Chapter 13
Black Hole Entropy in the Presence of Chern–Simons Term and Holography

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Abstract We propose a manifestly covariant formulation of the differential Noether charge for higher-dimensional Chern–Simons terms. With our differential Noether charge, we provide a covariant proof of the black hole entropy formula for gravitational and mixed $U(1)$-gravitational Chern–Simons terms. By evaluating the charge on the rotating charged AdS black hole background constructed by the fluid/gravity derivative expansion, we show that the Chern–Simons contribution to black hole entropy agrees with the anomaly-induced entropy current in the dual conformal field theory at the leading order of the derivative expansion.

13.1 Chern–Simons Term and Anomaly Polynomial

The setup we are interested in here is $(2n+1)$-dimensional Einstein–Maxwell–Chern–Simons theory with a negative cosmological constant. We consider gravitational and/or mixed $U(1)$-gravitational Chern–Simons terms which, as will be seen later, contribute nontrivially to black hole entropy. For example, the five-dimensional mixed $U(1)$-gravitational Chern–Simons term and seven-dimensional (double trace) gravitational Chern–Simons term are given respectively by

$$A \wedge \mathrm{tr}\,(R \wedge R)\ , \qquad \mathrm{tr}\left(\Gamma \wedge R - \frac{1}{3}\Gamma \wedge \Gamma \wedge \Gamma\right) \wedge \mathrm{tr}\,(R \wedge R)\ . \tag{13.1}$$

Here $A = A_a \mathrm{d}x^a$ is the $U(1)$ gauge potential one-form, $\Gamma^a{}_b = \Gamma^a{}_{bc}\mathrm{d}x^c$ is the Christoffel connection one-form and $R^a{}_b = (1/2)R^a{}_{bcd}\mathrm{d}x^c \wedge \mathrm{d}x^d$ is the Riemann curvature two-form (A_a: $U(1)$ gauge potential, $\Gamma^a{}_{bc}$: Christoffel connection, $R^a{}_{bcd}$: Riemann curvature). In general, the Chern–Simons term I_{CS} depends on covariant quantities, F and $R^a{}_b$, as well as non-covariant quantities, A and $\Gamma^a{}_b$ ($F = \mathrm{d}A$:

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P. Nicolini et al. (eds.), *2nd Karl Schwarzschild Meeting on Gravitational Physics*, Springer Proceedings in Physics 208,
https://doi.org/10.1007/978-3-319-94256-8_13

field strength two-form for A). In addition to this, the Chern–Simons term has the following three important properties:

1. *Non-Covariance of Chern–Simons Term*
Under gauge transformation/diffeomorphism labeled by $\chi = \{\Lambda, \xi\}$, the Chern–Simons term transforms covariantly up to a total derivative term:
$$\delta_\chi I_{CS} = \mathcal{L}_\xi I_{CS} + \mathrm{d}(\ldots)\,, \tag{13.2}$$
where $\mathcal{L}_\xi$ is the Lie derivative generated by ξ.
2. *Anomaly Polynomial*
For each Chern–Simons term, one can define a formal $(2n+2)$-form called the anomaly polynomial. The anomaly polynomial is defined by taking the derivative of I_{CS}: $P_{\mathrm{anom}} = \mathrm{d}I_{CS}$. The anomaly polynomial depends only on covariant quantities, F and $R^a{}_b$, and thus is covariant. For example, the anomaly polynomials corresponding to the Chern–Simons terms in (13.1) are
$$F \wedge \mathrm{tr}(R \wedge R)\,, \qquad \mathrm{tr}(R \wedge R) \wedge \mathrm{tr}(R \wedge R)\,. \tag{13.3}$$
3. *Covariance of Equation of Motion*
Although the Chern–Simons term is not covariant under gauge transformation/diffeomorphism, Chern–Simons contribution to the equation of motion is written in terms of (a covariant derivative of) a derivative of the anomaly polynomial. Therefore, the equation of motion is covariant even in the presence of the Chern–Simons term in the Lagrangian.

