Generalised Twinning Property

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Abstract. In this paper we consider the problem of sequentialisation of rational functions $f : \Sigma^* \to \mathcal{M}$. We introduce a class of monoids that includes infinitary groups, free monoids, tropical monoids and is closed under Cartesian Product. For this class of monoids we provide a sequentialisation construction for transducers and appropriately generalise the notion of Twinning Property. We provide a construction to test the Twinning Property for transducers over the considered class of monoids and prove that it is a necessary and sufficient condition for the sequentialisation construction to terminate.

Keywords: Sequential functions · Transducers · Sequentialisation
Monoid

1 Introduction

Finite State Transducers (FST) provide a natural effective way to represent a large class of relations, called *rational relations*, applied in Natural Language Processing [12,14–17]. In their essence the FST's are formal devices that generalise the classical Finite State Automata (FSA).

Aiming at linear on-line algorithms for processing words, one prefers the Deterministic FSA to the general FSA. In the case of FSA it is well known that both formalisms are equivalent in their expressive power, [11]. However, for the FST's and the deterministic, called *sequential*, FST's this is not the case, [2,13, 15]. The constraint for an FST to deterministically process an input word clearly implies that it represents a graph of a function $f : \Sigma^* \to \mathcal{M}$. But it is by far not sufficient that an FST to be *functional* to be turned into a sequential FST. Functions recognised by some sequential FST are called *sequential functions*.

The problem we are looking at in this paper is to recognise if a given transducer $\mathcal{T}$ represents a sequential function and if so to construct a sequential transducer equivalent to $\mathcal{T}$.

In the case where $\mathcal{M}$ is a free monoid this problem has been solved by Choffrut [2]. The case where $\mathcal{M}$ is the tropical monoid was solved by Mohri [14, 15]. For a survey see also [13]. In [9] a class of monoids, sequentiable structures,

C. Câmpeanu (Ed.): CIAA 2018, LNCS 10977, pp. 173–185, 2018.
https://doi.org/10.1007/978-3-319-94812-6_15

has been introduced and the results from [2,13–15] have been generalised. The case where $\mathcal{M}$ is an infinitary group was solved in [3]. In this paper we consider a class of monoids that contains the free monoids, the tropical monoids, the sequentiable structures, and infinitary groups and additionally is closed under Cartesian Product. In its essence the class of monoids that we consider is a subclass of mge monoids [10] and the monoids considered in [7] obtained by adding three more axioms. We formally introduce it in Sect. 3.

Typically, the problem for sequentialisation of an FST starts with a functionality test. This problem can be efficiently solved for free monoids, [1], and groups, [5]. These techniques were generalised to arbitrary mge monoids in [10]. The second step is usually to characterise the sequential functions as rational functions of *bounded variation*, [2,3,9,13,15]. The third step is to introduce an appropriate notion of *Twinning Property*, [2,3,13,18].

We generalise the notion of Twinning Property in Sect. 4.2, but we do not have an appropriate notion for bounded variation. Thus, we cannot follow the common way, [2,3,9,13,15,18], of proving the characterisation theorem in order (i) sequential; (ii) bounded variation; (iii) Twinning Property; (iv) termination of a power set construction. The proof in Sect. 4.3 skips (ii) and also requires a modification of the power set construction. The latter is presented in Sect. 4.1.

2 Preliminaries

The reader familiar with the main notions on monoids and automata[1], [4,18], may prefer to skip this section.

A *monoid* $\mathcal{M} = \langle M, \circ, e\rangle$ is a semigroup $\langle M, \circ\rangle$ with a unit element e. A special case of monoids are the free monoids Σ^* generated by a finite set Σ. The support of Σ^* is the set of all finite sequence over Σ, called *words*, the multiplication is the *concatenation* of words, and the unit element is *the empty word*, ε. For monoids $\mathcal{M}_i = \langle M_i, \circ_i, e_i\rangle$ for $i = 1, 2$, the Cartesian Product $\mathcal{M} = \mathcal{M}_1 \times \mathcal{M}_2$ is defined as $\mathcal{M} = \langle M_1 \times M_2, \circ, \langle e_1, e_2\rangle\rangle$ where:

$$\langle a_1, a_2\rangle \circ \langle b_1, b_2\rangle = \langle a_1 \circ_1 b_1, a_2 \circ_2 b_2\rangle\,.$$

It is straightforward to see that $\mathcal{M}$ is also a monoid, [4]. For an element $a \in M$ and set $S \subseteq M$, we use aS and Sa as abbreviations for:

$$aS = \{as \mid s \in S\} \text{ and } Sa = \{sa \mid s \in S\}.$$

A *finite automaton* over a monoid $\mathcal{M}$ is a tuple $\mathcal{A} = \langle \mathcal{M}, Q, s, F, \Delta, \iota, \Psi\rangle$ where Q is a finite set of *states*, $s \in Q$ is the *initial state*, $F \subseteq Q$ is the set of *final states*, $\Delta \subseteq Q \times M \times Q$ is a finite relation of *transitions*, $\iota \in M$, and $\Psi : F \to M$ is the *terminal function*.

A *non-trivial path* in an automaton $\mathcal{A}$ is a non-empty sequence of transitions $\pi = \langle p_0, m_1, p_1\rangle \ldots \langle p_{n-1}, m_n, p_n\rangle$. For each state $p \in Q$ we also have the *trivial*

[1] We consider one-letter transducers with unique initial state. It emits an initial output. Final states emit final outputs.

path $\pi = (p)$. A *path* is either a trivial or a non-trivial path. Each path π has a *source state*, $\sigma(\pi)$, a *terminal state*, $\tau(\pi)$, *label*, $\ell(\pi)$, and *length*, $|\pi|$. For a non-trivial path $\pi = \langle p_0, m_1, p_1\rangle \ldots \langle p_{n-1}, m_n, p_n\rangle$ they are defined as: $\sigma(\pi) = p_0$, $\tau(\pi) = p_n$, $\ell(\pi) = \prod_{i=1}^{n} m_i$, and $|\pi| = n$. For a trivial path $\pi = (p)$, $\sigma(\pi) = \tau(\pi) = p$, $\ell(\pi) = e$, $|\pi| = 0$.

