Chapter 1
Introduction

Berezin–Toeplitz operators appear in the study of the semiclassical limit of the quantisation of compact symplectic manifolds. They were introduced by Berezin [5], their microlocal analysis was initiated by Boutet de Monvel and Guillemin [33], and they have been studied by many authors since, see for instance [8, 9, 14, 23, 30, 49]. This list is of course far from exhaustive, and the very nice survey paper by Schlichenmaier [43] gives a review of the Kähler case and contains a lot of additional useful references.

Besides consolidation of the theory, the past twenty years have seen the development of applications of these operators to various domains of mathematics and physics, such as topics in Kähler and algebraic geometry [22, 29, 41], topological quantum field theory [1, 18, 19, 32] or the study of integrable systems [7]. Moreover, they constitute a natural setting to investigate the connection between symplectic rigidity results on compact manifolds and their quantum consequences, and have recently been used to this effect by Charles and Polterovich [20, 21, 37, 38]. For all these reasons, their importance is now comparable to the one of pseudodifferential operators. Yet, while many textbooks on the latter are available, there is still, to our knowledge, no single place for a graduate student getting started on the subject of Berezin–Toeplitz operators to quickly learn the basic material that they need.

These notes are a modest attempt at filling this gap and are designed as an introduction to the case of compact Kähler manifolds, for which the constructions are simpler than in the general case. Before detailing their contents, let us explain how they have been built.

1.1 Overview of the Book

The philosophy of this book is to give a short and—hopefully—simple introduction to Berezin–Toeplitz operators on compact Kähler manifolds. Here, the word "simple" means that it has been written with the purpose of being understandable to, at least, graduate students; therefore, we have tried not to assume any know-

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ledge in the advanced material used throughout the different chapters. Thus, the minimal requirement is some acquaintance with the basics of differential geometry. Nevertheless, this does not mean that these notes are self-contained; despite all our efforts, we sometimes had to sacrifice completeness on the altar of concision. Furthermore, there is one major blackbox at the heart of these notes, namely the description of the asymptotics of the Bergman kernel (Theorem 7.2.1). The reason is that this result is quite involved, and presenting a proof would require space, the introduction of more advanced material and would go against the spirit of the present manuscript, which is to remain as short and non-technical as possible. Nonetheless, we will briefly sketch one of the most direct proofs we are aware of. Besides, one can directly start with the explicit form of the Bergman kernel in the case of $\mathbb{CP}^n$ (see Exercise 7.2.7), check that it satisfies all the conclusions of Theorem 7.2.1, and follow the rest of these notes with this particular example in mind.

The choice to focus on the case of Kähler manifolds is motivated by the fact that this is the setting in which the constructions are the simplest to explain, and that most of the usual examples belong to this class anyway. In particular, we will describe several examples on symplectic surfaces, which are automatically Kähler. We should nevertheless mention, for the interested reader, that there are several approaches to Berezin–Toeplitz operators in the general compact symplectic case: via almost-holomorphic sections and Fourier integral operators of Hermite type [10, 33, 44], via spin^c-Dirac operators [30] or, more recently, via a direct construction of a candidate for the Szegő projector [17]. We will not go any further in the discussion of the details of these constructions.

As regards the Kähler case, the quantisation procedure, named geometric quantisation, and due to Kostant [28] and Souriau [45], requires the existence of a certain complex line bundle over the manifold, called a prequantum line bundle. The Hilbert space of quantum states is then constructed as the space of holomorphic sections of some tensor power of this line bundle; in fact, the power in question serves as a semiclassical parameter, and we eventually consider a family of Hilbert spaces indexed by this power. Roughly speaking, the first half of this book is devoted to the construction and study of this family of Hilbert spaces; for a different point of view on this part, we recommend the excellent textbook by Woodhouse [47]. The second half deals with Berezin–Toeplitz operators, which are particular families of operators acting on these quantum spaces.

1.2 Contents

Since the aim of these notes is to give a brief introduction to the topic at hand, there is obviously a lot of material that has been left untouched. Our choice of the subjects to discuss or discard has been guided by two imperatives. Firstly, the notes follow the general guidelines of the course they were designed to accompany; namely, to introduce Berezin–Toeplitz operators on compact Kähler manifolds and state those of their properties which are needed to explain the main results from the three papers [20, 37, 38], with an audience knowing little—or even nothing—on the topic. This means that we have tried to make the exposition as clear as possible, and to refrain from going into full generality when this was not necessary.

Secondly, we have made the choice of focusing on the practical side of the subject, by devoting an important part of these notes to examples and useful tools. By doing so, we want to encourage the reader to immediately start playing with concrete Berezin–Toeplitz operators and check by themself that the properties stated in this book are satisfied by these examples.

