Chapter 6
The Hamilton–Jacobi Formalism

In the preceding chapters we have studied convenient forms of expressing some systems of ordinary differential equations, most of them related to mechanical systems. In this chapter we shall see that each of these systems of equations can be translated into a single partial differential equation, known as the Hamilton–Jacobi equation, which is constructed out of the Hamiltonian. A complete solution of this equation is the generating function of a canonical transformation that relates the coordinates being employed with another set of canonical coordinates which are all constants of motion.

As we have seen in the preceding chapter, the canonical transformations can be employed to simplify the solution of the Hamilton equations. However, we do not have a systematic method to find a convenient canonical transformation for any given Hamiltonian. As we shall show in this chapter, by finding a complete solution of a certain first-order partial differential equation (the Hamilton–Jacobi equation, or HJ equation, for short) one obtains the generating function of a local canonical transformation such that the new Hamiltonian is equal to zero.

In Section 6.1 we present the HJ equation and we give several standard examples of its application, finding complete solutions of the HJ equation by means of the method of separation of variables. In Section 6.1.1 we study the relationship between different complete solutions of the HJ equation. In Section 6.1.2, we consider alternative expressions for the HJ equation, which are useful in some cases, but not usually discussed in the standard textbooks. In Section 6.1.3 we show that in some problems where the method of separation of variables is not applicable, it may be possible to obtain R-separable solutions, which are sums of a fixed function that may depend on all the variables, and separated one-variable functions. In Section 6.2 we give a simple proof and several applications of the Liouville theorem, which enables us to find complete solutions of the HJ equation, making use of an adequate set of constants of motion.

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In Section 6.3 we show how to map the solutions of the HJ equation corresponding to a Hamiltonian H into solutions of the HJ equation corresponding to the new Hamiltonian K obtained by a canonical transformation. This mapping is then applied in Section 6.3.1 to find the solutions of the HJ equation with a specified initial condition. In Section 6.4 we show that with any point transformation in the extended configuration space, in which the time may be also transformed, and any Hamiltonian, we can obtain new Hamiltonians such that the solutions of the corresponding HJ equations are related in a simple way.

In Section 6.5 we apply the Lagrangian and the Hamiltonian formalisms to the study of geometrical optics and we show that the HJ equation leads to the eikonal equation.

6.1 The Hamilton–Jacobi Equation

As we have seen in Section 5.2, any real-valued function of $2n + 1$ variables, $F_2(q_i, P_i, t)$, such that

$$\det\left(\frac{\partial^2 F_2}{\partial q_i \partial P_j}\right) \neq 0, \tag{6.1}$$

defines a canonical transformation, $Q_i = Q_i(q_j, p_j, t)$, $P_i = P_i(q_j, p_j, t)$, given implicitly by

$$p_i = \frac{\partial F_2}{\partial q_i}, \qquad Q_i = \frac{\partial F_2}{\partial P_i}. \tag{6.2}$$

Then, the Hamilton equations

$$\dot{q}_i = \frac{\partial H}{\partial p_i}, \qquad \dot{p}_i = -\frac{\partial H}{\partial q_i}, \tag{6.3}$$

for an arbitrary Hamiltonian, H, are equivalent to

$$\dot{Q}_i = \frac{\partial K}{\partial P_i}, \qquad \dot{P}_i = -\frac{\partial K}{\partial Q_i}, \tag{6.4}$$

where

$$K = H + \frac{\partial F_2}{\partial t}, \tag{6.5}$$

but Equations (6.4) need not be simpler than (6.3).

However, if we find a generating function F_2 such that the new Hamiltonian, K, is equal to zero, then the equations of motion (6.4) are trivially integrated, yielding $Q_i = \text{const.}$, $P_i = \text{const.}$ (that is, the new canonical coordinates, Q_i, P_i, are $2n$, locally defined, constants of motion) (cf. Proposition 5.42). In that case, by combining (6.5) with the first equations in (6.2), and denoting the generating function F_2 by S, one obtains

$$H\left(q_i, \frac{\partial S}{\partial q_i}, t\right) + \frac{\partial S}{\partial t} = 0. \tag{6.6}$$

Equation (6.6) is a first-order partial differential equation (PDE) for $S(q_i, P_i, t)$, known as the *Hamilton–Jacobi* (HJ) equation, and the function S will be called *Hamilton's principal function*. It should be noted that this equation does not contain the variables P_i explicitly, so that, in order to satisfy the condition

$$\det\left(\frac{\partial^2 S}{\partial q_i \partial P_j}\right) \neq 0 \tag{6.7}$$

[see Equation (6.1)], the function S must contain the n variables P_i as parameters. Any solution of the HJ equation satisfying (6.7) is called a *complete solution.* (A first-order linear PDE possesses a *general* solution that contains an arbitrary *function*. See, e.g., Example 6.29.)

Since the HJ equation does not contain S explicitly, but only its partial derivatives, given a solution, S, of the HJ equation, if c is an arbitrary constant, then $S+c$ is also a solution of the same equation. However, such a trivial constant cannot be one of the n parameters P_i contained in a complete solution because it would produce an entire row or column of zeroes in the matrix (6.7).

As we have seen in Proposition 5.42, making use of the explicit form of the solution of the Hamilton equations, one can find the generating function of a canonical transformation such that the new Hamiltonian is equal to zero (that is, a complete solution of the HJ equation) (see Examples 5.40 and 5.43). What is desirable is to find complete solutions of the HJ equation without knowing beforehand the solution of the Hamilton equations. Unfortunately, we do not have an alternative method to solve the HJ equation in general.

Once we have a complete solution, $S(q_i, P_i, t)$, of the HJ equation, we substitute it into the equations

$$p_i = \frac{\partial S}{\partial q_i}, \qquad Q_i = \frac{\partial S}{\partial P_i} \tag{6.8}$$

[see Equation (6.2)] in order to obtain the canonical transformation generated by S. If we make use of these equations to find Q_i and P_i in terms of (q_j, p_j, t), we obtain $2n$ (functionally independent) constants of motion, and if we use them to express q_i and p_i in terms of (Q_j, P_j, t), we obtain the solution of the Hamilton equations (6.3), using the fact that the solution of the Hamilton equations (6.4), with K being equal to zero, is $Q_i(t) = \text{const.}$, $P_i(t) = \text{const.}$

Example 6.1 (One-dimensional harmonic oscillator). By means of a direct substitution one can readily verify that

$$S(q, P, t) = -\left(\frac{P^2}{2m} + \frac{m\omega^2}{2}q^2\right)\frac{\tan \omega t}{\omega} + Pq \sec \omega t$$

is a solution of the HJ equation

$$\frac{1}{2m}\left(\frac{\partial S}{\partial q}\right)^2 + \frac{m\omega^2}{2}q^2 + \frac{\partial S}{\partial t} = 0, \tag{6.9}$$

which corresponds to the standard Hamiltonian of a one-dimensional harmonic oscillator

$$H = \frac{p^2}{2m} + \frac{m\omega^2}{2}q^2$$

[cf. Equation (6.6)]. (This solution of Equation (6.9) will be obtained in Example 6.18, below.) Then, Equations (6.8) yield

$$p = -m\omega q \tan \omega t + P \sec \omega t, \qquad Q = -\frac{P}{m\omega}\tan \omega t + q \sec \omega t$$

and, therefore, the canonical transformation generated by S is given by

$$Q = q \cos \omega t - \frac{p}{m\omega}\sin \omega t, \qquad P = m\omega q \sin \omega t + p \cos \omega t, \tag{6.10}$$

or

$$q = Q \cos \omega t + \frac{P}{m\omega}\sin \omega t, \qquad p = -m\omega Q \sin \omega t + P \cos \omega t. \tag{6.11}$$

According to the discussion above, Equations (6.10) give two constants of motion, while Equations (6.11) give the solution of the Hamilton equations (in this example, Q and P happen to be the values of q and p at $t = 0$, respectively). It may be noticed that, by virtue of the Hamilton equations, $p = m\dot{q}$ and, therefore, the second equation in (6.11) can be obtained by differentiating the first one with respect to the time, but this *is not necessary* (though not wrong, either); the canonical transformation generated by any complete solution of the HJ equation yields the entire solution of the Hamilton equations.

Separation of Variables

The method regularly employed to find complete solutions of the HJ equation (and in most textbooks the only one mentioned) is the method of *separation of variables*. In this method one looks for solutions of Equation (6.6) that can be written as the sum of $n+1$ one-variable functions, $S = S_1(q_1) + S_2(q_2) + \cdots + S_n(q_n) + S_{n+1}(t)$. When the method is applicable, one obtains $n+1$ first-order ODEs (for the functions $S_1, S_2, \ldots, S_{n+1}$), and in the process of separating the variables one has to introduce n constants of separation, which can be taken as the parameters P_i (see the examples below).

Example 6.2 (Particle in a uniform gravitational field). A very simple, but illustrative, example is given by the Hamiltonian

$$H = \frac{{p_x}^2 + {p_y}^2}{2m} + mgy, \tag{6.12}$$

corresponding to a particle of mass m in a uniform gravitational field. The HJ equation is given by

$$\frac{1}{2m}\left[\left(\frac{\partial S}{\partial x}\right)^2 + \left(\frac{\partial S}{\partial y}\right)^2\right] + mgy + \frac{\partial S}{\partial t} = 0 \tag{6.13}$$

and we look for a separable solution of (6.13), that is, a solution of the form

$$S = A(x) + B(y) + C(t), \tag{6.14}$$

where A, B, and C are real-valued functions of a single variable. Substituting (6.14) into (6.13) we obtain, after rearrangement of the terms,

$$\frac{1}{2m}\left[\left(\frac{\mathrm{d}A}{\mathrm{d}x}\right)^2 + \left(\frac{\mathrm{d}B}{\mathrm{d}y}\right)^2\right] + mgy = -\frac{\mathrm{d}C}{\mathrm{d}t},$$

which must hold for all values of x, y, and t, in some open subset of $\mathbb{R}^3$. The left-hand side of this last equation does not depend on t, while the right-hand side does not depend on x and y; hence, the two sides of the equation do not depend on x, y, or t, and therefore must be equal to some constant, P_1, say. Hence, up to an irrelevant constant term,

$$C(t) = -P_1 t \tag{6.15}$$

and

$$\frac{1}{2m}\left[\left(\frac{\mathrm{d}A}{\mathrm{d}x}\right)^2 + \left(\frac{\mathrm{d}B}{\mathrm{d}y}\right)^2\right] + mgy = P_1.$$

Rewriting the last equation in the form

$$\left(\frac{\mathrm{d}A}{\mathrm{d}x}\right)^2 = 2mP_1 - 2m^2gy - \left(\frac{\mathrm{d}B}{\mathrm{d}y}\right)^2,$$

we obtain an equation such that the left-hand side does not depend on y, and the right-hand side does not depend on x, thus, each side must be a constant. Hence,

$$A(x) = P_2 x, \tag{6.16}$$

where P_2 is a constant, and

$$\frac{\mathrm{d}B}{\mathrm{d}y} = \pm\sqrt{2mP_1 - P_2{}^2 - 2m^2gy}.$$

(In what follows there is no need to consider the two signs in the square root, since we only require *one* complete solution of the HJ equation.) In this manner, we have obtained a solution of the HJ equation (6.13),

$$S(x, y, P_1, P_2, t) = P_2 x + \int \sqrt{2mP_1 - P_2{}^2 - 2m^2gy}\,\mathrm{d}y - P_1 t, \tag{6.17}$$

that contains two parameters (the *constants of separation* P_1 and P_2) which have been identified with the new momenta, in order to emphasize the role of S as the (type F_2) generating function of a canonical transformation.

Making use of Equations (6.8) we have

$$p_x = P_2, \qquad p_y = \sqrt{2mP_1 - P_2{}^2 - 2m^2gy},$$

and

$$Q_1 = -t + \int \frac{m\,\mathrm{d}y}{\sqrt{2mP_1 - P_2{}^2 - 2m^2gy}} = -t - \frac{1}{mg}\sqrt{2mP_1 - P_2{}^2 - 2m^2gy},$$

$$Q_2 = x - \int \frac{P_2\,\mathrm{d}y}{\sqrt{2mP_1 - P_2{}^2 - 2m^2gy}} = x + \frac{P_2}{m^2g}\sqrt{2mP_1 - P_2{}^2 - 2m^2gy}.$$

By combining these last expressions, we obtain the coordinate transformation

$$Q_1 = -t - \frac{p_y}{mg}, \qquad Q_2 = x + \frac{p_x p_y}{m^2 g}, \qquad P_1 = \frac{p_x{}^2 + p_y{}^2}{2m} + mgy, \qquad P_2 = p_x, \tag{6.18}$$

and its inverse

$$x = Q_2 + \frac{P_2(t+Q_1)}{m}, \quad y = \frac{P_1}{mg} - \frac{{P_2}^2}{2m^2 g} - \frac{g}{2}(t+Q_1)^2,$$
$$p_x = P_2, \qquad\qquad p_y = -mg(t+Q_1). \tag{6.19}$$

Since the new Hamiltonian is equal to zero, Equations (6.18) give four constants of motion, while Equations (6.19) give the solution of the Hamilton equations in the original variables. (In this example not all of the Q_i and P_i coincide with the initial values of q_i and p_i; however, the constants of motion Q_i and P_i can be expressed in terms of the initial values of q_i and p_i by simply setting $t = 0$, or any other initial value of t, in Equations (6.18).)

For future convenience, it is useful to note that the functions $A(x)$ and $C(t)$, which depend on the variables that do not appear in the Hamiltonian (6.12), are *linear functions* [see Equations (6.15) and (6.16)]. One can convince oneself that this is a general rule: if a coordinate q_K does not appear in the Hamiltonian (but its conjugate momentum, p_K, does appear in H), then, in a separable solution of the HJ equation, the function depending on q_K must be a linear function. Similarly, if t does not appear in the Hamiltonian, then, in a separable solution of the HJ equation, the function depending on t must be a linear function of t.

Example 6.3 (Particle in a central field of force). One of the standard examples of the application of the HJ equation is that of a particle in a central field of force. Using the fact that the orbit must lie on a plane passing through the center of force, we consider a particle moving on the Euclidean plane under the influence of a potential $V(r)$, hence

$$H = \frac{1}{2m}\left({p_r}^2 + \frac{{p_\theta}^2}{r^2}\right) + V(r), \tag{6.20}$$

in terms of the polar coordinates (r, θ) [see (4.7)]. Thus, the HJ equation is given by

$$\frac{1}{2m}\left[\left(\frac{\partial S}{\partial r}\right)^2 + \frac{1}{r^2}\left(\frac{\partial S}{\partial \theta}\right)^2\right] + V(r) + \frac{\partial S}{\partial t} = 0 \tag{6.21}$$

and, taking into account that θ and t do not appear in the Hamiltonian (or in the HJ equation), we look for a separable solution of (6.21) of the form

$$S = A(r) + P_2\theta - P_1 t, \tag{6.22}$$

where A is a real-valued function of a single variable, and P_1, P_2 are separation constants. Substituting (6.22) into (6.21) we obtain

$$\frac{\mathrm{d}A}{\mathrm{d}r} = \pm\sqrt{2m\left[P_1 - V(r) - \frac{{P_2}^2}{2mr^2}\right]}.$$

Thus, we have a solution of the HJ equation (6.21),

$$S(r, \theta, P_1, P_2, t) = \int \sqrt{2m\left[P_1 - V(r) - \frac{{P_2}^2}{2mr^2}\right]}\,\mathrm{d}r + P_2\theta - P_1 t. \tag{6.23}$$

that contains two parameters (P_1 and P_2), identified with the new momenta.

