Chapter 1
The Epistemic View Upon Science

Abstract This chapter gives the background for the book. Its relation to other views on the foundation of quantum theory are clarified and discussed. The fundamental notion of an e-variable (epistemic conceptual variable) is explained, it is discussed and is related to the statistical parameter-concept. A quantum state is in some generality linked to a question-and-answer pair, and an experiment connected to such a question-and-answer pair is described for the case of a spin 1/2 particle. The two basic postulates of quantum theory are stated and discussed. The importance of inaccessible conceptual variables is stressed, and this is related to Bohr complementarity.

1.1 Introduction

The aim of science is to gain knowledge about the external world; this is what we mean by an epistemic process. In its most primitive form, the process of achieving knowledge can be described by what Brody (1993) called an epistemic cycle: "Act, and see what happens". Experiments in laboratories and observational studies done by scientists are usually much more sophisticated than this; they often require several epistemic cycles and also higher order epistemic cycles acting upon the first order cycles. An experiment or an observational study is always focused on some concrete system, it involves concrete experimental/observational questions and it is always done in a context, which might depend on conceptual formulations; in addition the context may be partly historical and partly chosen by the scientist himself, or depending upon the scientist.

In earlier years, experiments were often done by single scientists; now it is more and more common that people are working in teams. Also, results of experiments should be communicated to many people. This calls for a conceptual basis which is common to a whole culture of scientists. One problem, however, is that people from different scientific cultures have difficulties with communicating. They might not have a common language. The first purpose of this book is to develop a scientific language for achieving knowledge which is a synthesis of the languages that I have met in the three cultures that have been exposed to myself: (1) Mathematical statistics; (2) Quantum mechanics; (3) Applied statistics

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including simple applications. It is a hope that this investigation may lead to a deeper understanding of the epistemic process itself. It is also a hope that such an investigation may be continued in order to include more scientific cultures, say, machine learning and quantum computation.

Since statistical inference is used as a tool in very many experimental studies, also within physics, it is natural to take this culture as a point of departure. But I will add some elements which are not very common in the statistical literature:

1. I make explicit that every experimental investigation is made in a context.
2. A transformation group may be added to the statistical model.
3. Model reductions by means of such groups are introduced.
4. In order to address also the physicists, the parameter concept is introduced through the more general term *epistemic conceptual variable* (e-variable). Any variable which can be defined in words by a person or by a group of persons in an experimental situation is called a conceptual variable. The notion of an e-variable at the outset includes every conceptual variable involved in an epistemic process, and an e-variable can also be connected to a single unit (say to a single human being in a sociological or psychological investigation or to a single particle in physics). The basic aim of an epistemic process is to gain some knowledge about the relevant e-variables. It turns out, however, that even in the basic quantum mechanical situation, the term 'e-variable' might in principle from a statistical point of view be replaced by 'parameter'. The problem is that the latter word is so over-burdened in physics. It is of course important to stress that whenever I say 'e-variable' in connection to an ordinary statistical investigation, this term can be replaced by 'parameter'.
5. To give a conceptual basis for causality theory and ultimately for finding a link to quantum theory, I will also introduce *inaccessible conceptual variables*, that is, conceptual variables which cannot be estimated or given any value with arbitrary accuracy in any experiment. Versions of such unobservable variables can be found in counterfactual situations, but the notion is also relevant, say, in connection to regression models where the number of variables by necessity is larger than the number of units. We will see that this notion is crucial in quantum mechanics, where it can be linked to Niels Bohr's concept of complementarity.

Also, I have included the recent notion of confidence distributions, in order to allow both a frequentist and a Bayesian basis for any given experimental investigation.

This framework as further developed in the present book will lead to a non-formal way to discuss essential elements of quantum theory, a theme which occupies later chapters of this book, and is also discussed further in this chapter. The usual basis for quantum theory as developed by von Neumann (1932) was a great achievement, and the language that is implied by this basis is used for all further theoretical developments and for all discussions among physicists. It is a strong intrinsic part of the quantum mechanical culture, in fact, of the culture shared by the whole community of modern physicists. But since the traditional language is purely formal and has little or no intuitive basis for people outside the community of physicists and mathematicians, it seems to be of some interest to develop an alternative language for discussing aspects of quantum theory, even if this language is somewhat limited compared to the ordinary quantum language.

Several recent investigators in quantum foundations have reasoned that quantum mechanics should be interpreted as an epistemic science. I agree with this. But I see it as somewhat problematic that this notion of an epistemic science should be connected to one language in fundamental physics and a completely different language in the rest of empirical science. One purpose of this book is to argue for a new language concerning the concept of a quantum state. I will keep the notion of a quantum state defined as a unit vector (or ray) in a complex Hilbert space. But in many connections this notion can be replaced by the following: One poses a focused question about the system under consideration: 'What is the value of θ?' and obtains a definite answer: $\theta = u_k$. Here θ is an e-variable/ observable (to be further discussed below), and u_k is one of the values that θ can take. There are open ends of the present programme as far as quantum physics is concerned, but I will argue that the investigations can be carried on further along the same lines.

The sceptic might ask: What is the purpose of introducing a new language when this does not lead to anything new? My first answer is that a simple language may be of importance in communication between people. For those who know this paradox, it may be that Wigner's friend is ignorant of the formal language of quantum theory. Nevertheless, he might have an intuitive feeling of what it means that the spin component in direction a of a particle is $+1$, and from this he might be able to communicate his state notion to Wigner, and the two will then share a common state for the physical system.

