Chapter 9
Coherent States and Norm Correspondence

Finally, we prove the lower bound for the operator norm of a Berezin–Toeplitz operator. In order to do so, we use the so-called coherent states.

9.1 Coherent Vectors

Let $P \subset L$ be the set of elements $u \in L$ such that $\|u\| = 1$, and denote by $\pi\colon P \to M$ the natural projection.

Lemma 9.1.1. *Fix $u \in P$. For every $k \geq 1$, there exists a unique vector ξ_k^u in $\mathcal{H}_k$ such that*

$$\forall \phi \in \mathcal{H}_k, \quad \phi\big(\pi(u)\big) = \langle \phi, \xi_k^u \rangle_k u^k.$$

Definition 9.1.2. The vector $\xi_k^u \in \mathcal{H}_k$ is called the *coherent vector* at u.

Proof of Lemma 9.1.1. Consider the linear form F_k defined on $\mathcal{H}_k$ by

$$\forall \phi \in \mathcal{H}_k, \quad F_k(\phi) = h_k\big(\phi\big(\pi(u)\big), u^k\big).$$

Since $\mathcal{H}_k$ is finite-dimensional, F_k is continuous, so the Riesz representation theorem implies that there exists a unique vector $\xi_k^u \in \mathcal{H}_k$ such that $F_k(\phi) = \langle \phi, \xi_k^u \rangle_k$ for all ϕ in $\mathcal{H}_k$. But since u^k is an orthonormal basis of $L^k_{\pi(u)}$, we have $\phi\big(\pi(u)\big) = F_k(\phi) u^k$. □

Lemma 9.1.3. *Let T_k be an operator $\mathcal{C}^\infty(M, L^k) \to \mathcal{C}^\infty(M, L^k)$ with kernel $T_k(\cdot,\cdot)$ and such that $\Pi_k T_k \Pi_k = T_k$. Then*

(1) $\forall x \in M$, $(T_k \xi_k^u)(x) = T_k\big(x, \pi(u)\big) \cdot u^k$,
(2) $\langle T_k \xi_k^u, \xi_k^v \rangle_k = \bar{v}^k \cdot T_k\big(\pi(v), \pi(u)\big) \cdot u^k$,

where we recall that the dot stands for contraction with respect to h_k.

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Proof. Let $(\phi_i)_{1\leq i\leq d_k}$ be an orthonormal basis of $\mathcal{H}_k$. By proposition 6.3.3, we can write the Schwartz kernel of the restriction of T_k to $\mathcal{H}_k$ as

$$\forall x,y\in M,\quad T_k(x,y)=\sum_{i,j=1}^{d_k}\langle T_k\phi_i,\phi_j\rangle_k\phi_j(x)\otimes\overline{\phi_i(y)}.$$

Therefore, for $x\in M$ we have that

$$T_k\big(x,\pi(u)\big)\cdot u^k=\sum_{i,j=1}^{d_k}\langle T_k\phi_i,\phi_j\rangle_k h_k\Big(u^k,\phi_i(\pi(u))\Big)\phi_j(x),$$

which we can rewrite, because $h_k\left(u^k,\phi_i\big(\pi(u)\big)\right)=\langle\xi_k^u,\phi_i\rangle_k$, as

$$T_k\big(x,\pi(u)\big)\cdot u^k=\sum_{j=1}^{d_k}\langle T_k\left(\sum_{i=1}^{d_k}\langle\xi_k^u,\phi_i\rangle_k\phi_i\right),\phi_j\rangle_k\phi_j(x),$$

which yields that

$$T_k\big(x,\pi(u)\big)\cdot u^k=\sum_{j=1}^{d_k}\langle T_k\xi_k^u,\phi_j\rangle_k\phi_j(x)=(T_k\xi_k^u)(x).$$

This corresponds to the first claim. For the second claim, we use the first one to write for x in M that $h_k\big((T_k\xi_k^u)(x),\xi_k^v(x)\big)=h_k\big(T_k\big(x,\pi(u)\big)\cdot u^k,\xi_k^v(x)\big)$. Integrating this equality leads to

$$\langle T_k\xi_k^u,\xi_k^v\rangle_{\mathcal{H}_k}=\langle T_k\big(\,\cdot\,,\pi(u)\big),\xi_k^v\rangle_k,$$

but the right-hand side of this equation is equal to $h_k\left(T_k\big(\pi(v),\pi(u)\big)\cdot u^k,\xi_k^v(x)\right)$ by definition of ξ_k^v, and this term is in turn equal to $\bar{v}^k\cdot T_k\big(\pi(v),\pi(u)\big)\cdot u^k$. □

By taking $T_k=\Pi_k$ in this proposition, we immediately get the following properties.

