Least Squares Method with Interactive Fuzzy Coefficient: Application on Longitudinal Data

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Abstract. This work focus on the least square method to fit a fuzzy function to longitudinal data given by fuzzy numbers. In order to consider the intrinsic correlation of longitudinal data, we assume that there exits a linear relation among the involved fuzzy numbers that arises from the concept of a joint possibility distribution. We propose a numerical method to solve a fuzzy least square problem taking into account this linear correlation. To this end, we extend the classical least square method by means of the sup-J extension principle, which consists of a generalization of Zadeh's extension principle. Finally, we use our proposal method to fit a longitudinal dataset.

Keywords: Fuzzy least square method · Interactive fuzzy numbers
Joint possibility distribution · Longitudinal data

1 Introduction

The least squares methods are used, in general, to obtain a continuous function that best fit pairs of data in a dataset [1]. The fuzzy least squares method arises when the dataset is composed by fuzzy numbers. Tanaka *et al.* proposed a fuzzy least squares method based on fuzzy regression models [2]. This method was used to find fuzzy parameters of a fuzzy linear function from a fuzzy dataset. However, this approach converts the problem to a classic linear programming problem which may lead to losing the notion of close distance between the fuzzy data and the obtained solution.

Celmins [3] proceeded with the same methodology of [2] but considered a intrinsic relation among the dataset based on conical membership functions that

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are (geometrically) similar to joint possibility distributions. In addiction, the concept of interactive fuzzy numbers [4–6] was only considered in [7], which improved the approach presented by [3].

In contrast to these previous methods, Diamond [8] proposed a fuzzy least squares method based on distance between functions. He used projection theorems for cones in Banach spaces to find the fuzzy linear function that best fit a dataset.

It is worth noting that all these approaches were developed for data given only by triangular fuzzy numbers and for fitting only fuzzy linear functions. Nevertheless, these methods can be used to model many phenomenons, for example, in economy [9], psychology [10], medicine [11], and logistics [12].

Data correlations arise naturally in longitudinal datasets. A dataset is said to be longitudinal if it contains the same type of information on the same itens at multiple points in time. Therefore, longitudinal data is characterized by the fact that repeated observations are correlated [13]. In this work, we suppose that this correlation is given by the notion of completely correlated fuzzy numbers [5,14].

The method proposed here is based on the (sup-J) extension of classical numerical algorithm to the fuzzy context and does not take into account any distance between fuzzy numbers. Moreover, our method can be applied not only for triangular fuzzy numbers, but for any type of completely correlated fuzzy numbers, and it can approximate the dataset with higher orders functions.

In Sect. 2 we briefly recall the classical least squares method and some basic definitions and results from fuzzy set theory. In Sect. 3, we develop the extension of the classical least squares method for the case where dataset is composed by completely correlated fuzzy numbers. Finally, in Sect. 4, we apply the proposed method to fit a fuzzy function to the longitudinal dataset given in [15].

2 Mathematical Background

This section presents the least squares method [1] and some basic concepts of fuzzy set theory [16].

2.1 Least Square Method

Let $f : [c, d] \to \mathbb{R}$ be a continuous function. Given n functions $g_1, \ldots, g_n$, where $g_i : \mathbb{R} \to \mathbb{R}$ for $i = 1, \ldots, n$, we need to find n coefficients $a_1, \ldots, a_n \in \mathbb{R}$ such that the function $\varphi : \mathbb{R} \to \mathbb{R}$ given by

$$\varphi(x) = a_1 g_1(x) + \ldots + a_n g_n(x)$$

is the best approximation of the function f, *i.e.*, $\varphi \approx f$.

