Chapter 2
A Short Introduction to Kähler Manifolds

In this chapter, we recall some general facts about complex and Kähler manifolds. It is not an exhaustive list of such facts, but rather an introduction of objects and properties that we will need in the rest of the notes. The interested reader might want to take a look at some standard textbooks, such as [24, 35] for instance.

Let M be a smooth manifold (M will always be paracompact). The tangent (respectively cotangent) space at a point $m \in M$ will be denoted by $T_m M$ (respectively $T_m^* M$); the tangent (respectively cotangent) bundle will be denoted by TM (respectively T^*M). A vector field is a smooth section of the tangent bundle; the notation $\mathcal{C}^\infty(M, TM)$ will stand for the set of vector fields. Similarly, a differential form of degree p is a section of the exterior bundle $\Lambda^p(T^*M)$; we will use the notation $\Omega^p(M)$ for the set of degree p differential forms. We will write $i_X \alpha$ for the interior product of a vector field X with a differential form α.

2.1 Almost Complex Structures

Definition 2.1.1. An *almost complex structure* on M is a smooth field j of endomorphisms of the tangent bundle of M whose square is minus the identity:

$$\forall m \in M, \quad j_m^2 = -\operatorname{Id}_{T_m M}.$$

If such a structure exists, we say that (M, j) is an *almost complex manifold.*

By taking the determinant, we notice that if M is endowed with an almost complex structure, then its dimension is necessarily even. In what follows, we will denote this dimension by $2n$ with $n \geq 1$.

Example 2.1.2. Consider $M = \mathbb{R}^2$ with its standard basis, and let j be the endomorphism of $\mathbb{R}^2$ whose matrix in this basis is

$$J = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}.$$

Y. Le Floch, *A Brief Introduction to Berezin–Toeplitz Operators on Compact Kähler Manifolds*, CRM Short Courses,
https://doi.org/10.1007/978-3-319-94682-5_2

Then j is an almost complex structure on M; it corresponds to multiplication by i on $\mathbb{R}^2 \simeq \mathbb{C}$, $(x, y) \to x + \mathrm{i}y$. More generally, the endomorphism of $\mathbb{R}^{2n}$ whose matrix in the standard basis is block diagonal with blocks as above is an almost complex structure on $\mathbb{R}^{2n}$.

This example is a particular case of a more general fact: if M is a complex manifold, i.e. a manifold modelled on $\mathbb{C}^n$ with holomorphic transition functions, then it has an almost complex structure. Indeed, let U be a trivialisation open set, and let $z_1, \ldots, z_n$ be holomorphic coordinates on U. For $\ell \in [\![1, n]\!]$, we define the functions $x_\ell = \Re(z_\ell)$ and $y_\ell = \Im(z_\ell)$. Then $(x_1, y_1, \ldots, x_n, y_n)$ are real coordinates on M, and

$$\forall \ell \in [\![1, n]\!], \qquad j\partial_{x_\ell} = \partial_{y_\ell}, \quad j\partial_{y_\ell} = -\partial_{x_\ell}$$

defines an almost complex structure j on M; it does not depend on the choice of local coordinates because the differentials of the transition functions are $\mathbb{C}$-linear isomorphisms, which means that they commute with this local j.

The converse is not true in general: an almost complex structure does not necessarily come from a structure of complex manifold. When it occurs, the almost complex structure is said to be *integrable*. We will state some integrability criterion later.

2.2 The Complexified Tangent Bundle

Given an almost complex manifold (M, j), we would like to diagonalise j; since it obviously has no real eigenvalue, we introduce the complexified tangent bundle $TM \otimes \mathbb{C}$ of M. We extend all endomorphisms of TM to its complexification by $\mathbb{C}$-linearity. Then we can decompose the complexified tangent bundle as the direct sum of the eigenspaces of j.

Lemma 2.2.1. *The complexified tangent bundle can be written as the direct sum*

$$TM \otimes \mathbb{C} = T^{1,0}M \oplus T^{0,1}M$$

where

$$T^{1,0}M \coloneqq \ker(j - \mathrm{i}\,\mathrm{Id}) = \{X - \mathrm{i}jX \mid X \in TM\}$$

and

$$T^{0,1}M \coloneqq \ker(j + \mathrm{i}\,\mathrm{Id}) = \{X + \mathrm{i}jX \mid X \in TM\} = \overline{T^{1,0}M}.$$

We will denote by $Y^{1,0}$ (respectively $Y^{0,1}$) the component in $T^{1,0}M$ (respectively $T^{0,1}M$) of an element Y of the complexified tangent bundle in this decomposition. We have that

$$Y^{1,0} = \frac{Y - \mathrm{i}jY}{2}, \quad Y^{0,1} = \frac{Y + \mathrm{i}jY}{2}$$

for such a Y.

Proof. Since $j^2 = -\,\mathrm{Id}$, j is diagonalisable over $\mathbb{C}$, with eigenvalues $\pm\mathrm{i}$:

$$TM \otimes \mathbb{C} = \ker(j - \mathrm{i}\,\mathrm{Id}) \oplus \ker(j + \mathrm{i}\,\mathrm{Id}).$$

Since these two eigenspaces correspond to complex conjugate eigenvalues, they are complex conjugate. Thus, it only remains to show that

$$\ker(j - \mathrm{i}\,\mathrm{Id}) = \{X - \mathrm{i}jX \mid X \in TM\}.$$

A simple computation shows that if $Y = X - \mathrm{i}jX$ with $X \in TM$, then $jY = \mathrm{i}Y$. Conversely, let $Z \in \ker(j - \mathrm{i}\,\mathrm{Id})$, and let us write $Z = X + \mathrm{i}Y$ with $X, Y \in TM$. From the equality

$$jX + \mathrm{i}jY = \mathrm{i}X - Y,$$

it follows, by identification of the real parts, that $Y = -jX$. □

Let us assume that M is a complex manifold and that j is the associated complex structure introduced in the previous section. We consider some local complex coordinates $(z_1 = x_1 + \mathrm{i}y_1, \dots, z_n = x_n + \mathrm{i}y_n)$, and define for $\ell \in [\![1, n]\!]$

$$\partial_{z_\ell} = \tfrac{1}{2}(\partial_{x_\ell} - \mathrm{i}\partial_{y_\ell}), \quad \partial_{\bar{z}_\ell} = \tfrac{1}{2}(\partial_{x_\ell} + \mathrm{i}\partial_{y_\ell});$$

then $(\partial_{z_\ell})_{1\leq\ell\leq n}$ and $(\partial_{\bar{z}_\ell})_{1\leq\ell\leq n}$ are local bases of $T^{1,0}M$ and $T^{0,1}M$, respectively.

