Chapter 6
Symplectic Structure of Extremal Black Holes

K. Hajian and A. Seraj

Abstract We review the construction of phase space for the near horizon extremal geometries (NHEG) as solutions to Einstein gravity. We study the symplectic symmetries of this phase space and compute their corresponding conserved charges. We show that the symmetry algebra is an interesting generalization of Virasoro algebra. The analysis is based on covariant phase space method.

6.1 Introduction

Black holes (BHs) are solutions to theories of gravity, specified by having an event horizon in their geometry. They also usually have a singularity behind the horizon. Although these solutions were known from the early stages of development of general relativity by Karl Schwarzschild, their thermodynamic behaviors were unraveled in early 70s by seminal works of Bekenstein and Hawking [1, 2] in which entropy and temperature were associated to BHs. In Einstein-Hilbert theory of gravity, BH entropy is related to the area of the horizon $S = \frac{A}{4G}$, while Hawking temperature can be read from the surface gravity of the black hole κ, through the relation $T_{\text{H}} = \frac{\kappa}{2\pi}$. Also for stationary BHs in d-dimensions, with a number of commuting and compact $U(1)$ axial isometries, labelled by index i, one can associate conserved angular momenta J_i as well as mass M (due to time translation symmetry). Thermodynamic conjugates to the angular momenta are angular velocities of the horizon, denoted by Ω^i_{H} (index H for *Horizon*). In addition, dynamics of BHs satisfy laws which are analogous to usual laws of thermodynamic [3]. Specifically, the first law of BH thermodynamics is $\delta M = T_{\text{H}}\delta S + \Omega^i_{\text{H}}\delta J_i$ [3, 4]. During the past four decades, an active line of research aims at describing microstates underlying these thermodynamic behaviors. The present work would also be in the same line.

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Motivated by the microstate counting for usual thermodynamic systems (e.g. an ideal gas) which is based on their phase spaces, here we try to build the classical symplectic structure for the set of *extremal* (vanishing temperature) BHs. Interested reader can refer to the original papers [5, 6] for detailed analysis, or to [7] as a pedagogical text. In this analysis, we keep the spacetime dimension to be arbitrary d and the theory to be given by the Einstein-Hilbert Lagrangian $\mathscr{L} = \frac{1}{16\pi G} R$, where R is the Ricci scalar. In order to make the analysis simpler, we will concentrate on extremal BHs with $d-3$ number of commuting $U(1)$ axial isometries, denoted by $U(1)^{d-3}$.

Significantly, thermodynamic properties of the BHs are encoded in their near horizon region. The temperature and other chemical potentials, in addition to BHs conserved charges can be read directly from that region. Interestingly, Iyer and Wald have shown that BH entropy is the conserved charge associated to the Killing vector of the horizon, which is calculated on the horizon [4, 8]. Hence, one expects to find the microstates of black holes by focusing on their near horizon region. Therefore, we study the phase space of near horizon geometries of extremal black holes (NHEG). These solutions share some interesting features:

- Taking the near horizon limit of an extremal BH as a solution to a given theory leads to a near horizon extremal geometry (NHEG) which is a solution to the same theory [9] (because they are found by a limiting process instead of approximation process [7]).
- Stationarity of BH is enhanced to $SL(2,\mathbb{R})$ in NHEG, therefore the symmetries of NHEG with the above mentioned properties is $SL(2,\mathbb{R}) \times U(1)^{d-3}$.
- NHEGs are uniquely identified by $d-3$ number of angular momenta J_i [10].
- Under appropriate isometry and boundary conditions, perturbations on NHEGs are restricted (*upto infinitesimal diffeomophisms*) to parametric variations, i.e. infinitesimal variations of the solution identified by J_i to an adjacent solution identified by $J_i + \delta J_i$ [11].