The function φ is obtained by minimizing the distance between f and φ. More precisely, let $||\cdot||_2$ be the $\mathcal{L}^2$-norm defined on the class of the continuous functions from $[c, d]$ to $\mathbb{R}$ (denoted by $C([c, d])$) given by $||h||_2 = \left(\int_c^d |h(s)|^2 ds\right)^{1/2}$, $\forall h \in$

$C([c,d])$. The coefficients $a_1, \ldots, a_n$ of the function φ which produces the best fit to f are obtained by solving the following minimization problem:

$$\min_{a_1,\ldots,a_n \in \mathbb{R}} {}^1/_2 ||\varphi - f||_2^2.$$

In the case some values of f are known, say $D = \{f(x_1) = y_1, \ldots, f(x_m) = y_m\}$, the function φ must fit the data D, that is, $\varphi(x_i) \approx y_i$, for all $i = 1, \ldots, m$. Therefore the following minimization problem must be solved.

$$\min_{a_1,\ldots,a_n \in \mathbb{R}} {}^1/_2 ||(\varphi(x_1) - y_1, \ldots, \varphi(x_m) - y_m)||_2^2. \tag{2.1}$$

The real coefficients $a_1, \ldots, a_n$ that minimize the problem (2.1), *i.e.*, that produces the best approximation φ of f, are obtained by solving the following matrix equation called normal equation:

$$Ma = b,$$

where

$$M = \begin{bmatrix} \sum_{k=1}^{m} g_1(x_k)g_1(x_k) & \ldots & \sum_{k=1}^{m} g_1(x_k)g_n(x_k) \\ \vdots & \ddots & \vdots \\ \sum_{k=1}^{m} g_n(x_k)g_1(x_k) & \ldots & \sum_{k=1}^{m} g_n(x_k)g_n(x_k), \end{bmatrix},$$

$$a = \begin{bmatrix} a_1 \\ \vdots \\ a_n \end{bmatrix} \text{ and } b = \begin{bmatrix} \sum_{k=1}^{m} y_k g_1(x_k) \\ \vdots \\ \sum_{k=1}^{m} y_k g_n(x_k) \end{bmatrix}.$$

If the matrix M is non singular, say $P = M^{-1} = [p_{ij}]$, then the vector a is obtained by

$$a = Pb. \tag{2.2}$$

Thus, each parameter a_i is given by

$$\begin{aligned} a_i &= p_{i1}b_1 + p_{i2}b_2 + \ldots + p_{in}b_n \\ &= p_{i1}\left(\sum_{k=1}^{m} y_k g_1(x_k)\right) + \ldots + p_{in}\left(\sum_{k=1}^{m} y_k g_n(x_k)\right) \\ &= \left(\sum_{j=1}^{n} p_{ij} g_j(x_1)\right) y_1 + \ldots + \left(\sum_{j=1}^{n} p_{ij} g_j(x_m)\right) y_m \\ &= c_{i1}y_1 + \ldots + c_{im}y_m, \end{aligned}$$

where $c_{ik} = \sum_{j=1}^{n} p_{ij} g_j(x_k)$, for $i = 1, \ldots, n$ and $k = 1, \ldots, m$. In general case, the matrix P stands for the pseudoinverse of M.

Since the parameters of the function φ can be obtained by computing the matrix product (2.2), we rewrite the function φ in terms of $y_1, \ldots, y_m$ as follows:

$$\begin{aligned}
\varphi(x) &= a_1 g_1(x) + \ldots + a_n g_n(x) \\
&= (c_{11} y_1 + \ldots + c_{1m} y_m) g_1(x) + \ldots + (c_{n1} y_1 + \ldots + c_{nm} y_m) g_n(x) \\
&= \left(\sum_{j=1}^{n} g_j(x) c_{j1} \right) y_1 + \ldots + \left(\sum_{j=1}^{n} g_j(x) c_{jm} \right) y_m \\
&= s_1(x) y_1 + \ldots + s_m(x) y_m, \qquad (2.3)
\end{aligned}$$

where

$$s_i = \left(\sum_{j=1}^{n} g_j(x) c_{ji} \right)$$

for each $i = 1, \ldots, n$.

2.2 Fuzzy Set Theory

A fuzzy subset A of an universe X is characterized by a function $\mu_A : X \to [0, 1]$, called membership function [16], where $\mu_A(x)$, or simply $A(x)$, represents the membership degree of x in A, for all $x \in X$. The class of fuzzy sets of X is denoted by the symbol $\mathcal{F}(X)$. Each classical subset A of X is a particular fuzzy set whose membership function is given by its characteristic function $\chi_A : X \to \{0, 1\}$, *i.e.*, $\chi_A(x) = 1$ if and only if $x \in A$.