The following statement gives a necessary and sufficient condition for an almost complex structure to induce a genuine complex structure. Let us recall that a distribution $E \subset TM$ is integrable if and only if for any two vector fields $X, Y \in E$, the Lie bracket $[X, Y]$ belongs to E (this is actually equivalent to the usual definition as a consequence of the Frobenius integrability theorem, but we take it as a definition to simplify).

Theorem 2.2.2 (The Newlander–Nirenberg theorem). *Let (M, j) be an almost complex manifold. Then j comes from a complex structure if and only if the distribution $T^{1,0}M$ is integrable.*

A proof of this standard but rather involved result can be found in [25, Section 5.7] for instance.

2.3 Decomposition of Forms

By duality, the decomposition $TM \otimes \mathbb{C} = T^{1,0}M \oplus T^{0,1}M$ induces a decomposition of the complexified cotangent bundle:

$$T^*M \otimes \mathbb{C} = (T^*M)^{1,0} \oplus (T^*M)^{0,1}$$

where

$$(T^*M)^{1,0} = \{\alpha \in T^*M \mid \forall X \in T^{0,1}M, \alpha(X) = 0\},$$

and $(T^*M)^{0,1}$ is defined in the same way, replacing $T^{0,1}M$ by $T^{1,0}M$. Similarly to Lemma 2.2.1, we have the following description.

Lemma 2.3.1. *We have that*

$$(T^*M)^{1,0} = \{\alpha - \mathrm{i}\alpha \circ j \mid \alpha \in T^*M\}, \quad (T^*M)^{0,1} = \overline{(T^*M)^{1,0}}.$$

It is well-known that the exterior algebra of a direct sum of two vector spaces is isomorphic to the tensor product of both exterior algebras of the vector spaces, and that this isomorphism respects the grading. Consequently, we have that

$$\Lambda^k(T^*M) \otimes \mathbb{C} \simeq \bigoplus_{\ell=0}^{k} (\Lambda^{\ell,0}M \otimes \Lambda^{0,k-\ell}M)$$

with $\Lambda^{p,0}M := \Lambda^p\big((T^*M)^{1,0}\big)$ and $\Lambda^{0,q}M := \Lambda^q\big((T^*M)^{0,1}\big)$. This can be written as

$$\Lambda^k(T^*M) \otimes \mathbb{C} \simeq \bigoplus_{\substack{p,q\in\mathbb{N}\\ p+q=k}} \Lambda^{p,q}M$$

with $\Lambda^{p,q}M := \Lambda^{p,0}M \otimes \Lambda^{0,q}M$. Therefore, this induces a decomposition of the space of k-forms:

$$\Omega^k(M) \otimes \mathbb{C} = \bigoplus_{\substack{p,q\in\mathbb{N}\\ p+q=k}} \Omega^{p,q}(M)$$

where $\Omega^{p,q}(M)$ is the space of smooth sections of $\Lambda^{p,q}M$. An element of $\Omega^{p,q}(M)$ will be called a (p,q)-form. These forms can be characterised in the following way.

Lemma 2.3.2. *A k-form α belongs to $\Omega^{k,0}(M)$ if and only if for every vector field $X \in \mathcal{C}^\infty(M, T^{0,1}M)$, $i_X\alpha = 0$. More generally, a k-form α belongs to $\Omega^{p,q}(M)$ with $p+q=k$, $q \neq k$, if and only if for any $q+1$ vector fields $X_1, \dots, X_{q+1} \in \mathcal{C}^\infty(M, T^{0,1}M)$, $i_{X_1} \dots i_{X_{q+1}}\alpha = 0$.*

By applying complex conjugation, we deduce from this result that a k-form α belongs to $\Omega^{p,q}(M)$ with $p+q=k$, $p \neq k$, if and only if for any $p+1$ vector fields $Y_1, \dots, Y_{p+1} \in \mathcal{C}^\infty(M, T^{1,0}M)$, $i_{X_1} \dots i_{X_{p+1}}\alpha = 0$.

Proof. Let $\alpha \in \Omega^{k,0}(M)$. We can write α locally as a sum of terms of the form

$$\alpha_\ell = \beta_1 \wedge \cdots \wedge \beta_k$$

with $\beta_1, \dots, \beta_k \in \Omega^{1,0}(M)$. If $X \in \mathcal{C}^\infty(M, T^{0,1}M)$, by using the formula

$$i_X(\gamma \wedge \delta) = (i_X\gamma) \wedge \delta + (-1)^{\deg\gamma}\gamma \wedge (i_X\delta)$$

for differential forms γ, δ, and the fact that $\beta_j(X) = 0$, we obtain that $i_X\alpha = 0$. Conversely, let $\alpha \in \Omega^k(M) \otimes \mathbb{C}$ whose interior product with every $X \in \mathcal{C}^\infty(M, T^{0,1}M)$ vanishes. We write as