13.2 Noether Charge Formalism for Chern–Simons Term

In general, a black hole entropy formula depends on gravitational theory one is considering. Entropy of a black hole in Einstein gravity is given by the well-known Bekenstein-Hawking formula. For a general gravitational theory with a covariant Lagrangian L_{cov}, the Wald formalism provides a covariant way to derive the corresponding black hole entropy formula [1–4]. In this formalism, variation of black hole entropy is given as a differential Noether charge evaluated at the bifurcation horizon of the black hole. More precisely, for a stationary rotating black hole with the Killing vector $\xi = (\partial_t + \Omega_H \partial_\phi)/T_H$ (Here ∂_t and ∂_ϕ respectively are the Killing vectors corresponding to time translation and a rotation, Ω_H is the associated angular velocity at the bifurcation horizon and T_H is the Hawking temperature), variation of black hole entropy is given by
$$\delta S_{\mathrm{Wald}} = \int_{Bif} \delta Q_\chi\,. \tag{13.4}$$

Here the differential Noether charge δQ_χ associated with χ is defined by

$$\mathrm{d}\delta Q_\chi = -\delta\delta_\chi \Omega_{PS} - i_\xi \delta E - \delta(\star N_\chi)\,. \tag{13.5}$$

($\delta\delta\Omega_{PS}$: pre-symplectic current, δE: equation of motion, N_χ: Noether current associated with χ, i_ξ: interior product with ξ, $\star$: Hodge dual. For more details of the definitions, please refer to [5], for example.) From (13.4), the celebrated Wald formula is obtained:

$$S_{\text{Wald}} = 2\pi \int_{\text{Bif}} \sqrt{h} \frac{\delta L_{\text{cov}}}{\delta R_{abcd}} \varepsilon_{ab}\varepsilon_{cd}\,, \tag{13.6}$$

where $\sqrt{h}$ and ε_{ab} respectively are the area element and binormal at the bifurcation horizon. We note that the right hand side of equation (13.6) evaluated at a general horizon slice also gives the correct black hole entropy [4].

The original Wald formalism is not directly applicable to the Chern–Simons term because of its non-covariant transformation property under gauge transformation/diffeomorphism. In [6], this point is taken into account and a generalization of the Wald formalism to the Chern–Simons term is proposed. By using this formalism, the black hole entropy formula for the Chern–Simons term is derived (See also [7, 8]):

$$S_{CS} = (4\pi) \int_{\text{Bif}} \sum_{k=1}^{\infty} (2k) \Gamma_N \wedge R_N^{2k-2} \wedge \frac{\partial P_{\text{anom}}}{\partial \mathrm{tr}(R^{2k})}\,. \tag{13.7}$$

Here $\Gamma_N = (-1/2)\varepsilon^a{}_b \Gamma^b{}_a$ is the normal bundle connection one-form at the bifurcation horizon and $R_N = d\Gamma_N$ is the curvature two-form associated with it. However, [9] pointed out recently that, for gravitational and mixed gravitational-$U(1)$ Chern–Simons terms in more than three-dimensions, this black hole entropy formula can be obtained from the generalization of the Wald formalism in [6] only when gauge and coordinates are chosen appropriately. This implies that Tachikawa's formalism seems to break covariance somewhere.

In [5], we pointed out that the formalism breaks covariance at the level of the pre-symplectic current. This non-covariance is then inherited to the differential Noether charge, ending up with the subtle result pointed out in [9]. To overcome this subtlety and to realize a manifestly covariant formulation of the differential Noether charge for the Chern–Simons term, we provided a covariant way to construct a pre-symplectic current for the Chern–Simons term. First of all, let us recall the defining equation of the pre-symplectic current $\delta\delta\Omega_{PS}$ based on equation of motion δE:

$$\mathrm{d}(\delta_1\delta_2\Omega_{PS}) = \delta_1\delta_2 E - \delta_2\delta_1 E\,. \tag{13.8}$$

We note that, by definition, the pre-symplectic current has an ambiguity to add total derivative terms. One of the key points of the Wald formalism is that, by starting with the Lagrangian description and variation of the Lagrangian $\delta L_{\text{cov}} = \delta E + \mathrm{d}\delta\Theta$, this defining equation can be rewritten as $\mathrm{d}(\delta_1\delta_2\Omega_{PS}) = -\mathrm{d}(\delta_1\delta_2\Theta - \delta_2\delta_1\Theta)$. Then the total derivative ambiguity of the pre-symplectic current can be fixed as $\Omega_{PS}^{\text{Wald}} = -\delta_1\delta_2\Theta + \delta_2\delta_1\Theta$. This is the definition of the pre-symplectic current for the Wald formalism. The same definition of the pre-symplectic current is used in [6]. For the Chern–Simons terms, however, the pre-symplectic current defined in this way breaks covariance. Essentially, this breakdown originates in the non-covariance of the boundary term $\delta\Theta$ for the Chern–Simons term [5].

