Chapter 8
Proof of Product and Commutator Estimates

The aim of this chapter is to prove Theorems 5.2.2 and 5.2.3.

8.1 Corrected Berezin–Toeplitz Operators

Given a function $f \in \mathcal{C}^\infty(M, \mathbb{R})$, we introduce the corrected Berezin–Toeplitz quantisation of f:

$$T_k^{\mathrm{c}}(f) = \Pi_k\left(f + \frac{1}{\mathrm{i}k}\nabla^k_{X_f}\right) : \mathcal{H}_k \to \mathcal{H}_k \tag{8.1}$$

where X_f is the Hamiltonian vector field associated with f. The operator

$$P_k(f) = f + \frac{1}{\mathrm{i}k}\nabla^k_{X_f} : \mathcal{C}^\infty(M, L^k) \to \mathcal{C}^\infty(M, L^k)$$

is called the *Kostant–Souriau operator* associated with f. The Kostant–Souriau operators satisfy the following nice properties.

Lemma 8.1.1. *For any* $f, g \in \mathcal{C}^\infty(M, \mathbb{R})$,

$$P_k(fg) = P_k(f)P_k(g) - \frac{1}{\mathrm{i}k}\left(\{f, g\} + \frac{1}{\mathrm{i}k}\nabla^k_{X_f}\nabla^k_{X_g}\right).$$

Proof. Since $X_{fg} = fX_g + gX_f$, we have that

$$P_k(fg) = f\left(g + \frac{1}{\mathrm{i}k}\nabla^k_{X_g}\right) + g\left(\frac{1}{\mathrm{i}k}\nabla^k_{X_f}\right) = fP_k(g) + g\left(\frac{1}{\mathrm{i}k}\nabla^k_{X_f}\right).$$

We can rewrite this as

$$P_k(fg) = P_k(f)P_k(g) - \frac{1}{\mathrm{i}k}\nabla^k_{X_f}P_k(g) + g\left(\frac{1}{\mathrm{i}k}\nabla^k_{X_f}\right).$$

Y. Le Floch, *A Brief Introduction to Berezin–Toeplitz Operators on Compact Kähler Manifolds*, CRM Short Courses,
https://doi.org/10.1007/978-3-319-94682-5_8

Let us simplify the second term of the right-hand side; for $\phi \in \mathcal{C}^\infty(M, L^k)$, one has

$$\nabla^k_{X_f}(P_k(g)\phi) = (\mathcal{L}_{X_f} g)\phi + g\nabla^k_{X_f}\phi + \frac{1}{\mathrm{i}k}\nabla^k_{X_f}\nabla^k_{X_g}\phi.$$

Using that $\mathcal{L}_{X_f} g = \{f, g\}$, this implies that

$$P_k(fg) = P_k(f)P_k(g) - \frac{1}{\mathrm{i}k}\big(\{f, g\} + \frac{1}{\mathrm{i}k}\nabla^k_{X_f}\nabla^k_{X_g}\big). \qquad \square$$

This shows that $P_k(fg)$ differs from $P_k(f)P_k(g)$ by a remainder "of order k^{-1}". It turns out that for commutators, however, there is an exact (i.e. without remainder) correspondence principle for Kostant–Souriau operators.

Lemma 8.1.2. *For any $f, g \in \mathcal{C}^\infty(M, \mathbb{R})$,*

$$[P_k(f), P_k(g)] = \frac{1}{\mathrm{i}k} P_k(\{f, g\}).$$

Proof. Since $P_k(gf) = P_k(fg)$, the previous lemma yields

$$[P_k(f), P_k(g)] = \frac{1}{\mathrm{i}k}\left(\{f, g\} + \frac{1}{\mathrm{i}k}\nabla^k_{X_f}\nabla^k_{X_g} - \{g, f\} - \frac{1}{\mathrm{i}k}\nabla^k_{X_g}\nabla^k_{X_f}\right).$$

This can be rewritten as

$$[P_k(f), P_k(g)] = \frac{1}{\mathrm{i}k}\left(2\{f, g\} + \frac{1}{\mathrm{i}k}[\nabla^k_{X_f}, \nabla^k_{X_g}]\right).$$

Moreover, by definition of the curvature, we have that

$$[\nabla^k_{X_f}, \nabla^k_{X_g}] = \mathrm{curv}(\nabla^k)(X_f, X_g) + \nabla^k_{[X_f, X_g]},$$

which yields, since $\mathrm{curv}(\nabla^k) = -\mathrm{i}k\omega$, and since $[X_f, X_g]$ is the Hamiltonian vector field associated with $\{f, g\}$,

$$[\nabla^k_{X_f}, \nabla^k_{X_g}] = -\mathrm{i}k\{f, g\} + \nabla^k_{X_{\{f,g\}}}.$$

Putting all these equalities together, we finally obtain that

$$[P_k(f), P_k(g)] = \frac{1}{\mathrm{i}k}\left(\{f, g\} + \frac{1}{\mathrm{i}k}\nabla^k_{X_{\{f,g\}}}\right),$$

which was to be proved. $\square$

The idea behind the proof of Theorems 5.2.2 and 5.2.3 is to derive from the properties above some estimates for the corrected Berezin–Toeplitz operators and to take profit of these estimates by comparing the corrected operator $T^{\mathrm{c}}_k(f)$ with the usual Berezin–Toeplitz operator $T_k(f)$. In order to do so, we will need a result due to Tuynman [46], but let us first introduce some notation. Let $g = \omega(\cdot, j\cdot)$ be the Kähler metric on M, let μ_g be the associated volume form, let grad_g be the associated gradient, and let Δ be the associated Laplacian. We recall that for any $f \in \mathcal{C}^2(M)$,

$$\Delta f = \mathrm{div}_g(\mathrm{grad}_g f)$$

where the divergence $\mathrm{div}_g(X)$ of a vector field X on M is the function defined by the equality

$$\mathcal{L}_X \mu_g = \mathrm{div}_g(X)\mu_g.$$

Proposition 8.1.3 (Tuynman's lemma). *Let $X \in \mathcal{C}^1(M, TM \otimes \mathbb{C})$. Then*

$$\Pi_k \nabla^k_X \Pi_k = -\Pi_k \,\mathrm{div}_g(X^{1,0})\Pi_k,$$

where we recall that $X^{1,0} = (X - \mathrm{i}jX)/2$. Furthermore, if $f \in \mathcal{C}^2(M, \mathbb{R})$, then

$$\Pi_k \left(\frac{1}{\mathrm{i}k} \nabla^k_{X_f} \right) \Pi_k = -\frac{1}{2k} \Pi_k (\Delta f) \Pi_k.$$

The following corollary is immediate.

Corollary 8.1.4. *For every $X \in \mathcal{C}^1(M, TM \otimes \mathbb{C})$,*

$$\left\| \Pi_k \left(\frac{1}{\mathrm{i}k} \nabla^k_X \right) \Pi_k \right\| = O\big(k^{-1}\big) \|X\|_1.$$

In particular, for every $f \in \mathcal{C}^2(M, \mathbb{R})$,

$$\left\| \Pi_k \left(\frac{1}{\mathrm{i}k} \nabla^k_{X_f} \right) \Pi_k \right\| = O\big(k^{-1}\big) \|f\|_2.$$

Consequently, for every $f \in \mathcal{C}^2(M, \mathbb{R})$, $\|T^{\mathrm{c}}_k(f) - T_k(f)\| = O\big(k^{-1}\big)\|f\|_2$.