One key feature that arose by taking into account these two aspects is the presence of exercises throughout the text. Again, we encourage the reader to try to solve these exercises, which constitute most of the time a simple verification that some notion or example has been understood, or a straightforward generalisation of some result. For these reasons, we do not provide with any solution to these exercises.

Keeping these guidelines in mind, let us now go to the heart of the subject, and explain further the general idea of the book. We want to quantise a phase space which is a compact Kähler manifold (M, ω, j), that is a compact manifold endowed with a symplectic form ω and a complex structure j, these two structures being compatible in some sense that we will not precise here. Roughly speaking, this means that we want to construct a Hilbert space $\mathcal{H}$, the space of quantum states, and to associate to each classical observable $f \in \mathcal{C}^\infty(M, \mathbb{R})$ a quantum observable, that is a self-adjoint operator $T(f) \in \mathcal{L}(\mathcal{H})$, in a way that respects a certain number of principles (note that we avoid discussing problems coming from the possible unboundedness of $T(f)$, which is fine since we will see that the relevant $\mathcal{H}$ will be finite-dimensional). More precisely, the map sending f to $T(f)$ must be linear, send the constant function equal to one to the identity of $\mathcal{H}$ and satisfy the famous correspondence principle, which states that the commutator $[T(f), T(g)]$ should be related to the quantum observable $T(\{f, g\})$ associated with the Poisson bracket of f and g. Before giving more precisions, let us insist on the fact that we want a semiclassical theory, so we want this construction to depend on Planck's constant $\hbar$ and to investigate the limit $\hbar \to 0$. Hence, what we really want is a family of Hilbert spaces $(\mathcal{H}_\hbar)_{\hbar>0}$ and a family of maps $f \mapsto T_\hbar(f)$, $\hbar > 0$. The geometric quantisation procedure requires the existence of an additional structure at the classical level, a holomorphic line bundle $L \to M$ with certain properties; the desired Hilbert spaces are then built as spaces of holomorphic sections of tensor powers of this line bundle. Hence, in this theory, what will play the role of $\hbar$ is the inverse of a positive integer k, and we will consider the family $(\mathcal{H}_k)_{k\geq 1}$ of spaces of holomorphic sections of $L^{\otimes k} \to M$; the semiclassical limit corresponds to $k \to +\infty$. The next step is to construct the family of maps $f \mapsto T_k(f)$, and the main objective of these notes is to prove that these maps satisfy the following properties as k goes to infinity:

(1) $\|T_k(f)\| \sim \|f\|_\infty$ (norm estimate),
(2) $T_k(f)T_k(g) \sim T_k(fg)$ (product estimate),
(3) $[T_k(f), T_k(g)] \sim \big(1/(\mathrm{i}k)\big)T_k(\{f, g\})$.

A rigorous version of these estimates was derived in the fundamental article [8], and the second and third were discussed in [23] together with the existence of a star product $\star$ such that $T_k(f)T_k(g) = T_k(f \star g)$ up to a remainder whose norm is small with respect to every negative power of k. One of the results in [31] is the explicit computation of the second-order term in the asymptotic expansion of $f \star g$. For all these results, f and g were assumed to be smooth; however, it is sometimes relevant to quantise non-smooth functions, and this case has been studied in [4].

Here, our goal is to prove more precise versions of the three facts above, where we add remainders on the right-hand sides, and we aim at describing these remainders in terms of f, g and their derivatives, following the article [20]. This goal will be reached as follows. Chapters 2 and 3 contain the minimal knowledge required to understand geometric quantisation, that is, respectively, some properties of Kähler manifolds and some facts about complex line bundles with connections; both chapters constitute quick overviews of the essential material, but are of course far from a complete treaty on these two topics. The reader who is already familiar with these two aspects may want to start with Chapter 4, where we describe the geometric quantisation procedure and investigate the first properties of the associated quantum spaces, such as the computation of their dimensions. In Chapter 5, we define Berezin–Toeplitz operators and state their properties, such as the estimate of their norm, and the behaviour of their compositions and commutators. The rest of the book is devoted to the proof of these three properties, based on the standard ansatz for the Schwartz kernel of the projector from the space of square integrable sections of the k-th tensor power of the prequantum line bundle to the space of its holomorphic sections. Consequently, Chapter 6 is devoted to a brief discussion of integral operators on spaces of sections and their kernels, before we describe the aforementioned Schwartz kernel in Chapter 7.