My second answer is that I will show that my programme indeed leads to something essentially new, also within the science of quantum mechanics itself: The Born formula, which is the basis for all probability calculations in quantum physics, is taken as an independent axiom in textbooks. I will derive it from a set of intuitive assumptions. I am also able to discuss the problematic questions connected to Bell's inequalities by using an epistemic point of departure. Several so-called paradoxes can be resolved using the language of epistemic processes. Also questions around the derivation of the Schrödinger equation are discussed.

One aspect of my programme is to propose sort of a link between quantum theory and statistical inference, two cultures which until now have been completely separated. The study of scientific cultures is not common. An exception is the book by Knorr Cetina (1999), where the author describes from the inside epistemic cultures connected to two empirical groups: High energy physics experimenters at CERN and molecular biologists at a laboratory. Her arguments strongly depend upon the notion of knowledge societies. Of course I agree that the nature of knowledge is different in different scientific communities, but it is the process of *achieving knowledge* that I feel should have something in common, and it is this process I will focus upon in this book.

As already mentioned, the language for quantum mechanics used in the present book also implies a particular interpretation of the theory. Nearly since its introduction in the beginning of the previous century, the physical community has been divided on the question of the interpretation of quantum theory. On the one hand many has argued for an ontic interpretation of the quantum state: It is a real state of nature. But on the other hand other people has argued for an epistemic interpretation of the quantum state: It only describes an observer's knowledge of the state of nature. In my opinion some sort of a synthesis of the two views is called for. The phenomenon of collapse of the wave packet during a measurement and paradoxes such as that of the Schrödinger cat give strong arguments in order that the epistemic view should play an important part. This in itself calls for a thorough analysis of how the epistemic process can be, and this is part of the purpose of the present book. The observer and his context play an important role in this process. By verbal communication and with the help of time, several observers may develop a common context. The ontic state of a particular physical system is in this book identified with the hypothetical state that all potential ideal observers with a common context of relevance to this physical systems agree upon.

Recently, several related no-go theorems have appeared in the physical literature which have been taken as arguments that a pure epistemic view are inconsistent with the predictions of quantum mechanics. However, these theorems rely on certain specific assumptions. The epistemic toy model of Spekkens (2007), which reproduces very many aspects of quantum theory, shows that these assumptions are not necessarily satisfied in practice.

It is interesting that a specialization of my own theory is closely related the Spekkens toy model, and this in itself indicates that my theory is related to several aspects of quantum mechanics.

This book is the result of a long process. In Helland (2006) an approach towards quantum mechanics was made in a leading statistical journal. Part of this approach is implemented later in the present book. In Helland (2008) an approach was made in a good physical journal, but again I know now that the reasoning is not complete. In the book Helland (2010) it was attempted to have two cultures in the mind at the same time. The book contains some relatively deep results in group action theory and in mathematical statistics, but the attempts made there to prove a link to quantum theory are too simple. The main limitation of all these three references is that I there attempted to deduce quantum theory from a version of statistical theory. In the present book I just assume that the two theories can be seen to have a common basis.

1.2 Different Views on the Foundation of Quantum Mechanics

The ordinary textbook formulation of quantum mechanics is very abstract. Its starting point: 'The state of a physical system is a normalized vector in a separable Hilbert space' has lead to an extremely rich theory, a theory which has not been refuted by any experiment and whose predictions range over an extremely wide variety of situations. Nevertheless, it is still unclear how this state concept should be interpreted.

Many conferences on quantum foundation have been arranged in recent years, but this has only implied that the number of new interpretations have increased, and no one of the old have died out. In two of these conferences, a poll among the participants was carried out (Schlosshauer et al. 2013; Norsen and Nelson 2013). The result was an astonishing disagreement on several simple and fundamental questions. One of these questions was whether quantum theory should be interpreted as an objective theory of the world (the ontological interpretation) or if it only expresses our knowledge of the world (the epistemic interpretation). According to Webster's Unabridged Dictionary, the adjective 'epistemic' means 'of or pertaining to knowledge, or the conditions for acquiring it'.

Recently Fuchs (2010), Fuchs and Schack (2011), Fuchs et al. (2013) and others have argued for various versions of Quantum Bayesianism, a radical interpretation where the subjective observer plays an important part. See also the philosophical discussion by Timpson (2008), the popular account in von Baeyer (2013) and the recent book (von Baeyer 2016).

QBism is a way of thinking about science quite generally, not just quantum physics. To cite Mermin (2014):

'QBism maintains that my understanding of the world rests entirely on the experiences that the world has induced in me throughout the course of my life. Nothing beyond my personal experience underlies the picture that I have formed of my own external world.'

But on the other hand:

'Facile charges of solipsism miss the point. My experience of you leads to hypothesize that you are very much like myself, with your own private experience.'

In the communication between you and me—in general between human beings, scientists and others, we need commonly defined concepts. I will come back to this later in the book.

This book takes as a point of departure that in some sense or other the epistemic interpretation should be important for issues of the quantum world. The next question then arises: Can one find a new and more intuitive *foundation* of quantum theory, a foundation related to the epistemic interpretation? It is my view that such a foundation ought to have some relation to statistical inference theory, another scientific fundament, which gives tools for wide variety of empirical investigations, and which in its very essence is epistemic. It should also be a kind of decision theory, related to decisions taken in everyday life.