Proof of Proposition 8.1.3. Set $Y = X^{1,0}$. By virtue of Lemma 8.1.5 below, proving the first statement amounts to showing that for every $\phi \in \mathcal{H}_k$,

$$\langle \Pi_k \big(\nabla^k_X \phi\big), \phi \rangle_k = -\langle \Pi_k (\mathrm{div}_g(Y)\phi), \phi \rangle_k.$$

Using the facts that Π_k is self-adjoint and that $\Pi_k \phi = \phi$ whenever ϕ belongs to $\mathcal{H}_k$, we only need to prove that

$$\forall \phi \in \mathcal{H}_k, \quad \langle \nabla^k_X \phi, \phi \rangle_k = -\langle \mathrm{div}_g(Y)\phi, \phi \rangle_k. \tag{8.2}$$

Recall that $\mu_g = \mu$ the Liouville measure on M. We have that

$$\langle \mathrm{div}_g(Y)\phi, \phi \rangle_k = \int_M \mathrm{div}_g(Y) h_k(\phi, \phi)\, \mu_g = \int_M h_k(\phi, \phi)\, \mathcal{L}_Y \mu_g. \tag{8.3}$$

Now, by integrating the equality

$$\mathcal{L}_Y (h_k(\phi, \phi)\mu_g) = \mathcal{L}_Y \big(h_k(\phi, \phi)\big)\mu_g + h_k(\phi, \phi)\mathcal{L}_Y \mu_g,$$

we obtain that

$$\int_M h_k(\phi,\phi)\,\mathcal{L}_Y\mu_g = -\int_M \mathcal{L}_Y\big(h_k(\phi,\phi)\big)\mu_g.$$

Indeed, by Cartan's formula, and using the fact that $h_k(\phi,\phi)\mu_g$ is closed, we have that $\mathcal{L}_Y(h_k(\phi,\phi)\mu_g) = \mathrm{d}\big(i_Y(h_k(\phi,\phi)\mu_g)\big)$, thus its integral on M vanishes. Coming back to (8.3), this yields

$$\langle \mathrm{div}_g(Y)\phi,\phi\rangle_k = -\int_M \mathcal{L}_Y\big(h_k(\phi,\phi)\big)\mu_g = -\int_M \Big(h_k\big(\nabla^k_Y\phi,\phi\big) + h_k\big(\phi,\nabla^k_{\overline{Y}}\phi\big)\Big)\mu_g,$$

where the second equality comes from the fact that ∇^k and h_k are compatible. But $\overline{Y}$ is a section of $T^{0,1}M$, and ϕ is a holomorphic section of L^k, so $\nabla^k_{\overline{Y}}\phi = 0$, which implies that $\nabla^k_X\phi = \nabla^k_Y\phi$ since $X = Y + \overline{Y}$, and (8.2) is proved.

We now want to apply this to X_f where f belongs to $\mathcal{C}^2(M,\mathbb{R})$. Observe that

$$\mathrm{div}_g\big(X_f^{1,0}\big) = \tfrac{1}{2}\big(\mathrm{div}_g(X_f) - \mathrm{i}\,\mathrm{div}_g(jX_f)\big).$$

We claim that $\mathrm{div}_g(X_f) = 0$; indeed, since $\mu_g = \mu$, we have that

$$\mathrm{div}_g(X_f)\mu_g = \mathcal{L}_{X_f}\mu_g = \mathcal{L}_{X_f}\mu = 0.$$

Consequently, $\mathrm{div}_g(X_f^{1,0}) = -(\mathrm{i}/2)\,\mathrm{div}_g(jX_f)$. Thanks to Lemma 2.6.1, this yields

$$\mathrm{div}_g(X_f^{1,0}) = \frac{\mathrm{i}}{2}\,\mathrm{div}(\mathrm{grad}_g f) = \frac{\mathrm{i}}{2}\Delta f,$$

and the second statement follows. □

Lemma 8.1.5. *Let T be a bounded operator acting on a complex Hilbert space $\mathcal{H}$. If $\langle T\xi,\xi\rangle = 0$ for every $\xi\in\mathcal{H}$, then $T = 0$.*

Proof. This is a standard exercise but we still prove it. Let $\xi,\eta\in\mathcal{H}$. Then

$$0 = \langle T(\xi+\eta),\xi+\eta\rangle = \langle T\xi,\xi\rangle + \langle T\xi,\eta\rangle + \langle T\eta,\xi\rangle + \langle T\eta,\eta\rangle$$

which yields

$$\langle T\xi,\eta\rangle = -\langle T\eta,\xi\rangle.$$

Replacing η by $\mathrm{i}\eta$, this implies that

$$-\mathrm{i}\langle T\xi,\eta\rangle = -\mathrm{i}\langle T\eta,\xi\rangle,$$

and combining these two equalities yields $\langle T\xi,\eta\rangle = 0$. □

8.2 Unitary Evolution of Kostant–Souriau Operators

The goal of this section is to give an alternate, more geometric proof of Lemma 8.1.2, and to use this as an excuse to address the topic of the Schrödinger equation for these operators. More precisely, given a function $f \in \mathcal{C}^\infty(M, \mathbb{R})$, we want to look for solutions of

$$\frac{\mathrm{d}\Psi_t}{\mathrm{d}t} = -\mathrm{i}kP_k(f)\Psi_t, \quad t \in \mathbb{R}, \tag{8.4}$$

where Ψ_t is a smooth section of $L^k \to M$ and $\Psi_0 \in \mathcal{C}^\infty(M, L^k)$ is a given initial condition. We can solve this equation as follows. Given a path $\gamma\colon [0,T] \to M$, let

$$\mathcal{T}^k_\gamma \colon L^k_{\gamma(0)} \to L^k_{\gamma(T)}$$

be the parallel transport operator in L^k with respect to ∇^k. Moreover, let ϕ^t be the Hamiltonian flow of f at time t.

Proposition 8.2.1. *Given $\Psi_0 \in \mathcal{C}^\infty(M, L^k)$, the family of sections $\Psi_t \in \mathcal{C}^\infty(M, L^k)$ defined as*

$$\Psi_t\big(\phi^t(m)\big) = \exp\big(-\mathrm{i}ktf(m)\big)\mathcal{T}^k_{(\phi^s(m))_{s\in[0,t]}}\big(\Psi_0(m)\big)$$

for every $m \in M$, is a solution of (8.4) with initial condition Ψ_0.

This defines an operator $U_k(t)\colon \mathcal{C}^\infty(M, L^k) \to \mathcal{C}^\infty(M, L^k)$ sending Ψ_0 to Ψ_t, which describes the prequantum evolution of the system.

Proof. We fix $m \in M$ and $\Psi_0 \in \mathcal{C}^\infty(M, L^k)$. We claim that it is enough to prove the proposition for t so small that for every $s \in [-t, t]$, the point $\phi^s(m)$ belongs to a trivialisation open set V for L. This is because the operator $U_k(t)$ satisfies the semigroup relation $U_k(t_1 + t_2) = U_k(t_2)U_k(t_1)$.