A path π is called *successful* if $\sigma(\pi) = s$ and $\tau(\pi) \in F$. In these notions, the *language* of a finite automaton $\mathcal{A} = \langle \mathcal{M}, Q, s, F, \Delta, \iota, \Psi\rangle$ is:

$$\mathcal{L}(\mathcal{A}) = \{\iota \circ \ell(\pi) \circ \Psi(\tau(\pi)) \,|\, \pi \text{ is a successful path in } \mathcal{A}\}.$$

We also denote $\Delta^* = \{\langle \sigma(\pi), \ell(\pi), \tau(\pi)\rangle \mid \pi \text{ is a path in } \mathcal{A}\}$. A state p is called *accessible* if there exists a path π with $\sigma(\pi) = s$ and $\tau(\pi) = p$. A state p is called *co-accessible* if there exists a path π with $\sigma(\pi) = p$ and $\tau(\pi) \in F$. We say that an automaton is *trimmed* if all its states are both accessible and co-accessible. For a state $p \in Q$ we denote with $\mathcal{A}_p = \langle \mathcal{M}, Q, p, F, \Delta, e, \Psi\rangle$ and we set $\mathcal{L}(p) = \mathcal{L}(\mathcal{A}_p)$.

$Rng(f)$ stays for the range of a function, f. Given a finite set Σ and a monoid $\mathcal{M}$, a *finite state transducer* is an automaton $\mathcal{T} = \langle \Sigma^* \times \mathcal{M}, Q, s, F, \Delta, \iota, \Psi\rangle$. If:

$$\Delta \subseteq Q \times ((\Sigma \cup \{\varepsilon\}) \times M) \times Q,\ \iota \in \{\varepsilon\} \times M \text{ and } Rng(\Psi) \subseteq \{\varepsilon\} \times M,$$

then $\mathcal{T}$ is called *one-letter transducer*. Clearly, an FST over some monoid, $\mathcal{M}$, is equivalent to a one-letter transducer, [4]. We denote one-letter transducers like $\mathcal{T} = \langle \Sigma \times \mathcal{M}, Q, s, F, \Delta, \iota, \Psi\rangle$ and we tacitly identify $\iota = \langle \varepsilon, \iota_2\rangle$ with ι_2 and, similarly, with $\Psi(f) = \langle \varepsilon, \Psi_2(f)\rangle$ we intend $\Psi(f) = \Psi_2(f) \in M$. By definition, a one-letter transducer recognises a relation $\mathcal{L}(\mathcal{T}) \subseteq \Sigma^* \times \mathcal{M}$. We say that $\mathcal{T}$ is *functional* if $\mathcal{L}(\mathcal{T})$ is a graph of a function $\mathcal{O}_\mathcal{T} : \Sigma^* \to M$. If $\mathcal{T}_p$ is functional, we use $\mathcal{O}_\mathcal{T}^{(p)}$ to denote the function corresponding to $\mathcal{T}_p$.

A special class of functional one-letter transducers are the *sequential transducers*. Formally, these are one-letter transducers, $\mathcal{T} = \langle \Sigma \times \mathcal{M}, Q, s, F, \Delta, \iota, \Psi\rangle$ such that there exist functions $\delta : Q \times \Sigma \to Q$ and $\lambda : Q \times \Sigma \to M$ with $Dom(\delta) = Dom(\lambda)$ satisfying: $\Delta = \{\langle p, \langle a, \lambda(p,a)\rangle, \delta(p,a)\rangle \mid \langle p, a\rangle \in Dom(\delta)\}$. To stress these particularities of the sequential transducers, we denote them as $\mathcal{T} = \langle \Sigma \times \mathcal{M}, Q, s, F, \delta, \lambda, \iota, \Psi\rangle$. As usual, $\delta^* : Q \times \Sigma^* \to Q$ and $\lambda^* : Q \times \Sigma^* \to M$ denote the natural extensions of δ and λ with $Dom(\lambda^*) = Dom(\delta^*)$ s.t.:

$$\Delta^* = \{\langle p, \langle w, \lambda^*(p,w)\rangle, \delta^*(p,w)\rangle \mid \langle p, w\rangle \in Dom(\delta^*)\}.$$

With these notions we can express the function $\mathcal{O}_\mathcal{T} : \Sigma^* \to M$ as:

$$\mathcal{O}_\mathcal{T}(w) = \iota \circ \lambda^*(s,w) \circ \Psi(f), \text{ where } f = \delta^*(s,w).$$

3 Classes of Monoids

In this section we define the class of monoids that we shall be interested in. It represents a subclass of the monoids considered in [7]. Similarly to the monoids considered in [7], it contains the free monoids, the tropical monoid, and sequentiable structures, [8,9], and it is closed under Cartesian Product. It also contains the infinitary groups, [3].

In the first paragraph, below, we revisit the basic notions from [7] and summarise the results obtained there. In the second paragraph, we introduce the new concepts that are important for the outline in next section.

3.1 MGE Monoids with LSL- and GCLF-axioms

First, we generalise the notions of a prefix and longest common prefix to monoids:

Definition 1. *For a monoid $\mathcal{M}$ and elements $a, b \in M$ we say that $a \leq_M b$ if there is an element $c \in M$ with $a \circ c = b$. We use $a \sim_M b$ as an abbreviation for the induced equivalence relation, $a \leq_M b \,\&\, b \leq_M a$. For a set $S \subseteq \mathcal{M}$, we define the sets $low(S)$ and $up(S)$ of lower and upper bounds for S, resp. as follows:*

$$low(S) = \{a \in M \mid \forall s \in S(a \leq_M s)\} \quad up(S) = \{b \in M \mid \forall s \in S(s \leq_M b)\}.$$

