Chapter 7
Weakly Consistent Regularisation Methods for Ill-Posed Problems

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Abstract This Chapter takes its origin in the lecture notes for a 9 h course at the Institut Henri Poincaré in September 2016. The course was divided in three parts. In the first part, which is not included herein, the aim was to first recall some basic aspects of stabilised finite element methods for convection-diffusion problems. We focus entirely on the second and third parts which were dedicated to ill-posed problems and their approximation using stabilised finite element methods. First we introduce the concept of conditional stability. Then we consider the elliptic Cauchy-problem and a data assimilation problem in a unified setting and show how stabilised finite element methods may be used to derive error estimates that are consistent with the stability properties of the problem and the approximation properties of the finite element space. Finally, we extend the result to a data assimilation problem subject to the heat equation.

7.1 Introduction

In these notes we will give an overview of some recent work on finite element methods for ill-posed problems. For well-posed problems it is known that, in the presence of non-symmetric operators, approximation using Galerkin finite element methods may have poor accuracy, due to the lack of H^1-coercivity. A popular remedy is then to add some stabilising terms that should be balanced in such a way that they cure the stability issue, but vanish quickly enough under mesh-refinement so that optimal error estimates can be obtained. For ill-posed problems on the other hand the state of the art is to add some regularising terms on the continuous level to obtain a well-posed continuous problem that can then typically be discretised using standard finite element methods.

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Here our aim is to make the ideas from the former class of problems carry over to the ill-posed case, using weakly consistent regularisation that is defined on the discrete level. Indeed prior to discretisation no regularisation is applied, instead the ill-posed problem and associated data are discretised in the form of a minimisation problem, where some suitable distance between the discrete solution and the measured data is minimised under the constraint of the discrete form of the partial differential equation. Regularisation terms may then be devised that are in some sense the minimal choice necessary to achieve a well-posed discrete system. To analyse the resulting approximation we rely on conditional stability estimates for the continuous problem, typically obtained through Carleman estimates.

Compared to the state of the art methods such as the quasi-reversibility method by Lattès and Lions (and the recent improvements on this technique by Bourgeois et al. [7, 8, 21]) or the penalty method by Kohn and Vogelius [4, 30], the present framework has some attractive features. Since no regularised continuous problem is involved the only (nontrivial) regularisation parameter present is the mesh size. This is not the case for more traditional methods where the discretisation parameter and the regularisation parameter must be matched carefully, or as is usually assumed, the mesh size is chosen substantially smaller than the regularisation parameter. Maybe more importantly, in the present framework, the regularisation is independent of the stability of the underlying physical problem while still having a convergence order with respect to the mesh size that is consistent with the stability of the physical problem. On the contrary, balancing regularisation and discretisation errors in the framework of conventional Tikhonov regularisation appears to inevitably lead to a nontrivial relation between the regularisation, the mesh size and the specific form of the stability of the physical problem.

With the recent increased understanding of the stability properties of ill-posed problems, in particular, in the context of inverse and data assimilation problems, we believe that these considerations are important. For instance, data assimilation problems with Hölder, or even Lipschitz, stability will have that precise order reproduced for the convergence order of the approximation error. To the best of our knowledge, apart from the work reviewed here, there exists no results in the literature reporting on such estimates even in Lipschitz stable cases that allow error estimates as good as those for classical well-posed problems. For other work on regularized methods for the Cauchy problem we refer to [2, 3, 6, 29].

The paper consists of two main parts. In the first we consider stationary ill-posed elliptic problems, such as the elliptic Cauchy problem and the so-called data assimilation problem, where measured data is available in some subdomain of the bulk, but not on the boundary. For these problems interior estimates with Hölder stability are known to hold and we show how to make these estimates translate into error estimates for the computational method. In the second chapter we consider the extension of these ideas to a data assimilation problem subject to the heat equation. In this case a Lipschitz-continuous stability estimate holds for the reconstruction of the solution away from the (unknown) initial datum. Also in this case we show, in a space semi-discretised framework, error estimates that reflect the stability of the physical problem. In this second case the estimates obtained are optimal with respect to the approximation order of the finite element space.

7.2 Preliminary Results

In this section we will introduce the geometrical setting of the problems that we will consider, the associated finite element spaces and some technical results, including discrete inequalities and approximation results. We will stay in the simplest of settings, considering only piecewise affine finite element spaces.

Let $\Omega \subset \mathbb{R}^d$, $d = 2, 3$, be a convex polygonal (polyhedral) domain, with boundary $\partial\Omega$ and outward pointing normal n. By $\mathcal{T}$ we denote a quasi-uniform decomposition of Ω in simplices T such that the intersection of two simplices in $\mathcal{T}$ is either the empty set, a shared vertex, a shared face or a shared edge. We also introduce the mesh parameter associated to $\mathcal{T}$, $h_T = \text{diam}(T)$ where the diameter of T is defined as the diameter of the smallest ball circumscribing T. Setting $h = \max_{T\in\mathcal{T}} h_T$ we consider the family of tesselations $\{\mathcal{T}\}_h$ indexed by h. The simplices are shape regular in the sense that the ratio between the smallest circumscribed ball and the largest inscribed ball of any $T \in \mathcal{T}$ is bounded uniformly, with a constant independent of h. The boundary of T will be denoted ∂T with outward pointing normal n_T. We denote the set of element faces by $\mathcal{F}$ and let $\mathcal{F}_i$ and $\mathcal{F}_b$ denote the set of faces in the interior of Ω and on its boundary, respectively. To each interior face we associate a normal n_F that is fixed, but with arbitrary orientation. The normal on faces on the boundary will be chosen pointing outwards.

We define the finite dimensional space

$$\mathbb{V}_h = \{v_h \in H^1(\Omega) : v|_T \in \mathbb{P}_1(T),\ \forall T \in \mathcal{T}\},$$

with $\mathbb{P}_1(T)$ the set of polynomials of degree less than or equal to 1. For a subspace $V \subset H^1(\Omega)$, we denote by V_h the intersection $\mathbb{V}_h \cap V$. In particular, we use the notation $V^0 = H^1_0(\Omega)$ and

$$V_h^0 := \mathbb{V}_h \cap V^0.$$

We will denote the L^2 scalar product over a set Ξ by

$$(v, w)_\Xi := \int_\Xi xy \,\mathrm{d}\Xi, \quad \forall v, w \in L^2(\Omega),$$

and the associated norm by

$$\|x\|_\Xi := (x, x)_\Xi^{\frac{1}{2}}.$$

The subscript will be dropped whenever $\Xi \equiv \Omega$.

7.2.1 Inequalities

We will need a few auxiliary results on how different norms or semi norms are related. In particular we will need the following so-called inverse inequality and trace inequalities (see for instance [22])

$$\|\nabla v_h\|_T \leq C_i h_T^{-1} \|v_h\|_T \quad \forall v_h \in \mathbb{P}_k(T),\ k \geq 0 \tag{7.1}$$

$$\|v\|_{\partial T} \leq C_t h_T^{-1/2} (\|v_h\|_T + h_T \|\nabla v\|_T), \quad \forall v \in H^1(T) \tag{7.2}$$

$$\|v_h\|_{\partial T} \leq C_t h_T^{-1/2} \|v_h\|_T, \quad \forall v_h \in \mathbb{P}_k(T),\ k \geq 0. \tag{7.3}$$

We also define the broken norm

$$\|v\|_h := \left(\sum_{T \in \mathcal{T}} \|v\|_T^2 \right)^{\frac{1}{2}}.$$

7.2.2 Interpolants and Approximation

We will use an interpolant $i_h : H^1(\Omega) \to \mathbb{V}_h$, that preserves homogeneous boundary conditions and satisfies the following estimates [33]

$$\|u - i_h u\| + h\|\nabla(u - i_h u)\| \leq C h^s \|u\|_{H^s(\Omega)}, \quad s = 1, 2. \tag{7.4}$$

Combining (7.4) and (7.2) allows us to prove the estimates

$$\|h^{-\frac{1}{2}}(u - i_h u_h)\|_{\mathcal{F}} + \|h^{\frac{1}{2}} \nabla(u - u_h)\|_{\mathcal{F}} \leq C h^{s-1} \|u\|_{H^s(\Omega)}, \quad s = 1, 2. \tag{7.5}$$

We will also make use of the H^1-projection $\pi_h : H^1_0(\Omega) \to V_h^0$ defined by

$$(\nabla \pi_h u, \nabla v_h) = (\nabla u, \nabla v_h), \quad \forall v_h \in V_h^0. \tag{7.6}$$

We note that under the assumption of quasi uniformity and convexity of the domain also this approximation satisfies (7.4) and (7.5).

7.3 Ill-Posed Problems

It is well known that instabilities may cause suboptimality for approximations of convection-diffusion equations when the standard Galerkin method is applied. Examples of how stabilised methods can improve on the situation include the Galerkin Least Squares method [10, 27], subgrid viscosity [26] or the continuous interior penalty method [15]. This is an example of a problem that is well-posed on the continuous level, but where the discrete system may be ill-conditioned and produce poor quality approximations, unless all the scales of the problem have been resolved, something which may be difficult to achieve in practice. The arguments to analyse such methods use the positivity of the bilinear operator $a(\cdot, \cdot)$ defining the problem.

In many practical cases however the problem is indefinite, for instance, this is the case for Helmholtz equation and for non-coercive convection-diffusion. Then the bilinear form does not satisfy such a positivity property, and the inf-sup condition that underpins well-posedness on the continuous level can be difficult to reproduce on the discrete level. This led the first author to develop a method which does not rely on coercivity or inf-sup stability for its analysis [11]. As the method does not rely on the well-posedness structure for its design, it can also be applied to ill-posed problems. This case was then analysed in [12] and applied to a series of different ill-posed problems in [13, 16, 17, 19].

In this section we will discuss how to apply stabilised finite elements to the approximation of ill-posed problems. Of course the class of ill-posed problems is very large and most of these problems are not tractable to the type of high resolution methods that we wish to apply here, so first we will discuss what type of ill-posed problems we are interested in and give some examples. For readers interested in delving deeper into the theory of inverse and ill-posed problems and their regularisation, we refer to [5, 24, 28, 31, 34].