Let u be a local non-vanishing section of L over V, and let $\varphi = hu^k$ for some $h \in \mathcal{C}^\infty(V, \mathbb{R})$. Moreover, let α be the differential form such that $\nabla s = -\mathrm{i}\alpha \otimes s$. Then we can write $P_k(f)\varphi = (\widetilde{P}_k(f)h)u^k$ with

$$\widetilde{P}_k(f)h = (f - i_{X_f}\alpha)h + \frac{1}{\mathrm{i}k}\mathcal{L}_{X_f}h. \tag{8.5}$$

Moreover, a standard computation yields

$$\mathcal{T}^k_{(\phi^s(m))_{s\in[0,t]}}\big(\varphi(m)\big) = \exp\left(\mathrm{i}k\int_0^t (\phi^s)^*(i_{X_f}\alpha)\,\mathrm{d}s\right)h(m)u^k\big(\phi^t(m)\big),$$

and consequently, if $\Psi_0 = h_0 s^k$ on V, then $\Psi_t = h_t s^k$ on V where

$$h_t(m) = \exp\left(\mathrm{i}k\left(\int_{-t}^0 (\phi^s)^*(i_{X_f}\alpha)\,\mathrm{d}s - tf(m)\right)\right)h_0\big(\phi^{-t}(m)\big).$$

for every $m \in M$. We only need to compare the time derivative of h_t and $\widetilde{P}_k(f)h_t$. To simplify notation, we will write

$$\theta(t,m) = \int_{-t}^{0} (\phi^s)^*(i_{X_f}\alpha)(m)\,\mathrm{d}s - tf(m).$$

On the one hand,

$$\frac{\mathrm{d}h_t}{\mathrm{d}t} = \exp\bigl(\mathrm{i}k\theta(t,\cdot)\bigr)\bigl(-(\phi^{-t})^*(\mathcal{L}_{X_f}h_0) + \mathrm{i}k\bigl((\phi^{-t})^*(i_{X_f}\alpha) - f\bigr)(\phi^{-t})^*h_0\bigr).$$

On the other hand, we have that

$$\mathcal{L}_{X_f}h_t = \exp\bigl(\mathrm{i}k\theta(t,\cdot)\bigr)\left((\phi^{-t})^*(\mathcal{L}_{X_f}h_0) + \mathrm{i}k\bigl((\phi^{-t})^*h_0\bigr)\int_{-t}^{0}\mathcal{L}_{X_f}\bigl((\phi^s)^*(i_{X_f}\alpha)\bigr)\,\mathrm{d}s\right).$$

Using Cartan's formula, we have that

$$\mathrm{d}\bigl((\phi^s)^*(i_{X_f}\alpha)\bigr) = (\phi^s)^*\bigl(\mathrm{d}(i_{X_f}\alpha)\bigr) = (\phi^s)^*(\mathcal{L}_{X_f}\alpha) - (\phi^s)^*(i_{X_f}\,\mathrm{d}\alpha).$$

Since $\mathrm{d}\alpha = \mathrm{i}\,\mathrm{curv}(L) = \omega$ and $(\phi^s)^*(\mathcal{L}_{X_f}\alpha) = \mathrm{d}(\phi^s)^*\alpha/\mathrm{d}s$, we can write

$$\int_{-t}^{0}\mathcal{L}_{X_f}\bigl((\phi^s)^*(i_{X_f}\alpha)\bigr)\,\mathrm{d}s = i_{X_f}\alpha - (\phi^{-t})^*(i_{X_f}\alpha),$$

therefore we finally obtain that

$$\widetilde{P}_k(f)h_t = \exp\bigl(\mathrm{i}k\theta(t,\cdot)\bigr)\left(\bigl(f - (\phi^{-t})^*(i_{X_f}\alpha)\bigr)(\phi^{-t})^*h_0 + \frac{1}{\mathrm{i}k}(\phi^{-t})^*(\mathcal{L}_{X_f}h_0)\right),$$

which yields the desired formula $-\mathrm{i}k\widetilde{P}_k(f)h_t = (\mathrm{d}h_t/\mathrm{d}t)$. □

One can check that $U_k(t)$ extends to a unitary operator on $L^2(M, L^k)$. It turns out that the Kostant–Souriau operators satisfy an exact version of Egorov's theorem (Theorem 5.3.2).

Proposition 8.2.2. *Let $f \in \mathcal{C}^\infty(M,\mathbb{R})$ and let $U_k(t)$ be the evolution operator associated with $P_k(f)$. Then*

$$U_k(t)^* P_k(g) U_k(t) = P_k(g \circ \phi^t)$$

for every $g \in \mathcal{C}^\infty(M,\mathbb{R})$, where ϕ^t is the Hamiltonian flow of f at time t.

Proof. Again, we can work in a trivialisation open set for L, since

$$U_k(t_1+t_2)^* P_k(g) U_k(t_1+t_2) = U_k(t_2)^* U_k(t_1)^* P_k(g) U_k(t_1) U_k(t_2),$$
$$g \circ \phi^{t_1+t_2} = g \circ \phi^{t_1} \circ \phi^{t_2}.$$

Hence we keep the same notation as in the proof of the previous proposition. If $U_k(t)\Psi_0 = h_t u^k$ on V, the computations performed in this proof yield

$$\mathrm{d}h_t = \exp\big(\mathrm{i}k\theta(t,\cdot)\big) \\ \times \left((\phi^{-t})^*(dh_0) + \mathrm{i}k\left(\alpha - (\phi^{-t})^*\alpha - \int_{-t}^{0} (\phi^s)^*(i_{X_f}\,\mathrm{d}\alpha)\,\mathrm{d}s - t\,\mathrm{d}f\right)(\phi^{-t})^* h_0\right).$$

We can simplify this further because

$$(\phi^s)^*(i_{X_f}\,\mathrm{d}\alpha) = (\phi^s)^*(i_{X_f}\omega) = -(\phi^s)^*(\,\mathrm{d}f) = -\,\mathrm{d}\big((\phi^s)^* f\big) = -\,\mathrm{d}f,$$

hence we obtain that

$$\mathcal{L}_{X_g} h_t = \exp\big(\mathrm{i}k\theta(t,\cdot)\big)\Big((\phi^{-t})^*(\mathcal{L}_{X_g} h_0) + \mathrm{i}k\Big(i_{X_g}\alpha - i_{X_g}\big((\phi^{-t})^*\alpha\big)\Big)(\phi^{-t})^* h_0\Big).$$

Therefore, (8.5) yields

$$\widetilde{P}_k(g) h_t = \exp\big(\mathrm{i}k\theta(t,\cdot)\big)\left(\frac{1}{\mathrm{i}k}(\phi^{-t})^*(\mathcal{L}_{X_g} h_0) + \Big(g - i_{X_g}\big((\phi^{-t})^*\alpha\big)\Big)(\phi^{-t})^* h_0\right).$$

