Chapter 14
A Quantum Cosmic Conjecture

R. Casadio and O. Micu

Abstract For a quantum mechanically Gaussian shaped, electrically charged, massive particle, we compute the Horizon Wave-function(s) in order to study (a) the existence of the inner Cauchy horizon of the corresponding Reissner–Nordström space-time when the charge-to-mass ratio $0 < \alpha < 1$ and (b) the survival of a naked singularity when the charge-to-mass ratio $\alpha > 1$. Our results suggest that any semi-classical instability one expects near the inner horizon may not occur in quantum black holes, with a mass around the Planck scale, and that no states with charge-to-mass ratio greater than a critical value (of the order of $\sqrt{2}$) should exist.

14.1 Introduction

There is a general consensus that black holes might play the same role in quantum gravity as the hydrogen atom does in the quantum theory of ordinary matter. In fact, a black hole realises the strongest non-perturbative effect that gravity can have, namely a total causal confinement. In order to investigate this feature in a quantum context, the Horizon Wave Function (HWF) formalism was proposed and developed in [1, 2], which is built on the quantised Einstein equation relating the size of the gravitational radius to the (quantum) state of matter.

The construction of the HWF starts from the spectral decomposition of the quantum mechanical state for a spherically symmetric matter source which is localised in space and static in time. By expressing the energy in terms of the gravitational

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(Schwarzschild) radius, the spectral decomposition then directly yields the (unnormalised) HWF. The normalised HWF supplies the probability for an observer to detect a gravitational radius of a certain size (areal radius) around the source in the quantum state that was used in the first place. The gravitational radius can then be interpreted as a horizon if the probability of finding the particle inside of it is reasonably high. According to this quantum picture, the horizon appears necessarily a fuzzy location in space, precisely for the same reason the position of the particle that sources the geometry is intrinsically uncertain. This formalism has been applied to several case studies [3, 4], yielding sensible results in agreement with (semi)classical expectations, and there is therefore hope that it will help our understanding of the quantum nature of black holes.

In this talk, we will in particular summarise the results obtained for charged sources in [5, 6].

14.2 Electrically Charged Spherical Source

We start from the Reissner–Nordström metric,

$$\mathrm{d}s^2 = -f\,\mathrm{d}t^2 + f^{-1}\,\mathrm{d}r^2 + r^2\left(\mathrm{d}\theta^2 + \sin^2\theta\,\mathrm{d}\phi^2\right) , \tag{14.1}$$

with $f = 1 - \frac{2\,\ell_{\mathrm{p}}\,M}{m_{\mathrm{p}}\,r} + \frac{Q^2}{r^2}$, where M and Q respectively represent the ADM mass and charge of the source. It is convenient to introduce the specific charge $\alpha = |Q|\,m_{\mathrm{p}}/\ell_{\mathrm{p}}\,M$. The case $\alpha = 0$ then reduces to the neutral Schwarzschild metric with one horizon of radius $R_{\mathrm{H}} = 2\,\ell_{\mathrm{p}}\,M/m_{\mathrm{p}}$. For $0 < \alpha \leq 1$, the space-time contains two horizons, namely

$$R_{\pm} = \ell_{\mathrm{p}}\,\frac{M}{m_{\mathrm{p}}}\left(1 \pm \sqrt{1-\alpha^2}\right) , \tag{14.2}$$

which overlap for the extremal case $\alpha = 1$. Finally, for $\alpha > 1$ no horizon exists and the central singularity is therefore "naked", or accessible to outer observers.

We next consider a spherically symmetric Gaussian source

$$\psi_{\mathrm{S}}(r) = \frac{e^{-\frac{r^2}{2\,\ell^2}}}{\ell^{3/2}\,\pi^{3/4}} , \tag{14.3}$$

whose width ℓ is assumed to be the minimum compatible with the Heisenberg uncertainty principle,

$$\ell = \lambda_m \simeq \ell_{\mathrm{p}}\,\frac{m_{\mathrm{p}}}{m} , \tag{14.4}$$

where λ_m is the Compton length of the particle of rest mass m [2]. The spectral decomposition of (14.3) is easily obtained from assuming the relativistic mass-shell relation in flat space, $M^2 = p^2 + m^2$, and by going to momentum space,

$$\psi_{\mathrm{S}}(p) = \frac{e^{-\frac{p^2}{2\Delta^2}}}{\Delta^{3/2}\,\pi^{3/4}} \ , \tag{14.5}$$

where $p^2 = \mathbf{p}\cdot\mathbf{p}$ is the square modulus of the spatial momentum, and the width $\Delta = m_{\mathrm{p}}\,\ell_{\mathrm{p}}/\ell \simeq m$.