Ill-posed problems are those problems that fail to be well-posed in the sense of the definition due to Hadamard. In order to make this precise we introduce the abstract problem

$$\mathcal{K}u = \mathfrak{f} \tag{7.7}$$

where $\mathcal{K} : V \to \mathcal{X}$ is a linear map between two Hilbert (or Banach) spaces and $\mathfrak{f} \in \mathcal{X}$.

Definition 7.1 (Well-Posed Problem) The problem (7.7) is well-posed if

1. For every $\mathfrak{f} \in \mathcal{X}$ there exists $u \in V$ satisfying (7.7). This means that $\mathcal{X}$ is the range of $\mathcal{L}$.
2. The solution u is unique in V. That is, $\mathcal{L}^{-1}$ exists.
3. The solution u depends continuously on data.

$$\|u\|_V \leq C\|\mathfrak{f}\|_{\mathcal{X}}.$$

Definition 7.2 (Ill-Posed Problem) The problem (7.7) is said to be ill-posed if at least one of the three points in Definition 7.1 fails.

It was recognised by Tikhonov that some ill-posed problems are better behaved than others, and conditionally stable problems are an important class of such problems. We give a definition that is a variation of [28, Def. 4.3].

Definition 7.3 (Conditionally Stable Problem) The problem (7.7) is said to be conditionally stable with respect to a semi-norm $|\cdot|$ on V if

1. For all $\mathfrak{f}$ in the range of $\mathcal{K}$ the solution u of (7.7) is unique.
2. There is a non-decreasing function $C_E : [0, \infty) \to [0, \infty)$ and a modulus of continuity $\Phi : [0, \infty) \to [0, \infty)$ such that for all $\mathfrak{f}$ in the range of $\mathcal{K}$,

$$|u| \le C_E(\|u\|_V)\Phi(\|\mathfrak{f}\|_{\mathcal{X}}).$$

   Here Φ being a modulus of continuity means that it is continuous and satisfies $\Phi(0) = 0$.

We restrict our attention to conditionally stable problems where $\mathcal{K}$ and $\mathcal{X}$ consist of two components

$$\mathcal{K} = (\mathcal{L}, \mathcal{R}), \quad \mathcal{X} = W' \times \mathcal{M}.$$

Here, for the Sobolev spaces V and W, W' is the dual of W and $\mathcal{L}$ is a differential operator, mapping V to W' when interpreted in weak form. For the part related to data we let $\mathcal{R} : V \to \mathcal{M}$ denote a restriction operator, possibly composed with a differential operator. To summarize, we will consider problems of the form

$$\mathcal{L}u = \tilde{f}, \quad \mathcal{R}u = \tilde{q} \tag{7.8}$$

where it is assumed that $(\tilde{f}, \tilde{q})$ is in a neighbourhood of the range of $\mathcal{K}$. We will prove estimates that depend on the distance

$$\|\delta f\|_{W'} + \|\delta q\|_{\mathcal{M}}, \quad \delta f = \tilde{f} - f,\ \delta q = \tilde{q} - q,$$

where (f, q) is in the range of $\mathcal{K}$. Observe that this means that we do not assume that the problem (7.8) admits a unique solution, we only assume that it can be solved for some point in a neighbourhood of the data $(\tilde{f}, \tilde{q})$. This allows for perturbed data to be used.

We will now proceed to give examples of problems that are conditionally stable in the above sense.

Example 7.1 (The Elliptic Cauchy Problem and Its Ill-Posedness) Let $\mathcal{L} = -\Delta + \sigma$ where $\sigma \in \mathbb{R}$ and assume that the boundary of Ω consists of two parts Γ and Γ'. Consider the problem of finding $u \in H^1(\Omega)$ such that

$$\mathcal{L}u = f \text{ in } \Omega \tag{7.9}$$

$$u = g \text{ on } \Gamma \tag{7.10}$$

$$\nabla u \cdot n = \psi \text{ on } \Gamma. \tag{7.11}$$

For simplicity, we consider below only the case $g = 0$, and refer to [14] for the case with non-vanishing g. Then

$$\mathcal{R}u = \nabla u \cdot n|_{\Gamma}, \quad \mathcal{M} = H^{-1/2}(\Gamma). \tag{7.12}$$

Following a classical counter-example by Hadamard, let us exemplify the failure of continuous dependence for this problem. Let $\Omega := \{(x, y) \in \mathbb{R}^2 : x > 0\}$ and $\Gamma = \{(x, y) \in \mathbb{R}^2 : x = 0\}$, $\sigma = 0$, $f = 0$, $g = 0$ and

$$\psi(y) = \frac{1}{n} \sin(ny).$$

It is easy to verify that the solution in that case is

$$u(x, y) = \frac{1}{2n^2} \sin(ny)(e^{nx} - e^{-nx}).$$

Clearly as n becomes large $\|\psi\|_{L^\infty(\Gamma)}$ goes to zero, but $u(x, y)$ blows up for any $x > 0$ and any y outside a countable set, showing the failure of continuous dependence.

Example 7.2 (The Elliptic Data Assimilation Problem and Its Uniqueness) Let $\mathcal{L} = -\Delta$ and assume that measurements u_M of u are available in some open subset of Ω, $\omega \subset \Omega$, then we can formulate the data assimilation problem as

$$\mathcal{L}u = f \text{ in } \Omega \tag{7.13}$$

$$u = u_M \text{ in } \omega. \tag{7.14}$$

Here we choose

$$\mathcal{R}u = u|_{\omega}, \quad \mathcal{M} = L^2(\omega). \tag{7.15}$$

This problem is often called also a unique continuation problem.

Assume that u_M, f are such that there exists a solution $u \in H^1(\Omega)$ to (7.13)–(7.14). Then this solution is unique which can be proven by using elementary properties of harmonic functions. Indeed, assume that there exists two solutions and let v be their difference. Then

$$\mathcal{L}v = 0 \text{ in } \Omega \tag{7.16}$$

$$v = 0 \text{ in } \omega. \tag{7.17}$$

This means that ν is a harmonic function in Ω and hence real analytic. But ν vanishes in the non-empty open set ω, and hence by analytic continuation, $\nu \equiv 0$ in Ω.

Remark 7.1 For the problem (7.13)–(7.14) to have a solution, it is of course necessary that the compatibility condition $\mathcal{L}u_M|_\omega = f|_\omega$ is satisfied. Using this one may show that, for sufficiently smooth f, (7.13)–(7.14) is equivalent to the Cauchy problem

$$\mathcal{L}u = f \text{ in } \Omega \setminus \omega \tag{7.18}$$

$$u = u_M \text{ on } \partial\omega \tag{7.19}$$

$$\nabla u \cdot n = \nabla u_M \cdot n \text{ on } \partial\omega. \tag{7.20}$$

The conditional stability for the problems in Examples 7.1 and 7.2 is classical, and we discuss it further in Sect. 7.3.2 below. Let us now turn to weak formulation of these problems on which the associated finite element methods will be based.

7.3.1 Weak Formulations of the Model Problems

Let us first consider the Cauchy problem in Example 7.1 and introduce the spaces

$$V^\Gamma := \{v \in H^1(\Omega) : v|_\Gamma = 0\} \quad \text{and} \quad W^\Gamma := \{v \in H^1(\Omega) : v|_{\Gamma'} = 0\}(= V^{\Gamma'}).$$

Now observe that the solution of (7.9)–(7.11), with $g = 0$, can be sought in V^Γ. Multiply (7.9) by $v \in W^\Gamma$ and integrate by parts to obtain

$$(\mathcal{L}u, v) = (\nabla u, \nabla v) + (\sigma u, v) - \int_\Gamma \underbrace{\nabla u \cdot n}_{=-\psi} v \,\mathrm{d}s - \int_{\Gamma'} \nabla u \cdot n \underbrace{v}_{=0} \,\mathrm{d}s$$

By defining

$$a(u, v) := (\nabla u, \nabla v) + (\sigma u, v)$$

we arrive at the weak formulation: find $u \in V^\Gamma$ such that

$$a(u, v) = (f, v) + (\psi, v)_\Gamma, \quad \forall v \in W^\Gamma. \tag{7.21}$$

This weak formulation looks deceptively like the weak formulation for the Poisson problem, but observe that the choice $v = u$ is not allowed since $u \notin W^\Gamma$.

Let us now turn to the data assimilation problem in Example 7.2. Recall from Sect. 7.2 that $V^0 = H^1_0(\Omega)$, and observe that we may multiply (7.13) with $v \in V^0$

to obtain

$$(\mathcal{L}u, v) = (\nabla u, \nabla v) - \int_{\partial\Omega} \nabla u \cdot n \underbrace{v}_{=0} \, \mathrm{d}s.$$

This time we define

$$a(u, v) := (\nabla u, \nabla v)$$

and obtain the weak formulation: find $u \in H^1(\Omega)$ such that $u|_\omega = u_M$ and

$$a(u, v) = (f, v) \quad \forall v \in V^0. \tag{7.22}$$

Once again it is not allowed to take $v = u$ due to the different choices of spaces.

7.3.2 Conditional Stability

To unify the treatment of the two examples, we will write V for the primal space and W for the test space. That is, $V = V^\Gamma$ and $W = W^\Gamma$ in the case of Example 7.1, and $V = H^1(\Omega)$ and $W = V^0$ in the case of Example 7.2. Observe that $W' = H^{-1}(\Omega)$ in the case of Example 7.2.

We refer to the review paper [1] for thorough discussion of conditional stability estimates for the two example problems. In particular, the following conditional stability estimate can be deduced from the paper.

