The Total Ancestor Potential in Singularity Theory

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Contents

Abstract This is an extended version of a long lecture given on the workshop "Pedagogical workshop on B-model" held at the University of Michigan, Ann Arbor on 3–7 March 2014. The main goal is to prove that the total ancestor potential in singularity theory depends analytically on the deformation parameters.

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1 Introduction

Motivated by quantum cohomology and Gromov–Witten theory Dubrovin invented the notion of a Frobenius manifold [4]. Furthermore, he noticed that the Frobenius manifolds satisfying certain semi-simplicity condition play a key role in the theory of integrable hierarchies. This lead to the remarkable discovery that every semi-simple Frobenius manifold gives rise to an integrable hierarchy [5]. Partially motivated by Dubrovin's work, Givental discovered a certain higher-genus reconstruction formalism in Gromov–Witten (GW) theory which lead him to introduce

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the notion of a total ancestor potential in the abstract settings of an arbitrary semi-simple Frobenius manifold S (see [8]). The potential is defined for each semi-simple point $s \in S$ and it has the form

$$\widetilde{\mathcal{A}}_s(\hbar;\mathbf{t}) = \exp\Big(\sum_{g=0}^{\infty} \hbar^{g-1}\widetilde{\mathcal{F}}_s^{(g)}(\mathbf{t})\Big)$$

where $\widetilde{\mathcal{F}}_s^{(g)}(\mathbf{t})$ is a formal power series in some formal vector variables $\mathbf{t} = (t_0, t_1, t_2, \dots)$. Let us denote by $B \subset S$ the subset of non-semisimple points. It is known that B is an analytic hypersurface in S and that the coefficients of the formal power series $\widetilde{\mathcal{F}}_s^{(g)}(\mathbf{t})$ depend analytically in s for all $s \in S \setminus B$.

Givental conjectured that if S is the quantum cohomology of some compact Kahler manifold, then under the semi-simplicity assumption, the total ancestor potential of the Frobenius structure is a generating function for the so called ancestor GW invariants (see Sect. 2.4 for more details). Givental's conjecture was proved by Teleman [21] in the more general settings of semi-simple Cohomological Field Theories (CohFT).

On the other hand, most of the CohFT that we would like to compute satisfy the semi-simplicity condition only after we deform them, so in order to use Givental's higher genus reconstruction it is important to determine whether the total ancestor potential $\mathcal{A}_s(\hbar;\mathbf{t})$ of a given semi-simple Frobenius structure extends analytically through the non-semisimple locus. For example, if S is the orbit space of the Weyl group of a non-simply laced simple Lie algebra (i.e., types B, C, F, or G), then there is a natural Frobenius structure on S (see [4, 19]), but the total ancestor potential *does not* extend analytically. It is a very interesting question to determine whether the total ancestor potential of the Frobenius structures in that case has a geometric origin, i.e., it is related in some way to some CohFT of Fan-Jarvis-Ruan-Witten (FJRW) [7]. In fact, some progress in this direction was recently made by Liu-Ruan-Zhang [15].

One of the most important examples of a semi-simple Frobenius structure, that plays a crucial role in mirror symmetry, is Saito's flat structure [18]. Motivated by the classical theory of period integrals, K. Saito introduced the notion of a *primitive form*. Let S be the base of the universal unfolding of the germ of a holomorphic function $f \in \mathcal{O}_{\mathbb{C}^{n+1},0}$ with an isolated critical point at $0 \in \mathbb{C}^{n+1}$. A primitive form is the germ of a holomorphic volume form on $\mathbb{C}^{n+1}$, possibly depending on the deformation parameters $s \in S$, with some very special properties. Spelling out the precise definition is quite difficult, but the main idea is that a primitive form and its covariant derivatives with respect to the Gauss–Manin connection, provide a frame for the vanishing cohomology bundle in which the Gauss–Manin connection turns into a Dubrovin's connection. In particular, the base S inherits a natural Frobenius structure, which is always semi-simple, because the critical values provide canonical coordinates (see [11, 20]). The goal in these notes is to prove the following theorem.

Theorem 1 *Let S be the base of the universal unfolding of some $f \in \mathcal{O}_{\mathbb{C}^{n+1},0}$ equipped with a Frobenius structure corresponding to a primitive form; then the coefficients of the total ancestor potential $\mathcal{A}_s(\hbar;\mathbf{t})$ in front of the monomials in* $\mathbf{t}$ *and $\hbar$ extend analytically across B to analytic functions on the entire Frobenius manifold S.*

Theorem 1 motivates the following question. Given a singularity $f \in \mathcal{O}_{\mathbb{C}^{n+1},0}$ and a primitive form, can we identify the total ancestor potential of the singularity with the generating function of invariants of some CohFT. For example, if the germ f can be represented by an invertible weighted-homogeneous polynomial and the primitive form is chosen appropriately, then there is a conjecture that the appropriate CohFT is a FJRW-CohFT [7].[1]

The proof of Theorem 1 follows the argument from [17]. We will try to keep the exposition as self-contained as possible. In particular, up to some linear algebra exercises, we give an introduction to Givental's higher-genus reconstruction, define and prove the properties of the so called propagators from [2], and finally give a proof of the local Eynard–Orantin recursion [6, 16]. The only requirements for reading this text is the knowledge of a Frobenius structure (see [4]). However, it might be useful also to refer from time to time to Givental's work [10], where the period integrals were introduced and some of their most fundamental properties were established.

2 Givental's Total Ancestor Potential

Let S be a complex semi-simple Frobenius manifold and $B \subset S$ be the analytic hypersurface consisting of non-semisimple points. Motivated by Gromov–Witten theory, Givental has defined the total ancestor potential $\mathcal{A}_s(\hbar;\mathbf{q})$ of the Frobenius manifold S for every semi-simple point $s \in S \setminus B$. The goal in this section is to recall Givental's construction.

2.1 Givental's Symplectic Loop Space Formalism

Let H be a complex vector space equipped with a non-degenerate bi-linear pairing $(\ ,\ )$ and with a distinguished vector $\mathbf{1} \in H$. By definition, Givental's symplectic loop space $\mathcal{H} = H(\!(z^{-1})\!)$ is the space of formal Laurent series in z^{-1} with coefficients in H, equipped with the following symplectic structure:

$$\Omega(f(z), g(z)) = \mathrm{Res}_{z=0}(f(-z), g(z))dz,$$

the residue is interpreted formally as the coefficient in front of z^{-1}.

[1]One of the pleasant outcomes of the workshop was that this conjecture was confirmed by generalizing the approach of [14].

The vector space $\mathcal{H}$ viewed as an abelian Lie algebra has a natural central extension $\mathcal{H} \oplus \mathbb{C}$ in which the symplectic form coincides with the cocycle defining the extension,

$$[v_1, v_2] := \Omega(v_1, v_2), \quad v_1, v_2 \in \mathcal{H}.$$

Since $\mathcal{H} \oplus \mathbb{C}$ is a Heisenberg Lie algebra it has a standard Fock space representations. In our case the construction is as follows. Let us fix bases $\{\phi_a\}_{a=1}^N$ and $\{\phi^a\}_{a=1}^N$ of H dual with respect to $(\, , \,)$, then

$$\Omega(\phi^a(-z)^{-n-1}, \phi_b z^m) = \delta_{a,b}\delta_{n,m}.$$

Let us fix a sequence of formal vector variables $\mathbf{q} = (q_0, q_1, q_2, \dots)$, where $q_k = \sum_{a=1}^N q_{k,a}\phi_a$. We will be interested in the Fock space of formal power series

$$\mathbb{C}_\hbar[\![q_0, q_1 + \mathbf{1}, q_2, \dots]\!],$$

where $\mathbb{C}_\hbar$ is the field of formal Laurent series in $\hbar^{\frac{1}{2}}$. The shift by $\mathbf{1}$ is known as the *dilaton shift*. The linear operator of the Fock space representing $v \in \mathcal{H} \oplus \mathbb{C}$ will be denote by $\widehat{v}$ or $(v)^\wedge$. The representation of the Heisenberg algebra on the Fock space is uniquely defined by

$$(\phi_a z^m)^\wedge := \hbar^{\frac{1}{2}} \partial_{q_{m,a}}, \quad (\phi^a(-z)^{-m-1})^\wedge := -\hbar^{-\frac{1}{2}} q_{m,a},$$

where $1 \le a \le N$ and $m \ge 0$.

2.1.1 Quantization of Quadratic Hamiltonians

Note that the map

$$\mathcal{H} \to \mathcal{H}^*, \quad v \mapsto \Omega(\, , v)$$

induces an isomorphism of Lie algebras

$$\mathcal{H} \oplus \mathbb{C} \cong \mathcal{H}^* \oplus \mathbb{C},$$

where the RHS is the vector space of constant and linear functions on $\mathcal{H}$ and the Lie bracket is the Poisson bracket corresponding to the symplectic form Ω. On the other hand, a linear operator A on $\mathcal{H}$ is an infinitesimal symplectic transformation if and only if the map $v \mapsto Av$ is a Hamiltonian vector field. Moreover, the Hamiltonian is given by the quadratic function $h_A(v) = \frac{1}{2}\Omega(Av, v)$. Put

$$p_{m,a} = \Omega(\, , \phi_a z^m), \quad q_{m,a} = -\Omega(\, , \phi^a(-z)^{-m-1}),$$

then h_A is a quadratic expression in $p_{m,a}$ and $q_{m,a}$. We define the quantization $\widehat{A} := \widehat{h}_A$ by

$$(p_{m,a}p_{n,b})^\wedge = \hbar\partial_{q_{m,a}}\partial_{q_{n,b}},$$
$$(p_{m,a}q_{n,b})^\wedge = (q_{n,b}p_{m,a})^\wedge = q_{n,b}\partial_{q_{m,a}},$$
$$(q_{m,a}q_{n,b})^\wedge = \hbar^{-1}q_{m,a}q_{n,b}.$$

We leave it as an exercise to verify the following properties

$$[\widehat{A}, \widehat{v}\,] = (Av)^\wedge, \quad \{h_A, h_B\} = h_{[A,B]},$$

for all $v \in \mathcal{H}$ and all infinitesimal symplectic transformations A and B, where $\{\ ,\ \}$ is the Poisson bracket.

