Chapter 3
Rigid Bodies

Another interesting application of the Lagrangian formalism is found in the motion of a rigid body. A rigid body can be defined as a collection of point particles such that the distances between them are constant. Even though, in essence, this example is similar to those already considered, the expression of the kinetic energy of a rigid body involves a more elaborate process and the definition of a new object (the inertia tensor).

This chapter differs from the other chapters of this book by the extensive use of objects with indices. A more elementary approach is based on the use of the vector algebra. The treatment given here highlights the use of the Lagrangian formalism.

3.1 The Configuration Space of a Rigid Body with a Fixed Point

We shall restrict ourselves to the study of the motion of a rigid body assuming that there exists a fixed point (with respect, of course, to some inertial frame). We shall also assume that the particles forming the rigid body are not all collinear (this means that there are at least three particles). Under these conditions, the system has three degrees of freedom (see below).

In order to study the motion of a rigid body with a fixed point, following a standard approach, we consider two sets of Cartesian axes, the first one, with coordinates x, y, z, assumed inertial, and the second one, with coordinates x', y', z', fixed in the rigid body. The origins of both sets of Cartesian axes coincide with the fixed point of the body (see Figure 3.1). It is convenient to denote x, y, z as x_1, x_2, x_3, and, similarly, x', y', z' as $x_{1'}, x_{2'}, x_{3'}$. Then, any point of the rigid body has a position vector

$$\mathbf{r} = x_{1'}\mathbf{e}_{1'} + x_{2'}\mathbf{e}_{2'} + x_{3'}\mathbf{e}_{3'},$$

G. F. Torres del Castillo, *An Introduction to Hamiltonian Mechanics*, Birkhäuser Advanced Texts Basler Lehrbücher, https://doi.org/10.1007/978-3-319-95225-3_3

where the unit vectors $\mathbf{e}_{i'}$ ($i' = 1', 2', 3'$) form an orthonormal basis fixed in the body, and, at the same time,

$$\mathbf{r} = x_1\mathbf{e}_1 + x_2\mathbf{e}_2 + x_3\mathbf{e}_3,$$

where the unit vectors $\mathbf{e}_i$ ($i = 1, 2, 3$) form an orthonormal basis associated with the inertial frame.

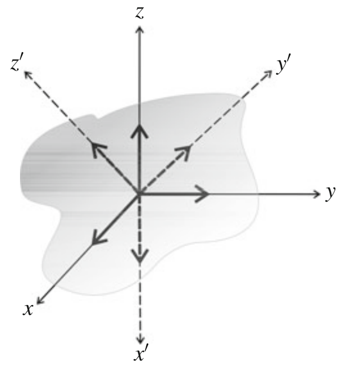

Fig. 3.1 The Cartesian axes with coordinates x', y', z' are fixed in the body and rotate about the origin of the Cartesian axes with coordinates x, y, z, which belong to an inertial frame. Both sets of axes are right-handed

Since the vectors $\mathbf{e}_i$ form a basis, there exist nine real numbers, $a_{ij'}$, which may depend on the time only, such that

$$\mathbf{e}_{i'} = a_{ji'}\mathbf{e}_j \tag{3.1}$$

(note the position of the indices, the order is purely conventional). With these numbers we can form a 3×3 matrix, $A = (a_{ij'})$, in the usual manner, using the first subscript to label rows and the second subscript to label columns, that is

$$A = \begin{pmatrix} a_{11'} & a_{12'} & a_{13'} \\ a_{21'} & a_{22'} & a_{23'} \\ a_{31'} & a_{32'} & a_{33'} \end{pmatrix},$$

so that the i-th column of this matrix contains the components of the vector $\mathbf{e}_{i'}$ with respect to the basis formed by the vectors $\mathbf{e}_j$. For instance, if the vectors $\mathbf{e}_{i'}$ are obtained from the vectors $\mathbf{e}_i$ by means of a rotation through an angle ϕ about the z-axis, then the matrix A is (see Figure 3.2)

$$A = \begin{pmatrix} \cos\phi & -\sin\phi & 0 \\ \sin\phi & \cos\phi & 0 \\ 0 & 0 & 1 \end{pmatrix}. \tag{3.2}$$

Using the fact that both bases are orthonormal we have

$$\delta_{i'j'} = \mathbf{e}_{i'} \cdot \mathbf{e}_{j'} = a_{ki'}\mathbf{e}_k \cdot a_{lj'}\mathbf{e}_l = a_{ki'}a_{lj'}\delta_{kl} = a_{ki'}a_{kj'},$$

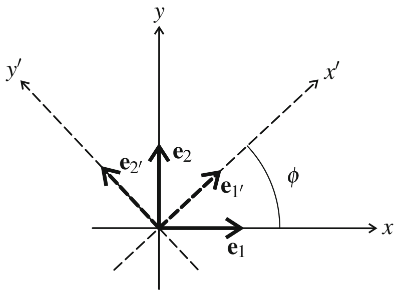

Fig. 3.2 The vectors $\mathbf{e}_{i'}$ are obtained from the vectors $\mathbf{e}_i$ by a rotation through an angle ϕ about the z-axis. The figure shows that $\mathbf{e}_{1'} = \cos\phi\, \mathbf{e}_1 + \sin\phi\, \mathbf{e}_2$, $\mathbf{e}_{2'} = -\sin\phi\, \mathbf{e}_1 + \cos\phi\, \mathbf{e}_2$, $\mathbf{e}_{3'} = \mathbf{e}_3$, which leads to the matrix (3.2)

i.e.,

$$\delta_{i'j'} = a_{ki'} a_{kj'}. \tag{3.3}$$

These conditions mean that the matrix $(a_{ij'})$ is orthogonal ($A^{\mathrm{t}} A = I$, where A^{t} is the transpose of A, and I is the unit matrix), that is, its inverse is equal to its transpose and, since any matrix commutes with its inverse, Equation (3.3) is equivalent to

$$\delta_{ij} = a_{ik'} a_{jk'}. \tag{3.4}$$

Once we have chosen the basis vectors $\mathbf{e}_i$ and $\mathbf{e}_{i'}$, the matrix $A = (a_{ij'})$ determines the configuration of the rigid body and, therefore, for a rigid body with a fixed point, the configuration space can be identified with the set of all the real 3×3 orthogonal matrices with positive determinant (so that the orientation, or handedness, of the basis vectors is not inverted). This set of matrices is, in fact, a group, which is denoted by SO(3). The set of equations (3.3) constitute six algebraically independent conditions on the nine entries of A (since both sides of the equation are symmetric in the indices i', j'); hence, the 3×3 orthogonal matrices can be parameterized by three coordinates (for instance, the three Euler angles presented in Section 3.3, below).

Usually, in the so-called tensor notation, the indices labeling the components of an object (e.g., a vector or a tensor) determine the way in which these components transform under a change of the basis vectors. For that reason, here we need two different kinds of indices: the unprimed and the primed ones, because we can perform two different kinds of changes of bases. We can replace the orthonormal basis $\mathbf{e}_i$ by another orthonormal basis (related to the first one by means of a constant orthogonal matrix) and, *independently*, we can replace the orthonormal basis $\mathbf{e}_{i'}$ by another orthonormal basis, also fixed in the rigid body (and the two orthonormal bases fixed with respect to the body are also related by some constant orthogonal matrix, see, e.g., Equation (3.19), below). The equations developed here must maintain their form under the independent changes of the two bases.

It may be remarked that in all the other examples in this book, we start by choosing some coordinates to represent the configuration of the mechanical system (which, in some cases, are replaced afterwards). By contrast, in the case of the motion of a rigid body we can postpone this choice and establish several results without having to write down the explicit expression of the Lagrangian in terms of coordinates.