Theorem 7.1 *Let $u \in V$ be such that, with $l \in W'$,*

$$a(u, v) = l(v).$$

Let $\mathcal{R} : V \to \mathcal{M}$ be defined by (7.12) for the Cauchy problem in Example 7.1, and by (7.15) for the data assimilation problem in Example 7.2. Write $u_M = \mathcal{R}u$ in both the cases. Then for every open simply connected $\omega' \subset \Omega$ such that dist$(\partial\omega', \partial\Omega) > 0$ there holds

$$\|u\|_{\omega'} \leq C_E(\|u\|)\Phi(|u_M|_{\mathcal{M}} + \|l\|_{W'}),$$

where $C_E(R) = CR^{1-\tau}$ and $\Phi(\eta + \varepsilon) = (\eta + \varepsilon)^\tau$. Here $C > 0$ and $\tau \in (0, 1)$ are constants that depend on ω'.

For a proof of this result with full detail on involved constants see [1, Theorem 1.7] for the Cauchy problem and [1, Theorem 4.4] for the data assimilation case. Let us remark that we state the conditions on ω' in slightly simplified form, for more precise conditions on ω' see [1]. Note that here $\|\cdot\|_{\omega'}$ is viewed as a semi-norm on V.

Remark 7.2 A similar result for global stability of u on the form

$$\|u\|_\Omega \leq C_E(\|u\|_V)\Phi(|u_M|_{\mathcal{M}} + \|l\|_{W'}),$$

with $\Phi(\eta + \varepsilon) = |\log(\eta + \varepsilon)|^{-\tau}$, $\tau \in (0, 1)$, is also derived in [1] and may be used to derive global error estimates using the techniques below.

Remark 7.3 Conditional stability has been used before to tune the regularisation parameters for Tikhonov regularisation methods see for instance [20]. What is new in the approach that we advocate is that it does not depend on the form of the modulus of continuity Φ, but still allows us to obtain the best possible accuracy with respect to the approximation error and the actual form of Φ.

7.4 Finite Element Approximation of Ill-Posed Problems

The aim of the present section is present a finite element method that draws on our experience of stabilised FEM for convection-diffusion equations. The ideas that are presented below are mainly taken from [13, 19].

We wish to attempt to discretise a conditionally stable ill-posed problem of the form: find $u \in V$ such that

$$a(u, v) = l(w), \quad \forall w \in W \tag{7.23}$$

$$|u - u_M|_{\mathcal{M}} = 0. \tag{7.24}$$

Let us consider, for the moment, the case of Cauchy problem and suppose that l is such that there exists a solution $u \in V$ to (7.23).

Recall the notation defined in Sect. 7.2, and define the finite element spaces

$$V_h^\Gamma := \mathbb{V}_h \cap V^\Gamma \quad \text{and} \quad W_h^\Gamma := \mathbb{V}_h \cap W^\Gamma.$$

We are assuming here that the mesh is fitted to the subsets of the boundary Γ and Γ'. We then have the discrete formulation of the Cauchy problem in Example 7.1: find $u_h \in V_h^\Gamma$ such that

$$a(u_h, w_h) = (f, w_h) + (\psi, w_h)_\Gamma, \quad \forall w_h \in W_h^\Gamma. \tag{7.25}$$

Observe that the corresponding linear system can not be invertible in general, because there is no reason that the system matrix is square. Indeed this only holds in the special case when the number of vertices in Γ is the same as the number of vertices in Γ'. Similarly the matrix corresponding to a naive finite element discretisation of the data assimilation problem in Example 7.2 is not square and in general the system is singular even if we impose $u_h|_\omega = 0$.

The idea is then to reformulate (7.23)–(7.24), on the discrete level, as the problem to *minimise* (7.24) under the *constraint* (7.23). This will allows us also to treat the case of perturbed data that is outside the range of the map $\mathcal{K} = (\mathcal{L}, \mathcal{R})$. In some cases $|\cdot|_{\mathcal{M}}$ may not be the most efficient choice for minimisation purposes and may be replaced by another norm $|\cdot|_{\mathcal{M}_h}$ that is equivalent on the discrete spaces. Then an additional step is required to show that the minimisation with respect to $|\cdot|_{\mathcal{M}_h}$ indeed leads to a bound in $|\cdot|_{\mathcal{M}}$.

Below we will mainly focus on the data assimilation problem in Example 7.2 and use

$$|u_h - \tilde{u}_M|^2_{\mathcal{M}_h} := \int_\omega h^\alpha (u_h - \tilde{u}_M)^2 \,\mathrm{d}x, \tag{7.26}$$

where α is a constant in the interval $[-2, 0]$. Here it is assumed that the mesh is fitted to the domain ω, which can always be easily achieved by replacing ω with a slightly smaller polygonal domain. For the Cauchy problem in Example 7.1, we can take

$$|u_h - \tilde{u}_M|^2_{\mathcal{M}_h} := \int_\Gamma h(\nabla u_h \cdot n - \tilde{\psi})^2 \,\mathrm{d}s. \tag{7.27}$$

In what follows it is important that, in both the cases and for all $\alpha \in [-2, 0]$, there holds for $u \in H^2(\Omega)$ that

$$|u - i_h u|_{\mathcal{M}_h} \leq Ch|u|_{H^2(\Omega)}.$$

We form the tentative Lagrangian

$$\text{Ł}(u_h, z_h) := \frac{1}{2}\gamma_M |u_h - \tilde{u}_M|^2_{\mathcal{M}_h} + a(u_h, z_h) - \tilde{l}(z_h),$$

where $\tilde{u}_M = u_M + \delta u$ is the perturbed data available and $\tilde{l}(z_h) = l(z_h) + \delta l(z_h)$ is a perturbed right hand side. Observe that if u is a solution to (7.23) and (7.24) then it will minimise the Lagrangian (if $\delta u = \delta l = 0$) with the associated multiplier $z = 0$. Unfortunately the associated minimisation problem may not be well-posed on the discrete level due to the ill-posedness of $a(\cdot, \cdot)$, even if the data of the continuous problem is in the range of $\mathcal{K}$. It follows that we need some regularisation.

7.4.1 Regularisation by Stabilisation

The classical way of obtaining a well-posed optimisation problem is through Tikhonov regularisation. In this case the natural choice would be to add regularising terms in the H^1-semi-norm for both the primal and the dual variable to obtain

$$Ł(u_h, z_h) := \frac{1}{2}\gamma_M |u_h - \tilde{u}_M|^2_{\mathcal{M}_h} + \gamma_1 \|\nabla u_h\|^2 - \gamma_2 \|\nabla z_h\|^2 + a(u_h, z_h) - \tilde{l}(z_h).$$

Computing the Euler-Lagrange equations for this Lagrangian we obtain the system: find $(u_h, z_h) \in V_h \times W_h$ such that

$$a(u_h, w_h) - \gamma_2(\nabla z_h, \nabla w_h) = \tilde{l}(w_h) \quad \forall w_h \in W_h \tag{7.28}$$

$$a(v_h, z_h) + \gamma_1(\nabla u_h, \nabla v_h) + \gamma_M(u_h, v_h)_{\mathcal{M}_h} = \gamma_M(\tilde{u}_M, v_h)_{\mathcal{M}_h} \quad \forall v_h \in V_h \tag{7.29}$$

Here it is assumed that the norm $|\cdot|_{\mathcal{M}_h}$ is associated to an inner product $(\cdot,\cdot)_{\mathcal{M}_h}$. This is of course the case for both (7.26) and (7.27).

Remark 7.4 This system bears a strong resemblance to the quasi-reversibility method for the Cauchy problem in the mixed form as proposed on the continuous level in [7]. Therein it was proven that if the exact solution exists, and the data are unperturbed, then letting $\gamma_1 \to 0$ for bounded γ_2 (that may tend to zero, but at a lower rate than γ_1) the regularised solution converges to the exact solution.

Drawing on our experience from stabilised finite element methods we would like to modify the regularisation terms, so that they vanish at an optimal rate in the limit $u_h \to u \in H^2(\Omega)$, $z_h \to 0$, while keeping the regularisation parameters γ_1 and γ_2 fixed. We therefore introduce the abstract regularisation operators $s : V_h \times V_h \mapsto \mathbb{R}$ and $s^* : W_h \times W_h \mapsto \mathbb{R}$ in the Lagrangian

$$Ł(u_h, z_h) := \frac{1}{2}\gamma_M |u_h - \tilde{u}_M|^2_{\mathcal{M}_h} + \frac{1}{2}s(u_h, u_h) - \frac{1}{2}s^*(z_h, z_h) + a(u_h, z_h) - \tilde{l}(z_h). \tag{7.30}$$

The corresponding Euler-Lagrange equations then reads

$$a(u_h, w_h) - s^*(z_h, w_h) = \tilde{l}(w_h) \tag{7.31}$$

$$a(v_h, z_h) + s(u_h, v_h) + \gamma_M(u_h, v_h)_{\mathcal{M}_h} = \gamma_M(\tilde{u}_M, v_h)_{\mathcal{M}_h}. \tag{7.32}$$

The primal stabilisation operator should be weakly consistent, in the sense that,

$$s(i_h u, i_h u)^{\frac{1}{2}} \leq Ch|u|_{H^2(\Omega)}. \tag{7.33}$$

We also require s to be bounded, $s(v_h, v_h) \leq C\|v_h\|^2_V$. The dual stabilisation on the other hand must be equivalent with the W norm

$$c_1(h)\|w_h\|^2_W \leq s^*(w_h, w_h) \leq C\|w_h\|^2_W,$$

where the lower bound is not required to be uniform in h. No condition analogous to (7.33) is required from s^*, the reason being that $z = 0$ is the solution to the unperturbed problem where data are such that a unique solution $u \in V$ exists. Thus any bilinear form s^* is weakly consistent in the sense that it vanishes in (7.31) when (u_h, z_h) is replaced by the solution to the unperturbed problem.

Anticipating the results in the next section we give the following examples of stabilisation operators,

$$s(v_h, v_h) := \gamma_1 \|h\sigma u_h\|^2 + \gamma_1 \sum_{F \in \mathcal{F}_i} (h_F [\![\nabla v_h]\!], [\![\nabla v_h]\!])_F =: \gamma_1 |v_h|_V^2 \tag{7.34}$$

$$s^*(v_h, v_h) := \gamma_2 (\nabla v_h, \nabla v_h)_\Omega =: \gamma_2 \|v_h\|_W^2. \tag{7.35}$$

We emphasize that, contrary to typical Tikhonov regularisation, the stabilisation parameters $\gamma_1, \gamma_2 > 0$ will not change during computation.