The metric of the considered NHEGs can be written in a suitable coordinate system as [12, 13]:

$$\mathrm{d}s^2 = \Gamma(\theta)\Big[-r^2\mathrm{d}t^2 + \frac{\mathrm{d}r^2}{r^2} + \mathrm{d}\theta^2 + \gamma_{ij}(\theta)(\mathrm{d}\varphi^i + k^i r\,\mathrm{d}t)(\mathrm{d}\varphi^j + k^j r\,\mathrm{d}t)\Big]. \tag{6.1}$$

$\Gamma(\theta)$ and $\gamma_{ij}(\theta)$ might be determined by imposing the equation of motion over the above ansatz, or by taking the near horizon limit of a given BH. In the coordinate which the metric is represented, the Killing vectors of $SL(2,\mathbb{R}) \times U(1)^{d-3}$ isometry are explicitly as

$$\xi_- = \partial_t\,, \qquad \xi_0 = t\partial_t - r\partial_r, \qquad \xi_+ = \frac{1}{2}\Big(t^2 + \frac{1}{r^2}\Big)\partial_t - tr\partial_r - \frac{k^i}{r}\partial_{\varphi^i}\,, \qquad \mathrm{m}_i = \partial_{\varphi^i}.$$

Their commutation relation is

$$[\xi_0, \xi_-] = -\xi_-, \qquad [\xi_0, \xi_+] = \xi_+, \qquad [\xi_-, \xi_+] = \xi_0\,, \qquad [\xi_a, m_i] = 0, \tag{6.2}$$

in which $a \in \{-, 0, +\}$ and $i \in \{1, \ldots, d-3\}$.

A significant property of NHEG geometry is that any surface of constant (t, r) is the bifurcation point of a Killing horizon [5, 6], which we denote by $\mathscr{H}$. Explicitly, the $d-2$ surface $\mathscr{H}$ determined by $t = t_{\mathscr{H}}, r = r_{\mathscr{H}}$, is the intersection of the following $d-1$-dimensional null hypersurfaces

$$\mathscr{N}_{\mathscr{H}+}:\ t + \frac{1}{r} = t_{\mathscr{H}} + \frac{1}{r_{\mathscr{H}}}\,, \qquad \mathscr{N}_{\mathscr{H}-}:\ t - \frac{1}{r} = t_{\mathscr{H}} - \frac{1}{r_{\mathscr{H}}}\,. \tag{6.3}$$

The magical Killing vector generating the above null two hypersurfaces is

$$\zeta_{\mathscr{H}} = n^a_{\mathscr{H}}\, \xi_a - k^i m_i\,, \tag{6.4}$$

in which

$$n^-_{\mathscr{H}} = -\frac{t^2_{\mathscr{H}} r^2_{\mathscr{H}} - 1}{2r_{\mathscr{H}}}\,, \qquad n^0_{\mathscr{H}} = t_{\mathscr{H}} r_{\mathscr{H}}\,, \qquad n^+_{\mathscr{H}} = -r_{\mathscr{H}}\,. \tag{6.5}$$

Therefore $\mathscr{N} = \{\mathscr{N}_{\mathscr{H}+} \cup \mathscr{N}_{\mathscr{H}-}\}$ is a Killing horizon and their intersection $\mathscr{H}$ is the bifurcation surface. It is shown [14] that entropy of the NHEG (which is equal to the entropy of original BH) is conserved charge associated to the $\zeta_{\mathscr{H}}$, calculated on $\mathscr{H}$.

6.2 A Review on Covariant Phase Space Method

By definition, a phase space is a manifold consisting of a set of allowed configurations of a system equipped with a symplectic form Ω_{ab} (i.e a nondegenerate closed 2 form). For a given gauge theory like Einstein gravity, and a given collection of geometries viewed as an abstract manifold, there is a well established method for defining the symplectic structure over this manifold, and thereby obtain a phase space [15]. This is known as covariant phase space method, since the construction of phase space does not involve with breaking of covariance of the theory (unlike what happens in ADM construction). Here we give the general method for construction of the symplectic structure. In next section, we specify exactly what is the set of geometries relevant for the construction of NHEG phase space (Fig. 6.1).