3.2 The Instantaneous Angular Velocity and the Inertia Tensor

Assuming that the rigid body is made out of N point particles with position vectors $\mathbf{r}_\alpha$ ($\alpha = 1, 2, \ldots, N$), each of these vectors is represented by three real numbers, $x_i^{(\alpha)}$ ($i = 1, 2, 3$), with respect to the inertial frame defined by the basis vectors $\mathbf{e}_i$, and by three real numbers, $x_{i'}^{(\alpha)}$ ($i' = 1', 2', 3'$), with respect to the frame fixed in the body, defined by the basis vectors $\mathbf{e}_{i'}$, that is,

$$\mathbf{r}_\alpha = x_i^{(\alpha)}\mathbf{e}_i = x_{i'}^{(\alpha)}\mathbf{e}_{i'}.$$

According to Equation (3.1), using the fact that the inverse of the matrix $(a_{ij'})$ is its transpose, these sets of coordinates are related by

$$x_i^{(\alpha)} = a_{ij'}x_{j'}^{(\alpha)}, \qquad x_{i'}^{(\alpha)} = a_{ji'}x_j^{(\alpha)}, \tag{3.5}$$

with the *same matrix* $(a_{ij'})$ for all the particles of the body (that is, Equations (3.5) hold for $\alpha = 1, 2, \ldots, N$). (Note that the components of *any* vector with respect to the bases formed by the vectors $\mathbf{e}_i$ and $\mathbf{e}_{i'}$ are related in this form.)

Since the basis vectors $\mathbf{e}_{i'}$ are fixed with respect to the body, the coordinates $x_{i'}^{(\alpha)}$ cannot vary with the time. On the other hand, the coordinates $x_i^{(\alpha)}$ will vary with the time as a consequence of the rotation of the body, hence, making use of the first equation in (3.5),

$$\dot{\mathbf{r}}_\alpha = \dot{x}_j^{(\alpha)}\mathbf{e}_j = \dot{a}_{ji'}x_{i'}^{(\alpha)}\mathbf{e}_j$$

and, therefore, the kinetic energy of the body (with respect to the inertial frame) is

$$\begin{aligned} T &= \sum_{\alpha=1}^{N} \frac{1}{2} m_\alpha \dot{\mathbf{r}}_\alpha \cdot \dot{\mathbf{r}}_\alpha \\ &= \sum_{\alpha=1}^{N} \frac{1}{2} m_\alpha (\dot{a}_{ji'}x_{i'}^{(\alpha)})(\dot{a}_{jk'}x_{k'}^{(\alpha)}) \\ &= \dot{a}_{ji'}\dot{a}_{jk'} \sum_{\alpha=1}^{N} \frac{1}{2} m_\alpha x_{i'}^{(\alpha)} x_{k'}^{(\alpha)}. \end{aligned} \tag{3.6}$$

In this manner, the kinetic energy is expressed in terms of the time derivative of the matrix $(a_{ij'})$, which depends on how the body moves, and of the nine *constant* real numbers $\sum_{\alpha=1}^{N} \frac{1}{2} m_\alpha x_{i'}^{(\alpha)} x_{k'}^{(\alpha)}$, ($i', k' = 1', 2', 3'$) which are determined by the positions and masses of the particles forming the rigid body. As we shall see, the product of time derivatives $\dot{a}_{ji'}\dot{a}_{jk'}$, appearing in (3.6), can be written in terms of a single vector, which corresponds to the body's angular velocity and, instead of the sums $\sum_{\alpha=1}^{N} \frac{1}{2} m_\alpha x_{i'}^{(\alpha)} x_{k'}^{(\alpha)}$, it will be more convenient to use the components of the inertia tensor, to be defined below.

Differentiating both sides of Equation (3.3) with respect to the time we obtain

$$0 = a_{ki'} \dot{a}_{kj'} + \dot{a}_{ki'} a_{kj'}.$$

Since the factors appearing in the last equation are real-valued functions, we have $a_{ki'} \dot{a}_{kj'} = \dot{a}_{kj'} a_{ki'}$ and, therefore,

$$\dot{a}_{ki'} a_{kj'} = -\dot{a}_{kj'} a_{ki'}$$

which shows that the product $\dot{a}_{ki'} a_{kj'}$ is antisymmetric in the indices i' and j'. This is equivalent to say that there exist three real-valued functions of the time, $\omega_{i'}$, such that

$$\dot{a}_{ki'} a_{kj'} = \varepsilon_{i'j's'} \omega_{s'}, \tag{3.7}$$

where $\varepsilon_{i'j'k'}$ is the Levi-Civita symbol, defined by

$$\varepsilon_{i'j'k'} = \begin{cases} 1 \text{ if } i'j'k' \text{ is an even permutation of } 1'2'3' \\ -1 \text{ if } i'j'k' \text{ is an odd permutation of } 1'2'3' \\ 0 \text{ otherwise.} \end{cases} \tag{3.8}$$

The $\omega_{i'}$ are the components of the angular velocity of the rigid body in the basis $\mathbf{e}_{i'}$.

In terms of the matrix notation, Equation (3.7) is equivalent to

$$\dot{A}^{\mathrm{t}} A = \begin{pmatrix} 0 & \omega_{3'} & -\omega_{2'} \\ -\omega_{3'} & 0 & \omega_{1'} \\ \omega_{2'} & -\omega_{1'} & 0 \end{pmatrix}. \tag{3.9}$$

For instance, in the case of the matrix (3.2) we find that the product $\dot{A}^{\mathrm{t}} A$ is

$$\dot{\phi} \begin{pmatrix} -\sin\phi & \cos\phi & 0 \\ -\cos\phi & -\sin\phi & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} \cos\phi & -\sin\phi & 0 \\ \sin\phi & \cos\phi & 0 \\ 0 & 0 & 1 \end{pmatrix} = \dot{\phi} \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

which shows that the only nonzero component $\omega_{i'}$ is $\omega_{3'} = \dot{\phi}$, as one would expect.

The Levi-Civita symbol (3.8) is invariant under cyclic permutations of the indices, that is,

$$\varepsilon_{i'j'k'} = \varepsilon_{j'k'i'} = \varepsilon_{k'i'j'} \tag{3.10}$$

and satisfies the relation

$$\varepsilon_{i'j'k'} \varepsilon_{i'l'm'} = \delta_{j'l'} \delta_{k'm'} - \delta_{j'm'} \delta_{k'l'}. \tag{3.11}$$

The Levi-Civita symbol is very useful owing to its relationship with the determinant. If $B = (b_{ij})$ is a 3×3 matrix, then from the definition (3.8) it follows that

$$\varepsilon_{ijk} b_{ip} b_{jq} b_{kr} = (\det B) \varepsilon_{pqr}. \tag{3.12}$$

A related result is that the components of the vector product of two vectors can be conveniently expressed with the aid of the Levi-Civita symbol. If a_i and b_i are the components of two vectors, **a** and **b**, respectively, with respect to some right-handed orthonormal basis, then the components of the vector product $\mathbf{c} = \mathbf{a} \times \mathbf{b}$ with respect to this basis are given by

$$c_i = \varepsilon_{ijk} a_j b_k. \tag{3.13}$$

Making use of Equations (3.4), from (3.7) we have

$$\varepsilon_{i'j's'} \omega_{s'} a_{rj'} = \dot{a}_{ki'} a_{kj'} a_{rj'} = \dot{a}_{ki'} \delta_{kr} = \dot{a}_{ri'}, \tag{3.14}$$

hence, with the aid of (3.3)

$$\begin{aligned}
\dot{a}_{ji'} \dot{a}_{jk'} &= \varepsilon_{i'l's'} \omega_{s'} a_{jl'} \varepsilon_{k'n'r'} \omega_{r'} a_{jn'} = \varepsilon_{i'l's'} \omega_{s'} \varepsilon_{k'n'r'} \omega_{r'} \delta_{l'n'} = \varepsilon_{i'l's'} \omega_{s'} \varepsilon_{k'l'r'} \omega_{r'} \\
&= \varepsilon_{l's'i'} \omega_{s'} \varepsilon_{l'r'k'} \omega_{r'} = (\delta_{s'r'} \delta_{i'k'} - \delta_{s'k'} \delta_{i'r'})\, \omega_{s'} \omega_{r'}.
\end{aligned}$$