$$\alpha = \alpha^{(k,0)} + \alpha^{(k-1,1)} + \cdots + \alpha^{(0,k)}$$

the decomposition of α in the direct sum $\Omega^k(M) = \Omega^{k,0}(M) \oplus \cdots \oplus \Omega^{0,k}(M)$. For $X \in \mathcal{C}^\infty(M, T^{0,1}M)$, one has

$$0 = i_X\alpha = i_X\alpha^{(k-1,1)} + \cdots + i_X\alpha^{(0,k)}$$

since $i_X\alpha^{(k,0)} = 0$ by the first part of the proof. It is easy to check that $i_X\alpha^{(k-p,p)}$ belongs to $\Omega^{k-p,p-1}(M)$ for $1 \le p \le k$. Therefore, the previous equality yields that $i_X\alpha^{(k-p,p)} = 0$ for every $p \in [\![1,k]\!]$. Now, we take a local basis $\beta_1, \dots \beta_n$ of $(T^*M)^{1,0}$ and write

$$\alpha^{(k-p,p)} = \sum_{\substack{L=\{\ell_1,\dots,\ell_{k-p}\}\subset[\![1,n]\!]\\ \ell_1<\cdots<\ell_{k-p}}} \sum_{\substack{M=\{m_1,\dots,m_p\}\subset[\![1,n]\!]\\ m_1<\cdots<m_p}} f_{L,M}\beta_{\ell_1}\wedge\cdots\wedge\beta_{\ell_{k-p}} \wedge \bar\beta_{m_1}\wedge\cdots\wedge\bar\beta_{m_p}$$

for some smooth functions $f_{L,M}$. Then

$$i_X\alpha^{(k-p,p)} = \sum_L\sum_M\sum_{r=1}^{p} \pm f_{L,M}\bar\beta_{\ell_r}(X)\beta_{\ell_1}\wedge\cdots\wedge\beta_{\ell_{k-p}} \wedge\bar\beta_{m_1}\wedge\cdots\wedge\bar\beta_{m_{r-1}}\wedge\bar\beta_{m_{r+1}}\wedge\cdots\wedge\bar\beta_{m_p},$$

thus $f_{L,M}\bar\beta_{m_r}(X) = 0$ for every L, M, m_r and every $X \in \mathcal{C}^\infty(M, T^{0,1}M)$, which finally yields $\alpha^{(k-p,p)} = 0$. Therefore $\alpha = \alpha^{(k,0)}$ belongs to $\Omega^{k,0}(M)$.

The second statement can be proved by induction on q (the first statement is the $q = 0$ case). □

We would like to understand the action of the exterior derivative (extended by $\mathbb{C}$-linearity) with respect to this decomposition. It turns out that it behaves "nicely" if and only if j is induced by a structure of complex manifold on M. Before explaining this, let us introduce one more object.

Definition 2.3.3. The *Nijenhuis tensor* N_j of j is defined as follows: for any vector fields $X, Y \in \mathcal{C}^\infty(M, TM)$,

$$N_j(X,Y) = [X,Y] + j[jX,Y] + j[X,jY] - [jX,jY].$$

This tensor allows one to express the integrability condition in the Newlander–Nirenberg theorem in a more algebraic way.

Proposition 2.3.4. *Let (M, j) be an almost complex manifold. The following assertions are equivalent*:

(1) *j comes from a complex structure*,
(2) $\mathrm{d}\big(\Omega^{1,0}(M)\big) \subset \Omega^{2,0}(M) \oplus \Omega^{1,1}(M)$,
(3) $\forall p, q \in \mathbb{N},\ \mathrm{d}\big(\Omega^{p,q}(M)\big) \subset \Omega^{p+1,q}(M) \oplus \Omega^{p,q+1}(M)$,
(4) $N_j = 0$.

Proof. (1) $\Leftrightarrow$ (4): the Newlander–Nirenberg theorem states that j comes from a complex structure if and only if $[\mathcal{C}^\infty(M,T^{1,0}M),\mathcal{C}^\infty(M,T^{1,0}M)] \subset \mathcal{C}^\infty(M,T^{1,0}M)$. So let $X,Y \in \mathcal{C}^\infty(M,TM)$; a straightforward computation yields

$$[X - \mathrm{i}jX, Y - \mathrm{i}jY] = [X,Y] - [jX,jY] - \mathrm{i}[X,jY] - \mathrm{i}[jX,Y],$$

which implies that

$$[X - \mathrm{i}jX, Y - \mathrm{i}jY]^{0,1} = \tfrac{1}{2}\big(N_j(X,Y) + \mathrm{i}jN_j(X,Y)\big).$$

Therefore, $[X-\mathrm{i}jX, Y-\mathrm{i}jY]$ belongs to $\mathcal{C}^\infty(M,T^{1,0}M)$ if and only if $N_j(X,Y)=0$.

(1) $\Leftrightarrow$ (2): let $\alpha \in \Omega^{1,0}(M)$, and let $X,Y \in \mathcal{C}^\infty(M,T^{0,1}M)$. Then

$$\mathrm{d}\alpha(X,Y) = \mathcal{L}_X\big(\alpha(Y)\big) - \mathcal{L}_Y\big(\alpha(X)\big) - \alpha([X,Y]) = -\alpha([X,Y])$$

because $\alpha(X) = 0 = \alpha(Y)$ by definition of $\Omega^{1,0}(M)$ (here $\mathcal{L}_X$ stands for the Lie derivative with respect to X). Therefore, $\mathrm{d}\alpha(X,Y) = 0$ if and only if $[X,Y]^{1,0} \in \ker\alpha$. This means that $\mathrm{d}\big(\Omega^{1,0}(M)\big) \subset \Omega^{2,0}(M) \oplus \Omega^{1,1}(M)$ if and only if for any $X,Y \in \mathcal{C}^\infty(M,T^{0,1}M)$, $[X,Y]^{1,0}=0$, i.e. $[X,Y]$ belongs to $\mathcal{C}^\infty(M,T^{0,1}M)$.