For the Chern–Simons term, in fact, we can construct a covariant pre-symplectic current in a much easier way. As mentioned above, the equation of motion for the Chern–Simons term is written in terms of (a covariant derivative of) a derivative of the anomaly polynomial. Thus, by starting from the equation of motion and the defining equation (13.8), one can directly carry out integration by part to rewrite the right hand side of equation (13.8) into a total derivative form. It is straightforward to keep track of covariance in the intermediate steps because of covariance of the equation of motion. We can then construct a manifestly covariant pre-symplectic current. Once this covariant pre-symplectic current is constructed, we can follow the same step as [6] to obtain a manifestly covariant differential Noether charge. By evaluating this manifestly covariant differential Noether charge for the Killing vector $\xi = (\partial_t + \Omega_H \partial_\phi)/T_H$ at the bifurcation horizon, the entropy formula (13.7) is proved covariantly [5].

13.3 Application: Replacement Rule from Fluid/Gravity Correspondence

One of the motivations to investigate higher-dimensional Chern–Simons terms is that these terms are needed to setup AdS/CFT duality for even-dimensional CFTs with some class of quantum anomalies. More precisely, to induce gravitational/mixed anomaly on the CFT side, the gravitational/mixed Chern–Simons term needs to be added on the dual gravity side. With deep connections to AdS/CFT, even-dimensional CFT at finite temperature with these anomalies has been investigated intensively in the hydrodynamic limit. As a consequence of these continuous anomalies in the underlying microscopic theory, hydrodynamic currents and stress tensor acquire extra terms called anomaly-induced transports [10–12]. Recently, it is proved by using an argument based on Euclidean thermal partition function in the hydrodynamic limit that the leading order anomaly-induced transport coefficients are completely determined from the corresponding anomaly polynomial through what-is-called the replacement rule [13–18]. In particular, the anomaly-induced entropy current is given by

$$s^{\mu}_{\text{anom}} = -\frac{\partial \mathscr{F}}{\partial T} V^{\mu} + \cdots , \quad \text{with} \quad \mathscr{F} = P_{\text{anom}}[F \to \mu,\ \mathrm{tr}(R^{2k}) = 2(2\pi T)^{2k}] , \tag{13.9}$$

where $V^{\mu} = \varepsilon^{\mu\alpha_1\alpha_2\ldots\alpha_{2n-1}} u_{\alpha_1}\omega_{\alpha_2\alpha_3}\omega_{\alpha_4\alpha_5}\ldots\omega_{\alpha_{2n-2}\alpha_{2n-1}}$ (u_α: fluid velocity, $\omega_{\alpha\beta}$: vorticity).

Now a natural question is whether we can reproduce the replacement rule for the anomaly-induced transport systematically from the dual gravity side. (Please refer to [19–23] for some analysis of anomaly-induced transports in the 4d CFT with mixed anomaly and its 5d gravity dual.) In [24], we started with Einstein-Maxwell-Chern–Simons theory with a negative cosmological constant as the simplest setup and carried out systematic analysis of anomaly-induced entropy current in even-dimensional CFTs from the dual gravity side. For Einstein-Maxwell theory with a negative cosmological constant, rotating charged AdS black hole can be constructed in the fluid/gravity derivative expansion and back reaction from the Chern–Simons terms is also taken into account [25]. By evaluating the differential Noether charge $\delta Q_\chi = \delta Q_\chi^{EM} + \delta Q_\chi^{CS}$ for the full theory at the horizon $r = r_H$ of this background, we have shown that the replacement rule for the anomaly-induced entropy current can be reproduced from the dual gravity side for any gravitational/mixed Chern–Simons term in any odd-dimensions [24]:

$$\int_{r=r_H} \delta Q_\chi|_{V^\mu-\text{linear}} = \int_{r=r_H} \delta Q_\chi^{EM}|_{V^\mu-\text{linear}} + \int_{r=r_H} \delta Q_\chi^{CS}|_{V^\mu-\text{linear}}\,, \tag{13.10}$$
$$\text{with} \qquad \int_H \delta Q_\chi^{EM}|_{V^\mu-\text{linear}} = 0\,, \quad \int_{r=r_H} \delta Q_\chi^{CS}|_{V^\mu-\text{linear}} = -\delta\int \frac{\partial\mathscr{F}}{\partial T} V_\mu\,.$$

Our result has a significant meaning for black hole microstate counting. Traditionally, black hole microstate counting based on AdS/CFT relies heavily on the Cardy(-like) formula in 2d CFT. One therefore needs to stick to some setups to which AdS_3/CFT_2 (or something similar) is applicable. Our result shows that, even for non-extremal black holes approaching asymptotically to higher-dimensional AdS, so far as we restrict ourselves to the Chern–Simons contribution, black hole entropy can be reproduced from the dual CFT side through the replacement rule. This is a new direction for black hole microstate counting business. We expect that our result is meaningful for the investigation of Cardy-like formulas in higher-dimensional CFTs and their application to black hole microstate counting.