In covariant phase space method, the manifold $\mathscr{M}$ is built up of a set of metric configurations $g_{\mu\nu}(x^\alpha)$. Therefore vectors tangent to the phase space are indeed perturbations of the metric. For Einstein gravity, the symplectic 2-form acting on two vectors $\delta_1 g, \delta_2 g$ is given by

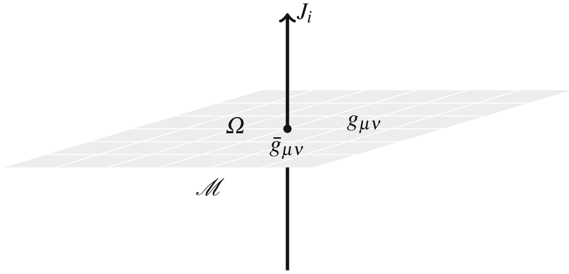

Fig. 6.1 A schematic of NHEG phase space, in terms of angular momenta **J**. The manifold $\mathscr{M}$ is comprised of some metric configurations $g_{\mu\nu}(x^\alpha)$. The point $\bar{g}_{\mu\nu}$ is the known NHEG solution. Symplectic 2-form Ω, is the Lee-Wald form, upto **Y** ambiguities

$$\Omega(\delta_1 g, \delta_2 g, g) = \int_\Sigma \boldsymbol{\omega}(\delta_1 g, \delta_2 g, g) \tag{6.6}$$

where the symplectic current $\boldsymbol{\omega}$ is

$$\boldsymbol{\omega}(\delta_1 g, \delta_2 g, g) \equiv \delta_1 \boldsymbol{\Theta}(\delta_2 g, g) - \delta_2 \boldsymbol{\Theta}(\delta_1 g, g)\,. \tag{6.7}$$

The integration surface Σ in (6.6) is a $d-1$-dim hypersurface. The $d-1$-form $\boldsymbol{\Theta}$ is defined through the variation of the Lagrangian (as a top form) after using the equations of motion, i.e $\delta \mathbf{L} \approx \mathrm{d}\boldsymbol{\Theta}$ (In this paper ≈ means on shell equality). For the Einstein gravity [8]

$$\boldsymbol{\Theta}(\delta g_{\mu\nu}, g_{\mu\nu}) = \frac{\sqrt{-g}}{(d-1)!}\, \varepsilon_{\mu\mu_1\cdots\mu_{d-1}} \frac{1}{16\pi G}(\nabla_\alpha h^{\mu\alpha} - \nabla^\mu h)\, \mathrm{d}x^{\mu_1} \wedge \cdots \wedge \mathrm{d}x^{\mu_{d-1}} \tag{6.8}$$

where $h^{\mu\nu} \equiv g^{\mu\sigma} g^{\nu\tau} \delta g_{\sigma\tau}$ and $h \equiv h^\alpha_{\ \alpha}$ and $\varepsilon_{\mu_1\cdots\mu_d}$ is the Levi-Civita symbol. If the variation δg satisfies the linearized equation of motion, then it can be shown that the symplectic current is closed on-shell, i.e. $\mathrm{d}\boldsymbol{\omega}(\delta_1 g, \delta_2 g, g) \approx 0$.

Now we turn to the definition of symmetries of the covariant phase space and their corresponding conserved charges. An infinitesimal symmetry of a phase space, is an infinitesimal coordinate transformation $x \to x - \xi$ such that any metric configuration in the phase space is sent to another configuration in the phase space. In other words, although the configurations are transformed under the symmetry action, but the whole phase space is closed under the symmetry action. Now the corresponding conserved charge H_ξ which is the generator of the symmetry transformation is defined through the contraction of $\mathscr{L}_\xi g$ with the symplectic form

$$\delta H_\xi \equiv \Omega(\delta g, \delta_\xi g, g) = \int_\Sigma \delta \boldsymbol{\Theta}(\mathscr{L}_\xi g, g) - \mathscr{L}_\xi \boldsymbol{\Theta}(\delta g, g). \tag{6.9}$$

It can be shown (e.g. see Appendix C.2 in [7]) that the integrand is on-shell an exact form $\mathrm{d}\boldsymbol{k}_\xi(\delta g, g)$. Therefore the charges can alternatively be defined (using Stoke's theorem) through the integration of $\boldsymbol{k}_\xi$ over $\partial\Sigma$ which is a codimension 2 closed surface. The latter is even more fundamental for geometries with more than one boundaries. In Einstein gravity $\boldsymbol{k}_\xi$ is [16]

$$\boldsymbol{k}_\xi(\delta g_{\mu\nu}, g_{\mu\nu}) = \frac{\sqrt{-g}}{(d-2)!\,2!}\ \varepsilon_{\mu\nu\mu_1\cdots\mu_{d-2}}\, k_\xi^{\mu\nu}\ \mathrm{d}x^{\mu_1}\wedge\cdots\wedge\mathrm{d}x^{\mu_{d-2}} \tag{6.10}$$