Corollary 9.1.4. *For every $u,v\in P$,*

(1) *for every x in M, $\xi_k^u(x)=\Pi_k\big(x,\pi(u)\big)\cdot u^k$,*
(2) $\langle\xi_k^u,\xi_k^v\rangle_k=\bar{v}^k\cdot\Pi_k\big(\pi(v),\pi(u)\big)\cdot u^k$, *so* $\Pi_k\big(\pi(v),\pi(u)\big)=\langle\xi_k^u,\xi_k^v\rangle_k v^k\otimes\bar{u}^k$,
(3) $\|\xi_k^u\|_k^2=\Pi_k\big(\pi(u),\pi(u)\big)$.

9.2 Operator Norm of a Berezin–Toeplitz Operator

In this section, we prove Theorem 5.2.1. By the above corollary and Theorem 7.2.1, we have that for every $u\in P$,

$$\|\xi_k^u\|_k^2 \sim \left(\frac{k}{2\pi}\right)^n$$

when k goes to infinity, the estimate being uniform in u. In particular, there exists $k_0 \geq 1$ such that for every $u \in P$, $\xi_k^u \neq 0$ whenever $k \geq k_0$. For $k \geq k_0$, we set $\xi_k^{u,\text{norm}} = \xi_k^u / \|\xi_k^u\|_k$. Observe also that this means that the class of ξ_k^u in the projective space $\mathbb{P}(\mathcal{H}_k)$ is well-defined. In fact, this class only depends on $\pi(u)$ (because for $\lambda \in \mathbb{C}$, $\xi_k^{\lambda u} = \lambda^k \xi_k^u$) and is called the *coherent state* at $x = \pi(u)$.

Proposition 9.2.1. *There exists $C > 0$ such that for every $x \in M$, for every $u \in P$ such that $x = \pi(u)$ and for every $f \in \mathcal{C}^2(M, \mathbb{R})$ having x as a critical point,*

$$\|T_k(f)\xi_k^{u,\text{norm}} - f(x)\xi_k^{u,\text{norm}}\|_k \leq Ck^{-1}\|f\|_2$$

for every $k \geq k_0$.

Proof. Let $(U_i)_{1 \leq i \leq m}$ be an open cover of M by trivialisation open sets, and let $(V_i)_{1 \leq i \leq m}$ be a refinement of $(U_i)_{1 \leq i \leq m}$ such that $\overline{V_i} \subset U_i$ is compact. Then it is enough to show that for every $i \in [\![1, m]\!]$, there exists $C_i > 0$ such that for every $x \in V_i$, for every $u \in P$ such that $x = \pi(u)$ and for every $f \in \mathcal{C}^2(M, \mathbb{R})$ having x as a critical point,

$$\|T_k(f)\xi_k^{u,\text{norm}} - f(x)\xi_k^{u,\text{norm}}\|_k \leq Ck^{-1}\|f\|_2$$

for every $k \geq k_0$. Indeed it will then suffice to take $C = \max_{1 \leq i \leq m} C_i$. So let us choose $i \in [\![1, d]\!]$ and let us take $x \in V_i$, and set $\lambda = f(x)$. Then

$$\|(f - \lambda)\xi_k^{u,\text{norm}}\|_k^2 = \int_{V_i} |f(y) - \lambda|^2 \ \ \|\xi_k^{u,\text{norm}}(y)\|^2 \mu(y) \\ + \int_{M \setminus V_i} |f(y) - \lambda|^2 \|\xi_k^{u,\text{norm}}(y)\|^2 \mu(y).$$