The α-cut of a fuzzy set A of X, denoted by $[A]^\alpha$, is defined as $[A]^\alpha = \{x \in X : A(x) \geq \alpha\}$, $\forall \alpha \in (0, 1]$. If X is also a topological space, then we can define the 0-cut of A by $[A]^0 = cl\{x \in X : A(x) > 0\}$ [17], where *cl* Y, $Y \subseteq X$, denotes the closure of Y.

Zadeh's extension principle [18] can be viewed as mathematical method to extend a function $f : X \to Y$ to a function $\hat{f} : \mathcal{F}(X) \to \mathcal{F}(Y)$.

Definition 1 *(Zadeh's extension principle* [17,18]*).* *Let $f : X \to Y$. The Zadeh's extension of f at $A \in \mathcal{F}(X)$ is the fuzzy set $\hat{f}(A) \in \mathcal{F}(Y)$ whose membership function is given by*

$$\hat{f}(A)(y) = \bigvee_{x \in f^{-1}(y)} A(x), \ \forall\, y \in Y,$$

where $f^{-1}(y) = \{x \in X : f(x) = y\}$ is the preimage of the function f at y and, by definition, $\bigvee \emptyset = 0$.

A fuzzy set $A \in \mathcal{F}(\mathbb{R})$ is called a fuzzy number if its α-cuts are closed, bounded and non-empty intervals for all $\alpha \in [0,1]$ [17]. Since each α-cut of a fuzzy number A is an interval that satisfies the previous properties, we can write $[A]^\alpha = [a_\alpha^-, a_\alpha^+]$. We denote the class of fuzzy numbers by the symbol $\mathbb{R}_\mathcal{F}$. The next theorem indicates when a family of subsets can be uniquely associated with a fuzzy number.

Theorem 1 *(Negoita-Ralescu's characterization theorem [19,20]).* *Given a family of subsets $\{A_\alpha : \alpha \in [0,1]\}$ that satisfies the following conditions*

(a) A_α is a non-empty, closed, and bounded interval for any $\alpha \in [0,1]$;
(b) $A_{\alpha_2} \subseteq A_{\alpha_1}$, for all $0 \leq \alpha_1 \leq \alpha_2 \leq 1$;
(c) For any sequence α_n which converges from below to $\alpha \in (0,1]$ we have

$$\bigcap_{n=1}^{\infty} A_{\alpha_n} = A_\alpha;$$

(d) For any sequence α_n which converges from above to 0 we have

$$A_0 = cl\left(\bigcup_{n=1}^{\infty} A_{\alpha_n}\right).$$

Then there exists a unique $A \in \mathbb{R}_\mathcal{F}$, such that $[A]^\alpha = A_\alpha$, for each $\alpha \in [0,1]$.

Conversely, let $A \in \mathbb{R}_\mathcal{F}$, if $A_\alpha = [A]^\alpha$ for all $\alpha \in [0,1]$ then the family of subsets $\{A_\alpha : \alpha \in [0,1]\}$ satisfies the conditions (a)–(d).

An example of fuzzy number is a triangular fuzzy number that is denoted by the triple $(a;b;c)$, with $a \leq b \leq c$. In view of Theorem 1, the triangular fuzzy number can be defined in terms of its α-cuts as follows:

$$[A]^\alpha = [a + \alpha(b-a), c - \alpha(c-b)],\ \forall \alpha \in [0,1].$$

Note that a real number a is a particular case of triangular fuzzy number since we have $a \equiv (a;a;a)$.

A fuzzy relation R over $X = X_1 \times \ldots \times X_n$ is any fuzzy subset of $X_1 \times \ldots \times X_n$. Thus, a fuzzy relation R is associated with a membership function $R : X_1 \times \ldots \times X_n \to [0,1]$, where $R(x_1, \ldots, x_n) \in [0,1]$ represents the degree of relationship among $x_1, \ldots, x_n$ with respect to R [17].