Since the classical Copenhagen interpretation, which can be made precise in slightly different ways, several groups of researchers have proposed different interpretations of quantum mechanics. An extreme view on this was recently given by Tammaro (2014), who claimed that all these interpretations were deficient. More precisely, Tammaro stated that no current interpretation was consistent with experiment, resolved the measurement problem, and was completely free from logical deficiencies or fine-tuning problems.

Central to this discussion was the measurement problem, in particular the reconciliation of the two possible modes of change that the wave function can take: (1) Discontinuous, indeterministic time evolution sending $|\psi\rangle$ into an eigenstate $|o_i\rangle$ of observable O as a result of measurement of O. (2) Unitary time evolution governed by the Schrödinger equation.

Tammaro claimed to demonstrate that these two processes are inconsistent. His argument considered the state of the observer plus system after a measurement by the observer has been made. Process (1) then generates a mixed state, while Process (2) generates a pure state. One possible view is as follows: That this represents an inconsistency, may be related to the assumption that this quantum state really represents some objective reality for the system plus observer. According to QBism, a quantum state is always connected to some agent, and at each time it represents the subjective reality of that agent. In particular, the above system plus observer may be observed by another agent, and the wave function of this 'Wigner' (relative to the first agent, 'Wigner's friend') at time t may represent the belief or knowledge that he has at time t. This may depend upon how he obtains knowledge about the system, the agent, and the system plus agent.

Discussions in the literature of the EPR paradox (Einstein et al. 1935) and of Bell's theorem (Bell 1987), usually end up with the statement that quantum mechanics must violate the assumption of local realism. Since the locality assumption is inherent in relativity theory, my view is that it is the assumption of realism which must be discussed further. The quantum state represents the subjective reality of an agent or by a group of communicating agents, and this is in principle all there is to it. When all real and imagined agents agree on some observation at time t, this may be regarded as an objective reality at time t.

This is a rather radical view upon what reality should mean to us, but it is not inconsistent with the fact that people in complex macroscopic situations also may have different world views, and that these different world views may be extremely difficult to reconcile. This analogy should not be taken too far, though. The agents of quantum theory are ideal observers, while the humans of the macro world are far from perfect. The QBism interpretation of quantum mechanics is still only held by a minority of physicists.

My own views on quantum mechanics follow the views of the QBists a long way, but I differ in two respects: (1) For the purpose of a given experiment, the single observer can be replaced by a group of communicating observers. Thus language and the forming of concepts are important issues. (2) As formulated originally, QBism was closely tied to the philosophy of Bayesianism. I want to look upon the inference from observation in physics as related to inference in statistics, and I want to allow also other philosophies behind statistical inference. The ideal observer might well be a Bayesian, but since we humans are imperfect, we must also be allowed to use frequentist methods.

In any case, both the QBists and I agree that epistemic processes are of importance in the understanding of quantum mechanics. The simplest epistemic process consists of at least two decisions by the relevant agent (or communicating agents): First a decision on what question to nature to focus upon. Then the experiment or collection of data itself. Then finally a decision on how to deduce from these data the answer to the question originally posed.

Statistical theory and practice are almost solely concentrated on the last decision here. The decisions connected to focusing are attempted included in the present book.

An interesting discussion of various interpretations of quantum mechanics can be found in Khrennikov (2014). Khrennikov (2016) discusses QBism from the point of view of general decision making.

1.3 Theory of Decisions: Focusing—Context

Classical decision theory has been used with great success in a variety of fields like economics, medicine and politics. It is the basis for much of statistical inference theory. Yukalov and Sornette (2008, 2010, 2011, 2014) have in a series of papers tried to challenge this tradition with their Quantum Decision Theory (QDT). QDT is based upon the formalism of separable Hilbert spaces. It is parallel to quantum theory in many respects, but this does not imply that the decision maker is a quantum object. QDT is a way to avoid dealing with hidden variables, but at the same time reflecting the complexity of nature. The authors demonstrate that several paradoxes of classical decision theory can be resolved within QDT, and they claim that QDT covers both conscious and unconscious decisions.

This is not a place to describe QDT in detail, but the main idea is a mindspace $\mathcal{M}$ spanned by states corresponding to elementary prospects e_n. These elementary prospects are intersections of intended actions. Both prospect states $|\pi_j\rangle$, describing possible future actions, and strategic states $|\psi_s(t)\rangle$, describing the actor at time t, are vectors in $\mathcal{M}$. Prospect probabilities are given as $p(\pi_j) = |\langle\pi_j|\psi_s(t)\rangle|^2$, and rational decision makers maximize these probabilities.

QDT is presented as a formalism by Yukalov and Sornette, but there is also some discussion in the articles cited above of possible intuitive reasons behind this formalism. It is interesting in this connection that there is a large recent literature on various quantum models in psychology and cognitive science in books and articles; see Khrennikov (2010), Busemeyer and Bruza (2012), Bagarello (2013), Haven and Khrennikov (2013), Yukalov and Sornette (2009), Sornette (2014), Ashtiani and Azgomi (2015), Haven and Khennikov (2016) and in particular the review and discussion article (Pothos and Busemeyer 2013).

