Combinatorial and Graph-Theoretical Problems and Augmenting Technique

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1 Introduction

Berge's lemma [3] states that a matching M (a set of edges without common vertices) of a graph G is maximum (contains the largest number of edges) if and only if there is no augmenting path (a path that starts and ends on free (unmatched) vertices, and alternates between edges in and not in the matching) with M. Edmonds [11] used this idea to develop *Blossom Algorithm* for this problem. This idea was used first for the Maximum Independent Set problem, i.e. the problem asks for a largest number of vertices set without edges among them, by Sbihi [33] and Minty [28]. Clearly, a matching in a graph G corresponds to an independent set in the line graph of G. Hence, we can use Edmonds' algorithm to find Maximum Independent Set for line graphs. Sbihi and Minty extended this idea for a more general graph class, say claw-free graph, by showing that an independent set S of a graph G is maximum if and only if there is no augmenting path (a path that starts and ends on vertices not lies in the independent set and alternates between vertices in and not in the independent set) with S. This technique was extended for more general graph classes by using the augmenting graph concept as described as follows.

Definition 1 ([17]) Given a graph G and an independent set S, an induced bipartite subgraph $H = (W, B, E)$ of G is called an augmenting graph for S if (**i**) $W \subseteq S$, $B \subseteq V(G)\backslash S$, (**ii**) $N(B) \cap (S\backslash W) = \emptyset$, and (**iii**) $|B| > |W|$.

An augmenting graph H is called minimal if it does not contain any augmenting graph as a proper induced subgraph.

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B. Goldengorin (ed.), *Optimization Problems in Graph Theory*, Springer Optimization and Its Applications 139,
https://doi.org/10.1007/978-3-319-94830-0_9

Theorem 1 ([17]) *An independent set S in a graph G is maximum if and only if there is no augmenting graph for S.*

This theorem suggests the following general approach to find a maximum independent set in a graph G. Begin with any independent set S (may be empty) in G and as long as S admits an augmenting graph H, exchange white and black vertices of H. Clearly, the problem of consecutively finding augmenting graphs and of applying these augmentations is generally NP-hard, as the MIS problem is NP-hard. Moreover, we can restrict ourselves in minimal augmenting graph only. Hence, for a polynomial time solution to some graph class, one has to solve the two following problems:

(P1) Find a complete list of (minimal) augmenting graphs.
(P2) Develop polynomial time algorithms for detecting (minimal) augmenting graphs.

So far, characterizations of (minimal) augmenting graphs mainly followed the two following directions. In the first approach, augmenting graphs in ($S_{1,2,k}$,banner)-free graphs are characterized based on the observation that a banner-free bipartite graph is either C_4-free or complete. The most general result follows this direction described by Lozin and Milanič [23] for ($S_{1,2,5}$,banner)-free graphs.

In the second approach, augmenting graphs of subclasses of P_5-free graphs are characterized based on the observation showed indepedently by many researchers (e.g., [29]) that every connected P_5-free bipartite graph is a *bipartite-chain* graph, i.e. the vertices of each part can be ordered under inclusion of their neighborhood. Based on this property, polynomial solutions were obtained for some subclasses of P_5-free graphs [4, 16, 25, 29, 30]. It is also worth to notice that the MIS problem is shown polynomially solvable in P_5-free graphs [22].

In this paper, we try to combine the two above approaches to a subclass of (banner$_2$,domino)-free graphs (see Figure 1). In particular, we obtain the following theorem.

Theorem 2 *Given integers m, l, the MIS problem is polynomially solvable in ($S_{2,2,5}$,banner$_2$,domino,M_m, $K_{m,m} - e$,R_l^1, R_l^2, R_l^3)-free graphs. (See Figure 4.)*

Obviously, banner and domino are two natural generalizations of P_5 and banner (banner$_1$), R_l^1, R_l^2, R_l^3 are generalizations of $S_{1,2,5}$. Hence, our result is a generalization of some previous known results for ($S_{1,2,5}$, banner)-free graphs [23], (P_5, $K_{3,3} - e$)-free graphs [16, 25] for (P_5, $K_{2,m} - e$)-free graphs, and some subclasses of $S_{1,2,2}$-free graphs [20].

The organization of the paper is as follows. Augmenting graphs for some subclasses of $S_{2,2,l}$-free graphs are characterized in Section 2, i.e. to solve Problem P1. Methods for finding such augmenting graphs are described in Section 3, i.e. to solve Problem V2. In Section 4, we summarize some results in using technique for other combinatorial and graph-theoretical problem. Section 5 is a discussion about the issue. Many of long proofs are put in the Appendix part.

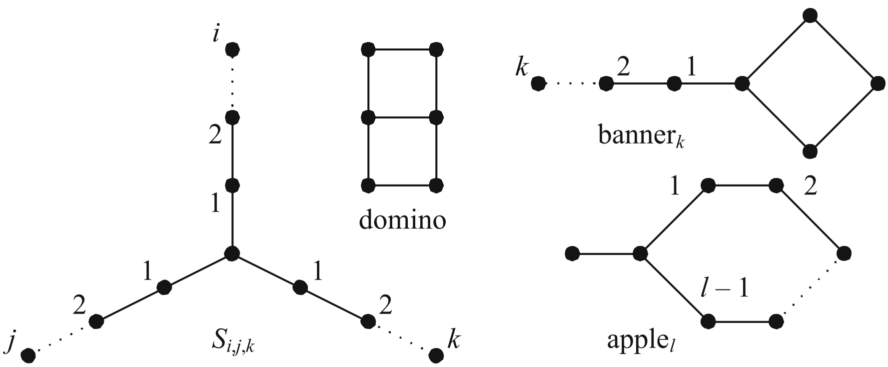


Fig. 1 $S_{i,j,k}$, domino, banner$_k$, and apple$_l$

Here, we want to collect most of the terminology and notations used in the paper. For those not given here, they will be defined when needed. For those not given, we refer the readers to [5]. Given a graph $G = (V, E)$, for a vertex u, we denote by $N(u) := \{v \in V : uv \in E\}$ the *neighborhood* of u in G. For a subset $U \subset V(G)$, we denote by $N(U) := (\bigcup_{u \in U} N(u)) \setminus U$ the neighborhood of U. If W, U are two vertex subsets of G, then $N_U(W) := N(W) \cap U$. Also, $N_U(v) := N(v) \cap U$ for a vertex v. Given a graph $G = (V, E)$ and a vertex subset U, we denote by $G - U$ the graph obtained from G by deleting all vertices (together with adjacent edges) in U. For two vertices $u, v \in V$, we write $u \sim v$ if $uv \in E$. For a vertex u, we denote by $d(u) := |N(u)|$, the degree of u in G. We also denote by $G[U] := G - (V(G) \setminus U)$, the subgraph of G induced by U.

2 Augmenting Graphs in Subclasses of ($S_{2,2,l}$,banner$_l$)-Free Graphs

Hertz and Lozin [17] obtained the following observation about minimal augmenting graphs.

Lemma 1 ([17]) *If $H = (B, W, E)$ is a minimal augmenting graph for an independent set S of a graph G, then*

1. *H is connected;*
2. *$|W| = |B| - 1$;*
3. *for every subset $U \subseteq W$, $|U| < |N_B(U)|$.*

2.1 Redundant Sets and Reduction Sets

Let us report from Section 3 of [23] a general observation on the problem of finding augmenting graphs and let us slightly extend it according to the Remark of Section 3 of [23]. Given an augmenting graph class $\mathscr{A}$, a graph G, and an independent set S, let Problem Augmentation ($\mathscr{A}$) denote the problem of finding augmenting graphs if S admits an augmenting graph in $\mathscr{A}$. Lozin and Milanič [23] showed that in ($S_{1,2,5}$,banner)-free graphs, the problem can be reduced to finding augmenting graphs of the form tree^1,...,tree^6 (see Figure 2) by using redundant set concept. We extend this concept as follows.

Definition 2 In an augmenting graph $H = (W, B, E)$, a vertex subset U is called redundant if

1. $|U \cap W| = |U \cap B|$ and
2. for every vertex $b \in B \backslash U$, $N_{W \backslash U}(U \cap B) \subseteq N_{W \backslash U}(b)$.

Theorem 3 *Let $\mathscr{A}_1$ and $\mathscr{A}_2$ be two classes of augmenting graphs. If there exists a constant k such that, for every augmenting graph $H = (W, B, E) \in \mathscr{A}_2$, there exists a redundant subset U of size at most k such that $H - U \in \mathscr{A}_1$, then Problem Augmentation($\mathscr{A}_2$) is polynomially reducible to the problem Augmentation($\mathscr{A}_1$).*

Proof Assume that Algorithm $Augment_1(G, S)$ outputs a subset $V' \subseteq V(G)$ such that $G[V']$ is augmenting for S whenever S admits an augmenting graph from $\mathscr{A}_1$ (and perhaps even if this is not the case). The procedure also returns $\emptyset$ if no augmenting graph is found.

Assume that S admits an augmenting graph $H = (B, W, E) \in \mathscr{A}_2$. Then by the theorem's assumption, H contains a redundant set U of size at most k such that $H - U \in \mathscr{A}_1$. It is obvious that the graph $H - U$ is augmenting for $S \backslash U$. Moreover, since U is redundant, G'' contains every vertex of $H - U$, i.e. Steps 1 and 2 have not removed any vertex of $H - U$. Therefore, Algorithm $Augment_1$ must output a non-empty set T. Consequently, Algorithm $Augment_2$ also outputs a non-empty set $U \cup T$.

We show that $G[U \cup T]$ is augmenting for S. Indeed, by Step 1, $G[U \cup T]$ is a bipartite graph. Since T is augmenting for $S \backslash U$ in G'', $|T \cap S \backslash U| < |T \cap V(G'')|$. Moreover, since $|U \cap S| = |U \cap V(G) \backslash S|$, $|(T \cup U) \cap S| < |(T \cup U) \cap V(G) \backslash S|$. By Step 2, $N_S(U \backslash S) \subseteq T \cap S$, i.e. $N_S((T \cup U) \backslash S) \subseteq (T \cup U) \cap S$. Hence, the graph $G[U \cup T]$ is augmenting for S, even if $G[T]$ does not coincide with $H - U$. Therefore, whenever S admits an augmenting graph in $\mathscr{A}_2$, Algorithm $Augment_2$ finds an augmenting graph.

To this end, the procedure inspects polynomially many subsets of vertices of the input graph, which results in polynomially many calls of Algorithm $Augment_1$. The construction of the graph G'' also is performed in polynomial time. Hence, Problem Augmentation($\mathscr{A}_2$) is polynomially reducible to Problem Augmentation ($\mathscr{A}_1$).

Note that Problem Augmentation($\mathscr{A}_1$) becomes Problem (P2) when $\mathscr{A}_1$ is the class of all (possible) augmenting graphs.

Algorithm 1 $Augment_2(G, S)$ (Version 1)

Input: A graph G and an independent set S of G

Output: A subset $V' \subseteq V(G)$ such that $G[V']$ is augmenting for S whenever S admits an augmenting graph from $\mathscr{A}_2$. Return $\emptyset$ if no augmenting graph is found.

for all $U \subseteq V(G)$ of size at most k such that

1. $B_0 := U \cap (V(G) \backslash S)$ is independent in G,
2. $|B_0| = |U \cap S|$

do

  $G' := G - N_G(B_0) \cap (V(G) \backslash S)$ {Remove the (black) neighbors of B_0 in $V(G) \backslash S$};

  $G'' := G' - \{b \in V(G') \backslash S : N_{S \backslash U}(B_0) \backslash N_{S \backslash U}(b) \neq \emptyset\}$ {Remove the (black) vertices of $V(G') \backslash S$ whose neighborhood in $S \backslash U$ does not cover the neighborhood of B_0 in $S \backslash U$};

  $T := Augment_1(G'' - U, S \backslash U)$;

  if $T \neq \emptyset$ **then**

    return $U \cup T$ {We have an augmenting graph for S}

  end if

end for

return $\emptyset$

Moreover, we can also extend the redundant set concept further as follows. If Algorithm $Augment_1$ starts with some initialization process (see Algorithm 2), which computes some finite vertex set C such that $N_{S \backslash U}(U \backslash S) \subseteq N_S(C \backslash S)$, then we can process this initialization procedure in $Augment_2$ as in Version 2 and remove the condition that every neighbor in $S \backslash U$ of black vertices in $B \backslash U$ covers the neighbor of U in $S \backslash U$ (see Algorithm 3). More precisely, we have the following definition.

Definition 3 Let $\mathscr{A}_1$ and $\mathscr{A}_2$ be the two augmenting graph classes. Given an integer k. Assume that there exists a polynomial time procedure finding an augmenting graph in $\mathscr{A}_1$ (or deciding such augmenting graph does not exist) and such a procedure has a form as in Algorithm 2, i.e. starts by generating some candidates and from each candidate C, builds up augmenting graphs ($Generate_1(C, G, S)$). In an augmenting graph $H = (B, W, E) \in \mathscr{A}_2$, a vertex subset U is called a reduction set associated with some key set $B^* \subseteq B \cap C$ if $|U \cap B| = |U \cap W|$ and $N_{W \backslash U}(U \cap B) \subseteq N_{W \backslash U}((B^* \backslash U) \cap B)$.

And by the above arguments, we have the following observation.

Theorem 4 *Let $\mathscr{A}_1$ and $\mathscr{A}_2$ be the two augmenting graph classes. Then Problem Augmentation($\mathscr{A}_2$) is polynomially reducible to Problem Augmentation($\mathscr{A}_1$) if there are two integers k_1, k_2 such that for every augmenting graph $H = (B, W, E) \in \mathscr{A}_2$, there is a reduction set U of size at most k_1 associated with a key set B^* of size at most k_2 such that $H - U \in \mathscr{A}_1$.*

Algorithm 2 $Augment_1(G, S)$

Input: A graph G and an independent set S of G

Output: A subset $V' \subseteq V(G)$ such that $G[V']$ is augmenting for S whenever S admits an augmenting graph from $\mathscr{A}_1$. Return $\emptyset$ if no augmenting graph is found.

Generate Candidates;
for all Candidates C **do**
  $T := Generate_1(C, G, S)$;
  if $T \neq \emptyset$ **then**
    return T {We have an augmenting graph for S}
  end if
end for
return $\emptyset$

Algorithm 3 $Augment_2(G, S)$ (Version 2)

Input: A graph G and an independent set S of G.

Output: A subset $V' \subseteq V(G)$ such that $G[V']$ is augmenting for S whenever S admits an augmenting graph from $\mathscr{A}_2$. Return $\emptyset$ if no augmenting graph is found.

for all $U \subseteq V(G)$ of size at most k such that

1. $B_0 := U \cap (V(G) \backslash S)$ is independent in H,
2. $|B_0| = |U \cap S|$

do
  $G' := G - N_G(B_0) \cap (V(G) \backslash S)$ {Remove the (black) neighbors of B_0 in $V(G) \backslash S$};
  Generate Candidates;
  for all Candidates C of G' such that $N_{S \backslash U}(B_0) \subseteq N_{S \backslash U}(C \cap (V(G' - U) \backslash S))$ **do**
    $T := Generate_1(C, G' - U, S \backslash U)$;
    if $T \neq \emptyset$ **then**
      return $U \cup T$ {We have an augmenting graph for S}
    end if
  end for
end for
return $\emptyset$

2.2 Augmenting Graphs in Subclasses of $S_{2,k,l}$-Free Graphs

The following corollary is a consequence of Lemma 1 and was obtained in [23].

Corollary 1 ([23]) *Let $H = (B, W, E)$ be a minimal augmenting graph for an independent set S of a graph G. Then for every vertex $b \in M$, there exists a perfect matching between $B \backslash \{b\}$ and W in H, i.e. a matching consists of every vertex of $B \backslash \{b\}$ and W.*

Remark 1 By the above corollary, from now on, given a minimal augmenting graph $H = (B, W)$ and a black vertex $b \in B$, we denote by M such a perfect matching and for every vertex u of H different from b and by $\mu(u)$ the matched vertex of u in M. For a subset $U \subseteq V(H)$, we also denote $\mu(U) := \{\mu(u) : u \in U\}$.

Corollary 2 *Let $H = (B, W)$ be a minimal augmenting graph. Then every white vertex of H is of degree at least two.*

We say that G is an (k, m)-extended-chain if G is a tree and contains two vertices a, b such that there exists an induced path $P \subset G$ connecting a, b, every vertex of $G - P$ is of distance at most $k - 1$ from either a or b, and every vertex of $G - P$ has no neighbor in P except possibly a or b and every vertex of G is of degree at most $m - 1$. The following observation is an extension of Theorem 8 of [17]. The result was announced in [21] without full proof.

Lemma 2 ([21]) *For any three integers k, l, and m such that $4 \leq 2k \leq l$ and $m \geq 3$, in ($S_{2,2k,l}$,apple$^l_4$, apple$^l_6$,. . .,apple$^l_{2k+2}$,$K_{1,m}$)-free graphs, there are only finitely many minimal augmenting graphs different from augmenting $(2k, m)$-extended-chains and not of the form apple$_{2p}$. Moreover, if H is of the form augmenting $(2k, m)$-extended-chain, then every white vertex is of degree two.*

Note that in an augmenting graph of the form apple$_{2p}$ (or augmenting apple for short), the vertex of degree three is white. However, given an augmenting apple $H = (B, W, E(H))$, where b is the black vertex of degree one and w is the white vertex of degree three. Then $U := \{b, w\}$ is a redundant set such that $H - U$ is an augmenting chain, a special case of augmenting (k, m)-extended-chain.

2.3 Augmenting Graphs in Subclasses of $S_{2,2,5}$-Free Graphs

Now, we try to omit $K_{1,m}$ from the list of forbidden induced subgraphs by considering ($S_{2,2,5}$,banner$_2$,domino)-free augmenting graphs. We extend the consideration of Section 4 in [23]

Lemma 3 *Given a graph G and an ($S_{2,2,5}$,banner$_2$,domino)-free minimal augmenting graph $H = (B, W, E)$ for an independent set S, at least one of the following statements is true:*

1. *H belongs to some finite set of augmenting graphs;*
2. *H is an augmenting chain or an augmenting apple (see Figure 1);*
3. *H is an augmenting graph of the form tree^1, tree^2, . . . , tree^7 (see Figure 2) or can be reduced by a redundant set containing at most 32 vertices to an augmenting graph of the form tree^1, tree^2, . . . , tree^7;*
4. *there is a vertex $b \in B$ such that b is adjacent to all vertices of W.*

Such b of Case 4 is called the *augmenting vertex* of S, as in [29, 30]. We also call augmenting graphs of the form tree^1, tree^2, . . . , tree^7 as *augmenting trees*. For Case 4 of Lemma 3, we show that under some restrictions, these augmenting graphs have structural properties similar to P_5-free augmenting graphs, i.e. being a bipartite-chain by the following observation.