The projection of fuzzy relation $R \in \mathcal{F}(X_1 \times \ldots \times X_n)$ onto X_i, for $i \in \{1, \ldots, n\}$, is the fuzzy set Π_R^i of X_i given by

$$\Pi_R^i(y) = \bigvee_{x \in X : x_i = y} R(x_1, \ldots, x_n).$$

A fuzzy relation $J \in \mathcal{F}(\mathbb{R}^n)$ is said to be a joint possibility distribution of $A_1, \ldots, A_n \in \mathbb{R}_\mathcal{F}$ if

$$A_i(y) = \Pi_J^i(y) = \bigvee_{x \in X : x_i = y} J(x_1, \ldots, x_n),$$

for all $y \in \mathbb{R}$ and for all $i = 1, \ldots, n$.

Given a t-norm t, that is, a commutative, associative, and increasing operator $t : [0,1]^2 \to [0,1]$ satisfying $t(x,1) = x\ t\ 1 = x$ for all $x \in [0,1]$. A fuzzy relation J_t given by

$$J_t(x_1, \dots, x_n) = A_1(x_1)\ t\ \dots\ t\ A_n(x_n) \tag{2.4}$$

is said to be a t-norm-based joint possibility distribution of $A_1, \dots, A_n \in \mathbb{R}_{\mathcal{F}}$ [4]. Well-known example of t-norm include the minimum t-norm "$\wedge$". In particular, when $J = J_\wedge$, that is, J is given by (2.4) with $t = \wedge$, we say that $A_1, \dots, A_n$ are non-interactive. Otherwise, $J \neq J_\wedge$, we say that $A_1, \dots, A_n$ are interactive [5,18,21].

Thus, the notion of interactivity between fuzzy numbers is given by means of joint possibility distributions. Carlsson *et al.* [5] introduced a possible type of interactivity relation between two fuzzy numbers that is not based on t-norms. Specifically, two fuzzy numbers A and B are said to be completely correlated if there exist $q, r \in \mathbb{R}$ with $q \neq 0$ such that the corresponding joint possibility distribution $J_{\{q,r\}}$ is given by

$$\begin{aligned} J_{\{q,r\}}(x_1, x_2) &= A(x_1)\chi_{\{qu+r=v\}}(x_1, x_2) \\ &= B(x_2)\chi_{\{qu+r=v\}}(x_1, x_2), \end{aligned} \tag{2.5}$$

where $\chi_{\{qu+r=v\}}$ stands for the characteristic function of the set $\{(u,v) \in \mathbb{R}^2 : qu + v = r\} \subset \mathbb{R}^2$. In addition, if $q > 0$ ($q < 0$) then A and B are said to be completely positively (negatively) correlated. Since $q \neq 0$ in Eq. (2.5), the membership function of B can be written as $B(qx + r) = A(x)$ for all $x \in \mathbb{R}$, and consequently $[B]^\alpha = q[A]^\alpha + \{r\}$ for all $\alpha \in [0,1]$. Moreover, for each $\alpha \in [0,1]$, the α-cut of the joint possibility distribution $J_{\{q,r\}}$ is given by [5]:

$$[J_{\{q,r\}}]^\alpha = \{(x, qx + r)\ :\ x \in [A]^\alpha\}.$$

Remark 1. Note that if the fuzzy numbers A and B are completely correlated by the line $qu + r_1 = v$, and we choose $r_2 = q(a^-_\alpha + a^+_\alpha) + r_1$, then A and B are also completely correlated if we consider $J_{\{-q,r_2\}}$, that is, A and B are also completely correlated with respect to the line $-qu + r_2 = v$. Therefore, the distribution J is not unique.

The next definition is a generalization of Zadeh's extension principle (cf. Definition 1).

Definition 2 *(Sup-J Extension Principle [6]). Let $J \in \mathcal{F}(\mathbb{R}^n)$ be a joint possibility distribution of $A_1, \dots, A_n \in \mathbb{R}_{\mathcal{F}}$ and let $f : \mathbb{R}^n \to \mathbb{R}$. The* $\sup -J$ *extension of f at $(A_1, \dots, A_n)$ is defined by*

$$f_J(A_1, \dots, A_n)(y) = \hat{f}(J)(y) = \bigvee_{(x_1,\dots,x_n) \in f^{-1}(y)} J(x_1, \dots, x_n),$$

where $f^{-1}(y) = \{(x_1, \dots, x_n) \in \mathbb{R}^n : f(x_1, \dots, x_n) = y\}$.