(2) $\Leftrightarrow$ (3): the implication (3) $\Rightarrow$ (2) is obvious. Thus, let us assume that statement (2) holds. By complex conjugation, this implies that

$$\mathrm{d}\big(\Omega^{0,1}(M)\big) \subset \Omega^{1,1}(M) \oplus \Omega^{0,2}(M)$$

as well. Let $\gamma \in \Omega^{p,q}(M)$; we can write locally γ as a sum of elements $\tilde\gamma$ of the form

$$\tilde\gamma = \alpha_1 \wedge \cdots \wedge \alpha_p \wedge \beta_1 \wedge \cdots \wedge \beta_q$$

with $\alpha_1,\ldots,\alpha_p \in \Omega^{1,0}(M)$, $\beta_1,\ldots,\beta_q \in \Omega^{0,1}(M)$. Then by the Leibniz rule for forms

$$\mathrm{d}\tilde\gamma = \mathrm{d}\alpha_1 \wedge \hat\gamma_{\alpha_1} + \cdots + \mathrm{d}\alpha_p \wedge \hat\gamma_{\alpha_p} + \mathrm{d}\beta_1 \wedge \hat\gamma_{\beta_1} + \cdots + \mathrm{d}\beta_q \wedge \hat\gamma_{\beta_q}.$$

where $\hat\gamma_{\alpha_j} = \alpha_1 \wedge \cdots \wedge \alpha_{j-1} \wedge \alpha_{j+1} \wedge \cdots \wedge \alpha_p \wedge \beta_1 \wedge \cdots \wedge \beta_q$ and $\hat\gamma_{\beta_j}$ is defined in the same way. In particular, $\hat\gamma_{\alpha_j}$ belongs to $\Omega^{p-1,q}(M)$. Moreover, since α_j is a $(1,0)$-form, $\mathrm{d}\alpha_j$ belongs to $\Omega^{2,0}(M) \oplus \Omega^{1,1}(M)$; therefore $\mathrm{d}\alpha_j \wedge \hat\gamma_{\alpha_j}$ belongs to $\Omega^{p+1,q}(M) \oplus \Omega^{p,q+1}(M)$. It follows from a similar reasoning that $\mathrm{d}\beta_k \wedge \hat\gamma_{\beta_k}$ also belongs to this direct sum. Consequently, $\mathrm{d}\tilde\gamma$ belongs to $\Omega^{p+1,q}(M) \oplus \Omega^{p,q+1}(M)$, and so does $\mathrm{d}\gamma$. □

Observe that, as a consequence of this result, an almost complex structure on a surface always comes from a complex structure. Indeed, if M is a surface, then $\Omega^{2,0}(M) = \Omega^{0,2}(M) = \{0\}$, and therefore the exterior derivative of a $(1,0)$-form always lies in $\Omega^{1,1}(M)$.

2.4 Complex Manifolds

Let us now assume that M is a complex manifold and that j is the induced almost complex structure. Let $(z_k = x_k + \mathrm{i}y_k)_{1 \leq k \leq n}$ be some local complex coordinates on an open subset $U \subset M$. We get complex-valued forms

$$\mathrm{d}z_k = \mathrm{d}x_k + \mathrm{i}\mathrm{d}y_k \in \Omega^{1,0}(U), \quad \mathrm{d}\bar{z}_k = \mathrm{d}x_k - \mathrm{i}\mathrm{d}y_k \in \Omega^{0,1}(U)$$

which form local bases $(\mathrm{d}z_1, \ldots, \mathrm{d}z_n)$, $(\mathrm{d}\bar{z}_1, \ldots, \mathrm{d}\bar{z}_n)$ of $\Omega^{1,0}(M)$ and $\Omega^{0,1}(M)$, respectively; $(\mathrm{d}z_1, \ldots, \mathrm{d}z_n, \mathrm{d}\bar{z}_1, \ldots, \mathrm{d}\bar{z}_n)$ is a local basis of $\Omega^1(M) \otimes \mathbb{C}$ which is dual to the local basis $(\partial_{z_1}, \ldots, \partial_{z_n}, \partial_{\bar{z}_1}, \ldots, \partial_{\bar{z}_n})$ introduced above. Therefore, a local basis of $\Omega^{p,q}(M)$ is given by the family

$$(\mathrm{d}z_{k_1} \wedge \cdots \wedge \mathrm{d}z_{k_p} \wedge \mathrm{d}\bar{z}_{\ell_1} \wedge \cdots \wedge \mathrm{d}\bar{z}_{\ell_q})_{1 \leq k_1 < \cdots < k_p, \ell_1 < \cdots < \ell_q \leq n}.$$

This immediately provides one with another proof of the fact that, in this case, the image of $\Omega^{p,q}(M)$ by the exterior derivative is included in $\Omega^{p+1,q}(M) \oplus \Omega^{p,q+1}(M)$. Because of this fact, we can write $\mathrm{d} = \partial + \bar{\partial}$ where

$$\partial\colon \Omega^{p,q}(M) \to \Omega^{p+1,q}(M), \quad \bar{\partial}\colon \Omega^{p,q}(M) \to \Omega^{p,q+1}(M).$$

The operators $\partial, \bar{\partial}$ satisfy the Leibniz rule

$$\partial(\alpha \wedge \beta) = \partial\alpha \wedge \beta + (-1)^{\deg(\alpha)}\alpha \wedge \partial\beta, \quad \bar{\partial}(\alpha \wedge \beta) = \bar{\partial}\alpha \wedge \beta + (-1)^{\deg(\alpha)}\alpha \wedge \bar{\partial}\beta,$$

which we can prove by writing the Leibniz rule for d and identifying the types.

Let $(U_k, \varphi_k)_{k \in I}$ be a holomorphic atlas of M. A function $f\colon M \to \mathbb{C}$ is called holomorphic if and only if for every $k \in I$, the function $f \circ \varphi_k^{-1}\colon \mathbb{C}^n \to \mathbb{C}$ is holomorphic.