References

1. J. Lee, R.M. Wald, Local symmetries and constraints. J. Math. Phys. **31**, 725 (1990)
2. R.M. Wald, Black hole entropy is the Noether charge, Phys. Rev. D **48** (8), R3427 (1993)
3. V. Iyer, R.M. Wald, Some properties of Noether charge and a proposal for dynamical black hole entropy. Phys. Rev. D **50**, 846 (1994)
4. T. Jacobson, G. Kang, R.C. Myers, On black hole entropy. Phys. Rev. D **49**, 6587 (1994)
5. T. Azeyanagi, R. Loganayagam, G.S. Ng, M.J. Rodriguez, Covariant Noether charge for higher dimensional Chern-Simons terms. JHEP **1505**, 041 (2015)
6. Y. Tachikawa, Black hole entropy in the presence of Chern–Simons terms. Class. Quantum Gravity **24**, 737 (2007)

7. P. Kraus, F. Larsen, Holographic gravitational anomalies. JHEP **0601**, 022 (2006)
8. S.N. Solodukhin, Holography with gravitational Chern–Simons. Phys. Rev. D **74**, 024015 (2006)
9. L. Bonora, M. Cvitan, P.D. Prester, S. Pallua, I. Smolic, Gravitational Chern–Simons terms and black hole entropy. Global aspects, JHEP **1210**, 077 (2012)
10. S. Bhattacharyya, S. Lahiri, R. Loganayagam, S. Minwalla, Large rotating AdS black holes from fluid mechanics. JHEP **0809**, 054 (2008)
11. J. Erdmenger, M. Haack, M. Kaminski, A. Yarom, Fluid dynamics of R-charged black holes. JHEP **01**, 055 (2009)
12. D.T. Son, P. Surowka, Hydrodynamics with triangle anomalies. Phys. Rev. Lett. **103**, 191601 (2009)
13. R. Loganayagam, Anomaly induced transport in arbitrary dimensions, arXiv:1106.0277
14. R. Loganayagam, P. Surowka, Anomaly/transport in an ideal Weyl gas. JHEP **1204**, 097 (2012)
15. K. Jensen, R. Loganayagam, A. Yarom, Thermodynamics, gravitational anomalies and cones, arXiv:1207.5824 [hep-th]
16. R. Loganayagam, Anomalies and the Helicity of the thermal state, arXiv:1211.3850
17. K. Jensen, R. Loganayagam, A. Yarom, Anomaly inflow and thermal equilibrium, arXiv:1310.7024
18. K. Jensen, R. Loganayagam, A. Yarom, Chern–Simons terms from thermal circles and anomalies. JHEP **1405**, 110 (2014)
19. K. Landsteiner, E. Megias, F. Pena-Benitez, Gravitation al anomaly and transport. Phys. Rev. Lett. **107**, 021601 (2011)
20. K. Landsteiner, E. Megias, L. Melgar, F. Pena-Benitez, Holographic gravitational anomaly and chiral vortical effect. JHEP **1109**, 121 (2011)
21. E. Megias, F. Pena-Benitez, Fluid/gravity correspondence, second order transport and gravitational anomaly, arXiv:1307.7592
22. D.E. Kharzeev, H.-U. Yee, Anomalies and time reversal invariance in relativistic hydrodynamics: the second order and higher dimensional formulations, Phys. Rev. **D84** (2011)
23. S. Chapman, Y. Neiman, Y. Oz, Fluid/gravity correspondence, local wald entropy current and gravitational anomaly. JHEP **1207**, 128 (2012)
24. T. Azeyanagi, R. Loganayagam, G.S. Ng, Anomalies, Chern–Simons terms and black hole entropy. JHEP **1509**, 121 (2015)
25. T. Azeyanagi, R. Loganayagam, G.S. Ng, M.J. Rodriguez, Holographic thermal helicity. JHEP **1408**, 040 (2014)