We define the sets of infimums and supremums for S as:

$$\inf S = low(S) \cap up(low(S)) \text{ and } \sup S = up(S) \cap low(up(S)).$$

Definition 2. *Let $\mathcal{T} = \langle \Sigma \times \mathcal{M}, Q, s, F, \Delta, \iota, \Psi \rangle$ be a one-letter transducer. We say that $\mathcal{T}$ is onward if for every accessible $p \in Q$ it holds $e \in \inf Rng(\mathcal{L}(p))$.*

Definition 3. *An mge monoid is a monoid $\mathcal{M}$ with the following properties:*

1. *(LC, left cancellation) for all $a, b \in M$ there is at most one element $c = \frac{b}{a}$ with $a \circ c = b$.*
2. *(RC, right cancellation) for all $a, b \in M$ there is at most one element $c = b - a$ with $c \circ a = b$.*
3. *(RMGE, right most general equaliser) for all $a, b \in M$ s.t. $up(\{a, b\}) \neq \emptyset$, there is an element $a \vee b \in \sup\{a, b\}$.*

An mge monoid $\mathcal{M}$ is called effective, if $\mathcal{M}$ is effective and the functions $\frac{a}{b}$, $a - b$, and $a \vee b$ are computable and their domains are recursive.

Theorem 1 ([10]). *Let $\mathcal{M}$ be an effective mge monoid. Then it is decidable given a one-letter $\Sigma - \mathcal{M}$-transducer $\mathcal{T}$ whether $\mathcal{T}$ is functional.*

Definition 4. *We say that a monoid $\mathcal{M}$ satisfies the* left semi-lattice *and* greatest common left factor *axioms, respectively, if:*

1. *the LSL-axiom[2] iff for all $a, b \in M$ there is an element $a \sqcap b \in \inf\{a, b\}$.*
2. *the GCLF-axiom[3] iff for all $a, b, c \in M$, $b \leq_M c$ and $b \leq_M ac$ imply $b \leq_M ab$.*

Theorem 2 ([7]). *Let $\mathcal{M}$ be an (effective) mge monoid with LSL- and GCLF-axioms. Then there is an (effective) construction that for every one-letter $\Sigma - \mathcal{M}$-transducer produces an equivalent onward transducer with the same states and input[4] transitions.*

Remark 1 ([7]). Groups, free monoids, and tropical monoids are all mge monoids with LSL- and GCLF-axioms. Furthermore the mge monoids and mge monoids with LSL- and GCLF-axioms are closed under Cartesian Product.

[2] LSL stays for *lower semi-lattice*.

[3] GCLF stays for *greatest common left factor*.

[4] That is, the only difference in the transitions is their $\mathcal{M}$-coordinate.

3.2 Sequentialisation Axioms

In this section we define some new notions that will be used in the constructions and proofs to come in the subsequent paragraphs.

Definition 5. *Let $\mathcal{M}$ be a monoid. For a natural number $n \in \mathbb{N}$ we define the relation $\equiv_M^{(n)} \subseteq M^n \times M^n$ as:*

$$\mathbf{a} \equiv_M^{(n)} \mathbf{b} \iff \exists u \in M(\forall i \leq n(u\mathbf{a}_i = \mathbf{b}_i) \text{ and } u \text{ is invertible}).$$

Lemma 1. *For each $n \in \mathbb{N}$, the relation $\equiv_M^{(n)}$ is an equivalence relation.*

The following definition is the symmetric variant[5] of the RMGE-axiom. In terms of free monoids, it requires that if two words are suffixes of the same word, then there is a shortest word with this property.

Definition 6. *A monoid $\mathcal{M}$ satisfies the Left Most General Equaliser Axiom (LMGE-axiom) if:*

$$\forall a, b \in M(Ma \cap Mb \neq \emptyset \Rightarrow \exists c \in \mathcal{M}(Ma \cap Mb = Mc)).$$

Definition 7. *A monoid $\mathcal{M}$ is an (effective) 2mge-monoid if it is an (effective) mge monoid and satisfies the LMGE-axiom.*

Definition 8. *Let $\mathcal{M}$ be a monoid. A left equaliser for $\mathbf{u} \in M^n$ is an n-tuple $\mathbf{a} \in M^n$ such that $a_i u_i = a_j u_j$ for all $i, j \leq n$. An element $\mathbf{u} \in M^n$ is called left equalisable if it admits a left equaliser. We say that $\mathbf{a}$ is a left mge for $\mathbf{u}$ if both:*

1. *$\mathbf{a}$ is a left equaliser for $\mathbf{u}$,*
2. *for every left equaliser, $\mathbf{b}$, for $\mathbf{u}$ there is $c \in \mathcal{M}$ such that: $\mathbf{b}_j = c\mathbf{a}_j$ for $j \leq n$.*

Lemma 2. *If $\mathcal{M}$ is a 2mge-monoid, and $\mathbf{u} \in \mathcal{M}^n$ is left equalisable, then $\mathbf{u}$ admits a unique up to equivalence w.r.t. $\equiv_M^{(n)}$ left mge $\mathbf{a} \in \mathcal{M}^n$.*

Definition 9. *A monoid $\mathcal{M}$ satisfies the Conjugate Closeness Axiom (CC) if:*

$$\forall u, r \in M(\exists k \geq 1(ru^k \in Mr)) \Rightarrow ru \in Mr.$$

Next definition captures the property that is characteristic for infinitary groups.

Definition 10. *A monoid $\mathcal{M}$ satisfies the Prime Root Axiom (PR) if:*

$$\forall u, v \in M(\exists k \geq 1(u^k = v^k)) \Rightarrow u = v.$$

Lemma 3. *LMGE, CC, and PR-axioms hold for free and tropical monoids*[6]*.*

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[5] Note that $up(\{a, b\}) = aM \cap bM$.

[6] The result extends to sequentiable structures, for the definition see [9].