14.2.1 Inner Horizon and Mass Inflation

For $0 < \alpha < 1$, one can write a HWF for each of the two horizons (14.2), namely [5]

$$\psi_{\mathrm{H}}(R_\pm) = \mathscr{N}_\pm\,\Theta\,(R_\pm - R_{\mathrm{min}\pm})\exp\left\{-\frac{m_{\mathrm{p}}^2\,R_\pm^2}{2\,\Delta^2\,\ell_{\mathrm{p}}^2\,(1\pm\sqrt{1-\alpha^2})^2}\right\} \ , \tag{14.6}$$

where the step function accounts for the minimum energy $M = m$, corresponding to $R_{\mathrm{min}\pm} = \ell_{\mathrm{p}}\,\frac{m}{m_{\mathrm{p}}}\left(1\pm\sqrt{1-\alpha^2}\right)$, and the normalisations $\mathscr{N}_\pm$ are fixed by using the Schrödinger scalar product.

The probability density that the particle lies inside its own gravitational radius of size $r = R_\pm$ can now be calculated starting from the wave-functions (14.6) as

$$\mathscr{P}_{<\pm}(r < R_\pm) = P_{\mathrm{S}}(r < R_\pm)\,\mathscr{P}_{\mathrm{H}}(R_\pm) \ , \tag{14.7}$$

where $P_{\mathrm{S}}(r < R_\pm) = 4\,\pi\,\int_0^{R_\pm}|\psi_{\mathrm{S}}(r)|^2\,r^2\,\mathrm{d}r$ is the probability that the particle is inside the sphere $r = R_\pm$, and $\mathscr{P}_{\mathrm{H}}(R_\pm) = 4\,\pi\,R_\pm^2\,|\psi_{\mathrm{H}}(R_\pm)|^2$ is the probability density that the sphere $r = R_\pm$ is the gravitational radius. Finally, one can integrate (14.7) over all possible values of R_+ to find the probability that the particle is a BH, namely

$$P_{\mathrm{BH}+} = \int_{R_{\mathrm{min}+}}^{\infty}\mathscr{P}_{<+}(r < R_+)\,\mathrm{d}R_+ \ . \tag{14.8}$$

The analogous quantity for R_-,

$$P_{\mathrm{BH}-} = \int_{R_{\mathrm{min}-}}^{\infty}\mathscr{P}_{<-}(r < R_-)\,\mathrm{d}R_- \ , \tag{14.9}$$

will instead be viewed as the probability that the particle lies further inside its inner horizon. It is obvious that $P_{\mathrm{BH}-} < P_{\mathrm{BH}+}$, and that only when $P_{\mathrm{BH}-} \simeq 1$ we can say that both horizons are physically realised at $\langle\,\hat{R}_-\,\rangle$ and $\langle\,\hat{R}_+\,\rangle$.

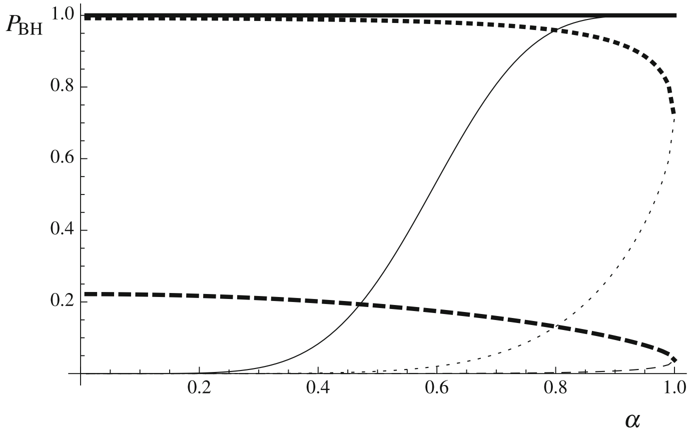

Fig. 14.1 Probabilities $P_{\rm BH+}$ (thick lines) and $P_{\rm BH-}$ (thin lines) as functions of α for $m = 2\,m_{\rm p}$ (continuous line), $m = m_{\rm p}$ (dotted line) and $m = 0.5\,m_{\rm p}$ (dashed line)