From Definition 2, we can define arithmetic operations among n fuzzy numbers by taking the sup-J extension of the corresponding arithmetic operator. For example, let $f(x_1, \ldots, x_n) = x_1 + \ldots + x_n$ for all $x_1, \ldots, x_n \in \mathbb{R}$. If $J_\wedge$ is defined as in (2.4) with $t = \wedge$, then $f_{J_\wedge}(A_1, \ldots, A_n)$ boils down to Zadeh's extension of f at $(A_1, \ldots, A_n)$, *i.e.*,

$$\widehat{f}(A_1, \ldots, A_n)(y) = \bigvee_{(x_1,\ldots,x_n)\in f^{-1}(y)} A_1(x_1) \wedge \ldots \wedge A_n(x_n), \quad \forall\, y \in \mathbb{R}.$$

The next proposition ensures that the completely correlation is a transitive relation of interactivity between fuzzy numbers. Moreover, under some conditions, the sup-$J_{q,r}$ extensions of the addition operator, denoted by the symbol $+_L$, satisfies the associative property.

Proposition 1 [22]**.** *Let A, B, $C \in \mathbb{R}_\mathcal{F}$. If A and B are completely correlated with respect to $J_{\{q_1,r_1\}}$ and B and C are completely correlated with respect to $J_{\{q_2,r_2\}}$, then there are real numbers q_3 and r_3 such that A and C are completely correlated with respect to $J_{\{q_3,r_3\}}$.*

Moreover, if each A, B, $C \in \mathbb{R}_\mathcal{F}$ is completely correlated to $D \in \mathbb{R}_\mathcal{F} \backslash \mathbb{R}$, then the associative property holds true, i.e., $A +_L (B +_L C) = (A +_L B) +_L C$.

The notion of completely correlation can be extended to n fuzzy numbers as follows.

Definition 3. *The fuzzy numbers $A_1, \ldots, A_n \in \mathbb{R}_\mathcal{F}$ are said completely correlated if the joint possibility distribution J is given by*

$$\begin{aligned} J(x_1, \ldots, x_n) &= \chi_U(x_1, \ldots, x_n) A_1(x_1) \\ &= \chi_U(x_1, \ldots, x_n) A_2(x_2) \\ &\;\vdots \\ &= \chi_U(x_1, \ldots, x_n) A_n(x_n), \end{aligned} \tag{2.6}$$

where $U = \{(u, q_2 u + r_2, \ldots, q_n u + r_n) : u \in \mathbb{R}\}$, $q_i, r_i \in \mathbb{R}$, with $q_i \neq 0$, $\forall i = 1, \ldots, n$.

From (2.5) and (2.6), one can see that A_1 and A_i, $i > 1$, are also completely correlated since we have $[A_i]^\alpha = q_i[A_1]^\alpha + \{r_i\}$, for all $i = 2, \ldots, n$. This implies that, for each $\alpha \in [0, 1]$, the α-cut of J is given as follows

$$[J]^\alpha = \{(x, q_2 x + r_2, \ldots, q_n x + r_n) \;:\; x \in [A_1]^\alpha\} \tag{2.7}$$

Remark 2. From Eq. (2.7), we can note that the α-cuts of the joint possibility distribution J can be expressed in terms of α-cuts of A_1 and the parameters q_i and r_i, for all $i = 2, \ldots, n$.

Theorem 2 [23,24]**.** *Let $f : \mathbb{R}^n \to \mathbb{R}$ be a continuous function and $J \in \mathcal{F}(\mathbb{R}^n)$. We have that*

$$[\widehat{f}_J(A_1, \ldots, A_n)]^\alpha = f([J]^\alpha),\ \forall \alpha \in [0, 1].$$

By Theorem 2, if the sup-J extension of f at $(A_1,\dots,A_n)$ is a fuzzy number, then the α-cuts of $\widehat{f}_J(A_1,\dots,A_n) = \widehat{f}(J)$ can be written as follows:

$$[\widehat{f}(J)]^\alpha = \left[\bigwedge_{(x_1,\dots,x_n)\in[J]^\alpha} f(x_1,\dots,x_n) \quad \bigvee_{(x_1,\dots,x_n)\in[J]^\alpha} f(x_1,\dots,x_n) \right]. \tag{2.8}$$

In the next section, we consider the problem given in (2.1) for the case where the known values y_i are interactive fuzzy numbers.