where

$$k_\xi^{\mu\nu} = \frac{1}{16\pi G}\left(\left[\xi^\nu\nabla^\mu h - \xi^\nu\nabla_\sigma h^{\mu\sigma} + \xi_\sigma\nabla^\nu h^{\mu\sigma} + \frac{1}{2}h\nabla^\nu\xi^\mu - h^{\rho\nu}\nabla_\rho\xi^\mu\right] - [\mu\leftrightarrow\nu]\right). \tag{6.11}$$

6.3 The NHEG Phase Space

Now in order to construct the NHEG phase space, we need to specify the set of relevant geometrie. The rough idea is that phase space configurations can serve as the microstates of extremal black hole. According to the uniqueness of dynamical perturbations, the set of relevant geometries are obtained by coordinate transformations of the background NHEG geometry. These transformations are infinitesimally obtained by a vector field χ through $x \to x - \chi$. We refer the interested reader to the original papers for the arguments for determination of χ, and state the result here. The vector field χ is given by

$$\chi[\varepsilon(\boldsymbol{\varphi})] = -\mathbf{k}\cdot\boldsymbol{\partial}_\varphi\varepsilon\left(\frac{1}{r}\partial_t + r\partial_r\right) + \varepsilon\mathbf{k}\cdot\boldsymbol{\partial}_\varphi\,, \tag{6.12}$$

where ε can be any periodic smooth function of the coordinates φ^i. Hence, χ generates the infinitesimal perturbations tangent to the phase space around the background by $\delta g[\varepsilon(\boldsymbol{\varphi})] = \mathcal{L}_\chi\bar{g}$. Exponentiation of this infinitesimal transformation produces the finite coordinate transformations which transfer $\bar{g}_{\mu\nu}$ to arbitrary configurations $g_{\mu\nu}$ of the phase space $\mathscr{M}$. The finite coordinate transformation is

$$\bar{t} = t - \frac{1}{r}(e^\Psi - 1), \qquad \bar{r} = re^{-\Psi}, \qquad \bar{\theta} = \theta\,, \qquad \bar{\varphi}^i = \varphi^i + k^i F\,. \tag{6.13}$$

We call $F(\boldsymbol{\varphi})$ the *wiggle function* which is periodic in all its arguments and Ψ is given by $e^\Psi = 1 + \mathbf{k}\cdot\boldsymbol{\partial}_\varphi F$. Therefore, corresponding to any function F, a configuration over $\mathscr{M}$ with the following metric is identified

$$\mathrm{d}s^2 = \Gamma(\theta)\left[-(\boldsymbol{\sigma} - \mathrm{d}\Psi)^2 + \left(\frac{\mathrm{d}r}{r} - \mathrm{d}\Psi\right)^2 + \mathrm{d}\theta^2 + \gamma_{ij}(\mathrm{d}\tilde{\varphi}^i + k^i\boldsymbol{\sigma})(\mathrm{d}\tilde{\varphi}^j + k^j\boldsymbol{\sigma})\right], \tag{6.14}$$

in which $\boldsymbol{\sigma} = e^{-\Psi}r\,\mathrm{d}(t + \frac{1}{r}) + \frac{\mathrm{d}r}{r}$ and $\tilde{\varphi}^i = \varphi^i + k^i(F - \Psi)$.

By construction, the infinitesimal transformations generated by χ are symmetries of the NHEG phase space. However, χ has also another important significance, i.e that χ is the *symplectic symmety* of the NHEG phase space.[1] The notion of symplectic symmetry is defined as

Definition 6.1 The vector field χ is the generator of a *symplectic symmetry generators* iff [5]

1. $\boldsymbol{\omega}(\delta g, \delta_\chi g, g) \approx 0 \qquad \forall g \in \mathscr{M}$ and $\delta g \in T\mathscr{M}$,
2. δH_χ be integrable, and H_χ be finite over the $\mathscr{M}$,

Thanks to the properties of diffeomorphisms, any point of the phase space has complete $SL(2,\mathbb{R}) \times U(1)^{d-3}$ isometry. It can be checked that any configuration has the same angular momenta **J** and entropy S as the NHEG metric $\bar{g}_{\mu\nu}$ background.