Substituting this expression into (3.6) we find that the kinetic energy of the rigid body can also be written in the form

$$\begin{aligned}
T &= (\delta_{s'r'} \delta_{i'k'} - \delta_{s'k'} \delta_{i'r'})\, \omega_{s'} \omega_{r'} \sum_{\alpha=1}^{N} \frac{1}{2} m_\alpha x_{i'}^{(\alpha)} x_{k'}^{(\alpha)} \\
&= \frac{1}{2} \omega_{s'} \omega_{r'} \sum_{\alpha=1}^{N} m_\alpha (\delta_{s'r'} \delta_{i'k'} - \delta_{s'k'} \delta_{i'r'})\, x_{i'}^{(\alpha)} x_{k'}^{(\alpha)} \\
&= \frac{1}{2} \omega_{s'} \omega_{r'} \sum_{\alpha=1}^{N} m_\alpha \left(\delta_{s'r'} \mathbf{r}_\alpha{}^2 - x_{s'}^{(\alpha)} x_{r'}^{(\alpha)} \right).
\end{aligned}$$

The nine real numbers

$$I_{j'k'} \equiv \sum_{\alpha=1}^{N} m_\alpha \left(\delta_{j'k'} \mathbf{r}_\alpha{}^2 - x_{j'}^{(\alpha)} x_{k'}^{(\alpha)} \right) \tag{3.15}$$

are the components of the *inertia tensor* of the rigid body (with respect to the basis vectors $\mathbf{e}_{i'}$) so that the kinetic energy of the rigid body is expressed as

$$T = \tfrac{1}{2} I_{j'k'} \omega_{j'} \omega_{k'}. \tag{3.16}$$

If we consider a continuous distribution of matter, with a mass density ρ, the components of the inertia tensor are given by

$$I_{j'k'} \equiv \int \rho(\mathbf{r})(\delta_{j'k'}\mathbf{r}^2 - x_{j'}x_{k'})\,\mathrm{d}v', \tag{3.17}$$

with $\mathbf{r} = x_{i'}\mathbf{e}_{i'}$ and $\mathrm{d}v' = \mathrm{d}x_{1'}\mathrm{d}x_{2'}\mathrm{d}x_{3'}$. The definitions (3.15) and (3.17) show that the inertia tensor is symmetric, $I_{i'j'} = I_{j'i'}$, and therefore it has six independent components only.

Expression (3.16) can also be obtained in a more elementary manner. However, the procedure followed above yields a useful expression for the angular velocity of the rigid body in terms of the matrix $(a_{ji'})$ [Equation (3.7)].

Example 3.1 (Inertia tensor of a homogeneous cylinder). We shall calculate the inertia tensor of a homogeneous right circular cylinder. The height of the cylinder will be denoted by h, its radius by a, and its mass by M. Then its density, assumed constant, is $\rho = M/(\pi a^2 h)$. We take the fixed point, O, at the center of the cylinder and the $x_{3'}$-axis will coincide with the axis of the cylinder (see Figure 3.3). From Equation (3.17) we have, making use of cylindrical coordinates (that is, $x_{1'} = \rho\cos\phi$, $x_{2'} = \rho\sin\phi$, $x_{3'} = z$),

$$\begin{aligned}
I_{1'1'} &= \frac{M}{\pi a^2 h}\int_0^a \mathrm{d}\rho \int_{-h/2}^{h/2}\mathrm{d}z \int_0^{2\pi} \rho\mathrm{d}\phi\,({x_{2'}}^2 + {x_{3'}}^2)\\
&= \frac{M}{\pi a^2 h}\int_0^a \mathrm{d}\rho \int_{-h/2}^{h/2}\mathrm{d}z \int_0^{2\pi} \rho\mathrm{d}\phi\,(\rho^2\sin^2\phi + z^2)\\
&= \frac{M}{\pi a^2 h}\int_0^a \mathrm{d}\rho \int_{-h/2}^{h/2}\mathrm{d}z\,\left(\pi\rho^3 + 2\pi\rho z^2\right)\\
&= \frac{M}{a^2 h}\int_0^a \mathrm{d}\rho\left(\rho^3 h + \frac{1}{6}\rho h^3\right)\\
&= M\left(\frac{a^2}{4} + \frac{h^2}{12}\right).
\end{aligned}$$

Similarly we find that

$$I_{2'2'} = \frac{M}{\pi a^2 h}\int_0^a \mathrm{d}\rho \int_{-h/2}^{h/2}\mathrm{d}z \int_0^{2\pi} \rho\mathrm{d}\phi\,({x_{1'}}^2 + {x_{3'}}^2) = M\left(\frac{a^2}{4} + \frac{h^2}{12}\right),$$

and

$$I_{3'3'} = \frac{M}{\pi a^2 h}\int_0^a \mathrm{d}\rho \int_{-h/2}^{h/2}\mathrm{d}z \int_0^{2\pi} \rho\mathrm{d}\phi\,({x_{1'}}^2 + {x_{2'}}^2) = \frac{Ma^2}{2}.$$

Making use of the parity of the integrands one finds that $I_{1'2'}$, $I_{1'3'}$, and $I_{2'3'}$ are equal to zero. Hence,

$$(I_{i'j'}) = \frac{M}{12}\begin{pmatrix} 3a^2 + h^2 & 0 & 0\\ 0 & 3a^2 + h^2 & 0\\ 0 & 0 & 6a^2 \end{pmatrix}. \tag{3.18}$$

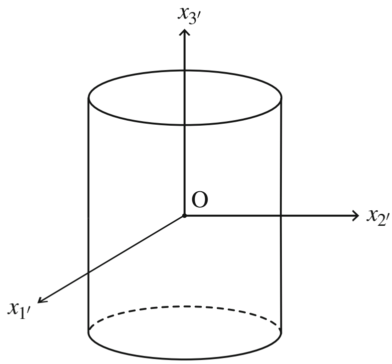

Fig. 3.3 The mass density of the cylinder is uniform. The origin is placed at the geometric center of the cylinder, which coincides with the center of mass. The $x_{3'}$-axis coincides with the axis of the cylinder

Principal Moments of Inertia

The components of the inertia tensor depend on the mass distribution of the body, but also on the choice of the coordinate axes fixed in the body. If we replace the orthonormal basis $\mathbf{e}_{i'}$ by another orthonormal basis $\tilde{\mathbf{e}}_{j'}$ also fixed in the body, then there exists a (time-independent) orthogonal matrix, $B = (b_{j'k'})$, such that

$$\tilde{\mathbf{e}}_{j'} = b_{k'j'}\mathbf{e}_{k'} \tag{3.19}$$

(again, note the position of the indices). According to this definition, the columns of the orthogonal matrix

$$B = \begin{pmatrix} b_{1'1'} & b_{1'2'} & b_{1'3'} \\ b_{2'1'} & b_{2'2'} & b_{2'3'} \\ b_{3'1'} & b_{3'2'} & b_{3'3'} \end{pmatrix}$$

are the components of the vectors $\tilde{\mathbf{e}}_{i'}$ with respect to the basis formed by the vectors $\mathbf{e}_{j'}$. Then, the Cartesian coordinates of the α-th particle with respect to the new axes fixed in the body are $\tilde{x}_{k'}^{(\alpha)} = b_{j'k'}x_{j'}^{(\alpha)}$ and, therefore, with respect to these new axes, the components of the inertia tensor are [see (3.15)]

$$\tilde{I}_{i'j'} = b_{r'i'}b_{s'j'}I_{r's'}. \tag{3.20}$$

In terms of matrices, this last equation amounts to

$$\tilde{\mathscr{I}} = B^{\mathrm{t}}\mathscr{I}B,$$

where $\tilde{\mathscr{I}} \equiv (\tilde{I}_{i'j'})$, $\mathscr{I} \equiv (I_{i'j'})$, and B is the orthogonal matrix defined above.