Other approaches to decision making using quantum theory methods are Aerts et al. (2014) and Eichberger and Pirner (2017).

It is clear that decisions can be made by single actors or by groups of communicating actors. Therefore language and a common set of concepts must be of some importance in a theory of decisions. In some cases decision making may take time; in other cases one does not have so long time in making decisions. Decisions may take into account selected experiences with past events, and may have a view towards future events. All decisions are made in a context. This context may be physical, historical, conceptual or constituting properties of the decision makers themself. In particular all these considerations are relevant for decisions made under an epistemic process.

1.4 The PBR Theorem. A Toy Model

Recently, Pusey et al. (2012) proved that the wave function must be ontic (i.e. a state of reality) in a broad class of realistic approaches to quantum theory. Two assumptions are made in that paper: (1) A system has a real physical state, not necessarily completely described by quantum theory, but objective and independent of the observer. (2) Systems that are prepared independently have independent physical states.

The assumption (1) goes to the roots of the traditional physicist's world view. I will claim that this state concept is unclear to many people outside the physical community. Why should one always be able to talk about a state independent of the observer? Of course the world itself exists independent of any observer, but the state of the world, what is that? As is discussed in Chap. 6 below, different people experience the world differently, also in macroscopic cases.

The Pusey, Barrett and Rudolph (PBR) theorem, related theorems and arguments connected to the theorem have been thoroughly reviewed by Leifer (2014). In particular it is discussed in detail there what is meant by a realistic approach. It is admitted that there are views of quantum mechanics that are not realistic, and that the Pusey, Barrett, Rudolph theorem does not apply to such interpretations. The approach discussed in the present book belongs to this class, broadly characterized in Leifer (2014) as neo-Copenhagen views. A recent argument against a realistic interpretation of the wave function is given by Rovelli (2016). A new criticism of the PBR theorem, discussing the ontological models framework, is given in Charrakh (2017).

The different interpretations of quantum theory were recently attempted classified by Cabello (2015). First, the interpretations were divided into two types: Type I (intrinsic realism) and Type II (participatory realism). For further discussion of the concept of participatory realism, see Fuchs (2016). The Type II interpretations were further divided into those concerned about knowledge and the one concerned about belief (QBism). As will become clear, this book is concentrating on Type II interpretations concerned about knowledge. According to Cabello (2015), this includes among others the classical Copenhagen interpretation, the approach by Zeilinger (1999) and even the no 'interpretation' approach by Fuchs and Peres (2000).

As a possible motivation behind epistemic views of quantum mechanics, the toy model of Spekkens (2007) is based on a principle that restricts the amount of knowledge an observer can have about reality. A wide variety of quantum phenomena were found to have analogues within this toy theory, and this can be taken as an argument in favour of the epistemic view of quantum states.

In the simplest version of the toy model, we have one elementary system. This system can be in one of the four ontic states 1, 2, 3 or 4, but our knowledge of this is in principle restricted. We can only know one of the following six epistemic states: (a) The ontic state is 1 or 2; (b) it is 3 or 4; (c) it is 1 or 3; (d) it is 2 or 4; (e) it is 1 or 4; or (f) it is 2 or 3. These are the epistemic states of maximal knowledge.

The ontic base of the state a) is $\{1, 2\}$ etc. If the intersection of the ontic bases of a pair of epistemic states is empty, then those states are said to be disjoint. Thus (a) and (b) are disjoint, (c) and (d) are disjoint, and (e) and (f) are disjoint. There is a correspondence with certain basis vectors of the two-dimensional complex Hilbert space, where disjointness corresponds to orthogonality in the Hilbert space. For those who knows the Bloch sphere representation of that Hilbert space, the pairs of disjoint epistemic states can be pictured on the intersections of three orthogonal axes with that sphere.

Transformations of the epistemic states correspond to permutations of the ontic states. Thus the underlying group is the permutation group of four symbols, which has 24 elements. Each permutation induces a map between the epistemic states. In the Hilbert space correspondence, the even permutations correspond to unitary transformations, and the odd permutations correspond to anti-unitary transformations.

The toy model of Spekkens (2007) is generalised in several directions in Spekkens (2014). The generalisations are called epirestricted theories, and are showed to be equivalent to subtheories of quantum mechanics. An epirestricted theory consists of three steps. One starts with a classical ontological theory. Then one constructs a statistical theory over these ontic states. Finally one postulates a restriction on what sorts of statistical distributions that can describe an agent's knowledge of the system. The theory of the present paper is related both to this and to the QBism school, but there are important differences, as will be seen from the discussion below.

1.5 Epistemic Processes

The Quantum Bayesianism is founded on an observer's belief, quantified by a Bayesian probability. I want to relate my state concept also to the notion of *certain* belief, which I call knowledge. The knowledge will be associated with an agent or with a group of communicating agents, and his/her/their knowledge will be knowledge about what I will call an e-variable.

An epistemic process is any process under which an agent or a group of communicating agents obtain knowledge about a physical system. In general there are many ways by which one can obtain knowledge about the world or about aspects of the world. In a given situation the observer has some background, in terms of his history, in terms of his physical environment, and in terms of the concepts that he is able to use in analysing the situation. This is called the context of the observer, and the context may limit his ability to obtain knowledge.