Consequently, if $U_k(t)^* P_k(g) U_k(t) = q_t u^k$ on V, we finally obtain that

$$q_t = \frac{1}{\mathrm{i}k}\mathcal{L}_{X_{g\circ\phi^t}} h_0 + \big(g\circ\phi^t - i_{X_{g\circ\phi^t}}\alpha\big) h_0 = \widetilde{P}_k(g\circ\phi^t) h_0. \qquad \square$$

In order to reprove Lemma 8.1.2 with the help of these two results, it suffices to write the time derivative of $\phi_k(t) = U_k(t)^* P_k(g) U_k(t) \Psi_0$, for $\Psi_0 \in \mathcal{C}^\infty(M, L^k)$, in two different ways. On the one hand, by definition of U_k,

$$\left.\frac{\mathrm{d}\phi_k}{\mathrm{d}t}\right|_{t=0} = \mathrm{i}k[P_k(f), P_k(g)]\Psi_0.$$

On the other hand, since $\phi_k(t) = P_k(g\circ\phi^t)\Psi_0$, Lemma 5.3.3 implies that

$$\left.\frac{\mathrm{d}\phi_k}{\mathrm{d}t}\right|_{t=0} = P_k(\{f,g\})\Psi_0,$$

and we conclude by comparing these two equalities that the Kostant–Souriau operators satisfy the exact correspondence principle.

8.3 Product Estimate

We will need the following result, of which we will give a proof in Section 8.5.

Theorem 8.3.1. *There exists $C > 0$ such that for every $f \in \mathcal{C}^2(M, \mathbb{R})$,*

$$\|[P_k(f), \Pi_k]\| \le C k^{-1} \|f\|_2.$$

This estimate is fundamental and allows us to obtain product and commutator estimates. We now use it to prove Theorem 5.2.2. We compute the difference

$$T_k(f)T_k(g) - T_k(fg) = \Pi_k f[\Pi_k, g]\Pi_k = \Pi_k f[\Pi_k, P_k(g)]\Pi_k - \Pi_k f\left[\Pi_k, \frac{1}{\mathrm{i}k}\nabla^k_{X_g}\right]\Pi_k.$$

Thanks to Theorem 8.3.1, we know that $\|\Pi_k f[\Pi_k, P_k(g)]\Pi_k\| = O(k^{-1})\|f\|_0\|g\|_2$. The other term can be estimated by writing it as

$$\Pi_k f\left[\Pi_k, \frac{1}{\mathrm{i}k}\nabla^k_{X_g}\right]\Pi_k = \Pi_k f \Pi_k\left(\frac{1}{\mathrm{i}k}\nabla^k_{X_g}\right)\Pi_k - \Pi_k\left(\frac{1}{\mathrm{i}k}\nabla^k_{fX_g}\right)\Pi_k.$$

Both terms can be estimated using Corollary 8.1.4. The first one satisfies

$$\left\|\Pi_k f \Pi_k\left(\frac{1}{\mathrm{i}k}\nabla^k_{X_g}\right)\Pi_k\right\| = O(k^{-1})\|f\|_0\|g\|_2,$$

whereas the second one satisfies

$$\left\|\Pi_k\left(\frac{1}{\mathrm{i}k}\nabla^k_{fX_g}\right)\Pi_k\right\| = O(k^{-1})\|fX_g\|_1 = O(k^{-1})(\|f\|_0\|g\|_2 + \|f\|_1\|g\|_1).$$

This proves the first estimate of the theorem. To derive the second one, observe that $T_k(fg)$ is self-adjoint and that the adjoint of $T_k(f)T_k(g)$ is $T_k(g)T_k(f)$, and use the fact that the operator norm of the adjoint of an operator is the same as the norm of the operator.

8.4 Commutator Estimate

We first prove commutator estimates for corrected Berezin–Toeplitz operators.

Proposition 8.4.1. *For any $f, g \in \mathcal{C}^2(M, \mathbb{R})$,*

$$\left\|[T^{\mathrm{c}}_k(f), T^{\mathrm{c}}_k(g)] - \frac{1}{\mathrm{i}k}T^{\mathrm{c}}_k(\{f, g\})\right\| = O(k^{-2})\|f\|_2\|g\|_2.$$

Proof. We will compare $[T^{\mathrm{c}}_k(f), T^{\mathrm{c}}_k(g)]$ with $[P_k(f), P_k(g)]$. In order to do so, we compute:

$$\Pi_k[\Pi_k, P_k(f)][\Pi_k, P_k(g)]\Pi_k = \Pi_k P_k(f)[\Pi_k, P_k(g)]\Pi_k - \Pi_k P_k(f)\Pi_k[\Pi_k, P_k(g)]\Pi_k.$$

Expanding the first term on the right-hand side of this equality, we get

$$\Pi_k P_k(f)[\Pi_k, P_k(g)]\Pi_k = \Pi_k P_k(f)\Pi_k P_k(g)\Pi_k - \Pi_k P_k(f)P_k(g)\Pi_k$$

and the second term satisfies

$$\varPi_k P_k(f)\varPi_k[\varPi_k, P_k(g)]\varPi_k = \varPi_k P_k(f)\varPi_k P_k(g)\varPi_k - \varPi_k P_k(f)\varPi_k P_k(g)\varPi_k = 0.$$

Therefore, we have that

$$\varPi_k[\varPi_k, P_k(f)][\varPi_k, P_k(g)]\varPi_k = T_k^{\mathrm{c}}(f)T_k^{\mathrm{c}}(g) - \varPi_k P_k(f)P_k(g)\varPi_k.$$

Thanks to Theorem 8.3.1, the left-hand side is a $\mathcal{O}(k^{-2})\|f\|_2\|g\|_2$, thus

$$[T_k^{\mathrm{c}}(f), T_k^{\mathrm{c}}(g)] = \varPi_k[P_k(f), P_k(g)]\varPi_k + \mathcal{O}(k^{-2})\|f\|_2\|g\|_2$$

which yields, using Lemma 8.1.2,

$$[T_k^{\mathrm{c}}(f), T_k^{\mathrm{c}}(g)] = \frac{1}{\mathrm{i}k}T_k^{\mathrm{c}}(\{f,g\}) + \mathcal{O}(k^{-2})\|f\|_2\|g\|_2. \qquad \square$$

We now prove Theorem 5.2.3. Thanks to Proposition 8.1.3, we have that

$$T_k(f) = T_k^{\mathrm{c}}(f) + \frac{1}{2k}T_k(\varDelta f),$$

and similarly for g. Consequently, $[T_k(f), T_k(g)] = [T_k^{\mathrm{c}}(f), T_k^{\mathrm{c}}(g)] + R_k$, with

$$R_k = \frac{1}{2k}[T_k(\varDelta f), T_k^{\mathrm{c}}(g)] + \frac{1}{2k}[T_k^{\mathrm{c}}(f), T_k(\varDelta g)] + \frac{1}{4k^2}[T_k(\varDelta f), T_k(\varDelta g)].$$

Let us estimate R_k. Firstly, we have that

$$[T_k(\varDelta f), T_k^{\mathrm{c}}(g)] = [T_k(\varDelta f), T_k(g)] - \frac{1}{2k}[T_k(\varDelta f), T_k(\varDelta g)].$$