We will estimate both integrals. Let us introduce some coordinates $y_1, \ldots, y_{2n}$ on U_i such that $x = (0, \ldots, 0)$, and set $q(y) = \sum_{j=1}^{2n} y_j^2$. By Taylor's formula, there exists a constant $\alpha > 0$, not depending on f, such that $|f(y) - \lambda| \leq \alpha \|f\|_2 q(y)$ for every $y \in V_i$. Therefore,

$$\int_{V_i} |f(y) - \lambda|^2 \|\xi_k^{u,\text{norm}}(y)\|^2 \mu(y) \leq \alpha^2 \|f\|_2^2 \int_{V_i} \|\xi_k^{u,\text{norm}}(y)\|^2 q(y)^2 \, \mu(y).$$

In order to estimate this integral, we write:

$$\|\xi_k^{u,\text{norm}}(y)\| = \frac{\|\xi_k^u(y)\|}{\|\xi_k^u\|_k} = \frac{\|\Pi_k(y, x) \cdot u^k\|}{\|\xi_k^u\|_k}.$$

We claim that $\|\Pi_k(y, x) \cdot u^k\| = \|\Pi_k(y, x)\|$. This is easily proved by fixing y, taking $v \in L_y$ with unit norm, and writing $\Pi_k(y, x)$ in the orthonormal basis $v^k \otimes \bar{u}^k$ of $L_y^k \otimes \bar{L}_x^k$. But it follows from (8.7) that there exists $\beta > 0$ such that for every $y \in V_i$,

$\|E(y,x)\| \leq \exp\bigl(-\beta q(y)\bigr)$. Therefore, using Theorem 7.2.1 and remembering that $\|\xi_k^u\|_k^2 \sim \bigl(k/(2\pi)\bigr)^n$, we obtain that there exists $\gamma > 0$ independent of f, x and u such that

$$\forall y \in V_i, \quad \|\xi_k^{u,\mathrm{norm}}(y)\|^2 \leq \gamma k^n \exp\bigl(-2\beta k q(y)\bigr).$$

Now, on U_i we can write $\mu = g\,\mathrm{d}y_1 \wedge \cdots \wedge \mathrm{d}y_{2n}$ for some smooth function g. So, if $\delta = \max_{\overline{V_i}}|g|$, we have that

$$\int_{V_i} \|\xi_k^{u,\mathrm{norm}}(y)\|^2 q(y)^2 \mu(y) \leq \gamma\delta k^n \int_{\mathbb{R}^{2n}} \exp\bigl(-2\beta k q(y)\bigr) q(y)^2\,\mathrm{d}y.$$

By performing the change of variable $w = \sqrt{k}\;y$, we finally obtain that

$$\int_{V_i} \|\xi_k^{u,\mathrm{norm}}(y)\|^2 q(y)^2 \mu(y) \leq \varepsilon k^{-2}$$

for some $\varepsilon > 0$, not depending on f, x, u. Consequently,

$$\int_{V_i} |f(y) - \lambda|^2 \|\xi_k^{u,\mathrm{norm}}(y)\|^2 \mu(y) \leq \alpha^2 \varepsilon \|f\|_2^2 k^{-2}.$$

It remains to estimate the integral on $M \setminus V_i$. Since for every $y \in M$, we have that $|f(y) - \lambda| \leq 2\|f\|_0 \leq 2\|f\|_2$, we immediately obtain that

$$\int_{M\setminus V_i} |f(y) - \lambda|^2 \|\xi_k^{u,\mathrm{norm}}(y)\|^2 \mu(y) \leq 4\|f\|_2^2 \int_{M\setminus V_i} \|\xi_k^{u,\mathrm{norm}}(y)\|^2\, \mu(y).$$