If the columns of the matrix B are three mutually orthogonal unit eigenvectors of the matrix $(I_{i'j'})$, then $(\tilde{I}_{i'j'})$ is diagonal (see below).

Since the matrix $(I_{r's'})$ is real and symmetric, we can always find three mutually orthogonal eigenvectors of $(I_{r's'})$. Recall that $v_{s'}$ is an eigenvector of $(I_{r's'})$, with eigenvalue λ, if

$$\begin{pmatrix} I_{1'1'} & I_{1'2'} & I_{1'3'} \\ I_{2'1'} & I_{2'2'} & I_{2'3'} \\ I_{3'1'} & I_{3'2'} & I_{3'3'} \end{pmatrix} \begin{pmatrix} v_{1'} \\ v_{2'} \\ v_{3'} \end{pmatrix} = \lambda \begin{pmatrix} v_{1'} \\ v_{2'} \\ v_{3'} \end{pmatrix} \tag{3.21}$$

or, equivalently,

$$I_{r's'} v_{s'} = \lambda v_{r'}. \tag{3.22}$$

The eigenvalue λ is a root of the characteristic polynomial of $(I_{r's'})$, that is, $\det(I_{r's'} - \lambda\delta_{r's'}) = 0$. Since the components $I_{r's'}$ are real, (3.22) is equivalent to

$$I_{r's'} \overline{v_{s'}} = \overline{\lambda} \overline{v_{r'}}, \tag{3.23}$$

where the bar denotes complex conjugation. By combining Equations (3.22) and (3.23), using the symmetry of $I_{r's'}$, we obtain

$$I_{i'j'} v_{j'} \overline{v_{i'}} = \lambda v_{i'} \overline{v_{i'}}$$

and

$$I_{i'j'} v_{j'} \overline{v_{i'}} = v_{j'} I_{j'i'} \overline{v_{i'}} = v_{j'} \overline{\lambda} \overline{v_{j'}} = \overline{\lambda} v_{i'} \overline{v_{i'}},$$

thus, $(\lambda - \overline{\lambda}) v_{i'} \overline{v_{i'}} = 0$, which means that λ is real ($v_{i'} \overline{v_{i'}}$ is equal to zero only if $v_{i'} = 0$, which is excluded from the definition of eigenvector). Furthermore, the eigenvectors corresponding to different eigenvalues are orthogonal to each other: if $w_{i'}$ is an eigenvector of $I_{i'j'}$ with eigenvalue μ, $I_{i'j'} w_{j'} = \mu w_{i'}$, then, proceeding as above,

$$I_{i'j'} v_{i'} w_{j'} = v_{i'} \mu w_{i'},$$

and

$$I_{i'j'} v_{i'} w_{j'} = w_{j'} I_{j'i'} v_{i'} = w_{j'} \lambda v_j,$$

which leads to $(\lambda - \mu) v_{i'} w_{i'} = 0$, showing that if $\lambda \neq \mu$ then $v_{i'} w_{i'} = 0$, i.e., the vectors $v_{i'}$ and $w_{i'}$ are orthogonal to each other. Thus, if the three eigenvalues of $(I_{i'j'})$ are distinct, then the corresponding unit eigenvectors form an orthonormal basis.

When only two eigenvalues of $(I_{r's'})$ coincide, the corresponding eigenvectors form a two-dimensional plane, and any pair of orthogonal unit vectors of this plane will be part of an orthonormal basis formed by eigenvectors of $(I_{r's'})$. When the three eigenvalues of $(I_{r's'})$ coincide, then $(I_{r's'})$ is a multiple of the identity matrix and any orthonormal basis is formed by eigenvectors of $(I_{r's'})$.

In conclusion, in all cases we can find an orthonormal basis formed by eigenvectors of $(I_{r's'})$, and from (3.22) it follows that if the columns of the matrix B are three mutually orthogonal unit eigenvectors of the matrix $(I_{i'j'})$, then $(\tilde{I}_{i'j'})$ is diagonal.

If the matrix $(\tilde{I}_{i'j'})$ is diagonal, the directions defined by the basis vectors $\tilde{\mathbf{e}}_{i'}$ are called *principal axes* at O and the entries of $(\tilde{I}_{i'j'})$ along the diagonal are called *principal moments of inertia* [hence, the principal moments of inertia are the eigenvalues of $(I_{i'j'})$].

In Example 3.1 the matrix $(I_{r's'})$ is already diagonal, which means that the basis vectors $\mathbf{e}_{i'}$ point along the principal axes. The entries along the diagonal of $(I_{r's'})$ are the principal moments of inertia and, therefore, at least two principal moments of inertia coincide (the three principal moments of inertia coincide when $h = \sqrt{3}\,a$).

On the other hand, if we place the origin at the base of the cylinder, with the axes as shown in Figure 3.4, the matrix $(I_{i'j'})$ is given by (see Equation (3.28), below)

$$(I_{i'j'}) = \frac{M}{12}\begin{pmatrix} 3a^2+4h^2 & 0 & -6ah \\ 0 & 15a^2+4h^2 & 0 \\ -6ah & 0 & 18a^2 \end{pmatrix}.$$

The eigenvalues of $(I_{i'j'})$ (and, hence, the principal moments of inertia) are the roots of the polynomial

$$\begin{vmatrix} \frac{M}{12}(3a^2+4h^2)-\lambda & 0 & -\frac{M}{2}ah \\ 0 & \frac{M}{12}(15a^2+4h^2)-\lambda & 0 \\ -\frac{M}{2}ah & 0 & \frac{3M}{2}a^2-\lambda \end{vmatrix} = 0,$$

thus, the principal moments of inertia are

$$\frac{M}{12}(15a^2+4h^2), \qquad \frac{M}{24}\left(21a^2+4h^2 \pm \sqrt{225a^4+24a^2h^2+16h^4}\right).$$

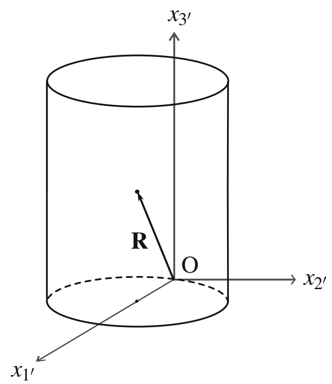

Fig. 3.4 The fixed point, O, is at the edge of the base of the cylinder. The $x_{1'}$-axis passes through the center of the base. The vector **R** goes from the origin to the center of mass

In order to simplify the expressions below, we shall consider the specific case where $h = \sqrt{3}\,a$, then, the principal moments of inertia are

$$\frac{9Ma^2}{4}, \qquad \frac{9Ma^2}{4}, \qquad \frac{Ma^2}{2}$$

and the matrix $(I_{i'j'})$ becomes

$$(I_{i'j'}) = \frac{Ma^2}{4}\begin{pmatrix} 5 & 0 & -2\sqrt{3} \\ 0 & 9 & 0 \\ -2\sqrt{3} & 0 & 6 \end{pmatrix}.$$

The eigenvectors of this matrix corresponding to the (repeated) eigenvalue $9Ma^2/4$ are determined by the homogeneous system of linear equations [see (3.21)]

$$\frac{Ma^2}{4}\begin{pmatrix} 5 & 0 & -2\sqrt{3} \\ 0 & 9 & 0 \\ -2\sqrt{3} & 0 & 6 \end{pmatrix}\begin{pmatrix} v_{1'} \\ v_{2'} \\ v_{3'} \end{pmatrix} = \frac{9Ma^2}{4}\begin{pmatrix} v_{1'} \\ v_{2'} \\ v_{3'} \end{pmatrix},$$

which gives $2v_{1'} + \sqrt{3}\,v_{3'} = 0$, with $v_{2'}$ arbitrary. These conditions define a two-dimensional plane, and two solutions of these conditions are $v_{1'} = -\sqrt{3/7}$, $v_{2'} = 0$, $v_{3'} = 2/\sqrt{7}$, and $v_{1'} = 0$, $v_{2'} = 1$, $v_{3'} = 0$, which correspond to two mutually orthogonal unit vectors. Hence, the two unit vectors

$$\tilde{\mathbf{e}}_{2'} \equiv \mathbf{e}_{2'}, \qquad \tilde{\mathbf{e}}_{3'} \equiv -\sqrt{\frac{3}{7}}\,\mathbf{e}_{1'} + \frac{2}{\sqrt{7}}\,\mathbf{e}_{3'}$$

point along principal axes (the labeling of the vectors $\tilde{\mathbf{e}}_{i'}$ is completely arbitrary).