2.1.2 Quantization of Symplectic Transformations

Let us assume that the operator series

$$R(z) = 1 + R_1 z + R_2 z^2 + \cdots, \quad R_k \in \mathrm{End}(H)$$

is a symplectic transformation. It will be convenient to identify the sequence $\mathbf{q} = (q_0, q_1, \ldots)$ with the series $q_0 + q_1 z + q_2 z^2 + \cdots$, then the natural action of $R(z)$ on $H[z]$ induces an action on the formal sequence: $\mathbf{q}(z) \mapsto R(z)\mathbf{q}(z)$, or in components

$$q_n \mapsto R_0 q_n + R_1 q_{n-1} + \cdots + R_n q_0.$$

Let us also define $V_{k\ell} \in \mathrm{End}(H)$ by the identity

$$\sum_{k,\ell=0}^{\infty} V_{k\ell} z^k w^\ell = \frac{R^T(z)R(w) - 1}{z + w}, \tag{1}$$

where ^{T} is transposition with respect to the bi-linear form $(\ ,\ )$. We can define formally $A(z) = \log R(z)$, so that $R(z) = e^{A(z)}$. By definition the quantization $\widehat{R} := e^{\widehat{A}}$. The action of $\widehat{R}$ on the Fock space is not well defined in general. We have the following Lemma.

Lemma 2 *If $\mathcal{F}(\hbar; \mathbf{q})$ is a formal power series in the Fock space and $\widehat{R}\mathcal{F}$ is well defined, then*

$$\widehat{R}\mathcal{F}(\hbar; \mathbf{q}) = \left(e^{\hbar V(\partial,\partial)/2}\mathcal{F}\right)(\hbar; R^{-1}\mathbf{q}),$$

where $V(\partial, \partial)$ *is the following 2nd order differential operator*

$$V(\partial, \partial) = \sum_{k,\ell=0}^{\infty} \sum_{a,b=1}^{N} (\phi^a, V_{k\ell}\phi^b)\partial_{q_{k,a}}\partial_{q_{\ell,b}}.$$

2.1.3 Tame Asymptotical Functions

We will be interested in the so called *tame* functions. To define them let us introduce first another sequence of formal vector variables $\mathbf{t} = (t_0, t_1, t_2, \dots)$, so that $t_k = q_k + \delta_{k,1}\mathbf{1}$. Formal power series in the Fock space of the type

$$\mathcal{A}(\hbar; \mathbf{q}) = \exp\Big(\sum_{g=0}^{\infty} \hbar^{g-1}\mathcal{F}^{(g)}(\mathbf{t})\Big), \quad \mathcal{F}^{(g)} \in \mathbb{C}[[\mathbf{t}]], \quad \mathcal{F}^{(g)}(0) = 0$$

are called formal asymptotical functions. We say that a formal asymptotical function is tame if its Taylor's coefficients satisfy the $3g-3+r$-jet condition

$$\left.\frac{\partial^r \mathcal{F}^{(g)}}{\mathrm{d}t_{k_1,a_1}\cdots \mathrm{d}t_{k_r,a_r}}\right|_{\mathbf{t}=0} = 0 \quad \text{if} \quad k_1 + \cdots + k_r > 3g-3+r.$$

Let us recall the following result from [10].

Lemma 3 *If* $\mathcal{A}$ *is a tame asymptotical function, then* $\widehat{R}\mathcal{A}$ *is a well defined tame asymptotical function.*

2.1.4 Symplectic Loop Space for a Frobenius Manifold

We fix a flat coordinate system $\{t_a\}_{a=1}^N$ on S, s.t., the Euler vector field takes the form

$$E = \sum_{a=1}^{N} d_a\, t_a \partial_a + \sum_{b:d_b=1} r_b \partial_b,$$

where $d_1, d_2, \dots, d_N$ and r_b $(b : d_b = 1)$ are some constants and $\partial_a := \partial/\partial t_a$. For simplicity, let us assume that S is simply connected, so that the flat vector fields give a trivialization of the tangent and the cotangent bundle. More precisely, let us denote by H the tangent space at some reference point, then we have

$$T^*S \cong TS \cong S \times H,$$

where the first isomorphism is via the non-degenerate bi-linear form of the Frobenius structure and the second one is via parallel transport with respect to the corresponding Levi–Civita connection. The vector space H can be viewed also as the vector space of flat vector fields on S. Note that $\phi_a = \partial_a$ and $\phi^a := dt^a$ form bases of H dual with respect to the Frobenius pairing. Recalling Givental's symplectic loop space formalism applied for H with $(\ , \ )$ being the Frobenius pairing, and **1** the unit vector field we get a symplectic loop space and a Fock space equipped with a representation of the Heisenberg algebra and a *projective* representation of the Poisson algebra of quadratic Hamiltonians on H.

2.2 Canonical Coordinates

Let us fix a semi-simple point $s \in S \setminus B$. By definition, there exists a coordinate system $\{u_i\}_{i=1}^N$ defined locally near s in which both the Frobenius multiplication and the flat metric are diagonal, i.e.,

$$\partial_{u_i} \bullet \partial_{u_j} = \delta_{i,j}\partial_{u_j}, \quad (\partial_{u_i}, \partial_{u_j}) = \frac{\delta_{i,j}}{\Delta_j},$$

where $\{\Delta_j\}_{j=1}^N$ are some functions analytic with no zeros in a neighborhood of s. Coordinates $\{u_i\}$ with the above properties are called *canonical*. They are unique up to permutation and a constant shift.

Let us denote by U the diagonal matrix of size $N \times N$ whose diagonal entries are $U_{i,i} = u_i$. We need also the $N \times N$ matrix Ψ corresponding to the linear map

$$\Psi : \mathbb{C}^N \to T_sB \cong H, \quad e_i \mapsto \sqrt{\Delta_i}\partial_{u_i}.$$

The matrix of Ψ is constructed by using the standard basis $\{e_i\}_{i=1}^N$ of $\mathbb{C}^N$ and the flat basis $\{\phi_a\}_{a=1}^N$ of H, so that the entries of Ψ are

$$\Psi_{a,i} = \sqrt{\Delta_i}\frac{\partial t_a}{\partial u_i}, \quad 1 \le a, i \le N.$$

Let us summarize some of the basic properties of the matrix Ψ. The proofs follow immediately from the definitions, so they will be left as an exercise.

Proposition 4 *The matrix Ψ has the following properties:*

(1) *If $g = (g_{a,b})$, $g_{a,b} = (\phi_a, \phi_b)$ is the matrix of the flat pairing, then*

$$\Psi\,\Psi^T = g^{-1},$$

where ^{T} is the usual transposition of matrices.

(2) *Let* $A = \sum_{a=1}^{N} A_a dt_a$ *be the connection 1-form on* S *where* A_a *is the linear operator of Frobenius multiplication by* ∂_a*, then*

$$\Psi^{-1} A \Psi = dU.$$

(3) *The Euler vector field has the form* $E = \sum_{i=1}^{N} u_i \partial_{u_i}$. *In particular,*

$$\Psi^{-1}(E\bullet)\Psi = U,$$

where $E\bullet$ *is the linear operator of multiplication by the Euler vector field* E.

Recall the Dubrovin's connection ∇ on the trivial bundle $S \times \mathbb{C}^* \times H \to S \times \mathbb{C}^*$. In flat coordinates

$$\nabla = d - Az^{-1} + \Big(- \theta z^{-1} + (E\bullet)z^{-2}\Big)dz,$$

where θ is the so called *Hodge grading operator* defined by

$$\theta : H \to H, \quad \theta(\phi_a) = \Big(\frac{D}{2} + d_a - 1\Big)\phi_a, \quad 1 \le a \le N.$$

Proposition 5 *Dubrovin's connection has an irregular singularity at* $z = 0$ *and it has a unique formal asymptotical solution of the form*

$$\Psi(1 + R_1 z + R_2 z^2 + \cdots)e^{U/z}. \tag{2}$$

Proof Using Proposition 4 we get

$$\Psi^{-1}\nabla\Psi = d + \Psi^{-1}d\Psi - dUz^{-1} + (Vz^{-1} + Uz^{-2})dz,$$

where $V := -\Psi^{-1}\theta\Psi$. The asymptotical series (2) is a solution to the Dubrovin's connection if and only if $\{R_k\}_{k=0}^{\infty}$ (we set $R_0 = 1$) satisfies the following system of differential equations

$$dR_k + (\Psi^{-1}d\Psi)R_k = [dU, R_{k+1}], \quad \forall k \ge 0 \tag{3}$$

and

$$kR_k + [U, R_{k+1}] = -VR_k, \quad \forall k \ge 0. \tag{4}$$

We have to prove that the above system has a unique solution. Arguing by induction on ℓ we will prove that there is a unique sequence $R_1, \dots, R_\ell$ satisfying (3) and (4) for all $k \le \ell - 1$, the diagonal part of (4) for $k = \ell$, and $E(R_k) = -kR_k$ for all $k \le \ell$.

Let us first prove the statement for $\ell = 1$. Using (3) with $k = 0$ and comparing the (i, j)-th entries of the matrices with $i \neq j$ we get

$$(\Psi^{-1}d\Psi)_{i,j} = (du_i - du_j)(R_1)_{i,j}.$$

The flatness of ∇ implies that $[dU, \Psi^{-1}d\Psi] = 0$. In particular, $(du_i - du_j) \wedge (\Psi^{-1}d\Psi)_{i,j} = 0$, which by the de Rham lemma implies that $(\Psi^{-1}d\Psi)_{i,j} = \alpha_{i,j}(du_i - du_j)$ for some function $\alpha_{i,j}$ analytic in a neighborhood of s. Hence $(R_1)_{i,j} = \alpha_{i,j}$. Comparing the diagonal entries in (4) for $k = 1$ we get

$$(R_1)_{i,i} = -\sum_{p \neq i} V_{i,p}(R_1)_{p,i},$$

so R_1 is uniquely determined. Let us check that the R_1 satisfies (4) with $k = 0$. We need only to compare the off-diagonal entries. Fix $i \neq j$, then by definition we have

$$(\Psi^{-1}\partial_{u_p}\Psi)_{i,j} = 0, \quad p \neq i, j,$$

and

$$(R_1)_{i,j} = (\Psi^{-1}\partial_{u_i}\Psi)_{i,j} = -(\Psi^{-1}\partial_{u_j}\Psi)_{i,j},$$

hence

$$[U, R_1]_{i,j} = (u_i - u_j)(R_1)_{i,j} = (\Psi^{-1}E(\Psi))_{i,j},$$

where $E = \sum_{i=1}^{N} u_i\partial_{u_i}$ is the Euler vector field. Since by definition $\mathrm{Lie}_E(\ ,\ ) = (2 - D)(\ ,\ )$ we get that $E(\Delta_i) = D\,\Delta_i$ and

$$E(\Psi_{a,i}) = \Big(\frac{D}{2} + \deg(t_a) - 1\Big)\Psi_{a,i} = \theta_{a,a}\Psi_{a,i}.$$

In other words $\Psi^{-1}E(\Psi) = \Psi^{-1}\theta\Psi = -V$. Finally, note that $E(U) = U$ and $E(V) = 0$, so the identity $[U, R_1] = -V$ implies that $E(R_1) = -R_1$.