Applying Theorem 5.2.2 to $\varDelta f \in \mathcal{C}^1(M,\mathbb{R})$ and $g \in \mathcal{C}^3(M,\mathbb{R})$, we obtain that

$$[T_k(\varDelta f), T_k(g)] = O\big(k^{-1}\big)(\|f\|_2\|g\|_2 + \|f\|_3\|g\|_1) = O\big(k^{-1}\big)\|f,g\|_{1,3}.$$

Moreover, Lemma 5.1.2 implies that

$$[T_k(\varDelta f), T_k(\varDelta g)] = O\big(1\big)\|\varDelta f\|_0\|\varDelta g\|_0 = O\big(1\big)\|f\|_2\|g\|_2. \tag{8.6}$$

It follows from these estimates that

$$\frac{1}{2k}[T_k(\varDelta f), T_k^{\mathrm{c}}(g)] = O\big(k^{-2}\big)\|f,g\|_{1,3}.$$

A similar reasoning leads to

$$\frac{1}{2k}[T_k^{\mathrm{c}}(f), T_k(\varDelta g)] = O\big(k^{-2}\big)\|f,g\|_{1,3}.$$

These two results combined with (8.6) imply that $R_k = O\big(k^{-2}\big)\|f,g\|_{1,3}$. Now, thanks to the previous proposition, we have that

$$[T_k^c(f), T_k^c(g)] = \frac{1}{ik} T_k^c(\{f,g\}) + O(k^{-2}) \|f,g\|_{1,3}.$$

Therefore,

$$[T_k(f), T_k(g)] = \frac{1}{ik} T_k(\{f,g\}) + \frac{i}{2k^2} T_k(\Delta\{f,g\}) + O(k^{-2}) \|f,g\|_{1,3},$$

and we conclude thanks to the estimate

$$T_k(\Delta\{f,g\}) = O(1) \|\Delta\{f,g\}\|_0 = O(1) \|f,g\|_{1,3},$$

which follows from Lemma 5.1.2.

8.5 Fundamental Estimates

This section, which follows the same lines as in the article [20], is devoted to the proof of Theorem 8.3.1; this strongly relies on the asymptotic expansion of the Schwartz kernel of the projector given by Theorem 7.2.1. Let $E \in \mathcal{C}^\infty(M \times \overline{M}, L^k \boxtimes \bar{L}^k)$ be as in this theorem, that is, satisfying the properties stated in Proposition 7.1.1. Let $U \subset M^2$ be the open set where E does not vanish; observe that U contains the diagonal Δ_M of M^2. Define as before a function $\varphi_E \in \mathcal{C}^\infty(U)$ and a differential form $\alpha_E \in \Omega^1(U) \otimes \mathbb{C}$ by the formulas

$$\varphi_E = -2\log\|E\|, \quad \widetilde{\nabla} E = -i\alpha_E \otimes E,$$

where we recall that $\widetilde{\nabla}$ is the connection induced by ∇ on $L \boxtimes \bar{L}$. The function φ_E vanishes along Δ_M and is positive outside Δ_M. We derived the following properties of φ_E and α_E in Lemmas 7.1.3 and 7.1.4:

(1) α_E vanishes along Δ_M,
(2) φ_E vanishes to second order along Δ_M,
(3) for every $x \in M$, the kernel of the Hessian of φ_E at (x,x) is equal to $T_{(x,x)}\Delta_M$, and this Hessian is positive definite on the complement of $T_{(x,x)}\Delta_M$.

In what follows, we will need the following additional property.

Lemma 8.5.1. *Let $f \in \mathcal{C}^2(M, \mathbb{R})$, and let $g \in \mathcal{C}^2(U, \mathbb{R})$ be defined by the formula $g(x,y) = f(x) - f(y)$. Then the function*

$$u = g - \alpha_E(X_f, X_f)$$

vanishes to second order along Δ_M.

Proof. It is clear that u vanishes along Δ_M since g and α_E do. Now, let Y and Z be two vector fields on M; we compute for $(y,z) \in U$

$$(\mathcal{L}_{(Y,Z)} u)(y,z) = (\mathcal{L}_Y f)(y) - (\mathcal{L}_Z f)(z) - \mathcal{L}_{(Y,Z)}(\alpha_E(X_f, X_f))(y,z).$$

As before, set $\widetilde{\omega} = p_1^*\omega - p_2^*\omega$ with p_1, p_2 the natural projections $M^2 \to M$. Therefore the first two terms in the above equation satisfy

$$\big(\mathcal{L}_Y f\big)(y) - \big(\mathcal{L}_Z f\big)(z) = \widetilde{\omega}\big((Y,Z),(X_f,X_f)\big)(x,y).$$

Moreover, since $\mathrm{d}\alpha_E = \mathrm{i}\,\mathrm{curv}(\widetilde{\nabla}) = \widetilde{\omega}$, the last term in the previous equation can be written as

$$\mathcal{L}_{(Y,Z)}\big(\alpha_E(X_f,X_f)\big) = \widetilde{\omega}\big((Y,Z),(X_f,X_f)\big) + \alpha_E([(Y,Z),(X_f,X_f)]) \\ + \mathcal{L}_{(X_f,X_f)}\big(\alpha_E(Y,Z)\big).$$

Thus we finally obtain that

$$\mathcal{L}_{(Y,Z)}u = \alpha_E([(X_f,X_f),(Y,Z)]) - \mathcal{L}_{(X_f,X_f)}\big(\alpha_E(Y,Z)\big).$$

The first term vanishes along Δ_M because α_E does. The second term vanishes along Δ_M because α_E vanishes along Δ_M and (X_f,X_f) is tangent to Δ_M. □

These properties yield the following result. For $u \in \mathcal{C}^0(M^2,\mathbb{R})$, let $Q_k(u)$ be the operator acting on $\mathcal{C}^0(M,L^k)$ with Schwartz kernel $F_k(u) = \big(k/(2\pi)\big)^n E^k u$.

Lemma 8.5.2. *Taking a smaller U still containing Δ_M if necessary, for every compact subset $K \subset U$ and for every $p \in \mathbb{N}$, there exists a constant $C_{K,p} > 0$ such that for any $u \in \mathcal{C}^0(M^2,\mathbb{R})$ with support contained in K, and for every $k \geq 1$,*

$$\|Q_k(u)\| \leq C_{K,p}|u|_{K,p}k^{-p/2}$$

where $|u|_{K,p}$ is the supremum of $|u|\varphi_E^{-p/2}$ on $K \setminus \Delta_M$, which may be infinite.