A *conceptual variable* is any variable related to the physical system, defined by an agent or by a group of communicating agents. The variable may be a scalar, a vector or belong to a larger space.

An *e-variable* or epistemic conceptual variable θ is a conceptual variable associated with an epistemic process: Before the process the agent (or agents) has (have) no knowledge about θ; after the process she/they has/have some knowledge, in the simplest case full knowledge: $\theta = u_k$. Here u_k is one of the possible values that θ can take. In this book it is mostly assumed for simplicity that θ is discrete, which it will be in the elementary quantum setting below. For a continuous variable θ, knowledge on the e-variable will be taken to mean a statement to the effect that θ belongs to some given set, an interval if θ is a scalar.

The e-variable concept is a generalization of the parameter concept as used in statistical inference, introduced by Fisher (1922), and today incorporated in nearly all applications of statistics. In statistics, a parameter θ is usually an index in the statistical model for the observations, and the purpose of an empirical investigation is to obtain statements about θ, in terms of point estimation, confidence interval estimation or conclusion from the testing of hypotheses. The parameter is often associated with a hypothetical infinite population. My e-variable will also be allowed to be associated with a finite physical system, a particle or a set of particles. But, in the same way as with a parameter, the purpose of any empirical investigation will be to try to conclude with some statement, a statement expressed in terms of an e-variable θ.

1.5.1 E-Variables in Simple Epistemic Questions

The point of departure is that we ask a question to nature, a question in order to achieve increased knowledge. The e-variable is the particular conceptual variable that is connected to such a question.

To give a very simple example, let us assume that we are given some object A, and ask 'What is the weight of object A?'. Then $\mu =$'weight of A' is an e-variable. We can use a scale to obtain a very accurate estimate of μ. Or we can use several independent measurements, and use the mean of those as a more accurate estimate. In the latter case it is common to introduce a statistical model where μ is a parameter of that model. But in my view the e-variable concept is a more fundamental notion. The variable μ exists before any statistical model is introduced. Most people will agree that μ exists in some sense. Thus in this example the e-variable has some ontic basis, but my claim is that this need not always be the case in all epistemic processes. Even in this case the existence of μ as a real number may be discussed. For instance, the question 'Is μ rational or irrational?' is rather meaningless.

In some cases we can obtain very precise knowledge about some e-variable after some time. As an example, ask 'What will be the number of sun hours tomorrow?' To to answer this question today, we need huge computer models and expert advice from meteorologists. Tomorrow, it is just a question about counting the sun hours.

Similar situations occur when doing simple causal inference. We may for instance ask 'What is the causal effect of medicine C on individual A?'. The e-variable we might have in mind, might be 'Time to recovery for A' compared to time to recovery without any medicine. If we have no earlier experience with A having the disease in question, we have a counterfactual situation. If we have observed A earlier with the same disease when he did not take any medicine, we can at least have a tentative answer within a few days.

1.5.2 E-Variables in Statistics

The most important kinds of e-variables in statistics are statistical parameters. The concept of a statistical parameter was introduced by Fisher (1922), and is today used in nearly all cases where statistical inference is applied. A statistical model is a probability model of the data given the parameters, and the purpose of inference is to obtain information about the parameters. Sometimes only a subset of the parameters are of interest, and the epistemic question is then 'What are the values of these parameters?' Usually one will not be able to get complete information. Partial information, given the data, can be expressed in terms of confidence regions of credibility regions, concepts that will be discussed later in this book.

A completely different kind of e-variables—here called simple e-variables—occur in prediction problems. When we want to predict a random variable Y, the epistemic question is again 'What is the value of Y?' The answer is often sought via a statistical model by first estimating the parameters of the model, and then formulating a prediction equation from this. This gives a predicted value $\widehat{Y}$, again incomplete information from the data. But prediction problems can also be seen as of different nature than estimation problems, requiring separate techniques.

1.5.3 E-Variables in Causal Inference

A causal model is different from a statistical model, as stressed by Pearl (2009). Statistical concepts are correlation, regression, conditional independence, association, likelihood etc., all concepts that can be related to a statistical model. Causal concepts are randomization, influence, effect, confounding etc.. A causal model is defined in terms of what is called a directed acyclic graph, and again these models contain parameters. The epistemic question is again 'What is the value of these parameters?' Sometimes, but not always, incomplete answers can be given by means of data.

Simple e-variable also occur in causal inference. Again we can look at the example 'What is the causal effect of medicine C on individual A?', and the e-variable might be 'Time to recovery for A'.

1.5.4 E-Variables in Quantum Mechanics

E-variables as statistical parameters connected to an infinite population may also occur in quantum mechanics. An example is given in Wootters (1980). Consider a photon which has just emerged from a polarizing filter and which is about to encounter a Nicol prism. The filter can have a continuum of possible orientations, each characterized by a polarization angle θ. But when the photon encounters the Nicol prism, it is required to choose between exactly two possible actions: (1) to go straight through the Nicol prism, or (2) to be deflected in a direction uniquely determined by the orientation of the prism. A straightforward epistemic problem is to estimate θ form an ensemble of photons using the probability law $p(\theta) = \cos^2\theta$. Wootters was investigating the much deeper problem whether this probability law is the best possible in some sense.