Assume that we have constructed $R_1, \ldots, R_\ell$. We would like to construct $R_{\ell+1}$ so that the inductive assumption holds. Note that since ∇ is flat we have

$$\Big(d + \Psi^{-1}d\Psi\Big)^2 = \Psi^{-1}d^2\Psi = 0, \quad [dU, d + \Psi^{-1}d\Psi] = 0,$$

so

$$[dU, dR_\ell + \Psi^{-1}d\Psi R_\ell] = (d + \Psi^{-1}d\Psi)[dU, R_\ell] = (d + \Psi^{-1}d\Psi)^2 R_{\ell-1} = 0.$$

Now the same argument that we used to construct R_1 can be used to construct $R_{\ell+1}$. The details are straightforward and will be left as an exercise. □

2.3 The Total Ancestor Potential

Let us begin first with the case when $S = \mathbb{C}$ is equipped with the natural Frobenius structure corresponding to the standard multiplication of complex numbers and the pairing is $(1, 1) = 1$. The total ancestor potential in this case is defined through the intersection theory on the Delign–Mumford moduli spaces $\overline{\mathcal{M}}_{g,n}$ of stable curves. Let us denote by ψ_i the 1st Chern class of the orbifold line bundle on $\overline{\mathcal{M}}_{g,n}$ corresponding to the cotangent lines at the i-th marked points. Put

$$\langle \psi_1^{k_1}, \ldots, \psi_n^{k_n} \rangle_{g,n} := \int_{\overline{\mathcal{M}}_{g,n}} \psi_1^{k_1} \cdots \psi_n^{k_n}. \tag{5}$$

The Witten-Kontsevich tau-function is a formal series in $\mathbf{t} = (t_0, t_1, \ldots)$ defined by

$$\widetilde{\mathcal{A}}_{\mathrm{pt}}(\hbar; \mathbf{t}) = \exp\Big(\sum_{g=0}^{\infty} \hbar^{g-1} \widetilde{\mathcal{F}}^{(g)}(\mathbf{t}) \Big),$$

where the *genus-g potential*

$$\widetilde{\mathcal{F}}^{(g)}(\mathbf{t}) = \sum_{n=0}^{\infty} \frac{1}{n!} \langle \mathbf{t}(\psi_1), \ldots, \mathbf{t}(\psi_n) \rangle_{g,n}$$

is defined as follows. We identify $\mathbf{t}$ with the formal series $\mathbf{t}(z) := t_0 + t_1 z + \cdots$ and the n-point genus-g correlator is expanded multilinearly in $t_0, t_1, \ldots,$ so that the correlators are reduced to expressions of the type (5). The total ancestor potential $\mathcal{A}_{\mathrm{pt}}$ is obtained from $\widetilde{\mathcal{A}}_{\mathrm{pt}}$ via the dilaton shift: $\mathbf{t}(z) = \mathbf{q}(z) + z$, or in components $t_k = q_k + \delta_{k,1}, \quad k = 0, 1, 2, \ldots,$ i.e.,

$$\mathcal{A}_{\mathrm{pt}}(\hbar; \mathbf{q}) = \widetilde{\mathcal{A}}_{\mathrm{pt}}(\hbar; \mathbf{q}(z) + z) \quad \in \quad \mathbb{C}_\hbar[\![q_0, q_1 + 1, q_2, \ldots]\!].$$

Note that in this case $B = \emptyset$ and that by definition $\mathcal{A}_{\mathrm{pt}}$ is independent of the choice of a semi-simple point.

If S is an arbitrary simply connected semi-simple Frobenius manifold, then we fix a reference tangent space H with a basis $\{\phi_a\}_{a=1}^N$ that gives rise to a flat coordinate system $t = \{t_a\}_{a=1}^N$. In a neighborhood of a fixed semi-simple point $s \in S \setminus B$ we pick canonical coordinates $\{u_i\}_{i=1}^N$ and fix a branch of $\sqrt{\Delta_i}$, so that the matrix Ψ is uniquely defined. The total ancestor potential is defined by

$$\mathcal{A}_s(\hbar;\mathbf{q}) := \widehat{\Psi}\,\widehat{R}\,e^{(U/z)^\wedge}\prod_{i=1}^{N}\mathcal{D}_{\mathrm{pt}}(\hbar\Delta_i;{}^i\mathbf{q}(z)\sqrt{\Delta_i})\in\mathbb{C}_\hbar[\![q_0,q_1+\mathbf{1},q_2\ldots]\!], \tag{6}$$

where ${}^i\mathbf{q}(z)=\sum_{k=0}^{\infty}{}^i\mathbf{q}_k z^k$. The expression preceding $\widehat{\Psi}$ is a formal series in the variables ${}^i\mathbf{q}_k$ $(1\le i\le N,\ k\ge 0)$. The quantization $\widehat{\Psi}$ is interpreted as the change of variables

$$\sum_{i=1}^{N}{}^i q_\ell\Psi(e_i)=\sum_{a=1}^{N}q_{\ell,a}\phi_a,$$

i.e., $\widehat{\Psi}$ transforms a formal series in ${}^i q_\ell$ into a formal series in $q_{\ell,a}$ via the substitution

$${}^i q_\ell=\sum_{a=1}^{N}(\Psi^{-1})_{i,a}q_{\ell,a}.$$

Proposition 6 *The coefficients in the formal series expansion of $\mathcal{A}_s(\hbar;\mathbf{q})$ as a series in $q_0,q_1+1,q_2,\ldots$ are Laurent series in $\hbar$, whose coefficients extend analytically to the open subset $S\setminus B$ of semi-simple points.*

Proof In order to prove that the coefficients extend analytically along any path in $S\setminus B$, it is enough to prove that the canonical coordinates u_i have this property. Let us denote by $L\subset T^*S$ the characteristic variety of the Frobenius multiplication. Namely, L is defined as the zero locus of the sheaf of ideals $\mathcal{I}$ on T^*S generated by the kernel of the map

$$\mathrm{Sym}(\mathcal{T}_S)\to\mathcal{T}_S,\quad v_1\ldots v_k\mapsto(v_1\bullet\cdots\bullet v_k). \tag{7}$$

Here we are using that there is a natural map $\pi^*\,\mathrm{Sym}(\mathcal{T}_S)\to\mathcal{O}_{T^*S}$, where $\pi: T^*S\to S$ is the projection, so that the kernel of the map (7) can be mapped to $\mathcal{O}_{T^*S}$ and it makes sense to define the ideal $\mathcal{I}$ generated by the image.

If s is a semi-simple point, then we can choose canonical coordinates $(u_1,\ldots,u_N)$ around s and fiberwise coordinates $x_1,\ldots,x_N$ on T^*S, so that all 1-forms in a neighborhood of s are given by $\sum_{i=1}^{N}x_i du_i$. In the local coordinates $(u_1,\ldots,u_N,x_1,\ldots,x_N)$ the characteristic variety L is given by the equations

$$x_ix_j-\delta_{i,j}x_j=0,\quad 1\le i,j\le N.$$

It follows that over a neighborhood of s the subvariety L is a N-sheet covering and the N sections of T^*S that define L are precisely the 1-forms du_i $(1\le i\le N)$.

It is not hard to see from here that the projection π induces a branched covering $L \to S$ of degree N and moreover the set B of non-semi-simple points coincides with the branching locus, i.e., with the support of the sheaf of relative differentials $\Omega^1_{L/S}$. Since L induces a regular covering on $S \setminus B$ the differential forms du_i extend along any path in $S \setminus B$, which proves that u_i also extends.

The analytic continuation along a closed loop in $S \setminus B$ acts as a permutation on the sequence $(u_1, \ldots, u_N)$, while on the sequence $(\sqrt{\Delta_1}, \ldots, \sqrt{\Delta_N})$ the action is given by the same permutation, but with possible sign changes of $\sqrt{\Delta_i}$. It remains only to check that formula (6) is independent of the choices of signs in $\sqrt{\Delta_i}$ and invariant under the permutations of the canonical coordinates. This follows easily from the definitions. □

2.4 The Ancestor Correlators

In order to motivate our definition of correlators, let us first recall the definition in the geometric settings, following [9]. For a given projective manifold V, let us denote by $\overline{\mathcal{M}}_{g,n}(V,d)$ the moduli space of degree-d stable maps from a genus-g nodal Riemann surface, equipped with n marked points, to V. The ancestor correlator functions are defined by the following intersection numbers:

$$\langle \phi_{a_1}\overline{\psi}_1^{k_1}, \ldots, \phi_{a_n}\overline{\psi}_n^{k_n}\rangle_{g,n}(t) := \sum_{m=0}^{\infty}\sum_d \frac{Q^d}{m!}\int_{[\overline{\mathcal{M}}_{g,n+m}(V,d)]^{\mathrm{virt}}} \mathrm{ev}^*(\phi_{a_1}\otimes\cdots\otimes\phi_{a_n}\otimes t^{\otimes m})\prod_{a=1}^{n}\overline{\psi}_a^{k_a},$$

where the notation is as follows. The classes $\{\phi_{a_s}\}_{s=1}^n$ and t are cohomology classes on V, the 2nd sum is over all effective curve classes $d \in H_2(V;\mathbb{Z})$ and Q^d is an element of the Novikov ring. Furthermore, evaluating the stable map at the marked points gives rise to the evaluation map

$$\mathrm{ev} : \overline{\mathcal{M}}_{g,n+m}(V,d) \to V^{n+m},$$

while the operation forgetting the last m marked points, the stable map, and stabilizing (i.e. contracting the unstable components) gives a map $\mathrm{ft} : \overline{\mathcal{M}}_{g,n+m}(V,d) \to \overline{\mathcal{M}}_{g,n}$. The cohomology classes $\overline{\psi}_s := \mathrm{ft}^*(\psi_s)$ $(1 \le s \le n)$. Finally, $[\overline{\mathcal{M}}_{g,n+m}(V,d)]^{\mathrm{virt}}$ is the virtual fundamental cycle. Let us point out that if $\overline{\mathcal{M}}_{g,n}$ is empty, i.e., $2g-2+n \le 0$, then the ancestor correlator is by definition 0. The total ancestor potential of V has the form

$$\widetilde{\mathcal{A}}_t(\hbar;\mathbf{t}) = \exp\Big(\sum_{g=0}^{\infty}\hbar^{g-1}\widetilde{\mathcal{F}}_t^{(g)}(\mathbf{t})\Big), \tag{8}$$

where $t \in H := H^*(V;\mathbb{C})$, $\mathbf{t} = \{t_{k,a}\}$ is a set of formal variables and

$$\widetilde{\mathcal{F}}_t^{(g)}(\mathbf{t}) = \sum_{n=0}^{\infty} \frac{1}{n!} \langle \mathbf{t}(\overline{\psi}_1), \ldots, \mathbf{t}(\overline{\psi}_n) \rangle (t)_{g,n}$$

is the so called genus-g ancestor potential, where $\mathbf{t}(z) = \sum_{k,a} t_{k,a}\phi_a z^k$ and the definition of the correlator is extended mult-linearly.