Proof. Assume first that $K \subset V^2$, where $V \subset M$ is a trivialisation open set for M, with coordinates $x_1,\dots,x_{2n}$, such that $V^2 \subset U$. So we may identify V with a subset of $\mathbb{R}^{2n}$ and assume that we are working in a subset of $\mathbb{R}^{4n}$. Since φ_E vanishes to second order along Δ_M, Taylor's formula with integral remainder yields

$$\varphi_E(x,y) = \frac{1}{2}\,\mathrm{d}^2\varphi_E(x,x)(v,v) + \int_0^1 \frac{(1-t)^2}{2}\,\mathrm{d}^3\varphi_E\big((1-t)(x,x)+t(x,y)\big)(v,v,v)\,\mathrm{d}t$$

with $v = (0, y-x)$. The last term is a $O\big(|x-y|^3\big)$ uniformly on K. Since $\mathrm{d}^2\varphi_E(x,x)$ is positive definite on the orthogonal of $\{x = y\} \subset \mathbb{R}^{4n}$, we have that

$$\lambda_{\min}(x)\|v\|^2 \leq \mathrm{d}^2\varphi_E(x,x)(v,v) \leq \lambda_{\max}(x)\|v\|^2$$

whenever $y \neq x$, where $\lambda_{\min}(x)$ (respectively $\lambda_{\max}(x)$) is the smallest (respectively largest) positive eigenvalue of $\mathrm{d}^2\varphi_E(x,x)$. Therefore, there exists $C > 0$ such that

$$\frac{\|x-y\|^2}{C} \leq \varphi_E(x,y) \leq C\|x-y\|^2 \tag{8.7}$$

for every $(x, y) \in K$. Now, let $u \in \mathcal{C}^0(M^2, \mathbb{R})$ be compactly supported in K. The previous estimate shows that for every $(x, y) \in K$, $x \neq y$,

$$\frac{|u(x,y)|}{\varphi_E(x,y)^{p/2}} \geq \frac{|u(x,y)|}{C^{p/2}\|x-y\|^p},$$

thus $|u(x,y)| \leq C^{p/2}|u|_{K,p}\|x-y\|^p$ on V^2. If $|u|_{K,p}$ is infinite, the result is obvious. If not, since $\|E\| = \exp(-\varphi_E/2)$, we have that

$$\int_M \|F_k(u)(x,y)\| \, \mathrm{d}x \leq \left(\frac{k}{2\pi}\right)^n C^{p/2}|u|_{K,p} \int_V \exp\left(-\frac{k\|x-y\|^2}{2C}\right)\|x-y\|^p \, \mathrm{d}x.$$

The integral on V is smaller that the integral on $\mathbb{R}^{2n}$ of the same integrand. The change of variable $v = \sqrt{k/C}(x-y)$ yields

$$\int_M \|F_k(u)(x,y)\| \, \mathrm{d}x \leq \frac{C^{p+n}}{(2\pi)^n} k^{-p/2}|u|_{K,p} \int_{\mathbb{R}^{2n}} \exp\left(-\frac{\|v\|^2}{2}\right)\|v\|^p \, \mathrm{d}v,$$

which implies that $\int_M \|F_k(u)(x,y)\| \, \mathrm{d}x \leq C^1_{K,p} k^{-p/2}|u|_{K,p}$. A similar computation leads to $\int_M \|F_k(u)(x,y)\| \, \mathrm{d}y \leq C^2_{K,p} k^{-p/2}|u|_{K,p}$ for some $C^2_{K,p} > 0$. It follows from the Schur test that

$$\|Q_k(u)\| \leq C_{K,p} k^{-p/2}|u|_{K,p}$$

for some $C_{K,p} > 0$.

Let us now turn to the general case. Taking a smaller U, still containing the diagonal, if necessary, let $(V_i)_{1 \leq i \leq d}$ be a finite family of trivialisation sets of M such that $K \subset \bigcup_{i=1}^d V_i^2 \subset U$. Choose a partition of unity $\eta, (\eta_i)_{1 \leq i \leq d}$ subordinate to the open cover $M^2 \subset (M^2 \setminus K) \cup (\bigcup_{i=1}^d V_i^2)$. Let $u \in \mathcal{C}^0(M^2, \mathbb{R})$ be compactly supported in K; then

$$u = \sum_{i=1}^d \eta_i u, \quad Q_k(u) = \sum_{i=1}^d Q_k(\eta_i u).$$

It follows from the first part of the proof that

$$\|Q_k(\eta_i u)\| \leq C_{K,p,i} k^{-p/2}|\eta_i u|_{K,p} \leq C_{K,p,i} k^{-p/2}|u|_{K,p}$$

for some constants $C_{K,p,i} > 0$. We conclude by applying the triangle inequality. □

Proposition 8.5.3. *For every $p \in \mathbb{N}$, for every $u \in \mathcal{C}^\infty(M^2, \mathbb{R})$ supported in U and vanishing to order p along Δ_M, there exists $C_u > 0$ such that for every $f \in \mathcal{C}^2(M, \mathbb{R})$,*

$$\|Q_k(u)\| \leq C_u k^{-p/2}, \quad \|[P_k(f), Q_k(u)]\| \leq C_u k^{-p/2-1}\|f\|_2,$$

where $P_k(f) = f + \big(1/(\mathrm{i}k)\big)\nabla^k_{X_f} : \mathcal{C}^\infty(M, L^k) \to \mathcal{C}^\infty(M, L^k)$ is the Kostant–Souriau operator associated with f.

Before proving this result, let us state several lemmas.

Lemma 8.5.4. *Let $u \in \mathcal{C}^\infty(M^2,\mathbb{R})$ be compactly supported in U, and let $f \in \mathcal{C}^2(M,\mathbb{R})$. Let $g \in \mathcal{C}^2(M^2,\mathbb{R})$ be defined by the formula $g(x,y) = f(x) - f(y)$ as before, and define the vector field $Y_f = (X_f, X_f)$ on M^2. Then*

$$[P_k(f), Q_k(u)] = Q_k\Big(\big(g - \alpha_E(Y_f)\big)u\Big) + \frac{1}{\mathrm{i}k} Q_k\big(\mathcal{L}_{Y_f} u\big).$$

Proof. We start by writing

$$[P_k(f), Q_k(u)] = fQ_k(u) - Q_k(u)f + \frac{1}{\mathrm{i}k}\big(\nabla^k_{X_f} \circ Q_k(u) - Q_k(u) \circ \nabla^k_{X_f}\big).$$

The Schwartz kernel of $fQ_k(u) - Q_k(u)f$ is equal to $f(x)F_k(u)(x,y) - F_k(u)(x,y)f(y)$. By Lemma 6.4.3, the Schwartz kernel of $\nabla^k_{X_f} \circ Q_k(u)$ is equal to $\big(\nabla^k_{X_f} \boxtimes \mathrm{id}\big)F_k(u)$. By Lemma 6.4.4, the Schwartz kernel of $Q_k(u) \circ \nabla^k_{X_f}$ is equal to $-\big(\mathrm{id} \boxtimes \nabla^k_{X_f}\big)F_k(u)$ since $\mathrm{div}(X_f) = 0$. Therefore, the Schwartz kernel of $[P_k(f), Q_k(u)]$ is given by

$$\left(f \boxtimes \mathrm{id} - \mathrm{id} \boxtimes f + \frac{1}{\mathrm{i}k}\widetilde{\nabla}^k_{(X_f,X_f)}\right)F_k(u).$$

Remembering the definition of α_E, and since u has support in U, we have that

$$\widetilde{\nabla}^k_{(X_f,X_f)}(E^k u) = u\widetilde{\nabla}^k_{Y_f}E^k + (\mathcal{L}_{Y_f}u)E^k = \big(-\mathrm{i}k\alpha_E(Y_f)u + \mathcal{L}_{Y_f}u\big)E^k.$$

Consequently, the Schwartz kernel of $[P_k(f), Q_k(u)]$ is equal to

$$F_k\Big(\big(g - \alpha_E(Y_f)\big)u\Big) + \frac{1}{\mathrm{i}k}F_k\big(\mathcal{L}_{Y_f}u\big);$$

in other words, $[P_k(f), Q_k(u)] = Q_k\left(\big(g - \alpha_E(Y_f)\big)u\right) + \big(1/(\mathrm{i}k)\big)Q_k(\mathcal{L}_{Y_f}u)$. □

In order to prove Proposition 8.5.3, we will investigate the two terms in the right-hand side of the equality obtained in this lemma. The following result will help us dealing with the first term.