Observe that for $u \in H^2(\Omega)$ there holds $s(u, v_h) = \gamma_1 (h^2 \sigma^2 u, v_h)_\Omega$ for all $v_h \in V_h$, since the jump term vanishes when applied to sufficiently smooth functions. The remaining L^2-term, is weakly consistent to the right order for piecewise affine elements. For higher order polynomial approximation of order k, the primal stabilisation operator in the Lagrangian (7.30) must be replaced by a strongly consistent residual based stabilisation of the form

$$s(v_h, v_h) := \|h^k \nabla v_h\|_\Omega^2 + \gamma_1 \|h(f + \Delta v_h - \sigma v_h)\|_h^2 + \gamma_1 \sum_{F \in \mathcal{F}_i} (h_F [\![\nabla v_h]\!], [\![\nabla v_h]\!])_F, \tag{7.36}$$

for details see the discussion in [13]. The weak consistency takes a different form in this case, since the presence of the source term f leads to a contribution on the form $\sum_{K \in \mathcal{T}_h} (f, h^2(-\Delta v_h + \sigma v_h))_K$ in the right hand side of (7.32). Observe also that s defines a semi-norm on $V_h + H^2(\Omega)$ but that s^* defines a norm on W.

Let us now introduce the mesh dependent norm

$$|||(u_h, z_h)|||^2 := \gamma_M |u_h|_{\mathcal{M}_h}^2 + \gamma_1 |u_h|_V^2 + \gamma_2 \|z_h\|_W^2 + \min(\gamma_1, \gamma_M) h^2 \|u_h\|_{H^1(\Omega)}^2. \tag{7.37}$$

As the parameters $\gamma_M, \gamma_1, \gamma_2$ are fixed we could omit including them in the above norm, however, we will keep track of the dependence of the constants in Proposition 7.2 below on these parameters, and for this reason it is convenient to include the parameters in the above norm.

Observe that using (7.4) and (7.5) it is straightforward to prove the interpolation inequality

$$|||(u - i_h u, 0)||| \le Ch|u|_{h^2(\Omega)}. \tag{7.38}$$

To include the last term in the definition (7.37) we can apply a discrete Poincaré inequality.

Lemma 7.1 (Discrete Poincaré Inequality) *There exists $c_p > 0$ such that for all $v_h \in V_h$ there holds*

$$c_P h \|u_h\|_{H^1(\Omega)} \leq |u_h|_{\mathcal{M}_h} + |u_h|_V.$$

In the case of the Cauchy problem where $|\cdot|_{\mathcal{M}_h}$ is defined by (7.27) and $u_h|_\Gamma = 0$ this is a consequence of the Poincaré inequalities of [9] and for the data assimilation case where $|\cdot|_{\mathcal{M}_h}$ is defined by (7.26) the result was proved in [19].

The system (7.31)–(7.32) can be cast on the compact form, find $(u_h, z_h) \in V_h \times W_h$ such that

$$A_h[(u_h, z_h), (v_h, w_h)] = \tilde{l}(w_h) + \gamma_M(\tilde{u}_M, v_h)_{\mathcal{M}_h}, \quad \forall (v_h, w_h) \in V_h \times W_h, \tag{7.39}$$

where

$$A_h[(u_h, z_h), (v_h, w_h)] := a(u_h, w_h) - s^*(z_h, w_h) + a(v_h, z_h) + s(u_h, v_h) + \gamma_M(u_h, v_h)_{\mathcal{M}_h}.$$

Proposition 7.1 *The system (7.39) admits a unique unique solution $(u_h, z_h) \in V_h \times W_h$.*

Proof By construction, for all (v_h, w_h)

$$\gamma_M |v_h|^2_{\mathcal{M}_h} + \gamma_1 |v_h|^2_V + \gamma_2 \|w_h\|^2_W = A_h[(v_h, w_h), (v_h, -w_h)]$$

and therefore by Lemma 7.1 there exists $C > 0$ such that

$$|||(v_h, w_h)|||^2 \leq C\, A_h[(v_h, w_h), (v_h, -w_h)]. \tag{7.40}$$

The linear system (7.39) is square, and by the above positivity there are no zero eigenvalues. We conclude that the system is invertible.

Comparing with the exact problem (7.23)–(7.24) and assuming that $u \in H^2(\Omega)$, we see that the formulation (7.39) satisfies the following consistency relation

$$A_h[(u_h - u, z_h), (v_h, w_h)] = \delta l(w_h) + \gamma_M(\delta u, v_h)_{\mathcal{M}_h}, \quad \forall (v_h, w_h) \in V_h \times W_h. \tag{7.41}$$

7.4.2 Error Analysis Using Conditional Stability

First we will introduce some continuity properties of the bilinear form using the stabilisations. Assume that $u \in H^2(\Omega)$, then there holds

$$a(u - i_h u, v_h) \leq C h |u|_{H^2(\Omega)} \|v_h\|_W \tag{7.42}$$

and for all $u_h \in V_h$ and all $w \in W$, $i_h w \in W_h$

$$a(u_h, w - i_h w) \leq (Ch\|u\|_{H^2(\Omega)} + |||(u - u_h, 0)|||)\|w\|_W, \tag{7.43}$$

where and the constants are allowed to depend on the parameters $\gamma_1,\ \gamma_2$ and γ_M.

For the data assimilation problem Eq. (7.42) follows by an application of the Cauchy-Schwarz inequality and (7.4), and (7.43) follows by the integration by parts followed by (7.4) and (7.5) leading to

$$\begin{aligned} a(u_h, w - i_h w) &\leq |(\sigma u_h, w - i_h w)| + \sum_{F \in \mathcal{F}_i} \int_F |h^{\frac{1}{2}} [\![\nabla u_h]\!]| h^{-\frac{1}{2}} |w - i_h w| \, \mathrm{d}s \\ &\leq C\gamma_1^{-\frac{1}{2}} (|u - u_h|_V + \|\sigma h u\|)\|w\|_W. \end{aligned}$$

The results for the Cauchy problem are obtained in a similar fashion and we refer to [14] for the details.

We are now ready to prove a first error estimate that holds independently of the stability properties of the continuous model.

Proposition 7.2 *If* (u_h, z_h) *is the solution of* (7.39) *and* $u \in H^2(\Omega)$ *is the solution of* (7.23)–(7.24) *then there holds*

$$|||(u - u_h, z_h)||| \leq C_\gamma h |u|_{H^2(\Omega)} + \delta_\gamma \tag{7.44}$$

where $\delta_\gamma := \gamma_2^{-1/2}\|\delta l\|_{W'} + \gamma_M^{1/2}|\delta u|_{\mathcal{M}_h}$ *and* $C_\gamma := C(1 + \gamma_1^{\frac{1}{2}} + \gamma_2^{-\frac{1}{2}})$.

Proof To prove (7.44) we observe that by (7.38) and the triangle inequality it is enough to consider the discrete error $\xi_h = i_h u - u_h$. By (7.40) we have

$$|||(\xi_h, z_h)|||^2 \leq C\, A_h[(\xi_h, z_h), (\xi_h, -z_h)].$$

Using the Galerkin orthogonality (7.41) we may write

$$A_h[(\xi_h, z_h), (\xi_h, -z_h)] = A_h[(i_h u - u, 0), (\xi_h, -z_h)] - \delta l(z_h) + \gamma_M(\delta u, \xi_h)_{\mathcal{M}_h}.$$

By the continuity (7.42) there holds

$$\begin{aligned} A_h[(i_h u - u, 0), (\xi_h, -z_h)] &= a(u - i_h u, z_h) + s(i_h u - u, \xi_h) + \gamma_M (i_h u - u, \xi_h)_{\mathcal{M}_h} \\ &\leq Ch\gamma_2^{-\frac{1}{2}} |u|_{H^2(\Omega)} \gamma_2^{\frac{1}{2}} \|z_h\|_W + \underbrace{\gamma_1^{\frac{1}{2}} |i_h u - u|_V}_{\leq Ch\gamma_1^{\frac{1}{2}} |u|_{H^2(\Omega)}} \gamma_1^{\frac{1}{2}} |\xi_h|_V + \gamma_M |i_h u - u|_{\mathcal{M}_h} |\xi_h|_{\mathcal{M}_h}. \end{aligned}$$

Bounding also the perturbation terms

$$\delta l(w_h) \leq \gamma_2^{-\frac{1}{2}} \|\delta l\|_{W'} \gamma_2^{\frac{1}{2}} \|z_h\|_W$$

and

$$(\delta u, \xi_h)_{\mathcal{M}_h} \leq |\delta u|_{\mathcal{M}_h} |\xi_h|_{\mathcal{M}_h}$$

we arrive at

$$A_h[(\xi_h, z_h), (-\xi_h, z_h)] \leq C_\gamma h |u|_{H^2(\Omega)} |||(\xi_h, z_h)||| + \delta_\gamma |||(\xi_h, z_h)|||.$$

We conclude by dividing by $|||(\xi_h, z_h)|||$.

This proof is insufficient to show error estimates. However for unperturbed data and $u \in H^2(\Omega)$, it may be used to show that $u_h \to u$ as $h \to 0$, by a compactness argument.

Remark 7.5 Note that δ_γ may depend on h via the quantity $|\delta u|_{\mathcal{M}_h}$. This is the case, for instance, when $|\cdot|_{\mathcal{M}_h}$ is chosen as in (7.26) with $\alpha \neq 0$, and then error in data is amplified for small h.

To prove error estimates we must rely on the conditional stability estimates in Theorem 7.1. The idea behind the argument is to consider the error $e = u - u_h$ and observe that this error satisfies

$$a(e, w) = l(w) - a(u_h, w) =: r(w), \quad \forall w \in W. \tag{7.45}$$

We will then use Proposition 7.2 to get bounds for $\|r\|_{W'}$, $|e|_{\mathcal{M}_h}$ and $\|e\|$, so that the conditional stability can be applied to e.

