# Chapter 5
# Rotating Black Hole Solutions in $f(R)$-Gravity

**M. De Laurentis and R. Farinelli**

**Abstract** We present a strategy to get axially symmetric solutions in $f(R)$ gravity by starting from spherically symmetric space-times. To do so, we assume the validity of a complex coordinate transformation, which acts on the spherically symmetric metric and permits one to infer the corresponding $f(R)$ modification. The consequences of this recipe are here described, giving particular emphasis to define a class of compatible axially symmetric solutions, which fairly well describes the motion in cylindrical geometries in the field of $f(R)$, in two different classes of coordinates. We demonstrate that our approach is general and may be applied for several cases of interest. We also show that our treatment is compatible with the standard approach of general relativity, evaluating the motion of a freely falling particle in the context of our metric.

## 5.1 Introduction

Alternative theories of gravity pose the problem to recover or extend the well-established results of General Relativity (GR) as the initial value problem, the stability of solutions and, in particular, the issue of finding out new solutions [1]. As it is well known, beside cosmological solutions, spherically and axially symmetric solutions play a fundamental role in several astrophysical problems ranging from black holes to active galactic nuclei. Alternative gravities, to be consistent with results of GR,

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should comprise solutions like Schwarzschild and Kerr ones but present, in general, new solutions that could be physically interesting. Due to this reason, methods to find out exact and approximate solutions are particularly relevant in order to check if observations can be framed in Extended Theories of Gravity [2].

Recently, the interest in spherically and axially symmetric solutions of $f(R)$-gravity is growing up [3–6].

In this paper, we want to seek for a general method to find out axially symmetric solutions by performing a complex coordinate transformation. Newman and Janis showed that it is possible to obtain an axially symmetric solution (like the Kerr metric) by making an elementary complex transformation on the Schwarzschild solution [7]. This same method has been used to obtain a new stationary and axially symmetric solution known as the Kerr-Newman metric [8]. The Kerr-Newman space-time is associated to the exterior geometry of a rotating massive and charged black-hole. For a review on the Newman-Janis method to obtain both the Kerr and Kerr-Newman metrics see [9].

By means of very elegant mathematical arguments, Schiffer et al. [10] have given a rigorous proof to show how the Kerr metric can be derived starting from a complex transformation on the Schwarzschild solution. We will not go into the details of this demonstration, but point out that the proof relies on two main assumptions. The first is that the metric belongs to the same algebraic class of the Kerr-Newman solution, namely the Kerr-Schild class [11]. The second assumption is that the metric corresponds to an empty solution of the Einstein field equations. Gürses and Gürsey, in 1975 [12], showed that if a metric can be written in the Kerr-Schild form, then a complex transformation "is allowed in General Relativity." In this paper, we will show that such a transformation can be extended to $f(R)$-gravity.

The paper is structured as follows. In Sect. 5.2, we describe the method and we highlight its fundamental properties. To do so, we consider the general treatment and we specialize it to the case of pure spherically symmetric solutions. We therefore obtain the corresponding modifications to the standard Kerr metric in the context of $f(R)$ gravity and we describe some dynamical properties of this solution, by means of circular orbits in the framework of the Hamiltonian formalism. We therefore demonstrate that our strategy is general and may be extended to the case of fourth order gravities without stability problems. In Sect. 5.3, we summarize our results and we propose possible perspectives of our method.

## 5.2 From Spherical Symmetry to Axially Symmetric Solutions in $f(R)$ Gravity

In the framework of $f(R)$ gravity, the action takes the simple form

$$S = \int d^4x\sqrt{-g}\Big[f(R) + \mathscr{X}\mathscr{L}_m\Big].$$

By varying it, in terms of the metric $g_{\mu\nu}$, one argues the corresponding field equations:

$$f'(R)R_{\mu\nu} - \frac{1}{2}f(R)g_{\mu\nu} - f'(R)_{;\mu\nu} + g_{\mu\nu}\Box f'(R) = \mathscr{X}T_{\mu\nu}\,,$$
$$3\Box f'(R) + f'(R)R - 2f(R) = \mathscr{X}T\,, \tag{5.1}$$

where $T_{\mu\nu}$ represents the standard energy-momentum tensor for dust-like matter, which can be expressed in the form: $T_{\mu\nu} = \frac{-2}{\sqrt{-g}}\frac{\delta(\sqrt{-g}\mathscr{L}_m)}{\delta g^{\mu\nu}}$. The constant $\mathscr{X}$ contains the gravitational constant $G$, since $\mathscr{X} = \frac{8\pi G}{c^4}$, while $g$ is the metric determinant.