Lemma 4 *Given a (domino,banner$_2$)-free graph G, an integer $m \geq 3$, and an M_m-free (see Figure 3) minimal augmenting graph $H = (B, W, E)$ for an independent set S such that there exists some black vertex $b \in B$ adjacent to every white vertex of W, and $|W| \geq 2m + 1$, at least one of the following statements is true.*

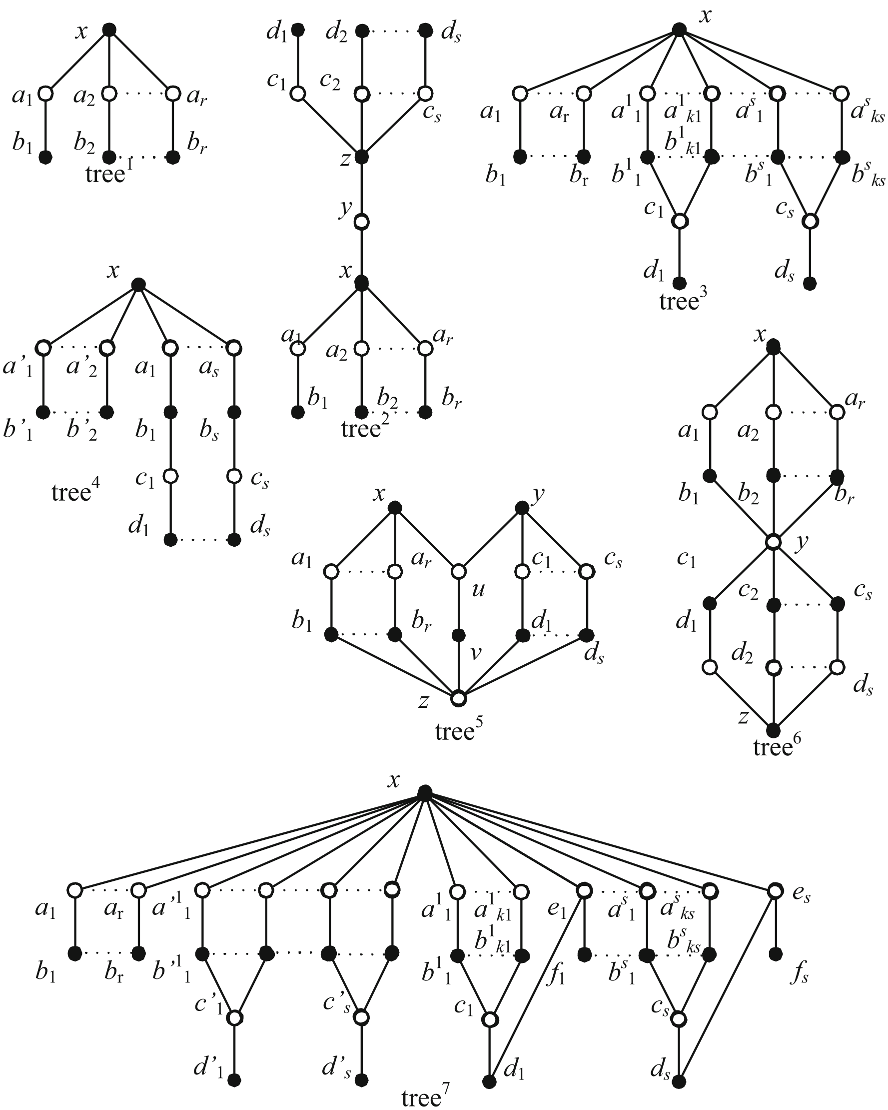


Fig. 2 Augmenting trees

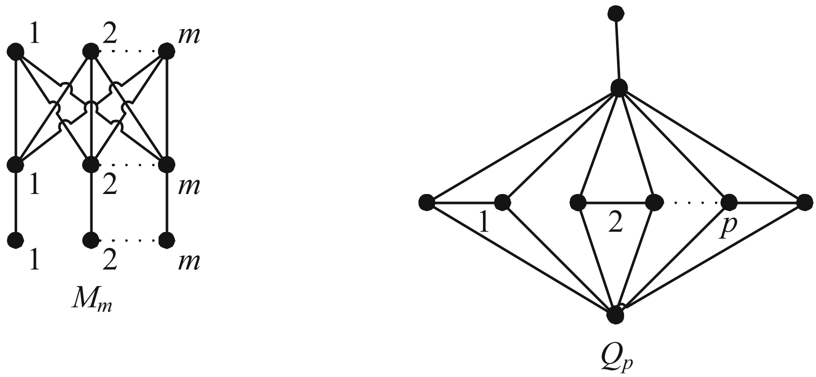


Fig. 3 M_m and Q_p

1. *H is of the form tree*[1] *or there exists a reduction set U of size at most* $2m - 2$ *associated with a key set of size one such that* $H - U$ *is of the form tree*[1].
2. *H is a bipartite-chain or there exists a redundant set U of size at most* $2m - 2$ *such that* $H - U$ *is a bipartite-chain.*

Proof We refer to Lemma 10 for the procedure finding tree[1] and note that such a procedure starts by finding a candidate containing b, i.e. b is adjacent to every white vertex in the augmenting tree[1] and we have the key set $B^* := \{b\}$.

Let $B = \{b, b_1, \ldots, b_q\}$, b be the vertex b in Corollary 1, p be the integer p in Lemma 8 such that $N_W(b_i) \supseteq N_W(b_j)$ for every $1 \leq i \leq p$, $i < j \leq q$ and $|N_W(b_i)| = 1$ for every $i \geq p + 1$.

If $p \leq m - 1$, then $U = \{b_1, \ldots, b_p, \mu(b_1), \ldots, \mu(b_p)\}$ is a reduction set of size at most $2m - 2$ associated with B^* such that $H - U$ is of the form tree[1].

If $p \geq q - m + 1$, then $U = \{b_{p+1}, \ldots, b_q, \mu(b_{p+1}), \ldots, \mu(b_q)\}$ is a redundant set of size at most $2m - 2$ such that $H - U$ is a bipartite-chain.

If $m \leq p \leq q - m$, then $\{b, b_1, \ldots, b_{k-1}, b_{q-k+1}, \ldots, b_q, \mu(b_{q-k+1}), \ldots, \mu(b_q)\}$ induces an M_m, a contradiction.

The following observation is a generalization of Lemma 10 in [4] and Theorem 1 in [16] about augmenting graphs in (P_5, $K_{2,m} - e$)-free graphs and (P_5, $K_{3,3,} - e$)-free graphs, respectively.

Lemma 5 *Given a graph G, an independent set S of G, an integer m, and a* ($K_{m,m} - e$)*-free minimal augmenting bipartite-chain* $H = (B, W, E)$*, either*

1. *H has at most* $2m - 2$ *white vertices; or*
2. *H is of the form* $K_{l,l+1}$ *or there is a redundant set of size at most* $2m - 4$ *such that* $H - U$ *is of the form* $K_{l,l+1}$*, for some l.*

Proof Assume that $|W| = p \geq 2m - 1$. Let $W = \{w_1, w_2, \ldots, w_p\}$ and $B = \{b_1, b_2, \ldots, b_p, b_{p+1}\}$. Assume that $N_W(b_i) \subseteq N_W(b_j)$ for $i < j$. Moreover, by Corollary 1, there exists a perfect matching between $B \backslash \{b_{p+1}\}$ and W. Without loss of generality, assume that $b_i \sim w_i$ for $1 \le i \le p$. Then we have $|N_W(b_i)| \ge i$ for $i = 1, 2, \ldots$.

Now, $b_i \sim w_j$ for every $b_i \in B$ and $w_j \in W$ such that $p - m + 4 \ge i \ge m - 1$ and $p - m + 3 \ge j \ge i + 1$, otherwise $\{b, b_p, \ldots, b_{p-m+3}, b_i, w_j, w_{m-1}, \ldots, w_1\}$ induces a $K_{m,m} - e$, a contradiction.

Hence, $\{b, b_p, \ldots, b_{m-1}, w_{p-m+1}, \ldots, w_1\}$ induces a $K_{p-m+3,p-m+2}$ and $U := \{b_{m-2}, \ldots, b_1, w_p, \ldots, w_{p-m+2}\}$ is a redundant of size $2m - 4$ such that $H - U$ is a $K_{p-m+3,p-m+2}$.

Note that if an augmenting graph contains at most $2m - 2$ white vertices, it contains at most $4m - 3$ vertices.

3 Finding Augmenting Graphs

Now, we consider Problem (P2), i.e. the problem of finding augmenting graphs characterized in Section 2. Remind that we can enumerate all augmenting graphs of bounded size in polynomial time. Moreover, Hertz and Lozin [17] described a method of finding augmenting graphs of the form $K_{m,m+1}$ in banner$_2$-free graphs. Besides, it is obvious that augmenting apples can be reduced to augmenting chains by a redundant set of size two. Hence, we have to find augmenting extended-chains and augmenting trees.

3.1 Augmenting Extended-Chain and Augmenting Trees

The method for finding augmenting chains in ($S_{1,2,j}$,banner)-free graphs has been described by Hertz, Lozin, and Schindl [18]. We have extended this method and obtain the following result, which was published in [21] without proof.

Lemma 6 ([21]) *Given integers l and m, where l is even, an ($S_{2,l,l}$,banner$_l$,R_l^1,R_l^2, R_l^3, R_l^4, R_l^5)-free graph G, and an independent set S in G, one can determine whether S admits an augmenting (l, m)-extended-chain in polynomial time (Figure 4).*

By extending the techniques presented in [23] (finding augmenting trees of the form tree$^1, \ldots,$ tree^6 in ($S_{1,2,5}$, banner)-free graphs), we obtain the following result.

Lemma 7 *An augmenting graph of the form tree^1, tree^2, ..., tree^7 can be found in ($S_{2,2,5}$,banner$_2$)-free graphs in polynomial time.*

Together with the method of Lozin and Hertz [17] for finding augmenting graphs of the form $K_{p,p+1}$ in banner$_2$-free graphs, it leads us to the following result.

Corollary 3 *Given an integer m, the MIS problem is polynomially solvable in ($S_{1,2,5}$,banner$_2$,domino,M_m, $K_{m,m} - e$)-free graphs.*

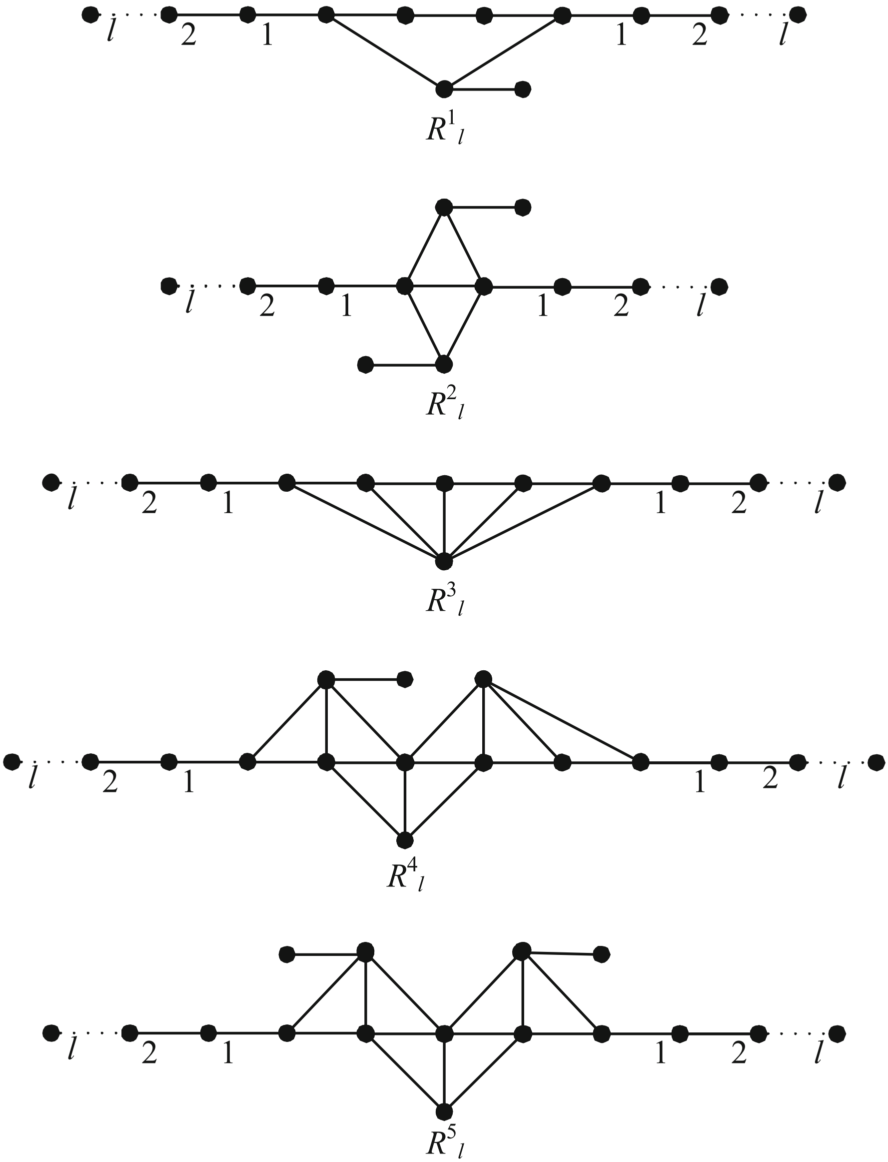


Fig. 4 R_l^1,R_l^2, R_l^3, R_l^4, and R_l^5

Theorem 42 (as well as Corollary 3) is a generalization of the results of Lozin and Milanič for ($S_{1,2,5}$, banner)-free graphs [23], of Lozin and Mosca for (P_5, $K_{3,3}-e$)-free graphs [25], of Gerber et al. [16] for (P_5, $K_{2,m}-e$)-free graphs, and some subclasses of $S_{1,2,2}$-free graphs [20]. Note that we used redundant set and reduction set to reduce "near" augmenting complete bipartite graphs to augmenting complete bipartite graphs. This technique is a generalization of method for augmenting $K^{+}_{m,m}$'s in [25].

3.2 *The Maximum Independent Set Problem in Further Subclasses of $S_{2,2,5}$-Free Graphs*

So far, for ($S_{2,2,5}$,banner$_2$,domino,R^1_l, R^2_l, R^3_l, M_m)-free graphs, we can find every (minimal) augmenting graph in polynomial time except for augmenting bipartite-chains. Mosca in [29] and then in [30] (see also [14, 16]) developed augmenting vertex technique for this issue, which we describe next.

Let S be an independent set of a graph $G = (V, E)$ and $v \in V\backslash S$. We denote as in [29], $H(v, S) := \{w \in V\backslash(S \cup \{v\} \cup N(v)) : N_S(w) \subset N_S(v)\}$. Given a graph $G = (V, E)$, an independent set S, and a vertex $v \in V\backslash S$, Mosca [29] defined that v is *augmenting* for S (and that S admits an augmenting vertex), if $G[H(v, S)]$ contains an independent set S_v such that $|S_v| \geq |N_S(v)|$. This implies that $H' := (S_v \cup \{v\}, N_S(v), E(H'))$ is an augmenting graph. Then by Theorem 1 and Lemma 3, we restrict ourselves in the following problem.

Here we use some notations in [30]. Let K be a graph, we denote as $K^{(h)}$ the graph obtained from K by adding $h + 1$ new vertices $v, s_1, \ldots, s_h$ such that $\{s_1, s_2, \ldots, s_h\}$ induce an independent set, s_i's dominate K, while v is adjacent only to s_i's.

By considering the problem of finding augmenting bipartite chains, we obtain the following result as an extension of a similar result in [30] for P_5-free graphs.

Theorem 5 *Given three integers h, l, m and a graph K, if the MIS problem in the ($S_{2,2,5}$,banner$_2$,domino,M_m, R^1_l, R^2_l, R^3_l, K)-free graph class is polynomially solvable, then so it is in the ($S_{2,2,5}$, banner$_2$,domino,M_m, R^1_l, R^2_l, R^3_l, $K^{(h)}$)-free graph class.*

Corollary 4 *Given two integers h, m and a graph K, if the MIS problem is polynomially solvable in ($S_{1,2,5}$,banner$_2$,domino,M_m, K)-free graph class, then so it is the ($S_{1,2,5}$,banner$_2$,domino,M_m,$K^{(h)}$)-free graph class.*

Especially, Theorem 5 leads to some interesting polynomially solvable graph classes of the MIS problem. Remind that the MIS problem was proved to be polynomially solvable in P_5-free graphs [22], (P_2+claw)-free graphs [24], $2P_3$-free graphs [26], and pK_2-free graphs [1], we have the following consequence.

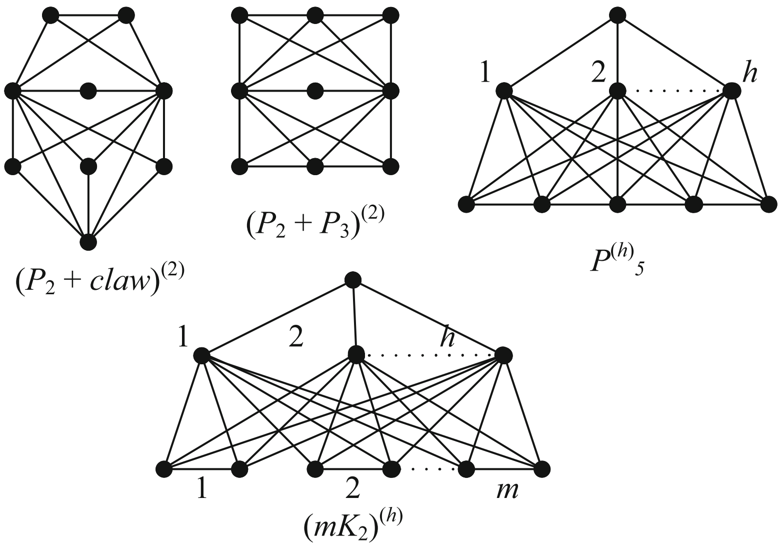


Fig. 5 Special graphs in Corollary 5

Corollary 5 *Given four integers h, l, m, p, the MIS problem is polynomially solvable in the following graph classes (see Figure 5):*

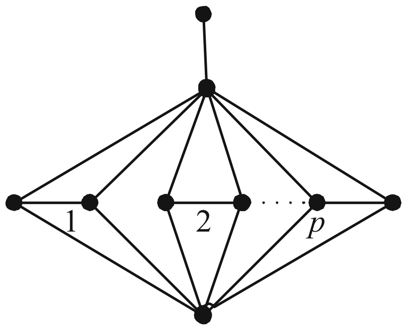
1. *($S_{2,2,5}$,banner$_2$,domino,M_m, R_l^1, R_l^2, R_l^3, $P_5^{(h)}$)-free graphs,*
2. *($S_{2,2,5}$,banner$_2$,domino,M_m, R_l^1, R_l^2, R_l^3, $(P_2+claw)^{(2)}$)-free graphs,*
3. *($S_{2,2,5}$,banner$_2$,domino,M_m, R_l^1, R_l^2, R_l^3, $(2P_3)^{(2)}$)-free graphs, and*
4. *($S_{2,2,5}$,banner$_2$,domino,M_m, R_l^1, R_l^2, R_l^3, $(pK_2)^{(h)}$).*

Let tree$_r$ be the graph of the form tree^1 with parameter r (Figure 2). Let $G = (V, E)$ be a graph, U be a subset of V and u be a vertex of G outside U. We say that u *distinguishes* U if u has both a neighbor and a non-neighbor in U. A subset $U \subseteq V(G)$ is called a *module* in G if it is indistinguishable for any vertex outside U. A module U is *trivial* if U is a single vertex or V itself, otherwise it is *non-trivial*. A graph whose each module is trivial is called *prime*. It has been shown (for example in [27]) that if the problem is polynomially solvable for every prime graph of a graph class $\mathscr{X}$, then it is also polynomial solvable in $\mathscr{X}$. Using the modular decomposition technique described in [6] for P_5-free graphs we can extend Case 4., the case $h = 2$ of the above corollary, as follows.

Corollary 6 *Given four integers l, m, p, and r, the MIS problem is polynomially solvable in ($S_{2,2,5}$,banner$_2$,domino,M_m, tree$_r$,R_l^1, R_l^2, R_l^3, Q_p)-free graph class (see Figure 3).*

Proof We show that a prime (Q_p,tree$_r$)-free graph is $((2p+r-2)K_2)^{(2)}$-free. Indeed, let G be a prime (Q_p,tree$_r$)-free graph, and suppose that G contains an induced subgraph Q' isomorphic to $((2p+r-2)K_2)^{(2)}$.

Fig. 6 Q_p



Let $T \subseteq V(G)$ be the subset of vertices of G adjacent to every vertex of the $(2p + r - 2)K_2$ of Q'. Since T contains at least two non-adjacent vertices, $\bar{G}[T]$, the complement subgraph of G induced by T, contains a non-trivial component C. Since G is prime, C is not a module. Hence, there exists a vertex $v \in V(G)\backslash C$ distinguishing C, i.e. $v \sim c_1$ and $v \nsim c_2$ for some vertices c_1, c_2 in C. Moreover, since $\bar{G}[C]$ is connected, we can substitute c_1, c_2 by two vertices of the path connecting them and can assume that $c_1 \nsim c_2$ in G.

If v is adjacent to every vertex of the $(2p+r-2)K_2$ of Q', then $v \in T$ and since $v \nsim c_2$, $v \in C$, a contradiction. Hence, there exists a vertex c' of the $(2p+r-2)K_2$ of Q' such that $c' \nsim v$.

Since G is tree$_r$-free, v distinguishes at most $r - 1$ edges of the $(2p + r - 2)K_2$ of Q'. Then we have the two following cases.