Let us return to the settings of an abstract semi-simple Frobenius manifold. It can be proved that the ancestor potential (6) still has the form (8). Motivated by Gromov–Witten theory we would like to define the analogues of the ancestor correlator functions, so that the ancestor potential can be written in the same way. Put

$$\langle \phi_{a_1}\psi^{k_1}, \ldots, \phi_{a_n}\psi^{k_n} \rangle_{g,n}(s;\mathbf{t}) := \partial_{t_{k_1,a_1}} \cdots \partial_{t_{k_n,a_n}} \widetilde{\mathcal{F}}_s^{(g)}(\hbar;\mathbf{t})$$

and

$$\langle \phi_{a_1}\psi^{k_1}, \ldots, \phi_{a_n}\psi^{k_n} \rangle_{g,n}(s) := \langle \phi_{a_1}\psi^{k_1}, \ldots, \phi_{a_n}\psi^{k_n} \rangle_{g,n}(s;0), \tag{9}$$

then by the Taylor's formula we have

$$\widetilde{\mathcal{A}}_s(\hbar;\mathbf{t}) = \exp\Big(\sum_{g,n=0}^{\infty} \frac{\hbar^{g-1}}{n!} \langle \mathbf{t}(\psi), \ldots, \mathbf{t}(\psi) \rangle_{g,n}(s) \Big),$$

where by extending multi-linearly the definition (9) we allow the insertions of the correlator to be any formal power series from $H[\![\psi]\!]$. We will refer to (9) as the *ancestor correlators* of the Frobenius structure. According to Proposition 6 they are analytic functions in $s \in S \setminus B$.

3 The Local Eynard–Orantin Recursion

Let us assume that S is a semi-simple Frobenius manifold. The goal in this section is to derive a recursion for the ancestor correlators.

3.1 Virasoro Constraints for the Point

Recall that the Witten–Kontsevich tau-function is a vacuum vector for a certain representation of the Virasoro algebra. The representation can be constructed as follows. Put

$$I_{A_1}^{(k)}(u,\lambda) = (-1)^k \frac{(2k-1)!!}{2^{k-1/2}} (\lambda - u)^{-k-1/2}, \quad k \geq 0$$
$$I_{A_1}^{(-k-1)}(u,\lambda) = 2 \frac{2^{k+1/2}}{(2k+1)!!} (\lambda - u)^{k+1/2}, \quad k \geq 0.$$

These functions are known to be the periods of the A_1-singularity. They satisfy the following crucial property

$$\partial_\lambda I_{A_1}^{(n)}(u,\lambda) = I_{A_1}^{(n+1)}(u,\lambda). \tag{10}$$

We form the generating series

$$\mathbf{f}_{A_1}(u,\lambda;z) = \sum_{n\in\mathbb{Z}} I_{A_1}^{(n)}(u,\lambda)\,(-z)^n$$

and define

$$L_{A_1}(u,\lambda) := \frac{1}{4} : (\partial_\lambda \mathbf{f}_{A_1}(u,\lambda;z)^\wedge (\partial_\lambda \mathbf{f}_{A_1}(u,\lambda;z)^\wedge : + \frac{1}{16}(\lambda-u)^{-2} =: \sum_{m\in\mathbb{Z}} L_{A_1,m}(\lambda-u)^{-m-2},$$

where : : is the normal ordering which means that all differentiation operations precede all multiplication ones. The operators $L_{A_1,m}$ form a representation of the Virasoro algebra (with central charge 1) on the Fock space of the Frobenius manifold $S=\mathbb{C}$. The first few of them have the form

$$L_{A_1,-1} = \frac{q_0^2}{2\hbar} + \sum_{k=0}^{\infty} q_{k+1}\partial_{q_k},$$

$$L_{A_1,0} = \frac{1}{16} + \frac{1}{2}\sum_{k=0}^{\infty}(2k+1) q_k \partial_{q_k},$$

$$L_{A_1,1} = \frac{\hbar}{8}\frac{\partial^2}{\partial {q_0}^2} + \frac{1}{4}\sum_{k=0}^{\infty}(2k+3)(2k+1) q_k \frac{\partial}{\partial q_{k+1}},$$

$$L_{A_1,2} = \frac{3\hbar}{8}\frac{\partial^2}{\partial q_0 \partial q_1} + \frac{1}{8}\sum_{k=0}^{\infty}(2k+5)(2k+3)(2k+1) q_k \frac{\partial}{\partial q_{k+2}}.$$

It was conjectured by Witten [22] and proved by Kontsevich [13] that $\widetilde{\mathcal{D}}_{\mathrm{pt}}$ is a tau-function of the KdV hierarchy. In addition, $\widetilde{\mathcal{D}}_{\mathrm{pt}}$ satisfies the string equation. According to Kac and Schwarz [12] there exists a unique tau-function of KdV satisfying string equation, which can be characterized also as the vacuum vectors for the Virasoro algebra. In our notation the Virasoro constraints take the form

$$L_{A_1,m}\,\mathcal{A}_{\mathrm{pt}}(\hbar;\mathbf{q}) = 0, \quad m \geq -1.$$

3.2 Virasoro Constraints for the Total Ancestor Potential

Fix a neighborhood of a generic semi-simple point, so that the canonical coordinates $(u_1, \dots, u_N)$ are pairwise distinct, i.e., $u_i \neq u_j$ for $i \neq j$. Let us fix a sufficiently small disk D_i near each u_i, s.t, $D_i \cap D_j = \emptyset$ for $i \neq j$. Put

$$\mathbf{f}_i(s,\lambda;z) := \Psi\, R(z)\, e^{U/z}\mathbf{f}_{A_1}(0,\lambda;z)e_i = \Psi\, R(z)\,\mathbf{f}_{A_1}(u_i,\lambda;z)e_i,$$

where for the 2nd equality we used the translation property (10). Expanding in the powers of z we get

$$\mathbf{f}_i(s,\lambda;z) = \sum_{n\in\mathbb{Z}} I_i^{(n)}(s,\lambda)(-z)^n,$$

where each $I_i^{(n)}(s,\lambda)$ makes sense as a formal Laurent series in $\lambda - u_i$. However, using that $\Psi R e^{U/z}$ is a solution for the Dubrovin's connection, it is easy to prove that $I_i^{(n)}(s,\lambda)$ is a solution to the following system of ODEs

$$\begin{aligned}
\partial_a I^{(n)}(s,\lambda) &= -\phi_a \bullet I^{(n)}(s,\lambda)\\
\partial_\lambda I^{(n)}(s,\lambda) &= I^{(n+1)}(s,\lambda)\\
(\lambda - E\bullet)\partial_\lambda I^{(n)}(s,\lambda) &= \Big(\theta - n - \frac{1}{2}\Big) I^{(n)}(s,\lambda).
\end{aligned} \tag{11}$$

Equation (11) has regular singularities at $\lambda = u_i$ $(1 \leq i \leq N)$, which implies that the Laurent series representing $I_i^{(n)}(s,\lambda)$ is convergent for all $\lambda \in D_i$ and moreover we can analytically extend in λ along any path in $\mathbb{C} \setminus \{u_1, \dots, u_N\}$.

After a direct computation using Lemma 2 we get the following Lemma.

Lemma 7 *The following identities hold:*

$$(\mathbf{f}_i(s,\lambda;z))^\wedge\, \widehat{\Psi}\widehat{R} = \widehat{\Psi}\widehat{R}(\mathbf{f}_{A_1}(u_i,\lambda;z)e_i)^\wedge, \quad 1 \leq i \leq N.$$

The symplectic vector space $\mathcal{H} = \mathcal{H}_+ \oplus \mathcal{H}_-$, where $\mathcal{H}_+ = H[z]$ and $\mathcal{H}_- = H[[z^{-1}]]z^{-1}$ are Lagrangian subspaces. We denote by $\mathbf{f} \mapsto \mathbf{f}^+$ and $\mathbf{f} \mapsto \mathbf{f}^-$ the corresponding projections.

Lemma 8 *The symplectic pairing*

$$\Omega(\mathbf{f}^+_{A_1}(u_i,\lambda;z),\mathbf{f}^-_{A_1}(u_i,\mu;z)) = 2\log(\lambda-\mu) - 4\log((\lambda-u_i)^{1/2}+(\mu-u_i)^{1/2}),$$

where the RHS is expanded into a Laurent series at $\mu = u_i$, while keeping λ as a parameter.

The proof is an easy computation using the explicit formulas for $\mathbf{f}_{A_1}$. The proof of next Lemma is also a direct computation.

Lemma 9 *The symplectic pairing $\Omega(\mathbf{f}^+_i(s,\lambda;z),\mathbf{f}^-_i(s,\mu;z))$ coincides with*

$$\Omega(\mathbf{f}^+_{A_1}(u_i,\lambda;z),\mathbf{f}^-_{A_1}(u_i,\mu;z)) + \sum_{\ell',\ell''=0}^{\infty}(-1)^{\ell'+\ell''}(V_{\ell',\ell''}e_i,e_i)\, I^{(-\ell''-1)}_{A_1}(u_i,\lambda) I^{(-\ell'-1)}_{A_1}(u_i,\mu),$$

where $V_{\ell',\ell''}\in \mathrm{End}(\mathbb{C}^N)$ are defined in terms of R by (1).