According to the discussion above, the eigenvectors corresponding to the third eigenvalue, which is different from the first two, must be orthogonal to $\tilde{\mathbf{e}}_{2'}$ and $\tilde{\mathbf{e}}_{3'}$. Thus, we can find the third principal axis by means of the cross product $\tilde{\mathbf{e}}_{2'} \times \tilde{\mathbf{e}}_{3'}$. Letting $\tilde{\mathbf{e}}_{1'} \equiv \tilde{\mathbf{e}}_{2'} \times \tilde{\mathbf{e}}_{3'}$ we obtain the missing element of a positively oriented orthonormal basis such that the vectors $\tilde{\mathbf{e}}_{i'}$ point along principal axes. We find

$$\tilde{\mathbf{e}}_{1'} = \frac{2}{\sqrt{7}}\,\mathbf{e}_{1'} + \sqrt{\frac{3}{7}}\,\mathbf{e}_{3'}.$$

(One can readily verify that the vector given by $v_{1'} = 2/\sqrt{7}$, $v_{2'} = 0$, $v_{3'} = \sqrt{3/7}$ is indeed an eigenvector of $(I_{i'j'})$ with eigenvalue $Ma^2/2$.)

Exercise 3.2. Four particles of mass m are at the points $(a, 0, 0)$, $(0, a, 0)$, $(a, a, 0)$, and $(0, 0, 0)$, with respect to the Cartesian axes $x_{i'}$, where a is a positive constant. Find the principal axes and the principal moments of inertia.

Angular Momentum

From the elementary definition of the angular momentum of a particle ($\mathbf{L} = \mathbf{r} \times m\dot{\mathbf{r}}$) it follows that the Cartesian components of the angular momentum of the rigid body (with respect to the inertial frame) are given by

$$L_i = \sum_{\alpha=1}^{N} m_\alpha \varepsilon_{ijk} x_j^{(\alpha)} \dot{x}_k^{(\alpha)}.$$

According to (3.5) and (3.14), we have

$$\dot{x}_k^{(\alpha)} = \dot{a}_{ki'} x_{i'}^{(\alpha)} = \varepsilon_{i'j's'} \omega_{s'} a_{kj'} x_{i'}^{(\alpha)},$$

hence,

$$\begin{aligned} L_i &= \sum_{\alpha=1}^{N} m_\alpha \varepsilon_{ijk} x_j^{(\alpha)} \varepsilon_{i'j's'} \omega_{s'} a_{kj'} x_{i'}^{(\alpha)} \\ &= \sum_{\alpha=1}^{N} m_\alpha \varepsilon_{ijk} a_{jq'} x_{q'}^{(\alpha)} \varepsilon_{i'j's'} \omega_{s'} a_{kj'} x_{i'}^{(\alpha)}. \end{aligned}$$

Noting that, owing to (3.4) and (3.12),

$$\begin{aligned} \varepsilon_{ijk} a_{jq'} a_{kj'} &= \delta_{ip} \varepsilon_{pjk} a_{jq'} a_{kj'} \\ &= a_{ir'} a_{pr'} \varepsilon_{pjk} a_{jq'} a_{kj'} \\ &= a_{ir'} \varepsilon_{r'q'j'}, \end{aligned} \tag{3.24}$$

where we have used that the determinant of an orthogonal matrix that does not invert the orientation is equal to $+1$ (which follows from $1 = \det I = \det(A^{\mathrm{t}} A) = \det A^{\mathrm{t}} \det A = (\det A)^2$), with the aid of (3.10) and (3.11), we have

$$\begin{aligned} L_i &= \sum_{\alpha=1}^{N} m_\alpha a_{ir'} \varepsilon_{r'q'j'} \varepsilon_{i'j's'} \omega_{s'} x_{q'}^{(\alpha)} x_{i'}^{(\alpha)} \\ &= a_{ir'} \sum_{\alpha=1}^{N} m_\alpha (\delta_{r's'} \delta_{q'i'} - \delta_{r'i'} \delta_{q's'}) \omega_{s'} x_{q'}^{(\alpha)} x_{i'}^{(\alpha)} \\ &= a_{ir'} \sum_{\alpha=1}^{N} m_\alpha (\delta_{r's'} \mathbf{r}_\alpha{}^2 - x_{r'}^{(\alpha)} x_{s'}^{(\alpha)}) \, \omega_{s'} \\ &= a_{ir'} I_{r's'} \omega_{s'}. \end{aligned}$$

This means that $I_{r's'} \omega_{s'}$ is the r-th Cartesian component of the angular momentum of the body with respect to the basis vectors $\mathbf{e}_{i'}$,

$$L_{r'} = I_{r's'} \omega_{s'}. \tag{3.25}$$

Among other things, Equation (3.25) means that the angular velocity and the angular momentum may not be collinear, but when $\omega_{s'}$ is an eigenvector of the matrix $(I_{r's'})$ then the angular momentum and the angular velocity are collinear. Thus, the principal axes are the directions where the angular momentum and the angular velocity are collinear.

Exercise 3.3. Show that

$$A\dot{A}^{\mathrm{t}} = \begin{pmatrix} 0 & \omega_3 & -\omega_2 \\ -\omega_3 & 0 & \omega_1 \\ \omega_2 & -\omega_1 & 0 \end{pmatrix},$$

where the ω_i are the components of the angular velocity with respect to the basis vectors $\mathbf{e}_i$ (the inertial frame), that is,

$$a_{is'}\dot{a}_{js'} = \varepsilon_{ijk}\omega_k$$

[cf. Equation (3.7)].

Parallel Axes Theorem

Now we shall study the behavior of the inertia tensor under a parallel translation of the axes fixed in the body. To this end, it is convenient to consider two parallel sets of Cartesian axes fixed in the body, $x_{i'}$ and $y_{i'}$ (see Figure 3.5), defined in the following way. The origin of the axes $x_{i'}$ is located at an arbitrary point of the body, while the origin of the axes $y_{i'}$ is at the center of mass of the rigid body, this means that

$$\sum_{\alpha=1}^{N} m_\alpha y_{i'}^{(\alpha)} = 0, \qquad i' = 1', 2', 3', \tag{3.26}$$

where $(y_{1'}^{(\alpha)}, y_{2'}^{(\alpha)}, y_{3'}^{(\alpha)})$ are the Cartesian coordinates of the α-th particle of the body with respect to the axes with origin at the center of mass.