Case 1 v is adjacent to both end-vertices of at least p edges of the $(2p + r - 2)K_2$ of Q'. Then $\{v, c', c_2\}$ together with these p edges induce a Q_p, a contradiction.

Case 2 v is non-adjacent to both end-vertices of at least p edges of the $(2p + r - 2)K_2$ of Q'. Then $\{v, c_1, c_2\}$ together with these p edges induce a Q_p, a contradiction (Figure 6).

4 Augmenting Graphs in Other Problems

In [19], we have extended the augmenting graph approach for a more more general combinatorial and graph-theoretical problem, say Maximum Π-set Problem. Given a graph G, the problem asks for a maximum vertex subset such that the induced subgraph satisfies some give properties Π. Here are some examples for the property Π and related problems.

Maximum k-Independent Set. [13] Π: Every vertex is of degree at most $k - 1$. Note that the Maximum Independent Set is the case $k = 1$.

Maximum k-Path Free Set. Π: The graph contains no path (not necessarily induced) of k vertices ($k \geq 2$), also called k-path free. This problem is a dual version of the Minimum Vertex k-Path Cover problem [7].

Maximum Forest. Π: The graph contains no cycle. This problem is a dual version of the Minimum Feedback Vertex Cover problem [12].

Maximum Induced Bipartite Subgraph. Π: The graph contains no cycle of odd length.

Maximum k-Acyclic Set. Π: The graph contains no cycle of length at most k.

Maximum k-Chordal Set. Π: The graph contains no cycle of length larger than k.

Maximum k-Cycle Free Set. Π: The graph contains no cycle of length k ($k \geq 3$), also called k-cycle free. This problem is a dual version of the Minimum Vertex k-Cycle Cover problem.

Maximum Induced Matching. [8] Π: Every vertex is of degree one.

Maximum k-Regular Induced Subgraph. [9] Π: Every vertex is of degree k.

Maximum k-Regular Induced Bipartite Subgraph. [9] Π: The graph is bipartite and every vertex is of degree k.

Maximum Induced k-Cliques. Π: Every connected component is a k-clique. This problem is a generalization of Maximum Induced Matching problem ($k = 2$).

We have considered two special cases of the problem. First, the property Π is hereditary (i.e., if a graph G satisfies Π, then every induced subgraph of G satisfies Π) and additive (i.e., a graph G satisfies Π if and only if every connected component of G satisfies Π). Second, Π is of the form $\mathscr{F}$-induced subgraph, i.e. every connected component of G belongs to some graph set $\mathscr{F}$. In both cases, we have defined the augmenting graphs and the key theorem, says the Π-set is maximum if and only if there exists no augmenting graph. We also have considered a simple case, says the ($S_{1,2,l}$,banner$_l$,$K_{1,m}$)-free minimal augmenting graph either belongs to a finite set or is augmenting extended-chain. By showing that we can find augmenting extended-chain in polynomial time for the above problem, we obtained polynomial algorithms for these non-trivial problems in ($S_{1,2,l}$,banner$_l$,$K_{1,m}$)-free graphs.

5 Conclusion

In this paper, we have combined the methods applied for P_5-free graphs and ($S_{1,2,5}$,banner)-free graphs to generalize some known results. By extending the method of Lozin and Milanič [23] for ($S_{1,2,5}$,banner)-free graphs, we show that the problem can be restricted to finding augmenting chains and augmenting bipartite-chains in ($S_{2,2,5}$,banner$_2$,domino, M_m)-free graphs by using concepts of redundant sets (in the extended senses). It leads us to some generalizations of results about (P_5, $K_{2,m} - e$)-free graphs [4], (P_5, $K_{3,3} - e$)-free graphs [16], and augmenting vertex in P_5-free graphs [14, 16, 29, 30]. It also leads to some interesting results in ($S_{2,2,5}$,banner$_2$,domino,M_m)-free graphs, e.g. Corollaries 5 and 6.

Note that $S_{1,1,2}$ (fork) and $S_{0,1,3}$ (P_5) are the largest single known forbidden subgraphs, for which the MIS problem is polynomially solvable. For $S_{i,j,k}$ such that $i + j + k \geq 5$, even for subclasses, to the best of our knowledge, there are still not many known results except in some subclasses of P_6-free graphs, graphs of bounded maximum degree, planar graphs, ($S_{1,2,5}$,banner)-free graphs [23], ($S_{1,1,3}$, $K_{p,p}$)-free graphs [10], and ($S_{1,2,l}$,banner$_l$,$K_{1,m}$)-free graphs [19]. Combining different techniques is a potential approach helping us extend these results to tackling the general question about complexity of the MIS problem in $S_{i,j,k}$-free graphs.

Besides, by applying a technique, which has been used for P_5-free graphs, for a larger graph class, e.g. $S_{2,2,5}$-free graphs, we believe that it is possible to apply other techniques, which were used in P_5-free graphs, in $S_{2,2,l}$-free graphs.

The augmenting graph technique is also very potential in many other non-trivial combinatorial and graph-theoretical problems.

Acknowledgements This research is supported by National Foundation for Science and Technology Development (NAFOSTED) of Vietnam, project code: 101.99-2016.20. We would like to express our special thanks to an anonymous reviewer for his/her very useful suggestions and comments.

Appendix 1: Proof of Lemma 2

Proof (of Lemma 2) Let $H = (B, W, E)$ be a minimal augmenting graph. If $\Delta(H) = 2$, then H is a cycle or a chain. Since H is bipartite and $|B| = |W| + 1$, H cannot be a cycle. Now, assume that H is not a chain. We show that either (i) there exists some vertex a such that there is no vertex of distance $2k + l + 1$ from a or (ii) H is an augmenting extended-chain or augmenting apple. Note that, every vertex of H is of degree at most $m - 1$, otherwise an induced $K_{1,m}$ appears, a contradiction. Since H is connected, if we have (i), then

$$|V(H)| \leq \sum_{i=0}^{2k+l+1} (m-1)^i = \frac{1-(m-1)^{2k+l+2}}{2-m},$$

i.e., H belongs to some finite set of augmenting graphs.

If a white vertex $w \in W$ has two black neighbor b_1, b_2 of degree one, then $\{b_1, a, b_2\}$ is an augmenting P_3, a contradiction. Hence, we have the following observation.

Claim 1 Every white vertex of H has at most one black neighbor of degree one. In particular, if a white vertex w is of degree at least four, then there are at least three neighbors of w of degree two.

Claim 2 Either H contains a vertex, say a, of degree at least three and a has at least three neighbors of degree at least two or H is an augmenting apple.

Proof Since H is neither a chain or a cycle, there exists at least one vertex of degree at least three.

By Corollary 2, every white vertex of H is of degree at least two, i.e. every white neighbor of a black vertex has another black neighbor. Hence, if H contains a black vertex of degree three, then this vertex is a desired vertex a.

Hence, we assume that **(1)** every black vertex of H is of degree at most two. If there exist two black vertices of degree one, then by (1), the path connecting these two black vertices is an augmenting chain, a contradiction. Hence, we assume that **(2)** there exists at most one black vertex of degree one.

By Claim 1, there exists no white vertex of degree four or we have a desired vertex a. Moreover, if there exist two white vertices of degree three, then either one of them has three neighbors of degree two, i.e. we have a desired vertex a, or we have two black vertex of degree one.

Now, if every white vertex of H is of degree two except one of degree three whose one black neighbor is of degree one, then H is an augmenting apple.

Let a be a vertex in the conclusion of the above claim. Denote by V_i the subset of vertices of H of distance i from a. Let a_p be the vertex of maximum distance from a and assume that $p \geq 2k + l + 1$. Let $P := (a_0, a_1, \ldots, a_p)$, where $a_i \in V_i$, be a shortest path connecting $a = a_0$ and a_p. Let $V_1 = \{a_1, b_{1,1}, b_{1,2}, \ldots\}$, and $b_{i+1,j}$ be a vertex of $N_{V_{i+1}}(b_{i,j})$, if such one exists. By the assumption about a, $b_{2,1}$, and $b_{2,2}$ exist (note that they may coincide).

We show that $a_i \nsim b_{i+1,1}$ and $a_{i+1} \nsim b_{i,1}$ for $i = 1, 2, \ldots, 2k$ by induction. Note that it also implies that $b_{i,j} \neq a_i$ for every i, j.

If $a_2 \sim b_{1,1}$, then $\{b_{1,1}, a, a_1, a_2, a_3, \ldots, a_{l+2}\}$ induces a banner$_l$, a contradiction.

If $a_1 \sim b_{2,1}$, then either $\{b_{2,1}, b_{1,1}, a, a_1, a_2, \ldots, a_{l+1}\}$ or $\{b_{2,1}, a_1, a_2, a_3, a_4, \ldots, a_{l+3}\}$ induces a banner$_l$ depending on $a_3 \sim b_{2,1}$ or not, a contradiction.

Now, by induction hypothesis, consider $2 \leq i \leq k$. If $a_i \sim b_{i+1,1}$, then either $\{b_{i+1,1}, a_i, a_{i+1}, a_{i+2}, a_{i+3}, \ldots, a_{i+l+2}\}$ induces a banner$_l$ or $\{b_{i+1,1}, b_{i,1}, \ldots, b_{1,1}, a, a_1, \ldots, a_i, a_{i+1}, \ldots, a_{i+l}\}$ induces an apple$^l_{2i+2}$ depending on $a_{i+2} \sim b_{i+1}$ or not, a contradiction. If $a_{i+1} \sim b_{i,1}$ for $2 \leq i \leq k$, then $\{b_{i,1}, b_{i-1,1}, \ldots, b_{1,1}, a, a_1, a_2, \ldots, a_{i+1}, a_{i+2}, \ldots, a_{i+l+1}\}$ induces an apple$^l_{2i+2}$, a contradiction.

Again, by induction hypothesis, consider $k + 1 \leq i \leq 2k$. If $a_i \sim b_{i+1,1}$, then either $\{b_{i+1,1}, a_i, a_{i+1}, a_{i+2}, a_{i+3}, \ldots, a_{i+l+2}\}$ induces a banner$_l$ or $\{a_{i-1}, a_{i-2}, \ldots, a_1, a, b_{1,1}, b_{2,1}, \ldots, b_{i+1,1}, a_i, a_{i+1}, a_{i+2}, \ldots, a_{i+l}\}$ induces an $S_{2,2k,l}$ depending on $a_{i+1} \sim b_{i,1}$ or not, a contradiction. If $a_{i+1} \sim b_{i,1}$, then $\{a_i, a_{i-1}, a_{i-2}, \ldots, a_1, a, b_{1,1}, b_{2,1}, \ldots, b_{i,1}, a_{i+1}, a_{i+2}, \ldots, a_{i+l+1}\}$ induces an $S_{2,2k,l}$, a contradiction.

Hence, a_i has only one neighbor, say a_{i+1}, in V_{i+1} and only one neighbor, say a_{i-1}, for $i = 1, 2, \ldots, 2k$.

If $b_{i,1} \sim b_{i+1,2}$ for some $1 \leq i \leq 2k-1$ (if such two vertices exist), then $\{b_{1,1}, \ldots, b_{i,1}, b_{i+1,2}, b_{i,2}, \ldots, b_{1,2}, a, a_1, \ldots, a_l\}$ induces an apple^l_{2i+2}, a contradiction. Hence, $b_{i,j}$ (if such vertex exists) has at most one neighbor in V_{i+1} for $1 \leq i \leq 2k-1$. It also implies that $b_{i,j} \neq b_{i,k}$ for every $1 \leq i \leq 2k$ and $j \neq k$ if such vertices exist.

If V_{2k} contains at least two vertices, say a_{2k} and, without loss of generality, $b_{2k,1}$, then $\{b_{2,2}, b_{1,2}, a, b_{1,1}, b_{2,1}, \ldots, b_{2k,1}, a_1, a_2, \ldots, a_l\}$ induces an $S_{2,2k,l}$, a contradiction.

To summarize, $V_{2k} = \{a_{2k}\}$, every vertex of V_i has only one neighbor in V_{i-1}, for every $1 \leq i \leq p$.

Let T be the connected component of $H - a_1$ containing a. Then T is a tree by the above arguments. We show that a is black. Indeed, for contradiction, suppose that a is white. Let a_1 be the black vertex b of Corollary 1. Then there is a perfect matching between $B \cap T$ and $W \cap T$. Let b be a leaf of T. Then by Corollary 2, b is black and hence $\mu(b)$ be the (only) white neighbor of b. It also implies that $\mu(b)$ has only one neighbor being a leaf. Indeed, if $\mu(b)$ has another black neighbor being a leaf b', then there exists no $\mu(b')$, a contradiction. Then by induction on T, a has only one black neighbor in T, a contradiction to a is of degree at least three. Hence, we have the following claim.

Claim 3 If a is a vertex of the conclusion of Claim 2, then a is black. Moreover, there exists a neighbor w of a such that the connected component of $H - w$ containing a is a tree T, every vertex of T is of distance at most $2k-2$ to a, and every white vertex of T is of degree two.

Let a be the black vertex b of Corollary 1. Then there is a perfect matching between $B \cap T \backslash \{a\}$ and $W \cap T$, i.e. $|B \cap T| = |W \cap T| + 1$. Claims 1 and 3 lead to the following observation.

Claim 4 Every white vertex w of H is either of degree two or three. Moreover, in the latter case, exactly one black neighbor of w is of degree one.

Let j be the largest number such that $|V_j| \geq 2$. Then $2 \leq j \leq 2k-2$. Moreover, j is even, since every leaf of T is black.

Note that every black vertex a_q such that $2k - j < q < p - 2k$ is of degree two, otherwise a_q becomes a vertex of the conclusion of Claim 2 and there exist at least two vertices of degree $2k$ from a_q, a contradiction to Claim 3.

Let T_1 and T_2 be the two connected component of $H - a_{2k-j+1} - a_{p-2k-1}$ containing a_{2k-j} and a_{p-2k}, respectively. Then by Claim 3, T_1 and T_2 are trees such that the most distance between a vertex of T_1 (respectively, T_2) to a_{2k-j} (respectively, a_{p-2k}) is $2k-2$. Moreover $|W \cap (T_1 + T_2)| + 2 = |B \cap (T_1 + T_2)|$.

Now, every white vertex a_q, where $2k - j < q < p - 2k$, is of degree two or three, and in the later case a black neighbor of a_q different from a_{q-1} and a_{q+1} is of degree one. Hence, every such white vertex is degree two, otherwise we have a contradiction to $|W| + 1 = |B|$.

Thus, H is an augmenting $(2k-1, m)$-extended-chain.

Appendix 2: Proof of Lemma 3

We go through the Proof by first obtaining some results related to the cases when the considered augmenting graph contains a $K_{1,m}$ as an induced subgraph.

Lemma 8 *Let $G = (X, Y, E)$ be a bipartite graph such that there exists a vertex $x \in X$ and $N_Y(x) = Y$. Assume that $|X| = m+1$. Then at least one of the following statements is true.*

1. *H_2 contains a banner$_2$ or a domino.*
2. *We can linearly order $X = (x, x_1, x_2, \ldots, x_m)$ so that there exists a natural p, with $0 \le p \le m$, such that (i) $N_Y(x_1) \supseteq \ldots \supseteq N_Y(x_p)$ and $|N_Y(x_i)| = 1$ for every $i \ge p+1$. Moreover, if $p \ge m-1$, then G is a bipartite-chain.*

Proof First, assume that Case 1 does not happen. We linearly order X by the construction method.

Assume that we already have chosen $x_1, \ldots, x_p$. Let $U = X \backslash \{x, x_1, \ldots, x_p\}$. Let $x_{p+1} \in U$ be a vertex such that $|N_Y(x_{p+1})|$ is largest among vertices in U. Suppose that $|N_Y(x_{p+1})| \ge 2$ and there exists a vertex $x_i \in U\backslash\{x_{p+1}\}$ such that $x_i \sim y_i$ and $x_{p+1} \nsim y_i$ for some $y_i \in Y$. By the choice of x_{p+1}, $x_i \nsim y_j$ for some $y_j \in N_Y(x_{p+1})$. Then $\{x, y_k, y_i, y_j, x_{p+1}, x_j\}$ induces a domino or a banner$_2$ for some $y_k \in N_Y(x_{p+1})\backslash\{y_j\}$ depending on $x_i \sim y_k$ or not and x is a vertex of degree three in both cases, a contradiction.

If $p \ge m-1$, then $N_Y(x) \supseteq N_Y(x_i) \supseteq N_Y(x_j)$ for every $1 \le i < j \le m$. We show that for $y_i, y_j \in Y$, either $N_X(y_i) \subseteq N_X(y_j)$ or $N_X(y_j) \subseteq N_X(y_i)$. Indeed, suppose that $y_i \sim x_i$ and $y_j \sim x_j$ for some $x_i \in X\backslash N(y_j)$ and $x_j \in X\backslash N(y_i)$. Then $N_Y(x_i) \nsubseteq N_Y(x_j)$ and $N_Y(x_j) \nsubseteq N_Y(x_i)$, a contradiction.

Lemma 9 *If an ($S_{2,2,5}$,banner$_2$,domino)-free minimal augmenting graph H contains no black vertex of degree more than k ($k \ge 2$), then the degree of each white vertex is at most $k^2 + k + 2$.*

Proof Suppose that H contains a white vertex w of degree more than $k^2 + k + 2$. Denote by V_j the set of vertices of H at distance j from w. Hence, $|V_1| \ge k^2+k+3$.

Claim 5 $|V_2| \ge k^2 + k + 1$, V_2 contains at least $k^2 + 1$ vertices having only one neighbor in V_1, i.e. having a neighbor in V_3, and $|V_3| \ge k + 1$.

Proof Suppose that $V_3 = \emptyset$. Then by Lemma 1, $|V_2| = |V_1| - 2 \ge k^2 + k + 1$. Let p be the p in 2. of Lemma 8. Note that $p \le k$, otherwise there exists a black vertex in V_1 having at least k neighbors in H, a contradiction. Hence, by Lemma 8, there exists a white vertex in V_2 having only one neighbor in V_1, i.e. only one black neighbor. This contradiction (with Corollary 2) implies that $V_3 \ne \emptyset$.

Then $|V_2| \ge k^2 + k + 1$, otherwise $H[\{b\} \cup V_1 \cup V_2]$ is an augmenting graph, a contradiction. Again, by Lemma 8 and condition that there is no black vertex of degree larger than k, V_2 contains at least $k^2 + 1$ vertices having only one neighbor in V_1, i.e. having a neighbor in V_3 by Corollary 2. Since every black vertices of V_3 has at most k neighbors in V_2, $|V_3| \ge k + 1$. □

Claim 6 $V_4 = \emptyset$, i.e. $|V_3| + |V_1| = |V_2| + 2$.

Proof Suppose that V_4 contains a (white) vertex x and let y be its neighbor in V_3. Assume that $y \sim w_1 \in V_2$ and $w_1 \sim b_1 \in V_1$.

If $w_1 \sim b_2$ for some $b_2 \in V_1 \backslash \{b_1\}$, then $\{b_1, w, b_2, w_1, y, x\}$ induces a banner$_2$, a contradiction. Hence, $N_{V_1}(w_1) = \{b_1\}$.

By Corollary 2, x has at least one more black neighbor, named z ($z \in V_3$ or $z \in V_5$). Now, let b_1 be the b in Corollary 1. We have $|\mu(V_1 \backslash \{b_1, \mu(w)\})| \geq k^2 + k + 1$. Since $d(y), d(z) \leq k$, V_1 contains at least two vertices, named b_2, b_3 whose the neighbors, say $w_2 = \mu(b_2), w_3 = \mu(b_3) \in V_2$, respectively, adjacent to neither y nor z.

If $w_2 \sim b_1$, then $\{w, w_1, w_2, b_1, b_2, y\}$ induces a domino or a banner$_2$, depending on $y \sim w_2$ or not, a contradiction. If $w_2 \sim b_4$ for some $b_4 \in V_1 \backslash \{b_1, b_2\}$, then $\{b_2, w_2, b_4, w, b_1, w_1\}$ induces a banner$_2$, a contradiction. Hence, w_1, w_2, and w_3, each has only one neighbor in V_1. Moreover, $z \nsim w_1$, otherwise, $\{y, x, z, w_1, b_1, w\}$ induces a banner$_2$, a contradiction. Now, $\{w_3, b_3, w, w_2, b_2, b_1, w_1, y, z, x\}$ induces an $S_{2,2,5}$, a contradiction. Therefore, V_4 is empty and $|V_3| + |V_1| = |V_2| + 2$ by Lemma 1. □

Let $b \in V_3$ be the vertex b in Corollary 1. Since $\mu(w)$ has at most $k - 1$ neighbors in $\mu(V_3 \backslash \{b\})$, there exists a vertex $d_1 \in V_3$ such that $\mu(d_1) \nsim \mu(w)$.