In the data assimilation case we have $|e|_{\mathcal{M}} = \|e\|_\omega = h^{-\alpha/2}|e|_{\mathcal{M}_h} \leq |e|_{\mathcal{M}_h}$ so this quantity is immediately bounded by (7.44). For the Cauchy problem the continuous and discrete data matching terms are not the same, but one can prove that a suitable bound can be obtained for a perturbed error $\tilde{e}$ by adding a small perturbation to u_h in the interface zone such that

$$|\tilde{e}|_{\mathcal{M}} \leq |||e, 0|||. \tag{7.46}$$

The error analysis then uses the arguments below together with a perturbation argument for $\tilde{e}$, for details see [14]. We will not consider that case here, instead focussing on the data assimilation case.

Theorem 7.2 *Let u be the exact solution to* (7.23)–(7.24)*, with $l(w) := (f, w)$, $f \in L^2(\Omega)$, and $|\cdot|_{\mathcal{M}} = \|\cdot\|_\omega$. Let u_h be the solution of* (7.31)–(7.32) *with the stabilisation operators* (7.34)–(7.35)*. Then, for all $\omega' \subset \Omega$ satisfying the* *assumptions in Theorem 7.1 there holds*

$$\|u - u_h\|_{\omega'} \leq Ch^{\tau}(\|u\|_{H^2(\Omega)} + h^{-1}\delta_{\gamma}).$$

where the constant depends on the geometry and the constants γ_1, γ_2 and γ_M.

Proof As discussed above, the estimate is shown by applying Theorem (7.1) to the problem satisfied by the error. We know that e is a solution to (7.23) with $l(w) = r(w)$ as per Eq. (7.45). By Proposition 7.2 the following bounds hold

$$|e|_{\mathcal{M}_h} = \|e\|_{\omega} \leq C_{\gamma}h|u|_{H^2(\Omega)} + \delta_{\gamma} \tag{7.47}$$

and

$$\|e\|_V \leq C_{\gamma}|u|_{H^2(\Omega)} + h^{-1}\delta_{\gamma}. \tag{7.48}$$

Now observe that using Eq. (7.31) we have

$$r(w) = r(w-i_h w)+r(i_h w) = l(w-i_h w)-a(u_h, w-i_h w)-s^*(z_h, i_h w)-\delta l(i_h w).$$

We estimate the terms on the right hand side, assuming that $\|w\|_W = 1$,

$$l(w - i_h w) = (f, w - i_h w) \leq \|f\|\|w - i_h w\| \leq Ch\|f\|,$$

and using the inequality (7.43)

$$a(u_h, w - i_h w) \leq Ch\|u\|_{H^2(\Omega)} + |||(u - u_h, 0)|||.$$

Then applying Proposition 7.2 we obtain the bound

$$a(u_h, w - i_h w) \leq \gamma_1^{-\frac{1}{2}}(C_{\gamma}h\|u\|_{H^2(\Omega)} + \delta_{\gamma}).$$

The two remaining terms are handled using the Cauchy-Schwarz inequality in the first case and the duality pairing $H^{-1} \times H^1$ in the second, followed by the stability of the interpolant i_h in the W-norm,

$$s(z_h, i_h w) \leq \gamma_2\|z_h\|_W\|w\|_W \leq \gamma_2^{\frac{1}{2}}(C_{\gamma}h|u|_{H^2(\Omega)} + \delta_{\gamma})$$

$$\delta l(i_h w) \leq C\|\delta l\|_{W'}$$

Collecting the terms above we have for all $w \in W$ with $\|w\|_W = 1$,

$$r(w) \leq Ch\|f\| + (\gamma_1^{-\frac{1}{2}} + \gamma_2^{\frac{1}{2}})(C_{\gamma}h\|u\|_{H^2(\Omega)} + \delta_{\gamma}) + C\|\delta l\|_{W'}. \tag{7.49}$$

But then

$$\|r\|_{W'} = \sup_{w \in W: \|w\|_W = 1} r(w)$$

satisfies the same bound. Note also that $\|f\| \leq C\|u\|_{H^2(\Omega)}$. We conclude that e satisfies the assumptions of Theorem 7.1 by with

$$R = \|e\|_V \leq C_\gamma |u|_{H^2(\Omega)} + h^{-1}\delta_\gamma, \quad \eta = |e|_{\mathcal{M}_h} \leq Ch|u|_{H^2(\Omega)} + \delta_\gamma,$$
$$\varepsilon = \|r\|_{W'} \leq C(h\|u\|_{H^2(\Omega)} + \delta_\gamma)$$

c.f. (7.47)–(7.49). In the last step we dropped the dependence on the constants γ_1, γ_2 and γ_M, but it can be traced in the proof.

Remark 7.6 We detailed Theorem 7.2 only in the case of the data assimilation problem, but the same arguments also leads to an analysis for the Cauchy problem, under the assumption (7.46).

Remark 7.7 One may prove Theorem 7.2 for the data assimilation problem if s^* is defined by (7.34). In this case an additional factor h^{-1} multiplies the term measuring perturbations in data.

7.4.3 A Numerical Example

We consider the problem in Example 7.1 on the unit square Ω. The exact solution is $u = 30.0 * x * (1 - x) * y * (1 - y)$, with $f = \mathcal{L}u$, and the data domain ω is defined by

$$\omega := \{(x, y) \in \Omega : |x - 0.5| < 0.25; |y - 0.5| < 0.25\}.$$

We use the formulation (7.31)–(7.32) with $s(\cdot, \cdot)$ given by (7.34) for piecewise affine approximation and (7.36) for piecewise quadratic approximation. The adjoint stabiliser $s^*(\cdot, \cdot)$ was defined by (7.35), and the norm $|\cdot|_{\mathcal{M}_h}$ by (7.26) with $\alpha = 0$ or -2. (Observe that if $\alpha = 0$ then γ_M must have the unit of the square of an inverse length for the equations to be dimensionally correct.)

We chose $\gamma_2 = \gamma_M = 1$ and $\gamma_1 = 10^{-3}$ for all computations. The latter value is similar to that used for computations in the well-posed case. We meshed the domain using structured meshes that were made to fit the boundary of ω. We performed computations on a sequence of meshes with `nele`= 40, 80, 160, 320, elements on each side of the square, using piecewise affine and piecewise quadratic elements. In Fig. 7.1, left graphic, we show a computational mesh and on the right graphic we illustrate the domains ω (the inner square) and ω' (the middle square). In Fig. 7.2, left plot, we show the contourlines of an approximate solution and in the right plot

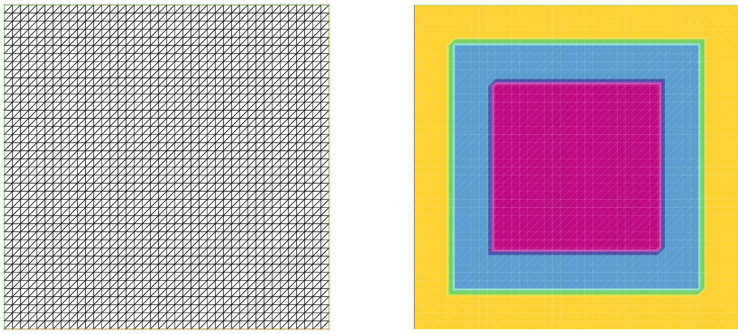

Fig. 7.1 Left: computational mesh, `nele`=40. Right: the different subdomains ω and ω'

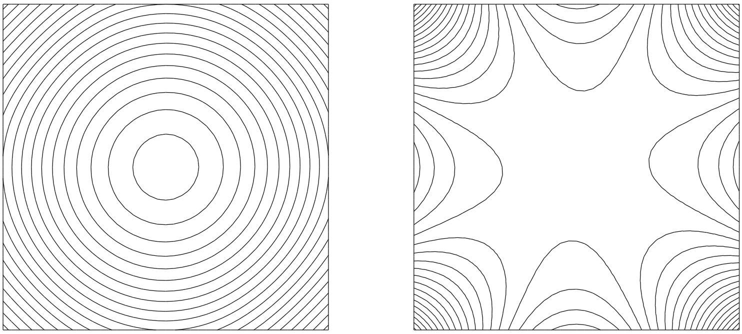

Fig. 7.2 Left: contour lines of approximate solution, `nele`=40. Right: contour lines of the computational error

the contour lines of the computational error. Observe that the error has a form that is similar to Hadamard's counter-example discussed in Example 7.1, but growing exponentially in the radial direction and oscillating in the direction tangential to the boundary of ω.