3 Least Squares Method for Interactive Fuzzy Data

In this paper, we deal with least squares method to fit uncertain data given by interactive fuzzy numbers. In particular, we focus on the case where these fuzzy numbers are completely correlated. A typical example of correlated data are the well-known longitudinal data, which are widely studied in the statistical area [13].

Let $D = \{(x_1, Y_1),\dots,(x_m, Y_m)\} \subset \mathbb{R} \times \mathbb{R}_\mathcal{F}$ such that $Y_1,\dots,Y_m$ are completely correlated fuzzy numbers, with respect to a joint possibility distribution J as in (2.6), and let $F : \mathbb{R} \to \mathbb{R}_\mathcal{F}$ be a function that satisfies $F(x_i) = Y_i$ for $i = 1,\dots,m$. We produce a function $\Phi : \mathbb{R} \to \mathbb{R}_\mathcal{F}$ that approximates F given by means of the sup-J extension principle of a function $\varphi : \mathbb{R} \to \mathbb{R}$ of the form

$$\varphi(x) = a_1 g_1(x) + \ldots + a_n g_n(x),$$

where $a_1,\dots,a_n \in \mathbb{R}$ and $g_1,\dots,g_n$ are real-valued-functions. More precisely, we define the function Φ in terms of the sup-J extension principle of (2.3) at $(Y_1,\dots,Y_m)$. Since Eq. (2.3) is continuous with respect to $y_1,\dots,y_m$, from Theorem 2 and Eq. (2.7), we have that α-cuts of the fuzzy number $\Phi(x)$ is given by

$$\begin{aligned} [\Phi(x)]^\alpha &= \{s_1(x)y_1 + \ldots + s_m(x)y_m : (y_1,\dots,y_m) \in [J]^\alpha\} \\ &= \{s_1(x)y + s_2(x)(q_2 y + r_2) + \ldots + s_m(x)(q_m y + r_m)y : y \in [Y_1]^\alpha\}. \end{aligned} \tag{3.9}$$

Since the interval $[Y_1]^\alpha = [y_{1\alpha}^-, y_{1\alpha}^+]$ can be rewritten as the set of all convex combination of $y_{1\alpha}^-$ and $y_{1\alpha}^+$, that is, $[Y_1]^\alpha = \{(1-\lambda)y_{1\alpha}^- + \lambda y_{1\alpha}^+ : \lambda \in [0,1]\}$, the α-cut of J can also be expressed in terms of a parameter $\lambda \in [0,1]$ as follows:

$$[J]^\alpha = \{(1-\lambda)Y_\alpha^- + \lambda Y_\alpha^+ : \lambda \in [0,1]\},$$

where $Y_\alpha^- = (y_{1\alpha}^-, q_2 y_{1\alpha}^- + r_2,\dots, q_m y_{1\alpha}^- + r_m)$ and $Y_\alpha^+ = (y_{1\alpha}^+, q_2 y_{1\alpha}^+ + r_2,\dots, q_m y_{1\alpha}^+ + r_m)$. Thus, Eq. (3.9) can be expressed as

$$[\Phi(x)]^\alpha = \{(1-\lambda)\langle S(x), Y_\alpha^-\rangle + \lambda\langle S(x), Y_\alpha^+\rangle : \lambda \in [0,1]\} \tag{3.10}$$

where $\langle\cdot,\cdot\rangle$ denotes the usual inner product of $\mathbb{R}^m$ and $S(x) = (s_1(x), s_2(x),\dots, s_m(x))$, $x \in \mathbb{R}$.