Lemma 4. *If $\mathcal{G}$ is a group then it satisfies the LMGE- and the CC-axiom. Furthermore, $\mathcal{G}$ is an infinitary group if and only if $\mathcal{G}$ satisfies the PR-axiom.*

Lemma 5. *Let $\mathcal{M}_1$ and $\mathcal{M}_2$ be monoids. If $A \in \{LMGE, CC, PR\}$ and $\mathcal{M}_i$ satisfies A for $i = 1, 2$, then so does $\mathcal{M} = \mathcal{M}_1 \times \mathcal{M}_2$.*

Lemma 6. *Let $\mathcal{M}$ be an mge-monoid satisfying the CC-axiom and the PR-axiom. If $r_1, r_2, u_1, u_2, t \in M$ and $k \geq 1$ are such that: $r_i u_i^k = t r_i$ for $i \in \{1, 2\}$, then there is $s \in M$ with $r_i u_i = s r_i$ for $i \in \{1, 2\}$.*

4 Sequentialisation

In this section we will be interested in the Sequentialisation Problem:

`Given:`$\mathcal{M}$ `effective 2mge-monoid with LSL and GCLF`
$\mathcal{T} = \langle \Sigma \times \mathcal{M}, Q, i, F, \Delta, \iota, \Psi \rangle$ `transducer`
`Output:`$\mathcal{T}_D$`, sequential transducer with` $\mathcal{O}_{\mathcal{T}_D} = \mathcal{O}_{\mathcal{T}}$`, if such exists.`
`No, alternatively.`

In view of Theorems 1 and 2 this problem is equivalent to the following Restricted Sequentialisation Problem:

`Given:`$\mathcal{M}$ `effective 2mge-monoid with LSL and GCLF`
$\mathcal{T} = \langle \Sigma \times \mathcal{M}, Q, i, F, \Delta, \iota, \Psi \rangle$ `trimmed, functional, onward`
`Output:`$\mathcal{T}_D$`, sequential transducer with` $\mathcal{O}_{\mathcal{T}_D} = \mathcal{O}_{\mathcal{T}}$`, if such exists.`
`No, alternatively.`

4.1 Sequentialisation Construction

We start by providing a natural semi-decision construction for the Restricted Sequentialisation Problem. It specialises the classical power-set construction of Choffrut, [2]. Under additional assumptions for the monoid $\mathcal{M}$, namely the PR- and CC-axioms, we are going to give necessary and sufficient condition for this procedure to halt.

First, note that since $\mathcal{T}$ is functional and trimmed whereas $\mathcal{M}$ satisfies LC- and RC-axioms, every cycle $\langle p, \langle \varepsilon, m \rangle, p \rangle \in \Delta^*$ satisfies $m = e$. The sequentialisation of $\mathcal{T}$ proceeds stepwise and constructs a sequence of sequential transducers: $\mathcal{T}_k = \langle \Sigma, \mathcal{M}, Q_k, s, F_k, \delta_k, \lambda_k, \iota, \Psi_k \rangle$. The states, Q_k, are sets of pairs, $Q_k \subseteq 2^{Q \times \mathcal{M}}$. The initial state is defined as $s = \{\langle p, m \rangle \mid \langle i, \langle \varepsilon, m \rangle, p \rangle \in \Delta^*\}$.

The main difference of our construction from the classical constructions [2, 13, 16, 18] lies in the special cares in Step 2.(c), below. Intuitively, they aim at preventing the unnecessary creation of equivalent states w.r.t. $\equiv_M$.

1. At step $k = 0$, set $Q_0 = \{s\}$, $Q_{-1} = \emptyset$, and $\mathcal{T}_0 = \langle \Sigma, \mathcal{M}, \{s\}, s, \emptyset, \emptyset, \emptyset, \iota, \emptyset \rangle$.
2. If $Q_k = Q_{k-1}$, then $\mathcal{T}_D = \mathcal{T}_k$ and stop. Otherwise, set $\delta_{k+1} = \delta_k$, $\lambda_{k+1} = \lambda_k$:
   (a) $F_{k+1} = F_k \cup \{P \in Q_k \setminus Q_{k-1} \mid \exists \langle p, v \rangle \in P (p \in F)\}$.

(b) $\Psi_{k+1}(P) = v \circ \Psi(f)$ s.t. there is $\langle f, v\rangle \in P$ with $f \in F$.
(c) for each $P \in Q_k \setminus Q_{k-1}$ and each character $a \in \Sigma$:
  i. compute the monoid element and the set of pairs:

$$\ell(P, a) = \bigsqcap\{v \circ m \mid \langle p, v\rangle \in P \text{ and } \exists q \in Q(\langle p, \langle a, m\rangle, q\rangle \in \Delta^*)\}$$
$$\partial(P, a) = \left\{\left\langle q, \frac{v \circ m}{\ell(P, a)}\right\rangle \mid \langle p, v\rangle \in P \text{ and } (\langle p, \langle a, m\rangle, q\rangle \in \Delta^*)\right\}$$

  Denote $\partial(P, a) = \{\langle q_k, u_k\rangle\}_{k=1}^{K}$.
  ii. check if there is already a state $P' \in Q_k \cup Rng(\delta_{k+1})$ satisfying:
    A. $P' = \{\langle q_k, u'_k\rangle\}_{k=1}^{K}$ for some $u'_k \in \mathcal{M}$,
    B. $\langle u_1, u_2, \ldots, u_K\rangle \equiv_M^{(K)} \langle u'_1, u'_2, \ldots, u'_K\rangle$.
  If such a state P' exists, set $u = u'_1 - u_1$ if $K \geq 1$ and $u = e$ otherwise.
  iii. Update:

$$\langle \delta_{k+1}(P, a), \lambda_{k+1}(P, a)\rangle = \begin{cases} \langle P', \ell(P, a) \circ u\rangle & \text{if } u \text{ is defined} \\ \langle \partial(P, a), \ell(P, a)\rangle & \text{otherwise.} \end{cases}$$

(d) $Q_{k+1} = Q_k \cup Rng(\delta_{k+1})$ and increase k to $k + 1$. Goto 2.