In the tables below we report the error in the normalised global L^2-error, the normalised local error for the subset

$$\omega' := \{(x, y) \in \mathbb{R}^2 : |x - 0.5| < 0.375; |y - 0.5| < 0.375\},$$

the data assimilation term, $|u - u_h|_\omega$, and the size of the weakly consistent regularisation

$$|(u - u_h, z)|_s := \sqrt{s(u - u_h, u - u_h) + s^*(z_h, z_h)}. \tag{7.50}$$

The experimental convergence rates are given in parenthesis, where appropriate. We report the results for unperturbed data and $\alpha = 0$ in Tables 7.1 and 7.5 and for $\alpha = -2$ in Tables 7.2 and 7.6. In all cases we observe the expected $O(h^k)$ convergence

Table 7.1 Computed quantities for the data assimilation problem using piecewise affine approximation, $\alpha = 0$ and unperturbed data

nele	$\|u - u_h\|$	$\|u - u_h\|_{\omega'}$	$\|u - u_h\|_{\omega}$	$\|(u - u_h, z)\|_s$
40	0.211594 (–)	0.050922 (–)	0.00816074 (–)	0.0289235 (–)
80	0.175512 (0.3)	0.0407488 (0.3)	0.00618422 (0.4)	0.0147585 (1.0)
160	0.113346 (0.6)	0.0235298 (0.8)	0.00337103 (0.9)	0.00791309 (0.9)
320	0.0672893 (0.75)	0.0102456 (1.2)	0.00119201 (1.5)	0.0042852 (0.9)
640	0.0510429 (0.4)	0.00529074 (1.0)	0.000342379 (1.8)	0.00221974 (0.9)

Table 7.2 Computed quantities for the data assimilation problem using piecewise affine approximation, $\alpha = -2$ and unperturbed data

nele	$\|u - u_h\|$	$\|u - u_h\|_{\omega'}$	$\|u - u_h\|_{\omega}$	$\|(u - u_h, z)\|_s$
40	0.0476335 (–)	0.00481282 (–)	0.000333429 (–)	0.0352793 (–)
80	0.0403148 (0.2)	0.00312934 (0.6)	8.0272e−05 (2.0)	0.0179655 (1.0)
160	0.0304957 (0.4)	0.00188862 (0.7)	1.998e−05 (2.0)	0.00911884 (1.0)
320	0.0227619 (0.4)	0.0009549 (1.0)	4.71016e−06 (2.1)	0.00464924 (1.0)
640	0.0200062 (0.2)	0.000642748 (0.6)	1.15698e−06 (2.0)	0.00234456 (1.0)

Table 7.3 Computed quantities for the data assimilation problem using piecewise affine approximation, $\alpha = 0$ and 2.5% perturbation in data

nele	$\|u - u_h\|$	$\|u - u_h\|_{\omega'}$	$\|u - u_h\|_{\omega}$	$\|(u - u_h, z)\|_s$
40	0.206909	0.0490942	0.0148287	0.0289287 (-)
80	0.176546	0.0409112	0.013946	0.0146984 (1.0)
160	0.119693	0.0267951	0.0131763	0.0077906 (0.9)
320	0.0793605	0.0180773	0.0125264	0.00416117 (0.9)
640	0.0640708	0.0158747	0.0124993	0.00214582 (1.0)

of the stabilising terms (7.50), with $k = 1$ for piecewise affine approximation and $k = 2$ in the quadratic case. We also observe that consistently with theory we have $\|u-u_h\|_\omega = O(h^{k-\alpha/2})$. The convergence of the data term is more even for $\alpha = -2$. For the global and local L^2-norms we see that the error is a factor $5-10$ larger when $\alpha = 0$ compared with the case where $\alpha = -2$. Most likely this is due to the fact that the missing length-scale present for $\alpha = 0$ is not well represented when $\gamma_M = 1.0$. Indeed the weak penalty does not impose the data sufficiently well compared to the other terms, when $\alpha = -2$ on the other hand the data penalty term is so strong that the data error is very small already on coarse meshes leading to improved local and global errors. We observe convergence compatible with Hölder stability for all quantities, indicating that possibly we are not yet in the asymptotic regime on these scales. Only on the finest meshes in Table 7.6 we see clearly the decreasing orders characteristic for logarithmic convergence in the global error.

We then make the same sequence of computations but adding a perturbation of 2.5% to the data in ω in the piecewise affine case and 1% in the quadratic case. The results are reported for affine approximation in Tables 7.3 ($\alpha = 0$) and 7.4 ($\alpha = -2$). We observe that although the data assimilation term stagnates, the local

Table 7.4 Computed quantities for the data assimilation problem using piecewise affine approximation, $\alpha = -2$ and 2.5% perturbation in data

nele	$\|u - u_h\|$	$\|u - u_h\|_{\omega'}$	$\|u - u_h\|_{\omega}$	$\|(u - u_h, z)\|_s$
40	0.0520752	0.0145883	0.0124714	0.03529
80	0.0507222	0.014398	0.0125092	0.0186372
160	0.0502568	0.0143645	0.0127194	0.0142032
320	0.0537505	0.0143083	0.0125169	0.0224315
640	0.0427351	0.0138826	0.0125888	0.0434341

Table 7.5 Computed quantities for the data assimilation problem using piecewise quadratic approximation, $\alpha = 0$ and unperturbed data

nele	$\|u - u_h\|$	$\|u - u_h\|_{\omega'}$	$\|u - u_h\|_{\omega}$	$\|(u - u_h, z)\|_s$
20	0.0113854 (–)	0.0020353 (–)	0.000272026 (–)	0.00263335 (–)
40	0.00701791 (0.7)	0.000668735 (1.6)	4.36798e−05 (2.6)	0.00067804 (2.0)
80	0.00630128 (0.16)	0.000458704 (0.54)	1.0293e−05 (2.1)	0.000171095 (2.0)
160	0.00457823 (0.5)	0.000278068 (0.72)	5.50828e−06 (1.0)	4.33632e−05 (2.0)
320	0.00275223 (0.7)	9.14176e−05 (1.6)	7.11806e−07 (2.8)	1.10465e−05 (2.0)

Table 7.6 Computed quantities for the data assimilation problem using piecewise quadratic approximation, $\alpha = -2$ and unperturbed data

nele	$\|u - u_h\|$	$\|u - u_h\|_{\omega'}$	$\|u - u_h\|_{\omega}$	$\|(u - u_h, z)\|_s$
20	0.00594613 (–)	0.000454428 (–)	1.92029e−05 (–)	0.00269387 (–)
40	0.00364274 (0.7)	0.000194766 (1.2)	3.21386e−06 (-2.6)	0.00069238 (–)
80	0.0023773 (0.6)	6.52831e−05 (1.6)	2.95005e−07 (3.4)	0.000176426 (2.0)
160	0.00159176 (0.6)	2.93421e−05 (1.2)	3.91486e−08 (2.9)	4.45628e−05 (2.0)
320	0.00118008 (0.4)	1.27615e−05 (1.2)	4.3179e−09 (3.2)	1.12277e−05 (2.0)

Table 7.7 Computed quantities for the data assimilation problem using piecewise quadratic approximation, $\alpha = 0$ and 1% perturbation in data

nele	$\|u - u_h\|$	$\|u - u_h\|_{\omega'}$	$\|u - u_h\|_{\omega}$	$\|(u - u_h, z)\|_s$
20	0.0146381	0.00619699	0.00510402	0.00260206
40	0.0137215	0.00593519	0.00492976	0.00066236 (2.0)
80	0.0135235	0.00594218	0.00498009	0.000167333 (2.0)
160	0.0110434	0.00593666	0.00497521	4.82896e−05 (1.8)
320	0.00982659	0.0058722	0.00497389	1.23888e−05 (2.0)

and global errors decrease under refinement for $\alpha = 0$. In this case the stabilisation norm also converges to optimal order in spite of the perturbation. When $\alpha = -2$ only the error in the stabilisation semi-norm show any decrease under refinement. On the finest scale we see that both the global error and the error in the stabilisation semi-norm has started to grow. For piecewise affine approximation it appears that the choice $\alpha = -2$ is superior both for perturbed and unperturbed data (at least for the choice $\gamma_M = 1$) (Tables 7.5 and 7.6).

For quadratic approximation the results are reported in Tables 7.7 ($\alpha = 0$) and 7.8 ($\alpha = -2$). Here the effect of the perturbation is present already on the coarsest mesh

Table 7.8 Computed quantities for the data assimilation problem using piecewise quadratic approximation, $\alpha = -2$ and 1% perturbation in data

nele	$\|u - u_h\|$	$\|u - u_h\|_{\omega'}$	$\|u - u_h\|_\omega$	$\|(u - u_h, z)\|_s$
20	0.0177247	0.00638777	0.00513258	0.00275637
40	0.026475	0.00628408	0.00495361	0.00164336
80	0.0503314	0.00644259	0.00500485	0.002676516
160	0.159728	0.0079909	0.0050097	0.00510579
320	0.335852	0.00962178	0.0050035	0.0101055

and the amplification of the error clearly much stronger for $\alpha = -2$. Indeed whereas for $\alpha = 0$ all error quantities still decrease under mesh refinement, the errors for $\alpha = -2$ all stagnate or increase. For the stabilisation norm we clearly see that the error doubles under mesh refinement on finer meshes, which is consistent with theory. In this case it appears that for resolutions where the mesh-size is of similar order as the perturbation it is advantageous to take $\alpha = 0$, also in accordance with theory.

7.5 Time Dependent Problems: Data Assimilation

In this section we consider the extension of the methods in the previous section to the time dependent case, where several interesting new features appear. In particular we can consider a problem which has Lipschitz stability and prove that our method can exploit this in the form of error estimates that are optimal compared to approximation. We consider a data assimilation problem for the heat equation

$$\partial_t u - \Delta u = f, \quad \text{in } (0, T) \times \Omega, \tag{7.51}$$

with homogeneous Dirichlet conditions. Here $T > 0$ and $\Omega \subset \mathbb{R}^n$ is an open convex polyhedral domain. Let $\omega \subset \Omega$ be open and let $0 < T_1 < T$. The data assimilation problems is of the following form: determine the restriction $u|_{(T_1,T)\times\Omega}$ of a solution to the heat equation (7.51) given f and the restriction $u|_{(0,T)\times\omega}$. In this case we have the following stability estimate due to Imanuvilov [23], see also [17, 32, 35] for variations of the estimate.

Theorem 7.3 *Let $\omega \subset \Omega$ be open and non-empty, and let $0 < T_1 < T$. Then there is $C > 0$ such that for all u in the space*

$$H^1(0, T; H^{-1}(\Omega)) \cap L^2(0, T; H_0^1(\Omega)), \tag{7.52}$$

it holds that

$$\|u\|_{C(T_1,T;L^2(\Omega))} + \|u\|_{L^2(T_1,T;H^1(\Omega))} + \|u\|_{H^1(T_1,T;H^{-1}(\Omega))}$$
$$\leq C(\|u\|_{L^2((0,T)\times\omega)} + \|Lu\|_{(0,-1)}),$$

where $L = \partial_t - \Delta$ *and* $\|\cdot\|_{(0,-1)} = \|\cdot\|_{L^2(0,T;H^{-1}(\Omega))}$.