Our formalism involves the use of spherically symmetric space-time as starting point. In fact, we set up our treatment by assuming the most general spherically symmetric space-time below:

$$\mathrm{d}s^2 = g_{tt}(t,r)\mathrm{d}t^2 - g_{rr}(t,r)\mathrm{d}r^2 - r^2\mathrm{d}\Omega\,, \tag{5.2}$$

in which $\mathrm{d}\Omega$ represents the solid angle. The basic demands consists in employing on it a transformation that maps (5.2), providing that the off-diagonal terms vanish. Hence, the spherically symmetric space-time may be obtained by assuming that (5.2) satisfies particular cosmic symmetries. Here, we consider the Noether symmetries and so, after several calculations, we can write down the simplest spherically symmetric space-time as:

$$\mathrm{d}s^2 = (\alpha + \beta r)\mathrm{d}t^2 - \frac{1}{2}\frac{\beta r}{\alpha + \beta r}\mathrm{d}r^2 - r^2\mathrm{d}\Omega\,, \tag{5.3}$$

where we assumed $\alpha$ as a combination of auxiliary constants, e.g. $\Sigma_0$ and $k$ and $\beta = k_1$ [4].

Here, we demonstrate how it is possible to get an axially symmetric solution adopting the Newman-Janis procedure, extending their treatment in the context of $f(R)$ gravities and going beyond the standard usage of using the Newman-Janis procedure in general relativity only. To this end, as we already stressed before, we employ the existence of Noether symmetries which make the $f(R)$ model consistent with the corresponding field equations. For our purposes, let us recast the spherically symmetric metric as $\mathrm{d}s^2 = e^{2\phi(r)}\mathrm{d}t^2 - e^{2\lambda(r)}\mathrm{d}r^2 - r^2\mathrm{d}\Omega$, with $g_{tt}(t,r) = e^{2\phi(r)}$ and $g_{rr}(t,r) = e^{2\lambda(r)}$. Hereafter, our convention is to refer to time-like components as $tt$ or 00, whereas space-like as $rr$ or $ii$, with $i$ running from $i = 0$–3.

Considering the suitable Eddington–Finkelstein coordinates, i.e. $(u, r, \theta, \phi)$, which represent a viable choice for our coordinate representation, after simple algebra, we definitively get $\mathrm{d}s^2 = e^{2\phi(r)}\mathrm{d}u^2 \pm 2e^{\lambda(r)+\phi(r)}\mathrm{d}u\mathrm{d}r - r^2\mathrm{d}\Omega$. Thus, the matrix associated to the metric is rewritable in terms of a null tetrad as:

$$g^{\mu\nu} = l^\mu n^\nu + l^\nu n^\mu - m^\mu\bar{m}^\nu - m^\nu\bar{m}^\mu\,, \tag{5.4}$$

where $l^\mu$, $n^\mu$, $m^\mu$ and $\bar{m}^\mu$ should satisfy

$$l_\mu l^\mu = m_\mu m^\mu = n_\mu n^\mu = 0\,, \tag{5.5}$$

$$l_\mu n^\mu = -m_\mu \bar{m}^\mu = 1\,, \tag{5.6}$$

$$l_\mu m^\mu = n_\mu m^\mu = 0\,, \tag{5.7}$$

where we assumed the bars as indication of the complex conjugation.

In our case, a generic space-time event becomes

$$x^\mu \rightarrow \tilde{x}^\mu = x^\mu + i y^\mu(x^\sigma)\,, \tag{5.8}$$

in which we notice that $y^\mu(x^\sigma)$ are functions of the real coordinates $x^\sigma$. Analogously, the null tetrad vectors $Z_a^\mu = (l^\mu, n^\mu, m^\mu, \bar{m}^\mu)$, with $a = 1, 2, 3, 4$, should satisfy

$$Z_a^\mu \rightarrow \tilde{Z}_a^\mu(\tilde{x}^\sigma, \bar{\tilde{x}}^\sigma) = Z_a^\rho \frac{\partial \tilde{x}^\mu}{\partial x^\rho}\,. \tag{5.9}$$

All this procedure provides a net effect which consists in generating a new metric. The component of such a space-time are real and depend upon complex variables. We have:

$$g_{\mu\nu} \rightarrow \tilde{g}_{\mu\nu} : \tilde{\mathbf{x}} \times \tilde{\mathbf{x}} \mapsto \mathbb{R}\,, \tag{5.10}$$

where we consider:

$$\tilde{Z}_a^\mu(\tilde{x}^\sigma, \bar{\tilde{x}}^\sigma)|_{\mathbf{x}=\tilde{\mathbf{x}}} = Z_a^\mu(x^\sigma)\,. \tag{5.11}$$