In order to characterize the endpoints of each α-cut of $\Phi(x)$, we define the auxiliary function h by

$$h(x,\alpha,\lambda) = (1-\lambda)B_1(x,\alpha) + \lambda B_2(x,\alpha),\ \forall x \in \mathbb{R} \text{ and } \forall \alpha, \lambda \in [0,1],$$

where

$$B_1(x,\alpha) = \langle S(x), Y_\alpha^- \rangle \text{ and } B_2(x,\alpha) = \langle S(x), Y_\alpha^+ \rangle.$$

By Eqs. (3.10) and (2.8), we have that

$$\begin{aligned}[\Phi(x)]^\alpha &= \{h(x,\alpha,\lambda) : \lambda \in [0,1]\} \\ &= \left[\bigwedge_{\lambda \in [0,1]} h(x,\alpha,\lambda), \bigvee_{\lambda \in [0,1]} h(x,\alpha,\lambda) \right]. \end{aligned} \tag{3.11}$$

Note that if $B_1(x,\alpha) \le B_2(x,\alpha)$, then the function $h(x,\alpha,\cdot)$ assumes the minimum and the maximum values at $\lambda = 0$ and $\lambda = 1$, respectively. On the other hand, if $B_1(x,\alpha) > B_2(x,\alpha)$ then the minimum and maximum values of $h(x,\alpha,\cdot)$ are achieved at $\lambda = 1$ and $\lambda = 0$, respectively. In other words, the global minimizer and maximizer of $h(x,\alpha,\lambda)$ for $\lambda \in [0,1]$ are given at $\lambda = 0$ or $\lambda = 1$. Therefore, for each $x \in \mathbb{R}$, the α-cuts of the fuzzy solution φ is given by

$$[\Phi(x)]^\alpha = [\min\{h(x,\alpha,0), h(x,\alpha,1)\}, \max\{h(x,\alpha,0), h(x,\alpha,1)\}], \tag{3.12}$$

where

$$h(x,\alpha,0) = B_1(x,\alpha) = \langle S(x), Y_\alpha^- \rangle$$

and

$$h(x,\alpha,1) = B_2(x,\alpha) = \langle S(x), Y_\alpha^+ \rangle.$$

In the next section we illustrate this proposed method by means of an example.

4 Application of Least Squares Method for Completely Correlated Fuzzy Data

In this section we apply the proposed method to determine a function that fits longitudinal data obtained from [15]. The authors discussed the association between children mortality and air pollution in São Paulo, Brazil, from 1994 to 1997. In their study were collected longitudinal data of sulfur dioxide (SO_2), carbon monoxide (CO), inhalable particulate (PM_{10}) and ozone (O_3). Here, we focus on the ozone dataset.

For simplicity, suppose that the longitudinal data are given by completely correlated triangular fuzzy numbers of the form $(M-\sigma; M; M+\sigma)$, where M and σ are the mean and the standard deviation of the collected data in each year, respectively. Recall that the proposed method is not restricted to triangular fuzzy numbers, then other types of fuzzy number can be considered.

Let $D = \{(x_1, Y_1), (x_2, Y_2), (x_3, Y_3), (x_4, Y_4)\} \subset \mathbb{R} \times \mathbb{R}_{\mathcal{F}}$ be the fuzzy dataset given in Table 1. The values $x_1 = 1$, $x_2 = 2$, $x_3 = 3$, and $x_4 = 4$ represent respectively the years 1994, 1995, 1996, and 1997. The fuzzy numbers $Y_1 = (17.6; 57; 96.4)$, $Y_2 = (25.3; 60.7; 96.1)$, $Y_3 = (34.8; 76.3; 117.8)$, and $Y_4 = (29.5; 63; 96.5)$ are completely correlated with respect to joint possibility distribution J, whose membership function is given by

$$J(v_1, v_2, v_3, v_4) = \chi_U(v_1, v_2, v_3, v_4) Y_1(v_1), \ \forall \ (v_1, v_2, v_3, v_4) \in \mathbb{R}^4,$$

where

$$U = \{(u, 0.8985u + 9.4855, 1.0533u + 16.2619, 0.8502u + 14.5386) \ : \ u \in \mathbb{R}\}. \tag{4.13}$$

Table 1. Fuzzy dataset D

x:	1	2	3	4
Y:	$(17.6; 57; 96.4)$	$(25.3; 60.7; 96.1)$	$(34.8; 76.3; 117.8)$	$(29.5; 63; 96.5)$

Note that Eq. (4.13) suggests that Y_1 and Y_2 are positively completely correlated, as well as Y_1 and Y_3, Y_1 and Y_4, since $q_i > 0$, for all $i = 2, 3, 4$.