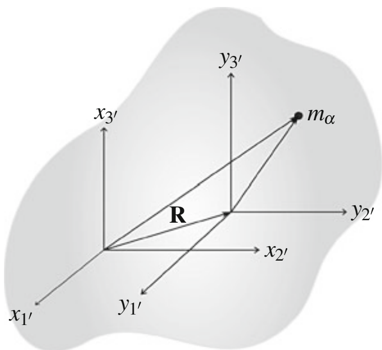

Fig. 3.5 The Cartesian axes $x_{i'}$ have their origin at an arbitrary point fixed in the body, and the origin of the Cartesian axes $y_{i'}$ is at the center of mass. The vector **R** is the position vector of the center of mass with respect to the axes $x_{i'}$. The axes $y_{i'}$ are parallel to the axes $x_{i'}$

If $(R_{1'}, R_{2'}, R_{3'})$ are the coordinates of the center of mass with respect to the axes $x_{i'}$, we have (see Figure 3.5)

$$x_{i'}^{(\alpha)} = R_{i'} + y_{i'}^{(\alpha)}, \qquad \alpha = 1, 2, \ldots, N;\ i' = 1', 2', 3'.$$

Hence, making use of the definition of the (components of the) inertia tensor (3.15) and Equation (3.26), we have

$$\begin{aligned}
I_{j'k'} &= \sum_{\alpha=1}^{N} m_\alpha \big(\delta_{j'k'} x_{i'}^{(\alpha)} x_{i'}^{(\alpha)} - x_{j'}^{(\alpha)} x_{k'}^{(\alpha)}\big) \\
&= \sum_{\alpha=1}^{N} m_\alpha \big[\delta_{j'k'} (R_{i'} + y_{i'}^{(\alpha)})(R_{i'} + y_{i'}^{(\alpha)}) - (R_{j'} + y_{j'}^{(\alpha)})(R_{k'} + y_{k'}^{(\alpha)})\big] \\
&= (R_{i'} R_{i'} \delta_{j'k'} - R_{j'} R_{k'}) \sum_{\alpha=1}^{N} m_\alpha + \sum_{\alpha=1}^{N} m_\alpha \big(\delta_{j'k'} y_{i'}^{(\alpha)} y_{i'}^{(\alpha)} - y_{j'}^{(\alpha)} y_{k'}^{(\alpha)}\big) \\
&= M(\mathbf{R}^2 \delta_{j'k'} - R_{j'} R_{k'}) + I_{j'k'}^{\mathrm{CM}}, \qquad (3.27)
\end{aligned}$$

where M is the total mass of the body, $\mathbf{R}^2 = R_{i'} R_{i'}$ is the square of the norm of the vector $(R_{1'}, R_{2'}, R_{3'})$, and the $I_{j'k'}^{\mathrm{CM}}$ are the components of the inertia tensor, taking the center of mass as the fixed point of the rigid body. This result is known as the parallel axes theorem.

For instance, considering again the homogeneous circular cylinder of Example 3.1, with the aid of Equation (3.27) we can readily obtain the components of the inertia tensor taking one point at the edge of the base of the cylinder as the fixed point (see Figure 3.4). If the $x_{1'}$-axis lies along a diameter of the base of the cylinder, then $(R_{1'}, R_{2'}, R_{3'}) = (a, 0, h/2)$ and from Equations (3.27) and (3.18) we find

$$\begin{aligned}
(I_{i'j'}) &= \frac{M}{4} \begin{pmatrix} h^2 & 0 & -2ah \\ 0 & 4a^2 + h^2 & 0 \\ -2ah & 0 & 4a^2 \end{pmatrix} + \frac{M}{12} \begin{pmatrix} 3a^2 + h^2 & 0 & 0 \\ 0 & 3a^2 + h^2 & 0 \\ 0 & 0 & 6a^2 \end{pmatrix} \\
&= \frac{M}{12} \begin{pmatrix} 3a^2 + 4h^2 & 0 & -6ah \\ 0 & 15a^2 + 4h^2 & 0 \\ -6ah & 0 & 18a^2 \end{pmatrix}. \qquad (3.28)
\end{aligned}$$

Exercise 3.4. Show that if the line joining O and the center of mass is parallel to one of the principal axes at the center of mass, then this line is also parallel to a principal axis at O. Furthermore, any principal axis at the center of mass orthogonal to the line is parallel to a principal axis at O.

Coordinate-Free Expression of the Lagrange Equations. The Euler Equations
So far, we have not required the introduction of coordinates to parameterize the configuration of the rigid body, and as we shall see below and in Section 4.2, there are some results that can be obtained without giving an explicit expression for the matrix elements $a_{ij'}$ in terms of coordinates.

Assuming that the $a_{ij'}$ are parameterized by some coordinates q_s, from (3.7) and the chain rule we have

$$\frac{\partial a_{ki'}}{\partial q_r}\dot{q}_r a_{kj'} = \varepsilon_{i'j's'}\omega_{s'},$$

then, making use of (3.11),

$$\omega_{s'} = \frac{1}{2}\varepsilon_{i'j's'}\frac{\partial a_{ki'}}{\partial q_r}\dot{q}_r a_{kj'} = M_{s'r}\dot{q}_r, \tag{3.29}$$

where we have introduced the functions

$$M_{s'r} \equiv \frac{1}{2}\varepsilon_{i'j's'}\frac{\partial a_{ki'}}{\partial q_r}a_{kj'}, \tag{3.30}$$

which depend on the coordinates q_s only, and relate the angular velocity with the generalized velocities $\dot{q}_i$. The last equation is equivalent to the relation

$$\frac{\partial a_{ki'}}{\partial q_r} = a_{kj'}\varepsilon_{i'j's'}M_{s'r}. \tag{3.31}$$

According to Equations (3.31), (3.10), and (3.11) the second partial derivatives of $a_{ki'}$ are given by

$$\begin{aligned}
\frac{\partial^2 a_{ki'}}{\partial q_m \partial q_r} &= \frac{\partial a_{kj'}}{\partial q_m}\varepsilon_{i'j's'}M_{s'r} + a_{kj'}\varepsilon_{i'j's'}\frac{\partial M_{s'r}}{\partial q_m} \\
&= a_{kp'}\varepsilon_{j'p'n'}M_{n'm}\varepsilon_{i'j's'}M_{s'r} + a_{kj'}\varepsilon_{i'j's'}\frac{\partial M_{s'r}}{\partial q_m} \\
&= (\delta_{p's'}\delta_{n'i'} - \delta_{p'i'}\delta_{n's'})a_{kp'}M_{n'm}M_{s'r} + a_{kj'}\varepsilon_{i'j's'}\frac{\partial M_{s'r}}{\partial q_m} \\
&= a_{ks'}M_{i'm}M_{s'r} - a_{ki'}M_{s'm}M_{s'r} + a_{kj'}\varepsilon_{i'j's'}\frac{\partial M_{s'r}}{\partial q_m}.
\end{aligned}$$

Then, the commutativity of the partial derivatives of $a_{ki'}$ is equivalent to

$$\begin{aligned}
a_{kj'}\varepsilon_{i'j's'}\left(\frac{\partial M_{s'r}}{\partial q_m} - \frac{\partial M_{s'm}}{\partial q_r}\right) &= a_{ks'}(M_{i'r}M_{s'm} - M_{i'm}M_{s'r}) \\
&= a_{kj'}(M_{i'r}M_{j'm} - M_{i'm}M_{j'r}),
\end{aligned}$$

hence,

$$\varepsilon_{i'j's'}\left(\frac{\partial M_{s'r}}{\partial q_m} - \frac{\partial M_{s'm}}{\partial q_r}\right) = M_{i'r}M_{j'm} - M_{i'm}M_{j'r}.$$

With the aid of (3.11) one finds that the last equation is equivalent to

$$\frac{\partial M_{s'r}}{\partial q_m} - \frac{\partial M_{s'm}}{\partial q_r} = \varepsilon_{i'j's'} M_{i'r} M_{j'm}. \tag{3.32}$$

Note that these equations must hold for any choice of the coordinates q_i. (It turns out that Equations (3.32) are related to the structure of the rotation group itself.)