We claim that this last integral is a $O\bigl(k^{-2}\bigr)$. This comes again from the fact that $\|\xi_k^{u,\mathrm{norm}}(y)\| = \|\varPi_k(y,x)\|/\|\xi_k^u\|_k$, since there exists $r < 1$ such that $\|E(y,x)\| \leq r$ whenever y belongs to $M \setminus V_i$. So we finally get that

$$\|(f-\lambda)\xi_k^{u,\mathrm{norm}}\|_k \leq C_i \|f\|_2 k^{-1}$$

for some $C_i > 0$ independent of f, x, u. Since the operator norm of $\varPi_k$ is smaller than one, this yields

$$\|(T_k(f) - \lambda)\xi_k^{u,\mathrm{norm}}\|_k = \|\varPi_k (f-\lambda)\xi_k^{u,\mathrm{norm}}\|_k \leq C_i \|f\|_2 k^{-1},$$

which concludes the proof. □

To prove Theorem 5.2.1, we assume that the maximum of $|f|$ is $f(x_0)$ for some $x_0 \in M$ (otherwise, we work with $-f$), and we apply the previous result to x_0 and $u \in L_{x_0}$. This gives

$$\|T_k(f)\xi_k^{u,\mathrm{norm}} - \|f\|\xi_k^{u,\mathrm{norm}}\| \leq C k^{-1} \|f\|_2.$$

This implies that the distance between $\|f\|$ and the spectrum of $T_k(f)$ satisfies $\mathrm{dist}\,\bigl(\|f\|, \mathrm{Sp}\bigl(T_k(f)\bigr)\bigr) \leq C k^{-1}\|f\|_2$. Indeed, it is an easy consequence of the spectral theorem that if A is a bounded self-adjoint operator acting on a Hilbert space, then

$$\|(A-\lambda)^{-1}\| \leq \frac{1}{\operatorname{dist}\big(\lambda, \operatorname{Sp}(A)\big)}$$

for every $\lambda \notin \operatorname{Sp}(A)$. So there exists $\lambda \in \operatorname{Sp}\big(T_k(f)\big)$ such that $\lambda \geq \|f\| - Ck^{-1}\|f\|_2$. Therefore, we have that

$$\|T_k(f)\| = \max_{\mu \in \operatorname{Sp}(T_k(f))} |\mu| \geq \|f\| - Ck^{-1}\|f\|_2.$$

9.3 Positive Operator-Valued Measures

Let us show how the coherent states that we have introduced can be used to describe Berezin–Toeplitz operators in terms of integrals against a positive operator-valued measure. Firstly, let us recall what this term means. Let $\mathcal{H}$ be a complex Hilbert space, and let $\mathcal{S}(\mathcal{H})$ be the space of bounded self-adjoint operators on $\mathcal{H}$. Let X be a set endowed with a σ-algebra $\mathcal{C}$.

Definition 9.3.1. A *positive operator-valued measure* on X with values in $\mathcal{S}(\mathcal{H})$ is a map $G \colon \mathcal{C} \to \mathcal{S}(\mathcal{H})$ which satisfies the following properties:

(1) for every $A \in \mathcal{C}$, $G(A)$ is a positive operator, i.e. $\langle A\xi, \xi \rangle \geq 0$ for every $\xi \in \mathcal{H}$,
(2) $G(\varnothing) = 0$ and $G(X) = \mathrm{Id}$,
(3) G is σ-additive: for any sequence $(A_j)_{j\geq 1}$ of disjoint elements of $\mathcal{C}$, $G\big(\bigcup_{j\geq 1} A_j\big) = \sum_{j\geq 1} G(A_j)$.

Such a positive operator-valued measure defines a probability measure μ_ξ on X for every $\xi \in \mathcal{H}$, by the formula $\mu_\xi(A) = \langle G(A)\xi, \xi \rangle$ for $A \in \mathcal{C}$. Given a bounded measurable function $f : X \to \mathbb{R}$, we define an operator $\int_X f dG \in \mathcal{S}(\mathcal{H})$ characterised by the following property:

$$\forall \xi \in \mathcal{H}, \quad \left\langle \left(\int_X f \, \mathrm{d}G \right) \xi, \xi \right\rangle = \int_X f \, \mathrm{d}\mu_\xi.$$