From the transformed null tetrad vectors, a new metric is therefore obtained. So, assuming the covariant form, we can list the corresponding metric components as:

$$\begin{aligned}
g_{00} &= e^{2\phi(\tilde{r},\theta)}\,,\\
g_{01} &= e^{\lambda(\tilde{r},\theta)+\phi(\tilde{r},\theta)}\,,\\
g_{03} &= a e^{\phi(\tilde{r},\theta)}[e^{\lambda(\tilde{r},\theta)} - e^{\phi(\tilde{r},\theta)}]\sin^2\theta\,,\\
g_{13} &= -a e^{\phi(\tilde{r},\theta)+\lambda(\tilde{r},\theta)}\sin^2\theta\,,\\
g_{22} &= -\Sigma^2\,,\\
g_{33} &= -[\Sigma^2 + a^2\sin^2\theta e^{\phi(\tilde{r},\theta)}(2e^{\lambda(\tilde{r},\theta)} - e^{\phi(\tilde{r},\theta)})]\sin^2\theta.
\end{aligned}$$

where we assumed that all the other components, i.e. the components that we did not report above, are zero.

This procedure is circumscribed to the use of the particular choice of coordinates. However, one can also perform the Newman-Janis algorithm on any static spherically symmetric solutions, by means of the more practically Boyer-Lindquist coordinates. So, evaluating the same steps performed above and the analogous strategy to get the tetrad null vectors in the case of axially symmetric space-time, we simply obtain:

$$
\begin{aligned}
\mathrm{d}s^2 = & \frac{r(\alpha+\beta r)+a^2\beta\cos^2\theta}{\Sigma}\mathrm{d}t^2 + 2\frac{a(-2\alpha r - 2\beta\Sigma^2+\sqrt{2\beta}\Sigma^{3/2})\sin^2\theta}{2\Sigma}\mathrm{d}t\mathrm{d}\phi + \\
& - \frac{\beta\Sigma^2}{2\alpha r+\beta(a^2+r^2+\Sigma^2)}\mathrm{d}r^2 \\
& - \Sigma^2\mathrm{d}\theta^2 - \left[\Sigma^2 - \frac{a^2(\alpha r+\beta\Sigma^2-\sqrt{2\beta}\Sigma^{3/2})\sin^2\theta}{\Sigma}\right]\sin^2\theta\mathrm{d}\phi^2
\end{aligned}
$$

As in standard general relativity, our treatment should be compatible with the motion of a freely falling particle. Hence, we can treat a physical example which accounts for a freely falling particle moving in our so-obtained metric. To do so, we make extensive use of the Hamiltonian formalism, which has the advantage not to show any sign ambiguity which may come from turning points in the orbits [13]. The reduced Hamiltonian, linearly reported in terms of momenta, is:

$$
H == \left[\frac{p_i g^{0i}}{g^{00}} + \left[\left(\frac{p_i g^{0i}}{g^{00}}\right)^2 - \frac{m^2+p_i p_j g^{ij}}{g^{00}}\right]^{1/2}\right], \tag{5.12}
$$

providing $H = -p_0$ and even satisfying the following motion equations:

$$
\frac{\mathrm{d}x^i}{\mathrm{d}t} = \frac{\partial H}{\partial p_i}, \quad \frac{\mathrm{d}p_i}{\mathrm{d}t} = -\frac{\partial H}{\partial x^i}, \tag{5.13}
$$

which permit to numerically obtain the requested orbits. In particular, in the equatorial plane, which corresponds to the case $\theta = \frac{\pi}{2}$, $\dot{\theta} = 0$, we conventionally employ $\alpha = 10$ and $\beta = 5$, without losing generality and we consider the dependence on $\phi$ and on the conjugate momentum $p_\phi$, which represents an integral of motion. As a consequence, we find out that the coupled equations for $\{r, \theta, \phi, p_r, p_\theta\}$ may be numerically integrated, giving compatible trajectories with respect to the ones inferred from the standard Kerr space-time. To better clarify this statement, we explicitly report below the geodesic equations:

$$
\frac{\mathrm{d}x^\mu}{\mathrm{d}\lambda} = \frac{\partial \mathscr{H}}{\partial p_\mu} = g^{\mu\nu}p_\nu = p^\mu, \tag{5.14}
$$

$$
\frac{\mathrm{d}p_\mu}{\mathrm{d}\lambda} = -\frac{\partial \mathscr{H}}{\partial x\mu} = -\frac{1}{2}\frac{\partial g^{\alpha\beta}}{\partial x^\mu}p_\alpha p_\beta = g^{\gamma\beta}\Gamma^\alpha_{\mu\gamma}p_\alpha p_\beta, \tag{5.15}
$$

In Fig. 5.1, the relative trajectories are sketched.