The symplectic symmetries form a closed algebra. To see this we expand χ in its Fourier modes

$$\chi_{\mathbf{n}} = -e^{-i(\mathbf{n}\cdot\boldsymbol{\varphi})}\left(i(\mathbf{n}\cdot\mathbf{k})\left(\frac{1}{r}\partial_t + r\partial_r\right) + \mathbf{k}\cdot\boldsymbol{\partial}_\varphi\right). \tag{6.15}$$

Then, the commutator of these vectors is

$$[\chi_{\mathbf{n}}, \chi_{\mathbf{m}}] = i\,\mathbf{k}\cdot(\mathbf{n}-\mathbf{m})\,\chi_{\mathbf{n}+\mathbf{m}} \tag{6.16}$$

which is a nice generalization of Witt algebra. It can be shown that the corresponding Hamiltonian generators are [5, 6]

$$H_{\mathbf{n}} = \oint_{\mathscr{H}} \boldsymbol{\varepsilon}_{\mathscr{H}}\ T[\Psi]e^{-i\mathbf{n}\cdot\boldsymbol{\varphi}}, \tag{6.17}$$

where

$$T[\Psi] = \frac{1}{16\pi G}\Big((\Psi')^2 - 2\Psi'' + 2e^{2\Psi}\Big) \tag{6.18}$$

and primes are directional derivatives along the vector **k**. The function $T[\Psi]$ transforms under infinitesimal phase space transformations as

$$\delta_\varepsilon T = \varepsilon T' + 2\varepsilon' T - \frac{1}{8\pi G}\varepsilon'''. \tag{6.19}$$

Therefore the function $T[\Psi]$ resembles a Liouville type stress tensor.

The Poisson bracket of conserved charges have the same commutation relations as (6.16) up to a central extension. Significantly, the central extension turns out to be the entropy of the NHEG. Explicitly [5, 6]

$$\{H_{\mathbf{m}}, H_{\mathbf{n}}\} = -i\mathbf{k}\cdot(\mathbf{m}-\mathbf{n})H_{\mathbf{m}+\mathbf{n}} - i(\mathbf{k}\cdot\mathbf{m})^3\frac{S}{2\pi}\,\delta_{\mathbf{m}+\mathbf{n},0}. \tag{6.20}$$

[1] The definition of a consistent symplectic form on NHEG phase space, however involves a suitable fixing of ambiguities in the symplectic current. For details, see [5, 6].

Using the Dirac quantization rules $\{\ \ \} \rightarrow \frac{1}{i}[\ \ ]$ and $H_{\mathbf{n}} \rightarrow L_{\mathbf{n}}$, the symmetry algebra promotes to an operator algebra, the *NHEG algebra* $\widehat{\mathscr{V}_{\mathbf{k},S}}$

$$[L_{\mathbf{m}}, L_{\mathbf{n}}] = \mathbf{k} \cdot (\mathbf{m} - \mathbf{n}) L_{\mathbf{m}+\mathbf{n}} + \frac{S}{2\pi} (\mathbf{k} \cdot \mathbf{m})^3 \delta_{\mathbf{m}+\mathbf{n},0} \,. \tag{6.21}$$

The J_i and H_{ξ_a} commute with $L_{\mathbf{n}}$, and are therefore central elements of the NHEG algebra $\widehat{\mathscr{V}_{\mathbf{k},S}}$. Also by Definition 6.1, they are symplectic symmetry generators. Hence, the *full symplectic symmetry of the phase space* is

$$\text{NHEG Symplectic Symmetry Algebra} = \widehat{\mathscr{V}_{\mathbf{k},S}} \oplus \ \ \mathrm{sl}(2,\mathbb{R}) \underbrace{\oplus \ \ \mathrm{u}(1)}_{(d-3 \text{ times})} \,. \tag{6.22}$$

We stress again that all geometries in the phase space have vanishing $SL(2,\mathbb{R})$ charges, and $U(1)$ charges equal to J_i.

Yet many different mathematical and physical aspects of the NHEG phase space and its algebra are yet to be understood and analyzed.

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