The canonical transformation generated by S is implicitly given by [see Equations (6.8)]

$$p_r = \frac{\partial S}{\partial r} = \sqrt{2m\left[P_1 - V(r) - \frac{{P_2}^2}{2mr^2}\right]}, \qquad p_\theta = \frac{\partial S}{\partial \theta} = P_2 \tag{6.24}$$

and

$$Q_1 = \frac{\partial S}{\partial P_1} = -t + \int \frac{m\,\mathrm{d}r}{\sqrt{2m\left[P_1 - V(r) - \frac{{P_2}^2}{2mr^2}\right]}}, \tag{6.25}$$

$$Q_2 = \frac{\partial S}{\partial P_2} = \theta - \int \frac{P_2}{\sqrt{2m\left[P_1 - V(r) - \frac{{P_2}^2}{2mr^2}\right]}}\frac{\mathrm{d}r}{r^2}. \tag{6.26}$$

From Equations (6.24) we see that the new momenta are related to the original coordinates by

$$P_1 = \frac{1}{2m}\left({p_r}^2 + \frac{{p_\theta}^2}{r^2}\right) + V(r), \qquad P_2 = p_\theta,$$

that is, P_1 and P_2 are the Hamiltonian and the angular momentum about the origin, respectively. (We already knew that these two quantities are conserved because the Hamiltonian does not depend on t or θ.) Equation (6.26) yields the equation of the orbit.

Equations (6.25) and (6.26) are essentially Equations (2.14) and (2.15), respectively.

Example 6.4 (Kepler problem in parabolic coordinates). The Hamiltonian for the two-dimensional Kepler problem, expressed in parabolic coordinates, is given by [see Equation (4.43)]

$$H = \frac{1}{2m} \frac{{p_u}^2 + {p_v}^2}{u^2 + v^2} - \frac{2k}{u^2 + v^2}.$$

Hence, the corresponding HJ equation is

$$\frac{1}{2m(u^2 + v^2)} \left[\left(\frac{\partial S}{\partial u} \right)^2 + \left(\frac{\partial S}{\partial v} \right)^2 \right] - \frac{2k}{u^2 + v^2} + \frac{\partial S}{\partial t} = 0. \tag{6.27}$$

A separable solution of this equation has the form

$$S = A(u) + B(v) - P_1 t,$$

where A and B are functions of one variable, and P_1 is a separation constant. Substituting this last expression into the HJ equation (6.27), after some rearrangement we obtain

$$\left(\frac{\mathrm{d}A}{\mathrm{d}u} \right)^2 - 2mk - 2mP_1 u^2 = - \left(\frac{\mathrm{d}B}{\mathrm{d}v} \right)^2 + 2mk + 2mP_1 v^2.$$

Since the left-hand side does not depend on v and the right-hand side does not depend on u, both sides must be equal to some constant, P_2, say. Hence,

$$S = \int \sqrt{P_2 + 2mk + 2mP_1 u^2}\, \mathrm{d}u + \int \sqrt{-P_2 + 2mk + 2mP_1 v^2}\, \mathrm{d}v - P_1 t,$$

is a separable solution of the HJ equation which leads to the expressions

$$p_u = \sqrt{P_2 + 2mk + 2mP_1 u^2}, \qquad p_v = \sqrt{-P_2 + 2mk + 2mP_1 v^2}.$$

By combining these two equations one readily finds that $P_1 = H$ and that

$$v^2 {p_u}^2 - u^2 {p_v}^2 = (u^2 + v^2) P_2 + 2mk(v^2 - u^2),$$

hence, making use of (4.40), (4.41), and (4.51),

$$\begin{aligned} P_2 &= \frac{v^2 {p_u}^2 - u^2 {p_v}^2 + 2mk(u^2 - v^2)}{u^2 + v^2} \\ &= \frac{v^2 (up_x + vp_y)^2 - u^2 (-vp_x + up_y)^2}{u^2 + v^2} + 2mk \frac{x}{r} \\ &= -2x{p_y}^2 + 2yp_x p_y + 2mk \frac{x}{r}, \end{aligned}$$

i.e., the constant of motion P_2 is equal to $-2A_1$, where A_1 is the x-component of the Laplace–Runge–Lenz vector (4.52).

The equation of the orbit is obtained from

$$Q_2 = \frac{\partial S}{\partial P_2} = \int \frac{\mathrm{d}u}{2\sqrt{P_2 + 2mk + 2mP_1u^2}} - \int \frac{\mathrm{d}v}{2\sqrt{-P_2 + 2mk + 2mP_1v^2}}, \tag{6.28}$$

using the fact that Q_2 is a constant of motion, and the dependence of the coordinates on the time is determined by

$$Q_1 = \frac{\partial S}{\partial P_1} = \int \frac{mu^2\mathrm{d}u}{\sqrt{P_2 + 2mk + 2mP_1u^2}} + \int \frac{mv^2\mathrm{d}v}{\sqrt{-P_2 + 2mk + 2mP_1v^2}} - t, \tag{6.29}$$

using the fact that Q_1 is a constant of motion.

In order to obtain the solution of the equations of motion, it is convenient to introduce an auxiliary parameter, τ, in the following way. Since Q_2 is a constant of motion, Equation (6.28) is equivalent to the ODE

$$\frac{\mathrm{d}u}{\sqrt{P_2 + 2mk + 2mP_1u^2}} = \frac{\mathrm{d}v}{\sqrt{-P_2 + 2mk + 2mP_1v^2}}.$$

Introducing the parameter τ by means of

$$\frac{\mathrm{d}\tau}{m} = \frac{\mathrm{d}u}{\sqrt{P_2 + 2mk + 2mP_1u^2}} = \frac{\mathrm{d}v}{\sqrt{-P_2 + 2mk + 2mP_1v^2}}, \tag{6.30}$$

where the constant factor $1/m$ is included in order to get agreement with the definition given in Section 4.3, from (6.29) we have

$$\mathrm{d}t = \frac{mu^2\mathrm{d}u}{\sqrt{P_2 + 2mk + 2mP_1u^2}} + \frac{mv^2\mathrm{d}v}{\sqrt{-P_2 + 2mk + 2mP_1v^2}} = (u^2 + v^2)\,\mathrm{d}\tau, \tag{6.31}$$

which coincides with Equation (4.47).

From Equations (6.30) one can readily get u and v as functions of τ and substituting the expressions thus obtained into (6.31) one obtains the relation between t and τ (see Section 4.3.1).

When the Hamiltonian does not depend explicitly on the time, the HJ equation (6.6) admits partially separable solutions of the form

$$S(q_i, t) = W(q_i) - Et,$$

where E is a separation constant and $W(q_i)$ obeys the equation

$$H\left(q_i, \frac{\partial W}{\partial q_i}, t\right) = E.$$

The function W is known as *Hamilton's characteristic function* and the equation satisfied by W is usually called time-independent Hamilton–Jacobi equation.

Example 6.5 (Charged particle in the field of a point electric dipole). Another well-known example of a Hamiltonian that leads to a separable HJ equation is the one corresponding to a charged particle in the field of a point electric dipole, expressed in spherical coordinates (r, θ, ϕ),

$$H = \frac{1}{2m}\left({p_r}^2 + \frac{{p_\theta}^2}{r^2} + \frac{{p_\phi}^2}{r^2 \sin^2\theta}\right) + \frac{k\cos\theta}{r^2},$$

where k is a constant. The HJ equation is

$$\frac{1}{2m}\left[\left(\frac{\partial S}{\partial r}\right)^2 + \frac{1}{r^2}\left(\frac{\partial S}{\partial \theta}\right)^2 + \frac{1}{r^2\sin^2\theta}\left(\frac{\partial S}{\partial \phi}\right)^2\right] + \frac{k\cos\theta}{r^2} + \frac{\partial S}{\partial t} = 0 \qquad (6.32)$$

and, taking into account that ϕ and t do not appear explicitly in H [but the partial derivatives of S with respect to ϕ and t do appear in (6.32)], we look for a separable solution of this equation of the form

$$S = A(r) + B(\theta) + P_2\phi - P_1 t,$$

where P_1 and P_2 are separation constants. Substituting this expression into (6.32) and multiplying by $2mr^2$ we obtain

$$r^2\left(\frac{\mathrm{d}A}{\mathrm{d}r}\right)^2 + \left(\frac{\mathrm{d}B}{\mathrm{d}\theta}\right)^2 + \frac{{P_2}^2}{\sin^2\theta} + 2mk\cos\theta - 2mP_1r^2 = 0.$$

Hence,

$$\left(\frac{\mathrm{d}B}{\mathrm{d}\theta}\right)^2 + \frac{{P_2}^2}{\sin^2\theta} + 2mk\cos\theta = P_3 \qquad (6.33)$$

and

$$r^2\left(\frac{\mathrm{d}A}{\mathrm{d}r}\right)^2 - 2mP_1r^2 = -P_3, \qquad (6.34)$$

where P_3 is a third separation constant. Thus, the HJ equation admits separable solutions given by

$$S = \int\sqrt{2mP_1 - \frac{P_3}{r^2}}\,\mathrm{d}r + \int\sqrt{P_3 - \frac{{P_2}^2}{\sin^2\theta} - 2mk\cos\theta}\,\mathrm{d}\theta + P_2\phi - P_1 t,$$

and the canonical transformation generated by S is implicitly given by

$$p_r = \sqrt{2mP_1 - \frac{P_3}{r^2}}, \qquad p_\theta = \sqrt{P_3 - \frac{{P_2}^2}{\sin^2\theta} - 2mk\cos\theta}, \qquad p_\phi = P_2, \tag{6.35}$$

and

$$Q_1 = -t + \int \frac{m\,\mathrm{d}r}{\sqrt{2mP_1 - \frac{P_3}{r^2}}}, \tag{6.36}$$

$$Q_2 = \phi - \int \frac{P_2\,\mathrm{d}\theta}{\sin^2\theta\sqrt{P_3 - \frac{{P_2}^2}{\sin^2\theta} - 2mk\cos\theta}}, \tag{6.37}$$

$$Q_3 = -\int \frac{\mathrm{d}r}{2r^2\sqrt{2mP_1 - \frac{P_3}{r^2}}} + \int \frac{\mathrm{d}\theta}{2\sqrt{P_3 - \frac{{P_2}^2}{\sin^2\theta} - 2mk\cos\theta}}. \tag{6.38}$$

From Equations (6.35) we find that the new momenta, P_i, are the constants of motion

$$P_1 = H, \qquad P_2 = p_\phi, \qquad P_3 = {p_\theta}^2 + \frac{{p_\phi}^2}{\sin^2\theta} + 2mk\cos\theta.$$

The conservation of P_1 and P_2 are related to the obvious symmetries of the Hamiltonian (i.e., t and ϕ do not appear in the Hamiltonian), while the conservation of P_3 is related to a "hidden" symmetry of H (cf. Exercise 4.22).

With Equations (6.36)–(6.38) the solution of the equations of motion has been reduced to quadratures. It should be kept in mind that Equations (6.35)–(6.38) only give a coordinate transformation, and that the equations of motion are $\dot{Q}_i = 0 = \dot{P}_i$. Thus, for example, Equation (6.36) amounts to the equation of motion

$$\frac{\mathrm{d}r}{\mathrm{d}t} = \frac{\sqrt{2mP_1 - \frac{P_3}{r^2}}}{m},$$

which makes sense also in the case where the constants of motion P_1 (the total energy) and P_3 are equal to zero.

Exercise 6.6. As shown in Exercise 4.18, the Hamiltonian for a particle of mass m moving on the Euclidean plane subject to the gravitational attraction of two fixed centers separated by a distance $2c$ can be written as [see Equation (4.57)]

$$H = \frac{{p_u}^2 + {p_v}^2}{2mc^2(\cosh^2 u - \cos^2 v)} - \frac{(k_1 + k_2)\cosh u + (k_1 - k_2)\cos v}{c\,(\cosh^2 u - \cos^2 v)}. \tag{6.39}$$

Show that the HJ equation admits separable solutions in these coordinates, find the explicit expressions of two constants of motion and reduce to quadratures the equation of the orbit.

Exercise 6.7. Show that the HJ equation for a two-dimensional isotropic harmonic oscillator can be solved by separation of variables in elliptic (or confocal) coordinates (see Exercise 4.18) and identify the constants of motion P_1, P_2.

Exercise 6.8. Show that the HJ equation for the Hamiltonian

$$K = \frac{p^2}{2t} + \frac{q^6}{6t} + \frac{pq}{2t},$$

which is related to the Emden–Fowler equation (see Example 5.17), can be solved by separation of variables and reduce the solution of the Hamilton equations to quadratures. (See also Example 5.18.)

Exercise 6.9. Show that the HJ equation for a Hamiltonian of the form

$$H = \frac{1}{2}\frac{\mathscr{P}{p_x}^2 + \mathscr{Q}{p_y}^2}{X+Y} + \frac{\xi + \eta}{X+Y},$$

where $\mathscr{P}$, X, ξ are functions of x only, and $\mathscr{Q}$, Y, η are functions of y only, can be solved by separation of variables.

A Multiplicatively Separable Solution

In some exceptional cases, the HJ equation admits multiplicatively separable complete solutions. A simple example of this is provided by the HJ equation

$$\frac{1}{2m}\left(\frac{\partial S}{\partial q}\right)^2 + \frac{\partial S}{\partial t} = 0, \tag{6.40}$$

which corresponds to a free particle. Looking for a solution of the form $S(q,t) = A(q)B(t)$ we obtain

$$\frac{1}{2mA}\left(\frac{\mathrm{d}A}{\mathrm{d}q}\right)^2 = -\frac{1}{B^2}\frac{\mathrm{d}B}{\mathrm{d}t}$$

and, in the usual manner, we conclude that both sides of the last equation must be equal to some constant, a, say. Solving the resulting ODEs, we readily obtain

$$S(q,t) = \frac{ma}{2}(q+b)^2\frac{1}{at+c},$$

where b and c are integration constants. However, rewriting the solution thus obtained in the equivalent form

$$S(q,t) = \frac{m(q+b)^2}{2(t+c/a)}, \tag{6.41}$$

we see that it only depends on two arbitrary constants (b and c/a).