Lemma 7. *Let $\mathcal{T}$ be an onward functional transducer with unique initial state. Let $k \in \mathbb{N}$ and $\alpha \in \Sigma^*$ be such that $P = \delta_k^*(s, \alpha)$ is defined. Then $\lambda_k^*(s, \alpha) = u$ is defined and:*

1. *if $P \neq \emptyset$, then $\bigsqcap\{v \mid \exists p \in Q(\langle p, v\rangle \in P)\} \sim_M e$.*
2. *for each $p \in Q$ and $v \in \mathcal{M}$ it holds: $\langle p, v\rangle \in P \iff \langle i, \langle \alpha, uv\rangle, p\rangle \in \Delta^*$.*

Proof. The proof follows by a straightforward induction on the length of α. □

As a corollary we get:

Corollary 1. *If $Q_{k-1} = Q_k$, then $\mathcal{T}_k$ is a sequential transducer and $\mathcal{O}_{\mathcal{T}_k} = \mathcal{O}_{\mathcal{T}}$.*

4.2 Squared Automaton and Twinning Property

Let $\mathcal{T} = \langle \Sigma \times \mathcal{M}, Q, i, F, \Delta\rangle$ be an onward trimmed functional transducer over a regular 2mge-monoid. We denote with $\mathcal{A}^2$ the squared automaton for $\mathcal{T}$:

$$\mathcal{A}^2 = \left\langle \Sigma \times \mathcal{M}^2, Q^2, \langle i, i\rangle, F^2, \Delta_2, e, \mathbf{e}\right\rangle, \text{ where}$$
$$\Delta_2 = \{\langle\langle p_1, p_2\rangle, \langle a, \langle m_1, m_2\rangle\rangle, \langle p'_1, p'_2\rangle\rangle \mid a \in \Sigma, \left\langle p_j, \langle a, m_j\rangle, p'_j\right\rangle \in \Delta \text{ for } j \leq 2\}$$
$$\cup\{\langle\langle p_1, p_2\rangle, \langle \varepsilon, \langle m_1, e\rangle\rangle, \langle p'_1, p_2\rangle\rangle \mid \langle p_1, \langle \varepsilon, m_1\rangle, p'_1\rangle \in \Delta\}$$
$$\cup\{\langle\langle p_1, p_2\rangle, \langle \varepsilon, \langle e, m_2\rangle\rangle, \langle p_1, p'_2\rangle\rangle \mid \langle p_2, \langle \varepsilon, m_2\rangle, p'_2\rangle \in \Delta\}$$

The squared automaton $\mathcal{A}^2$ has the following structural property:

Lemma 8. *Let $\mathbf{q}, \mathbf{q}' \in Q^2$ be arbitrary. Then for a word $\alpha \in \Sigma^*$ and $\mathbf{m} \in \mathcal{M}^2$ the following are equivalent:*

1. $\langle \mathbf{q}, \langle \alpha, \mathbf{m} \rangle, \mathbf{q}' \rangle \in \Delta_2^*$,
2. *for each* $i \leq 2$, $\langle \mathbf{q}_i, \langle \alpha, \mathbf{m}_i \rangle, \mathbf{q}_i' \rangle \in \Delta^*$,

Proof. The implication $\underline{\mathbf{1} \Rightarrow \mathbf{2}}$ follows by induction on the length of the generalised transition, $\langle \mathbf{q}, \langle \alpha, \mathbf{m} \rangle, \mathbf{q}' \rangle$. In turn, the implication $\underline{\mathbf{2} \Rightarrow \mathbf{1}}$ follows by induction on the sum of the lengths of the generalised transitions, $\langle \mathbf{q}_i, \langle \alpha, \mathbf{m}_i \rangle, \mathbf{q}_i' \rangle$ □

Next, we introduce the *advance action*. It generalises the *delay of runs*, [1,3], by factorising w.r.t. the equivalence relation $\equiv_M^{(2)}$. Let $t = \langle \mathbf{q}, a, \mathbf{m}, \mathbf{q}' \rangle \in \Delta_2$ be a transition. We introduce $adv_t : \mathcal{M}^2 \to \mathcal{M}^2$ as:

$$adv_t(v) = \left\langle \frac{(\mathbf{v}_1 \circ \mathbf{m}_1)}{m}, \frac{(\mathbf{v}_2 \circ \mathbf{m}_2)}{m} \right\rangle, \text{ where } m = (\mathbf{v}_1 \circ \mathbf{m}_1) \sqcap (\mathbf{v}_2 \circ \mathbf{m}_2).$$

For a path $\pi = t_1 t_2 \dots t_n$ in $\mathcal{A}^2$, we denote with $adv^{(\pi)} : \mathcal{M}^2 \to \mathcal{M}^2$ the function:

$$adv^{(\pi)} = adv_{t_1} \circ adv_{t_2} \circ \dots \circ adv_{t_n}.$$

Next we list some useful properties of the advance action.

Lemma 9. *Let* $v \leq_M \mathbf{v}_1$ *and* $v \leq_M \mathbf{v}_2$, *then* $adv_t(\langle \mathbf{v}_1, \mathbf{v}_2 \rangle) \equiv_M^{(2)} adv_t(\langle \frac{\mathbf{v}_1}{v}, \frac{\mathbf{v}_2}{v} \rangle)$.

Corollary 2. *Let* $\mathbf{v}' \equiv_M^{(2)} \mathbf{v}''$ *and* $t \in \Delta_2$, *then* $adv_t(\mathbf{v}') \equiv_M^{(2)} adv_t(\mathbf{v}'')$.

Proof. Since $\mathbf{v}' \equiv_M^{(2)} \mathbf{v}''$ there is an invertible element c with $\frac{\mathbf{v}_j'}{c} = \mathbf{v}_j''$ for $j = 1, 2$. Now the result follows by the previous lemma. □

Corollary 3. *Let* π_1 *and* π_2 *be paths in* $\mathcal{A}^2$ *that start at* $\langle i, i \rangle$ *and terminate in the same state* $\mathbf{q}$ *be such that:* $adv^{(\pi_1)}(e, e) \equiv_M^{(2)} adv^{(\pi_2)}(e, e)$. *Then for any path* π *in* $\mathcal{A}^2$ *that starts at* $\mathbf{q}$ *it holds that:* $adv^{(\pi_1 \pi)}(e, e) \equiv_M^{(2)} adv^{(\pi_2 \pi)}(e, e)$.