Coming back to the context of Berezin–Toeplitz operators, we consider $X = M$ with the σ-algebra generated by its Borel sets, and $\mathcal{H} = \mathcal{H}_k = H^0(M, L^k)$. As before, for $x \in M$ and $u \in L_x$ with unit norm, let ξ_k^u be the coherent vector at u. Recall that there exists $k_0 \geq 1$ such that $\xi_k^u \neq 0$ whenever $k \geq k_0$. We claim that the function

$$\rho_k \colon M \to \mathbb{R}, \quad x \mapsto \|\xi_k^u\|_k^2$$

is well-defined, i.e. only depends on x. Indeed, if v is another unit vector in L_x, then $v = \lambda u$ for some $\lambda \in \mathbb{S}^1$. But then we have that $\xi_k^v = \lambda^k \xi_k^u$, so $\|\xi_k^v\|_k^2 = \|\xi_k^u\|_k^2$. For $k \geq k_0$, ρ_k is a positive function. Furthermore, the projection

$$P_k^x \colon \mathcal{H}_k \to \mathcal{H}_k, \quad \phi \mapsto \frac{\langle \phi, \xi_k^u \rangle_k}{\|\xi_k^u\|_k^2} \xi_k^u$$

is also only dependent on x.

Lemma 9.3.2. *For $k \geq k_0$, the map G_k such that $\mathrm{d}G_k = \rho_k(x) P_k^x \mu$ defines a positive operator-valued measure on M.*

Proof. The positivity and σ-additivity are immediate from the form of G_k. Let us prove the fact that $G_k(M) = \mathrm{Id}$. Let $\phi \in \mathcal{H}_k$ and $y \in M$; we have that

$$(G_k(M)\phi)(y) = \int_M \rho_k(x)(P_k^x \phi)(y)\mu(x).$$

Recall that $\xi_k^u(y) = \Pi_k(y,x) \cdot u^k$. Thus,

$$\rho_k(x)(P_k^x \phi)(y) = \langle \phi, \xi_k^u \rangle_k \xi_k^u(y) = \Pi_k(y,x) \cdot \big(\langle \phi, \xi_k^u \rangle_k u^k \big).$$

But ξ_k^u satisfies the reproducing property (7.2), hence $\langle \phi, \xi_k^u \rangle_k u^k = \phi(x)$. So finally

$$(G_k(M)\phi)(y) = \int_M \Pi_k(y,x) \cdot \phi(x)\mu(x) = (\Pi_k \phi)(y) = \phi(y). \qquad \square$$

Proposition 9.3.3. *Let $k \geq k_0$. For any $f \in \mathcal{C}^\infty(M, \mathbb{R})$, $T_k(f) = \int_M f \, \mathrm{d}G_k$.*

Proof. Let $S_k(f) = \int_M f \, \mathrm{d}G_k$, and let $\phi \in \mathcal{H}_k$. Then by definition,

$$\langle S_k(f)\phi, \phi \rangle_k = \int_M f(x) \langle P_k^x \phi, \phi \rangle_k \rho_k(x) \mu(x).$$

We claim that for every $x \in M$, $\langle P_k^x \phi, \phi \rangle_k \rho_k(x) = h_k\big(\phi(x), \phi(x)\big)$. Indeed, on the one hand, since ξ_k^u satisfies the reproducing property (7.2), we have that $\phi(x) = \langle \phi, \xi_k^u \rangle_k u^k$. Therefore

$$h_k\big(\phi(x), \phi(x)\big) = |\langle \phi, \xi_k^u \rangle_k|^2 h_k(u^k, u^k) = |\langle \phi, \xi_k^u \rangle_k|^2.$$

But on the other hand, we have that

$$\langle P_k^x \phi, \phi \rangle_k = \frac{|\langle \phi, \xi_k^u \rangle_k|^2}{\|\xi_k^u\|_k^2} = \frac{|\langle \phi, \xi_k^u \rangle_k|^2}{\rho_k(x)},$$

which proves the claim. Consequently,

$$\langle S_k(f)\phi, \phi \rangle_k = \int_M h_k(\big(f(x)\phi(x), \phi(x)\big)\mu(x) = \langle T_k(f)\phi, \phi \rangle_k,$$

which proves the result. $\square$

9.4 Projective Embeddings

The coherent states construction gives a way to embed M into a complex projective space. Remember that given a unit vector $u \in L$, the coherent state $\xi_k^u \in \mathcal{H}_k$ at u is the holomorphic section of $L^k \to M$ given by