Setting $c/a = 0$ or $b = 0$ (but not both) in (6.41) we obtain a complete solution of the HJ equation (6.40). This shows that the frequently found assertion that any additional constant in a complete solution of the HJ equation must be an additive constant is wrong. (See also Example 6.10, below.)

6.1.1 Relation Between Complete Solutions of the HJ Equation

The HJ equation for a given Hamiltonian, as any other first-order PDE, possesses an infinite number of complete solutions. As we shall show now, any two complete solutions of the HJ equation are related by means of a time-independent canonical transformation [cf. Equation (5.99)]. Indeed, if $S(q_i, P_i, t)$ is a complete solution of the HJ equation corresponding to a Hamiltonian $H(q_i, p_i, t)$, we have

$$p_i \mathrm{d}q_i - H\mathrm{d}t + Q_i \mathrm{d}P_i = \mathrm{d}S$$

[see Equation (5.54)]. Similarly, if $\tilde{S}(q_i, \tilde{P}_i, t)$ is any other complete solution of the same equation (in the same coordinates q_i), then

$$p_i \mathrm{d}q_i - H\mathrm{d}t + \tilde{Q}_i \mathrm{d}\tilde{P}_i = \mathrm{d}\tilde{S},$$

hence,

$$Q_i \mathrm{d}P_i - \tilde{Q}_i \mathrm{d}\tilde{P}_i = \mathrm{d}F, \tag{6.42}$$

where

$$F \equiv S - \tilde{S}. \tag{6.43}$$

Equation (6.42) explicitly shows that (Q_i, P_i) and $(\tilde{Q}_i, \tilde{P}_i)$ are related by means of a time-independent canonical transformation [cf. Equation (5.46)]. *If* the set $(P_i, \tilde{P}_i)$ is functionally independent, then the function F defined in (6.43) is a generating function of this canonical transformation.

Since $p_i = \partial S/\partial q_i$ and, also, $p_i = \partial \tilde{S}/\partial q_i$, we have

$$\frac{\partial (S - \tilde{S})}{\partial q_i} = 0. \tag{6.44}$$

Making use of these n conditions one can eliminate the q_i from the right-hand side of (6.43) (the dependence on t automatically disappears as a consequence of (6.44); no additional conditions come from $\partial(S - \tilde{S})/\partial t = 0$ since, by hypothesis, S and $\tilde{S}$ satisfy the same HJ equation).

Similarly, given a complete solution, $S(q_i, P_i, t)$, of the HJ equation for some Hamiltonian, and a function $F(P_i, \tilde{P}_i)$ that defines a canonical transformation,

$$\tilde{S}(q_i, \tilde{P}_i, t) = S(q_i, P_i, t) - F(P_i, \tilde{P}_i), \tag{6.45}$$

is also a complete solution of the same HJ equation. The dependence on the parameters P_i is eliminated from the expression on the right-hand side of (6.45) with the aid of the n conditions

$$\frac{\partial (S - F)}{\partial P_i} = 0. \tag{6.46}$$

(Cf. Calkin [2, pp. 148–150].)

Example 6.10. As pointed out in Example 6.1, the function

$$S(q, P, t) = -\left(\frac{P^2}{2m} + \frac{m\omega^2}{2} q^2\right) \frac{\tan \omega t}{\omega} + Pq \sec \omega t$$

is a complete solution of the HJ equation in the case of the standard Hamiltonian of a one-dimensional harmonic oscillator. With the aid of the function

$$F(P, \tilde{P}) = \frac{P^2}{2m\omega} \tan \tilde{P},$$

we can obtain another complete solution of the same HJ equation. Indeed, the condition (6.46) reads

$$0 = \frac{\partial (S - F)}{\partial P} = -\frac{P}{m\omega} \tan \omega t + q \sec \omega t - \frac{P}{m\omega} \tan \tilde{P},$$

that is,

$$P = \frac{m\omega q \sec \omega t}{\tan \omega t + \tan \tilde{P}}$$

and, therefore,

$$\tilde{S} = S - F = \frac{m\omega q^2}{2}\cot(\omega t + \tilde{P})$$

is also a complete solution of the HJ equation for the standard Hamiltonian of a one-dimensional harmonic oscillator. (It may be noticed that this solution is the *product* of separated functions of q and t.)

Exercise 6.11. Find a generating function of the canonical transformation that leads from the complete, separable, solution of the HJ equation

$$S(x, y, P_1, P_2, t) = P_1 x + P_2 y - \frac{{P_1}^2 + {P_2}^2}{2m} t$$

to the non-separable complete solution

$$\tilde{S}(x, y, \tilde{P}_1, \tilde{P}_2, t) = \frac{m}{2t}\big[(x - \tilde{P}_1)^2 + (y - \tilde{P}_2)^2\big].$$

What is the Hamiltonian?

Since any complete solution of the HJ equation leads to the solution of the Hamilton equations, it is not necessary to find a second complete solution of the HJ equation. In the context of classical mechanics, we make use of a complete solution of the HJ equation only as a means to find the solution of the Hamilton equations. However, in geometrical optics the function S is interesting in itself and it is highly relevant to find different solutions of the appropriate version of the HJ equation, which correspond to different trains of wavefronts (see Section 6.5, below).

Other Special Generating Functions

In the same manner as we can look for a canonical transformation that produces a new Hamiltonian equal to zero, we can also look for canonical transformations that take, locally, a given Hamiltonian H into any other specified Hamiltonian K (the only restriction is that the number of degrees of freedom in both Hamiltonians be the same). Making use again of Equations (6.2) we find that the required generating function must satisfy the PDE

$$K\left(\frac{\partial F_2}{\partial P_i}, P_i, t\right) = H\left(q_i, \frac{\partial F_2}{\partial q_i}, t\right) + \frac{\partial F_2}{\partial t} \tag{6.47}$$

and the condition (6.1). For instance, in the case of the Hamiltonians

$$H(q, p, t) = \frac{p^2}{2m} + mgq, \qquad K(Q, P, t) = \frac{P^2}{2m},$$

where g is a constant, one can verify that the function

$$F_2(q, P, t) = (P - mgt)q + \frac{1}{2}gt^2 P - \frac{m}{6}g^2 t^3$$

satisfies (6.47) and (6.1). In fact, substituting this expression into (6.2) one finds that F_2 generates the canonical transformation

$$q = Q - \tfrac{1}{2}gt^2, \qquad p = P - mgt, \tag{6.48}$$

and that H and K are related by (6.5). Hence, the coordinate transformation (6.48) maps the phase space trajectories of a free particle into those of a particle in free fall. Cf. Exercise 5.6. It may be noticed that, when $g = 0$, the transformation (6.48) reduces to the identity. (The coordinate transformation (6.48) gives the relation between two reference frames, one of which has an acceleration equal to g with respect to the other.)

These transformations can be constructed by finding separately solutions of the HJ equations for H and K, and combining them or composing the coordinate transformations generated by them (see also Section 6.3).

6.1.2 Alternative Expressions of the HJ Equation

It should be clear that, in order to find new canonical coordinates with a Hamiltonian equal to zero, instead of a (type F_2) generating function $S(q_i, P_i, t)$, that depends on the n original coordinates q_i, we can also look for generating functions that depend on other combinations of the original variables, replacing one or several coordinates q_i by its conjugate momentum p_i, which gives a total of 2^n different possibilities. Some of these alternatives are sometimes mentioned [e.g., Corben and Stehle ([4], Sect. 61), Greenwood [9, Sect. 6–1]], without actually using them, claiming that they are of little interest. However, in some cases, the dependence of H on the coordinates may be simpler than that on the momenta.

Example 6.12 (Particle in a uniform gravitational field). Let us consider again the simple case corresponding to the Hamiltonian

$$H = \frac{{p_x}^2 + {p_y}^2}{2m} + mgy,$$

where m and g are constants. We look for a type F_4 generating function (which we shall denote also by S) such that $K = 0$. According to Equations (5.57), S must be a complete solution of

$$\frac{{p_x}^2 + {p_y}^2}{2m} - mg\frac{\partial S}{\partial p_y} + \frac{\partial S}{\partial t} = 0, \tag{6.49}$$

and the canonical transformation is implicitly given by

$$x = -\frac{\partial S}{\partial p_x}, \qquad y = -\frac{\partial S}{\partial p_y}, \qquad Q_i = \frac{\partial S}{\partial P_i}. \tag{6.50}$$

Since H does not depend on t, we look for a solution of (6.49) of the form

$$S = W(p_x, p_y) - P_1 t,$$

where P_1 is a separation constant. Then, the "characteristic function," W, has to satisfy

$$\frac{\partial W}{\partial p_y} = -\frac{P_1}{mg} + \frac{{p_x}^2 + {p_y}^2}{2m^2 g}.$$

The *general* solution of this PDE is readily found:

$$W = -\frac{P_1 p_y}{mg} + \frac{{p_x}^2 p_y}{2m^2 g} + \frac{{p_y}^3}{6m^2 g} + f(p_x),$$

where $f(p_x)$ is an *arbitrary function* of p_x only. Choosing $f(p_x) = P_2 p_x$, where P_2 is a constant, we obtain

$$S(p_x, p_y, P_1, P_2, t) = -\frac{P_1 p_y}{mg} + \frac{{p_x}^2 p_y}{2m^2 g} + \frac{{p_y}^3}{6m^2 g} + P_2 p_x - P_1 t.$$

It may be noticed that this solution of the HJ equation (6.49) is not the sum of separate functions of p_x, p_y and t. (This is an example of an R-separable solution, to be discussed in Section 6.1.3.) According to (6.50), S generates the canonical transformation given by

$$x = -\frac{p_x p_y}{m^2 g} - P_2, \qquad y = \frac{P_1}{mg} - \frac{{p_x}^2 + {p_y}^2}{2m^2 g}, \qquad Q_1 = -\frac{p_y}{mg} - t, \qquad Q_2 = p_x,$$

i.e.,

$$Q_1 = -\frac{p_y}{mg} - t, \quad Q_2 = p_x, \quad P_1 = \frac{{p_x}^2 + {p_y}^2}{2m} + mgy, \quad P_2 = -\frac{p_x p_y}{m^2 g} - x,$$

which are the constants of motion obtained in Example 6.2. (See also Example 6.24.) The solution of the Hamilton equations in the original variables is obtained writing x, y, p_x, p_y in terms of Q_i, P_i.

6.1.3 R-Separable Solutions of the HJ Equation

As pointed out above, the method commonly employed to solve the HJ equation is the method of separation of variables, but, in many cases, this method may not work. For example, in the case of the Hamiltonian

$$H = \frac{p^2}{2m} - ktq, \tag{6.51}$$

which corresponds to a particle of mass m subjected to a variable force kt, where k is some constant, the HJ equation

$$\frac{1}{2m}\left(\frac{\partial S}{\partial q}\right)^2 - ktq + \frac{\partial S}{\partial t} = 0 \tag{6.52}$$

does *not* admit separable solutions owing to the presence of the term ktq. However, noting that the last two terms on the left-hand side of this equation can be written as

$$-ktq + \frac{\partial S}{\partial t} = \frac{\partial}{\partial t}\left(-\frac{kt^2 q}{2} + S\right),$$

we introduce $\widetilde{S} \equiv S - kt^2 q/2$, and we find that (6.52) amounts to

$$\frac{1}{2m}\left(\frac{\partial \widetilde{S}}{\partial q} + \frac{kt^2}{2}\right)^2 + \frac{\partial \widetilde{S}}{\partial t} = 0.$$

By contrast with (6.52), this last equation admits separable solutions, and since q does not appear explicitly in the equation, its separable solutions are of the form $\widetilde{S} = Pq + F(t)$, where P is a separation constant, with

$$\frac{1}{2m}\left(P + \frac{kt^2}{2}\right)^2 + \frac{\mathrm{d}F}{\mathrm{d}t} = 0.$$

Thus,

$$S = \frac{kt^2 q}{2} + Pq - \frac{1}{2m}\left(P^2 t + \frac{1}{3}Pkt^3 + \frac{1}{20}k^2 t^5\right) \tag{6.53}$$

is a (complete) solution of (6.52). This solution is the sum of a function of q and t (the term $kt^2 q/2$), that does not contain the separation constant P, and one-variable functions. Such solutions are called *R-separable solutions*.

Thus, we have the generating function of a canonical transformation implicitly given by [see Equations (6.8)],

$$Q = \frac{\partial S}{\partial P} = q - \frac{Pt}{m} - \frac{kt^3}{6m}, \qquad p = \frac{\partial S}{\partial q} = \frac{kt^2}{2} + P.$$

The original variables are given by

$$p = P + \frac{kt^2}{2}, \qquad q = Q + \frac{Pt}{m} + \frac{kt^3}{6m}.$$

Since Q and P are constants of motion, these expressions constitute the solution of the Hamilton equations. From these expressions we see that Q and P correspond to the values of q and p at $t = 0$, respectively. (Cf. Example 5.40.)

