N(x)$ for all $x \in [0,1]$, then

$$S_2(y,y) = S_2(N(N^{-1}(y)), N(N^{-1}(y))) = N(N^{-1}(y)) = y$$

for all $y \in [0,1]$, so $S_2 = S_M$. On the other hand, for $x = 1$ and $z = y$, $N(T(1, N(S_1(y,y)))) = S_2(N(T(1, N(y))), N(T(1, N(y))))$ for all $y \in [0,1]$, so by (T4),

$$N(N(S_1(y,y))) = S_2(N(N(y)), N(N(y))) \stackrel{S_2=S_M}{=} N(N(y)),$$

for all $y \in [0,1]$. Thus, $S_1(y,y) = y$ for all $y \in [0,1]$, since N is strict. Therefore, $S_1 = S_M$.

Corollary 3. *Let N be a strict negation and T be a t-norm. Then, I_T^N satisfies Eq. (10) if and only if $S_1 = S_2 = S_M$.*

Proposition 7. *Let I_T^N be a (T, N)-implication and T_1 and T_2 be t-norms. Then:*

(i) If $T_1 = T_2 = T_M$ then, for any t-norm T and any negation N, I_T^N satisfies Eq. (11);
(ii) If I_T^N satisfies Eq. (11) with respect to t-norms T_1 and T_2, then:
(1) The range of N is a subset of the idempotent elements of T_2 and
(2) If N is strict then $T_1 = T_2 = T_M$.

Proof. (i) For all $x, y, z \in [0,1]$, if $y \leq z$ then $T_M(y,z) = y$ and, by (N1) and (T3), $I_T^N(x,y) \leq I_T^N(x,z)$, so

$$T_M(I_T^N(x,y), I_T^N(x,z)) = I_T^N(x,y) = I_T^N(x, T_M(y,z)).$$

Therefore, I_T^N satisfies Eq. (11). Similarly, if $y > z$ the result follows.

(ii) (1) As I_T^N satisfies Eq. (11) then, in particular for $y = z = 0$,

$$N(T(x, N(T_1(0,0)))) = T_2(N(T(x, N(0))), N(T(x, N(0)))),$$

so by (T4), $N(x) = T_2(N(x), N(x))$, for all $x \in [0,1]$. (2) Since N is strict and the range of N a subset of the idempotent elements of T_2, we have that $T_2(x,x) = T_2(N(N^{-1}(x)), N(N^{-1}(x))) = N(N^{-1}(x)) = x$. On the other hand, for $x = 1$ and $z = y$, $N(T(1, N(T_1(y,y)))) = T_2(N(T(1, N(y))), N(T(1, N(y))))$, so by (T4),

$$N(N(T_1(y,y))) = T_2(N(N(y)), N(N(y))) \overset{T_2 = T_M}{=} N(N(y)),$$

for all $y \in [0,1]$. Thus, $T_1(y,y) = y$ for all $y \in [0,1]$, since N is strict. Therefore, $T_1 = T_M$.

There are other conditions for t-norms and negations that imply that a (T, N)-implication satisfies Eq. (11). The following example ensures that if we take $T_1 = T_M$ and the crisp negation N, given by $N = N_\alpha$ with $\alpha \in [0,1)$, then, independently from t-norms T and T_2, I_T^N satisfies Eq. (11).

Example 3. Take the crisp negation N given by $N = N_\alpha$ and take T_1 as the minimum t-norm, so

$$\begin{aligned} T_2(\, I_T^{N_\alpha}(x,y),\, I_T^{N_\alpha}(x,z)) &= \\ &= \quad T_2(N_\alpha(T(x, N_\alpha(y))), N_\alpha(T(x, N_\alpha(z)))) \\ &= \quad \begin{cases} N_\alpha(T(x, N_\alpha(y))), & \text{if } z > \alpha \\ T_2(N_\alpha(T(x, N_\alpha(y))), N_\alpha(x)), & \text{if } z \leq \alpha \end{cases} \\ &= \quad \begin{cases} 1, & \text{if } z > \alpha \text{ and } y > \alpha \\ N_\alpha(x), & \text{if } z > \alpha \text{ and } y \leq \alpha \\ N_\alpha(x), & \text{if } z \leq \alpha \text{ and } y > \alpha \\ T_2(N_\alpha(x), N_\alpha(x)), & \text{if } z \leq \alpha \text{ and } y \leq \alpha \end{cases} \\ &= \quad \begin{cases} 0, & \text{if } x > \alpha \text{ and } T_1(y,z) \leq \alpha \\ 1, & \text{otherwise} \end{cases} \end{aligned}$$

and,

$$
\begin{aligned}
I_T^{N_\alpha}(x, T_1(y,z)) &= N_\alpha(T(x, N_\alpha(T_1(y,z)))) \\
&= \begin{cases} 1, & \text{if } T_1(y,z) > \alpha \\ N_\alpha(x), & \text{if } T_1(y,z) \leq \alpha \end{cases} \\
&= \begin{cases} 1, & \text{if } T_1(y,z) > \alpha \\ 0, & \text{if } T_1(y,z) \leq \alpha \text{ and } x > \alpha \\ 1, & \text{if } T_1(y,z) \leq \alpha \text{ and } x \leq \alpha \end{cases} \\
&= \begin{cases} 0, & \text{if } T_1(y,z) \leq \alpha \text{ and } x > \alpha \\ 1, & \text{otherwise} \end{cases}
\end{aligned}
$$

thus, I_T^N satisfies Eq. (11).

4 Final Remarks

In this work, we carried on the study on (T, N)-implications and presented some results considering functional equations, namely the law of importation and properties related to distributivity. It is well-known that fuzzy implications can used to construct many types of measures such as fuzzy subsethood measures, penalty functions and fuzzy entropy [9,14,25,26], which are useful for several practical applications. Thus, similarly to the works mentioned previously, we believe that (T, N)-implications can also be used to construct fuzzy subsethood measures. Besides that, we are willing to investigate other operators to define different classes of implications, for instance, functions given by the composition of overlaps and negations yielding what we call (O, N)-implication, that possibly can be related to (G, N)-implications [12], (R, O)-Implications [13] and (O, G, N)-implications [14].

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