Let us define the *propagator*

$$P_{i,i}(s,\lambda,\mu) := \partial_\lambda\partial_\mu\Omega(\mathbf{f}^+_i(s,\lambda;z),\mathbf{f}^-_i(s,\mu;z)). \tag{12}$$

where the RHS is interpreted as a Laurent series in $(\mu-u_i)$ whose coefficients are Laurent series in $(\lambda-u_i)$. In fact, using Lemmas 8 and 9 we get that the propagator has the form of a singular term $2(\lambda-\mu)^{-2}$ plus a Laurent series in $(\lambda-u_i)$ and $(\mu-u_i)$. Furthermore, we define

$$P^{(0)}_{i,i}(s,\lambda) := \frac{1}{2!}\,\partial^2_\mu\Big((\lambda-\mu)^2P_{i,i}(s,\lambda,\mu)\Big)\Big|_{\mu=\lambda}.$$

It is convenient to define

$$\phi_j(s,\lambda;z) := \partial_\lambda\mathbf{f}_j(s,\lambda;z),\quad \widehat{\phi}_j(s,\lambda) := (\phi_j(s,\lambda;z))^\wedge.$$

Put

$$L_i(s,\lambda) := :\widehat{\phi}_i(s,\lambda)^2: + P^{(0)}_{i,i}(s,\lambda).$$

Proposition 10 *The following formula holds*

$$L_i(s,\lambda)\,\widehat{\Psi}\widehat{R} = 4\,\widehat{\Psi}\widehat{R}\,L_{A_1}(u_i,\lambda).$$

Proof Put

$$P_{A_1,A_1}(u_i,\lambda,\mu) := \partial_\lambda\partial_\mu\Omega(\mathbf{f}_{A_1}(u_i,\lambda;z)_+,\mathbf{f}_{A_1}(u_i,\mu;z)_-),$$

and define

$$P^{(0)}_{A_1,A_1}(u_i,\lambda) := \frac{1}{2!}\,\partial_\mu^2\Big((\lambda-\mu)^2 P_{A_1,A_1}(u_i,\lambda,\mu)\Big)\Big|_{\mu=\lambda} = \frac{1}{4}(\lambda-u_i)^{-2}.$$

Note that

$$4L_{A_1}(u_i,\lambda) =: \widehat{\phi}_{A_1}(u_i,\lambda)^2 : + P^{(0)}_{A_1,A_1}(s,\lambda).$$

After this observation the proof is straightforward. Namely, according to Lemma 7 we have

$$\widehat{\phi}_i(s,\lambda)\widehat{\phi}_i(s,\mu)\,\widehat{\Psi}\widehat{R} = \widehat{\Psi}\widehat{R}\,\widehat{\phi}_{A_1}(u_i,\lambda)\widehat{\phi}_{A_1}(u_i,\mu). \tag{13}$$

On the other hand

$$\widehat{\phi}_i(s,\lambda)\widehat{\phi}_i(s,\mu) = :\widehat{\phi}_i(s,\lambda)\widehat{\phi}_i(s,\mu): + P_{i,i}(s,\lambda,\mu)$$

and

$$\widehat{\phi}_{A_1}(u_i,\lambda)\widehat{\phi}_{A_1}(u_i,\mu) = :\widehat{\phi}_{A_1}(u_i,\lambda)\widehat{\phi}_{A_1}(u_i,\mu): + P_{A_1,A_1}(u_i,\lambda,\mu).$$

Also

$$P_{i,i}(s,\lambda,\mu) = \frac{2}{(\lambda-\mu)^2} + P^{(0)}_{i,i}(s,\lambda) + O(\lambda-\mu)$$

and

$$P_{A_1,A_1}(u_i,\lambda,\mu) = \frac{2}{(\lambda-\mu)^2} + P^{(0)}_{A_1,A_1}(u_i,\lambda) + O(\lambda-\mu).$$

Hence after subtracting the singular term $2(\lambda-\mu)^{-2}$ from both sides in (13) we can set $\mu=\lambda$. We get precisely the identity that we wanted to prove. □

Corollary 11 *Let*

$$L_i(s,\lambda) = \sum_{m\in\mathbb{Z}} L_{i,m}(\lambda-u_i)^{-m-2}$$

be the Laurent series expansion at $\lambda = u_i$*; then*

$$L_{i,m}\,\mathcal{A}_s(\hbar;\mathbf{q}) = 0, \quad 1\le i\le N, \quad m\ge -1.$$

3.3 The Local Eynard–Orantin Recursion

By definition, the ancestor potential does not have non-zero correlators in the unstable range $(g, n) = (0, 0), (0, 1), (0, 2)$ and $(1, 0)$. It is convenient however, to extend the definition in the unstable range as well in the following two cases:

$$\Big\langle \phi_j^+(s,\lambda;\psi_1), \mathbf{t}\Big\rangle_{0,2} := \Omega(\mathbf{t}(z), \phi_j^-(s,\lambda;z)), \tag{14}$$

$$\Big\langle \phi_j^+(s,\lambda;\psi_1), \phi_j^+(s,\lambda;\psi_1)\Big\rangle_{0,2} := P^{(0)}_{j,j}(s,\lambda). \tag{15}$$

Theorem 12 *The ancestor correlators satisfy the following recursion*

$$\Big\langle \phi_a\psi_1^m, \mathbf{t},\dots,\mathbf{t}\Big\rangle_{g,n+1}(s) =$$

$$\frac{1}{4}\sum_{j=1}^N \mathrm{Res}_{\lambda=u_j} \frac{\Omega(\phi_a z^m, \mathbf{f}_j(s,\lambda;z)_-)}{(I_j^{(-1)}(s,\lambda), \mathbf{1})} \times \Bigg(\Big\langle \phi_j^+(s,\lambda;\psi_1), \phi_j^+(s,\lambda;\psi_2), \mathbf{t},\dots,\mathbf{t}\Big\rangle_{g-1,n+2}(s) +$$

$$\sum_{\substack{g'+g''=g\\ n'+n''=n}} \binom{n}{n'} \Big\langle \phi_j^+(s,\lambda;\psi_1), \mathbf{t},\dots,\mathbf{t}\Big\rangle_{g',n'+1}(s) \Big\langle \phi_j^+(s,\lambda;\psi_1), \mathbf{t},\dots,\mathbf{t}\Big\rangle_{g'',n''+1}(s) \Bigg),$$

for all stable pairs $(g, n+1)$, *i.e.,* $2g-2+n \geq 0$, *where all unstable correlators on the RHS are set to* 0, *except for the ones of the type (14) and (15).*

Proof We will prove that the recursion is equivalent to the Virasoro constraints stated in Corollary 11. To begin with let us write the generating series $L_j(s,\lambda)$ explicitly as the sum of the following three terms

$$\sum_{k',k''=0}^{\infty}\sum_{a,b=1}^{N} (-1)^{k'+k''} (I_j^{(k'+1)}(s,\lambda), \phi^a)\, (I_j^{(k''+1)}(s,\lambda), \phi^b)\, \hbar\partial_{q_{k',a}}\partial_{q_{k'',b}}, \tag{16}$$

$$\sum_{k',k''=0}^{\infty}\sum_{a,b=1}^{N} 2(-1)^{k''+1} (I_j^{(-k')}(s,\lambda), \phi_a)\, (I_j^{(k''+1)}(s,\lambda), \phi^b)\, q_{k',a}\partial_{q_{k'',b}}, \tag{17}$$

$$P^{(0)}_{i,i}(s,\lambda) + \sum_{k',k''=0}^{\infty}\sum_{a,b=1}^{N} (I_j^{(-k')}(s,\lambda), \phi_a)\, (I_j^{(-k'')}(s,\lambda), \phi_b)\, \hbar^{-1} q_{k',a}\, q_{k'',b}. \tag{18}$$

Note that the double sum in (18) is analytic at $\lambda = u_j$, so the sum does not contribute to the Virasoro constraints, which means that it can be ignored. Let us undo the dilaton shift, i.e., switch to the variables $t_{k,a} = q_{k,a} - \delta_{k,1}\delta_{a,1}$, where for simplicity we assume that $\phi_1 = \mathbf{1}$. Note that the only term affected by the change is (see (17) when $k' = a = 1$)

$$\sum_{k=0}^{\infty} 2(-1)^{k+1} \, (I_j^{(-1)}(s,\lambda), \mathbf{1}) \, (I_j^{(k+1)}(s,\lambda), \phi^b) \, (t_{1,1}+1)\partial_{t_{k,b}}.$$

Now we need the following identity:

$$\sum_{j=1}^{N} \mathrm{Res}_{\lambda=u_j} \, (I_{\beta_j}^{(k')}(s,\lambda), \phi_a) \, (I_{\beta_j}^{(k'')}(s,\lambda), \phi^b) \, d\lambda = 2(-1)^{k'} \delta_{a,b}\delta_{k'+k'',0},$$

for all $k', k'' \in \mathbb{Z}$ and $a, b = 1, 2, \dots, N$. The proof follows from the definitions, so it is left as an exercise. Fix $m \geq 0$ and $a \in \{1, 2, \dots, N\}$, then

$$\frac{1}{4}\sum_{j=1}^{N} \mathrm{Res}_{\lambda=u_j} \frac{(I_j^{(-m-1)}(s,\lambda), \phi_a)}{(I_j^{(-1)}(s,\lambda), \mathbf{1})} L_j(s,\lambda) d\lambda =$$

$$\frac{\partial}{\partial t_{m,a}} + \frac{1}{4}\sum_{j=1}^{N} \mathrm{Res}_{\lambda=u_j} \frac{(I_j^{(-m-1)}(s,\lambda), \phi_a)}{(I_j^{(-1)}(s,\lambda), \mathbf{1})} \Big(L_j(s,\lambda)\big|_{\mathbf{q}=\mathbf{t}} \Big) d\lambda. \qquad (19)$$

The Virasoro constraints for the ancestor potential can be stated equivalently as $L_j(s,\lambda)\mathcal{A}_s(\hbar;\mathbf{q})$ is analytic at $\lambda = u_j$ for all $j = 1, 2, \dots, N$. Hence the operator (19) annihilates $\widetilde{\mathcal{A}}_s(\hbar;\mathbf{t})$. Comparing the coefficients in front of the monomial expressions in $\mathbf{t}$ and $\hbar$ of fixed degree n and genus $g-1$ we get the recursion that we wanted to prove. □

Remark 13 The recursion in Theorem 12 is the same as the local Eynard–Orantin recursion introduced in [6].