Example 6.13 (Charged particle in a uniform magnetic field). The Hamiltonian

$$H = \frac{1}{2m}\left[\left(p_x + \frac{eB_0 y}{2c}\right)^2 + \left(p_y - \frac{eB_0 x}{2c}\right)^2 + {p_z}^2\right], \tag{6.54}$$

corresponds to a charged particle of mass m and electric charge e in a uniform magnetic field $\mathbf{B} = B_0\mathbf{k}$, if the vector potential is chosen according to the rule $\mathbf{A} = \frac{1}{2}\mathbf{B} \times \mathbf{r}$, which is applicable for a uniform magnetic field $\mathbf{B}$ [see (4.14)]. The resulting HJ equation

$$\frac{1}{2m}\left[\left(\frac{\partial S}{\partial x} + \frac{eB_0 y}{2c}\right)^2 + \left(\frac{\partial S}{\partial y} - \frac{eB_0 x}{2c}\right)^2 + \left(\frac{\partial S}{\partial z}\right)^2\right] + \frac{\partial S}{\partial t} = 0 \tag{6.55}$$

does not admit separable solutions but, letting $\widetilde{S} \equiv S + eB_0 xy/2c$, we obtain

$$\frac{1}{2m}\left[\left(\frac{\partial \widetilde{S}}{\partial x}\right)^2 + \left(\frac{\partial \widetilde{S}}{\partial y} - \frac{eB_0 x}{c}\right)^2 + \left(\frac{\partial \widetilde{S}}{\partial z}\right)^2\right] + \frac{\partial \widetilde{S}}{\partial t} = 0. \tag{6.56}$$

Since y, z, and t do not appear explicitly in this equation, it admits separable solutions of the form $\widetilde{S} = F(x) + P_2 y + P_3 z - P_1 t$, where P_1, P_2, and P_3 are constants, and F satisfies the separated equation

$$\frac{1}{2m}\left[\left(\frac{\mathrm{d}F}{\mathrm{d}x}\right)^2 + \left(P_2 - \frac{eB_0 x}{c}\right)^2 + {P_3}^2\right] = P_1.$$

Thus,

$$S = -\frac{eB_0 xy}{2c} + P_2 y + P_3 z - P_1 t + \int \sqrt{2mP_1 - {P_3}^2 - \left(P_2 - eB_0 x/c\right)^2}\,\mathrm{d}x$$

and making use of Equations (6.8) we obtain

$$p_x = -\frac{eB_0 y}{2c} + \sqrt{2mP_1 - P_3{}^2 - \left(P_2 - eB_0 x/c\right)^2}, \quad p_y = -\frac{eB_0 x}{2c} + P_2, \quad p_z = P_3,$$

and

$$\begin{aligned} Q_1 &= -t + \int \frac{m\,\mathrm{d}x}{\sqrt{2mP_1 - P_3{}^2 - \left(P_2 - eB_0 x/c\right)^2}}, \\ Q_2 &= y - \int \frac{(P_2 - eB_0 x/c)\,\mathrm{d}x}{\sqrt{2mP_1 - P_3{}^2 - \left(P_2 - eB_0 x/c\right)^2}}, \\ Q_3 &= z - \int \frac{P_3\,\mathrm{d}x}{\sqrt{2mP_1 - P_3{}^2 - \left(P_2 - eB_0 x/c\right)^2}}. \end{aligned} \tag{6.57}$$

By combining these equations one obtains the constants of motion

$$P_1 = H, \qquad P_2 = p_y + \frac{eB_0 x}{2c}, \qquad P_3 = p_z, \qquad Q_2 = -\frac{c}{eB_0}\left(p_x - \frac{eB_0 y}{2c}\right).$$

With the aid of the change of variable

$$P_2 - \frac{eB_0 x}{c} = \sqrt{2mP_1 - P_3{}^2}\,\cos\theta,$$

from the first two equations in (6.57) we obtain

$$x = \frac{cP_2}{eB_0} - \frac{c}{eB_0}\sqrt{2mP_1 - P_3{}^2}\,\cos\omega_c(t + Q_1),$$

where $\omega_c \equiv eB_0/mc$ (the cyclotron frequency), and

$$y = Q_2 + \frac{c}{eB_0}\sqrt{2mP_1 - P_3{}^2}\,\sin\omega_c(t + Q_1),$$

respectively. These last expressions show that the projection of the orbit on the xy-plane is a circle whose center and radius are given in terms of the constants of motion Q_i and P_i (cf. Example 1.19).

The vector potential $\mathbf{A}' = B_0 x\,\mathbf{j}$ also yields the uniform magnetic field $\mathbf{B} = B_0\,\mathbf{k}$, and leads to a separable HJ equation. (In fact, the difference $\mathbf{A}' - \mathbf{A} = B_0 x\,\mathbf{j} - \frac{1}{2}B_0(-y\,\mathbf{i} + x\,\mathbf{j}) = \frac{1}{2}B_0(y\,\mathbf{i} + x\,\mathbf{j})$, which is the gradient of $\frac{1}{2}B_0 xy$.) (See also the discussion at the end of this section.)

Exercise 6.14. Show that the HJ equation for a charged particle in a uniform magnetic field, with $\mathbf{A} = \frac{1}{2}\mathbf{B} \times \mathbf{r}$, can be solved by separation of variables in circular cylindrical coordinates and identify the new momenta.

Example 6.15. We consider the HJ equation

$$\frac{1}{2m}\left[\left(\frac{\partial S}{\partial x}\right)^2 + \left(\frac{\partial S}{\partial y}\right)^2\right] + mgy + \frac{\partial S}{\partial t} = 0, \tag{6.58}$$

which *does* admit separable solutions as a consequence of the fact that x and t do not appear explicitly in the equation (see Example 6.2). Noting that

$$mgy + \frac{\partial S}{\partial t} = \frac{\partial}{\partial t}(mgyt + S),$$

we introduce $\widetilde{S} \equiv S + mgyt$, and we have

$$\frac{1}{2m}\left[\left(\frac{\partial \widetilde{S}}{\partial x}\right)^2 + \left(\frac{\partial \widetilde{S}}{\partial y} - mgt\right)^2\right] + \frac{\partial \widetilde{S}}{\partial t} = 0. \tag{6.59}$$

This equation admits separable solutions as a consequence of the fact that x and y do not appear explicitly in it. Indeed, looking for solutions of the form

$$\widetilde{S} = P_1 x + P_2 y + F(t),$$

where P_1 and P_2 are constants, we obtain

$$\frac{1}{2m}\left[{P_1}^2 + (P_2 - mgt)^2\right] + \frac{\mathrm{d}F}{\mathrm{d}t} = 0.$$

Hence,

$$F(t) = -\frac{1}{2m}\left[({P_1}^2 + {P_2}^2)t - P_2 mgt^2 + \frac{1}{3}m^2 g^2 t^3\right]$$

and, therefore,

$$S = -mgyt + P_1 x + P_2 y - \frac{1}{2m}\left[\left({P_1}^2 + {P_2}^2\right)t - P_2 mgt^2 + \frac{1}{3}m^2 g^2 t^3\right] \tag{6.60}$$

is a complete R-separable solution of (6.58). Thus, the HJ equation (6.58) admits both separable and R-separable solutions in the Cartesian coordinates (x, y).

Substitution of (6.60) into Equations (6.8) yields the canonical transformation

$$p_x = \frac{\partial S}{\partial x} = P_1, \qquad p_y = \frac{\partial S}{\partial y} = -mgt + P_2,$$

and

$$Q_1 = \frac{\partial S}{\partial P_1} = x - \frac{P_1 t}{m}, \qquad Q_2 = \frac{\partial S}{\partial P_2} = y - \frac{P_2 t}{m} + \frac{1}{2}gt^2.$$

That is, we have four constants of motion

$$Q_1 = x - \frac{tp_x}{m}, \qquad Q_2 = y - \frac{tp_y}{m} - \frac{1}{2}gt^2, \qquad P_1 = p_x, \qquad P_2 = p_y + mgt,$$

and the solution of the Hamilton equations

$$x = Q_1 + \frac{tP_1}{m}, \qquad y = Q_2 + \frac{tP_2}{m} - \frac{1}{2}gt^2, \qquad p_x = P_1, \qquad p_y = P_2 - mgt.$$

The constants of motion Q_1, Q_2, P_1, and P_2 correspond to the values at $t = 0$ of x, y, p_x, and p_y, respectively.

It may be noticed that finding R-separable solutions of the HJ equation for a Hamiltonian H is equivalent to finding *separable* solutions of the HJ equation for another Hamiltonian, H', obtained from H by means of a canonical transformation of the form

$$q'_i = q_i, \qquad p'_i = p_i + \frac{\partial R}{\partial q_i}, \tag{6.61}$$

where R is a function of q_i and t only [see (5.67)]. For instance, Equation (6.59) is the HJ equation corresponding to the Hamiltonian (5.69), which is obtained from (5.68) by means of the canonical transformation (6.61) with $R = mgyt$.

Exercise 6.16. Show that if the Hamiltonian has the form

$$H = \frac{p^2}{2m} - \phi(t)q,$$

where $\phi(t)$ is a given function of t only, then the corresponding HJ equation admits R-separable complete solutions. This result is applicable to the problem of a rocket in a uniform gravitational field, for which the Hamiltonian can be taken as

$$H = \frac{p^2}{2} + \left(u\frac{\mathrm{d}\ln m}{\mathrm{d}t} + g\right)q,$$

where $m(t)$ is the mass of the rocket at time t and u is the speed of the exhaust gases with respect to the rocket (see Example 2.13).

6.2 The Liouville Theorem on Solutions of the HJ Equation

Apart from the method of separation of variables, there exist some other methods for solving first-order PDEs (see, e.g., Sneddon [14]). In one of these lesser-known methods, when applied to the HJ equation, one has to express the canonical momenta in terms of the coordinates and n constants of motion; a complete solution, S, of the HJ equation can then be obtained from $\mathrm{d}S = p_i \mathrm{d}q_i - H\mathrm{d}t$. However, it turns out that $p_i \mathrm{d}q_i - H\mathrm{d}t$ is an exact differential if and only if the constants of motion employed in this process are in involution, that is, their Poisson brackets are all equal to zero, and this result is known as Liouville's Theorem. In the case where there is only one degree of freedom, the Liouville Theorem can be applied making use of a single arbitrary constant of motion, since the Poisson bracket of a function with itself is trivially equal to zero.

The application of the Liouville theorem requires the knowledge of n constants of motion in involution, but is not linked to some specific coordinate system; the complete solutions of the HJ equation obtained in this manner need not be separable or R-separable.

Proposition 6.17 (Liouville's Theorem). *If $P_i = P_i(q_j, p_j, t)$, $i = 1, 2, \ldots, n$, are n functionally independent constants of motion in involution such that the momenta p_i can be written in terms of q_j, P_j, and t, then, locally, there exists a function $S(q_i, t)$, depending parametrically on the P_i, such that*

$$p_i(q_j, P_j, t)\,\mathrm{d}q_i - H\big(q_i, p_i(q_j, P_j, t), t\big)\,\mathrm{d}t = \mathrm{d}S \tag{6.62}$$

and S is a complete *solution of the HJ equation.*

Note that if the momenta p_i can be written in terms of q_j, P_j, and t, then the constants of motion $P_i = P_i(q_j, p_j, t)$, $i = 1, 2, \ldots, n$, have to be functionally independent.

Proof. According to the hypotheses, from the expressions $P_i = P_i(q_j, p_j, t)$, we can find the p_i as functions of q_j, P_j, and t, hence

$$\det\left(\frac{\partial P_i}{\partial p_j}\right) \neq 0. \tag{6.63}$$

Substituting

$$\mathrm{d}p_k = \frac{\partial p_k}{\partial q_j}\mathrm{d}q_j + \frac{\partial p_k}{\partial P_j}\mathrm{d}P_j + \frac{\partial p_k}{\partial t}\mathrm{d}t$$

into

$$\mathrm{d}P_i = \frac{\partial P_i}{\partial q_j}\mathrm{d}q_j + \frac{\partial P_i}{\partial p_j}\mathrm{d}p_j + \frac{\partial P_i}{\partial t}\mathrm{d}t$$

one obtains

$$\mathrm{d}P_i = \frac{\partial P_i}{\partial q_j}\mathrm{d}q_j + \frac{\partial P_i}{\partial p_k}\left(\frac{\partial p_k}{\partial q_j}\mathrm{d}q_j + \frac{\partial p_k}{\partial P_j}\mathrm{d}P_j + \frac{\partial p_k}{\partial t}\mathrm{d}t\right) + \frac{\partial P_i}{\partial t}\mathrm{d}t$$

which implies the *identities*

$$\frac{\partial P_i}{\partial q_j} = -\frac{\partial P_i}{\partial p_k}\frac{\partial p_k}{\partial q_j}, \qquad \frac{\partial P_i}{\partial t} = -\frac{\partial P_i}{\partial p_k}\frac{\partial p_k}{\partial t} \tag{6.64}$$

and

$$\frac{\partial P_i}{\partial p_k}\frac{\partial p_k}{\partial P_j} = \delta_{ij}. \tag{6.65}$$

(Note that in the partial derivatives of the P_i, P_i is a function of q_j, p_j, and t, that is

$$\frac{\partial P_i}{\partial q_j} = \left(\frac{\partial P_i}{\partial q_j}\right)_{q,p,t}, \qquad \frac{\partial P_i}{\partial p_j} = \left(\frac{\partial P_i}{\partial p_j}\right)_{q,p,t},$$

while in the partial derivatives of the p_i, p_i is a function of q_j, P_j, and t,

$$\frac{\partial p_i}{\partial P_j} = \left(\frac{\partial p_i}{\partial P_j}\right)_{q,P,t}, \qquad \frac{\partial p_i}{\partial q_j} = \left(\frac{\partial p_i}{\partial q_j}\right)_{q,P,t}.$$

Equations (6.64) and (6.65) can also be derived with the aid of the chain rule.)

Making use of the first equation in (6.64) we find that

$$\begin{aligned}\{P_i, P_j\} &= \frac{\partial P_i}{\partial q_k}\frac{\partial P_j}{\partial p_k} - \frac{\partial P_j}{\partial q_k}\frac{\partial P_i}{\partial p_k} \\ &= -\frac{\partial P_i}{\partial p_m}\frac{\partial p_m}{\partial q_k}\frac{\partial P_j}{\partial p_k} + \frac{\partial P_j}{\partial p_m}\frac{\partial p_m}{\partial q_k}\frac{\partial P_i}{\partial p_k} \\ &= -\frac{\partial P_i}{\partial p_m}\frac{\partial p_m}{\partial q_k}\frac{\partial P_j}{\partial p_k} + \frac{\partial P_j}{\partial p_k}\frac{\partial p_k}{\partial q_m}\frac{\partial P_i}{\partial p_m} \\ &= \frac{\partial P_i}{\partial p_m}\frac{\partial P_j}{\partial p_k}\left(\frac{\partial p_k}{\partial q_m} - \frac{\partial p_m}{\partial q_k}\right).\end{aligned}$$

Thus, taking into account (6.63), it follows that $\{P_i, P_j\} = 0$ if and only if

$$\frac{\partial p_k}{\partial q_m} = \frac{\partial p_m}{\partial q_k}. \tag{6.66}$$

On the other hand, making use of the fact that each P_i is a constant of motion, and of (6.64) and (6.66),

$$
\begin{aligned}
0 &= \frac{\partial P_i}{\partial t} + \frac{\partial P_i}{\partial q_j}\frac{\partial H}{\partial p_j} - \frac{\partial H}{\partial q_j}\frac{\partial P_i}{\partial p_j} \\
&= -\frac{\partial P_i}{\partial p_k}\frac{\partial p_k}{\partial t} - \frac{\partial P_i}{\partial p_k}\frac{\partial p_k}{\partial q_j}\frac{\partial H}{\partial p_j} - \frac{\partial H}{\partial q_j}\frac{\partial P_i}{\partial p_j} \\
&= -\frac{\partial P_i}{\partial p_k}\left(\frac{\partial p_k}{\partial t} + \frac{\partial p_k}{\partial q_j}\frac{\partial H}{\partial p_j} + \frac{\partial H}{\partial q_k}\right) \\
&= -\frac{\partial P_i}{\partial p_k}\left(\frac{\partial p_k}{\partial t} + \frac{\partial p_j}{\partial q_k}\frac{\partial H}{\partial p_j} + \frac{\partial H}{\partial q_k}\right).
\end{aligned}
$$

As a consequence of (6.63) and the chain rule, this amounts to

$$
\frac{\partial p_k}{\partial t} = -\frac{\partial H}{\partial q_k} - \frac{\partial H}{\partial p_j}\frac{\partial p_j}{\partial q_k} = -\frac{\partial}{\partial q_k} H\big(q_j, p_j(q_k, P_k, t), t\big),
$$

and these conditions together with (6.66) imply that the left-hand side of (6.62) is locally exact.