$$\xi_k^u(y) = \Pi_k\big(y, \pi(u)\big) \cdot u^k,$$

and that there exists $k_0 \geq 1$ such that for every $k \geq k_0$ and for every unit vector $u \in L$, $\xi_k^u \neq 0$. Hence for $k \geq k_0$ (from now on, we will assume that it is the case), the class $[\xi_k^u]$ of ξ_k^u in $\mathbb{P}(\mathcal{H}_k)$ is well-defined, and we saw that this class only depends on $\pi(u)$ where π is the projection from L to M. Thus we obtain a map

$$\Phi_{\mathrm{coh}} \colon M \to \mathbb{P}(\mathcal{H}_k), \qquad x \mapsto [\xi_k^u], \quad u \in \pi^{-1}(x).$$

Since $\Pi(\cdot,\cdot)$ is anti-holomorphic on the right variable, this map is anti-holomorphic. To get a holomorphic map, we consider

$$\Phi_{\mathrm{hol}} \colon M \to \mathbb{P}(\mathcal{H}_k^*), \qquad x \mapsto \big[\langle\, \cdot\,, \xi_k^u\rangle_k\big], \quad u \in \pi^{-1}(x).$$

By Lemma 9.1.1, we have the alternative expression $\Phi_{\mathrm{hol}}(x) = [\alpha_u]$ for any $u \in \pi^{-1}(x)$ with norm one, where $\alpha_u(\phi) = \phi(x) \cdot \bar{u}^k$ for every $\phi \in \mathcal{H}_k$.

In order to identify $\mathbb{P}(\mathcal{H}_k)$ with $\mathbb{CP}^{d_k}$, let us choose an orthonormal basis $(\varphi_j)_{0 \leq j \leq d_k}$ of $\mathcal{H}_k$, $d_k = \dim(\mathcal{H}_k) - 1$, and let us write for any unit vector $u \in L$

$$\xi_k^u = \sum_{j=0}^{d_k} \lambda_j(u)\varphi_j$$

for some complex numbers $\lambda_0(u), \dots, \lambda_{d_k}(u)$. Then, using homogeneous coordinates,

$$\Phi_{\mathrm{coh}}(x) = [\lambda_0(u) : \cdots : \lambda_{d_k}(u)], \quad \Phi_{\mathrm{hol}}(x) = \big[\overline{\lambda_0(u)} : \cdots : \overline{\lambda_{d_k}(u)}\big].$$

The latter is obtained by decomposing $\langle\, \cdot\,, \xi_k^u\rangle$ in the dual basis $(\varphi_j^*)_{0 \leq j \leq d_k}$.

Proposition 9.4.1. *The maps Φ_{coh} and Φ_{hol} are embeddings for k large enough.*

Proof. Since L^k is very ample for k large enough because L is positive, this follows from the fact that Φ_{hol} is the embedding considered in Kodaira's embedding theorem [24, Section 5.3]. Indeed, for $j \in [\![0, d_k]\!]$ and $x \in M$, we have that for any unit vector $u \in \pi^{-1}(x)$:

$$\varphi_j(x) = \langle \varphi_j, \xi_k^u\rangle_k u^k = \overline{\lambda_j(u)}u^k. \qquad \square$$

As before, let $\rho_k \colon M \to \mathbb{R}$ be the function sending $x \in M$ to $\|\xi_k^u\|_k^2$ for any $u \in L_x$ with norm one. This function is often called *Rawnsley's function*, since it was introduced in [40] (see also [39]); however, the reader may encounter this terminology for a slightly different function, since many authors work with elements $u \neq 0 \in L$ instead of unit vectors.

Proposition 9.4.2. *The pullback of the Fubini–Study form by Φ_{hol} is given by*

$$\Phi_{\text{hol}}^* \omega_{\text{FS}} = k\omega + \mathrm{i}\partial\bar{\partial} \log \rho_k.$$

Proof. As in Example 2.5.9, introduce, for $j \in [\![1, d_k]\!]$, the open subset

$$U_j = \{[z_0 : \cdots : z_{d_k}] \in \mathbb{CP}^{d_k} \mid z_j \neq 0\}$$

of $\mathbb{CP}^{d_k}$. Then on U_j,

$$\omega_{\text{FS}} = \mathrm{i}\partial\bar{\partial} \log \left(\sum_{m=0}^{d_k} \left| \frac{z_m}{z_j} \right|^2 \right).$$