The plot in Fig. 14.1 shows the probabilities $P_{\rm BH\pm}$ as functions of α for values of the particle mass above, equal to and below the Planck mass. One can notice that $P_{\rm BH+}$ stays very close to one for masses larger than the Planck scale, whereas, for $m < m_{\rm p}$, it clearly decreases as the specific charge increases to one. For instance, if $m = 0.5\,m_{\rm p}$ ($\ell = 2\,\ell_{\rm p}$), $P_{\rm BH+} \simeq 0.2$ for a sizable range of α, and it only decreases below 0.1 when $\alpha \to 1$ and the source is nearly maximally charged. The situation is however different for the inner horizon. The probability $P_{\rm BH-}$ starts out negligible for small values of the charge-to-mass ratio and increases with α – the larger m, the smaller the value of α for which the probability becomes significant. There is a considerable range of α for which the probability for the inner horizon to exist is approximately zero, while $P_{\rm BH+} \simeq 1$ and the object is a black hole. Figure 14.2 shows the probabilities $P_{\rm BH\pm}$ as functions of the mass m for $\alpha = 0.3$, 0.8 and 1. It is clear that for smaller values of α, the probability $P_{\rm BH+}$ starts to increase from zero to one at smaller values of m, but the opposite occurs for $P_{\rm BH-}$. For $\alpha = 0.3$, it is only around a particle mass $m \simeq 6\,m_{\rm p}$ that $P_{\rm BH-} \simeq 1$, while $P_{\rm BH+} \simeq 1$ already around $m_{\rm p}$. This means that for $m_{\rm p} \le m \le 6\,m_{\rm p}$, the probability $P_{\rm BH+} \simeq 1$ while $P_{\rm BH-} \ll 1$. This mass range broadens up even more for smaller values of α, but decreases to zero in the maximally charged limit $\alpha = 1$. Our main finding is therefore that there exists a considerable parameter space for m (around the Planck scale) and $\alpha < 1$ in which

$$P_{\rm BH+} \simeq 1 \quad \text{and} \quad P_{\rm BH-} \simeq 0\,. \tag{14.10}$$

In this range the particle is (most likely) a black hole, but the inner horizon at $r = \langle\,\hat{R}_-\,\rangle$ is suppressed by quantum mechanical fluctuations. This conclusion is important in light of the instability usually referred to as the "mass inflation" [7].

Let us conclude this part by deriving a generalised uncertainty relation analogous to the neutral case [1]. We first note that the expectation value

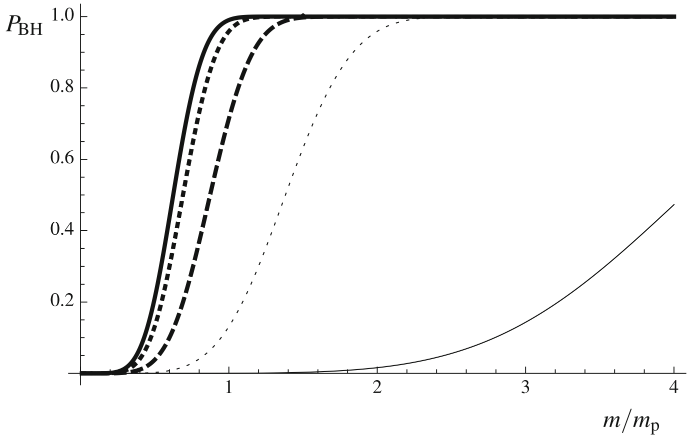

Fig. 14.2 Probabilities $P_{\mathrm{BH}+}$ (thick lines) and $P_{\mathrm{BH}-}$ (thin lines) as functions of the mass for $\alpha = 0.3$ (continuous line), $\alpha = 0.8$ (dotted line) and $\alpha = 1$ (dashed line)