Claim 7 $\mu(d_1)$ has a neighbor a_1 in V_1 such that $\mu(a_1)$ has no neighbor in V_1 other than a_1.

Proof Let a_1 be a neighbor of $\mu(d_1)$ in V_1, i.e. $\mu(a_1) \neq w$. If $\mu(a_1)$ has no neighbor in V_1 other than a_1, then we have the statement of the claim. Now, let a_2 be a neighbor of $\mu(a_1)$ in V_1. Then $\mu(d_1) \sim a_2$, otherwise $\{w, a_2, \mu(a_1), a_1, \mu(d_1), d_1\}$ induces a domino or a banner$_2$ depending on $d_1 \sim \mu(a_1)$ or not, a contradiction. It implies that $\mu(a_2) \neq w$. We continue considering $\mu(a_2)$. Since V_1 is finite, this process must stop, i.e. we have the claim. □

Note that $d_1 \nsim \mu(a_1)$, otherwise $\{\mu(a_1), d_1, \mu(d_1), a_1, w, \mu(w)\}$ induces a domino or a banner$_2$ depending on $\mu(w) \sim \mu(a_1)$ or not, a contradiction. Since $\mu(a_1)$ has no neighbor in V_1 other than a_1, by Corollary 2, $\mu(a_1)$ has a neighbor $d_2 \in V_3$.

Since $|N_{V_2}(\{d_1, d_2\})| \leq 2k$, there is a vertex $a \in V_1 \backslash \{\mu(w)\}$ such that $\mu(a)$ is not adjacent to d_1, d_2. Then $\mu(a) \nsim a_1$, otherwise $\{w, a, \mu(a), a_1, \mu(a_1), d_2\}$ induces a banner$_2$, a contradiction. If $\mu(a) \sim a_2$ for some $a_2 \in V_1$, then $\{a, \mu(a), a_2, w, a_1, \mu(a_1)\}$ induces a banner$_2$, a contradiction. Hence, $\mu(a)$ has only one neighbor in V_1 and has a neighbor, named $d_3 \in V_3$, by Corollary 2.

Then $\mu(d_1) \nsim a$, otherwise $\{w, a_1, a, \mu(d_1), d_3, \mu(a)\}$ induces a domino or a banner$_2$ depending on $d_3 \sim \mu(d_1)$ or not, a contradiction. Moreover $\mu(d_3) \nsim a$, otherwise $\{w, a_1, a, \mu(a), d_3, \mu(d_3)\}$ induces a domino or a banner$_2$ depending on $a_1 \sim \mu(d_3)$ or not, a contradiction.

We show that $d_1 \nsim \mu(d_3)$. Indeed, if $d_1 \sim \mu(d_3)$, then $\mu(d_1) \nsim d_3$, otherwise $\{\mu(d_3), d_1, \mu(d_1), d_3, \mu(a), a\}$ induces a banner$_2$, a contradiction. If $\mu(d_1) \sim a_2$ for some $a_2 \in V_1 \backslash \{a_1\}$, then $\{w, a_1, \mu(d_1), a_2, d_1, \mu(d_3)\}$ induces a domino or a banner$_2$ depending $\mu(d_3) \sim a_2$ or not, a contradiction. If $\mu(d_3)$ has two neighbors $a_2, a_3 \in V_1 \backslash \{a_1\}$, then $\{a_2, w, a_3, \mu(d_3), d_1, \mu(d_1)\}$ induces a banner$_2$, a contradiction. Hence, $\mu(d_1)$ has only one neighbor in V_1 and $\mu(d_3)$ has at most one neighbor in V_1 different from a_1. Thus, because $|N_{V_2}(d_1, d_2)| \le 2k$, there exist two vertices $b_1, b_2 \in V_1 \backslash \{\mu(w)\}$ such that $\mu(b_1), \mu(b_2)$ each has only one neighbor in V_1 and is not adjacent to d_1, d_3. Now, $\{\mu(b_1), b_1, w, b_2, \mu(b_2), a_1, \mu(d_1), d_1, \mu(d_3), d_3\}$ induces an $S_{2,2,5}$, a contradiction.

Similarly, d_3 is not adjacent to $\mu(d_1)$, $\mu(a_1)$, and $\mu(d_3) \nsim d_2$. Moreover $\mu(d_1) \nsim d_2$, otherwise $\{w_1, d_2, \mu(d_1), a_1, w, \mu(w)\}$ induces a banner$_2$, a contradiction. Similarly, $\mu(a_1) \nsim d_1$.

Now, $\{d_2, \mu(a_1), a_1, \mu(d_1), d_1, w, a, \mu(a), d_3, \mu(d_3)\}$ induces an $S_{2,2,5}$, a contradiction. □

Proof (of Lemma 3) We proof by contradiction. Let $b \in B$ such that $|N_W(b)|$ is largest. If every black vertex is of degree one, then H is an augmenting P_3. If $N_W(b) = W$, then we have 4. By Lemma 9, if every black vertex of H is of degree bounded by a given number k, then every white vertex of H is of degree bounded by $k^2 + k + 2$, i.e. H is $K_{1,m}$-free for $m = k^2 + k + 3$. In this case, by Lemma 2, we have 1. or 2.

Now, we assume that $10 \le |N_W(b)| \le |W| - 1$. Let b be the vertex b of Corollary 1. Let $A = N(b) = \{w_1, w_2, \ldots, w_k\}$ $(k \ge 10)$, $C = W \backslash A$, i.e. $C \ne \emptyset$. Let $b_i = \mu(w_i)$. Let C_1 denote the set of vertices in C having at least one neighbor in $\mu(A)$ and $C_0 = C \backslash C_1$. By the connectivity of H, one can choose $\mu(A)$ in order that $C_1 \ne \emptyset$. We have the following observations.

Claim 8 $H[A \cup \mu(A)]$ is an induced sub-matching of M.

Proof We show that $b_i \nsim w_j$ for every pair i, j such that $i \ne j$, $1 \le i, j \le k$. Let $z \in C_1$ and without loss of generality, assume that $z \sim b_1 \in \mu(A)$.

By the choice of b, b_1 is not adjacent to all w_i's, without loss of generality, assume that $b_1 \nsim w_2$.

Now, $b_2 \nsim w_1$, otherwise $\{b, b_1, b_2, w_1, w_2, z\}$ induces a domino or a banner$_2$ depending on $b_2 \sim z$ or not, a contradiction.

Moreover, $b_2 \nsim w_i$ for every $i > 2$, otherwise $\{b, b_1, b_2, w_1, w_2, w_i\}$ induces a domino or a banner$_2$ depending on $b_1 \sim w_i$ or not, a contradiction.

Now, $b_1 \nsim w_i$, for every $i > 2$, otherwise $\{w_1, b_1, w_i, b, w_2, b_2\}$ induces a banner$_2$, a contradiction.

Hence, $b_i \nsim w_1$ for $i > 2$, otherwise $\{b, w_i, b_i, w_1, b_1, z\}$ induces a domino or a banner$_2$, depending on $z \sim b_i$ or not, a contradiction.

Thus, $b_i \nsim w_2$ for $i > 2$, otherwise $\{w_2, b_i, w_i, b, w_1, b_1\}$ induces a banner$_2$, a contradiction.

Moreover $b_i \nsim w_j$, for any $j \ne i$ and $i, j > 2$, otherwise $\{w_j, b_i, w_i, b, w_1, b_1\}$ induces a banner$_2$, a contradiction.

Claim 9 There exists no vertex pair $z_1, z_2 \in C_1$ sharing two neighbors in $\mu(A)$.

Proof Suppose that there exists a vertex pair $z_1, z_2 \in C_1$ sharing two neighbors in $\mu(A)$, without loss of generality, assume that they are b_1, b_2. Then $\{z_1, b_2, z_2, b_1, w_1, b\}$ induces a banner$_2$, a contradiction. □

Claim 10 Given $z \in C_1$, $z \sim b_j$ for some $b_j \in \mu(A)$, a black neighbor c of z different from b_j, a black neighbor $\mu(t)$ of z for some $t \in C$, and another white neighbor $y \in C$ of $\mu(t)$ different from z, the following statements are true:

1. $c \nsim w_j$;
2. $y \nsim b_j$ and $\mu(y) \nsim z$; and
3. if $\mu(t) \sim w_i$ for some $i \neq j$, then y, z are not adjacent to b_i and $\mu(y) \nsim w_i$;
4. in particular, $\mu(y)$ and $\mu(t)$ cannot share a same neighbor in A.

Proof Suppose that $c \sim w_j$. Then $c \sim w_i$ for every $i \neq j$, otherwise $\{b_j, z, c, w_j, b, w_i\}$ induces a banner$_2$, a contradiction. But now, we have a contradiction to the choice of b.

Now, $y \nsim b_j$, otherwise $\{z, \mu(t), y, b_j, w_j, b\}$ induces a banner$_2$, a contradiction. Moreover, $\mu(y) \nsim z$, otherwise $\{w_j, b_j, z, \mu(t), y, \mu(y)\}$ induces a domino or a banner$_2$ depending on $\mu(y) \sim w_j$ or not, a contradiction.

Assume that $\mu(t) \sim w_i$ for some $i \neq j$. Then $z \nsim b_i$, otherwise $\{\mu(t), w_i, b_i, z, b_j, w_j\}$ induces a banner$_2$, a contradiction. Hence, $y \nsim b_i$, otherwise $\{b_i, y, \mu(t), w_i, b, w_j\}$ induces a banner$_2$, a contradiction. Now, $\mu(y) \nsim w_i$, otherwise $\{w_i, \mu(y), y, \mu(t), z, b_j\}$ induces a banner$_2$, a contradiction. □

Claim 11 Every black vertex different from b has at most one neighbor in A.

Proof Clearly, every black vertex of $\mu(A)$ has only one neighbor in A by Claim 8. Now, suppose that there exists some black vertex $y \in B \backslash (\{b\} \cup \mu(A))$ having two neighbors, without loss of generality, assume that they are $w_1, w_2 \in A$. Then y is adjacent to every vertex $w_i \in A \backslash \{w_1, w_2\}$, otherwise $\{w_1, y, w_2, b, w_i, b_i\}$ induces a banner$_2$, contradiction. Now, y is adjacent to every vertex of A and $\mu(y)$, a contradiction to the choice of b. □

Claim 12 There exists no vertex $b_j \in \mu(A)$ having two neighbors $z_1, z_2 \in C_1$ sharing another black neighbor, named $c \neq b_j$.

Proof Indeed, otherwise, by Claim 10, $c \nsim w_j$, then $\{z_1, c, z_2, b_j, w_j, b\}$ induces a banner$_2$, a contradiction. □

Claim 13 Given a vertex $b_j \in \mu(A)$, let $C(b_j)$ be the set of vertices of C_1 adjacent to b_j. Then $H[C(b_j) \cup \mu(C(b_j))]$ is an induced sub-matching of M.

Proof For contradiction, without loss of generality, suppose that $z_1, z_2 \in C$ are two neighbors of b_j and $z_1 \sim \mu(z_2)$. By Claim 10, $\mu(z_2) \nsim w_j$. Hence, $\{z_1, \mu(z_2), z_2, b_j, w_j, b\}$ induces a banner$_2$, a contradiction. □

Claim 14 If H contains a vertex $y \in C_1$ adjacent to at least $k - 3$ vertices of $\mu(A)$, then either H is of the form tree^5 or tree^6 or H contains a redundant set U of size at most 32, such that $H - U$ is of the form either tree^1, tree^4, tree^5, or tree^6.

Proof Let D_1 be the subset of vertices of C_1 sharing some neighbor in $\mu(A)$ with y, A_1 be the vertex subset of A such that $\mu(A_1) = N_{\mu(A)}(y)$, $A_2 = A\backslash A_1$, E_1 be the vertices subset of C_1 adjacent to some vertex in $\mu(A_2)$. Without loss of generality, assume that $w_1, w_2, \ldots, w_{k-3} \in A_1$. We have the following observations.

(**1**) y has no neighbor in $\mu(D_1)$ and $\mu(y)$ has no neighbor in $A_1 \cup D_1$. Indeed, by Claim 10, $\mu(y)$ has no neighbor in A_1. If for some $z \in D_1$, without loss of generality, assume that $z \sim b_1$, $y \sim \mu(z)$, then $y \nsim b_1$, by Claim 10, a contradiction. Moreover, since $\mu(y) \nsim w_1$, $\mu(y) \nsim z$, otherwise $\{z, \mu(y), y, b_1, w_1, b\}$ induces a banner$_2$, a contradiction.

(**2**) By Claim 9, every vertex of D_1 has exactly one neighbor in $\mu(A_1)$. In particular, every vertex of $C_1\backslash\{y\}$ has at most four neighbors in $\mu(A)$. Moreover, there exists only one vertex $y \in C_1$ adjacent to at least $k-3$ vertices in $\mu(A)$.

(**3**) Any two vertices of D_1 have different neighbors in $\mu(A_1)$. Indeed, without loss of generality, suppose that $z_1, z_2 \in D_1$ both are adjacent to b_1. By Claim 11, and since $|A_1| = k-3 \geq 7$, there exist $w_i, w_j \in A_1$ different from w_1 and not adjacent to $\mu(z_1), \mu(z_2)$. By (2) and Claim 13, $\{\mu(z_1), z_1, b_1, z_2, \mu(z_2), y, b_i, w_i, b, w_j\}$ induces an $S_{2,2,5}$, a contradiction.

(**4**) Similar to Claim 13, let $C(y)$ be the subset of vertices of C_0 adjacent to $\mu(y)$. Then $H[C(y) \cup \mu(C(y))]$ is an induced sub-matching of M.

(**5**) Similarly to (3) (using (4)), there is at most one vertex of C_0 adjacent to $\mu(y)$.

(**6**) $H[(C_1\backslash\{y\}) \cup \mu(C_1\backslash\{y\})]$ is an induced sub-matching of M. Indeed, suppose that for a couple of vertices $z_1, z_2 \in C_1\backslash\{y\}$, $z_1 \sim \mu(z_2)$. Without loss of generality, assume that z_1, z_2 are adjacent to $b_{i_1}, b_{i_2} \in \mu(A)$, respectively. Then by Claim 10, $\mu(z_2) \nsim w_{i_2}$. Hence, $z_1 \nsim b_{i_2}$, otherwise $\{z_2, \mu(z_2), z_1, b_{i_2}, w_{i_2}, b\}$ induces a banner$_2$, a contradiction. By (**2**) and Claim 11, there exists a pair of vertices $b_i, b_j \in \mu(A)$ not adjacent to z_1, z_2 such that w_i and w_j are not adjacent to $\mu(z_1)$, $\mu(z_2)$. Now, $\{b_i, w_i, b, w_j, b_j, w_{i_2}, b_{i_2}, z_2, \mu(z_2), z_1\}$ induces an $S_{2,2,5}$, a contradiction.

(**7**) There exists no vertex $t \in C\backslash\{y\}$ having a neighbor in $\mu(C_1\backslash\{y, \mu(t)\})$. Indeed, if $t \in C$ is adjacent to $\mu(z)$ for some $z \in C_1\backslash\{y, t\}$, then for the vertex b_j adjacent to z, $t \nsim b_j$ by Claim 10. By (2) and Claim 13, there exists a pair of vertices w_i, w_l non-adjacent to $\mu(z)$ such that b_i, b_l non-adjacent z, t. Now, $\{b_i, w_i, b, w_l, b_l, w_j, b_j, z, \mu(z), t\}$ induces an $S_{2,2,5}$, a contradiction.

(**8**) Similarly, there exists no vertex $t \in C_1\backslash\{y\}$ having a neighbor in $\mu(C\backslash\{y, \mu(t)\})$.

(**9**) If $C_0 = \{z\}$, then $z \sim \mu(y)$. If $|C_0| \geq 2$, then there exists a vertex $x \in C_0$ such that $x \sim \mu(z)$. For every such vertex x, the following statements are true: $y \sim \mu(x)$, $\mu(x) \nsim z$, and $\mu(x) \nsim w_i$ for $w_i \in A_1$. Moreover, if $|C_0| \geq 2$, then $A_2 = \emptyset$, i.e. y is adjacent to every vertex of $\mu(A)$.

Indeed, if $C_0 \neq \emptyset$, then by (7) and the minimality of H, there exists a vertex $z \in C_0$ such that $z \sim \mu(y)$, otherwise $|C_0| = |N_H(C_0)|(= |\mu(C_0)|)$, a contradiction. Moreover, no other vertex of C_0 is adjacent to $\mu(y)$ by (5). Hence, if $|C_0| \geq 2$, then, again by (7) and the minimality of H, there exists a vertex $x \in C_0$ such that $x \sim \mu(z)$.

Let $x \in C_0$ such that $x \sim \mu(z)$. Since $\mu(z) \nsim y$ by Claim 10, $x \nsim \mu(y)$, otherwise $\{z, \mu(z), x, \mu(y), y, b_1\}$ induces a banner$_2$, a contradiction. Thus, $\mu(x) \nsim z$, otherwise $\{y, \mu(y), z, \mu(z), x, \mu(x)\}$ induces a domino or a banner$_2$, depending on $\mu(x) \sim y$ or not, a contradiction. Now, if $y \nsim \mu(x)$, then by Claim 11, there exists a pair of vertices $b_i, b_j \in \mu(A_1)$ such that w_i and w_j are not adjacent to $\mu(x), \mu(z)$ and $\{w_i, b_i, y, b_j, w_j, \mu(y), z, \mu(z), x, \mu(x)\}$ induces an $S_{2,2,5}$, a contradiction. Then $\mu(x) \nsim w_i$ for any $w_i \in A_1$, otherwise $\{y, b_i, w_i, \mu(x), x, \mu(t)\}$ induces a banner$_2$, a contradiction.

Assume that $|C_0| \geq 2$, we show that $A_2 = \emptyset$. Indeed, without loss of generality, assume that $y \nsim b_k$. Let $x \in C_0$ be a vertex such that $x \sim \mu(z)$. Then $\mu(y)$ or $\mu(z)$ is not adjacent to w_k, otherwise since $z \nsim w_k$ by Claim 10, $\{z, \mu(z), w_k, \mu(y), y, b_1\}$ induces a banner$_2$, a contradiction. Similarly, $\mu(x)$ or $\mu(z)$ is not adjacent to w_k. Now, $\mu(y) \nsim w_k$, otherwise since there exists a pair of vertices $w_i, w_j \in A_1$ not adjacent to $\mu(y), \mu(z)$ by Claim 11, $\{b_i, w_i, b, w_j, b_j, w_k, \mu(y), z, \mu(z), x\}$ induces an $S_{2,2,5}$, a contradiction. By similar reasons, $\mu(x) \nsim w_k$. Now, by Claim 11, there exists a vertex $w_i \in A_1$ not adjacent to $\mu(x)$ and $\{z, \mu(y), y, \mu(x), x, b_i, w_i, b, w_k, b_k\}$ induces an $S_{2,2,5}$, a contradiction.

(**10**) If $|D_1| \geq 2$, then no vertex of $\mu(D_1)$ has a neighbor in A. Indeed, by (3), without loss of generality, let $z_1, z_2 \in D_1$ be adjacent to b_1, b_2, respectively. To the contrary, suppose that $\mu(z_1)$ has a neighbor $w_i \in A$. By Claim 10, $w_i \neq w_1$. If $w_i = w_2$, then by (1), (6), and Claims 10, 11, $\{z_2, b_2, w_2, b, w_j, \mu(z_1), z_1, b_1, y, \mu(y)\}$ induces an $S_{2,2,5}$ for some vertex $w_j \neq w_1, w_2$ such that $w_j \nsim \mu(z_1)$, a contradiction. If $w_i \neq w_1, w_2$, then by (1) and (6), $\{w_2, b, w_i, \mu(z_2), z_2, \mu(z_1), z_1, b_1, y, \mu(y)\}$ induces an $S_{2,2,5}$ in the case that $\mu(z_2) \sim w_i$, or $\{\mu(z_2), z_2, b_2, y, \mu(y), w_2, b, w_i, \mu(z_1), z_1\}$ induces an $S_{2,2,5}$ in the case that $\mu(z_2) \nsim w_i$, a contradiction.