Finally, from (6.62) it follows that S is a solution of the HJ equation, which is complete by virtue of (6.63), in fact,

$$
\det\left(\frac{\partial^2 S}{\partial P_i \partial q_j}\right) = \det\left(\frac{\partial p_j}{\partial P_i}\right) = \left[\det\left(\frac{\partial P_i}{\partial p_j}\right)\right]^{-1} \neq 0.
$$

□

In some textbooks this result is called *Liouville's integrability theorem.*

Example 6.18. One can readily verify that the function $P = m\omega q \sin\omega t + p\cos\omega t$ is a constant of motion for the one-dimensional harmonic oscillator, that is, if

$$
H = \frac{p^2}{2m} + \frac{m\omega^2}{2}q^2.
$$

Since, $p = P\sec\omega t - m\omega q\tan\omega t$, expressing $p\mathrm{d}q - H\mathrm{d}t$ in terms of q, P, t, and treating P as a parameter, we obtain

$$
\begin{aligned}
&p\mathrm{d}q - H\mathrm{d}t \\
&\quad = \big(P\sec\omega t - m\omega q\tan\omega t\big)\mathrm{d}q - \left[\frac{(P\sec\omega t - m\omega q\tan\omega t)^2}{2m} + \frac{m\omega^2}{2}q^2\right]\mathrm{d}t \\
&\quad = \mathrm{d}\left(Pq\sec\omega t - \frac{1}{2}m\omega q^2\tan\omega t\right) - \omega Pq\sec\omega t\tan\omega t\,\mathrm{d}t + \frac{1}{2}m\omega^2 q^2\sec^2\omega t\,\mathrm{d}t
\end{aligned}
$$

$$- \left[\frac{(P \sec \omega t - m\omega q \tan \omega t)^2}{2m} + \frac{m\omega^2}{2} q^2 \right] \mathrm{d}t$$
$$= \mathrm{d} \left(Pq \sec \omega t - \frac{1}{2} m\omega q^2 \tan \omega t - \frac{P^2}{2m\omega} \tan \omega t \right).$$

According to Proposition 6.17, the expression inside the parenthesis must be a complete solution of the HJ equation for the Hamiltonian H and, following the standard procedure, it can be used to find the solution of the equations of motion.

Example 6.19. The HJ equation for the Kepler problem in two dimensions, which corresponds to the Hamiltonian

$$H = \frac{p_x{}^2 + p_y{}^2}{2m} - \frac{k}{\sqrt{x^2 + y^2}},$$

expressed in Cartesian coordinates x, y, where m is the mass of the particle and k is a positive constant, is separable in polar and parabolic coordinates (see Examples 6.3 and 6.4, respectively) but *is not* separable in Cartesian coordinates.

Since H is time-independent and invariant under rotations about the origin,

$$P_1 \equiv H, \qquad P_2 \equiv x p_y - y p_x$$

(the total energy and the angular momentum about the origin) are constants of motion, which are in involution (as can be seen from the fact that the angular momentum is a constant of motion). Inverting these expressions one finds

$$p_x = \frac{-P_2 y \pm x\sqrt{2mP_1 r^2 + 2mkr - P_2{}^2}}{r^2}, \qquad p_y = \frac{P_2 x \pm y\sqrt{2mP_1 r^2 + 2mkr - P_2{}^2}}{r^2},$$

where the signs in front of the square roots have to be chosen both plus or both minus and $r^2 \equiv x^2 + y^2$, which give the p_i in terms of q_j and P_j. Thus, the left-hand side of Equation (6.62) becomes

$$P_2 \frac{(-y\mathrm{d}x + x\mathrm{d}y)}{r^2} \pm \frac{\sqrt{2mP_1 r^2 + 2mkr - P_2{}^2}}{r^2} (x\mathrm{d}x + y\mathrm{d}y) - P_1 \mathrm{d}t$$

or, equivalently,

$$P_2 \, \mathrm{d} \left(\arctan \frac{y}{x} \right) \pm \sqrt{2mP_1 + \frac{2mk}{r} - \frac{P_2{}^2}{r^2}} \, \mathrm{d}r - P_1 \mathrm{d}t.$$

This last expression is indeed, *locally*, the differential of a function, which must be a complete solution of the HJ equation. It may be noticed that this function turns out to be the sum of separate functions of the polar coordinates θ, r, though we started with the Hamiltonian in Cartesian coordinates.

Example 6.20. Another simple example is given by the Hamiltonian

$$H = \frac{{p_x}^2 + {p_y}^2}{2m} + mgy, \tag{6.67}$$

for which all constants of motion are readily obtained (here m and g are constants). In fact, the HJ equation is separable in the coordinates (x, y) of the configuration space, and the separation constants are the values of H and p_x, which are constants of motion as a consequence of the fact that t and x do not appear in the Hamiltonian (see Example 6.2).

Another constant of motion (which is related to a "hidden" symmetry of the Hamiltonian) is

$$\frac{p_x p_y}{m} + mgx \tag{6.68}$$

(see, e.g., Example 5.35), therefore, H and $\frac{1}{m} p_x p_y + mgx$ are in involution and can be taken as P_1 and P_2, respectively. A straightforward computation leads to the expressions

$$\begin{aligned} p_x + p_y &= \pm\sqrt{2m(P_1 + P_2) - 2m^2 g\,(x + y)}, \\ p_x - p_y &= \pm\sqrt{2m(P_1 - P_2) + 2m^2 g\,(x - y)}, \end{aligned} \tag{6.69}$$

where the signs in front of the square roots have to be chosen both plus or both minus. Hence, by writing Equation (6.62) in the form

$$\tfrac{1}{2}(p_x + p_y)\,\mathrm{d}(x + y) + \tfrac{1}{2}(p_x - p_y)\,\mathrm{d}(x - y) - P_1 \mathrm{d}t = \mathrm{d}S, \tag{6.70}$$

and taking into account that $p_x \pm p_y$ is a function of $x \pm y$ only, we see that the function S is the sum of three one-variable functions that depend on $x + y$, $x - y$, and t (with P_1 and P_2 being treated as parameters). In other words, the HJ equation corresponding to the Hamiltonian (6.67) admits separable solutions in the coordinates (u, v) defined by

$$u \equiv x + y, \qquad v \equiv x - y,$$

and the separation constants are the values of H and $\frac{1}{m} p_x p_y + mgx$.

Exercise 6.21. Find a complete solution of the HJ equation corresponding to the Hamiltonian (6.67) making use of the constants of motion in involution $P_1 = p_x$ and $P_2 = p_y + mgt$.

Exercise 6.22. Making use of the fact that the coordinates P_1 and P_2 defined in Example 5.45 are constants of motion in involution, find a complete solution of the HJ equation for a charged particle in a uniform magnetic field and use it to find a second pair of constants of motion.

Exercise 6.23 (Toda lattice). In Example 5.81 the system with Hamiltonian

$$H = \tfrac{1}{2}({p_1}^2 + {p_2}^2) + e^{q_1 - q_2},$$

was considered and it was shown that the functions

$$P_1 \equiv p_1 + p_2, \qquad P_2 \equiv p_1 p_2 - e^{q_1 - q_2}$$

are constants of motion. Prove that these two functions are in involution, find a complete solution of the HJ equation and use it to solve the equations of motion.

Example 6.24. It should be clear that in order to find a complete solution of the HJ equation starting from n functionally independent constants of motion in involution, $P_1, P_2, \ldots, P_n$, it is not indispensable to solve the equations $P_i = P_i(q_j, p_j, t)$ for $p_1, p_2, \ldots, p_n$; instead of p_i we can employ the corresponding conjugate coordinate q_i (that is, in place of, say, p_5, we can make use of q_5, and so on). For instance, in the case of the Hamiltonian (6.67), the functions

$$P_1 = p_y + mgt, \qquad P_2 = \frac{p_x p_y}{m} + mgx \tag{6.71}$$

are two functionally independent constants of motion in involution. Even though we can solve (6.71) for p_x and p_y, we shall make use of x instead of p_x. From (6.71) we obtain

$$x = \frac{P_2}{mg} - \frac{(P_1 - mgt)p_x}{m^2 g}, \qquad p_y = P_1 - mgt$$

and instead of (6.62), we have

$$-x\, dp_x + p_y dy - H dt = dS,$$

with x and p_y expressed in terms of y, p_x, P_1, P_2, and t. Hence,

$$\begin{aligned} dS &= -\left(\frac{P_2}{mg} - \frac{P_1 p_x}{m^2 g} + \frac{t p_x}{m}\right) dp_x + \big(P_1 - mgt\big) dy - \left[\frac{{p_x}^2}{2m} + \frac{(P_1 - mgt)^2}{2m} + mgy\right] dt \\ &= d\left[-\frac{P_2 p_x}{mg} + \frac{P_1 {p_x}^2}{2m^2 g} - \frac{t {p_x}^2}{2m} + P_1 y - mgty + \frac{(P_1 - mgt)^3}{6m^2 g}\right] \end{aligned}$$

and we can take

$$S(y, p_x, P_1, P_2, t) = -\frac{P_2 p_x}{mg} + \frac{P_1 {p_x}^2}{2m^2 g} - \frac{t {p_x}^2}{2m} + P_1 y - mgty + \frac{(P_1 - mgt)^3}{6m^2 g}.$$

Note that the function S thus obtained is R-separable and is a complete solution of the HJ equation

$$\frac{1}{2m}\left[{p_x}^2 + \left(\frac{\partial S}{\partial y}\right)^2\right] + mgy + \frac{\partial S}{\partial t} = 0,$$

which does not contain the partial derivative of S with respect to p_x.

Exercise 6.25. Show that, in the case of the constants of motion considered in Example 6.20, the coordinates x and y can be expressed in terms of p_x, p_y, P_1, P_2, and t, and use those expressions to find a (type F_4) complete solution of the appropriate HJ equation.

Exercise 6.26. Show that

$$Q = q - \frac{pt}{m} + \frac{kt^3}{3m}$$

is a constant of motion if the Hamiltonian is given by

$$H = \frac{p^2}{2m} - ktq.$$

Use the expression for q in terms of p and Q to find a (type F_3) complete solution, $S(p, Q, t)$, of the HJ equation

$$H\left(-\frac{\partial S}{\partial p}, p, t\right) + \frac{\partial S}{\partial t} = 0$$

and use it to find a second constant of motion.

Exercise 6.27. Show that

$$P = p - \int^t \phi(u)\,\mathrm{d}u$$

is a constant of motion if the Hamiltonian is given by

$$H = \frac{p^2}{2m} - \phi(t)q,$$

where $\phi(t)$ is a given function of t only, and use it to find a complete solution of the HJ equation (cf. Exercise 6.16).

6.3 Mapping of Solutions of the HJ Equation Under Canonical Transformations

The form of the Hamiltonian of a given system can be modified by a canonical transformation and, therefore, the expression of the HJ equation and its solutions can also be modified by these transformations. As we shall see now, there is a simple way of relating a solution of the HJ equation corresponding to a Hamiltonian H with a solution of the HJ equation corresponding to the Hamiltonian K, obtained by means of a canonical transformation. We begin by pointing out a relation between solutions of the HJ equation and certain subsets of the extended phase space [10, 17].

Proposition 6.28. *Any solution, $S(q_i, t)$, of the HJ equation defines a surface (a submanifold), N, of the extended phase space, given by the n equations*

$$p_i = \frac{\partial S}{\partial q_i} \tag{6.72}$$

($i = 1, 2, \ldots, n$), on which the linear differential form $p_i \mathrm{d}q_i - H\mathrm{d}t$ is exact; in fact,

$$p_i \mathrm{d}q_i - H\mathrm{d}t = \mathrm{d}S, \qquad \text{on } N. \tag{6.73}$$

Conversely, an $(n+1)$-dimensional submanifold, N, of the extended phase space, on which the differential form $p_i \mathrm{d}q_i - H\mathrm{d}t$ is exact, defines (up to an additive constant) a solution of the HJ equation. (The solution in question is the function S determined by Equation (6.73).)

The function S appearing in Equations (6.72) and (6.73) may contain some parameters (as in the case of a complete solution), but this is not essential at this point. For example, if the Hamiltonian is taken as

$$H = \frac{p^2}{2m} + \frac{m\omega^2 q^2}{2}, \tag{6.74}$$

then, on the two-dimensional submanifold of the extended phase space defined by

$$p = -m\omega q \tan \omega t,$$

we have

$$\begin{aligned} p\mathrm{d}q - H\mathrm{d}t &= -m\omega q \tan \omega t \, \mathrm{d}q - \left(\frac{m^2\omega^2 q^2 \tan^2 \omega t}{2m} + \frac{m\omega^2 q^2}{2} \right) \mathrm{d}t \\ &= -m\omega \tan \omega t \, \mathrm{d}\left(\frac{q^2}{2}\right) - \frac{m\omega^2}{2} q^2 \sec^2 \omega t \, \mathrm{d}t \\ &= \mathrm{d}\big(-\tfrac{1}{2} m\omega q^2 \tan \omega t\big). \end{aligned}$$

Hence, the function

$$S = -\tfrac{1}{2} m\omega q^2 \tan \omega t$$

is a solution of the HJ equation corresponding to the Hamiltonian (6.74), which does not contain arbitrary parameters.

However, for each value of the parameter P, the equation

$$p = -m\omega q \tan \omega (t + P) \tag{6.75}$$

defines a two-dimensional submanifold of the extended phase space, on which the differential form $p\mathrm{d}q - H\mathrm{d}t$ is exact. In fact, one finds that [on the surface defined by (6.75)]

$$p\mathrm{d}q - H\mathrm{d}t = \mathrm{d}\big[-\tfrac{1}{2} m\omega q^2 \tan \omega (t + P)\big]$$

and this time we have a complete solution of the HJ equation

$$S(q, P, t) = -\tfrac{1}{2} m\omega q^2 \tan \omega (t + P), \tag{6.76}$$

which is not (additively) separable. The completeness of the solution (6.76) is related to the fact that the family of submanifolds defined by (6.75) fills the extended phase space.

Returning to the problem of finding the effect of a canonical transformation on the solutions of the HJ equation, we recall that if the coordinate transformation

$$Q_i = Q_i(q_j, p_j, t), \qquad P_i = P_i(q_j, p_j, t), \tag{6.77}$$

is canonical, then

$$p_i \mathrm{d}q_i - H\mathrm{d}t - (P_i \mathrm{d}Q_i - K\mathrm{d}t) = \mathrm{d}F_1,$$

for some real-valued function F_1 defined in a [$(2n + 1)$-dimensional] region of the extended phase space [see Equation (5.46)]. By contrast with the differential form $p_i \mathrm{d}q_i - H\mathrm{d}t$ (and, similarly, $P_i \mathrm{d}Q_i - K\mathrm{d}t$), which is exact only on some submanifolds of the extended phase space, the combination $p_i \mathrm{d}q_i - H\mathrm{d}t - (P_i \mathrm{d}Q_i - K\mathrm{d}t)$ is exact everywhere (or in some open neighborhood of each point of the extended phase space). Hence, if $p_i \mathrm{d}q_i - H\mathrm{d}t$ is an exact differential on some submanifold of the extended phase space, then $P_i \mathrm{d}Q_i - K\mathrm{d}t$ is also exact on that submanifold.