Assuming that the applied forces on the body are derivable from a potential $V(q_i)$, the equations of motion of the rigid body can be obtained substituting the Lagrangian [see Equation (3.16)]

$$L = \tfrac{1}{2} I_{j'k'} \omega_{j'} \omega_{k'} - V \tag{3.33}$$

into the Lagrange equations. Making use of the symmetry of $I_{j'k'}$, (3.25), (3.29), and (3.32), we obtain

$$\begin{aligned}
0 &= \frac{\mathrm{d}}{\mathrm{d}t} \frac{\partial}{\partial \dot{q}_i} \left(\tfrac{1}{2} I_{j'k'} \omega_{j'} \omega_{k'} \right) - \frac{\partial}{\partial q_i} \left(\tfrac{1}{2} I_{j'k'} \omega_{j'} \omega_{k'} - V \right) \\
&= \frac{\mathrm{d}}{\mathrm{d}t} \left(I_{j'k'} \omega_{k'} \frac{\partial \omega_{j'}}{\partial \dot{q}_i} \right) - I_{j'k'} \omega_{k'} \frac{\partial \omega_{j'}}{\partial q_i} + \frac{\partial V}{\partial q_i} \\
&= \frac{\mathrm{d}}{\mathrm{d}t} \left(L_{j'} M_{j'i} \right) - L_{j'} \frac{\partial M_{j'k}}{\partial q_i} \dot{q}_k + \frac{\partial V}{\partial q_i} \\
&= M_{j'i} \frac{\mathrm{d}L_{j'}}{\mathrm{d}t} + L_{j'} \frac{\partial M_{j'i}}{\partial q_k} \dot{q}_k - L_{j'} \frac{\partial M_{j'k}}{\partial q_i} \dot{q}_k + \frac{\partial V}{\partial q_i} \\
&= M_{j'i} \frac{\mathrm{d}L_{j'}}{\mathrm{d}t} + L_{j'} \left(\frac{\partial M_{j'i}}{\partial q_k} - \frac{\partial M_{j'k}}{\partial q_i} \right) \dot{q}_k + \frac{\partial V}{\partial q_i} \\
&= M_{j'i} \frac{\mathrm{d}L_{j'}}{\mathrm{d}t} + L_{j'} \varepsilon_{r's'j'} M_{r'i} M_{s'k} \dot{q}_k + \frac{\partial V}{\partial q_i} \\
&= M_{r'i} \left(\frac{\mathrm{d}L_{r'}}{\mathrm{d}t} + \varepsilon_{r's'j'} \omega_{s'} L_{j'} \right) + \frac{\partial V}{\partial q_i},
\end{aligned}$$

that is,

$$\frac{\mathrm{d}L_{r'}}{\mathrm{d}t} + \varepsilon_{r's'j'} \omega_{s'} L_{j'} = -(M^{-1})_{ir'} \frac{\partial V}{\partial q_i}, \tag{3.34}$$

where the $(M^{-1})_{ir'}$ are the entries of the inverse of the matrix $(M_{r'i})$. The right-hand side of (3.34) is the r-th component of the torque on the rigid body, $\tau_{r'}$, with respect to the basis fixed in the body. In fact, with the aid of Equations (3.14) and (3.10) we find that

$$\begin{aligned}\frac{\mathrm{d}L_i}{\mathrm{d}t} &= \frac{\mathrm{d}(a_{ir'}L_{r'})}{\mathrm{d}t} \\ &= a_{ir'}\frac{\mathrm{d}L_{r'}}{\mathrm{d}t} + \varepsilon_{r'k's'}\omega_{s'}a_{ik'}L_{r'} \\ &= a_{ir'}\left(\frac{\mathrm{d}L_{r'}}{\mathrm{d}t} + \varepsilon_{r's'k'}\omega_{s'}L_{k'}\right),\end{aligned}$$

which shows that, indeed, the left-hand side of (3.34) is the r-th component of the torque on the rigid body with respect to the basis fixed in the body. Equations (3.34), written in the form

$$\frac{\mathrm{d}L_{r'}}{\mathrm{d}t} + \varepsilon_{r's'j'}\omega_{s'}L_{j'} = \tau_{r'},$$

are known as the Euler equations for a rigid body with a fixed point. As we have shown, these equations are equivalent to the Lagrange equations for the Lagrangian (3.33).

A particular case corresponds to the motion of the rigid body with the torque equal to zero. Then, the Euler equations reduce to

$$\frac{\mathrm{d}L_{r'}}{\mathrm{d}t} + \varepsilon_{r's'j'}\omega_{s'}L_{j'} = 0.$$

If the matrix $(I_{i'j'})$ is diagonal, these equations expressed in terms of the components of the angular velocity $\omega_{i'}$ take the form [see (3.25)]

$$\begin{aligned}I_1\frac{\mathrm{d}\omega_{1'}}{\mathrm{d}t} + (I_3 - I_2)\omega_{2'}\omega_{3'} &= 0, \\ I_2\frac{\mathrm{d}\omega_{2'}}{\mathrm{d}t} + (I_1 - I_3)\omega_{3'}\omega_{1'} &= 0, \\ I_3\frac{\mathrm{d}\omega_{3'}}{\mathrm{d}t} + (I_2 - I_1)\omega_{1'}\omega_{2'} &= 0,\end{aligned} \tag{3.35}$$

where the I_i are the principal moments of inertia ($I_1 \equiv I_{1'1'}$, $I_2 \equiv I_{2'2'}$, $I_3 \equiv I_{3'3'}$).

Exercise 3.5. A rigid body is symmetric if two of its principal moments of inertia coincide. Solve Equations (3.35) for a symmetric rigid body. Note that this solution only gives the angular velocity as a function of time; in order to find the configuration of the body we would still have to solve another system of ODEs [e.g., (3.14) or (3.38)].

Exercise 3.6. Making use of the Euler equations (3.35), show that the kinetic energy and the total angular momentum of the rigid body are constants of motion, that is,

$$\tfrac{1}{2}(I_1{\omega_{1'}}^2 + I_2{\omega_{2'}}^2 + I_3{\omega_{3'}}^2) \qquad \text{and} \qquad {I_1}^2{\omega_{1'}}^2 + {I_2}^2{\omega_{2'}}^2 + {I_3}^2{\omega_{3'}}^2$$

are conserved [see (3.16) and (3.25)]. (With the aid of these two constants of motion, Equations (3.35) can be reduced to a single first-order ODE whose solution, in general, involves elliptic functions.)

3.3 The Euler Angles

A usual and convenient set of generalized coordinates for a rigid body with a fixed point is given by the so-called Euler angles. The matrix $A = (a_{ij'})$, representing the configuration of a rigid body, is expressed in the form

$$A = R_z(\phi) R_x(\theta) R_z(\psi), \tag{3.36}$$

where $R_z(\phi)$ is the 3×3 orthogonal matrix (3.2), corresponding to a rotation about the z-axis through an angle ϕ and, similarly,

$$R_x(\theta) \equiv \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & -\sin\theta \\ 0 & \sin\theta & \cos\theta \end{pmatrix}$$

corresponds to a rotation about the x-axis through an angle θ. The angles ϕ, θ, ϕ are called Euler angles and are restricted by $0 \leqslant \phi \leqslant 2\pi$, $0 \leqslant \theta \leqslant \pi$, $0 \leqslant \psi \leqslant 2\pi$. (A slightly different definition, which is especially convenient in the study of the rotation group in quantum mechanics, is given by $A = R_z(\phi) R_y(\theta) R_z(\psi)$, where $R_y(\theta)$ corresponds to a rotation about the y-axis through an angle θ.)