(**11**) If there exist two vertices $z_1, z_2 \in C_1$ sharing a neighbor in $\mu(A_2)$, then either H is of the form tree^5 or there is a redundant set U containing at most four vertices such that $H - U$ is of the form tree^2 or tree^5.

First, since $A_2 \neq \emptyset$, $|C_0| \leq 1$ by (9). Without loss of generality, assume that z_1, z_2 share a neighbor $b_k \in \mu(A_2)$.

If z_2 has another neighbor, say $b_l \in \mu(A)$, then since by (2), there exists a pair of vertices $b_i, b_j \in \mu(A_1)$ not adjacent to z_1, z_2, one has that $\{b_i, w_i, b, w_j, b_j, w_l, b_l, z_2, b_k, z_1\}$ induces an $S_{2,2,5}$, a contradiction. Thus, b_k is the only one neighbor in $\mu(A)$ for any vertex $z \in C_1$ adjacent to b_k.

Note that, for any such z, $\mu(z) \nsim w_k$ by Claim 10. Moreover, $\mu(z) \nsim w_j \in A$ for $w_j \neq w_k$, otherwise $\{b_i, w_i, b, b_l, w_l, w_j, \mu(z), z, b_k, z'\}$ induces an $S_{2,2,5}$ for z' be another neighbor of b_k in C_1 different from z; by Claim 11 and (2), b_i, b_l not adjacent to z, z'; and w_i, w_l not adjacent to $\mu(z)$, a contradiction.

Now, y is adjacent to at least one vertex among $\mu(z_1), \mu(z_2)$, otherwise by (6), $\{\mu(z_1), z_1, b_k, z_2, \mu(z_2), w_k, b, w_1, b_1, y\}$ induces an $S_{2,2,5}$, a contradiction. Without loss of generality, assume that $y \sim \mu(z_1)$. Then $y \sim \mu(z_2)$, otherwise by

(6), $\{w_1, b_1, y, b_2, w_2, \mu(z_1), z_1, b_k, z_2, \mu(z_2)\}$ induces an $S_{2,2,5}$, a contradiction. Hence, y is adjacent to every vertex $z \in C_1$ adjacent to b_k.

That also implies that y has no other non-neighbor than b_k in $\mu(A)$. Indeed, without loss of generality, suppose that $y \nsim b_{k-1}$. Then $\{z_1, \mu(z_1), y, \mu(z_2), z_2, b_1, w_1, b, w_{k-1}, b_{k-1}\}$ induces an $S_{2,2,5}$, a contradiction.

Moreover, $\mu(y) \nsim z$ for every vertex $z \in C_1$ adjacent to b_k, otherwise $\{\mu(y), z, \mu(z), y, b_1, w_1\}$ induces a banner$_2$, a contradiction.

Besides, $D_1 = \emptyset$. Indeed, without loss of generality, suppose that there exists some vertex $t \in D_1$ such that $t \sim b_1$. Then $t \nsim b_k$, otherwise an $S_{2,2,5}$ arises. Moreover, $t \nsim \mu(z)$ for any $z \in C_1$ adjacent to b_k, otherwise $\{t, \mu(z), y, b_1, w_1, b\}$ induces a banner$_2$, a contradiction. Now, by (6), $\{\mu(z_1), z_1, b_k, z_2, \mu(z_2), w_k, b, w_1, b_1, t\}$ induces an $S_{2,2,5}$, a contradiction.

We consider the two following cases.

Case 1. $C_0 = \emptyset$. Then

$$U := \{y, \mu(y)\}$$

is a redundant set of size two such that $H - U$ is of the form tree^2 in the case that $\mu(y) \nsim w_k$, or H is of the form tree^5 in the case that $\mu(y) \sim w_k$.

Case 2. $C_0 = \{x\}$ and $x \sim \mu(y)$ by (9). Then $\mu(x) \nsim w_k$, otherwise $\{x, \mu(x), w_k, \mu(y), y, b_1\}$ induces a banner$_2$ or $\{w_1, b_1, y, b_2, w_2, \mu(y), x, \mu(x), w_k, b_k\}$ induces an $S_{2,2,5}$ depending on $\mu(y) \sim w_k$ or not, a contradiction. Thus, $\mu(x) \nsim z$ for any $z \in C_1$ adjacent to b_k, otherwise, by Claim 11, there exists a pair of vertices $w_i, w_j \neq w_k$ not adjacent to $\mu(x)$ and hence, $\{b_i, w_i, b, w_j, b_j, w_k, b_k, z, \mu(x), x\}$ induces an $S_{2,2,5}$, a contradiction. Moreover, $\mu(x) \nsim w_i$ for any $w_i \in A_1$, otherwise $\{z_1, \mu(z_1), y, \mu(z_2), z_2, \mu(y), x, \mu(x), w_i, b\}$ induces an $S_{2,2,5}$, a contradiction. Now,

$$U := \{y, \mu(y), x, \mu(x)\}$$

is a redundant set of size at most four such that $H - U$ is of the form tree^2, in the case that $\mu(y) \nsim w_k$, or

$$U := \{x, \mu(x)\}$$

is a redundant set of size at most two such that $H - U$ is of the form tree^5, in the case that $\mu(y) \sim w_k$.

From now on, we assume the following statement.

(11') Two different vertices in $C_1 \backslash \{y\}$ share no common neighbor in $\mu(A)$. This also implies that $|E_1| \leq 3$.

(12) If $D_1 = \emptyset$, then there exists a redundant set U of size at most 24 such that $H - U$ is of the form tree^1. Indeed, if in addition, $C_0 = \emptyset$, then by Claim 11,

$$U := \{y, \mu(y)\} \cup A_2 \cup \mu(A_2) \cup E_1 \cup \mu(E_1) \cup N_A(\mu(E_1)) \cup \mu(N_A(\mu(E_1)))$$

is a redundant set of size at most 20 such that $H - U$ is of the form tree^1. Now, we consider the two following cases.

Case 1. $C_0 = \{z\}$. Then by (9) and Claim 11,

$$U := \{y, \mu(y), z, \mu(z)\} \cup A_2 \cup \mu(A_2) \cup E_1 \cup \mu(E_1) \cup \\ \cup N_A(\mu(E_1) \cup \{\mu(z)\}) \cup \mu(N_A(\mu(E_1) \cup \{\mu(z)\}))$$

is a redundant set of size at most 24 such that $H - U$ is of the form tree^1.

Case 2. $|C_0| \geq 2$. Then y is adjacent to every vertex of $\mu(A)$ by (2). Let z be the (only) vertex of C_0 adjacent to $\mu(y)$. Denote by C_0' the set of vertices of $C_0\backslash\{z\}$ adjacent to $\mu(z)$ and let $C_0'' := C_0\backslash(C_0' \cup \{z\})$. Then $C_0' \neq \emptyset$, otherwise $|C_0\backslash\{z\}| = |N_H(C_0\backslash\{z\})|$, a contradiction to the minimality of H. Moreover, for every $x \in C_0'$, $\mu(x) \sim y$, $\mu(x)$ is not adjacent to any vertex of A_1, and $x \nsim \mu(y)$ by (9).

2.1. $C_0'' = \emptyset$. Then H is of the form tree^5 or tree^6 depending on $\mu(z)$ has a neighbor in A or not.

2.2. $C_0'' \neq \emptyset$. Then it must contain a vertex $t \sim \mu(x)$ for some $x \in C_0'$, otherwise $|N(C_0'')| = |C_0''|$, a contradiction to the minimality of H. Now, $\mu(t) \nsim x$, otherwise $\{z, \mu(z), x, \mu(x), t, \mu(t)\}$ induces a domino or a banner$_2$ depending on $\mu(t) \sim z$ or not, a contradiction. Thus, $\mu(t) \nsim y$, otherwise $\{y, \mu(t), t, \mu(x), x, \mu(z)\}$ induces a banner$_2$, a contradiction. Now, by Claim 11, there exists a pair of vertices w_i, w_j is not adjacent to $\mu(x), \mu(t), \mu(z)$ and hence, $\{\mu(t), t, \mu(x), x, \mu(z), y, b_i, w_i, b, w_j\}$ induces an $S_{2,2,5}$, a contradiction.

From now on, we assume the following statement.

(12') $D_1 \neq \emptyset$.
(13) If $|C_0| \geq 2$, then H contains a redundant set U of size at two such that $H - U$ is of the form tree^5.

By (9), y is adjacent to every vertex of $\mu(A)$. Let z be the (only) vertex of C_0 adjacent to $\mu(y)$ and $x \in C_0$ be adjacent to $\mu(z)$. Also by (9), for every such vertex x, $\mu(x) \sim y$, $\mu(x) \nsim z$. Moreover, by Claim 10, z has no neighbor in $\mu(A)$.

Since $D_1 \neq \emptyset$, without loss of generality, assume that there exists a vertex $z_1 \in D_1$ adjacent to b_1. Now, $\mu(z) \sim w_1$, otherwise $\{\mu(z_1), z_1, b_1, w_1, b, y, \mu(y), z, \mu(z), x\}$ induces an $S_{2,2,5}$, a contradiction. Moreover, by (3) and Claim 11, $D_1 = \{z_1\}$. We consider the two following cases.

Case 1. z has a neighbor $\mu(t) \in \mu(C_0)$ for some $t \in C_0$ different from z. Then by (7), (8), and Claim 10, $\mu(t) \sim w_1$, otherwise $\{\mu(z_1), z_1, b_1, w_1, b, y, \mu(y), z, \mu(t), t\}$ induces an $S_{2,2,5}$, a contradiction. But now, $\{\mu(z), w_1, \mu(t), z, \mu(y), y\}$ induces a banner$_2$, a contradiction.

Case 2. z has no neighbor in $\mu(C_0)$ other than $\mu(z)$. Let x be a vertex in C_0 adjacent to $\mu(z)$ and C_0' be the set of vertices of C_0 different from z and not adjacent to $\mu(z)$. If $C_0' \neq \emptyset$, then by (7) and (8), there exists a vertex $t \in C_0'$ adjacent to $\mu(x)$, otherwise $|C_0'| = |N_H(C_0')|$, a contradiction to the minimality of H. Now, $t \nsim \mu(z)$, otherwise $\{\mu(y), z, \mu(z), x, \mu(x), t\}$ induces a domino or a banner$_2$ depending on $t \sim \mu(y)$ or not, a contradiction. Now, by Claim 11, there exists a pair of vertices w_i, w_j different from w_1 not adjacent to $\mu(x)$ and hence, $\{b_i, w_i, b, w_j, b_j, w_1, \mu(z), x, \mu(x), t\}$ induces an $S_{2,2,5}$, a contradiction.

From above considerations, every vertex $x \in C_0$ different from z is adjacent to $\mu(z)$ and $\mu(x)$ is adjacent to y. Now,

$$U := \{z_1, \mu(z_1)\}$$

is a redundant set of size two, such that $H - U$ is of the form tree^5.

From now on, we assume the following statement.

(13') $|C_0| \leq 1$.

(14) If $|D_1| \geq 2$, then by (10) and (13'),

$$\begin{aligned} U := \{y, \mu(y)\} \cup C_0 \cup \mu(C_0) \cup E_1 \cup \mu(E_1) \cup \\ \cup N_A(\mu(E_1) \cup \mu(C_0)) \cup \mu(N_A(\mu(E_1) \cup \mu(C_0))) \cup \\ \cup N_{D_1}(\mu(N_A(\mu(E_1) \cup \mu(C_0)))) \cup \\ \cup \mu(N_{D_1}(\mu(N_A(\mu(E_1) \cup \mu(C_0))))) \end{aligned}$$

is a redundant set of size at most 26 such that $H - U$ is of the form tree^4.

(15) If $|D_1| = 1$, then

$$\begin{aligned} U := \{y, \mu(y)\} \cup C_0 \cup \mu(C_0) \cup D_1 \cup \mu(D_1) \cup E_1 \cup \mu(E_1) \cup \\ \cup N_A(\mu(D_1) \cup \mu(E_1) \cup \mu(C_0)) \cup \mu(N_A(\mu(D_1) \cup \mu(E_1) \cup \mu(C_0))) \cup \\ \cup N_{D_1}(\mu(N_A(\mu(D_1) \cup \mu(E_1) \cup \mu(C_0)))) \\ \cup \mu(N_{D_1}(\mu(N_A(\mu(D_1) \cup \mu(E_1) \cup \mu(C_0))))) \end{aligned}$$

is a redundant set of size at most 32 such that $H - U$ is of the form tree^1.

All the above observations ((1)–(15)) finish the proof of the claim. □

From now on, assume that every vertex of C_1 has at least four non-neighbors in $\mu(A)$.

Claim 15 $C_0 = \emptyset$, i.e. $C = C_1$.

Proof Suppose that $C_0 \neq \emptyset$. Then there exists some vertex $z \in C_1$, without loss of generality, assume that $z \sim b_1$, and $y \in C_0$ such that $y \sim \mu(z)$, otherwise $|C_0| = |N_H(C_0)|$, a contradiction to the minimality of H. Thus, $\{b_i, w_i, b, w_j, b_j, w_1, b_1, z, \mu(z), y\}$ induces an $S_{2,2,5}$, for b_i, b_j not adjacent to z and w_i, w_j not adjacent to $\mu(z)$, a contradiction. □

Claim 16 If $|C| \leq 4$, then H contains a redundant set U of size at most 16 such that $H - U$ is of the form tree^1.

Proof Assume that $|C| \leq 4$, i.e. $|\mu(C)| \leq 4$. Note that every (black) vertex of $\mu(C)$ has at most one neighbor in A by Claim 11, i.e. $|N_A(\mu(C))| \leq 4$. Then

$$U := C \cup \mu(C) \cup N_A(\mu(C)) \cup \mu(N_A(\mu(C)))$$

is a redundant set of size at most 16 such that $H - U$ is of the form tree^1. □

Claim 17 Assume that $|C| \geq 5$. Then the following statements are true.

Case 1. If there exist vertices $z_1, z_2 \in C$ sharing some neighbor in $\mu(A)$, then H is of the form tree^2.

Case 2. If for any two vertices $y, z \in C$, y, z share no neighbor in $\mu(A)$, then H is of the form tree^3 or tree^7 or H contains a redundant set U of size at most six such that $H - U$ is of the form tree^3.

Proof We consider the two above cases.

Case 1. Without loss of generality, assume that $z_1, z_2 \in C$ share a neighbor $b_1 \in \mu(A)$. Let us consider the following occurrences which are exhaustive by symmetry.

1.1. z_2 has another neighbor, say $b_2 \in \mu(A)$. Note that then $b_2 \nsim b_1$ since otherwise a banner$_2$ arises. Assume that there exist two vertices, without loss of generality, assume that they are b_3, b_4, not adjacent to z_1, z_2. Then $\{b_3, w_3, b, b_4, w_4, w_2, b_2, z_2, b_1, z_1\}$ induces an $S_{2,2,5}$, a contradiction. Hence, $|N_{\mu(A)}(\{z_1, z_2\})| \geq k - 1$. Since both z_1 and z_2 have at most $k - 4$ neighbors in $\mu(A)$, each of them has at least four neighbors in $\mu(A)$.

Let $z_3 \in C$ be adjacent to some vertex $b_i \in N_{\mu(A)}(\{z_1, z_2\})$. Then z_3 has at least four neighbors in $\mu(A)$. Hence, z_3 shares two neighbors in $\mu(A)$ with z_1 or z_2, a contradiction to Claim 9. So, there exists no other vertex in C (than z_1, z_2) having a neighbor in $N_{\mu(A)}(\{z_1, z_2\})$. Together with $|C| \geq 5$, this implies that $|N_{\mu(A)}(\{z_1, z_2\})| \leq k - 1$, i.e. $|N_{\mu(A)}(\{z_1, z_2\})| = k - 1$.

Without loss of generality, assume that z_1, z_2 are not adjacent to b_k. Since $|C| \geq 5$, there exist $z_3, z_4 \in C$ such that z_3, z_4 are adjacent to b_k. Moreover, z_3, z_4 have no other neighbor in $\mu(A)$. By Claim 11, there exists a vertex b_i such that $b_i \sim z_1$ and w_i is not adjacent to $\mu(z_3), \mu(z_4)$. Hence, by Claim 13, $\{\mu(z_3), z_3, b_k, z_4, \mu(z_4), w_k, b, b_i, w_i, z_1\}$ induces an $S_{2,2,5}$, a contradiction.

1.2. Every vertex of C adjacent to b_1 has only one neighbor (b_1) in $\mu(A)$. Note that, for every such vertex z, $\mu(z) \nsim w_1$ by Claim 10. Moreover, $\mu(z) \nsim w_i \in A$ for $w_i \neq w_1$, otherwise since by Claim 11, there exists a pair of vertices $w_j, w_l \neq w_1$ and non-adjacent to $\mu(z)$ and one has that $\{b_j, w_j, b, w_l, b_l, w_i, \mu(z), z, b_1, z'\}$ induces an $S_{2,2,5}$ for z' be another neighbor of b_1 in C different from z, a contradiction.

Now, let C_{11} be the set of vertices of C_1 adjacent to b_1 and $C_{12} := C_1 \backslash C_{11}$. If $C_{12} = \emptyset$, then H is of the form tree^2. Then assume that $C_{12} \neq \emptyset$ and let $y \in C_{12}$ and, without loss of generality, assume that $y \sim b_2 \in \mu(A)$. If y is not adjacent to two vertices, say $\mu(z_1), \mu(z_2) \in \mu(C_{11})$, then $\{\mu(z_1), z_1, b_1, z_2, \mu(z_2), w_1, b, w_2, b_2, y\}$ induces an $S_{2,2,5}$, a contradiction.

If y is adjacent to two vertices $\mu(z_1), \mu(z_2) \in \mu(C_{11})$, then y is adjacent to every vertex $b_i \in \mu(A)$ different from b_1, otherwise $\{z_1, \mu(z_1), y, \mu(z_2), z_2, b_2, w_2, b, w_i, b_i\}$ induces an $S_{2,2,5}$, a contradiction.

Now, y has at least $k-1$ neighbors in $\mu(A)$, a contradiction. Hence, $C_{11} = \{z_1, z_2\}$ and every vertex $y \in C_{12}$ is adjacent to exactly one vertex of $\mu(C_{11})$.

If $\mu(z_1)$ is adjacent to two vertices $y_1, y_2 \in C_{12}$, then $\{y_1, \mu(z_1), y_2, b_i, w_i, b\}$ induces a banner$_2$ in the case that y_1, y_2 share the same neighbor $b_i \in \mu(A)$ by Claim 10 or $\{b_{i_1}, y_1, \mu(z_1), y_2, b_{i_2}, z_1, b_1, w_1, b, w_i\}$ induces an $S_{2,2,5}$ for b_{i_1}, b_{i_2} be (different) neighbors of y_1, y_2 in $\mu(A)$, respectively, and $w_i \in A$ different from w_1, w_{i_1}, w_{i_2}, a contradiction. Hence, each $\mu(z_1), \mu(z_2)$ has at most one neighbor in C_{12}. It implies that $|C_{12}| \le 2$ and thus, $|C| \le 4$, a contradiction.

Case 2. If for every vertex $\mu(z) \in \mu(C_1)$, z is the only neighbor of $\mu(z)$, then H is of the form tree^3.

Then assume that there is a vertex $\mu(z) \in \mu(C_1)$ such that z is not the only neighbor of $\mu(z)$. First we show that for every pair $z_1, z_2 \in C$, $\mu(z_1) \nsim z_2$. Indeed, for contradiction, suppose that $\mu(z_1) \sim z_2$. Without loss of generality, assume that z_1, z_2 are adjacent to b_1, b_2, respectively. Then $\mu(z_2) \nsim z_1$, otherwise by Claim 10, $\{\mu(z_2), z_1, \mu(z_1), z_2, b_2, w_2\}$ induces a banner$_2$, a contradiction.

Moreover, $N_{\mu(A)}(\{z_1, z_2\}) \ge k-2$, otherwise by Claim 11, there exists a pair of vertices w_i, w_j not adjacent to $\mu(z)$ such that b_i, b_j not adjacent to z_1, z_2, and hence, $\{b_i, w_i, b, w_j, b_j, w_2, b_2, z_2, \mu(z_1), z_1\}$ induces an $S_{2,2,5}$, a contradiction.