4 Analyticity of the Total Ancestor Potential in Singularity Theory

Let us assume now that S is the base of the universal unfolding F of some function $f \in \mathcal{O}_{\mathbb{C}^{n+1},0}$ with an isolated critical point at 0. We may assume that S is a small ball in $\mathbb{C}^N$ with center at 0, where N is the multiplicity of the critical point. Furthermore, we may arrange that the domain of F is an appropriate small contractible Stein domain $X \subset S \times \mathbb{C}^{n+1}$, s.t., $(0,0) \in X$ and $F(0,x) = f(x)$ (see [1] for some background on singularity theory). Let us fix a primitive holomorphic volume form $\omega \in \Omega^{n+1}_{X/S}(X)$, so that S becomes a Frobenius manifold (see [11, 20]). Moreover, it can be proved that the critical values provide a canonical coordinate system and the non semi-simple points B are precisely those $s \in S$ for which at least one of the critical points of $F(s, x)$ is not of type A_1.

4.1 Period Integrals

The map

$$\varphi : X \to S \times \mathbb{C}, \quad (s, x) \mapsto (s, F(s, x)),$$

gives rise to a smooth fibration on $(S \times D)'$, called the *Milnor fibration*. Here $D \subset \mathbb{C}$ is a sufficiently small disk with center 0 and $'$ denotes removing the points (s, λ) for which the fiber $X_{s,\lambda} := \varphi^{-1}(s, \lambda)$ is singular. Let us fix a reference point $(s_0, \eta_0) \in S \times D$ and denote by $\mathfrak{h} := H_n(X_{s_0,\eta_0}\mathbb{C})$ and by $\Delta \subset \mathfrak{h}$ the set of vanishing cycles. The definition of a primitive form implies that the functions $I_j^{(k)}(s, \lambda)$ introduced in Sect. 3.2 can be identified with period integrals of the following type:

$$I_\alpha^{(k)}(s, \lambda) := -d\Big((2\pi)^{-\ell}\partial_\lambda^{k+\ell} \int_{\alpha_{s,\lambda}} d^{-1}\omega\Big) \quad \in \quad T_s^* S \cong H,$$

where $\ell := n/2$ (by stabilizing the singularity if necessary we may assume that n is even), $\alpha \in \mathfrak{h}$ is a cycle, and $d^{-1}\omega$ denotes an arbitrary n-form $\widetilde{\omega}$, holomorphic in a neighborhood of $X_{s,\lambda}$, s.t., $\omega = d\widetilde{\omega}$. The period is a multi-valued function on $(S \times D)'$ and its value depends on the choice of a path from the reference point to (s, λ). In particular, we denoted by $\alpha_{s,\lambda}$ the parallel transport of α along the path. If $s \in S \setminus B$ is semi-simple and λ is in a neighborhood D_j of the critical value u_j, then let us choose $\alpha \in \Delta$ to be a vanishing cycle and fix the path in such a way that $\alpha_{s,\lambda}$ becomes the cycle vanishing over $\lambda = u_j$, then the period integral coincides with $I_j^{(k)}(s, \lambda)$ (see [10]).

For each fixed $s \in S$, the period vectors $I_\alpha^{(n)}(s, \lambda)$ satisfy Fuchsian differential equation in λ with singularities only at the critical values of $F(s, x)$ and $\lambda = \infty$. Hence using analytic continuation we may assume that the period integrals are define on $(S \times \mathbb{C})'$. Equivalently, the cohomology groups $H^n(X_{s,\lambda}; \mathbb{C})$, $(s, \lambda) \in (S \times D)'$ form a vector bundle equipped with a flat Gauss–Manin connection and the primitive form determines an extension of this bundle to a vector bundle on $S \times \mathbb{P}^1$, s.t., the Gauss–Manin connection has a logarithmic singularity at $\lambda = \infty$.

Finally, let us discuss the so called *primitive direction*. The flat identity of the Frobenius structure is a vector field δ_1, called *primitive*, s.t., $\delta_1 F = 1$. We denote by $s \mapsto s + \lambda\mathbf{1}$ the time-λ flow of δ_1. Note that if $(s, \lambda) \in S \times D$ is such that $s - \lambda\mathbf{1} \in S$, then $X_{s,\lambda} = X_{s-\lambda\mathbf{1},0}$, so the periods have the following translation symmetry

$$I_\alpha^{(n)}(s, \lambda) = I_\alpha^{(n)}(s - \lambda\mathbf{1}, 0).$$

Therefore we can extend the Frobenius structure in the primitive direction as well, i.e., we may assume that S is invariant under the translations $s \mapsto s + \lambda\mathbf{1}$ for all $\lambda \in \mathbb{C}$.

4.2 Propagators and the Monodromy Representation

Recall the propagators (12). In this section we prove that they can be extended analytically along any path in $(S \times \mathbb{C})'$ and moreover the analytic extension is compatible with the monodromy action. To begin with let us introduce the following terminology. Given cycles $\alpha, \beta \in \mathfrak{h}$ we define a *propagator* on $(S \times \mathbb{C})'$ from α to β to be a Laurent series

$$P_{\alpha,\beta}(s,\lambda,\mu) = \frac{(\alpha|\beta)}{(\lambda-\mu)^2} + \sum_{k=0}^{\infty} P^{(k)}_{\alpha,\beta}(s,\mu)(\lambda-\mu)^k,$$

where $(\alpha|\beta)$ up to the sign $(-1)^\ell$ is the intersection pairing of the cycles α and β, satisfying the following properties.

(1) For every $(s,\mu) \in (S \times \mathbb{C})'$, the radius of convergence of the series is non-zero.
(2) The functions $P^{(k)}_{\alpha,\beta}(s,\mu)$ extend analytically along any path in $(S \times \mathbb{C})'$ and the analytic continuation is compatible with the monodromy representation.
(3) If (s,μ) is such that s is semi-simple, μ is sufficiently close to $u_i(s)$, and β is the cycle vanishing over $\mu = u_i(s)$, then

$$P_{\beta,\beta}(s,\lambda,\mu) = P_{i,i}(s,\lambda,\mu).$$

Property (2) means that if C is a closed loop in $(S \times \mathbb{C})'$ based at (s,μ) and w is the monodromy transformation of $\mathfrak{h}$ corresponding to the parallel transport of cycles along C, then the analytic continuation of $P^{(k)}_{\alpha,\beta}(s,\mu)$ along C is $P^{(k)}_{w(\alpha),w(\beta)}(s,\mu)$.

The key to constructing propagators is the so called *phase 1-form* (see [2, 10])

$$\mathcal{W}_{\alpha,\beta}(s,\xi) = I^{(0)}_\alpha(s,\xi) \bullet I^{(0)}_\beta(s,0) \quad \in \quad T^*_s S,$$

where the period vectors are interpreted as elements in $T^*_s S$ and the multiplication in $T^*_s S$ is induced by the Frobenius multiplication via the natural identification $T^*_s S \cong T_s S$. The dependence on the parameter ξ is in the sense of a germ at $\xi = 0$, i.e., we will be interested in the Taylor's series expansion about $\xi = 0$. The phase form is a power series in ξ whose coefficients are multivalued 1-forms on $S' := (S \times \{0\})'$.

Lemma 14 *We have*

$$(\alpha|\beta) = -\iota_E \mathcal{W}_{\alpha,\beta}(s,0) = -(I^{(0)}_\alpha(s,0), E \bullet I^{(0)}_\beta(s,0)).$$

This is a well known fact due originally to K. Saito [18].

Lemma 15 *The phase form is weighted-homogeneous of weight* 0*, i.e.,*

$$(\xi\partial_\xi + L_E)\mathcal{W}_{\alpha,\beta}(s,\xi) = 0,$$

where L_E is the Lie derivative with respect to the vector field E.

Proof Note that

$$\mathcal{W}_{\alpha,\beta}(s,\xi) = (I^{(0)}_\alpha(s,\xi), dI^{(-1)}_\beta(s,0)).$$

It is easy to check that $\mathcal{W}_{\alpha,\beta}$ is a closed 1-form, so using the Cartan's magic formula $L_E = d\iota_E + \iota_E d$, where ι_E is the contraction by the vector field E, we get

$$L_E\mathcal{W}_{\alpha,\beta} = d(I^{(0)}_\alpha(s,\xi), (\theta+1/2)I^{(-1)}_\beta(s,0)) = -d((\theta-1/2)I^{(0)}_\alpha(s,\xi), I^{(-1)}_\beta(s,0)).$$

We used that θ is skew-symmetric with respect to the residue pairing and that

$$\iota_E dI^{(-1)}_\beta(s,0) = EI^{(-1)}_\beta(s,0)) = (\theta+1/2)I^{(-1)}_\beta(s,0),$$

where the last equality comes from the differential equation (11) with $n=-1$ and $\lambda = 0$. Furthermore, using the Leibnitz rule we get

$$-((\theta-1/2)dI^{(0)}_\alpha(s,\xi), I^{(-1)}_\beta(s,0)) - ((\theta-1/2)I^{(0)}_\alpha(s,\xi), dI^{(-1)}_\beta(s,0)).$$

The first residue pairing, using the skew-symmetry of θ and the differential equation $dI^{(0)}_\alpha = -AI^{(1)}_\alpha$ becomes

$$(AI^{(1)}_\alpha(s,\xi), (\theta+1/2)I^{(-1)}_\beta(s,0)) = -(AI^{(1)}_\alpha(s,\xi), E\bullet I^{(0)}_\beta(s,0)). \tag{20}$$

Similarly, the 2nd residue pairing becomes

$$((\xi\partial_\xi + E)I^{(0)}_\alpha(s,\xi), dI^{(-1)}_\beta(s,0)) = \xi\partial_\xi\mathcal{W}_{\alpha,\beta}(s,\xi) + (E\bullet I^{(1)}_\alpha(s,\xi), AI^{(0)}_\beta(s,0)). \tag{21}$$

On the other hand, recall that $A = \sum_i(\phi_i\bullet)dt_i$ and that the Frobenius multiplication is commutative. In particular, $[A, E\bullet] = 0$, so the terms (20) and (21) add up to $\xi\partial_\xi\mathcal{W}_{\alpha,\beta}(s,\xi)$. The lemma follows. □

Given cycles $\alpha, \beta \in \mathfrak{h}$, then we define

$$P_{\alpha,\beta}(s,\lambda,\mu) := \partial_\lambda \partial_\mu \int_{s_0}^{s-\mu\mathbf{1}} \mathcal{W}_{\alpha,\beta}(s',\lambda-\mu), \tag{22}$$

where the integration is along a path C in $(S \times \mathbb{C})'$, s.t., s_0 is a generic point on the discriminant and the cycle $\beta_s \in H_n(X_{s,0};\mathbb{C})$ vanishes along C.