Thus, if $S(q_i, t)$ is a solution of the HJ equation, then, on the submanifold N defined by (6.72),

$$
\begin{aligned}
P_i \mathrm{d}Q_i - K\mathrm{d}t &= p_i \mathrm{d}q_i - H\mathrm{d}t - \mathrm{d}F_1 \\
&= \mathrm{d}(S - F_1),
\end{aligned}
$$

which means that

$$
S' = S - F_1 \tag{6.78}
$$

is a solution of the HJ equation corresponding to K, provided that it is expressed in terms of Q_i and t, making use of Equations (6.72) and (6.77). (Cf. Equations (6.45)–(6.46).) By construction, the solution of the Hamilton equations obtained from S' is the image under the canonical transformation (6.77) of the solution of the Hamilton equations obtained from S.

Example 6.29. A simple and illustrative example is given by the standard Hamiltonian of a one-dimensional harmonic oscillator

$$
H = \frac{p^2}{2m} + \frac{m\omega^2 q^2}{2}.
$$

The coordinate transformation

$$
q = \frac{1}{\omega}\sqrt{\frac{2Q}{m}}\,\cos\omega P, \qquad p = \sqrt{2mQ}\,\sin\omega P, \tag{6.79}
$$

is canonical and we can take $K = H$. In fact,

$$
\begin{aligned}
p\mathrm{d}q - P\mathrm{d}Q &= \sqrt{2mQ}\,\sin\omega P\left(-\sqrt{\frac{2Q}{m}}\,\sin\omega P\,\mathrm{d}P + \frac{\cos\omega P}{\omega\sqrt{2mQ}}\,\mathrm{d}Q\right) - P\mathrm{d}Q \\
&= -2Q\sin^2\omega P\,\mathrm{d}P + \frac{1}{\omega}\sin\omega P\,\cos\omega P\,\mathrm{d}Q - P\mathrm{d}Q \\
&= \mathrm{d}\left(-PQ + \frac{Q}{\omega}\sin\omega P\,\cos\omega P\right),
\end{aligned}
$$

hence, up to an additive trivial constant, $F_1 = -PQ + (Q/\omega)\sin\omega P\,\cos\omega P$.

Since $K = H = Q$ [see (6.79)], the HJ equation for K is given by

$$
Q + \frac{\partial S'}{\partial t} = 0, \tag{6.80}
$$

whose *general* solution is $S' = -Qt + f(Q)$, where $f(Q)$ is an *arbitrary* function of Q only. In order to simplify the computations below, we choose $f(Q) = t_0 Q$, where t_0 is a constant; thus

$$S' = -Q(t - t_0),$$

which constitutes a complete solution of the HJ equation (6.80). Then, from Equation (6.78), taking into account that $P = \partial S'/\partial Q = t_0 - t$, with the aid of the first equation in (6.79) we obtain

$$\begin{aligned} S &= S' + F_1 \\ &= -Q(t - t_0) - PQ + \frac{Q}{\omega} \sin \omega P \, \cos \omega P \\ &= \frac{Q}{\omega} \sin \omega P \, \cos \omega P \\ &= -\tfrac{1}{2} m\omega q^2 \tan \omega (t - t_0), \end{aligned}$$

which is, therefore, the complete solution of the HJ equation corresponding to H [cf. Equation (6.76)].

Example 6.30 (Damped harmonic oscillator). The Hamiltonian

$$H = \mathrm{e}^{-2\gamma t} \frac{p^2}{2m} + \mathrm{e}^{2\gamma t} \frac{m\omega^2}{2} q^2,$$

where γ is a positive constant, corresponds to a damped harmonic oscillator (see Example 2.6). Making use of (5.15) one finds that the coordinate transformation

$$Q = \mathrm{e}^{\gamma t} q, \qquad P = \mathrm{e}^{-\gamma t} p$$

is canonical (cf. Example 5.33) and that the new Hamiltonian can be taken as

$$K = \frac{P^2}{2m} + \frac{m\omega^2}{2} Q^2 + \gamma PQ,$$

with $F_1 = 0$. By contrast with H, the Hamiltonian K does not depend explicitly on t and therefore the HJ equation for K admits separable solutions of the form

$$S' = -\tilde{P}t + f(Q),$$

where $\tilde{P}$ is a separation constant and f satisfies

$$\frac{\mathrm{d}f}{\mathrm{d}Q} = -m\gamma Q \pm \sqrt{2m\tilde{P} - m^2(\omega^2 - \gamma^2)Q^2}.$$

Thus,

$$S' = -\tilde{P}t - \tfrac{1}{2}m\gamma Q^2 + \int^{Q} \sqrt{2m\tilde{P} - m^2(\omega^2 - \gamma^2)u^2}\,\mathrm{d}u$$

is a complete solution of the HJ equation for K and, according to Equation (6.78), the function

$$S(q, \tilde{P}, t) = -\tilde{P}t - \tfrac{1}{2}m\gamma\, \mathrm{e}^{2\gamma t}q^2 + \int^{\mathrm{e}^{\gamma t}q} \sqrt{2m\tilde{P} - m^2(\omega^2 - \gamma^2)u^2}\,\mathrm{d}u$$

is the corresponding solution of the HJ equation for H. It may be noticed that this function is neither separable nor R-separable. (Note that in this example we are considering two canonical transformations; the first one relates the original coordinates, q, p, with a second set of canonical coordinates, Q, P. A second canonical transformation is generated by S', leading to a third set of canonical coordinates, $\tilde{Q}, \tilde{P}$, which are constants of motion.)

Hence,

$$p = -m\gamma\, \mathrm{e}^{2\gamma t}q + \mathrm{e}^{\gamma t}\sqrt{2m\tilde{P} - m^2(\omega^2 - \gamma^2)\, \mathrm{e}^{2\gamma t}q^2},$$

and from this equation we can obtain the constant of motion $\tilde{P}$ in terms of (q, p, t),

$$\tilde{P} = \mathrm{e}^{-2\gamma t}\frac{p^2}{2m} + \mathrm{e}^{2\gamma t}\frac{m\omega^2}{2}q^2 + \gamma pq$$

(which coincides with the Hamiltonian K). The second constant of motion,

$$\tilde{Q} = \frac{\partial S}{\partial \tilde{P}} = -t + \int^{\mathrm{e}^{\gamma t}q} \frac{m\,\mathrm{d}u}{\sqrt{2m\tilde{P} - m^2(\omega^2 - \gamma^2)u^2}},$$

gives q as a function of the time (and the constants of motion $\tilde{P}$ and $\tilde{Q}$). For instance, in the case where $\gamma < \omega$ (the so-called underdamped motion), one readily finds

$$q = \mathrm{e}^{-\gamma t}\sqrt{\frac{m(\omega^2 - \gamma^2)}{2\tilde{P}}}\cos\sqrt{\omega^2 - \gamma^2}\big(t + \tilde{Q}\big)$$

(cf. Example 5.13).

Exercise 6.31. The canonical transformation

$$Q = q + \tfrac{1}{2}gt^2, \qquad P = p + mgt$$

relates the Hamiltonians

$$H = \frac{p^2}{2m} + mgq \qquad \text{and} \qquad K = \frac{P^2}{2m}$$

[cf. Equations (6.48)]. Using the fact that

$$S' = \frac{m}{2t}(Q - a)^2,$$

where a is a constant, is a (complete) solution of the HJ equation for K, find the corresponding solution for the HJ equation for H.

Exercise 6.32. The canonical transformation

$$\begin{aligned} Q &= q(\cos\omega t + \omega t \sin\omega t) + \frac{p}{m\omega}(\omega t \cos\omega t - \sin\omega t), \\ P &= m\omega q \sin\omega t + p\cos\omega t \end{aligned}$$

relates the Hamiltonians

$$H = \frac{p^2}{2m} + \frac{m\omega^2}{2}q^2 \qquad \text{and} \qquad K = \frac{P^2}{2m}$$

(see Exercise 5.6). Making use of the fact that

$$S' = \frac{m}{2t}(Q - a)^2,$$

where a is a constant, is a solution of the HJ equation for K (corresponding to a free particle), find the corresponding solution of the HJ equation for H. (*Hint*: the results of Example 5.7 may be useful.)

Exercise 6.33. Consider the canonical transformation

$$Q = q - vt, \qquad P = p - mv,$$

where v is a constant and m is the mass of a particle. *Assuming* that the new Hamiltonian is given by

$$K = H - vp + \tfrac{1}{2}mv^2$$

(cf. Example 5.63), show that if $S(q, t)$ is a solution of the HJ equation for H, then

$$S'(Q, t) = S(Q + vt, t) - mvQ - \tfrac{1}{2}mv^2 t$$

is the corresponding solution of the HJ equation for K. (This relationship has an analog in quantum mechanics in the transformation of a wavefunction under a Galilean transformation, see, e.g., Torres del Castillo and Nájera Salazar [19].)

Exercise 6.34. Show that in the case of a (passive) translation, $Q = q - s$, $P = p$, assuming that $K = H$, we have $S'(Q, t) = S(Q + s, t)$. Similarly, show that for a translation in the momentum, $Q = q$, $P = p - s$, choosing $K = H$, it follows that $S'(Q, t) = S(Q, t) - sQ$. (Note that the assumption $K = H$ is consistent with the fact that the transformations considered here do not involve the time. Note also that H need not be invariant under these translations.)

Exercise 6.35 (Transformation of the principal function under gauge transformations). As shown in Section 5.2, a gauge transformation

$$\mathbf{A} \mapsto \mathbf{A} + \nabla\xi, \qquad \varphi \mapsto \varphi - \frac{1}{c}\frac{\partial\xi}{\partial t},$$

where ξ is some function of the coordinates and the time, corresponds to a canonical transformation given by

$$Q_i = q_i, \qquad P_i = p_i + \frac{\partial(e\xi/c)}{\partial q_i}.$$

Show that if $K = H - \partial(e\xi/c)/\partial t$, then $S' = S + e\xi/c$. (This result also has a well-known analog in quantum mechanics.)

Covariance of the HJ Equation

The HJ equation is a partial differential equation somewhat similar to other scalar PDEs of mathematical physics, such as the Laplace equation for the electrostatic potential, or the wave equation for the fractional change of the density of the air, in the case of the sound waves. However, apart from the fact that the HJ equation is of first order and not necessarily linear, an important difference between the HJ equation and the other equations just mentioned is that, under a change of coordinates of the configuration space, the solutions of the HJ equation may require an additional term [see Equation (6.78)].

However, according to the discussion presented in Example 5.36, a time-independent coordinate transformation in the configuration space,

$$Q_i = Q_i(q_j),$$

together with the implicit relation

$$p_i = P_j \frac{\partial Q_j}{\partial q_i}, \tag{6.81}$$

constitute a canonical transformation. If we choose $K = H$, i.e.,

$$K(Q_i, P_i, t) = H\big(q_i(Q_j), p_i(Q_j, P_j), t\big) = H\left(q_i(Q_j), P_j \frac{\partial Q_j}{\partial q_i}, t\right),$$

then the function F_1 can be taken equal to zero and, according to (6.78),

$$S'(Q_i, t) = S(q_i(Q_j), t) \tag{6.82}$$

is the solution of the HJ equation for the Hamiltonian K corresponding to a solution, $S(q_i, t)$, of the HJ equation for H.

Thus, if we have a (not necessarily complete) solution of the HJ equation, in terms of some coordinates, $S(q_i, t)$, by simply substituting the coordinates q_i by any other set of coordinates of the configuration space, $q_i = q_i(Q_j)$, we obtain a solution of the HJ equation for the same Hamiltonian, provided that the momenta are related by (6.81). This means that the HJ equation is *covariant* under this restricted class of coordinate transformations. (See also Section 6.4.)

6.3.1 The HJ Equation as an Evolution Equation

The HJ equation can be seen as an *evolution equation*, which determines the function $S(q_i, t)$ that reduces to a given function, $f(q_i)$, for $t = 0$ (or any other initial value, t_0, of t). According to the results of the previous section, if we have the solution of the Hamilton equations, we can find the solution of the HJ equation satisfying any initial condition, $S(q_i, t_0) = f(q_i)$, making use of the fact that the time evolution from $t = t_0$ to an arbitrary value of t is a canonical transformation, with the Hamiltonian corresponding to the initial coordinates equal to zero [see Equation (5.97)]. The initial condition $f(q_i)$ can be chosen arbitrarily because *any function*, $f(q_i)$, that does not depend on t, is trivially a solution of the HJ equation if the Hamiltonian is equal to zero.

In the following examples, we obtain the function F_1 appearing in Equation (6.78), corresponding to the time evolution, making use of the explicit solution of the Hamilton equations, while the function S' is the initial condition. If the function $f(q_i)$ contains arbitrary parameters, then the solution $S(q_i, t)$ of the HJ equation will also contain those parameters.

A different approach to the problem of finding the solution of a PDE passing through a given curve or surface can be found, e.g., in Sneddon [14, Sect. 12]; one advantage of the method presented there is that the initial condition need not be the value of S at some particular value of t.

Example 6.36. In the case of the Hamiltonian

$$H = \frac{{p_x}^2 + {p_y}^2}{2m} + mgy, \tag{6.83}$$

where m and g are constants, the solution of the corresponding Hamilton equations can be readily obtained and is given by

$$\begin{aligned} x &= Q_1 + \frac{P_1 t}{m}, \quad & y &= Q_2 + \frac{P_2 t}{m} - \frac{g t^2}{2}, \\ p_x &= P_1, & p_y &= P_2 - mgt, \end{aligned} \tag{6.84}$$

where Q_1, Q_2, P_1, and P_2 are the values of x, y, p_x, and p_y, respectively, at $t = 0$. As the initial condition we choose

$$S(x, y, 0) = \alpha_1 x + \alpha_2 y, \tag{6.85}$$

where α_1, α_2 are two arbitrary constants, that is, as the initial function in terms of the initial coordinates, we take

$$S'(Q_1, Q_2, 0) = \alpha_1 Q_1 + \alpha_2 Q_2. \tag{6.86}$$

Note that, as pointed out above, S' is a solution of the HJ equation for $K = 0$.