$$\langle \hat{R}_+ \rangle = 4\pi \int_{R_{\mathrm{min}+}}^{\infty} |\psi_{\mathrm{H}}(R_+)|^2 R_+^3 \, \mathrm{d}R_+ = R_+(\bar{M}) , \tag{14.11}$$

reproduces exactly the classical expression of R_+ in (14.2) for $\ell = \lambda_m \sim m^{-1}$ and $\bar{M} = 4\,m/[2 + e\sqrt{\pi}\,\mathrm{erfc}(1)] \simeq 1.45\,m$ (in agreement with the wave-function ψ_{S} containing energy contributions from momenta $p > 0$). From $\langle \hat{R}_+^2 \rangle \simeq R_+^2(\bar{M})$ one can then calculate the uncertainty

$$\Delta R_+ = \sqrt{\langle \hat{R}_+^2 \rangle - \langle \hat{R}_+ \rangle^2} \simeq R_+ \sim m , \tag{14.12}$$

which, like in the neutral Schwarzschild case, grows linearly with the mass m of the source.[1] If we now combine the horizon uncertainty (14.12) with the usual Heisenberg uncertainty in the radial size of the source, $\Delta r^2 \simeq \ell^2$, we finally obtain a total uncertainty

$$\Delta r = \sqrt{\langle \Delta r^2 \rangle} + \gamma \sqrt{\langle \Delta R_+^2 \rangle} \simeq \ell_{\mathrm{p}} \frac{m_{\mathrm{p}}}{\Delta p} + \gamma\, \ell_{\mathrm{p}} \frac{\Delta p}{m_{\mathrm{p}}} , \tag{14.13}$$

where γ is a coefficient of order one. We can therefore conclude that the outer horizon behaves qualitatively like the neutral Schwarzschild radius.

[1] Such objects would remain quantum mechanical even in astrophysical regimes, where we expect the horizon has a sharp location. This result therefore supports alternative models of black holes as extended quantum objects, like the ones in [4, 8].

14.2.2 Naked Singularity and Cosmic Censorship

We next move on to overcharged sources, represented by $\alpha > 1$ [6]. The Cosmic Censorship Conjecture [9] was formulated in order to forbid the existence of such naked singularities in the classical theory of gravity, so it is interesting to investigate whether quantum physics leads to any predictions therein. Our guiding principle will be to assume that the quantum states for $\alpha > 1$ can be obtained by extending continuously the HWF obtained for $\alpha < 1$. Of course, this is by no means a compelling choice, but we hope that it leads to consistent predictions for charges not too much larger than the classical limiting value of $\alpha = 1$.

The classical expressions of $R_\pm$ are complex for $\alpha > 1$, hence we lift only the real parts of $R_\pm$ to quantum observables. The modulus squared of the two HWFs (14.6), for $R_\pm > R_{\min\pm}$, then merge into

$$|\psi_{\rm H}(R)|^2 = \mathcal{N}^2 \exp\left\{-\frac{2-\alpha^2}{\alpha^4}\,\frac{m_{\rm p}^2 R^2}{\Delta^2 \ell_{\rm p}^2}\right\} , \tag{14.14}$$

where R has replaced $R_\pm$. This HWF is still normalizable if R is real and

$$1 < \alpha^2 < 2 . \tag{14.15}$$

We deduce that no normalisable quantum state with $\alpha^2 > 2$ is allowed. We must also consider what happens to the Heaviside function in (14.6) in the superextremal regime. First we note that the real parts of the minimum values of R_+ and R_- are again the same for $\alpha > 1$, and our continuity principle requires

$$R \geq R_{\min} = {\rm Re}\left[\ell_{\rm p}\,\frac{m}{m_{\rm p}}\left(1 \pm \sqrt{1-\alpha^2}\right)\right] = \ell_{\rm p}\,\frac{m}{m_{\rm p}} . \tag{14.16}$$