Proposition 16 *The integral in definition* (22) *is convergent, independent of the choice of path along which* β *vanishes, and the Laurent series expansion at* $\lambda = \mu$ *of* $P_{\alpha,\beta}(s,\lambda,\mu)$ *defines a propagator from* α *to* β *on* $(S \times \mathbb{C})'$.

Proof The integral can be computed explicitly in terms of the period integrals, because according to Lemma 15 we have

$$\partial_\lambda \mathcal{W}_{\alpha,\beta}(s',\lambda-\mu) = -d\left(\frac{1}{\lambda-\mu}\,\iota_E \mathcal{W}_{\alpha,\beta}(s',\lambda-\mu)\right),$$

which by definition is

$$d\left(\frac{1}{\lambda-\mu}\,(I^{(0)}_\alpha(s',\lambda-\mu),(\theta+1/2)I^{(-1)}_\beta(s',0)\right).$$

Using that $I^{(-1)}_\beta(s',0)$ vanishes as $s' \to s_0$, we get

$$P_{\alpha,\beta}(s,\lambda,\mu) = \partial_\mu\left(\frac{1}{\lambda-\mu}\,(I^{(0)}_\alpha(s,\lambda),(\theta+1/2)I^{(-1)}_\beta(s,\mu))\right).$$

The above series has a Laurent series expansion at $\lambda = \mu$ with a pole of order 2 and no residue. The leading order term is

$$\begin{aligned}&\frac{1}{(\lambda-\mu)^2}\,(I^{(0)}_\alpha(s,\mu),(\theta+1/2)I^{(-1)}_\beta(s,\mu))\\ &\quad= \frac{1}{(\lambda-\mu)^2}\,(I^{(0)}_\alpha(s,\mu),(\mu-E\bullet)I^{(0)}_\beta(s,\mu)) = \frac{(\alpha|\beta)}{(\lambda-\mu)^2},\end{aligned}$$

where the last equality follows from Saito's formula (see Lemma 14).

It remains only to prove that if s is a semi-simple point, λ and μ are sufficiently close to a critical value $u_i(s)$, and $\alpha = \beta$ is vanishing cycle vanishing over $\lambda = u_i(s)$, then $P_{\alpha,\beta}(s,\lambda,\mu) = P_{i,i}(s,\lambda,\mu)$. Since in definition (22) we can choose the generic point s_0 and the integration path as we wish, let us pick $s_0 = s - u_i(s)\mathbf{1}$ and integrate along the straight segment $[s_0, s-\mu\mathbf{1}]$. Using integration by parts together with

$$(I_\alpha^{(k)}(s',\lambda-\mu), dI_\beta^{(-k-1)}(s',0)) =$$
$$d(I_\alpha^{(k)}(s',\lambda-\mu), I_\beta^{(-k-1)}(s',0)) - (I_\alpha^{(k+1)}(s',\lambda-\mu), dI_\beta^{(-k-2)}(s',0))$$

it is easy to see that the integral in (22) coincides with the Laurent series expansion at $\mu = u_i(s)$ of the symplectic pairing

$$\Omega(\mathbf{f}_\alpha^+(s,\lambda;z), \mathbf{f}_\beta^-(s,\mu;z)) = \sum_{k=0}^{\infty}(-1)^{k+1}(I_\alpha^{(k)}(s,\lambda), I_\beta^{(-k-1)}(s,\mu)).$$

This completes the proof. □

Note that in the course of the proof we derived the following explicit formulas for the the coefficients $P_{\alpha,\beta}^{(k)}(s,\mu)$ of the Laurent series expansion in $(\lambda-\mu)$ of the propagator:

$$\frac{1}{(k+1)!}(I_\alpha^{(k+1)}(s,\mu), (\theta+1/2)I_\beta^{(0)}(s,\mu)) + \frac{1}{(k+2)!}(I_\alpha^{(k+2)}(s,\mu), (\theta+1/2)I_\beta^{(-1)}(s,\mu)).$$

In particular,

$$P_{\alpha,\beta}^{(0)}(s,\mu) = \frac{1}{2}\Big((\mu - E\bullet_s)I_\alpha^{(1)}(s,\mu), I_\beta^{(1)}(s,\mu)\Big) = \frac{1}{2}\Big((\theta-1/2)I_\alpha^{(0)}(s,\mu), I_\beta^{(1)}(s,\mu)\Big). \tag{23}$$

Note that the propagator $P_{\alpha,\beta}^{(0)}(s,\mu)$ is symmetric with respect to α and β.

4.3 Twisted Representation of the Heisenberg VOA

Let us denote by $\mathcal{F} = \mathrm{Sym}(\mathfrak{h}[\zeta^{-1}]\zeta^{-1})$. Given $a \in \mathfrak{h}$ it is convenient to put $a_{(-n-1)} := a\zeta^{-n-1}$, then every element in $\mathcal{F}$ is a linear combination of elements of the type

$$a = \alpha^1_{(-k_1-1)}\cdots\alpha^r_{(-k_r-1)}, \quad \alpha^i \in \mathfrak{h}, \quad k_i \geq 0.$$

Following [2] we define differential operators acting on the Fock space as follows. First we define

$$X_{s,\lambda}(\alpha) := \widehat{\phi}_\alpha(s,\lambda), \quad \alpha \in \mathfrak{h}, \tag{24}$$

where we identify $\alpha \in \mathfrak{h}$ with $\alpha_{(-1)} \in \mathcal{F}$ and put $\widehat{\phi}_\alpha(s,\lambda) = (\partial_\lambda \mathbf{f}_\alpha(s,\lambda;z))^\wedge$, then we set

$$X_{s,\lambda}(a) = \sum_J \Big(\prod_{(i,j)\in J} \partial_\lambda^{(k_j)}\, P^{(k_i)}_{\alpha^i,\alpha^j}(s,\lambda)\Big) :\Big(\prod_{l\in J'} \partial_\lambda^{(k_l)} X_{s,\lambda}(\alpha^l)\Big):, \tag{25}$$

where $\partial_\lambda^{(k)} := \frac{\partial_\lambda^k}{k!}$ and the sum is over all collections J of disjoint ordered pairs $(i_1, j_1), \ldots, (i_s, j_s) \subset \{1, \ldots, r\}$ such that $i_1 < \cdots < i_s$ and $i_l < j_l$ for all l, and $J' = \{1, \ldots, r\} \setminus \{i_1, \ldots, i_s, j_1, \ldots, j_s\}$. Although we are not going to use the theory of vertex algebras here, let us point out that formula (25) is obtained by the axioms of vertex operator algebra representations. Namely, the vector space $\mathcal{F}$ has a standard structure of a Heisenberg Vertex Operator Algebra (VOA) and the fields (24) are known to be local to each other. It was proved in [2], that the definition (24) extends uniquely to a σ-twisted representation of $\mathcal{F}$, where σ is the classical monodromy corresponding to a big loop that goes around the discriminant.

For $(s,\lambda) \in (S\times\mathbb{C})'$ and $c_1, \ldots, c_r \in \mathfrak{h}$ we define

$$\Omega^{(g)}_{c_1\cdots c_r}(s,\lambda;\mathbf{t}) \quad \in \quad \mathbb{C}[\![t_0, t_1, \ldots,]\!]$$

by the following equation

$$X_{s,\lambda}(c_1\cdots c_r)\mathcal{A}_s(\hbar;\mathbf{q}) =: \sum_{g=0}^{\infty} \hbar^{g-\frac{r}{2}}\, \Omega^{(g)}_{c_1\cdots c_r}(s,\lambda;\mathbf{q})\mathcal{A}_s(\hbar;\mathbf{q}), \tag{26}$$

where in order to define $\Omega^{(g)}_{c_1\cdots c_r}(s,\lambda;\mathbf{t})$ we replace $\mathbf{q}$ by $\mathbf{t}$ *without* using the dilaton shift. If we denote by $W \subset \mathrm{GL}(\mathfrak{h})$ the monodromy group, then W acts naturally on $\mathcal{F}$ via $w(a_{(-n-1)}) := (w(a))_{(-n-1)}$. Since both the generating fields (24) and the propagators are compatible with the monodromy representation we get that the analytic continuation of $X_{s,\lambda}(a)$ along a closed loop C in $(S\times\mathbb{C})'$ is $X_{s,\lambda}(w(a))$, where $w \in W$ is the monodromy transformation corresponding to the loop C. In particular, the analytic continuation in (s,λ) along C of $\Omega^{(g)}_{c_1\cdots c_r}(s,\lambda;\mathbf{t})$ is $\Omega^{(g)}_{w(c_1)\cdots w(c_r)}(s,\lambda;\mathbf{t})$.

4.4 Extension Through a Generic Non-semisimple Point

Let $b_0 \in B$ be a generic point, so that $F(b_0,x)$ has $N-2$ critical points of type A_1 and 1 critical point of type A_2. The critical values corresponding to the A_1-critical points will be denoted by $u_i(b_0)$ $(1 \le i \le N-2)$ and we will assume that they are pairwise distinct. All points $b \in B$ that do not satisfy the above property are points in some codimension 2 analytic subvariety of S. Therefore, according to Hartogue's extension theorem, in order to prove that the ancestor potential extends analytically, it is enough to prove that it extends analytically at $s = b_0$.

4.4.1 Fixing a Neighborhood of b_0

We fix pairwise disjoint sufficiently small disks D_i, $1 \le i \le N-1$, in $\mathbb{C}$, s.t., the center of D_i is the critical value $u_i(b_0)$, where $u_{N-1}(b_0)$ is the critical value of the A_2-critical point. Put

$$S_0 = \{s \in S \mid u_i(s) \in D_i (1 \le i \le N-2),\ (u_{N-1}(s), u_N(s)) \in (D_{N-1} \times D_{N-1})/\mathbb{Z}_2\},$$

where $\mathbb{Z}_2 = \mathbb{Z}/2\mathbb{Z}$ acts on $D_{N-1} \times D_{N-1}$ by permuting the coordinates. Note that $S_0 \subset S$ is an open neighborhood of b_0 homeomorphic to

$$D_1 \times \cdots \times D_{N-2} \times (D_{N-1} \times D_{N-1})/\mathbb{Z}_2.$$

Furthermore, if we restrict the Milnor fibration to $(S_0 \times D_0)'$, where $D_0 = \sqcup_{i=1}^{N-1} D_i$, then the vanishing cycles Δ_0 of the new fibration form a disjoint union

$$\Delta_0 = \Delta_1 \sqcup \cdots \sqcup \Delta_{N-1},$$

where Δ_i $(1 \le i \le N-1)$ is the set of cycles vanishing respectively over $\lambda = u_i(b_0)$ $(1 \le i \le N-1)$ along some path in $(S_0 \times D_0)'$. Note that Δ_{N-1} is a A_2-root system, while the remaining Δ_i's are A_1-root systems.