Hence, the non-neighbors of z_1, z_2 in $\mu(A)$ have at most two neighbors in C, i.e. $|C| \le 4$, a contradiction.

Then there exists some vertex $z \in C$, such that $\mu(z)$ is adjacent to some vertex of A. Without loss of generality, assume that $z \sim b_1$ and $\mu(z) \sim w_2$. Then $b_2 \nsim z$, by Claim 10. We consider the two following subcases.

2.1. $b_2 \sim y$ for some $y \in C$. Then for every $x \in C \backslash \{y, z\}$, $\mu(x) \sim w_2$, otherwise $\{z, \mu(z), w_2, b_2, y, b, w_i, b_i, x, \mu(x)\}$ induces an $S_{2,2,5}$ for $b_i \sim x$, a contradiction. By Claim 11, that also implies that $\mu(y)$ is not adjacent to any vertex $w_i \in A$ such that $b_i \sim x$ for some $x \in C_1$ different from y, otherwise $|C| = 2 < 5$, a contradiction. Now,

$$U := \{w_2, b_2, y, \mu(y)\} \cup N_A(\mu(y)) \cup \mu(N_A(\mu(y)))$$

is a redundant set containing at most six vertices such that $H - U$ is of the form tree^3.

2.2. $N_C(b_2) = \emptyset$. Assume that there exists some vertex $y \in C$, without loss of generality, assume that $y \sim b_3$ and $\mu(y) \sim w_2$. Then for every $x \in C$ different from y, z, $\mu(x) \sim w_2$, otherwise $\{z, \mu(z), w_2, \mu(y), y, b, w_i, b_i, x, \mu(x)\}$ induces an $S_{2,2,5}$ for $b_i \sim x$, a contradiction. Now,

$$U := \{w_2, b_2\}$$

is a redundant set of size two such that $H - U$ is of the form tree^3.

Now, if there exists no vertex pair $y, z \in C$, such that $\mu(y)$, $\mu(z)$ share the same neighbor in A, then H is of the form tree[7]. □

All above claims finish the proof.

Appendix 3: Proof of Lemma 6

Proof (of Lemma 6) To simplify the proof, we start with a pre-processing consisting in detecting augmenting (l, m)-extended-chains whose path-part is of length at most $2l$ since such an augmenting (l, m)-extended-chain contains at most $\frac{1-(m-1)^l}{2-m}+2l+1$ vertices and can be enumerated in polynomial time.

In order to determine whether S admits an augmenting (l, m)-extended-chain whose path-part is of length at least $2l + 2$, we first find a candidate, i.e. a pair (L, R), where L and R are disjoint trees consisting induced paths $x_0, x_1, \dots, x_l$ and $x_{2p-l}, x_{2p-l+1}, \dots, x_{2p}$, respectively ($p \geq l+1$) and every vertex outside that path of L (R, respectively) is of distance at most $l-1$ from x_0 (x_{2p}, respectively) and not adjacent to any vertices among $\{x_1, x_2, \dots, x_l, x_{2p-l}, x_{2p-l+1}, \dots, x_{2p}\}$. If such a candidate does not exist, then there is no augmenting (l, m)-extended-chain whose path-part is of length at least $2l + 2$ for S. Moreover, since such candidates contain only finite vertices, we can enumerate them in polynomial time.

Our purpose is to find an alternating chain connecting x_l and x_{2p-l}. Evidently, if there are no such chains, then there is no augmenting (l, m)-extended-chain whose path-part is of length at least $2l + 2$ for S containing L and R.

Having found a candidate (L, R), we have the following observations about vertices of G in the sense that the vertices not satisfying these assumptions can be simply removed from the graph, since they cannot occur in any valid alternating chain connecting x_l and x_{2p-l}. Let $P := (x_0, x_1, \dots, x_{2p})$ be the path part of a desired (l, m)-extended-chain.

Claim 18

1. Each white vertex has at least two black neighbors.
2. Each black vertex lying outside L and R has exactly two white neighbors.
3. No black vertex outside L and R has a neighbor in L or R.
4. No white vertex outside L and R has a neighbor in L or R, except such a neighbor is x_l or x_{2p-l}.

   Moreover, no white vertex outside P has a neighbor in P.

Proof 1. and 2. are obvious since a vertex not satisfying these conditions cannot occur in any augmenting extended-chain containing L and R as sub-extended-chains.

Note that x_l and x_{2p-l} are black vertices. Hence, if a black vertex outside L and R has a neighbor in L or R, then clearly such a vertex cannot belong to the desired augmenting chain, similar for a white vertex outside L and R.

If a white vertex outside P has a neighbor in P, then clearly such a neighbor is black and hence it has at least three white neighbors, a contradiction.

From the conditions of the above claim, we have the following observation.

Claim 19 If S admits an augmenting (l, m)-extended-chain containing L and R, then no vertex of $P \backslash (L \cup R)$ is the center of an induced claw.

Proof By contradiction, suppose that G contains a claw $G[C]$, where $C = \{a, b, c, d\}$, whose center a (i.e., the vertex of degree three) is a vertex x_j on P. Without loss of generality, we choose a claw such that $|\{b, c, d\} \backslash P|$ is minimal and, among such claws, choose a claw such that j is minimum. Note that, since there exists at least one vertex of $\{b, c, d\}$ lying outside P, together with 3. of Claim 18, $l + 1 \leq j \leq 2p - l - 1$. Moreover, since every black vertex of P has all its white neighbors lying in P, every vertex of $C \backslash P$ is black.

We shall use the following convention: for a black vertex v outside P, if only one of the two white neighbors of v is defined explicitly, then the other is denoted as $\bar{v}$. Also, for a vertex v of C not belonging to P such that $N(v) \cap P \neq \emptyset$, we denote by $r(v)$ the largest index in $\{j, j+1, \ldots, 2p-l-1\}$ and by $s(v)$ the smallest index in $\{l+1, l+2, \ldots, j\}$ such that v is adjacent to $x_{r(v)}, x_{s(v)}$.

We now analyze three cases: exactly one (C1), two (C2), or three (C3) vertex/vertices of $\{b, c, d\}$ do(es)n't belong to P.

Case (C1). Without loss of generality, assume that $b = x_{j-1}$ and $c = x_{j+1}$. Then we have the following observations.

(1) d is not adjacent to x_{j-2}, x_{j+2}. Indeed, if $d \sim x_{j-2}$ (similar for the case $d \sim x_{j+2}$), then $\{x_{j-2}, x_{j-1}, x_j, d, x_{r(d)}, x_{r(d)+1}, \ldots, x_{r(d)+l-1}\}$ induces a banner$_l$ in the case $r(d) \geq j+2$ or $\{d, x_{j-2}, x_{j-1}, x_j, x_{j+1}, \ldots, x_{j+l}\}$ induces a banner$_l$ in the case $r(d) = j$, a contradiction.
(2) $r(d) = j$ or $s(d) = j$. Indeed, by (1), suppose that $r(d) \geq j+3$ and $s(d) \leq j-3$. Then $\{x_{j-1}, x_j, d, x_{s(d)}, x_{s(d)-1}, \ldots, x_{s(d)-l+1}, x_{r(d)}, x_{r(d)+1}, \ldots, x_{r(d)+l-1}\}$ induces an $S_{2,l,l}$, a contradiction.
(3) $s(d) \geq j-3$ and $r(d) \leq j+3$. Indeed, suppose that $s(d) \leq j-4$ (similar for the case $r(d) \geq j+4$). Then by (2), $\{x_{j-2}, x_{j-1}, x_j, x_{s(d)}, x_{s(d)-1}, \ldots, x_{s(d)-l+1}, x_{j+1}, x_{j+2}, \ldots, x_{j+l-1}\}$ induces an $S_{2,l,l}$, a contradiction.
(4) $r(d) = s(d) = j$. Indeed, by (2) and (3), suppose that $r(d) = j+3$ and $s(d) = j$ (similar for the case $s(d) = j-3$ and $r(d) = j$). Among $\{x_j, x_{j+3}\}$, there exists at most one white vertex. Hence, $\{x_{j+2}, x_{j+1}, \bar{d}, d, x_{j+3}, x_{j+4}, x_{j+5}, \ldots, x_{j+l+3}, x_j, x_{j-1}, \ldots, x_{j-l}\}$ induces an R_l^1, a contradiction.

Now, since $r(d) = s(d) = j$, $\{\bar{d}, d, x_j, x_{j-1}, x_{j-2}, \ldots, x_{j-l}, x_{j+1}, x_{j+2}, \ldots, x_{j+l}\}$ induces an $S_{2,l,l}$, a contradiction.

Case (C2). Without loss of generality, assume that $b = x_{j-1}$ and c and d are outside P. Then we have the following observations.

(1) x_{j+1} is adjacent both to c and d to avoid (C1).

(2) Also to avoid (C1), c is adjacent to $x_{s(c)+1}, x_{r(c)-1}$, similarly for d.
(3) It cannot happen that $s(c) = s(d) \leq j-2$ or $r(c) = r(d) \geq j+2$. Indeed, say if $s(c) = s(d) \leq j-2$, then $\{c, x_{j+1}, d, x_{s(c)}, x_{s(c)-1}, \dots, x_{s(c)-l}\}$ induces a banner$_l$, a contradiction.
(4) Similarly, if $s(c) = s(d) = j$, then there exists no common neighbor x_i of c and d for $i \geq j+2$ and if $r(c) = r(d) = j+1$, then there exists no common neighbor x_i of c and d for $i \leq j-2$. And in both cases, c and d have no common neighbor outside P.
(5) c and d are not adjacent to x_{j-2}. Indeed, suppose that $c \sim x_{j-2}$ (similar for the case $d \sim x_{j-2}$). Then $r(c) = j+1$ (similarly, $r(d) = j+1$), otherwise $\{x_j, x_{j-1}, x_{j-2}, c, x_{r(c)}, x_{r(c)+1}, \dots, x_{r(c)+l-1}\}$ induces a banner$_l$, a contradiction, and $s(c) = j-3$, otherwise $\{x_j, x_{j-1}, x_{j-2}, c, x_{s(c)}, x_{s(c)-1}, \dots, x_{s(c)-l+1}\}$ induces a banner$_l$, a contradiction. Moreover, d is neither adjacent to x_{j-2} nor x_{j-3} also by (4). Hence, $s(d) = j$, otherwise $\{x_{j-1}, x_{j-2}, c, x_j, d, x_{s(d)}, x_{s(d)-1}, \dots, x_{s(d)-l+1}\}$ induces a banner$_l$, a contradiction. Now, among $\{x_j, x_{j+1}\}$, there exists exactly one white vertex. Moreover, $c \nsim \bar{d}$ by (4). Now, $\{d, \bar{d}, x_{j+1}, c, x_{j-3}, x_{j-4}, \dots, x_{j-l-2}, x_{j+2}, x_{j+3}, \dots, x_{j+l+1}\}$, induces an $S_{2,l,l}$, a contradiction.
(6) By (2) and (5), if $s(c) \leq j-3$, then $s(c) \leq j-4$.
(7) $s(c) = j$ or $r(c) = j+1$. Similarly, $s(d) = j$ or $r(d) = j+1$. Indeed, by (5) and (6), if $s(c) \leq j-4$ and $r(c) \geq j+2$, then $\{x_{j-1}, x_j, c, x_{s(c)}, x_{s(c)-1}, \dots, x_{s(c)-l+1}, x_{r(c)}, x_{r(c)+1}, \dots, x_{r(c)+l-1}\}$ induces an $S_{2,l,l}$, a contradiction.
(8) $s(c) = j$ or $r(d) = j+1$ (similarly, $s(d) = j$ or $r(c) = j+1$). Indeed, by (5) and (6), without loss of generality, suppose that $s(c) \leq j-4$ and $r(d) \geq j+2$. Then by (7), $r(c) = j+1$ and $s(d) = j$. Hence, $\{x_{j-2}, x_{j-1}, x_j, c, x_{s(c)}, x_{s(c)-1}, \dots, x_{s(c)-l+2}, d, x_{r(d)}, x_{r(d)+1}, \dots, x_{r(d)+l-2}\}$ induces an $S_{2,l,l}$, a contradiction.
(9) $s(c) = j$ or $s(d) = j$. Indeed, by (5) and (6), without loss of generality, suppose that $s(c), s(d) \leq j-4$. Then $r(c) = r(d) = j+1$, by (7). Now, by (3), without loss of generality, assume that $s(c) < s(d)$. Then by (4), $\{x_{s(d)+1}, d, x_{j+1}, c, x_{s(c)}, x_{s(c)-1}, \dots, x_{s(c)-l+2}, x_{j+2}, x_{j+3}, \dots, x_{j+l+1}\}$ induces an $S_{2,l,l}$, a contradiction.
(10) $r(c) = j+1$ or $r(d) = j+1$. Indeed, if $r(c), r(d) \geq j+2$, then by (7), $s(c) = s(d) = j$. Without loss of generality, by (2) and (4), assume that $r(c) > r(d)+1$. Then $\{x_{r(d)}, d, x_j, c, x_{r(c)}, x_{r(c)+1}, \dots, x_{r(c)+l-2}, x_{j-1}, x_{j-2}, \dots, x_{j-l}\}$ induces an $S_{2,l,l}$, a contradiction.
(11) $s(c) = s(d) = j$. Indeed, by (5) and (6), suppose that $s(c) \leq j-4$ (similar for the case that $s(d) \leq j-4$). Then by (9), (8), and (7), $s(d) = j, r(d) = r(c) = j+1$. Note that, among $\{x_j, x_{j+1}, x_{s(c)}, x_{s(c)+1}\}$, neighbors of c, there exist exactly two white vertices and hence, $c \nsim \bar{d}$. Now, $\{\bar{d}, d, x_{j+1}, c, x_{s(c)}, x_{s(c)-1}, \dots, x_{s(c)-l+2}, x_{j+2}, x_{j+3}, \dots, x_{j+l+1}\}$ induces an $S_{2,l,l}$, a contradiction.
(12) $r(c) = r(d) = j+1$. Indeed, by (10), suppose that $r(c) = j+1$ and $r(d) \geq j+2$. Among x_j, x_{j+1}, there exists only one white vertex and $d \nsim \bar{c}$ by (4).

Then $\{\bar{c}, c, x_j, x_{j-1}, x_{j-2}, \ldots, x_{j-l}, d, x_{r(d)}, x_{r(d)+1}, \ldots, x_{r(d)+l-2}\}$ induces an $S_{2,l,l}$, a contradiction.

Now, $\{\bar{c}, c, x_j, d, \bar{d}, x_{j-1}, x_{j-2}, \ldots, x_{j-l}, x_{j+1}, x_{j+2}, \ldots, x_{j+l+1}\}$ induces an R_l^2, a contradiction.

Case (C3). We have the following observations.

(1) First, note that, $r(b)$, $r(c)$, and $r(d)$ (and similarly, $s(b)$, $s(c)$, and $s(c)$) are three mutually different integers. Otherwise, suppose that $r(b) = r(c)$. Then we have the claw $\{x_{r(c)}, x_{r(c)+1}, b, c\}$, i.e. (C2).

(2) To avoid (C1), if $b \sim x_i$ for some i, then b is adjacent to at least one vertex among x_{i-1}, x_{i+1}. It implies b is adjacent to $x_{s(b)+1}, x_{r(b)-1}$. Similarly for c and d.

(3) Moreover, by the minimality of j and to avoid (C2), we know that x_{j-1} has exactly two neighbors in $\{b, c, d\}$, say b and c. To avoid (C1) and (C2), we conclude that x_{j+1} is adjacent to d and has at least one neighbor in $\{b, c\}$, say c. Moreover, $b \nsim x_{j+1}$. Indeed, if $b \sim x_{j+1}$, then $r(b), r(c), r(d) \leq j + 2$, otherwise $\{x_{j-1}, b, x_{j+1}, c, x_{r(c)}, x_{r(c)+1}, \ldots, x_{r(c)+l-1}\}$ or $\{x_{j-1}, c, x_{j+1}, b, x_{r(b)}, x_{r(b)+1}, \ldots, x_{r(b)+l-1}\}$ or $\{b, x_{j-1}, c, x_{j+1}, d, x_{r(d)}, x_{r(d)+1}, \ldots, x_{r(d)+l-2}\}$ induces a banner$_l$ depending on which is the largest index among $r(b)$, $r(c)$, $r(d)$, a contradiction. But now, $j + 1 \leq r(c), r(b), r(d) \leq j + 2$, a contradiction with the mutual difference of $r(b), r(c)$, and $r(d)$.

(4) It also implies that at least one of $s(b), s(c)$ is less than $j - 1$ and at least one of $r(d), r(c)$ is greater than $j + 1$.

(5) $b \nsim x_{j+1}$, together with $b \sim x_{r(b)-1}$, it implies that if $r(b) \geq j + 2$, then $r(b) \geq j + 3$. Similarly, if $s(d) \leq j - 2$, then $s(d) \leq j - 3$.

(6) In a pair of consecutive vertices of P, there is a black vertex and a white vertex. Hence, b, c, d are not adjacent to three pairs of consecutive vertices of P, otherwise we have a black vertex with three white neighbors, a contradiction. Together with c is adjacent to $x_{s(c)+1}$ and $x_{r(c)-1}$, it leads to either $r(c) \leq j+2$ or $s(c) \geq j - 2$. Moreover, if c is adjacent to x_{j-2}, x_{j+2}, then $s(c) = j - 2$ and $r(c) = j + 2$. Similarly, we have the following observations: $r(b) = j$ or $s(b) \geq j - 2$, $s(d) = j$ or $r(d) \leq j + 2$.

(7) c and b cannot share a neighbor x_i for some $i \leq j - 2$, otherwise $\{x_i, c, x_j, b, x_{r(b)}, \ldots, x_{r(b)+l-1}\}$, $\{b, x_i, c, x_j, d, x_{r(d)}, \ldots, x_{r(d)+l-2}\}$, or $\{x_i, b, x_j, c, x_{r(c)}, \ldots, x_{r(c)+l-1}\}$ induces a banner$_l$ depending on which is the largest index among $r(b)$, $r(c)$, $r(d)$ (note that at least one of these integers is bigger than $j + 1$ and they are mutually different by (1)), a contradiction. Moreover, b and c cannot share a neighbor x_i for some $i \geq j + 2$, otherwise $\{x_j, c, x_i, b, x_{s(b)}, x_{s(b)-1}, \ldots, x_{s(b)-l+1}\}$ or $\{x_j, b, x_i, c, x_{s(c)}, \ldots, x_{s(c)-l+1}\}$ induces a banner$_l$ depending on which one is larger among $s(b)$ and $s(c)$. Similarly, c and b cannot share a white neighbor outside P. By similar arguments, these properties are also true for the two pairs c, d and b, d.

(8) $s(c) \geq j-2$, similarly, $r(c) \leq j+2$. Moreover, if $s(c) = j-2$, then $r(c) = j+1$. Similarly, if $r(c) = j+2$, then $s(c) = j-1$. Indeed, suppose that $s(c) \leq j-4$. Then $c \sim x_{j-2}$, otherwise $\{x_{j-1}, x_{j-2}, x_{j-3}, c, x_{r(c)}, x_{r(c)+1}, \dots, x_{r(c)+l-1}\}$ induces a banner$_l$ or $\{x_{j-2}, x_{j-1}, c, x_{s(c)}, x_{s(c)-1}, \dots, x_{s(c)-l+1}, x_{r(c)}, x_{r(c)+1}, x_{r(c)+l-1}\}$ induces an $S_{2,l,l}$ depending on $c \sim x_{j-3}$ or not. But now, c is adjacent to $\{x_{s(c)}, x_{s(c)+1}, x_{j+1}, x_j, x_{j-1}, x_{j-2}\}$, a contradiction to (6). Now, if $s(c) = j-3$, then $c \sim x_{j-2}$ by (2) and $r(c) = j+1$ by (6). Hence, $\{c, x_{j-l-3}, \dots, x_{j-4}, x_{j-3}, \dots, x_{j+1}, x_{j+2}, \dots, x_{j+l+1}\}$ induces an R_l^3, a contradiction. Moreover, if $s(c) = j-2$ and $r(c) = j+2$, then $\{c, x_{j-l-2}, \dots, x_{j-3}, x_{j-2}, \dots, x_{j+1}, x_{j+2}, \dots, x_{j+l+2}\}$ induces an R_l^3, a contradiction.