But in quantum mechanics simple e-variables are most important. A quantum system can be given some preparation, and under this preparation e-variables like position, momentum, energy, spin, angular momentum may be investigated. The quantum formulation is introduced by associating each e-variable to a Hermitian operator in a Hilbert space (in the discrete case to be precise; for continuous variables a more general construction is needed, one approach is the rigged Hilbert space, see Ballentine 1998) In the present book I will mostly concentrate on discrete e-variables, at least to begin with, and these e-variables always correspond to Hermitian operators in some basic Hilbert space. The quantum states are in general given by vectors in the basic Hilbert space, but in this book special emphasis is often given to state vectors that are eigenvectors of some Hermitian operator. Given a quantum system and an e-variable for this system, a natural focused question will be ‘What is the value of this e-variable?’, and a simple (ideal) measurement will give the answer. But the epistemic question can also be to predict the e-variable before the measurement is done. Then an incomplete answer can be given, the probability distribution as found from the Born rule.

Many books and papers use the term ‘observable’ for what I have called a simple e-variable in quantum mechanics. Ballentine (1998), remarking on some ambiguity in the use of this term, prefers to use ‘dynamical variable’. Bell (1975) introduced the term ‘beable’ for a related concept, assuming some sort of reality of the dynamical variables. My own point is basically to relate the e-variable concept closely to an epistemic process.

The first basic postulate of the quantum formalism may now be written:

Postulate 1.1 *To each physical system there corresponds a Hilbert space. To each simple e-variable of this physical system there corresponds a unique Hermitian operator of the Hilbert space. The possible values of the e-variable are the eigenvalues of this operator.*

(This formulation is only valid for discrete e-variables, which will occupy most of this book. In general one needs a rigged Hilbert space if possible values as eigenvalues should be taken literally.)

There are several recent attempts to motivate this postulate from more intuitive assumptions, see references later. My own attempt, starting from what I call a symmetrical epistemic setting, generalizing the case of spin/ angular momentum, is contained in Chap. 4 below.

1.5.5 Real and Ideal Measurements in Quantum Mechanics

As emphasized by Ballentine (1998), it is important to distinguish between preparation and measurement of a physical system. In my notation from Chap. 3, the preparation gives part of the context τ for the measurement. The purpose of the measurement is to say something about an e-variable θ^a.

To be concrete, let us assume that we want to measure the z-component θ^z of the spin of a silver atom. (A similar experiment with a charged particle like electron will lead to practical difficulties due to the so-called Lorentz force acting upon the charge.) The spin component takes one of two possible values, say ± 1. The experiment is done by sending the atom in some direction, say the y-direction, through a magnetic field which is inhomogeneous in the z-direction. This will cause the atom to be deflected up in the z-direction if $\theta^z = +1$, down if $\theta^z = -1$.

How should this deflection be detected? One way would be to place a screen of detectors in the xz-plane after the silver atom has passed the magnetic field. Hopefully this will give a click in one detector in the positive z-direction if $\theta^z = +1$, or one in the negative z-direction if $\theta^z = -1$.

But such detectors are far from perfect. it may be that the silver atom goes through the screen without being detected. It may be that it clicks on two or more neighbouring detectors when passing through the screen. What we have as a result of the experiment is *data*, which is an array of 0's and 1's corresponding to the detectors: 1 if the detector clicks, 0 otherwise.

This kind of detection errors may seem like a nuisance, but they have fundamental physical importance. Similar detection errors have played a large role in the physical literature on Bell's inequality, which will be discussed in Sect. 5.8 below. The point here is that one on the background of such detection errors have proposed possible 'loopholes' in quantum mechanical experiments to test Bell's theorem. Very recent experiments have excluded such loopholes.

Another point about this simple experiment is the following: The same magnet and screen of detectors can be rotated so as to measure spin component θ^a in any direction a, and the same probability model can be used for the measurement. Thus there is strictly speaking no reason to use a superscript a on the data z^a of the experiment, but we will do it anyway to make it clear that this is the data for the experiment for the spin e-variable θ^a in the direction a.

Thus we have data z^a, and we may construct a statistical model for these data, a model depending on the true value of θ^a. The details of this model need not concern us now, but my important point is that we are in the generalized experiment situation discussed in detail in Sect. 3.2. The distribution of z^a, given the context τ, depends on the unknown e-variable θ^a, and relative to this distribution, τ is independent of θ^a. (A prior for θ^a may depend upon τ, however.)

So far, we have had only one particle, but a similar construction applies in the situation with a beam of n independent particles. Then we have a set of e-variables $\theta^a = (\theta_1^a, \theta_2^a, \ldots, \theta_n^a)$ with data $z^a = (z_1^a, z_2^a, \ldots, z_n^a)$. When $n \to \infty$, this approaches a probability distribution over the e-variable, and the data may without loss of generality be reduced to a frequency distribution of clicks over the detectors of the screen. Under suitable assumptions the probability distribution of the e-variable is given by the Born rule derived in Chap. 5, and the data reduction corresponds to a reduction by sufficiency, a concept discussed in Chap. 3.

In Chap. 3 three fundamental principles of statistical inference are discussed, the conditionality principle, the sufficiency principle and the likelihood principle, where the last one is shown to follow from the first two principles. This whole apparatus is now available for experiments of the quantum type. I will in fact use the likelihood principle later when deriving Born's rule, the basic probability rule of quantum theory. It is therefore important to me that a similar discussion with an imperfect apparatus in principle can be carried out in all quantum experiments. But this is in principle. Once the statistical principles are established, we can return to ideal experiments were data are in one-to-one correspondence with the e-variables of the experiment.