Consider the functions $g_1(x) = x^2$, $g_2(x) = x$ and $g_3(x) = 1$. From (3.12), for each $\alpha \in [0, 1]$ and $x \in [1, 4]$, the fuzzy function Φ is given by $[\Phi(x)]^\alpha = [\min\{h(x, \alpha, 0), h(x, \alpha, 1)\}, \max\{h(x, \alpha, 0), h(x, \alpha, 1)\}]$, where

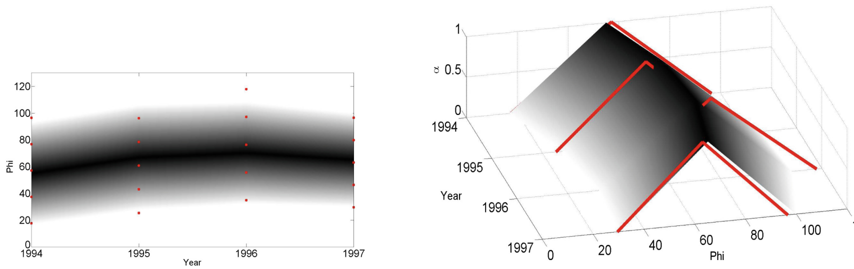
$$h(x, \alpha, 0) = -3.24x^2 + 20.76x - 0.75 + \alpha(-x^2 + 3.84x + 35.34)$$

and

$$h(x, \alpha, 1) = -5.24x^2 + 28.44x + 69.93 - \alpha(-x^2 + 3.84x + 35.34).$$

Figure 1 exhibits the fuzzy function Φ produced by our proposal. One can observe in Subfigure 1(a) fits the data of Table 1 which varies from 1994 to 1997. The red triangles and the gray-scale surface depicted in Subfigure 1(b) correspond to the membership functions of fuzzy data Y_i, $i = 1, \ldots, 4$, and fuzzy solution, respectively.

Note that $Y_1, \ldots, Y_4$ are completely correlated with respect to 2^3 different joint possibility distributions. Thus, we can obtain 2^3 fuzzy functions Φ. However, in general, the choice of a joint possibility distribution is not arbitrary and depends on the context. For example, if each object is measured m times with the same n measuring devices then we can assume that the obtained values depend only on the calibration of each equipment and not on the objects. This type of assumption induces the choice of specific parameters q_i and r_i in (2.7).



(a) (b)

Fig. 1. Subfigures (a) and (b) exhibit respectively the top and depiction views of the fuzzy solution Φ where the greatest and smallest membership values are represented respectively by the black and white colors. In Subfigure (a), the red dots represent the endpoints of the $\alpha-$cuts of the fuzzy data Y_i for $\alpha = 0, 0.5, 1$ and $i = 1, \ldots, 4$. Each fuzzy data Y_i is represented by red lines in Subfigure (b).

5 Conclusion

In this manuscript, we considered a fuzzy least squares problem based on dataset that has some type of correlation, for example a longitudinal dataset. We assumed that the dataset is composed by completely correlated fuzzy numbers [5]. In particular, we presented a method that provides a fuzzy function that fits a given fuzzy data. This fuzzy function depends on the choice of a joint possibility distributions as in (2.6).

The α-cut of the fuzzy solution given by means of the sup-J extension principle is a non-empty, bounded, closed interval whose endpoints are obtained by solving a minimization and maximization problems given in Eq. (3.11). Investigating this problem, we concluded that the endpoints of the α-cut of the proposed solution can be evaluated by taking the minimum and maximum of two associated real functions (see Eq. (3.12)).

Finally, we applied the proposed method to determine a fuzzy function which fits a longitudinal air polution dataset [15]. The fuzzy data in this dataset was modelled using triangular fuzzy numbers, but it can be done with other types of completely correlated fuzzy numbers. The fuzzy solution was calculated considering polynomial functions g_1, g_2, and g_3. For further works, we intend to investigate fuzzy least squares method for dataset with other intrinsic type of interactivity.

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