Lemma 2.4.1. *Let $f\colon M \to \mathbb{C}$ be a smooth function. The following statements are equivalent*

(1) *f is holomorphic,*
(2) *for every $Z \in \mathcal{C}^\infty(M, T^{0,1}M)$, $\mathcal{L}_Z f = 0$,*
(3) *$\bar{\partial} f = 0$.*

Proof. The equivalence of the last two statements is clear because $\bar{\partial} f = 0$ is equivalent to the fact that $\mathrm{d}f$ belongs to $\Omega^{1,0}(M)$. Now, let $(z_1, \ldots, z_n)$ be the local complex coordinates defined by φ_k; then f is holomorphic if and only if

$$\forall \ell \in [\![1, n]\!], \quad \frac{\partial f}{\partial \bar{z}_\ell} = 0$$

in these coordinates. This is equivalent to saying that $\mathrm{d}f(\partial_{\bar{z}_\ell}) = 0$ for every ℓ; since $(\partial_{\bar{z}_\ell})_{1 \leq \ell \leq n}$ is a local basis of $T^{0,1}M$, this amounts to $\mathrm{d}f \in \Omega^{1,0}(M)$, which in turn is equivalent to $\bar{\partial} f = 0$. □

Lemma 2.4.2. *The following identities hold*:

$$\partial^2 = 0, \quad \partial\bar\partial + \bar\partial\partial = 0, \quad \bar\partial^2 = 0.$$

Proof. This follows from the equality

$$0 = \mathrm{d}^2 = \partial^2 + \partial\bar\partial + \bar\partial\partial + \bar\partial^2$$

and the fact that $\partial^2 \colon \Omega^{p,q}(M) \to \Omega^{p+2,q}(M)$, $\partial\bar\partial + \bar\partial\partial \colon \Omega^{p,q}(M) \to \Omega^{p+1,q+1}(M)$ and $\bar\partial^2 \colon \Omega^{p,q}(M) \to \Omega^{p,q+2}(M)$.

Following the standard terminology for the exterior derivative, we say that a complex-valued form α is $\bar\partial$-*closed* if $\bar\partial\alpha = 0$, and $\bar\partial$-*exact* if there exists a differential form β such that $\alpha = \bar\partial\beta$. The operator $\bar\partial$ defines a cohomology, called Dolbeault cohomology. The cohomology groups are the quotients of $\bar\partial$-closed forms by $\bar\partial$-exact forms:

$$H^{p,q}(M) = \ker\big(\bar\partial_{|\Omega^{p,q}(M)}\big)/\bar\partial\big(\Omega^{p,q-1}(M)\big).$$

The following result is an analogue of the Poincaré lemma for the exterior derivative.

Lemma 2.4.3 (Dolbeault–Grothendieck lemma, or $\bar\partial$-Poincaré lemma). *A $\bar\partial$-closed form is locally $\bar\partial$-exact.*

For a proof, we refer the reader to standard textbooks, for instance [24, Proposition 1.3.8]. This result can be used to prove the following property of the operator $\mathrm{i}\partial\bar\partial$, which will be useful later.

Lemma 2.4.4 (The i$\partial\bar\partial$-lemma). *Let $\alpha \in \Omega^{1,1}(M)$ be a differential form of type $(1,1)$. Then α is closed and real-valued (i.e., $\alpha \in \Omega^{1,1}(M) \cap \Omega^2(M)$) if and only if every point $m \in M$ has an open neighbourhood U such that $\alpha = \mathrm{i}\partial\bar\partial\phi$ over U for some $\phi \in \mathcal{C}^\infty(U, \mathbb{R})$.*

Proof. Assume that $\alpha = \mathrm{i}\partial\bar\partial\phi$ over some open subset $U \subset M$ for some $\phi \in \mathcal{C}^\infty(U, \mathbb{R})$. Then

$$\mathrm{d}\alpha = \mathrm{i}(\partial^2\bar\partial\phi + \bar\partial\partial\bar\partial\phi) = -\mathrm{i}\partial\bar\partial^2\phi = 0,$$

which proves that α is closed. Moreover,

$$\bar\alpha = -\mathrm{i}\bar\partial\partial\bar\phi = -\mathrm{i}\bar\partial\partial\phi = \mathrm{i}\partial\bar\partial\phi = \alpha,$$

thus α is real-valued.

Conversely, assume that α is closed and real-valued. From the usual Poincaré lemma, there exists locally a real-valued one-form β such that $\alpha = \mathrm{d}\beta$. From the equality

$$\alpha = \mathrm{d}\beta = \partial\beta^{(1,0)} + \bar\partial\beta^{(1,0)} + \partial\beta^{(0,1)} + \bar\partial\beta^{(0,1)},$$

we deduce that $\alpha = \bar\partial\beta^{(1,0)} + \partial\beta^{(0,1)}$ and $\bar\partial\beta^{(0,1)} = 0$. Thanks to the Dolbeault–Grothendieck lemma, we can find a local function f such that $\beta^{(0,1)} = \bar\partial f$. Since β is real-valued, the components of β satisfy

$$\beta^{(1,0)} = \overline{\beta^{(0,1)}} = \partial \bar{f}.$$

We finally obtain that

$$\alpha = \bar{\partial}\partial \bar{f} + \partial\bar{\partial} f = \partial\bar{\partial}(f - \bar{f}) = \mathrm{i}\partial\bar{\partial}\phi$$

with $\phi = 2\,\Im(f)$. □

2.5 Kähler Manifolds

Let (M, j) be an almost complex manifold.

Definition 2.5.1. A Riemannian metric g on M is said to be *compatible* with j if

$$g(jX, jY) = g(X, Y)$$

for every $X, Y \in TM$.