Therefore, we have that, on U_j:

$$\Phi_{\text{hol}}^* \omega_{\text{FS}} = \mathrm{i}\partial\bar{\partial} \log \left(\sum_{m=0}^{d_k} \left| \frac{\lambda_m}{\lambda_j} \right|^2 \right) = \mathrm{i}\partial\bar{\partial} \log \rho_k - \mathrm{i}\partial\bar{\partial} \log |\lambda_j|^2. \tag{9.1}$$

Now, let u_j be a local section of L over U_j such that $u_j(x)$ is a unit vector of L_x for every $x \in U_j$. Then $\varphi_j(x) = \lambda_j\big(u_j(x)\big) u_j(x)^k$ is a local non-vanishing holomorphic section of L, thus, remembering the proof of Proposition (3.5.4), we get that

$$\nabla^k \varphi_j = \beta_j \otimes \varphi_j, \quad \beta_j = \partial \log H_j$$

on U_j, with $H_j = h_k(\varphi_j, \varphi_j) = |\lambda_j(u_j)|^2$. Therefore

$$-\mathrm{i}k\omega = \operatorname{curv}(\nabla^k) = \bar{\partial}\partial \log H_j = \bar{\partial}\partial \log |\lambda_j(u_j)|^2$$

on U_j, which, in view of (9.1), yields the result. □

Thus $\Phi_{\text{hol}}^* \omega_{\text{FS}} = k\omega$ whenever ρ_k is constant. In this case, applying Proposition 9.3.3 to $f = 1$, we get that

$$\dim \mathcal{H}_k = \int_M \rho_k \mu(x) = \operatorname{vol}(M) \rho_k,$$

therefore $\rho_k = \dim \mathcal{H}_k / \operatorname{vol}(M)$.

Example 9.4.3 (*The complex projective line*). Let us come back to Example 7.2.5. On $U_0 = \{[z_0 : z_1] | \ z_0 \neq 0\}$, we have the following expression for the kernel of Π_k:

$$\Pi_k(z, w) = \frac{k+1}{2\pi} (1 + z\overline{w})^k \, t_0^k(z) \otimes \bar{t}_0^k(w).$$

Considering the unit vector

$$u(z) = \frac{1}{h\big(t_0(z), t_0(z)\big)^{1/2}} t_0(z) = \sqrt{1+|z|^2}\, t_0(z),$$

we get that the coherent state at $u(z)$ has value at w

$$\begin{aligned}\xi_k^{u(z)}(w) &= \frac{k+1}{2\pi}\big(1+|z|^2\big)^{k/2}(1+\bar{z}w)^k h\big(t_0(z), t_0(z)\big)^k\, t_0^k(w) \\ &= \frac{k+1}{2\pi}\frac{\big(1+\bar{z}w\big)^k}{\big(1+|z|^2\big)^{k/2}}\, t_0^k(w).\end{aligned}$$

Exercise 9.4.4. Check that $\rho_k(z) = \|\xi_k^{u(z)}\|_k^2 = (k+1)/(2\pi)$.

To understand the coherent states embedding, we expand this coherent state to get a linear combination of the $e_\ell(w) = \sqrt{(k+1)\binom{k}{\ell}/(2\pi)}\, w^{k-\ell} t_0^k(w)$, $0 \le \ell \le k$:

$$\xi_k^{u(z)}(w) = \sqrt{\frac{(k+1)}{2\pi\big(1+|z|^2\big)^k}} \sum_{\ell=0}^{k} \sqrt{\binom{k}{\ell}}\, \bar{z}^\ell e_\ell(w).$$

This means that

$$\Phi_{\mathrm{coh}}(z) = \left[1 : \cdots : \sqrt{\binom{k}{\ell}}\ \bar{z}^\ell : \cdots : \bar{z}^k\right]$$

and finally

$$\Phi_{\mathrm{hol}}(z) = \left[1 : \cdots : \sqrt{\binom{k}{\ell}}\ z^\ell : \cdots : z^k\right]$$

is the Veronese embedding of $\mathbb{CP}^1$ into $\mathbb{CP}^k$.