Making use of the expressions (6.84) we obtain

$$p_i \mathrm{d}q_i - H \mathrm{d}t - P_i \mathrm{d}Q_i = \mathrm{d}\left(\frac{{p_x}^2 + {p_y}^2}{2m} t - mgty + gt^2 p_y + \frac{mg^2 t^3}{3} \right),$$

while from (6.86) it follows that $\mathrm{d}S' = \alpha_1 \mathrm{d}Q_1 + \alpha_2 \mathrm{d}Q_2$, that is, $P_1 = \alpha_1$, $P_2 = \alpha_2$. Then, from Equations (6.78) and (6.84), expressing all coordinates in terms of x, y, α_1, α_2, and t, we have

$$\begin{aligned} S &= S' + F_1 \\ &= \alpha_1 Q_1 + \alpha_2 Q_2 + \frac{{p_x}^2 + {p_y}^2}{2m} t - mgty + gt^2 p_y + \frac{mg^2 t^3}{3} \\ &= \alpha_1 \left(x - \frac{\alpha_1 t}{m} \right) + \alpha_2 \left(y - \frac{\alpha_2 t}{m} + \frac{gt^2}{2} \right) + \frac{{\alpha_1}^2 + (\alpha_2 - mgt)^2}{2m} t - mgty \\ &\quad + gt^2(\alpha_2 - mgt) + \frac{mg^2 t^3}{3}, \end{aligned}$$

i.e.,

$$S(x, y, t) = \alpha_1 x + \alpha_2 y - mgty - \frac{{\alpha_1}^2 t}{2m} + \frac{(\alpha_2 - mgt)^3 - {\alpha_2}^3}{6m^2 g} \tag{6.87}$$

[cf. Equation (6.60)]. The expression (6.87) is a (complete, R-separable) solution of the HJ equation that reduces to the specified function (6.85) for $t = 0$.

Exercise 6.37. Find the solution of the Hamilton equations for the time-dependent Hamiltonian

$$H = \frac{p^2}{2m} - ktq, \tag{6.88}$$

where m and k are constants, and use it to find the solution of the corresponding the HJ equation such that $S(q, 0) = aq$, where a is an arbitrary constant. Compare the result with (6.53).

We might consider expressions for $S(q_i, 0)$ more complicated than (6.86) and the one given in Exercise 6.37, but these simple expressions are enough to obtain complete solutions of the HJ equation and to illustrate the procedure.

Example 6.38. We shall consider again the HJ equation

$$\frac{1}{2m}\left[\left(\frac{\partial S}{\partial x}\right)^2 + \left(\frac{\partial S}{\partial y}\right)^2\right] + mgy + \frac{\partial S}{\partial t} = 0 \tag{6.89}$$

corresponding to the Hamiltonian (6.83) but, instead of (6.85), we take as the initial condition

$$S(x, y, 0) = k\big[(x - \alpha_1)^2 + (y - \alpha_2)^2\big], \tag{6.90}$$

where k is a constant with the appropriate dimensions, and α_1, α_2 are two arbitrary parameters. Thus,

$$S'(Q_1, Q_2, 0) = k\big[(Q_1 - \alpha_1)^2 + (Q_2 - \alpha_2)^2\big] \tag{6.91}$$

and, therefore,

$$\mathrm{d}S' = 2k\big[(Q_1 - \alpha_1)\,\mathrm{d}Q_1 + (Q_2 - \alpha_2)\,\mathrm{d}Q_2\big]$$

i.e., $P_1 = 2k(Q_1 - \alpha_1)$ and $P_2 = 2k(Q_2 - \alpha_2)$. Proceeding as above, from (6.78), (6.84), and (6.91) we have

$$\begin{aligned} S &= S' + F_1 \\ &= k\big[(Q_1 - \alpha_1)^2 + (Q_2 - \alpha_2)^2\big] + \frac{{p_x}^2 + {p_y}^2}{2m}t - mgty + gt^2 p_y + \frac{mg^2t^3}{3} \\ &= \frac{k(m + 2kt)}{m}\big[(Q_1 - \alpha_1)^2 + (Q_2 - \alpha_2)^2\big] - mgty - \frac{mg^2t^3}{6}. \end{aligned}$$

Hence, after the elimination of the Q_i we find that the solution of the HJ equation that reduces to (6.90) at $t = 0$, is given by

$$S(x, y, t) = \frac{km\big[(x-\alpha_1)^2 + (y-\alpha_2+gt^2/2)^2\big]}{m+2kt} - mgty - \frac{mg^2t^3}{6}. \tag{6.92}$$

Exercise 6.39. Show that the solution of the HJ equation corresponding to the standard Hamiltonian of a one-dimensional harmonic oscillator

$$H = \frac{p^2}{2m} + \frac{m\omega^2}{2}q^2$$

that satisfies the initial condition $S(q, 0) = \alpha q$, where α is an arbitrary constant, is given by

$$S(q, t) = -\left(\frac{\alpha^2}{2m} + \frac{m\omega^2}{2}q^2\right)\frac{\tan\omega t}{\omega} + \alpha q \sec\omega t.$$

Of course, if we already have the solution of the Hamilton equations, it does not seem necessary to find a complete solution of the HJ equation. However, the construction presented in this section explicitly shows that given the solution of the Hamilton equations, one can find any complete solution of the HJ equation, and that the general solution of the HJ equation involves an arbitrary function of n variables. On the other hand, in geometrical optics, each solution (complete or not) of the eikonal equation corresponds to a wavefront train and the procedure developed in this section allows us to find the evolution of a given wavefront (see Section 6.5, below).

6.4 Transformation of the HJ Equation Under Arbitrary Point Transformations

With the aid of Proposition 6.28 we can see that under an *arbitrary* point transformation,

$$q'_i = q'_i(q_j, t), \qquad t' = t'(q_j, t), \tag{6.93}$$

the HJ equation for a Hamiltonian $H(q_i, p_i, t)$ is transformed into the HJ equation for a Hamiltonian H', possibly different from H. (These transformations differ from those considered in Section 6.3, because in the latter the time is not transformed.) Indeed, $S(q_i, t)$ is a solution of the HJ equation for H (containing arbitrary parameters or not) if and only if

$$\mathrm{d}S = p_i\mathrm{d}q_i - H\mathrm{d}t$$

on the submanifold of the extended phase space defined by $p_i = \partial S/\partial q_i$. Inverting the formulas (6.93) we obtain q_i and t as functions of the q'_i and t', then if F_1 is any function of q'_i and t' only (or, equivalently, of q_i and t only), defining

$$S' \equiv S - F_1 \tag{6.94}$$

we have

$$\begin{aligned} \mathrm{d}S' &= p_i \left(\frac{\partial q_i}{\partial q'_j} \mathrm{d}q'_j + \frac{\partial q_i}{\partial t'} \mathrm{d}t' \right) - H \left(\frac{\partial t}{\partial q'_j} \mathrm{d}q'_j + \frac{\partial t}{\partial t'} \mathrm{d}t' \right) - \frac{\partial F_1}{\partial q'_j} \mathrm{d}q'_j - \frac{\partial F_1}{\partial t'} \mathrm{d}t' \\ &= \left(\frac{\partial q_i}{\partial q'_j} p_i - \frac{\partial t}{\partial q'_j} H - \frac{\partial F_1}{\partial q'_j} \right) \mathrm{d}q'_j - \left(\frac{\partial t}{\partial t'} H - \frac{\partial q_i}{\partial t'} p_i + \frac{\partial F_1}{\partial t'} \right) \mathrm{d}t'. \end{aligned}$$

Making use again of Proposition 6.28, the last equation shows that S' is a solution of the HJ equation for the Hamiltonian

$$H' = \frac{\partial t}{\partial t'} H - \frac{\partial q_i}{\partial t'} p_i + \frac{\partial F_1}{\partial t'} \tag{6.95}$$

with

$$p'_j = \frac{\partial q_i}{\partial q'_j} p_i - \frac{\partial t}{\partial q'_j} H - \frac{\partial F_1}{\partial q'_j} \tag{6.96}$$

[cf. Equations (5.65) and (5.66)]. In conclusion, given a Hamiltonian H and an arbitrary function, $F_1(q_i, t)$, any coordinate transformation (6.93), in which the time may be also transformed, leads to a new Hamiltonian (6.95) in such a way that any solution, S, of the HJ equation for H produces a solution, S', of the HJ equation for H' given by (6.94). If S contains arbitrary parameters, so will do S'.

Example 6.40. We shall consider the point transformation

$$q = q' \sec \omega t', \qquad t = \frac{\tan \omega t'}{\omega} \tag{6.97}$$

where ω is a constant. Then, from Equation (6.96), we obtain

$$p' = p \sec \omega t' - \frac{\partial F_1}{\partial q'},$$

which substituted into (6.95) gives

$$\begin{aligned} H' &= H \sec^2 \omega t' - p q' \omega \sec \omega t' \tan \omega t' + \frac{\partial F_1}{\partial t'} \\ &= H \sec^2 \omega t' - \omega q' \tan \omega t' \left(p' + \frac{\partial F_1}{\partial q'} \right) + \frac{\partial F_1}{\partial t'}. \end{aligned}$$

Hence, taking $H = p^2/2m$, and expressing the result in terms of the primed variables

$$\begin{aligned} H' &= \frac{p^2}{2m}\sec^2\omega t' - \omega q' \tan\omega t' \left(p' + \frac{\partial F_1}{\partial q'}\right) + \frac{\partial F_1}{\partial t'} \\ &= \frac{1}{2m}\left(p' + \frac{\partial F_1}{\partial q'}\right)^2 - \omega q' \tan\omega t' \left(p' + \frac{\partial F_1}{\partial q'}\right) + \frac{\partial F_1}{\partial t'} \\ &= \frac{p'^2}{2m} + \left(\frac{\partial F_1}{\partial q'} - m\omega q' \tan\omega t'\right)\frac{p'}{m} + \frac{1}{2m}\left(\frac{\partial F_1}{\partial q'}\right)^2 - \omega q' \tan\omega t' \frac{\partial F_1}{\partial q'} + \frac{\partial F_1}{\partial t'}. \end{aligned}$$

In order to eliminate the term linear in p' we take

$$\frac{\partial F_1}{\partial q'} = m\omega q' \tan\omega t',$$

which implies that $F_1 = \frac{1}{2}m\omega q'^2 \tan\omega t' + f(t')$, where $f(t')$ is some function of t' only. In this manner, H' reduces to

$$H' = \frac{p'^2}{2m} + \frac{m\omega^2}{2}q'^2 + \frac{\mathrm{d}f}{\mathrm{d}t'}.$$

Thus, choosing $f = 0$ it follows that if S is a solution for the HJ equation corresponding to the standard Hamiltonian of a free particle, then

$$S' = S - F_1 = S - \tfrac{1}{2}m\omega q'^2 \tan\omega t' \tag{6.98}$$

is a solution of the HJ equation corresponding to the standard Hamiltonian of a harmonic oscillator.

For instance,

$$S = \frac{m(q-a)^2}{2t},$$

where a is a constant, is a solution of the HJ equation for a free particle, which substituted into (6.98) yields

$$S' = \frac{m\omega[(q'^2 + a^2)\cos\omega t' - 2aq']}{2\sin\omega t'}.$$

Exercise 6.41. Apply the transformation

$$q' = q\mathrm{e}^s + \tfrac{1}{2}gt^2(\mathrm{e}^s - \mathrm{e}^{4s}), \qquad t' = t\mathrm{e}^{2s},$$

where g and s are constants, to the Hamiltonian

$$H = \frac{p^2}{2m} + mgq.$$

Show that by suitably choosing the function F_1 appearing in Equations (6.94)–(6.96) one obtains

$$H' = \frac{p'^2}{2m} + mgq'$$

(that is, the Hamiltonian is form-invariant). This means that, except for the substitution of the coordinates (q, t) by (q', t'), the HJ equation for H' is the same as that for H and, therefore, by means of (6.94), from a given solution of the HJ equation for H we obtain a possibly different solution of the same equation.

6.5 Geometrical Optics

The Hamiltonian formulation of classical mechanics arose from the study of geometrical optics (see, e.g., Whittaker [22, Chap. XI]) and, as we shall see in this section, it is very instructive to apply the formalism developed in this chapter to geometrical optics.

Fermat's Principle. The Ray Equation

In geometrical optics it is assumed that the light propagates along curves, which are called light rays. The basic equations of geometrical optics can be obtained from the *Fermat principle of least time*, which can be formulated in the following way. The speed of light in an isotropic medium, with refractive index n, is c/n, where c is the speed of light in vacuum; therefore, given two points of the three-dimensional Euclidean space, A and B, the time required for the light to go from A to B along a curve C is given by the integral

$$\frac{1}{c} \int_C n \, \mathrm{d}s, \tag{6.99}$$

where ds is the arclength element (see below). Of course, there are an infinite number of curves joining A and B; the Fermat principle states that the path actually followed by the light is the one that minimizes the integral (6.99). Since c is a constant, finding the curve corresponding to the least time is equivalent to finding the curve with the minimum *optical length* (or *optical path length*), defined as

$$\int_C n \, \mathrm{d}s. \tag{6.100}$$

If we consider curves that can be parameterized by one of the Cartesian coordinates, z say, the integral (6.100) can be expressed as

$$\int_{z_0}^{z_1} n(x, y, z) \left[1 + \left(\frac{\mathrm{d}x}{\mathrm{d}z} \right)^2 + \left(\frac{\mathrm{d}y}{\mathrm{d}z} \right)^2 \right]^{1/2} \mathrm{d}z, \tag{6.101}$$

where z_0 and z_1 are the values of the coordinate z at the points A and B, respectively (see Figure 6.1). Hence, the light rays are determined by the Euler–Lagrange equations for the Lagrangian

$$L(x, y, x', y', z) = n(x, y, z)\sqrt{1 + x'^2 + y'^2}, \tag{6.102}$$

where $x' \equiv \mathrm{d}x/\mathrm{d}z$ and $y' \equiv \mathrm{d}y/\mathrm{d}z$. However, instead of writing down these equations and attempting to solve them, we shall be mainly interested in the Hamiltonian description.

Exercise 6.42. Making use of the fact that $\mathrm{d}s = \sqrt{1 + x'^2 + y'^2}\, \mathrm{d}z$, show that the Euler–Lagrange equations for the Lagrangian (6.102) amount to

$$\frac{\mathrm{d}}{\mathrm{d}s}\left(n \frac{\mathrm{d}x}{\mathrm{d}s} \right) = \frac{\partial n}{\partial x}, \qquad \frac{\mathrm{d}}{\mathrm{d}s}\left(n \frac{\mathrm{d}y}{\mathrm{d}s} \right) = \frac{\partial n}{\partial y},$$

and that, making use of the identity (1.92), one obtains the equation

$$\frac{\mathrm{d}}{\mathrm{d}s}\left(n \frac{\mathrm{d}z}{\mathrm{d}s} \right) = \frac{\partial n}{\partial z}.$$

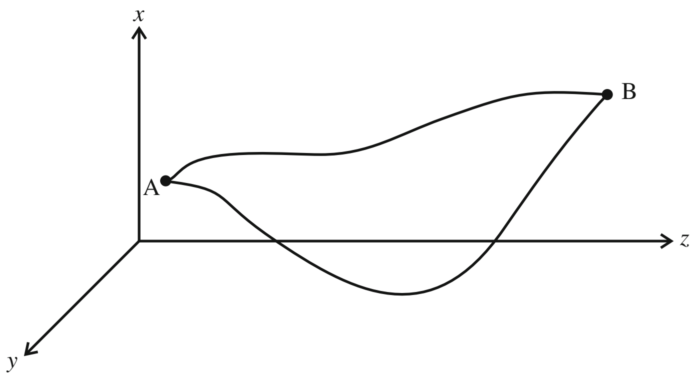

Fig. 6.1 The curves shown join the points A and B. In order to use z as a parameter for these curves, any plane $z =$ const must intersect each curve at most at one point

The last three equations are equivalent to the vector equation

$$\frac{\mathrm{d}}{\mathrm{d}s}\left(n\frac{\mathrm{d}\mathbf{r}}{\mathrm{d}s}\right) = \nabla n,$$

where $\mathbf{r} = (x, y, z)$ is the position vector of a point of the ray. This equation is known as the *ray equation*.