(9) $r(b) = j$ or $s(b) = j-1$, similarly, $r(d) = j+1$ or $s(d) = j$. Indeed, if $r(b) \geq j+3$ and $s(b) \leq j-2$, then $\{x_j, x_{j+1}, x_{j+2}, b, x_{s(b)}, x_{s(b)-1}, \dots, x_{s(b)-l+1}\}$ induces a banner$_l$ or $\{x_{j+1}, x_j, b, x_{s(b)}, x_{s(b)-1}, \dots, x_{s(b)-l+1}, x_{r(b)}, x_{r(b)+1}, \dots, x_{r(b)+l-1}\}$ induces an $S_{2,l,l}$ depending on $b \sim x_{j+2}$ or not, a contradiction.

(10) $s(b) \geq j-3$, similarly, $r(d) \geq j+3$. Indeed, suppose that $s(b) \leq j-4$. Then $r(b) = j$, by (9). Now b is not adjacent to x_{j-2} and x_{j-3} at the same time, otherwise either $\{b, x_{j-l-4}, \dots, x_{j-5}, x_{j-4}, \dots, x_j, x_{j+1}, \dots, x_{j+l}\}$ induces an R_l^3 or b is adjacent to three pairs of consecutive vertices of P, a contradiction to (6). Hence, $b \nsim x_{j-2}$, otherwise $\{x_{j-3}, x_{j-2}, b, x_{s(b)}, x_{s(b)-1}, \dots, x_{s(b)-l+1}, x_j, x_{j+1}, \dots, x_{j+l-1}\}$ induces an $S_{2,l,l}$, a contradiction. Suppose that $b \sim x_{j-3}$. Then $c \sim x_{j-2}$, otherwise $\{b, x_{j-3}, x_{j-2}, x_{j-1}, c, x_{r(c)}, x_{r(c)+1}, \dots, x_{r(c)+l-2}\}$ induces a banner$_l$, a contradiction. Now, $r(c) = j+1$ by (8), $r(d) \geq j+2$ by (1), and $s(d) = j$ by (9). Hence, $\{x_{j-2}, c, x_j, b, x_{s(b),x_{s(b)-1}}, \dots, x_{s(b)-l+2}, d, x_{r(d)}, x_{r(d)+1}, \dots, x_{r(d)+l-2}\}$ induces an $S_{2,l,l}$, a contradiction. Thus, $b \nsim x_{j-3}$. Now, $\{x_{j-3}, x_{j-2}, x_{j-1}, b, x_{s(b)}, \dots, x_{s(b)-l+2}, c, x_{r(c)}, \dots, x_{r(c)+l-2}\}$ induces an $S_{2,l,l}$, a contradiction.

(11) $r(b) = j$, similarly, $s(d) = j$. Indeed, suppose that $r(b) \geq j+3$. Then by (9), $s(b) = j-1$. Moreover, $s(c) = j-2$, $r(c) = j+1$, $r(d) \geq j+2$, and $s(d) = j$ by (1), (8), and (9). Now, $\{x_{r(b)-1}, b, x_j, c, x_{j-2}, x_{j-3}, \dots, x_{j-l}, d, x_{r(d)}, x_{r(d)+1}, \dots, x_{r(d)+l-2}\}$ or $\{x_{r(d)}, d, x_j, c, x_{j-2}, x_{j-3}, \dots, x_{j-l}, b, x_{r(b)}, x_{r(b)+1}, \dots, x_{r(b)+l-2}\}$ induces an $S_{2,l,l}$ depending on $r(d) > r(b)$ or $r(b) > r(d)$ (note that by (2) and (7), if $r(b) > r(d)$, then $r(b) > r(d)+1$).

(12) $s(c) = j-1$, similarly, $r(c) = j+1$. Indeed, suppose that $s(c) = j-2$. Then $r(c) = j+1$ by (8), $s(b) = j-1$ by (1), (2), and (7) and $r(d) \geq j+2$ by (1). Among x_j and x_{j-1}, there exists only one white vertex. Consider the other white neighbor of b, say $\bar{b}$. Then $\{\bar{b}, b, x_j, c, x_{j-2}, x_{j-3}, \dots, x_{j-l}, d, x_{r(d)}, x_{r(d)+1}, \dots, x_{r(d)+l-2}\}$ induces an $S_{2,l,l}$, a contradiction.

(13) x_j is black, otherwise $\{\bar{c}, c, x_j, b, x_{s(b)}, \ldots, x_{s(b)-l+2}, d, x_{r(d)}, \ldots, x_{r(d)+l-2}\}$ induces an $S_{2,l,l}$, a contradiction. Now, by the symmetry, we have three remaining cases, which are considered follows.

Case 3.1. b is adjacent to x_{j-2} and x_{j-3}, d is adjacent to x_{j+2} and x_{j+3}. Then $\{x_j, x_{j-l-2}, \ldots, x_{j-3}, b, x_{j-1}, c, x_{j+1}, d, x_{j+3}, \ldots, x_{j+l+2}\}$ induces an R_l^3, a contradiction.

Case 3.2. $s(b) = j-2$ and $r(d) = j+2$. Then $\{x_j, x_{j-l-1}, \ldots, x_{j-2}, \bar{b}, b, x_{j-1}, c, x_{j+1}, d, \bar{d}, x_{j+2}, \ldots, x_{j+l+1}\}$ induces an R_l^4, a contradiction.

Case 3.3. $s(b) = j-2$ and d is adjacent to x_{j+2} and x_{j+3}. Then $\{x_j, x_{j-l-1}, \ldots, x_{j-2}, \bar{b}, b, x_{j-1}, c, x_{j+1}, d, x_{j+2}, x_{j+3}, \ldots, x_{j+l+1}\}$ induces an R_l^5, a contradiction.

Our purpose here is to detect an augmenting extended-chain whose path-part is of length at least $2l + 2$. We first find candidates (L, R) as described above. Note that such candidates can be enumerated in polynomial time. Then perform Steps (a) through (d) for each such pair:

(a) remove all black vertices that have a neighbor in L or in R,
(b) remove the vertices of L and R except for x_l and x_{2p-l}, and
(c) remove all the vertices that are the center of a claw in the remaining graph,
(d) then in the resulting claw-free graph, determine whether there exists an alternating chain between x_l and x_{2p-l} by the method described in [28, 33].

For each candidate, Steps (a) through (d) can be implemented in time $O(n^4)$. Hence, we have the conclusion of the lemma.

Appendix 4: Proof of Lemma 7

The proof is consisted of the six following observations.

Lemma 10 *If G contains no augmenting P_3, then an augmenting tree*^{1} *(if any) can be found in time* $O(n^{17})$.

Proof Refer to Figure 2, tree^1 with parameter r. If $r = 1$, then tree^1 is a P_3. Assume that G contains an augmenting graph tree^1, for some $r \geq 2$. Therefore, G contains an induced $P_5 = (b_1, a_1, x, a_2, b_2)$, where $b_1, b_2 \in B^1$ and b_1, b_2 are non-adjacent to any vertex of $W\{a_1, x, a_2\}$. If G contains no such an initial structure, then it contains no augmenting tree^1. If such a structure exists, then we proceed as follows.

Let us denote $A = \{a \in W(x)\backslash\{a_1, a_2\} : a \nsim b_1, b_2\}$ and for $a \in A$, let $K(a)$ denote the set of black neighbors of a in B_1 not adjacent to any vertex of $\{x, a_1, a_2, b_1, b_2\}$. Notice that a desired augmenting tree exists only if $K(a) \neq \emptyset$ for every $a \in A$. Finally, let $V' = \bigcup_{a \in A} K(a)$. Since $K(a) \subseteq B^1$ for every $a \in A$, $K(a) \cap K(a') = \emptyset$ for every pair of distinct vertices $a, a' \in A$.

Consider any vertex $a \in A$, we show that $K(a)$ induces a clique for every $a \in A$. Indeed, suppose that $K(a)$ contains two non-adjacent vertices b_1, b_2. Then $\{b_1, a, b_2\}$ induces an augmenting P_3, a contradiction. It follows that a desired augmenting tree^1 exists if and only if $\alpha(G[V']) = |A|$.

We show that $G[V']$ must be P_5-free. Indeed, consider an induced $P_4 = (p_1, p_2, p_3, p_4)$ in $G[V']$ and let $a \in A$ be such that $p_1 \in K(a)$. Then none of the vertices p_3, p_4 is adjacent to a because $K(a)$ is a clique. Thus, $p_2 \in K(a)$, otherwise $\{b_1, a_1, x, a_2, b_2, a, p_1, p_2, p_3, p_4\}$ induces an $S_{2,2,5}$, a contradiction. Hence, if $G[V']$ induces a $P_4 = (p_1, p_2, p_3, p_4)$, then p_1 and p_2 have a common white neighbor, while p_2 and p_3 have no common white neighbor, a contradiction to when consider an induced $P_4 = (p_2, p_3, p_4, p_5)$ in the $P_5 = (p_1, p_2, p_3, p_4, p_5)$.

Since the P_5-free graph class is MIS-solvable in time $O(n^{12})$ [22], one can find a simple augmenting tree containing the P_5 (b_1, w_1, b, w_2, b_2) in $O(n^{12})$. With an exhaustive search, all candidate P_5 of augmenting trees can be found in time $O(n^5)$. For such candidates P_5's, V' can be built in $O(n^3)$. Hence, we have the conclusion of the lemma.

Lemma 11 *If G contains neither augmenting P_3 nor P_7, then an augmenting tree^2 (if any) can be found in time* $O(n^{14})$.

Proof Refer to Figure 2, tree^2 with parameter r and s. We may restrict ourselves to finding a tree^2 with $r, s \geq 2$, since any tree^2 with, say $r = 1$, either equals to P_7 or contains a redundant subset U of size two such that tree$^2 - U$ is of the form tree^1.

As a candidate, consider the subgraph of tree^2 (see Figure 2) induced by $\{a_1, a_2, b_1, b_2, c_1, c_2, d_1, d_2, x, y, z\}$ such that $b_1, b_2, d_1, d_2 \in B^1$ and x, z share no common white neighbor other than y.

Let us denote $A = (W(x) \cup W(z)) \setminus \{a_1, a_2, c_1, c_2, y\}$. For $a \in A$, let $K(a)$ denote the set of black neighbors of a in B^1 not adjacent to any vertex of $\{x, b_1, b_2, d_1, d_2\}$. Note that, by the assumption, every vertex of A is either adjacent to x or y. Notice that a desired augmenting tree exists only if $K(a) \neq \emptyset$ for every $a \in A$.

We show that $K(a)$ induces a clique. Indeed, suppose that $K(a)$ contains two non-adjacent vertices b_1, b_2. Then $\{b_1, a, b_2\}$ induces an augmenting P_3, a contradiction.

Since for every $a \in A$, $K(a) \in B^1$, $K(a) \cap K(a') = \emptyset$ for every pair of distinct vertices $a, a' \in A$.

Finally, let $V' = \bigcup_{a \in A} K(a)$. It follows that a desired augmenting tree^2 exists if and only if $\alpha(G[V']) = |A|$.

We now show that $G[V']$ is P_3-free. Suppose, to the contrary, that (p_1, p_2, p_3) is an induced P_3 in $G[V']$. Let $a \in A$ such that $p_1 \in K(a)$. Since $K(a)$ is a clique, p_3 is not adjacent to a. Assume that $p_3 \sim a'$. Then since $p_2 \in B^1$, p_2 is not adjacent to at least one vertex among a, a'. Without loss of generality, assume that $p_2 \nsim a$, and a is adjacent to x, but not to z. Then $\{d_2, c_2, z, c_1, d_1, y, x, a, p_1, p_2\}$ induces an $S_{2,2,5}$, a contradiction.

Hence, $G[V']$ is a disjoint union of cliques, i.e. a maximum independent set in $G[V']$ can be found in linear time. All candidates of the form tree^2 whose $r = s = 2$

can be found by an exhaustive search in time $O(n^{11})$. For such candidates P_5's, V' can be built in $O(n^3)$. Hence, we have the conclusion of the lemma.

Lemma 12 *If G contains neither augmenting P_3 nor P_5, then an augmenting tree^3 or an augmenting tree^4 (if any) can be found in time* $O(n^{31})$.

Proof First, note that tree^4 is a special case of tree^3. We refer to Figure 2, tree^3 for indices. Moreover, we may restrict ourselves to finding a tree^3 with $s \geq 3$, since any tree^3 with, say, $s \leq 2$ is either of the form tree^1 or contains a redundant subset U of size four such that tree$^3 - U$ is of the form tree^1.

As a candidate, consider the subgraph of tree^3 (see Figure 2) induced by $\{d_1, c_1, b_1^1, a_1^1, x, a_1^2, b_1^2, c_2, d_2, a_1^3, b_1^3, c_3, d_3\}$ such that $b_1^1, b_1^2, b_1^3 \in B^2$, $d_1, d_2, d_3 \in B^1$. Let us denote $A = W(x)\backslash\{a_1^1, a_1^2, a_1^3\}$. For $a \in A$, let $K(a)$ denote the set of black neighbors b of a in $B^1 \cup B^2$ and not adjacent to any vertex of $\{x, b_1^1, b_1^2, b_1^3, d_1, d_2, d_3\}$ such that if $b \in B^2$, then G contains a pair of adjacent vertices c_b and d_b such that $c_b \notin W(x)$, $W(b) = \{a, c_b\}$, $d_b \in B^1$, and d_b is not adjacent to any vertex of $\{x, b_1^1, b_1^2, b_1^3, d_1, d_2, d_3, b\}$ (note that d_b may coincide with d_1, d_2, or d_3). Let $V' = \bigcup_{a \in A} K(a)$. And again, by the existence of a desired augmenting tree^3, $K(a)$ is not empty for all $a \in A$. Note that by the assumption, $K(a) \cap K(a') = \emptyset$ for every pair of distinct vertices $a, a' \in A$.

Consider any vertex $a \in A$, we show that $K(a)$ induces a clique. Indeed, suppose that $K(a)$ contains two non-adjacent vertices b, b'. By the symmetry, we consider the three following cases.

Case 1. $b, b' \in B^1$. Then $\{b, a, b'\}$ induces an augmenting P_3, a contradiction.

Case 2. $b' \in B^1$ and $b \in B^2$. Then $\{b', a, b, c_b, d_b\}$ induces an augmenting P_5, a contradiction.

Case 3. $b, b' \in B^2$. Then $c_b \neq c_{b'}$, otherwise $\{b, c_b, b', a, x, a_1^1\}$ induces a banner$_2$, a contradiction. Now, $\{c_{b'}, b', a, b, c_b, x, a_1^i, b_1^i, c_i, d_i\}$ induces an $S_{2,2,5}$, for c_i is among c_1, c_2, c_3 different from $c_b, c_{b'}$, a contradiction.

It follows that a desired augmenting tree^3 exists if and only if $\alpha(G[V']) = |A|$.

Given $a, a' \in A$ and $b \in K(a) \cap B^2$, $b' \in K(a')$ such that $b \nsim b'$ and if $b' \in B^2$, assume that $d_b \neq d_{b'}$, we show that $b' \nsim d_b$. Indeed, suppose that $b' \sim d_b$. Then $b' \nsim c_b$, otherwise $c_{b'} = c_b$, and hence, $d_{b'} = d_b$, a contradiction. Thus, $\{b_1^1, a_1^1, x, a_1^2, b_1^2, a', b', d_b, c_b, b\}$ induces an $S_{2,2,5}$, a contradiction. Now, if $b' \in B^2$, then $d_b \nsim d_{b'}$, otherwise $\{b_1^1, a_1^1, x, a_1^2, b_1^2, a', b', c_{b'}, d_{b'}, d_b\}$ induces an $S_{2,2,5}$, a contradiction.

Hence, for every pair of non-adjacent vertices b, b' such that $b \in K(a) \cap B^2$, $b' \in K(a')$ for two distinct vertices $a, a' \in A$, $\{b, b', d(b)\}$ is independent. Moreover, if $b' \in B^2$, then $\{b, b', d_b, d_{b'}\}$ is independent.

Now, assume that B' is a maximum independent set of $G[V']$. Let $C' := \{c_b : b \in B' \cap B^2\}$, $D' := \{d_b : b \in B' \cap B^2\}$. Then by above arguments, $B' \cup D'$ is independent. And in the case that $|B'| = |A|$, $H := G[A \cup B' \cup C' \cup D']$ is an augmenting graph of the form tree^3 of G.

As in Lemma 10, we show that $G[V']$ is P_5 free. Indeed, consider an induced $P_4 = (p_1, p_2, p_3, p_4)$ in $G[V']$ and let $a \in A$ such that $p_1 \in K(a)$. Then none of the vertices p_3, p_4 is adjacent to a because $K(a)$ is a clique. But now, $p_2 \in K(a)$, otherwise $\{b_1^1, a_1^1, x, a_1^2, b_1^2, a, p_1, p_2, p_3, p_4\}$ induces an $S_{2,2,5}$, a contradiction. Hence, if $G[V']$ induces a $P_4 = (p_1, p_2, p_3, p_4)$, then p_1 and p_2 have a common white neighbor, while p_2 and p_3 have no common white neighbor, a contradiction to when consider an induced $P_4 = (p_2, p_3, p_4, p_5)$ in the $P_5 = (p_1, p_2, p_3, p_4, p_5)$.

All candidates can be found by an exhaustive search in time $O(n^{19})$. For such candidates, V' can be built in $O(n^3)$. Again, by the solution for the MIS problem in P_5-free graphs [22], we have the conclusion of the lemma.

Lemma 13 *An augmenting tree*^{5} *(if any) can be found in time* $O(n^{14})$.

Proof Refer to Figure 2, tree^5 with parameter r and s. We may restrict ourselves to finding a tree^5 with $r, s \geq 1$ and $r \geq 2$, since a tree^5 with, say, $r = 0$ contains a redundant set U of size four such that tree$^5 - U$ is of the form tree^1, and a tree^5 with $r = s = 1$ can be found in time $O(n^9)$.

As a candidate, consider the subgraph of tree^5 (see Figure 2) induced by $\{a_1, a_2, b_1, b_2, c_1, d_1, u, v, x, y, z\}$ such that $b_1, b_2, v, d_1 \in B^2$ and x, y share no common white neighbor other than u. Let us denote $A_x = W(x)\backslash\{a_1, a_2, u\}$ and $A_y = W(y)\backslash\{c_1, u\}$ and for $a \in A := A_x \cup A_y$, let $K(a)$ denote the set of common black neighbors of a and z in B^2 not adjacent to any vertex of $\{x, y, b_1, b_2, v, d_1\}$.

Note that by the assumption, every vertex of A is either adjacent to x or y. Since $K(a) \subseteq B^2$ for every $a \in A$, $K(a) \cap K(a') = \emptyset$, for every pair of distinct vertices $a, a' \in A$.

Consider a pair of distinct vertices $b, b' \in K(a)$ for some $a \in A$. If $b \nsim b'$, then $\{b, a, b', z, v, u\}$ induces a banner$_2$, a contradiction. Hence, $K(a)$ is a clique for all $a \in A$.

Now, let $V'(x) := \bigcup_{a \in A_x} (K(a))$, $V'(y) := \bigcup_{a \in A_y} (K(a))$, and $V' := V'(x) \cup V'_y$. Note that, $V'(x) \cap V'(y) = \emptyset$ by the definition. Then a desired augmenting tree^5 exists if and only if $K(a) \neq \emptyset$ for every $a \in A$ and $\alpha(G[V']) = |A|$.

As in Lemma 11, we show that $G[V']$ is P_3-free. Suppose, to the contrary, that (p_1, p_2, p_3) is an induced P_3 in $G[V']$. Let $a \in A$ such that $p_1 \in K(a)$. Since $K(a)$ is a clique, p_3 is not adjacent to a. Assume that $p_3 \sim a'$. Since $p_2 \in B^2$, p_2 is not adjacent to at least one vertex among a, a'. Without loss of generality, assume that $p_2 \nsim a$ and a is adjacent to y, but not to x. Then $\{b_2, a_2, x, b_1, a_1, u, y, a, p_1, p_2\}$ induces an $S_{2,2,5}$, a contradiction. Hence, a maximum independent set can be found in $G[V']$ in linear time.

All candidates can be found by an exhaustive search in time $O(n^{11})$. For such candidates, V' can be build in $O(n^3)$. Hence, we have the conclusion of the lemma.