According to Equation (3.16), in order to write the kinetic energy in terms of the Euler angles and their time derivatives, we need the explicit expression of the components of the angular velocity, $\omega_{i'}$, in terms of those variables. These expressions can be readily obtained with the aid of Equation (3.9) by calculating the product $\dot{A}^{\mathrm{t}} A$ (without having to resort to a geometrical image or to the consideration of "infinitesimal rotations"). Using the fact that $(AB)^{\mathrm{t}} = B^{\mathrm{t}} A^{\mathrm{t}}$ and that each matrix appearing in (3.36) is orthogonal, we find

$$\begin{aligned} \dot{A}^{\mathrm{t}} A &= \left[\dot{R}_z(\phi) R_x(\theta) R_z(\psi) + R_z(\phi) \dot{R}_x(\theta) R_z(\psi) + R_z(\phi) R_x(\theta) \dot{R}_z(\psi)\right]^{\mathrm{t}} \\ &\quad \times R_z(\phi) R_x(\theta) R_z(\psi) \\ &= R_z(\psi)^{\mathrm{t}} R_x(\theta)^{\mathrm{t}} \dot{R}_z(\phi)^{\mathrm{t}} R_z(\phi) R_x(\theta) R_z(\psi) + R_z(\psi)^{\mathrm{t}} \dot{R}_x(\theta)^{\mathrm{t}} R_x(\theta) R_z(\psi) \\ &\quad + \dot{R}_z(\psi)^{\mathrm{t}} R_z(\psi). \end{aligned} \tag{3.37}$$

The last term in Equation (3.37) was already calculated in Section 3.2; the result is

$$\dot{R}_z(\psi)^{\mathrm{t}} R_z(\psi) = \dot{\psi} \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

A similar computation gives

$$\dot{R}_x(\theta)^{\mathrm{t}} R_x(\theta) = \dot{\theta} \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{pmatrix}$$

and, therefore,

$$R_z(\psi)^{\mathrm{t}} \dot{R}_x(\theta)^{\mathrm{t}} R_x(\theta) R_z(\psi) = \dot{\theta} \begin{pmatrix} 0 & 0 & \sin\psi \\ 0 & 0 & \cos\psi \\ -\sin\psi & -\cos\psi & 0 \end{pmatrix}.$$

A more lengthy computation gives

$$\begin{aligned} &R_z(\psi)^{\mathrm{t}} R_x(\theta)^{\mathrm{t}} \dot{R}_z(\phi)^{\mathrm{t}} R_z(\phi) R_x(\theta) R_z(\psi) \\ &\qquad = \dot{\phi} \begin{pmatrix} 0 & \cos\theta & -\sin\theta\cos\psi \\ -\cos\theta & 0 & \sin\theta\sin\psi \\ \sin\theta\cos\psi & -\sin\theta\sin\psi & 0 \end{pmatrix}. \end{aligned}$$

Adding these expressions and comparing the result with (3.9) we conclude that the components of the angular velocity of the body, with respect to the axes fixed in the body, are

$$\begin{aligned} \omega_{1'} &= \dot{\phi}\sin\theta\sin\psi + \dot{\theta}\cos\psi, \\ \omega_{2'} &= \dot{\phi}\sin\theta\cos\psi - \dot{\theta}\sin\psi, \\ \omega_{3'} &= \dot{\phi}\cos\theta + \dot{\psi}. \end{aligned} \tag{3.38}$$

Thus, assuming that the matrix $(I_{i'j'})$ is diagonal, from (3.16) and (3.38) we have the expression for the kinetic energy in terms of the Euler angles

$$\begin{aligned} T = {} & \tfrac{1}{2}[I_1(\dot{\phi}\sin\theta\sin\psi + \dot{\theta}\cos\psi)^2 + I_2(\dot{\phi}\sin\theta\cos\psi - \dot{\theta}\sin\psi)^2 \\ & + I_3(\dot{\phi}\cos\theta + \dot{\psi})^2]. \end{aligned} \tag{3.39}$$

When two principal moments of inertia coincide, it is convenient to select the axes in such a way that $I_1 = I_2$ because then (3.39) reduces to

$$T = \tfrac{1}{2}[I_1(\dot{\phi}^2\sin^2\theta + \dot{\theta}^2) + I_3(\dot{\phi}\cos\theta + \dot{\psi})^2]. \tag{3.40}$$

Example 3.7 (Symmetric top in a uniform gravitational field). A commonly studied example is that of a symmetric top in a uniform gravitational field. This problem consists of an axially symmetric top with a fixed point, in a uniform gravitational field. Assuming that the x_3-axis points upwards and taking the $x_{3'}$-axis as the symmetry axis of the top, we have $I_1 = I_2$. Then, making use of (3.40) we find that the standard Lagrangian is

$$L = \tfrac{1}{2}[I_1(\dot{\phi}^2 \sin^2\theta + \dot{\theta}^2) + I_3(\dot{\phi}\cos\theta + \dot{\psi})^2] - Mgl\cos\theta, \tag{3.41}$$

where M is the mass of the top and l is the distance between the fixed point of the body (which is placed at the origin) and the center of mass. (The product $l\cos\theta$ is the height of the center of mass with respect to the origin since, according to (3.1), the components of the vector $\mathbf{e}_{3'}$ with respect to the basis formed by the vectors $\mathbf{e}_i$ appear in the third column of the matrix A [see (3.36)], which is given by

$$R_z(\phi)R_x(\theta)R_z(\psi)\begin{pmatrix}0\\0\\1\end{pmatrix} = R_z(\phi)R_x(\theta)\begin{pmatrix}0\\0\\1\end{pmatrix} = R_z(\phi)\begin{pmatrix}0\\-\sin\theta\\\cos\theta\end{pmatrix}$$
$$= \begin{pmatrix}\cos\phi & -\sin\phi & 0\\ \sin\phi & \cos\phi & 0\\ 0 & 0 & 1\end{pmatrix}\begin{pmatrix}0\\-\sin\theta\\\cos\theta\end{pmatrix} = \begin{pmatrix}\sin\theta\sin\phi\\-\sin\theta\cos\phi\\\cos\theta\end{pmatrix}$$

and the position vector of the center of mass is $l\mathbf{e}_{3'}$.)

As in previous examples, it is not convenient to obtain the equations of motion by substituting the Lagrangian (3.41) into the Lagrange equations and then try to solve them. It is preferable to use the fact that the coordinates ϕ and ψ are ignorable and that the Lagrangian does not depend on t; this implies that

$$\frac{\partial L}{\partial\dot{\phi}} = I_1\dot{\phi}\sin^2\theta + I_3\cos\theta(\dot{\phi}\cos\theta + \dot{\psi}), \qquad \frac{\partial L}{\partial\dot{\psi}} = I_3(\dot{\phi}\cos\theta + \dot{\psi})$$

as well as

$$\tfrac{1}{2}[I_1(\dot{\phi}^2 \sin^2\theta + \dot{\theta}^2) + I_3(\dot{\phi}\cos\theta + \dot{\psi})^2] + Mgl\cos\theta$$

are constants of motion. Denoting as a, b, and E, respectively, the values of these constants of motion, the combination of the foregoing expressions leads to the first-order ODE

$$E = \frac{1}{2}\left[I_1\dot{\theta}^2 + \frac{(a - b\cos\theta)^2}{I_1\sin^2\theta} + \frac{b^2}{I_3}\right] + Mgl\cos\theta,$$

which determines θ as a function of the time [cf. Equation (2.32)]. The substitution $u = \cos\theta$ yields the equivalent equation

$$\frac{1}{2}I_1\dot{u}^2 + \frac{(a - bu)^2}{2I_1} + \left(\frac{b^2}{2I_3} - E\right)(1 - u^2) + Mglu(1 - u^2) = 0. \tag{3.42}$$

Since the "effective potential,"

$$\frac{(a-bu)^2}{2I_1} + \left(\frac{b^2}{2I_3} - E\right)(1-u^2) + Mglu(1-u^2),$$

is a third-degree polynomial in u, the solution of (3.42) involves elliptic functions. Alternatively, one can find the qualitative behavior of the solutions with the aid of the graph of the effective potential.