The same kind of data and data model can be used for any choice of focused direction, focused measurement. Assume that the situation is such that the vector of different e-variables θ^a is inaccessible; see Sect. 1.5.8. Then this gives rise to a focused version of the likelihood principle. This is the kind of likelihood principle used to derive the Born rule. The details are in Chaps. 3, 4 and 5.

1.5.6 Quantum States, Their Interpretations, and a Link to the Ensemble Interpretation

Pure quantum states are defined formally as unit vectors in the Hilbert space. These vectors may or may not be eigenvectors of physical meaningful Hermitian operators. They are denoted as ket vectors $|k\rangle$ and may be assumed to form a complete set. For more information, see Sect. 5.1 and Appendix B.

Let first $|k\rangle$ be a unit eigenvector of a Hermitian operator A corresponding to a simple e-variable θ. Assume for simplicity that A do not have multiple eigenvalues. Then, from an epistemic process point of view, and from what can be seen as a simple observation from the quantum formalism, $|k\rangle$ may be identified by a question 'What is the value of θ?' together with a definite answer '$\theta = u_k$', where u_k is the corresponding eigenvalue.

Given a unit vector $|k\rangle$, there will in general be an infinity of Hermitian operators for which $|k\rangle$ is an eigenvector, and there may be many such operators which correspond to physically meaningful e-variables θ. Also, one must consider the case with multiple eigenvalues. Hence, given $|k\rangle$, its association to a question-and-answer pair is not unique in general. In the special case of the spin or angular momentum of a particle and in related cases, there is some uniqueness, however.

(a) For a spin 1/2 particle, or more generally for a qubit, a unit vector in a 2-dimensional Hilbert space, there is a unique question-and-answer pair for spin component corresponding to each unit vector; see Sect. 5.1.1 and Proposition 5.4 of Sect. 5.2.
(b) In the general case of spin or angular momentum component there will correspond a unique normalized ket vector corresponding to each question-and-answer pair; see Proposition 5.3 of Sect. 5.2. For dimension higher than 2 there are however unit vectors that do not correspond directly to a question-and-answer pair for spin/angular momentum components.
(c) In a general setting, related to, but weaker than that of spin/ angular momentum, one can prove without assuming a quantum formalism from the outset, that each question-and-answer pair under a certain technical assumption corresponds to a unique normalized ket vector of some Hilbert space. This is proved in Chap. 4. Unfortunately, one of the technical assumptions as stated there, seems to be too strong; it is argued that it should be possible to weaken this assumption.

A larger class of pure states than those corresponding directly to question-and-answer pairs for some given set of e-variables may be found by taking linear combinations, which may be motivated physically, or as solutions of the Schrödinger equation.

In most of this book I will discuss epistemic processes, processes to obtain knowledge about some system, physical or otherwise. Then the question-and-answer pair corresponds to two decisions: First a decision on which e-variable to focus upon. Then after the data are obtained, a decision on the value of this e-variable. In statistical theory, only the last of these decisions is discussed. It is very interesting that the quantum state concept turns out to be useful for more general decisions; from a qualitative point of view this will be discussed in Chap. 6 below.

To the ket vector $|k\rangle$ correspond the bra vector $\langle k|$ and the projection operator $|k\rangle\langle k|$.

The more general concept of a mixed state is defined from a probability distribution $\{\pi_k\}$ as follows:

$$\rho = \sum_k \pi_k |k\rangle\langle k|. \tag{1.1}$$

Depending upon the observator(s) and upon the physical situation, the probabilities π_k may be interpreted—and assessed/estimated—in three possible ways: (1) as Bayesian prior distributions; (2) as Bayesian posterior distributions; (3) as frequentist confidence distributions, see Schweder and Hjort (2016). From a statistical point of view, Bayesian probabilities are connected to credibility intervals, confidence distributions are derived from confidence intervals; see Chap. 2 below.

It is interesting that under specific symmetry assumptions, the confidence intervals and the credibility intervals coincide and are associated with the same probability; see Corollary 3.6.2 p. 93 in Helland (2010). A consequence of this is that the probabilities estimated under the interpretations (2) and (3) will be numerically equivalent under these symmetry assumptions.

In this section it will be convenient to use the confidence distribution interpretation (3), even though it is new and unknown, also largely among statisticians. The basic concept is that of a confidence interval with some variable confidence coefficient; details will be discussed in Chap. 2. The important point to us now is the interpretation of the confidence coefficient. It can be seen as a limiting frequency obtained from data in a large hypothetical set of epistemic processes were the same statistical method is used.

A similar idea carries over to the quantum situation: Ballentine (1998) makes an important point of the fact that probabilities in quantum mechanics must be interpreted with respect to a hypothetical ensemble. In Smolin (2011) it is proposed that this hypothetical ensemble is realised by all systems in the universe that occupy the same quantum state.

The state concept is the basis for all calculation of probabilities in quantum mechanics. A physical system is prepared in some given state, and probability distributions of measurements from this state are given by the Born rule.