Every almost complex manifold can be equipped with a compatible Riemannian metric. Indeed, take any Riemannian metric g on M and define

$$h(X, Y) := g(X, Y) + g(jX, jY)$$

for every $X, Y \in TM$; then h is compatible with j. Given a compatible Riemannian metric g on (M, j), one defines its *fundamental form* as

$$\omega(X, Y) := g(jX, Y)$$

for every $X, Y \in TM$.

Lemma 2.5.2. *The fundamental form ω is a real $(1,1)$-form, i.e., it belongs to $\Omega^{1,1}(M) \cap \Omega^2(M)$.*

Proof. Firstly, we check that ω belongs to $\Omega^2(M)$:

$$\omega(Y, X) = g(jY, X) = g(j^2Y, jX) = -g(Y, jX) = -g(jX, Y) = -\omega(X, Y)$$

for $X, Y \in TM$. Secondly, to prove that ω is of type $(1,1)$, it is enough, by Lemma 2.3.2, to show that it vanishes when applied to a pair of elements of $T^{1,0}M$. Therefore, let $X, Y \in \mathcal{C}^\infty(M, TM)$; then

$$\omega(X - \mathrm{i}jX, Y - \mathrm{i}jY) = \omega(X, Y) - \mathrm{i}\omega(X, jY) - \mathrm{i}\omega(jX, Y) - \omega(jX, jY).$$

But on the one hand

$$\omega(jX, jY) = g(j^2X, jY) = -g(X, jY) = -g(jX, j^2Y) = g(jX, Y) = \omega(X, Y)$$

and on the other hand

$$\omega(jX, Y) = g(j^2X, Y) = -g(X, Y) = -g(jX, jY) = -\omega(X, jY).$$

Consequently, $\omega(X - \mathrm{i}jX, Y - \mathrm{i}jY) = 0$. □

To illustrate this, let us assume for a moment that M is a complex manifold and that j is the induced almost complex structure. We choose some local holomorphic coordinates $(z_1, \ldots, z_n)$, and define the function

$$h_{\ell,m} := g(\partial_{z_\ell}, \partial_{\bar{z}_m})$$

where g has been extended to $TM \otimes \mathbb{C}$ by $\mathbb{C}$-bilinearity (and not sesquilinearity!). One can check that

$$\omega = \mathrm{i} \sum_{\ell,m=1}^{n} h_{\ell,m} \, \mathrm{d}z_\ell \wedge \mathrm{d}\bar{z}_m$$

in these coordinates.

Let (M, j) be an almost complex manifold, and let g be a compatible Riemannian metric. Since j is an isomorphism and g is non-degenerate, it is clear that ω is non-degenerate. Hence, if it is closed, it is a symplectic form.

Definition 2.5.3. A compatible Riemannian metric on an almost complex manifold is called a *Kähler metric* if j is integrable and the fundamental form ω is closed. A *Kähler manifold* (M, j, g) is an almost complex manifold (M, j) endowed with a Kähler metric g.

In this case, the fundamental form is a symplectic form on M. By Lemma 2.4.4, near each point $p \in M$, there exists a real-valued smooth function ϕ such that

$$\omega = \mathrm{i}\partial\bar{\partial}\phi.$$

This function ϕ is called a *Kähler potential*. In local coordinates, this gives

$$h_{\ell,m} = \frac{\partial^2 \phi}{\partial z_\ell \partial \bar{z}_m}$$

which means that the metric is determined locally by the Kähler potential.

In what follows, we will be more interested in the symplectic point of view. So let us start with a symplectic manifold (M, ω).

Definition 2.5.4. An almost complex structure j on M is said to be *compatible* with ω if

$$\omega(jX, jY) = \omega(X, Y)$$

for any $X, Y \in TM$ and

$$\omega(X, jX) > 0$$

for every $X \neq 0 \in TM$.

One readily checks that, given a compatible almost complex structure j on (M,ω), the spaces $T^{1,0}M$ and $T^{0,1}M$ are Lagrangian (when we extend ω to $TM\otimes\mathbb{C}$ by $\mathbb{C}$-bilinearity).

Assume that M is a complex manifold endowed with the induced complex structure j, and that ω belongs to $\Omega^{1,1}(M)$. In local coordinates $(z_\ell)_{1\le\ell\le n}$, the symplectic form is of the form

$$\omega = \mathrm{i}\sum_{\ell,m=1}^{n} h_{\ell,m}\,\mathrm{d}z_\ell\wedge\,\mathrm{d}\bar{z}_m$$

for some smooth functions $h_{\ell,m}$, $1\le\ell,m\le n$. Then ω is compatible with j if and only if all the matrices $\big(h_{\ell,m}(p)\big)_{1\le\ell,m\le n}$, $p\in M$ coming from such local expressions are positive definite Hermitian matrices.

A symplectic manifold always has a compatible almost complex structure. Indeed, take any Riemannian metric g on M. By the Riesz representation theorem, we have two isomorphisms

$$\widetilde{\omega}\colon TM\to T^*M,\quad X\mapsto i_X\omega \quad\text{and}\quad \tilde{g}\colon TM\to T^*M,\quad X\mapsto g(X,\cdot).$$

Consider $a=\tilde{g}^{-1}\circ\widetilde{\omega}\colon TM\to TM$; it is an isomorphism, which is moreover anti-symmetric, in the sense that $a^*=-a$ (a^* is the adjoint of a with respect to g). Indeed,

$$g(aX,Y)=\omega(X,Y)=-\omega(Y,X)=-g(aY,X)=g(X,-aY)$$

for any $X,Y\in TM$. Let

$$a=j(a^*a)^{1/2}$$

be the polar decomposition of a; j is unitary (with respect to g).