In what follows, we use the shorthand notations

$$H^{(k,m)} = H^k(0,T;H^m(\Omega)), \quad H_0^{(k,m)} = H^{(k,m)} \cap L^2(0,T;H_0^1(\Omega)),$$
$$\|u\|_{(k,m)} = \|u\|_{H^k(0,T;H^m(\Omega))}, \quad \|u\| = \|u\|_{(0,0)},$$

and denote by $\|u\|_\omega$ the norm of $L^2((0,T)\times\omega)$. Moreover, we use the following notation for the data of the problem

$$q = u|_{(0,T)\times\omega}, \quad f = Lu, \tag{7.53}$$

and write

$$a(u,z) = (\nabla u, \nabla z), \quad G_f(u,z) = (\partial_t u, z) + a(u,z) - \langle f, z\rangle, \quad G = G_0,$$

where $(\cdot,\cdot)$ is the inner product of $L^2((0,T)\times\Omega)$ and $\langle\cdot,\cdot\rangle$ is the dual pairing between $L^2(0,T;H^{-1}(\Omega))$ and $L^2(0,T;H_0^1(\Omega))$. Note that for $u \in H^1((0,T)\times\Omega)$, the equations

$$G_f(u,z) = 0, \quad z \in L^2(0,T;H_0^1(\Omega)), \tag{7.54}$$

give the weak formulation of $\partial_t u - \Delta u = f$.

7.5.1 Optimisation Based Finite Element Space Discretisation

We consider only the problem semi-discretised in space, and show that the time continuous dynamical system is well-posed for every fixed h. This section summarizes part of the analysis from [17], where also a problem with weaker stability, similar to that of the data assimilation problem in the previous section was considered. The analysis carries over to the fully discrete case, but the stabilisation operators are not the same. In particular in the fully discrete case, the adjoint stabilisation can be omitted (see reference [18] for details).

Since the problem is time dependent we introduce the spaces $\mathcal{V}_h$ and $\mathcal{W}_h$,

$$\mathcal{V}_h = H^1(0,T;V_h^0), \quad \mathcal{W}_h = L^2(0,T;V_h^0).$$

Observe that contrary to the developments in the previous section both spaces are equipped with Dirichlet conditions in space. The difference between the two spaces here is the regularity in time. Following the development in the previous sections our approach to solve the data assimilation problem is based on minimizing the Lagrangian functional

$$Ł_{q,f}(u,z) = \frac{1}{2}\|u-q\|_\omega^2 + \frac{1}{2}s(u,u) - \frac{1}{2}s^*(z,z), + G_f(u,z), \tag{7.55}$$

where the data q and f are fixed. Here $\|\cdot\|_\omega$ is the norm of $L^2((0,T)\times\omega)$, and s and s^* are the primal and dual stabilizers, respectively. Note that minimizing $Ł_{q,f}$ can be seen as fitting $u|_{(0,T)\times\omega}$ to the data q under the constraint (7.54), z can be interpreted as a Lagrange multiplier, and $s/2$ and $s^*/2$ as regularizing penalty terms.

Let $q \in L^2((0,T)\times\omega)$ and $f \in H^{(0,-1)}$. The Lagrangian $Ł_{q,f}$, defined by (7.55), satisfies

$$D_u Ł_{q,f} v = (u-q,v)_\omega + s(u,v) + G(v,z),$$
$$D_z Ł_{q,f} w = -s^*(z,w) + G(u,w) - \langle f, w\rangle,$$

and therefore the critical points $(u,z) \in \mathcal{V}_h \times \mathcal{W}_h$ of $Ł_{q,f}$ satisfy

$$A[(u,z),(v,w)] = (q,v)_\omega + \langle f, w\rangle, \quad (v,w) \in \mathcal{V}_h \times \mathcal{W}_h, \tag{7.56}$$

where A is the symmetric bilinear form

$$A[(u,z),(v,w)] = (u,v)_\omega + s(u,v) + G(v,z) - s^*(z,w) + G(u,w). \tag{7.57}$$

Note that

$$A[(u,z),(u,-z)] = s(u,u) + \|u\|_\omega^2 + s^*(z,z),$$

Herein we consider only semi-discretisations, that is, we minimize $Ł_{q,f}$ on a scale of spaces that are discrete in the spatial variable but not in the time variable. As before the spatial mesh size $h > 0$ will be the only parameter controlling the convergence of the approximation, and we use piecewise affine finite elements. For simplicity we have set all the auxiliary regularisation parameters $\gamma_1, \gamma_2, \gamma_M$ to one, and we consider only the case of unperturbed data.

7.5.2 A Framework for Stabilisation

Before proceeding to the analysis of the data assimilation problem, we introduce an abstract stabilisation framework.

Let s and s^* be bilinear forms on the spaces $\mathcal{V}_h$ and $\mathcal{W}_h$, respectively. Let $|\cdot|_{\mathcal{V}}$ be a semi-norm on $\mathcal{V}_h$ and let $\|\cdot\|_{\mathcal{W}}$ be a norm on $\mathcal{W}_h$. We relax (7.34) and (7.35) by requiring only that s and s^* are continuous with respect to $|\cdot|_{\mathcal{V}}$ and $\|\cdot\|_{\mathcal{W}}$, that is,

$$s(u,u) \leq C|u|_{\mathcal{V}}^2, \quad s^*(z,z) \leq C\|z\|_{\mathcal{W}}^2, \quad u \in \mathcal{V}_h,\ z \in \mathcal{W}_h,\ h > 0. \tag{7.58}$$

Let $\|\cdot\|_*$ be the norm of a continuously embedded subspace H^* of the energy space (7.52). The space H^* encodes the a priori smoothness. We assume that the stabilizations and norms introduced are such that the following continuities hold

$$G(u, z - \pi_h z) \leq C|u|_{\mathcal{V}}\|z\|_{(0,1)}, \qquad u \in \mathcal{V}_h,\ z \in H_0^{(0,1)}, \tag{7.59}$$

$$G(u - \pi_h u, z) \leq Ch\|z\|_{\mathcal{W}}\|u\|_*, \qquad u \in H^*,\ z \in \mathcal{W}_h, \tag{7.60}$$

where π_h is an interpolator satisfying

$$\pi_h : H_0^1(\Omega) \to V_h^0, \quad h > 0. \tag{7.61}$$

$$\|\pi_h u\|_{H^1(\Omega)} \leq C\|u\|_{H^1(\Omega)}, \qquad u \in H^1(\Omega),\ h > 0, \tag{7.62}$$

$$\|u - \pi_h u\|_{H^m(\Omega)} \leq Ch^{k-m}\|u\|_{H^k(\Omega)}, \qquad u \in H^k(\Omega),\ h > 0, \tag{7.63}$$

where $k = 1, 2$ and $m = 0, k-1$. We assume that the following upper bound holds

$$|\pi_h u|_{\mathcal{V}} \leq Ch\|u\|_*, \qquad u \in H^*, \tag{7.64}$$

and require that analogously to the stationary case

$$\|\pi_h z\|_{\mathcal{W}} \leq C\|z\|_{(0,1)}, \qquad z \in H_0^{(0,1)}. \tag{7.65}$$

We assume that

$$|||(u,z)||| = |u|_{\mathcal{V}} + \|u\|_\omega + \|z\|_{\mathcal{W}},$$

is a norm on $\mathcal{V}_h \times \mathcal{W}_h$. Finally, in the abstract setting, we assume that the s and s^* are sufficiently strong so that the following weak coercivity holds

$$|||(u,z)||| \leq C \sup_{(v,w)\in\mathcal{V}_h\times\mathcal{W}_h} \frac{A[(u,z),(v,w)]}{|||(v,w)|||}, \quad (u,z) \in \mathcal{V}_h \times \mathcal{W}_h \tag{7.66}$$

and for all $(v,w) \in \mathcal{V}_h \times \mathcal{W}_h$,

$$\sup_{\substack{(x,y)\in\mathcal{V}_h\times\mathcal{W}_h \\ x,y\neq 0}} |A[(x,y),(v,w)]| > 0. \tag{7.67}$$

The Babuska-Lax-Milgram theorem implies that Eq. (7.56) has a unique solution in $\mathcal{V}_h \times \mathcal{W}_h$. As we shall see below, these design criteria are sufficient to derive optimal error estimates in the transient case, provided the problem has a conditional stability property.

7.5.3 The Data Assimilation Problem

We will now proceed to a specific case. We choose the stabilizers and semi-norms as follows,

$$s(u, u) = \|h\nabla u(0, \cdot)\|^2_{L^2(\Omega)}, \quad s^* = a, \tag{7.68}$$

$$|u|_{\mathcal{V}} = s(u, u)^{1/2} + \|h\partial_t u\|, \quad \|z\|_{\mathcal{W}} = s^*(z, z)^{1/2}, \tag{7.69}$$

and we define $H^* = H_0^{(1,1)}$. To counter the lack of primal stabilisation on most of the cylinder $(0, T) \times \Omega$, we choose π_h to be the orthogonal projection $\pi_h : H_0^1(\Omega) \to W_h$ as defined in Sect. 7.2.2. As Ω is a convex polyhedron, it is well known that this choice satisfies (7.61)–(7.63), see e.g. [25, Th. 3.12–18].

Lemma 7.2 *The choices (7.68)–(7.69) satisfy the properties (7.58)–(7.64), (7.65) and (7.66). Moreover,* $|||\cdot|||$ *is a norm on* $\mathcal{V}_h \times \mathcal{W}_h$.

Proof It is clear that the continuities (7.58) hold. We begin with the lower bound (7.59). By the orthogonality of π_h,

$$G(u, z - \pi_h z) = (\partial_t u, z - \pi_h z) \le \|h\partial_t u\| h^{-1}\|z - \pi_h z\| \le C\|h\partial_t u\| \|z\|_{(0,1)}.$$

Towards the upper bound (7.60), we use the orthogonality as above,

$$G(u - \pi_h u, z) = (\partial_t u - \pi_h \partial_t u, z) \le Ch\|u\|_{(1,1)}\|z\|.$$

The bound (7.60) then follows from an application of the Poincaré inequality on $\|z\|$.