The Born rule is developed from my point of view in Sect. 5.6 below. A consequence of this rule is a simple formula which can be used to restate Ballentine's

Postulate 1.2 *To each state there corresponds a unique state operator ρ. The average value of an e-variable θ, represented by the operator A, in the virtual ensemble of events that may result from a preparation procedure for state represented by the operator ρ is*

$$\langle\theta\rangle = \mathrm{tr}(\rho A).$$

From a statistical point of view, the average value here is interpreted as an expectation. By extending this formula in a natural way, the probability distribution of θ under ρ may be derived; see again Ballentine (1998). In fact this probability distribution follows easily directly from my Theorem 5.6 in Sect. 5.6.

1.5.7 Quantum States for Spin 1/2 Particles

Consider again the silver atom of Sect. 1.5.5, and suppose that we want to measure the spin component in some direction a. Let us now assume that the measurement apparatus is perfect. Then the experiment has two possible outcomes: spin up or spin down. This corresponds to two possible quantum states:

$$|a;+\rangle \text{ or } |a;-\rangle. \tag{1.2}$$

Let now a be arbitrary. In Sect. 5.2 I will prove the important result: *All pure quantum states for spin 1/2 particles can be written in one of the forms (1.2) for some a.*

In concrete terms, this means that every pure quantum state for a spin 1/2 particle can be interpreted as a question: What is the spin component in direction a? together with a definite answer: $+1$ or -1. (Strictly speaking, this gives a double counting of quantum states, since, given some direction a, we can also choose the opposite direction $-a$; and $\theta^a = +1$, say, is equivalent to $\theta^{-a} = -\theta^a = -1$.)

It is of some related interest that the axioms of quantum mechanics recently were approached by Smilga (2017) by taking spin measurements in different directions as a point of departure, and using group representation theory.

1.5.8 Inaccessible Conceptual Variables and Complementarity

It is important that not all conceptual variables are e-variables. A conceptual variable ϕ is called *inaccessible* if there is no epistemic process by which one can get accurate knowledge about it. An example from the area of quantum mechanics is $\phi = (\xi, \pi)$, where ξ is the position of a particle, and π is the momentum. An example which will be discussed throughout the present book, is $\phi = (\lambda^x, \lambda^y, \lambda^z)$, where λ^a is the component of an angular momentum or a spin for a particle or a system of particles, the component in direction a. Here each λ^a is an accessible conceptual variable, an e-variable, but the vector ϕ is inaccessible. Also macroscopic examples abound, for instance connected to counterfactual situations in causal inference: Let θ^1 be the time to recovery for given patient A at time t when he is given treatment B, let θ^2 be his time to recovery when he is given treatment C, and let $\phi = (\theta^1, \theta^2)$.

In such cases, where a vector of e-variables is inaccessible, it is equivalent to say that the components are complementary. Since introduced by Bohr, the concept of complementarity has played a fundamental and important role in quantum mechanics.

It is essential to stress that the conceptual variables above are not hidden variables. Variables like ϕ are just mathematical variables, but variables upon which group actions may be defined. $\phi = (\xi, \pi)$ may be subject to Galilean transformations, time translations or changes of units, while $\phi = (\lambda^x, \lambda^y, \lambda^z)$ may be subject to rotations. This will of course also induce transformation of the components, the e-variables. Important transformations in the group of rotations of the last ϕ are: (1) Those leading to a change in the values of λ^x (or of any other fixed component); (2) Those leading to an exchange of λ^x and λ^y (or any other pair of components).

The (simple) e-variables are not hidden variables, but closely tied to the epistemic processes.

To illustrate the general view of this book, look at Fig. 1.1. Here ϕ is an inaccessible conceptual variable, the θ's are e-variables, the τ's are context variables

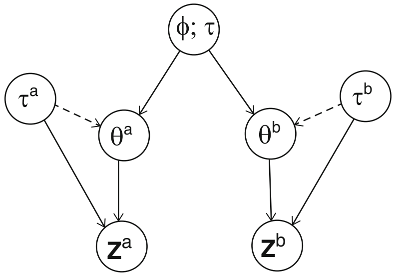

Fig. 1.1 A graphical picture, illustrating a general view upon quantum theory

and the z's are data. The upper arrows denote functional dependence, and the lower arrows denote conditional probability distributions of the data. The dotted arrows indicate that one may or may not have a prior for θ, given the context. The experimentalist has the choice between two mutually excluding experiments, denoted by a and b.

When does this situation lead to a quantum theory, which can alternatively be described by a Hilbert space formulation? A partial answer is given in Chap. 4 and in Sect. 5.2 below. The crucial concept is the context variable τ. One possibility is that this denotes the maximal symmetrical epistemic setting of Chap. 4 (here θ is replaced by λ), satisfying Assumptions 4.1–4.3 there. Another possibility is the corresponding general symmetrical epistemic setting, and a final situation is a spin/angular momentum situation. An open question is to find the most general conditions under which a quantum theory can come into being from the situation of Fig. 1.1. In the situations above, the θ's are discrete, which they are in elementary quantum theory, but we can also let the same figure illustrate a setting where the θ's are continuous; for a completely general formulation, see Sect. 5.15.

The inference on simple e-variables as presented in this book implies some connection between the quantum mechanical culture and the statistical culture. In the next chapter I will give a broad description of the science of statistical inference. Then I will give some general principles on inference on parameters/ e-variables when data are involved, and after that I will turn to quantum theory.

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