The reader must be warned once again that this description, stated in Theorem 7.2.1, is a highly non-trivial result, which led us to the choice of not proving it in these pages; this constitutes a major blackbox in these notes. The reason behind this choice is that we believe that adding a lengthy and technical proof would have reduced the clarity of the exposition, for almost no added value. Should the reader be interested in such a proof, we point to, and give a rough outline of, a very nice recent one [6], in Section 7.3; additionally, we very briefly explain in the Appendix how to derive Theorem 7.2.1 from a theorem of Boutet de Monvel and Sjöstrand on the Szegő projector of a strictly pseudoconvex domain (unfortunately, the latter is itself a difficult result). Alternatively, this kernel is explicitly computed in several examples, the most interesting ones being complex projective spaces, see Exercise 7.2.7. The computation in this case is accessible and can be easily checked by the reader, who can on the one hand obtain a complete derivation of all the results in these notes for projective spaces, and on the other hand convince themselves of the validity of the general result.

We investigate composition and commutators of Berezin–Toeplitz operators in Chapter 8. Finally, in Chapter 9, we explain how to estimate the norm of a Berezin–Toeplitz operator; to this effect, we introduce the so-called coherent states, and we use the rest of the chapter to discuss some nice properties of these states.

As a conclusion, we should warn the reader that they will not find anything new in these notes, but should rather see them as a convenient gathering of the folklore knowledge on the subject. We do not claim originality in any of the results contained in this manuscript.

1.3 Uncontents

As unbirthdays sometimes provide with more gifts and excitement than birthdays, the "uncontents" of these notes probably constitute the most interesting part of the topic, and it is worth mentioning the aspects that will not be evocated along these lines, if only to convince the interested reader that there is much more to learn about Berezin–Toeplitz operators. This will also allow us to point to a few references regarding these missing parts.

Perhaps the most important choice that we have made is to not talk about metaplectic correction. This first-order correction to quantisation is widely used, and in the context of geometric quantisation, it consists in working with holomorphic sections of $L^{\otimes k} \otimes \delta \to M$ instead of holomorphic sections of $L^{\otimes k} \to M$. Here $\delta \to M$ is a half-form bundle, that is a square root of the canonical bundle of M. Although this construction leads to nicer formulas, one example being the cancellation of the term of order k^{n-1} in the computation of the dimension of $\mathcal{H}_k$, the decision not to include it was not so complicated to make, because we felt that it would have led to a general obfuscation of the text and hindered the pedagogical writing that we have tried to use. Not only because replacing $L^{\otimes k}$ by $L^{\otimes k} \otimes \delta$ everywhere could have brought confusion to the reader, but also because such a half-form bundle may not exist globally over M, and this problem would have forced us to introduce some technical discussion. For more details on Berezin–Toeplitz operators within the framework of metaplectic correction, one can for instance look at the article by Charles on the subject [16].

A certain number of experts in Berezin–Toeplitz operators are used to working with circle bundles instead of line bundles. While we respect this choice, for semi-classical purposes, we have some good reasons to prefer using line bundles rather than circle bundles. However, a small number of proofs in these notes could have been simplified by adopting the circle bundle point of view, essentially in the section about the unitary evolution of Kostant–Souriau operators. We chose not to do so, since we realised that the gain would be small in comparison to the loss of efficiency induced by forcing the reader to digest a chapter on circle bundles. Nonetheless, for those who are interested in this aspect, and since we believe that it is useful to be able to easily pass from one theory to another, we have included an Appendix in which we compare the two points of view.

Besides these two major characters, there is a certain number of interesting topics that this book will not even allude to. In the product formula for Berezin–Toeplitz, one can go further than simply saying that the product $T_k(f)T_k(g)$ coincides with $T_k(fg)$ up to some small remainder. In fact, one can get a better approximation by comparing this product with $T_k\big(u(\cdot,k)\big)$ where u has a complete asymptotic expansion in negative powers of k (see, e.g. [23]). One can then talk about *a* subprincipal symbol, not *the* subprincipal symbol, since there are several choices of symbols. For more details about this and symbolic calculus, see for instance [14, 31]. A related question is the study of deformation quantisations on Kähler manifolds; it is discussed in [13, 23, 27, 42] for instance. We do not discuss Fourier integral operators in Kähler quantisation, and refer the reader to [15, 33, 48] for example. We do not mention the group theoretical aspects of geometric quantisation and Berezin–Toeplitz operators either, namely the quantisation of coadjoint orbits of compact Lie groups. Several references are available, but the original article by Kostant [28] constitutes a good starting place. Finally, as already explained, this book does not contain anything about the general symplectic case, and we invite the reader to have a look at the references listed above for this matter.

This list is of course not exhaustive (we could have cited spectral theory, trace formulae, etc.), and we hope that this introduction to Berezin–Toeplitz operators will give the reader the impulse to go through the looking glass and discover by themselves the wonders that lie on the other side.