Chapter 1
An Introduction to Recent Developments in Numerical Methods for Partial Differential Equations

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Abstract Numerical Analysis applied to the approximate resolution of Partial Differential Equations (PDEs) has become a key discipline in Applied Mathematics. One of the reasons for this success is that the wide availability of high-performance computational resources and the increase in the predictive capabilities of the models have significantly expanded the range of possibilities offered by numerical modeling.

Novel discretization methods, the solution of ill-posed and nonlinear problems, model reduction and adaptivity are main topics covered by the contributions of this volume. This introductory chapter provides a brief overview of the book and some related references.

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This volume comprises nine chapters reflecting many of the topics covered by the Ph.D. level courses given during the Thematic Quarter *Numerical Methods for PDEs*, held at the Institut Henri Poincaré in the fall 2016.[1] These chapters can be loosely organized into three groups: (1) novel discretisation methods; (2) nonlinear and ill-posed problems; (3) model reduction and adaptivity.

Over the last few years, new paradigms have appeared to devise discretization methods supporting polytopal elements and arbitrary approximation orders. One key motivation is that the use of general element shapes provides an unprecedented flexibility in mesh generation, which is often the most time-consuming step in numerical modeling. Two examples of polytopal, arbitrary-order discretization methods are treated in this volume. On the one hand, the Hybridizable Discontinuous Galerkin (HDG) methods introduced in [9], where one central idea is the devising of local spaces to approximate the flux and the primal variable using the notion of M-decompositions from [11]. On the other hand, the Hybrid High-Order (HHO) methods introduced in [12, 13], where one central idea is the devising of the stabilization operator within a primal formulation. HDG and HHO methods have been recently bridged in [10]. Another important paradigm for the development of discretization methods is to reproduce exactly at the discrete level the fundamental properties of the model problem at hand, leading to so-called mimetic (or compatible, or structure-preserving) discretizations. This field, which is at the crossroads of differential geometry, algebraic topology and numerical analysis, has seen a lot of activity over the last decades; recent reviews with an historical perspective can be found in [2, 3, 8]. The contributions gathered in the first four chapters of this volume concern the theory of M-decomposition and its application to hybridisable discontinuous Galerkin and mixed methods; Mimetic Spectral Element method where the metric- and material-dependent Hodge operator is built as a mass matrix from tensor-product polynomials on Cartesian and deformed grids; an introduction to Hybrid High-Order methods able to deal with generally polytopal grids, with applications to the p-Laplace and diffusion-reaction equations.

The second group of three chapters concerns nonlinear and ill-posed problems. The first contribution concerns a numerical investigation of the Distributed Lagrange Multiplier method for fluid-structure interaction. This method, which has close links with the Immersed Boundary method [17] as well as with Fictitious Domain methods with a distributed Lagrange multiplier [14], has been recently developed and analyzed in [4]. The second contribution deals with the approximation of the spectrum of an elliptic operator and addresses the benefits of combining isogeometric analysis [15] with blending quadrature rules [1]. A Pythagorean theorem linking eigenvalue and eigenfunction errors, together with numerical results, are presented. The third contribution considers ill-posed problems as encountered, for instance, in the context of inverse and data assimilation problems. While state-of-the-art methods typically rely on the introduction of a regularization at the

[1] http://imag.edu.umontpellier.fr/event/ihp-nmpdes.

continuous level, one introduces here only a weakly-consistent regularization at the discrete level. Using very recent ideas on finite element stabilization [5, 6] leads to error estimates that are compatible with the (modest, yet provable) stability of the continuous problem at hand.

The third group of three chapters highlights recent advances in reduced-order modeling and adaptivity. The increased complexity of the physical models and the need to use PDE simulators in many-query scenarios (optimisation, inverse problems, real-time, etc.) has prompted the study of model reduction techniques such as the Reduced Basis (RB) method [18]. The present contribution, which focuses on elasticity problems in affinely parameterised geometries with (non-)compliant output error control [19], describes the RB approximation of such problems and presents various numerical examples. Finally, the numerical resolution of complex problems is often feasible only if the computation resources are used judiciously. This has prompted the study of adaptive resolution algorithms, often based on a posteriori estimates of the approximation error. Important advances have been accomplished over the last decade, as discussed among others in [7, 16] and in the recent textbook [20]. The present contribution develops a relatively less explored question, namely the adaptive approximation of a given univariate target function using mesh refinement by bisection. The last chapter gives an introduction on the possible treatment of defective boundary conditions, which typically appear in the coupling of PDE problems posed in domains of different geometrical dimensions.

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