Lemma 8.5.5. *Let K be a compact subset of U. Then there exists $C > 0$ such that for every $f \in \mathcal{C}^2(M,\mathbb{R})$,*

$$|g - \alpha_E(Y_f)| \leq C\|f\|_2\varphi_E$$

on K, with $g(x,y) = f(x) - f(y)$ and $Y_f = (X_f, X_f)$ as above.

Proof. Assume first that $K \subset V^2$ where V is a trivialisation open set for M such that $V^2 \subset U$. Introduce some coordinates $x_1, \ldots, x_{2n}$ on V. By Taylor's formula and (8.7), there exist some functions $g_i \in \mathcal{C}^1(V,\mathbb{R})$, $1 \leq i \leq 2n$, such that for $x, y \in V$

$$g(x,y) = \sum_{i=1}^{2n} g_i(y)(y_i - x_i) + O\big(\varphi_E\big)\|f\|_2, \tag{8.8}$$

and the $O\big(\varphi_E\big)$ is uniform on K. Now, write

$$\alpha_E(x,y) = \sum_{j=1}^{2n} (\mu_j(x,y)\,\mathrm{d}x_j + \nu_j(x,y)\,\mathrm{d}y_j)$$

for some functions $\mu_j, \nu_j \in \mathcal{C}^\infty(V^2)$. Since α_E vanishes along Δ_M, so does μ_j. Therefore, by Taylor's formula, there exist some functions $\mu_{ji} \in \mathcal{C}^\infty(V)$, $1 \leq i \leq 2n$, such that

$$\mu_j(x,y) = \sum_{i=1}^{2n} \mu_{ji}(y)(y_i - x_i) + O\big(\varphi_E\big).$$

Similarly, there exist some functions $\nu_{ji} \in \mathcal{C}^\infty(V)$, $1 \leq i \leq 2n$, such that

$$\nu_j(x,y) = \sum_{i=1}^{2n} \nu_{ji}(y)(y_i - x_i) + O\big(\varphi_E\big).$$

Consequently, we have that

$$\alpha_E(x,y) = \sum_{i=1}^{2n} \left(\sum_{j=1}^{2n} \mu_{ji}(y)\,\mathrm{d}x_j + \nu_{ji}(y)\,\mathrm{d}y_j \right)(y_i - x_i) + O\big(\varphi_E\big) \sum_{j=1}^{2n} (\,\mathrm{d}x_j + \,\mathrm{d}y_j).$$

Now, by Taylor's formula, $\mathrm{d}x_j(X_f)(x) = \mathrm{d}x_j(X_f)(y) + O\big(\varphi_E^{1/2}\big)\|f\|_2$. Thus, the previous formula implies that

$$\alpha_E(Y_f)(x,y) = \sum_{i=1}^{2n} \kappa_i(y)(y_i - x_i) + O\big(\varphi_E\big)\|f\|_2 \tag{8.9}$$

for some smooth functions κ_i, and the $O\big(\varphi_E\big)$ is uniform on K. Since, by Lemma 8.5.1, the function $g - \alpha_E(Y_f)$ vanishes to second order along Δ_M, it follows from (8.8) and (8.9) that $g_i - \kappa_i = 0$ for every $i \in [\![1, 2n]\!]$. Therefore

$$g - \alpha_E(Y_f) = O\big(\varphi_E\big)\|f\|_2$$

uniformly on K.

To handle the general case, we use the same partition of unity argument that we have used at the end of the proof of Lemma 8.5.2. □

Finally, the following lemma will take care of the second term in the equality displayed in Lemma 8.5.4.

Lemma 8.5.6. *Let $u \in \mathcal{C}^\infty(M^2, \mathbb{R})$ be a function vanishing to order p along Δ_M. Then there exists $C > 0$ such that for any vector field X of M^2 of class $\mathcal{C}^1$ and tangent to Δ_M, we have that*

$$|\mathcal{L}_X u| \leq C\|X\|_1 \varphi_E^{p/2}.$$

Proof. We start by proving the lemma for vector fields which are compactly supported in V^2, where V is a trivialisation open set of M, endowed with coordinates $x_1, \dots, x_{2n}$. Write

$$\mathrm{d}u = \sum_{i=1}^{2n} \left(\frac{\partial u}{\partial x_i} \,\mathrm{d}x_i + \frac{\partial u}{\partial y_i} \,\mathrm{d}y_i \right) = \sum_{i=1}^{2n} \left(\frac{\partial u}{\partial y_i} (dy_i - \,\mathrm{d}x_i) + \left(\frac{\partial u}{\partial x_i} + \frac{\partial u}{\partial y_i} \right) \mathrm{d}x_i \right).$$

Since u vanishes to order p along Δ_M and the vector field $\partial_{x_i} + \partial_{y_i}$ is tangent to Δ_M, the function $\partial u/\partial x_i + \partial u/\partial y_i$ vanishes to order p along Δ_M, so by Taylor's formula, it is a $O\big(\varphi_E^{p/2}\big)$. Moreover, there exists $C_1 > 0$ such that for any $\mathcal{C}^1$ vector field X compactly supported in V^2, $|\,\mathrm{d}x_i(X)| \leq C_1 \|X\|_0$. Furthermore, $\partial u/\partial y_i$ vanishes to order $p-1$ along Δ_M, so it is a $O\big(\varphi_E^{(p-1)/2}\big)$. We claim that there exists $C_2 > 0$ such that for any $\mathcal{C}^1$ vector field X compactly supported in V^2 and tangent to Δ_M,

$$|(\,\mathrm{d}y_i - \,\mathrm{d}x_i)(X)| \leq C_2 \|X\|_1 \varphi_E^{1/2}.$$

Indeed, take any such vector field X and write it as

$$X = \sum_{i=1}^{2n} \alpha_i(x,y) \partial_{x_i} + \beta_i(x,y) \partial_{y_i},$$

where $\alpha_i(x,x) = \beta_i(x,x)$ since X is tangent to Δ_M. Now

$$(\,\mathrm{d}y_i - \,\mathrm{d}x_i)(X) = \beta_i(x,y) - \alpha_i(x,y) = \int_0^1 \mathrm{d}(\beta_i - \alpha_i)\big((1-t)(x,x) + t(x,y)\big) v \,\mathrm{d}t$$

with $v = (0, y-x)$, by Taylor's formula. This last term is smaller than a constant not depending on X times $\|X\|_1 \varphi_E^{1/2}$.

Combining all of the above estimates, we obtain the result for vector fields which are compactly supported in V^2. We prove the general case by using a partition of unity argument. □

Let us now show how to apply all of the above.