Proof. The proof follows by Corollary 2 and straightforward induction on the length of the path π. □

Lemma 10. *Let* π *be a path in* $\mathcal{A}^2$ *from* $\mathbf{i} = \langle i, i \rangle$ *to some state* $\mathbf{q} \in Q^2$. *Let* $\ell(\pi) = \langle \alpha, \mathbf{m} \rangle$ *be the label of* π, *and* $m = \mathbf{m}_1 \sqcap \mathbf{m}_2$, *then:*

$$adv^{(\pi)}(\langle e, e \rangle) \equiv_M^{(2)} \left\langle \frac{\mathbf{m}_1}{m}, \frac{\mathbf{m}_2}{m} \right\rangle.$$

Definition 11. *Let* $\mathcal{A}^2$ *be a squared automaton for a trimmed onward transducer with unique initial state. We say that* $\mathcal{A}^2$ *satisfies the Twinning Property iff for any two paths* π_1 *and* π_2 *in* $\mathcal{A}^2$ *such that* $\sigma(\pi_1) = \langle i, i \rangle$ *and* $\tau(\pi_1) = \sigma(\pi_2) = \tau(\pi_2)$, *i.e.* π_2 *is a cycle starting at* $\tau(\pi_1)$, *it holds:*

$$adv^{(\pi_1)}(e, e) \equiv_M^{(2)} adv^{(\pi_1 \pi_2)}(e, e).$$

We conclude this section by showing that the Twinning Property is decidable:

Lemma 11. *Given a squared automaton $\mathcal{A}^2$ over an effective 2mge-monoid, $\mathcal{M}$, with LSL- and GCLF-axioms we can effectively test whether $\mathcal{A}^2$ obeys the Twinning Property.*

Proof. (Idea) Let $n = |Q_2|$. We denote with Π_{2n} and $\mathcal{C}$ the sets:

$$\Pi_{2n} = \{\pi \text{ path in } \mathcal{A}^2 \mid \sigma(\pi) = \langle i, i\rangle\,,\ |\pi| < 2n\} \text{ and } \mathcal{C} = \{\pi \text{ simple cycle in } \mathcal{A}^2\}.$$

We say that $\mathcal{A}^2$ satisfies the *restricted* Twinning Property if and only if for every $\pi_1 \in \Pi_{2n}$ and any $\pi_2 \in \mathcal{C}$ such that $\tau(\pi_1) = \sigma(\pi_2)$:

$$adv^{(\pi_1)}(e,e) \equiv_M^{(2)} adv^{(\pi_1\pi_2)}(e,e).$$

Clearly, under the assumptions of the lemma, the restricted Twinning Property is decidable. It is also clear that the Twinning Property implies the restricted Twinning Property. The reverse is also true. This follows by induction, the Pigeonhole Principle, and Corollary 3. □

4.3 Twinning Property ⇔ Sequentialisation Algorithm Halts

The main result in this section is the following:

Theorem 3. *Assume that $\mathcal{M}$ is a 2mge-monoid satisfying the PR- and CC-axioms. Let $\mathcal{T} = \langle \Sigma \times \mathcal{M}, Q, i, F, \Delta, \iota, \Psi\rangle$ be an onward, trimmed, functional transducer and let $f = \mathcal{O}_{\mathcal{T}}$. Then the following are equivalent:*

1. *the sequentialisation procedure on $\mathcal{T}$ terminates.*
2. *f is sequential.*
3. *$\mathcal{A}^2$ satisfies the Twinning Property.*

Before we step to the formal proof of Theorem 3, we note the important consequence of this theorem:

Theorem 4. *Let $\mathcal{M}$ be an effective 2mge-monoid with PR-, CC-, LSL-, and GCLF-axioms. Then it is decidable given a transducer $\mathcal{T}$ over $\Sigma^* \times \mathcal{M}$ whether $\mathcal{T}$ represents a sequential function.*

Proof. Immediate from Theorem 1, Theorem 2, Lemma 11, and Theorem 3. □

The rest of this section is devoted to the proof of Theorem 3. The implication **1 ⇒ 2** is obvious and follows immediately from Corollary 1. The implication **3 ⇒ 1** is standard as it appropriately generalises the main ideas from [2,3,9,13,15,18]. Yet, the implication **2 ⇒ 3** is more involved since it has to surmount the lack of *Bounded Variation Property* that is usually the bridge between the sequential functions and the Twinning Property. This is also the only place in the proof where we need the PR- and the CC-axioms and more precisely their consequence Lemma 6. With these remarks we delve into the proof of Theorem 3:

Proof (of Theorem 3). **1 ⇒ 2**. Follows by Corollary 1.

2 ⇒ 3 Let the paths π_1, π_2, with $\mathbf{q} = \tau(\pi_1) = \sigma(\pi_2) = \tau(\pi_2)$ satisfy the premise of the Twinning Property. Let $\ell(\pi_1) = \langle \alpha, \mathbf{m} \rangle$ and $\ell(\pi_2) = \langle \beta, \mathbf{n} \rangle$. First consider the case where $\beta = \varepsilon$. Since, $\mathcal{T}$ is trimmed and functional over an mge monoid (LC- and RC-axiom), we conclude that $\mathbf{m} = \mathbf{e}$. Therefore $\ell(\pi_1) = \ell(\pi_1 \pi_2)$. Thus by Lemma 10 we deduce that:

$$adv^{(\pi_1)}(e, e) \equiv_M^{(2)} adv^{(\pi_1 \pi_2)}(e, e).$$

In the sequel we assume that $\beta \neq \varepsilon$. Let $\Gamma \subseteq \Sigma^*$ be the language[7] $\Gamma = \{\alpha\} \circ \bigcup_{j=1}^2 Dom(\mathcal{O}_{\mathcal{T}}^{(\mathbf{q}_j)})$. We set $g = f \restriction \Gamma$, i.e. the restriction of f to Γ. Since there is a sequential transducer for f and Γ is regular, it follows that there is also a sequential transducer for g. Let

$$m^{(k)} = (\mathbf{m}_1 \circ \mathbf{n}_1^k) \sqcap (\mathbf{m}_2 \circ \mathbf{n}_2^k) \text{ and } \mathbf{r}^{(k)} = \left\langle \frac{\mathbf{m}_1 \circ \mathbf{n}_1^k}{m^{(k)}}, \frac{\mathbf{m}_2 \circ \mathbf{n}_2^k}{m^{(k)}} \right\rangle.$$