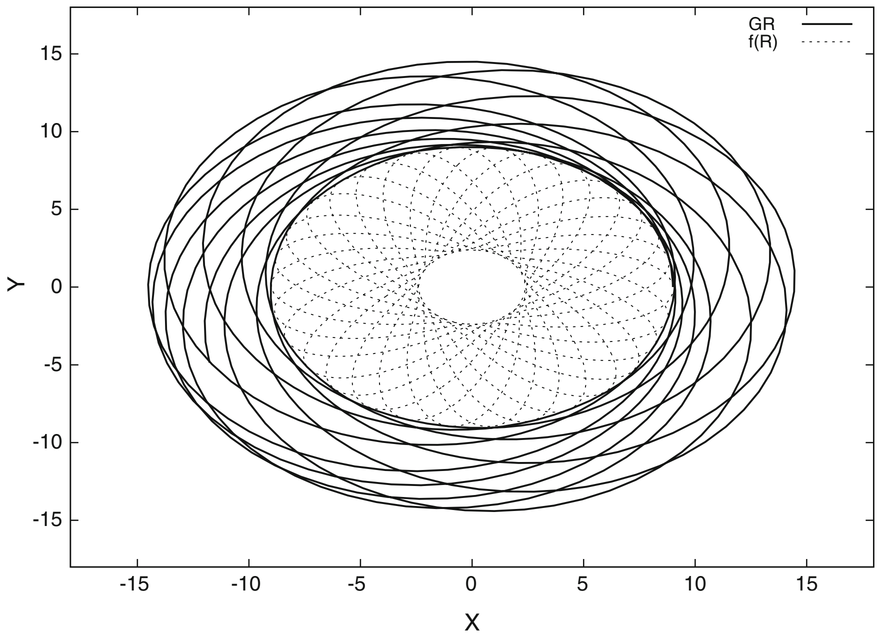


**Fig. 5.1** Example of massive particle equatorial trajectories in a Kerr and axially-simmetric $f(R)$ metric, obtained from the solution of the Hamilton-Jacboy equations (5.13). In both cases the BH spin is $a = 0.5$, and for $f(R)$ we employed $\alpha = 10$ and $\beta = 5$ for representative purposes. The test mass at the beginning has a pure tangential velocity component $\mathrm{d}\phi/\mathrm{d}t = 0.03$ and is placed at $9R_g$

## 5.3 Final Outlooks and Perspectives

In this paper, we considered the framework of $f(R)$ gravity to describe a technique able to get axially symmetric solutions from spherical ones. This treatment has been extensively described by Newman-Janis in a precise algorithm, which takes into account complex transformations. In particular, assuming a spherically symmetric expression for the space-time, we demonstrated that it is possible to extend the complex transformations in the context of $f(R)$ gravity. To do so, we evaluated the null tetrad associated to this method in two different classes of coordinates and we found out the corresponding axially symmetric metrics. In order to understand if the thus obtained space-time works well in the field of particle motion, we considered a freely falling particle and we showed that its motion is perfectly compatible with the expected standard Kerr metric, which corresponds to the simplest axially symmetric solution in general relativity. Further investigations will be carried forward in order to describe different symmetries by means of the Newman-Janis strategy. In particular, measuring possible corrections due to $f(R)$ around compact objects, e.g. evaluating possible discrepancies from the standard cases of accretion disks, one would constrain the $f(R)$ functions at astrophysical regimes. This would open new challenges for the problem of $f(R)$ reconstructions.

## References


1. S. Capozziello, M. De Laurentis, Phys. Rep. **509**, 167 (2011)
2. S. Capozziello, M. De Laurentis, Int. J. Geom. Methods Mod. Phys. **11**, 1460004 (2014)
3. S. CapozzielloA, A.Troisi Stabile, Class. Quant. Gravity **25**, 085004 (2008)
4. S. Capozziello, M. De Laurentis, A. Stabile, Class. Quantum Gravity **27**, 165008 (2010)
5. R. Farinelli, M. De Laurentis, S. Capozziello, S.D. Odintsov, MNRAS **440**(3), 2894 (2014)
6. S. Capozziello, M. De Laurentis, S.D. Odintsov, A. Stabile, Phys. Rev. D **83**, 064004 (2011)
7. E.T. Newman, A.I. Janis, J. Math. Phys. **6**, 915 (1965)
8. E.T. Newman, E. Couch, K. Chinnapared, A. Exton, A. Prakash, R. Torrence, J. Math. Phys. **6**, 918 (1965)
9. R. D'Inverno, *Introducing Einstein's Relativity* (Oxford University Press, New York, 1992)
10. M.M. Schiffer, R.J. Adler, J. Mark, C. Sheffield, J. Math. Phys. **14**, 52 (1973)
11. G.C. Debney, R.P. Kerr, A. Schild, J. Math. Phys. **10**, 1842 (1969)
12. M. Gürses, F. Gürsey, J. Math. Phys. **16**, 2385 (1975)
13. E. Berti (2014). arXiv:1410.4481 [gr-qc]