Exercise 6.43 (Spherically symmetric media). Show that if the refractive index is a function of the distance, r, to a fixed point (taken as the origin), $n = n(r)$, then

$$\mathbf{r} \times n\,\frac{\mathrm{d}\mathbf{r}}{\mathrm{d}s}$$

is constant along each ray and show that this implies that each ray lies on a plane passing through the origin.

The optical system defined by the spherically symmetric refractive index

$$n = \frac{a}{b + r^2}, \tag{6.103}$$

where a and b are real constants, with $a > 0$, is known as *Maxwell's fish eye*. Several properties of this system can be derived from its relationship with the Kepler problem. Show that, in this case,

$$\mathbf{r} \times \left(\mathbf{r} \times n\,\frac{\mathrm{d}\mathbf{r}}{\mathrm{d}s}\right) + \frac{a}{2}\frac{\mathrm{d}\mathbf{r}}{\mathrm{d}s}$$

is also constant along each ray and deduce from this that the rays are (arcs of) circles (just like the hodographs of the Kepler problem). (See also Exercise 6.45, below.)

The Eikonal Equation

The canonical momenta conjugate to x and y are

$$p_x = \frac{\partial L}{\partial x'} = \frac{nx'}{\sqrt{1 + x'^2 + y'^2}}, \qquad p_y = \frac{\partial L}{\partial y'} = \frac{ny'}{\sqrt{1 + x'^2 + y'^2}}, \tag{6.104}$$

respectively, and from these equations we obtain

$$\sqrt{n^2 - p_x{}^2 - p_y{}^2} = \frac{n}{\sqrt{1 + x'^2 + y'^2}}$$

and

$$x' = \frac{p_x}{\sqrt{n^2 - p_x{}^2 - p_y{}^2}}, \qquad y' = \frac{p_y}{\sqrt{n^2 - p_x{}^2 - p_y{}^2}}. \tag{6.105}$$

Thus, in this approach, we have a system with two degrees of freedom, with the coordinate z as the independent variable, and a Hamiltonian given by [see Equation (4.10)]

$$H = -\sqrt{n^2 - p_x{}^2 - p_y{}^2} \tag{6.106}$$

(hence, e.g., $\mathrm{d}p_x/\mathrm{d}z = -\partial H/\partial x$). Therefore, the corresponding HJ equation is

$$-\left[n^2 - \left(\frac{\partial S}{\partial x}\right)^2 - \left(\frac{\partial S}{\partial y}\right)^2\right]^{1/2} + \frac{\partial S}{\partial z} = 0, \tag{6.107}$$

which implies that

$$\left(\frac{\partial S}{\partial x}\right)^2 + \left(\frac{\partial S}{\partial y}\right)^2 + \left(\frac{\partial S}{\partial z}\right)^2 = n^2. \tag{6.108}$$

This last equation is known as the *eikonal equation* and, in this context, S is called the *eikonal* (or *eikonal function*). The eikonal equation can also be derived from the Huygens principle (see, e.g., Synge [15, Sect. 22]). Any complete solution of Equation (6.107) or (6.108) allows us to find all the light rays in the medium characterized by the refractive index n (that is, the solutions of the ray equation). Note that in the eikonal equation, the three coordinates (x, y, z) appear on an equal footing and it is optional which of them is taken as the independent variable.

For instance, if the refractive index is *constant*, Equation (6.107) admits separable solutions of the form

$$S(x, y, P_1, P_2, z) = P_1 x + P_2 y + \sqrt{n^2 - P_1{}^2 - P_2{}^2}\, z, \tag{6.109}$$

where P_1, P_2 are constants such that $P_1{}^2 + P_2{}^2 \leqslant n^2$. Making use of the standard formulas (6.8) we obtain the canonical transformation generated by (6.109) (treating again z as the independent variable)

$$p_x = P_1, \quad p_y = P_2, \quad Q_1 = x - \frac{P_1 z}{\sqrt{n^2 - P_1{}^2 - P_2{}^2}}, \quad Q_2 = y - \frac{P_2 z}{\sqrt{n^2 - P_1{}^2 - P_2{}^2}}. \tag{6.110}$$

The last two equations in (6.110) show that in this case the light rays are straight lines, as expected. In fact, in terms of the usual vector notation, we have

$$(x, y, z) = (Q_1, Q_2, 0) + \frac{z\,(P_1, P_2, \sqrt{n^2 - P_1{}^2 - P_2{}^2})}{\sqrt{n^2 - P_1{}^2 - P_2{}^2}}.$$

Thus, the constants P_1, P_2 determine the direction of the light ray, and $(Q_1, Q_2, 0)$ are the Cartesian coordinates of the intersection of the ray with the plane $z = 0$

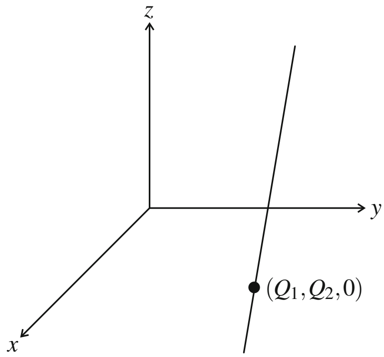

Fig. 6.2 Any light ray in a homogeneous isotropic medium is a straight line. With the exception of the light rays parallel to the xy-plane, any light ray can be specified by the four real numbers (Q_1, Q_2, P_1, P_2); P_1 and P_2 determine the direction of the ray, and $(Q_1, Q_2, 0)$ are the Cartesian coordinates of the intersection of the ray with the plane $z = 0$. The planes orthogonal to this straight line are the wavefronts defined by (6.109)

(see Figure 6.2). (Note also that the two last equations in (6.110) are obtained regardless of which of the coordinates (x, y, z) is taken as the independent variable.)

Exercise 6.44. In terms of the spherical coordinates (r, θ, ϕ), the optical length (6.100) is given by

$$\int_{r_1}^{r_2} n(r, \theta, \phi) \sqrt{1 + r^2\theta'^2 + r^2 \sin^2\theta \phi'^2}\, \mathrm{d}r,$$

with $\theta' \equiv \mathrm{d}\theta/\mathrm{d}r$, $\phi' \equiv \mathrm{d}\phi/\mathrm{d}r$, assuming that the curve C can be parameterized by r [cf. Equation (6.101)]. Starting from the Fermat principle, using this expression, show that the corresponding HJ equation leads to the equation

$$\left(\frac{\partial S}{\partial r}\right)^2 + \frac{1}{r^2}\left(\frac{\partial S}{\partial \theta}\right)^2 + \frac{1}{r^2 \sin^2\theta}\left(\frac{\partial S}{\partial \phi}\right)^2 = n^2 \tag{6.111}$$

which is just the eikonal equation (6.108) expressed in spherical coordinates. Even though we might expect this result, taking into account the meaning of the eikonal function, it does not follow from the discussion presented in the preceding sections (e.g., Section 6.3) because in the present case we are also changing the parameter of the light rays z in Equation (6.101), by r.

Exercise 6.45. Solving the eikonal equation, show that in the case of the Maxwell fish eye, the light rays are (arcs of) circles. (It is convenient to make use of the fact that each ray lies on a plane passing through the origin—see Exercise 6.43.)

Going back to the general case, where n is an arbitrary function, from Equations (6.105) and (6.107) we find that the vector with Cartesian components $(\mathrm{d}x/\mathrm{d}z, \mathrm{d}y/\mathrm{d}z, \mathrm{d}z/\mathrm{d}z)$, which is tangent to the light ray, is proportional to the gradient of S,

$$\left(\frac{\mathrm{d}x}{\mathrm{d}z}, \frac{\mathrm{d}y}{\mathrm{d}z}, \frac{\mathrm{d}z}{\mathrm{d}z}\right) = \frac{(p_x, p_y, \sqrt{n^2 - p_x{}^2 - p_y{}^2})}{\sqrt{n^2 - p_x{}^2 - p_y{}^2}} = \frac{1}{\partial S/\partial z}\left(\frac{\partial S}{\partial x}, \frac{\partial S}{\partial y}, \frac{\partial S}{\partial z}\right),$$

which means that the light rays intersect orthogonally the level surfaces $S(x, y, z) =$ const. (see Figure 6.3). The surfaces $S =$ const. constitute the *wavefronts*.

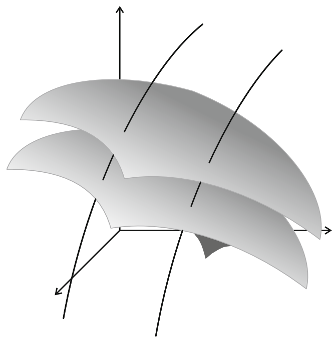

Fig. 6.3 The level surfaces of a solution $S(x, y, z)$ of the eikonal equation constitute a family of two-dimensional surfaces that fill the three-dimensional space. The curves orthogonal to these surfaces correspond to some of the possible rays of light in the medium

In the example (6.109), the wavefronts are the planes normal to the vector with Cartesian components $(P_1, P_2, \sqrt{n^2 - P_1{}^2 - P_2{}^2})$ (see Figure 6.2).

Exercise 6.46. Show that if $S(x, y, z)$ is a solution of the eikonal equation, which may not contain arbitrary parameters, then the curves orthogonal to the level surfaces $S =$ const. correspond to possible light rays. (*Hint*: if $x_i(s)$ are the Cartesian coordinates of a curve parameterized by its arclength, then the norm of its tangent vector, $(\mathrm{d}x/\mathrm{d}s, \mathrm{d}y/\mathrm{d}s, \mathrm{d}z/\mathrm{d}s)$, is equal to 1. On the other hand, if this curve is orthogonal to the level surfaces of S, then its tangent vector is proportional to ∇S, whose norm is equal to n.)

Exercise 6.47. Find the light rays determined by the eikonal function in two dimensions

$$S(x, y, P) = \tfrac{1}{2}a\big[(x^2 - y^2)\cos P + 2xy \sin P\big],$$

where a and P are constants. What is the refractive index?

Each solution, $S(x, y, z)$, of the eikonal equation defines a family of surfaces (its level surfaces) in such a way that if a_1 and a_2 are two real constants (such that the sets $\{(x, y, z) \in \mathbb{R}^3 \mid S(x, y, z) = a_1\}$ and $\{(x, y, z) \in \mathbb{R}^3 \mid S(x, y, z) = a_2\}$ are nonempty), the wavefront $S = a_2$ is obtained from $S = a_1$ by the propagation of the light by a time $(a_2 - a_1)/c$ (see Figure 6.4). In fact, if we consider a light ray, C, connecting a point belonging to the surface $S(x, y, z) = a_1$ with a point belonging to $S(x, y, z) = a_2$, assuming that this curve can be parameterized by z, we have [see Equations (6.107) and (6.104)]

$$
\begin{aligned}
a_2 - a_1 &= \int_C \mathrm{d}S = \int_{z_1}^{z_2} \left(\frac{\partial S}{\partial x} x' + \frac{\partial S}{\partial y} y' + \frac{\partial S}{\partial z} \right) \mathrm{d}z \\
&= \int_{z_1}^{z_2} \left(p_x x' + p_y y' + \sqrt{n^2 - p_x{}^2 - p_y{}^2} \right) \mathrm{d}z = \int_{z_1}^{z_2} \frac{n\ (x'^2 + y'^2 + 1)}{\sqrt{1 + x'^2 + y'^2}}\, \mathrm{d}z \\
&= \int_C n\, \mathrm{d}s.
\end{aligned}
$$

Thus, the set of level surfaces $S =$ const. represent the evolution of one of them (see Figure 6.4).

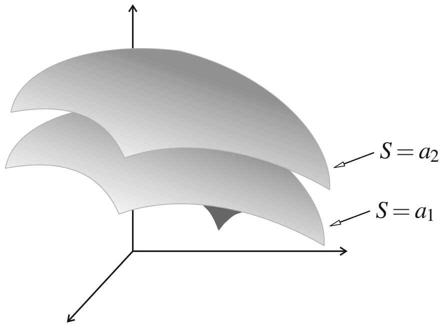

Fig. 6.4 Each level surface of $S(x, y, z)$ represents the wavefront at some particular time. The level surface $S = a_1$ evolves into the level surface $S = a_2$ after a time $(a_2 - a_1)/c$. The set of all the level surfaces of S represents the time evolution of anyone of them

Generation of Complete Solutions of the Eikonal Equation from a given Complete Solution

As shown in Section 6.1.1, from a given complete solution of the HJ equation one can obtain any other complete solution of the same equation. For instance, making use of the complete solution (6.109), and choosing the time-independent generating function $F(P_i, \tilde{P}_i) = P_1 \tilde{P}_1 + P_2 \tilde{P}_2$, according to Equations (6.46) we have

$$
0 = \frac{\partial (S - F)}{\partial P_1} = x - \frac{P_1 z}{\sqrt{n^2 - P_1{}^2 - P_2{}^2}} - \tilde{P}_1,
$$

hence,

$$
P_1 = \frac{(x - \tilde{P}_1)\sqrt{n^2 - P_1{}^2 - P_2{}^2}}{z},
$$

with a similar expression for P_2. Then, from the expressions thus obtained, we find that

$$P_1 = \frac{n\,(x - \tilde{P}_1)}{\sqrt{(x - \tilde{P}_1)^2 + (y - \tilde{P}_2)^2 + z^2}}, \qquad P_2 = \frac{n\,(y - \tilde{P}_2)}{\sqrt{(x - \tilde{P}_1)^2 + (y - \tilde{P}_2)^2 + z^2}}.$$

With the aid of these formulas we can eliminate the parameters P_i appearing in the right-hand side of (6.45) and in this manner we find a second complete solution of the eikonal equation

$$\tilde{S}(x, y, \tilde{P}_1, \tilde{P}_2, z) = n\sqrt{(x - \tilde{P}_1)^2 + (y - \tilde{P}_2)^2 + z^2}, \tag{6.112}$$

$\tilde{P}_1, \tilde{P}_2 \in (-\infty, \infty)$. The wavefronts $\tilde{S} =$ const. are spheres [centered at $(\tilde{P}_1, \tilde{P}_2, 0)$].