Lemma 14 *An augmenting tree*^{6} *(if any) can be found in time* $O(n^{27})$.

Proof Refer to Figure 2, tree^6 with parameter r and s. We may restrict ourselves to finding a tree^6 with $r, s \geq 2$, since a tree^6 with, say, $r = 1$, contains a redundant set U of size four such that tree$^6 - U$ is of the form tree^1.

As a candidate, consider the subgraph of tree[6] (see Figure 2) induced by $\{a_1, a_2, b_1, b_2, c_1, c_2, d_1, d_2, x, y, z\}$ such that $b_1, b_2, c_1, c_2 \in B^2$ and x, z share no common white neighbor.

Let us denote $A_x = W(x)\backslash\{a_1, a_2\}$ and $A_z = W(z)\backslash\{d_1, d_2\}$. For $a \in A := A_x \cup A_z$, let $K(a)$ denote the set of common black neighbors of a and y in B^2 and not adjacent to any vertex of $\{x, b_1, b_2, c_1, c_2, z\}$. Note that $A_x \cap A_z = \emptyset$ by the assumption. Since for every $a \in A$, $K(a) \subseteq B^2$, $K(a) \cap K(a') = \emptyset$ for every pair of distinct vertices $a, a' \in A$.

Consider a pair of distinct vertices $b, b' \in K(a)$ for some $a \in A$. If $b \nsim b'$, then $\{b, a, b', y, c_1, d_1\}$ induces a banner$_2$ in the case that $a \in A_x$ (similar for the case $a \in A_z$), a contradiction. Hence, $K(a)$ is a clique for all $a \in A$.

Now, let $V'(x) := \bigcup_{a \in A_x} (K(a))$, $V'(z) := \bigcup_{a \in A_z} (K(a))$, and $V' := V'(x) \cup V'_z$. Note that, $V'(x) \cap V'(z) = \emptyset$. Then a desired augmenting tree[6] exists if and only if $K(a) \neq \emptyset$ for every $a \in A$ and $\alpha(G[V']) = |A|$.

As in Lemma 10, we show that $G[V'_x]$ and $G[V'_z]$ are P_5-free. Indeed, consider an induced $P_4 = (p_1, p_2, p_3, p_4)$ in $G[V'_x]$ or $G[V'_z]$, let $a \in A$ be such that $p_1 \in K(a)$. Then none of the vertices p_3, p_4 is adjacent to a because $K(a)$ is a clique. But now, $p_2 \in K(a)$, otherwise $\{b_1, a_1, x, a_2, b_2, a, p_1, p_2, p_3, p_4\}$ or $\{c_1, d_1, z, d_2, c_2, a, p_1, p_2, p_3, p_4\}$ induces an $S_{2,2,5}$ depending on $a \in A_x$ or $a \in A_z$, a contradiction. Hence, if $G[V'_x]$ or $G[V'_z]$ induces a $P_4 = (p_1, p_2, p_3, p_4)$, then p_1 and p_2 have a common white neighbor, while p_2 and p_3 have no common white neighbor, a contradiction to when consider an induced $P_4 = (p_2, p_3, p_4, p_5)$ in the $P_5 = (p_1, p_2, p_3, p_4, p_5)$.

Moreover, assume that there exists a pair of vertices b, b' such that $b \in K(a)$, $b' \in K(a')$ for some $a \in A(x)$, $a' \in A_z$, and $b \sim b'$. Then $\{b_1, a_1, x, a_2, b_2, a, b, b', a', z\}$ induces an $S_{2,2,5}$, a contradiction. Hence, there is no edge connecting a vertex in $G[V'_x]$ and a vertex in $G[V'_z]$. So, $G[V']$ is P_5-free.

Note that all candidates can be found by an exhaustive search in time $O(n^{15})$. For such candidates, V' can be build in $O(n^3)$. Hence, by the result of Lokshtanov et al. [22] we have the conclusion of the lemma.

Lemma 15 *If G contains no augmenting P_3, nor P_5, nor P_7, then an augmenting tree*[7] *(if any) can be found in time $O(n^{19})$.*

Proof Refer to Figure 2 for indices. We may restrict ourselves to finding a tree[7] with $s \geq 3$, since a tree[7] with $s \leq 2$ is of the form tree[3] or contains a redundant set U of size at most eight such that tree[7] $- U$ is of the form tree[3].

As a candidate, consider the subgraph of tree[7] (see Figure 2) induced by $\{x, a_1^1, b_1^1, c_1, d_1, e_1, f_1, a_1^2, b_1^2, c_2, d_2, e_2, f_2, a_1^3, b_1^3, c_3, d_3, e_3, f_3\}$ such that $b_1^1, d_1 \in B^2$ and $f_1 \in B^1$. Let us denote $A = W(x)\backslash\{a_1^1, a_1^2, a_1^3, e_1, e_2, e_3\}$. For $a \in A$, let $K(a)$ denote the set of black neighbors b of a in $B^1 \cup B^2$ not adjacent to any vertex of $\{x, b_1^1, d_1, e_1, f_1, b_1^2, d_2, e_2, f_2, b_1^3, d_3, e_3, f_3\}$ and such that if $b \in B^2$, then G contains either

- two vertices c_b, d_b such that $c_b \notin W(x)$, $W(b) = \{a, c_b\}$, $d_b \in B^1$, and d_b is not adjacent to any vertex of $\{x, b_1^1, b_1^2, b_1^3, d_1, d_2, d_3, f_1, f_2, f_3, b\}$ or
- an induced alternating (black white vertices) P_4 (c_b, d_b, e_b, f_b) such that $e_b \in W(x)\backslash\{a_1^1, c_1, a_1^2, c_2, a_1^3, c_3\}$, $c_b \notin W(x)$, $W(b) = \{a, c_b\}$, $W(d_b) = \{c_b, e_b\}$, $W(f_b) = \{e_b\}$, and d_b, f_b are not adjacent to any vertex of $\{x, b_1^1, b_1^2, b_1^3, d_1, d_2, d_3, f_1, f_2, f_3, b\}$.

Let $V' = \bigcup_{a \in A} K(a)$.

By the existence of a desired augmenting tree[7], $K(a)$ is not empty for all $a \in A$. Note that, by assumption, $K(a) \cap K(a') = \emptyset$ for every pair of distinct vertices $a, a' \in A$.

Given a vertex $b \in K(a) \cap B^2$ for some $a \in A$, we show that $d_b \notin K(e_b)$. Indeed, suppose that $d_b \notin K(e_b)$. Since $d_b \in B^2$, $c_b = c_{d_b}$, $d_{d_b} = b$, and $e_{d_b} = a$. Hence, there exists some vertex $b' \in B^1$, such that $f_{d_b} = b'$, i.e. $b' \sim a$ and b' is not adjacent to b, d_b. Hence, $b' \nsim f_b$, otherwise $\{c_b, b, a, b', f_b, x, a_1^i, b_1^i, c_i, d_i\}$ induces an $S_{2,2,5}$, for c_i is a vertex among c_1, c_2, c_3 different from c_b, a contradiction. Now, $\{b', a, b, c_b, d_b, e_b, f_b\}$ induces an augmenting P_7, a contradiction.

Suppose that there exist two vertices b, b' such that $b \in K(a) \cap B^2$ and $b' \in K(a') \cap B^2$ for two distinct vertices $a, a' \in A$ and $d_b, d_{b'}$ are different and adjacent to some vertex $a'' \in W(x)\backslash\{a, a', a_1^1, a_1^2, a_1^3\}$ different from a, a'. Then $\{c_b, d_b, a'', d_{b'}, c_{b'}, x, a_1^i, b_1^i, c_i, d_i\}$ induces an $S_{2,2,5}$ where c_i is a vertex among c_1, c_2, c_3 different from $c_b, c_{b'}$, a contradiction. Hence, for every pair of vertices b, b' such that $b \in K(a) \cap B^2$, $b' \in K(a') \cap B^2$ for two distinct vertices $a, a' \in A$, $e_b \neq e_{b'}$.

Consider any vertex $a \in A$, we show that $K(a)$ induces a clique. Indeed, suppose that $K(a)$ contains two non-adjacent vertices b, b'. By the symmetry, we consider the three following cases.

Case 1. $b, b' \in B^1$. Then $\{b, a, b'\}$ induces an augmenting P_3, a contradiction.

Case 2. $b' \in B^1$ and $b \in B^2$. We have the three following subcases.

2.1. $d_b \in B^1$. Then $\{b', a, b, c_b, d_b\}$ induces an augmenting P_5, a contradiction.

2.2. $d_b \in B^2$ and $b' \nsim f_b$. Then $\{b', a, b, c_b, d_b, e_b, f_b\}$ induces an augmenting P_7, a contradiction.

2.3. $d_b \in B^2$ and $b' \sim f_b$. Then $\{f_b, b', a, b, c_b, x, a_1^i, b_1^i, c_i, d_i\}$ induces an $S_{2,2,5}$, for c_i is a vertex among c_1, c_2, c_3 different from c_b, a contradiction.

Case 3. $b, b' \in B^2$. Then $c_b \neq c_{b'}$, otherwise $\{b, c_b, b', a, x, a_1^1\}$ induces a banner$_2$, a contradiction. Now, $\{c_{b'}, b', a, b, c_b, x, a_1^i, b_1^i, c_i, d_i\}$ induces an $S_{2,2,5}$, for c_i is a vertex among c_1, c_2, c_3 different from $c_b, c_{b'}$, a contradiction.

It follows that a desired augmenting tree[7] exists if and only if $\alpha(G[V']) = |A|$.

Given $a, a' \in A$, $b \in K(a) \cap B^2$, and $b' \in K(a')$ such that $b \nsim b'$, if $b' \sim d_b$, then $b' \nsim c_b$, otherwise $c_{b'} = c_b$ and then $d_{b'} = d_b$, a contradiction. Then $\{b_1^1, a_1^1, x, a_1^2, b_1^2, a', b', d_b, c_b, b\}$ induces an $S_{2,2,5}$, a contradiction. Now, if $b' \in B^2$, then $d_b \nsim d_{b'}$, otherwise $\{b_1^i, a_1^i, x, a_1^j, b_1^j, a', b', c_{b'}, d_{b'}, d_b\}$ induces an $S_{2,2,5}$, for $i, j \in \{1, 2, 3\}$ such that c_b is different from c_i, c_j, a contradiction. Note that for every $b \in K(a) \cap B^2$ for some $a \in A$, $f_b \in K(e_b)$. Hence, for every pair of non-adjacent vertices b, b' such that $b \in K(a) \cap B^2$, $b' \in K(a')$ for two distinc vertices $a, a' \in A$, $\{b, b', d_b, f_b\}$ is independent. Moreover, if $b' \in B^2$, then $\{b, b', d_b, d_{b'}, f_b, f_{b'}\}$ is independent.

Now, assume that B' is a maximum independent set of $G[V']$. Let $C' := \{c_b : b \in B' \cap B^2\}$, $D' := \{d_b : b \in B' \cap B^2\}$. Then by above arguments, $B' \cup D'$ is independent. And in the case that $|B'| = |A|$, $H := G[A \cup B' \cup C' \cup D']$ is an augmenting graph of the form tree^7 of G. Hence, a maximum independent set of $G[V']$ in the case that $\alpha(G[V']) = |A|$ gives us an augmenting of the form tree^7.

As in Lemma 10, we show that $G[V']$ is P_5-free. Indeed, consider an induced $P_4 = (p_1, p_2, p_3, p_4)$ in $G[V']$, and let $a \in A$ be such that $p_1 \in K(a)$. Then none of the vertices p_3, p_4 is adjacent to a because $K(a)$ is a clique. But now, $p_2 \in K(a)$, otherwise $\{b_1^1, a_1^1, x, a_1^2, b_1^2, a, p_1, p_2, p_3, p_4\}$ induces an $S_{2,2,5}$, a contradiction. Hence, if $G[V']$ induces a $P_4 = (p_1, p_2, p_3, p_4)$, then p_1 and p_2 have a common white neighbor, while p_2 and p_3 have no common white neighbor, a contradiction to when consider an induced $P_4 = (p_2, p_3, p_4, p_5)$ in the $P_5 = (p_1, p_2, p_3, p_4, p_5)$.

All candidates can be found by an exhaustive search in time $O(n^{19})$. For such candidates, V' can be built in $O(n^3)$. By the result of Lokshtanov et al. [22], we have the conclusion of the lemma.

Appendix 5: Proof of Theorem 5

So, we modify the concept of augmenting vertex [30] as follows.

Definition 4 Let S be an independent set of a graph $G = (V, F)$ and $v \in V \backslash S$, $s \in N_S(v)$. We say that v is *augmenting* for S associated with s if $G[N(s) \cap H(v, S)]$ contains an independent set $S_{v,s}$ such that $|S_{v,s}| \geq |N_S(v)|$.

Moreover, with an addition assumption that a maximum independent set of $G[N(s) \cap H(v, S)]$ can be found in polynomial time for every $s \in N_S(v)$, we can also choose s such that $\alpha(G[N(s) \cap H(v, S)])$ is maximum.

Refer to Algorithm 4, where p is a constant defined as in Lemma 3, an extended version of Algorithm Alpha in [29], a maximal independent set of G can be found (say by some greedy method) in time $O(n^2)$. One can compute the set $H(v, S)$ in time $O(n^2)$. Note that an augmenting of at most $2m-1$ vertices can be found in time $O(n^{2m+1})$. Moreover, by Lemmas 6, 10, ..., 15, an augmenting graph of the forms mentioned in the **while** condition can be found in polynomial time. The **while** loop is repeated at most n time. Hence, we observe the following result, an extension of Theorem 7 in [29].

Lemma 16 *Given two integers l and m, an ($S_{2,2,5}$,banner$_2$,domino,M_m, R_l^3, R_l^4, R_l^5)-free graph $G = (V, E)$, a maximal independent set of G S, and $v \in V \backslash S$, if one can find a maximum independent set of $G[N(s) \cap H(v, S)]$ for every $s \in N_S(v)$ in polynomial time, then one can find a maximum independent set of G in polynomial time.*

Algorithm 4 MISAugVer(G)

Input: a ($S_{2,2,5}$,banner$_2$,domino,M_m)-free graph G
Output: S, A maximum independent set of G.
1: Find an arbitrary maximal independent set S in G;
2: **while** There exists an H-augmentations to S where H contains at most $2m-1$ vertices, or H is an augmenting $(4, p)$-extended-chain, an augmenting apple, or H is of the form tree^1, ..., tree^7 or can be reduced to such forms by some redundant set or some reduction set of size at most 32, or S admits an augmenting vertex v associated with some vertex s **do**
3: **if** S admits an H-augmentation **then**
4: Apply an augmenting H for S;
5: **end if**
6: **if** S admits an augmenting vertex v associated with s **then**
7: $S := (S \backslash N_S(v)) \cup \{v\} \cup S_{v,s}$;
8: **end if**
9: **end while**
10: **return** S

Let $G = (V, E)$ be an ($S_{2,2,5}$,banner$_2$,domino,M_m, R_l^3, R_l^4, R_l^5, $K^{(h)}$)-free graph with n vertices and S be a maximal independent set of G. Assume that one can solve the MIS problem for ($S_{2,2,5}$,banner$_2$,domino,M_m, R_l^3, R_l^4, R_l^5, K)-free graphs in polynomial time. The goal is to show that one can carry out Step 2 of Algorithm 4 in polynomial time. We use the technique described in [30]. Let us say that a vertex $v \in V$ is a *trivial augmenting vertex* for S if v is augmenting for S and $|N_S(v)| \leq h$. Then one can check if a vertex $v \in V$ is a trivial augmenting vertex for S in time $O(n^{h+1})$, by verifying if $G[H(v, S)]$ contains an independent set S^* of $|N_S(v)|$ vertices. Such S^* is called the independent set associated with the augmenting vertex v.

Assume that G admits no trivial augmenting vertex for S and that there exists $v \in V \backslash S$ augmenting for S (in particular, $h < |N_S(v)|$). Thus, $G[H(v, S)]$ contains an independent set T with $|N_S(v)| \leq |T|$. Since G is ($S_{2,2,5}$,banner$_2$,domino,M_m)-free together with an additional assumption that G contains no augmenting graph contains at most $2m-1$ vertices, no augmenting graph of the forms tree^1, ..., tree^7, no augmenting $(4, p)$-extended-chain, no augmenting apple, no augmenting graph that can be reduced to such forms by some redundant set or reduction set, by Lemmas 3 and 4, $H' := (T \cup \{v\}, N_S(v), E(H'))$ is an augmenting bipartite-chain.

Let us write $T = \{t_1, \ldots, t_r\}$ ($r \geq |N_S(v)| \geq h$), with $N_S(t_i) \subset N_S(t_{i+1})$ for any index i. Since G admits no trivial augmenting vertex for S, one has $|N_S(t_k)| \geq k$ for $k = 1, \ldots, h$. For any $t \in H(v; S)$, let us write $M(t) = \{w \in H(v, S) : N_S(w) \supset N_S(t), |N_S(w)| \geq h\}$. Then $T \subset \{t_1, \ldots, t_h\} \cup (M(t_h) \backslash N(\{t_1, \ldots, t_h\}))$. Note that $M(t_h)$ is K-free, otherwise $M(t_h) \cup \{s_1, s_2, \ldots, s_h\} \cup \{v\}$ induces a $K^{(h)}$ for $s_1, \ldots, s_h \in N_S(t_h)$, a contradiction.

Now, since Step 2 of Algorithm 4 considers all the vertices in $V \backslash S$, to check if S admits an augmenting vertex one has not to solve the MIS problem in $H(v, S)$ for every $v \in V \backslash S$. In fact, for every $v \in V \backslash S$, it is sufficient to verify: (i) if v is a trivial augmenting vertex for S, and then (ii) if v is augmenting, by assuming that S admit

Algorithm 5 Procedure Green (v)

Input: a vertex $v \in V \backslash S$

Output: a possible proof that v is augmenting associated with $T = \{t_1, \ldots, t_h\}$ and an independent set S^* associated with v.

```
S* := ∅; T := ∅;
if |N_S(v)| ≤ h then
    if H(v; S) contains an independent set Q of |N_S(v)| vertices then
        set S* := Q; {v is (trivially) augmenting for S};
    end if
else
    for all independent set U of h vertices of G[H(v, S)], i.e. U = {t_1, ..., t_h}, with N_S(t_i) ⊂
    N_S(t_{i+1}), and |N_S(t_i)| ≥ i do
        S' := MISAugVer(G[M(t_h)\N({t_1, ..., t_h})]);
        if |S' ∪ {t_1, ..., t_h}| > |S*| then
            S* := S' ∪ {t_1, ..., t_h}; T := {t_1, ..., t_h};
        end if
    end for
end if
if |S*| ≥ |N_S(v)| then
    return v is augmenting for S associated with T and S*
end if
```

no trivial augmenting vertex. That can be formalized by the procedure Algorithm 5 [30], whose input is any vertex v of $V \backslash S$ which can be executed in time $O(n^{h+d+1})$.

Note that, given an augmenting vertex v (for S), Procedure Green(v) could not recognize it as an augmenting vertex: that can happen whenever $H(v, S)$ contains a trivial augmenting vertex. Now, we give the new definition for augmenting vertex v as following.

Definition 5 Let S be an independent set of a graph $G = (V, E)$, h be an integer, and $v \in V \backslash S$, $t_1, t_2, \ldots, t_h \in H[v, S]$. We say that v is h-augmenting for S associated with $\{t_1, \ldots, t_h\}$, where $N_S(t_i) \subset N_S(t_{i+1})$ for every index i, if $G[M(t_h) \backslash N(\{t_1, \ldots, t_h\})]$ contains an independent set $S_{v,t_1,\ldots,t_h}$ such that $|S^*| \geq |N_S(v)|$ where $S^* := S_{v,t_1,\ldots,t_h} \cup \{t_1, t_2, \ldots, t_h\}$. S^* is called the independent set associated with the augmenting vertex v.

To summarize, in order to define an efficient method to solve the MIS problem in ($S_{2,2,5}$,banner$_2$,domino,M_m, $K^{(h)}$)-free graphs, one can rewrite Step 2 of Algorithm 4 as in Algorithm 6.

Algorithm 6 New Step 6

for all $v \in V \backslash S$ **do**
  Procedure Green(v);
  if v is augmenting for S associated with S^* **then**
    $S := (S \backslash N_S(v)) \cup S^*$; **stop**;
  end if
end for

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