Note that by Lemma 10 we have that $\mathbf{r}^{(k)} \equiv_M^{(2)} adv^{(\pi_1 \pi_2^k)}(e, e)$. To complete the proof we need the following:

Lemma 12. *If $\mathcal{T}$ is trimmed and onward, and $g = f \restriction \Gamma$ is sequential, then there is some $l \in \mathbb{N}$ with $\mathbf{r}^{(l)} \equiv_M \mathbf{r}^{(l+1)}$.*

Proof (Idea). First, using the sequential transducer for g we find two integers $k > l$ such that $\alpha\beta^k$ and $\alpha\beta^l$ lead to the same state in this transducer. Then, we establish the existence of $u' \sim_M m^{(l)}$ and $v' \sim_M m^{(k)}$ and a function $\widehat{g}' : \Sigma^* \to M$ such that for any $\gamma \in Dom(\mathcal{O}_{\mathcal{T}}^{(\mathbf{q}_1)}) \cup Dom(\mathcal{O}_{\mathcal{T}}^{(\mathbf{q}_2)})$ it holds:

$$g(\alpha\beta^l\gamma) = u' \circ \widehat{g}'(\gamma) \text{ and } g(\alpha\beta^k\gamma) = v' \circ \widehat{g}'(\gamma). \qquad (1)$$

The mere existence of u, v, and $\widehat{g}$ satisfying Eq. 1 can be easily derived from the sequential transducer for g. The onward property of the original transducer $\mathcal{T}$, allows us to conclude that $u \leq_M m^{(l)}$ and $v \leq_M m^{(k)}$. Using the RMGE-axiom it is then easy to construct $u' \sim_M m^{(l)}$, $v' \sim_M m^{(k)}$, and $\widehat{g}'$ satisfying Eqs. 1.

Next, the function $\widehat{g}'$ allows us to transfer information from the reduct, $\mathbf{r}^{(l)}$, to the reduct, $\mathbf{r}^{(k)}$, and obtain that $\mathbf{r}^{(l)} \equiv_M^{(2)} \mathbf{r}^{(k)}$. Finally, using that $\mathbf{m}_j \circ \mathbf{n}_j^k = m^{(k)} \mathbf{r}_j^{(k)}$ and $\mathbf{m}_j \circ \mathbf{n}_j^k = m^{(l)} \mathbf{r}_j^{(l)} \mathbf{n}_j^{k-l}$ and $\mathbf{r}^{(l)} \equiv_M \mathbf{r}^{(k)}$ we can see that $t\mathbf{r}_j^{(l)} = \mathbf{r}_j^{(l)} \mathbf{n}_j^{k-l}$ where t does not depend on $j = 1, 2$. Now, the result follows by Lemma 6. ◻

Back to the Proof of 2 ⇒ 3. Let l be such $\mathbf{r}^{(l)} \equiv_M^{(2)} \mathbf{r}^{(l+1)}$. Then by $\mathbf{r}^{(0)} \equiv_M^{(2)} adv^{(\pi_1)}(\langle e, e \rangle)$ and $\mathbf{r}^{(1)} \equiv_M^{(2)} adv^{(\pi_1 \pi_2)}(\langle e, e \rangle)$ Lemma 10 implies:

$$\mathbf{r}^{(l)} \equiv_M^{(2)} adv^{(\pi_2^l)}(\mathbf{r}^{(0)}) \text{ and } \mathbf{r}^{(l)} \equiv_M^{(2)} \mathbf{r}^{(l+1)} \equiv_M^{(2)} adv^{(\pi_2^l)}(\mathbf{r}^{(1)}).$$

[7] Note that $\mathcal{T}$ is functional and $\mathcal{M}$ satisfies the LC-axiom. Therefore $\mathcal{T}_q$ is functional for any accessible state q, hence $\mathcal{O}_{\mathcal{T}}^{(\mathbf{q}_j)}$ are well-defined.

This means that $y\mathbf{r}_j^{(l)} = \mathbf{r}_j^{(0)}\mathbf{n}_j^l$ and $w\mathbf{r}_j^{(l)} = \mathbf{r}_j^{(1)}\mathbf{n}_j^l$ for appropriate $y, w \in \mathcal{M}$ that are independent of $j = 1, 2$. This shows that for $j = 1, 2$ the pairs $\left\langle \mathbf{r}_j^{(l)}, \mathbf{n}_j^{(l)} \right\rangle$ are left equalisable. We conclude that both $\left\langle y, \mathbf{r}_j^{(0)} \right\rangle$ and $\left\langle w, \mathbf{r}_j^{(1)} \right\rangle$ are equalisers for this pair. Let $\langle a_j, b_j \rangle$ be a left mge for the pair $\left\langle \mathbf{r}_j^{(l)}, \mathbf{n}_j^{(l)} \right\rangle$. Therefore there are $c_j, d_j \in \mathcal{M}$ with:

$$y = c_j a_j \text{ and } w = d_j a_j, \quad \mathbf{r}_j^{(0)} = c_j b_j \text{ and } \mathbf{r}_j^{(1)} = d_j b_j$$