Lemma 2.5.5. *j is an almost complex structure which is compatible with ω.*

Proof. On the one hand, since j is unitary, $j^*j=\mathrm{Id}_{TM}$. On the other hand, since $(a^*a)^{1/2}$ is an isomorphism commuting with a and a is anti-symmetric, we have

$$j^*=\big(a(a^*a)^{-1/2}\big)^*=(a^*a)^{-1/2}a^*=-(a^*a)^{-1/2}a=-a(a^*a)^{-1/2}=-j,$$

thus $j^2=-\mathrm{Id}_{TM}$. It remains to check the compatibility between j and ω. Firstly, we have that

$$\omega(jX,jY)=g(ajX,jY)=g(jaX,jY)=g(aX,j^*jY)=g(aX,Y)=\omega(X,Y)$$

for any $X,Y\in TM$. Secondly, for every $X\neq 0\in TM$,

$$\omega(X,jX)=g(aX,jX)=g(j^*aX,X)=g\big((a^*a)^{1/2}X,X\big)>0$$

because $(a^*a)^{1/2}$ is positive definite. □

Observe that, given a symplectic form ω and a compatible almost complex structure j, the formula

$$g(X,Y) := \omega(X, jY)$$

defines a Riemannian metric on M, which is compatible with j and whose fundamental form is equal to ω. The latter is closed by definition; therefore we obtain an equivalent definition of Kähler manifolds.

Proposition 2.5.6. *A symplectic manifold (M,ω) is a Kähler manifold if and only if there exists an almost complex structure j which is compatible with ω and integrable.*

Note that an orientable surface is always a Kähler manifold. Indeed, by the discussion above, it can be endowed with an almost complex structure compatible with the symplectic (volume) form. But as we noticed earlier, an almost complex structure on a surface is always integrable.

Example 2.5.7. On $\mathbb{C}$ with its standard complex structure, the standard symplectic form

$$\omega = \frac{\mathrm{i}}{2}\sum_{\ell=1}^{n} \mathrm{d}z_\ell \wedge \mathrm{d}\bar{z}_\ell$$

is the fundamental form of the Kähler metric given by the standard scalar product on $\mathbb{R}^{2n}$. There is a globally defined Kähler potential given by

$$\phi(z_1,\dots,z_n,\bar{z}_1,\dots,\bar{z}_n) = \frac{1}{2}\sum_{\ell=1}^{n}|z_\ell|^2.$$

Example 2.5.8 (*The unit disc*). On the open unit disc $\mathbb{D}^n \subset \mathbb{C}^n$ (still with standard complex structure), we consider the function

$$\phi(z_1,\dots,z_n,\bar{z}_1,\dots,\bar{z}_n) = -\tfrac{1}{2}\log(1-\|z\|^2),$$

where $\|z\|^2 = \langle z,z\rangle = \sum_{\ell=1}^{n}|z_\ell|^2$ is the square of the norm of z with respect to the standard Hermitian product on $\mathbb{C}^n$, and introduce the form $\omega = \mathrm{i}\partial\bar{\partial}\phi$. This is a closed real $(1,1)$-form; we will show that it is compatible with the complex structure. We compute

$$\bar{\partial}\phi = \frac{-\bar{\partial}\bigl(1-\sum_{\ell=1}^{n}|z_\ell|^2\bigr)}{2(1-\|z\|^2)} = \frac{\sum_{\ell=1}^{n} z_\ell\,\mathrm{d}\bar{z}_\ell}{2(1-\|z\|^2)},$$

which yields

$$\partial\bar{\partial}\phi = \frac{1}{2}\left(\frac{\bigl(\sum_{k=1}^{n}\bar{z}_k\,\mathrm{d}z_k\bigr)\wedge\bigl(\sum_{\ell=1}^{n} z_\ell\,\mathrm{d}\bar{z}_\ell\bigr)}{(1-\|z\|^2)^2} + \frac{\sum_{\ell=1}^{n}\mathrm{d}z_\ell\wedge\mathrm{d}\bar{z}_\ell}{1-\|z\|^2}\right).$$

We finally obtain that

$$\omega = \frac{\mathrm{i}}{2(1-\|z\|^2)^2}\sum_{k,\ell=1}^{n}\bigl(\bar{z}_k z_\ell + \bigl(1-\|z\|^2\bigr)\delta_{k,\ell}\bigr)\,\mathrm{d}z_k\wedge\mathrm{d}\bar{z}_\ell;$$

we claim that the matrix $H = \big(\bar{z}_k z_\ell + \big(1 - \|z\|^2\big)\delta_{k,\ell}\big)_{1\le k,\ell\le n}$ is Hermitian positive definite for every $z \in \mathbb{D}^n$, which means that ω is compatible with the complex structure. Indeed, for a nonzero u in $\mathbb{C}^n$, we have

$$\langle Hu, u\rangle = \langle u, z\rangle\langle z, u\rangle + \big(1 - \|z\|^2\big)\|u\|^2 = |\langle z, u\rangle|^2 + \big(1 - \|z\|^2\big)\|u\|^2 > 0$$

since $1 - \|z\|^2 > 0$.

Example 2.5.9 (*The Fubini–Study structure*). Let $M = \mathbb{CP}^n$ with its standard open covering $\mathbb{CP}^n \subset \bigcup_{k=0}^n U_k$ where $U_k = \{[z_0 : \cdots : z_n] \in \mathbb{CP}^n \mid z_k \neq 0\}$ and charts

$$\varphi_k \colon U_k \to \mathbb{C}^n, \quad [z_0 : \cdots : z_n] \mapsto (w_1, \ldots, w_n) = \left(\frac{z_0}{z_k}, \ldots, \frac{z_{k-1}}{z_k}, \frac{z_{k+1}}{z_k}, \ldots, \frac{z_n}{z_k}\right).$$