Proof of Proposition 8.5.3. Let K denote the support of u. Since u vanishes to order p along the diagonal, it follows from Taylor's formula, (8.7) and a partition of unity argument that $|u|_{K,p}$ is finite. Consequently, the first estimate follows from Lemma 8.5.2.

To prove the second estimate, recall that it follows from Lemma 8.5.4 that

$$[P_k(f), Q_k(u)] = Q_k\Big(\big(g - \alpha_E(Y_f)\big)u\Big) + \frac{1}{\mathrm{i}k} Q_k\big(\mathcal{L}_{Y_f} u\big).$$

It follows from Lemma 8.5.5 that $|g - \alpha_E(Y_f)| \leq C\|f\|_2 \varphi_E$ for some constant $C > 0$ not depending on f. Moreover, since u vanishes to order p along Δ_M, u is a $O\big(\varphi_E^{p/2}\big)$. Thus, $\big(g - \alpha_E(Y_f)\big)u = O\big(\varphi_E^{(p+2)/2}\big)$, and by Lemma 8.5.2,

$$\left\| Q_k\Big(\big(g - \alpha_E(Y_f)\big)u \Big) \right\| = O\big(k^{-p/2-1}\big)\|f\|_2.$$

Similarly, it follows from Lemma 8.5.6 that $|\mathcal{L}_{Y_f} u| \le C'\|f\|_2 \varphi_E^{p/2}$ for some $C' > 0$ not depending on f. Therefore, Lemma 8.5.2 yields

$$\|Q_k\big(\mathcal{L}_{Y_f} u\big)\| = O\big(k^{-p/2}\big)\|f\|_2,$$

and the result follows. □

We are now ready to prove Theorem 8.3.1. Write as in Theorem 7.2.1

$$\Pi_k(x,y) = \left(\frac{k}{2\pi}\right)^n E^k(x,y)u(x,y,k) + R_k(x,y),$$

and let $u \sim \sum_{\ell \le 0} k^{-\ell} u_\ell$ be the asymptotic expansion of $u(\,\cdot\,,\cdot\,,k)$. Choose a compactly supported function $\chi \in \mathcal{C}^\infty(M^2,\mathbb{R})$ such that $\operatorname{supp}(\chi) \subset U$ and equal to one near Δ_M. Fixing $m \in \mathbb{N}$, we write

$$\Pi_k = \sum_{\ell=0}^m k^{-\ell} Q_k(\chi u_\ell) + \sum_{\ell=0}^m k^{-\ell} Q_k\big((1-\chi)u_\ell\big) + Q_k\left(u - \sum_{\ell=0}^m k^{-\ell} u_\ell\right) + R_k,$$

where R_k is the operator with Schwartz kernel $R_k(\cdot,\cdot)$. We only need to estimate the commutator of each of these terms with $P_k(f)$. Since χu_ℓ is compactly supported in U, it follows from Proposition 8.5.3 that $[P_k(f), Q_k(\chi u_\ell)] = O\big(k^{-1}\big)\|f\|_2$, so

$$\left[P_k(f), \sum_{\ell=0}^m k^{-\ell} Q_k(\chi u_\ell)\right] = O\big(k^{-1}\big)\|f\|_2.$$

For the second term, we use the following fact. Let V be a neighbourhood of Δ_M, and let $r = \sup_{M^2 \setminus V}\|E\| < 1$; then for any $v \in \mathcal{C}^0(M^2)$ vanishing in V, we have that

$$\|F_k(v)\| \le C k^n r^k \|v\|_0$$

for some $C > 0$ not depending on v. Therefore this Schwartz kernel is a $O\big(k^{-\infty}\big)\|v\|_0$ uniformly on M^2, and by Proposition 6.4.1, $Q_k(v) = O\big(k^{-\infty}\big)\|v\|_0$. Since $1-\chi$ vanishes in a neighbourhood of Δ_M, combining this fact with the equality

$$[P_k(f), Q_k\big((1-\chi)u_\ell\big)] = Q_k\big((1-\chi)\big(g - \alpha_E(Y_f)\big)u_\ell\big) + \frac{1}{\mathrm{i}k} Q_k\Big(\mathcal{L}_{Y_f}\big((1-\chi)u_\ell\big)\Big),$$

coming from Lemma 8.5.4, we obtain that

$$\left[P_k(f), \sum_{\ell=0}^m k^{-\ell} Q_k\big((1-\chi)u_\ell\big)\right] = O\big(k^{-1}\big)\|f\|_2.$$

It only remains to estimate the commutator $[P_k(f), S_k]$ where

$$S_k = Q_k\left(u - \sum_{\ell=0}^{m} k^{-\ell} u_\ell\right) + R_k.$$

The Schwartz kernel $S_k(\,\cdot\,,\cdot\,)$ of S_k is a $O\big(k^{n-(m+1)}\big)$. We conclude the proof by taking m large enough and using the following lemma.

Lemma 8.5.7. *There exists $C > 0$ such that for every $f \in \mathcal{C}^2(M, \mathbb{R})$,*

$$\|[P_k(f), S_k]\| \leq C k^{n-(m+1)} \|f\|_2.$$

Proof. By computing $\widetilde{\nabla}^k\big(F_k(u - \sum_{\ell=0}^{m} k^{-\ell} u_\ell)\big)$, we obtain that for every vector field X on M^2 of class $\mathcal{C}^0$, there exists $C_X > 0$ such that $\|\widetilde{\nabla}^k_X S_k\| \leq C_X k^{n-m}$. This implies that there exists $C > 0$ such that for every vector field X on M^2 of class $\mathcal{C}^0$, the inequality $\|\widetilde{\nabla}^k_X S_k\| \leq C k^{n-m} \|X\|_0$ holds. Indeed, let $(\eta_i)_{1 \leq i \leq q}$ be a partition of unity subordinate to an open cover $(U_i)_{1 \leq i \leq q}$ of M^2 by trivialisation open sets for TM^2, with a local basis $(Y_{ij})_{1 \leq j \leq 4n}$, and write

$$X = \sum_{i=1}^{q} \eta_i X = \sum_{i=1}^{q} \sum_{j=1}^{4n} \lambda_{ij} Y_{ij},$$

where λ_{ij} is a continuous function, which satisfies $\|\lambda_{ij}\|_0 \leq C' \|X\|_0$ for some $C' > 0$. Consequently,

$$\|\widetilde{\nabla}^k_X S_k\| = \left\| \sum_{i=1}^{q} \sum_{j=1}^{4n} \lambda_{ij} \widetilde{\nabla}^k_{Y_{ij}} S_k \right\| \leq C' (\max_{i,j} C_{Y_{ij}}) \|X\|_0.$$

To finish the proof, we obtain as in the proof of Lemma 8.5.4 that the Schwartz kernel of $[P_k(f), S_k]$ is equal to

$$\left(f \boxtimes \mathrm{id} - \mathrm{id} \boxtimes f + \frac{1}{\mathrm{i}k} \widetilde{\nabla}^k_{(X_f, X_f)} \right) S_k.$$

By the above estimate, $\|\widetilde{\nabla}^k_{(X_f,X_f)} S_k\| \leq C k^{n-m} \|f\|_1$, and the result follows. $\square$