Considering the first pair of equalities, we have that $\langle a_1, a_2 \rangle$ is left equalisable and $\langle c_1, c_2 \rangle$ and $\langle d_1, d_2 \rangle$ are left equalisers for this pair. Hence, if $\langle a_1', a_2' \rangle$ is the left mge for $\langle a_1, a_2 \rangle$, then there are c, d with $c_j = ca_j'$ and $d_j = da_j'$. This shows that $d \leq_M \mathbf{r}_j^{(1)}$ for $j = 1, 2$ and similarly, $c \leq_M \mathbf{r}_j^{(0)}$. Since $\mathbf{r}_1^{(0)} \sqcap \mathbf{r}_2^{(0)} \sim_M e$ and $\mathbf{r}_1^{(1)} \sqcap \mathbf{r}_2^{(1)} \sim_M e$ we conclude that c and d are invertible. Therefore $w \equiv_{\mathcal{M}} y$. Let $uw = y$ where u is invertible. Therefore:

$$u\mathbf{r}_j^{(0)}\mathbf{n}_j^l = uy\mathbf{r}_j^{(l)} = uy\mathbf{r}_j^{(l)} = w\mathbf{r}_j^{(l)} = \mathbf{r}_j^{(1)}\mathbf{n}_j^l$$

and by the RC-axiom, we derive that $\mathbf{r}_j^{(0)} = u\mathbf{r}_j^{(1)}$ for $j = 1, 2$ with $u \sim_M e$. Therefore $\mathbf{r}^{(0)} \equiv_M^{(2)} \mathbf{r}^{(1)}$, i.e. $adv^{(\pi_1)}(e, e) \equiv_{\mathcal{M}}^{(2)} adv^{(\pi_1\pi_2)}(e, e)$, as required.

3 ⇒ 1. By Corollary 1, it suffices to show that if $\mathcal{A}^2$ obeys the Twinning Property, then $Q_{k+1} = Q_k$ for some k. We set out to show that there are only finitely many tuples in $2^{Q \times \mathcal{M}}$ that can be generated by the algorithm. Let:

$$Adv(q_1, q_2) = \{[adv^{(\pi)}(\langle e, e \rangle)]_{\equiv_{\mathcal{M}}^{(2)}} \mid \pi \text{ is a path from } \langle i, i \rangle \text{ to } \langle q_1, q_2 \rangle \text{ in } \mathcal{A}^2\}$$

for $q_1, q_2 \in Q$. The Twinning Property implies that $Adv(q_1, q_2)$ is generated entirely by cycle-free paths, thus it is finite.

Next, consider a state $P = \{\langle p_j, v_j \rangle\}_{j=1}^J \in Q_{k-1}$ for some k. Let α be a word such that: $\delta_k^*(s, \alpha) = P$ and $\lambda_k^*(s, \alpha) = u$. By Lemma 7 for each j we have: $\langle i, \langle \alpha, uv_j \rangle, p_j \rangle \in \Delta^*$. Therefore in the squared automaton $\mathcal{A}^2$ there are paths:

$$\langle \langle i, i \rangle, \langle \alpha, \langle uv_{j_1}, uv_{j_2} \rangle \rangle, \langle p_{j_1}, p_{j_2} \rangle \rangle \in \Delta_2^*$$

for all $j_1, j_2 \leq J$. Hence there is an element $\mathbf{r}(j_1, j_2) = \langle \mathbf{r}_1(j_1, j_2), \mathbf{r}_2(j_1, j_2) \rangle$ such that $[\mathbf{r}(j_1, j_2)] \in Adv(p_{j_1}, p_{j_2})$ and:

$$uv_{j_i} = t(j_1, j_2)\mathbf{r}_i(j_1, j_2) \text{ for } i = 1, 2 \text{ where } t(j_1, j_2) = uv_{j_1} \sqcap uv_{j_2}.$$

Consider the sequence $\langle \mathbf{r}_1(1, j) \rangle_{j=1}^J$. It is left equalisable for $uv_1 = t(1, j)\mathbf{r}_1(1, j)$. Thus, by Lemma 2, it has a left mge $\langle a_1, a_2, \ldots, a_J \rangle$. In particular, there exists a t' s.t. $t(1, j) = t' \circ a_j$ for all $j \leq J$. Since $t' \sim_M \bigsqcap_j t(1, j) \sim_M \bigsqcap_j uv_j$ and, by construction, $\bigsqcap v_j \sim_M e$, we get that $t' \sim_M u$. Finally, the equalities $u \circ v_j = t(1, j)\mathbf{r}_2(1, j) = t' a_j \mathbf{r}_2(1, j)$, show that $v_j = \frac{t'}{u} a_j \mathbf{r}_2(1, j)$. Since $\frac{t'}{u}$ is invertible, $\langle v_j \rangle_{j=1}^J \equiv_M^{(J)} \langle a_j \mathbf{r}_2(1, j) \rangle_{j=1}^J$ is determined by $\{\mathbf{r}(1, j)\}_{j=1}^J$ up to equivalence w.r.t. $\equiv_M^{(J)}$. This is exactly what Step 2.(c) in our algorithm guards. This proves the existence of an injection between the states in $\bigcup_{k=0}^{\infty} Q_k$ and the subsets of $Q \times \left(\bigcup_{p,q\in Q} Adv(p,q)\right)$, which is finite. Since $Q_k \subseteq Q_{k+1}$ for all k, this implies that the algorithm halts. □

5 Conclusion

In this paper we described a general class of monoids and characterised the sequential functions w.r.t. this class in terms of and appropriately generalised Twinning Property. We consider that the axiomatisation approach should make it easier to strengthen these results or alternatively to recognise that some of these axioms are necessary.

Most of the axioms seem natural from algebraic point of view. Yet, the GCLF- and PR-axioms are odd. From [7] we know that there are mge monoids with LSL-axiom that violate the GCLF-axiom and that admit regular languages with $\inf L = \emptyset$. Yet, the GCLF-axiom is the only axiom from the mge and LSL-axioms not satisfied by the gcd monoids, [19]. Actually, both GCLF- and PR-axioms have the intrinsic property we need to surmount the cycles in the transducers and effectively reduce the infinite nature of the problem to a finite one. Can we relax them?

Notably, we have a characterisation of the sequential functions in terms of congruence relations, both for gcd monoids [19] and mge monoids with GCLF-axiom and additional (but rather tight) second order axiom, [6]. This challenges the necessity of all: LMGE-, PR-, and CC-axioms. Are there monoids that admit characterisation of sequential functions in terms of congruence relations but do not admit sequentialisation algorithm?

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