The bound (7.64) follows from the continuity of the trace

$$\|\nabla u(0, \cdot)\|_{L^2(\Omega)} \le \|u\|_{(1,1)}, \tag{7.70}$$

and the continuity of the projection π_h. The bound (7.65) follows immediately from the continuity of π_h.

We turn to the weak coercivity (7.66). The essential difference between the time dependent case and the stationary case is that in the latter case, the choice $w = u$

is prohibited. In this case it is allowed, but due to the time-derivative and the lack of initial condition it does not lead to stability. Instead we observe that $\partial_t u \in \mathcal{W}_h$ when $u \in \mathcal{V}_h$ so that this can be used as a test function $w = \partial_t u$ to obtain

$$A[(u, z), (0, \partial_t u)] = -s^*(z, \partial_t u) + G(u, \partial_t u) = \|\partial_t u\|^2 + a(u, \partial_t u) - a(z, \partial_t u),$$

and thus using bilinearity of A,

$$A[(u, z), (u, \alpha h^2 \partial_t u - z)] = s(u, u) + \alpha \|h \partial_t u\|^2 + \|u\|_\omega^2 + s^*(z, z) \\ + \alpha h^2 a(u, \partial_t u) - \alpha h^2 a(z, \partial_t u), \tag{7.71}$$

where $\alpha > 0$. We will establish (7.66) by showing that there is $\alpha \in (0, 1)$ such that

$$|||(u, w - z)||| \le C|||(u, z)|||, \tag{7.72}$$

$$|||(u, z)|||^2 \le CA[(u, z), (u, w - z)], \tag{7.73}$$

where $w = \alpha h^2 \partial_t u$.

Towards (7.72) we observe that

$$|||(u, w - z)|||^2 = |||(u, z)|||^2 - 2s^*(z, w) + s^*(w, w) \le 2|||(u, z)|||^2 + 2s^*(w, w).$$

We use the discrete inverse inequality (7.1) to bound the second term

$$s^*(w, w) = \alpha^2 h^4 \|\partial_t \nabla u\|^2 \le C\alpha^2 h^2 \|\partial_t u\|^2 \le C\alpha^2 |||(u, z)|||^2, \quad \alpha > 0.$$

It remains to show (7.73). Towards bounding the first cross term in (7.71) we observe that

$$2a(u, \partial_t u) = \int_0^T \partial_t \|\nabla u(t, \cdot)\|_{L^2(\Omega)}^2 dt = \|\nabla u(T, \cdot)\|_{L^2(\Omega)}^2 - \|\nabla u(0, \cdot)\|_{L^2(\Omega)}^2.$$

Hence $\alpha h^2 a(u, \partial_t u) \ge -\alpha s(u, u)/2$. We use the arithmetic-geometric inequality,

$$ab \le (4\epsilon)^{-1} a^2 + \epsilon b^2, \quad a, b \in \mathbb{R}, \ \epsilon > 0,$$

and the discrete inverse inequality (7.1) to bound the second cross term in (7.71),

$$\alpha h^2 a(z, \partial_t u) \le \alpha (4\epsilon)^{-1} a(z, z) + \alpha \epsilon h^4 \|\partial_t \nabla u\|^2 \le \alpha (4\epsilon)^{-1} a(z, z) + C\alpha\epsilon \|h \partial_t u\|^2.$$

Choosing $\epsilon = 1/(2C)$ we obtain

$$A[(u,z),(u,w-z)] \geq (1-\alpha/2)s(u,u)+\alpha\|h\partial_t u\|^2/2+\|u\|_\omega^2+(1-C\alpha/2)s^*(z,z),$$

and therefore (7.73) holds with small enough $\alpha > 0$.

The second condition (7.67) follows using the symmetry of A. Indeed, if $(v,w) \neq 0$, then $A[(x,y),(v,w)] = A[(v,w),(x,y)] > 0$ for some (x,y) by (7.66). Finally, using the Poincaré inequality, we see that $|||(u,z)||| = 0$ implies $z = 0$ and $u(0,\cdot) = 0$. As also $\partial_t u = 0$, we have $u = 0$, and therefore $|||\cdot|||$ is a norm.

7.5.4 Error Estimates

We are now in a situation to prove an error estimate using the abstract theory.

Theorem 7.4 *Let $\omega \subset \Omega$ be open and non-empty and let $0 < T_1 < T$. Suppose that (A2) holds. Let $u \in H^*$ and define $f = \partial_t u - \Delta u$ and $q = u|_\omega$. Suppose that the primal and dual stabilizers satisfy (7.58)–(7.64), (7.65) and (7.66). Then (7.56) has a unique solution $(u_h, z_h) \in \mathcal{V}_h \times \mathcal{W}_h$, and there exists $C > 0$ such that for all $h \in (0,1)$*

$$\begin{aligned}&\|u_h - u\|_{C(T_1,T;L^2(\Omega))} + \|u_h - u\|_{L^2(T_1,T;H^1(\Omega))} + \|u_h - u\|_{H^1(T_1,T;H^{-1}(\Omega))}\\ &\quad \leq Ch(\|u\|_* + \|f\|).\end{aligned}$$

Proof We begin again by showing the estimate

$$|||(u_h - \pi_h u, z_h)||| \leq Ch\|u\|_*. \tag{7.74}$$

The equations $\partial_t u - \Delta u = f$ and $u|_\omega = q$ are equivalent with

$$\begin{aligned} G(u,w) &= \langle f, w\rangle, \quad && w \in L^2(0,T;H_0^1(\Omega)),\\ (q-u,v)_\omega &= 0, && v \in L^2((0,T)\times\omega), \end{aligned} \tag{7.75}$$

and Eqs. (7.56) and (7.75) imply for all $v \in \mathcal{V}_h$ and $w \in \mathcal{W}_h$ that

$$A[(u_h - \pi_h u, z_h),(v,w)] = (u - \pi_h u, v)_\omega + G(u - \pi_h u, w) - s(\pi_h u, v). \tag{7.76}$$

The weak coercivity (7.66) implies that in order to show (7.74) it is enough bound the three terms on the right hand side of (7.76). For the first of them, that is, $(u - \pi_h u, v)_\omega$, we use (7.63). The upper bound (7.60) applies to the second term

$G(u - \pi_h u, w)$, and for the third one we use the continuity (7.58) and the upper bound (7.64),

$$s(\pi_h u, v) \leq C|\pi_h u|_{\mathcal{V}}|v|_{\mathcal{V}} \leq Ch\|u\|_*|v|_{\mathcal{V}}.$$

We define the residual r as follows. By taking $v = 0$ in (7.56) we get $G(u_h, w) = \langle f, w\rangle + s^*(z_h, w)$, $w \in \mathcal{W}_h$, and therefore

$$\begin{aligned}\langle r, w\rangle &= G(u_h, w) - \langle f, w\rangle - G(u_h, \pi_h w) + G(u_h, \pi_h w) \\ &= G(u_h, w - \pi_h w) - \langle f, w - \pi_h w\rangle + s^*(z_h, \pi_h w), \quad w \in H_0^{(0,1)}.\end{aligned} \tag{7.77}$$

We now wish to arrive to the estimate

$$\|r\|_{(0,-1)} \leq C(|u_h|_{\mathcal{V}} + \|z_h\|_{\mathcal{W}} + h\|f\|). \tag{7.78}$$

To show that (7.78) holds, it is enough to bound the three terms on the right hand side of (7.77). The upper bound (7.59) applies to the first term $G(u_h, w - \pi_h w)$, for the second term $(f, w - \pi_h w)$ we use (7.63), for the third term we use the continuity (7.58) and the upper bound (7.65)

$$s^*(z_h, \pi_h w) \leq C\|z_h\|_{\mathcal{W}}\|\pi_h w\|_{\mathcal{W}} \leq C\|z_h\|_{\mathcal{W}}\|w\|_{(0,1)}.$$

The inequalities (7.78), (7.74) and (7.64) imply

$$\|r\|_{(0,-1)} \leq C(|u_h - \pi_h u|_{\mathcal{V}} + |\pi_h u|_{\mathcal{V}} + \|z_h\|_{\mathcal{W}} + h\|f\|) \leq Ch(\|u\|_* + \|f\|).$$

Finally using the above bound on r, Theorem 7.3 implies that

$$\begin{aligned}&\|u_h - u\|_{C(T_1,T;L^2(\Omega))} + \|u_h - u\|_{L^2(T_1,T;H^1(\Omega))} + \|u_h - u\|_{H^1(T_1,T;H^{-1}(\Omega))} \\ &\quad \leq C\|u_h - u\|_\omega + Ch(\|u\|_* + \|f\|).\end{aligned}$$

The claim follows by using (7.74) and (7.63),

$$\|u_h - u\|_\omega \leq \|u_h - \pi_h u\|_\omega + \|\pi_h u - u\|_\omega \leq Ch\|u\|_*.$$

Here we used also the assumption that H^* is a continuously embedded subspace of the energy space (7.52), namely, this implies that the embedding $H^* \subset H^{(0,1)}$ is continuous.

Remark 7.8 If the data q, f is perturbed in this time-dependent case, the data assimilation problem behaves like a typical well posed problem, that is, the term

$$\|\delta q\|_{L^2(0,T;L^2(\omega))} + \|\delta f\|_{(0,-1)}$$

needs to be added on the right-hand side of the estimate in Theorem 7.4, but this time without any negative power of h. The proof is similar as in the stationary case and we omit it.

7.6 Conclusion

We have shown on some model problems how weakly consistent regularisation may be applied in the context of finite element approximation of ill-posed problems as a means to obtain approximations to the exact solution that are optimal with respect the approximation order of the finite element space and the (conditional) stability of the physical problem. We have only considered piecewise affine approximation here but the extension to high order polynomial approximation (and with associated enhanced accuracy for smooth solutions) is possible using the ideas from [13]. Ongoing work focuses on problems where the stability depends on the parameters of the physical problem in a more intricate way such as for the convection-diffusion equation or the Helmholtz equation. Further work will also address the extension to systems such as the linearised Navier-Stokes' equations.

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