The expectation value for $\hat{R}$ then matches exactly the corresponding expressions for $\alpha < 1$ [5],

$$\lim_{\alpha\searrow 1} \langle \hat{R} \rangle = \frac{4\,\ell_{\rm p}^2/\ell}{2 + e\sqrt{\pi}\,{\rm erfc}(1)} = \lim_{\alpha\nearrow 1} \langle \hat{R}_\pm \rangle , \tag{14.17}$$

like the uncertainty $\Delta R(\ell, \alpha > 1)$ matches the uncertainties $\Delta R_\pm(\ell, \alpha < 1)$ at $\alpha = 1$. We remark that, for $\alpha = 1$, the width $\ell > \langle \hat{R} \rangle$ for $m < \sqrt{2 + e\sqrt{\pi}\,{\rm erfc}(1)}\, m_{\rm p}/2 \simeq 0.8\, m_{\rm p}$, and quantum fluctuations of the source will dominate for masses $m \ll m_{\rm p}$ (like in the neutral case [1, 2]). It is important to further note that the ratio

$$\frac{\langle \hat{R} \rangle}{\ell} \simeq \frac{2^{5/4}\,\ell_{\rm p}^2}{\sqrt{\pi\left(\sqrt{2}-\alpha\right)}} \tag{14.18}$$

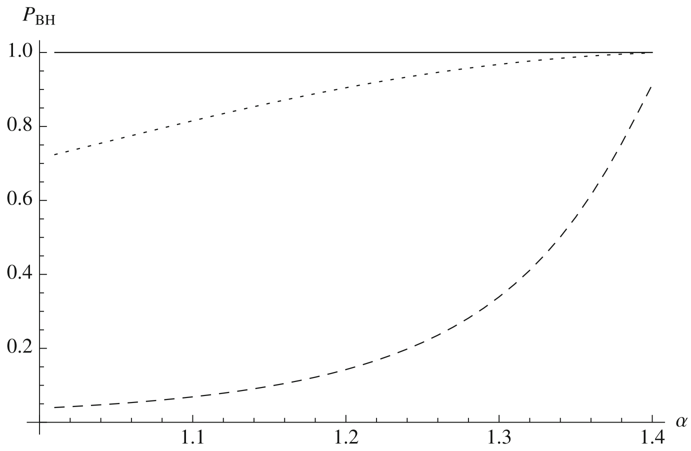

Fig. 14.3 P_{BH} as a function of α for $m = 2\, m_p$ (solid line), $m = m_p$ (dotted line) and $m = 0.5\, m_p$ (dashed line). Cases with $m \gg m_p$ are not plotted since they behave the same as $m = 2\, m_p$, i.e. an object with $1 < \alpha^2 < 2$ must be a BH

blows up for all values of the mass $m \sim 1/\ell$ in the limit $\alpha^2 \to 2$, and so does its uncertainty, since $\Delta R \simeq \sqrt{3\pi/8 - 1}\, \langle \hat{R} \rangle \simeq 0.4 \langle \hat{R} \rangle$.

Using (14.8), one can calculate the probability P_{BH} that the particle is a black hole for α in the range (14.15). Figure 14.3 shows that, for a mass above the Planck scale, $P_{BH} \simeq 1$ throughout the entire range of α (extending the similar result for $\alpha < 1$). Moreover, even for m significantly less than m_p, P_{BH} approaches one in the limit $\alpha^2 \to 2$. We recall that $P_{BH} \ll 1$ for small m is related to $\ell \gg \langle \hat{R} \rangle$, and quantum fluctuations in the source's size dominate. On the other end, since both $\langle \hat{R} \rangle$ and ΔR blow up at $\alpha^2 = 2$, the superextremal configurations with a significant probability of being black holes display strong quantum fluctuations in the horizon's size.

14.3 Conclusions and Outlook

We extended the HWF formalism from neutral to electrically charged sources and considered separately the analogues of the classical Reissner–Nordström space-times with two horizons or a naked singularity. In the former case, with $0 < \alpha \leq 1$, we have shown that quantum fluctuations can cover the inner horizon, thus helping to avoid the instability known as mass inflation, at least for black holes not much heavier than the Planck scale. In the latter, we have found that quantum black holes extend into the range of classical naked singularities $\alpha > 1$, but a quantum obstruction occurs at $\alpha^2 = 2$.

Future developments involve extending the HWF to spinning sources and black hole formation by colliding particles with a non-vanishing impact parameter.

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