On each U_k we can define a function

$$\phi_k = \log\left(\sum_{\ell=0}^n \left|\frac{z_\ell}{z_k}\right|^2\right) = \log\left(1 + \sum_{m=1}^n |w_m|^2\right),$$

which, as we will prove, is a local Kähler potential. We define real $(1,1)$-forms ω_k on each U_k by $\omega_k = \mathrm{i}\partial\bar{\partial}\phi_k$. Firstly, we check that this defines a global element $\omega \in \Omega^{1,1}(M) \cap \Omega^2(M)$, i.e. that

$$\omega_{k|U_k\cap U_\ell} = \omega_{\ell|U_k\cap U_\ell};$$

on $U_k \cap U_\ell$, we have

$$\phi_k = \log\left(\left|\frac{z_\ell}{z_k}\right|^2 \sum_{m=0}^n \left|\frac{z_m}{z_\ell}\right|^2\right) = \log\left(\left|\frac{z_\ell}{z_k}\right|^2\right) + \phi_\ell.$$

Hence, we only need to show that $\partial\bar{\partial}\log\big(|z_\ell/z_k|^2\big) = 0$ on $U_k \cap U_\ell$. This follows from the fact that on $\mathbb{C}$

$$\partial\bar{\partial}\log|w|^2 = \partial\left(\frac{w\,\mathrm{d}\overline{w}}{|w|^2}\right) = \partial\left(\frac{\mathrm{d}\overline{w}}{\overline{w}}\right) = 0.$$

Now, a computation similar to the one in the previous example yields

$$\omega_k = \frac{\mathrm{i}}{(1 + \|w\|^2)^2} \sum_{\ell,m=1}^n \big((1 + \|w\|^2)\delta_{\ell,m} - \overline{w}_\ell w_m\big)\,\mathrm{d}w_\ell \wedge \mathrm{d}\overline{w}_m.$$

Let $H = \big((1 + \|w\|^2)\delta_{\ell,m} - \overline{w}_\ell w_m\big)_{1\le\ell,m\le n}$ and consider $u \neq 0$ in $\mathbb{C}^n$; then

$$\langle Hu, u\rangle = \|u\|^2 + \|w\|^2\|u\|^2 - |\langle w, u\rangle|^2 \geq \|u\|^2 > 0$$

by the Cauchy–Schwarz inequality. Consequently, $\omega_{\mathrm{FS}} = \omega$ is a Kähler form, called the *Fubini–Study* form. Sometimes its definition involves a factor $\pm 1/(2\pi)$, so that the integral of ω_{FS} on $\mathbb{CP}^1 \subset \mathbb{CP}^n$ is equal to ± 1. In our setting, it is better not to include this factor, as we will see later.

2.6 A Few Useful Properties

Let (M, ω, j) be a Kähler manifold and let $g = \omega(\cdot, j\cdot)$ be the induced Kähler metric. The gradient with respect to g of a function f and the Hamiltonian vector field associated with f are related as follows.

Lemma 2.6.1. *Let* $f \in \mathcal{C}^1(M)$*. Then* $\operatorname{grad}_g f = -jX_f$*.*

Proof. On the one hand, by definition, the gradient of f is such that the equation $\mathrm{d}f = g(\cdot, \operatorname{grad}_g f) = \omega(\cdot, j \operatorname{grad}_g f)$ holds. But on the other hand, the Hamiltonian vector field of f satisfies $\mathrm{d}f + \omega(X_f, \cdot) = 0$.

Like any other Riemannian metric, the Kähler metric g induces a volume form μ_g on M. But the symplectic form ω also defines a volume form, namely the Liouville volume form $\mu = \omega^{\wedge n}/n!$. They are also related.

Lemma 2.6.2. *These two volume forms are equal:* $\mu = \mu_g$*.*

Proof. Let us use local complex coordinates $(z_1, \ldots, z_n)$ and let us introduce the real local coordinates $(x_1, y_1, \ldots, x_n, y_n)$ satisfying $z_\ell = x_\ell + \mathrm{i} y_\ell$ for every $\ell \in [\![1, n]\!]$. Then we can write

$$\omega = \mathrm{i} \sum_{\ell, m=1}^{n} h_{\ell,m} \, \mathrm{d}z_\ell \wedge \mathrm{d}\bar{z}_m$$

for some functions $h_{\ell,m}$ such that $H(p) = \big(h_{\ell,m}(p)\big)$ is a positive definite Hermitian matrix for every p. Consequently,

$$\mu = \mathrm{i}^n \det(H) \, \mathrm{d}z_1 \wedge \mathrm{d}\bar{z}_1 \wedge \cdots \wedge \mathrm{d}z_n \wedge \mathrm{d}\bar{z}_n = 2^n \det(H) \, \mathrm{d}x_1 \wedge \mathrm{d}y_1 \wedge \cdots \wedge \mathrm{d}x_n \wedge \mathrm{d}y_n.$$

Note that $2^n \det(H) = \sqrt{\det g}$; this is a consequence of the definition of H, because

$$h_{\ell,m} = \tfrac{1}{4} g(\partial_{x_\ell} - \mathrm{i}\partial y_\ell, \partial_{x_m} + \mathrm{i}\partial_{y_m}).$$

Therefore, we finally obtain that

$$\mu = \sqrt{\det g} \, \mathrm{d}x_1 \wedge \mathrm{d}y_1 \wedge \cdots \wedge \mathrm{d}x_n \wedge \mathrm{d}y_n = \mu_g,$$

which was to be proved. □

In what follows, we will also need the following result, which can be derived from the Hodge theory of compact Kähler manifolds. We do not want to spend time on this theory in these notes; therefore we will not give a proof of this result. It is a consequence of [24, Corollary 3.2.10] for example.

Lemma 2.6.3 (The global i$\partial\bar{\partial}$-lemma). *Let* (M, ω) *be a compact Kähler manifold. Let* α *be an exact, real-valued form of type* $(1, 1)$ *on* M*. Then there exists a function* $\phi \in \mathcal{C}^\infty(M, \mathbb{R})$ *such that* $\alpha = \mathrm{i}\partial\bar{\partial}\phi$*. This function is unique up to the addition of a constant.*