4.4.2 Extending the Recursion

Let us rewrite the Eynard–Orantin recursion in terms of the operators (25). By definition

$$X_{s,\lambda}(\beta_i^2) =: \widehat{\phi}_{\beta_i}(s,\lambda)^2 : + P^{(0)}_{\beta_i,\beta_i}(s,\lambda).$$

Let us denote by $\widehat{\phi}^{\pm}_{\alpha}(s,\lambda)$ the quantization of $\phi^{\pm}_{\alpha}(s,\lambda;z)$, then the above operator becomes

$$\widehat{\phi}^{+}_{\beta_i}(s,\lambda)^2 + 2\widehat{\phi}^{-}_{\beta_i}(s,\lambda)\widehat{\phi}^{+}_{\beta_i}(s,\lambda) + \widehat{\phi}^{-}_{\beta_i}(s,\lambda)^2 + P^{(0)}_{\beta_i,\beta_i}(s,\lambda). \tag{27}$$

Recalling the definition (26), we get that $\Omega^{(g)}_{\beta_i,\beta_i}(s,\lambda;\mathbf{t})$ can be written as a sum of 4 type of terms corresponding to the 4 summands in (27). Let us compare $\Omega^{(g)}_{\beta_i,\beta_i}(s,\lambda,\mathbf{t})$ with the sum of correlators that appear in the big brackets on the RHS of the local Eynard–Orantin recursion in Theorem 12. The contributions of the 1st summand in (27) coincide with the sum of all stable correlators, the 2nd summand in (27) corresponds to the sum of all products of an unstable correlator of type (14) and a stable correlator, the third summand depends analytically on $\lambda - u_i$ so it does not contribute to the residue, and finally the 4th summand corresponds to the contribution of the unstable correlator (15). Hence, the local Eynard-Orantin recursion stated in Theorem 12 can be written conveniently in the following way

$$\langle \phi_a\, \psi^m \rangle_{g,1}(s;\mathbf{t}) = -\frac{1}{4}\sum_{i=1}^{N} \mathrm{Res}_{\lambda=u_i}\, \frac{\Omega(\phi_a\, z^m, \mathbf{f}^-_{\beta_i}(s,\lambda;z))}{y_{\beta_i}(s,\lambda)}\, \Omega^{(g)}_{\beta_i,\beta_i}(s,\lambda;\mathbf{t})\, d\lambda, \tag{28}$$

where β_i is a cycle vanishing over $\lambda = u_i$ and

$$y_\beta(s,\lambda) := (I^{(-1)}_\beta(s,\lambda), 1).$$

Let $\{\alpha, \beta\}$ be a basis of simple roots for Δ_{N-1}. Put

$$\chi_1 = \frac{2}{3}\alpha + \frac{1}{3}\beta, \quad \chi_2 = -\frac{1}{3}\alpha + \frac{1}{3}\beta, \quad \chi_3 = -\frac{1}{3}\alpha - \frac{2}{3}\beta.$$

We refer to these as 1-point cycles. Note that the root system Δ_{N-1} consists of all differences $\chi_i - \chi_j$ for $i \neq j$. Motivated by the construction of Bouchard–Eynard [3] we introduce the following integral

$$-\frac{1}{2\pi\sqrt{-1}} \oint \sum_{c_1,\ldots,c_r} \frac{1}{(r-1)!}\, \frac{\Omega(\phi_a\, z^m, \mathbf{f}^-_{c_1}(s,\lambda;z))}{\prod_{k=2}^{r} y_{c_k-c_1}(s,\lambda)}\, \Omega^{(g)}_{c_1\cdots c_r}(s,\lambda;\mathbf{t})\, d\lambda, \tag{29}$$

where the integral is along a closed loop in D_{N-1} that goes once counterclockwise around the critical values $u_{N-1}(s)$ and $u_N(s)$ and the sum is over all $r = 2, 3$ and all $c_1, \ldots, c_r \in \{\chi_1, \chi_2, \chi_3\}$ such that $c_i \neq c_j$ for $i \neq j$. Note that the integrand is monodromy invariant (see Sect. 4.3), hence a single valued analytic 1-form in $D_{N-1} \setminus \{u_{N-1}(s), u_N(s)\}$, so the integral makes sense.

Theorem 17 *The integral* (29) *coincides with the sum of the last two summands in* (28) *corresponding to the residues at* $\lambda = u_{N-1}, u_N$.

4.4.3 Proof of Theorem 17

The proof relies on a certain identity that we would like to present first. Let us denote by $\mathfrak{h}_{\Delta_i}$ $(1 \leq i \leq N)$ the vector subspace of $\mathfrak{h}$ spanned by the root system Δ_i (we assume that $\Delta_N = \Delta_{N-1}$). Let u_i and u_j $(1 \leq i, j \leq N)$ be two of the critical values, $\beta := \beta_j$ be the cycle vanishing over u_j, and $a \in \mathfrak{h}_{\Delta_i}$. Let us fix some Laurent series

$$f(\lambda,\mu) \in (\lambda - u_i)^{1/2}\mathbb{C}((\lambda - u_i, \mu - u_j)) + \mathbb{C}((\lambda - u_i, \mu - u_j))$$

where $\mathbb{C}((\lambda - u_i, \mu - u_j))$ denotes the space of formal Laurent series. We will have to evaluate residues of the following form:

$$\mathrm{Res}_{\lambda=u_i} \mathrm{Res}_{\mu=u_j} \sum_{\text{all branches}} \frac{\Omega(\phi_a^+(s,\lambda;z), \mathbf{f}_\beta^-(s,\mu;z))}{y_\beta(s,\mu)} f(\lambda,\mu)\, d\mu, \tag{30}$$

where the sum is over all branches (2 of them) of the multivalued function that follows.

Lemma 18 *If $f(\lambda,\mu)$ does not have a pole at $\lambda = u_i$; then the residue (30) is non-zero only if $i = j$ and in the latter case it equals to*

$$(a|\beta)\, \mathrm{Res}_{\lambda=u_i} \sum_{\text{all branches}} \frac{f(\lambda,\lambda)}{y_\beta(s,\lambda)} d\lambda.$$

Proof Put $a = a' + (a|\beta_i)\beta_i/2$; then a' is invariant with respect to the monodromy around $\lambda = u_i$. From this we get that $\phi_{a'}^+(s,\lambda;z)$ is analytic at $\lambda = u_i$, so it does not contribute to the residue. In other words, it is enough to prove the lemma only for $a = \beta_i$. Let us assume that $a = \beta_i$. Recall that by definition $\beta = \beta_j$, then we get

$$\Omega(\phi_a^+(s,\lambda;z), \mathbf{f}_\beta^-(s,\mu;z)) = \Omega(\mathbf{f}_{\beta_j}^+(s,\mu;z), \phi_{\beta_i}^-(s,\lambda;z)) + \Omega(\phi_{\beta_i}(s,\lambda;z), \mathbf{f}_{\beta_j}(s,\mu;z)).$$

The first symplectic pairing on the RHS does not contribute to the residue, because $\phi_a^-(s,\lambda;z)$ has a pole of order at most $\frac{1}{2}$ so after taking the sum over all branches, the poles of fractional degrees cancel out and hence the 1-form at hands is analytic at $\lambda = u_i$. For the second symplectic pairing, recalling that $\mathbf{f}_{\beta_k}(s,\lambda;z) = \Psi R \mathbf{f}_{A_1}(u_k,\lambda;z)$ for $k = i, j$, we get

$$\Omega(\phi_{A_1}(u_i,\lambda;z)e_i, \mathbf{f}_{A_1}(u_j,\mu;z)e_j) = 2\delta_{i,j} \frac{(\mu-u_j)^{\frac{1}{2}}}{(\lambda-u_i)^{\frac{1}{2}}} \delta(\lambda-u_i, \mu-u_j),$$

where

$$\delta(x,y) = \sum_{n\in\mathbb{Z}} x^n y^{-n-1}$$

is the formal δ-function. It is an easy exercise to check that for every $f(y) \in \mathbb{C}((y))$ we have

$$\mathrm{Res}_{y=0}\, \delta(x,y) f(y) = f(x).$$

The lemma follows. □

The integral (29) can be written as a sum of two residues: $\mathrm{Res}_{\lambda=u_{N-1}}$ and $\mathrm{Res}_{\lambda=u_N}$. We claim that each of these residues can be reduced to the corresponding residue in the sum (28). Let us present the argument for $\lambda = u_{N-1}$. The other case is completely analogous.

Let $\alpha = \beta_{N-1}$ be the cycle vanishing over u_{N-1}. The summands in (29) for which $r = 2$ and $c_1, c_2 \in \{\chi_1, \chi_2\}$ give precisely

$$\mathrm{Res}_{\lambda=u_{N-1}} \frac{\Omega(\phi_a z^m, \mathbf{f}^-_{\chi_1-\chi_2}(s,\lambda;z))}{y_{\chi_1-\chi_2}(s,\lambda)}\, \Omega^{(g)}_{\chi_1,\chi_2}(s,\lambda;\mathbf{t})\, d\lambda.$$

On the other hand, using that $\alpha = \chi_1 - \chi_2$ we get

$$\Omega^{(g)}_{\chi_1,\chi_2}(s,\lambda;\mathbf{t}) = -\frac{1}{4}\,\Omega^{(g)}_{\alpha,\alpha}(s,\lambda;\mathbf{t}) + \frac{1}{4}\,\Omega^{(g)}_{\chi_1+\chi_2,\chi_1+\chi_2}(s,\lambda;\mathbf{t})$$

Since $(\chi_1 + \chi_2|\alpha) = 0$, the form $\Omega^{(g)}_{\chi_1+\chi_2,\chi_1+\chi_2}(s,\lambda;\mathbf{t})$ is analytic at $\lambda = u_{N-1}$, so it does not contribute to the residue. Therefore we obtain precisely the $(N-1)$-st residue in (28). It remain only to see that the remaining summands with $r = 2$ cancel out with the summand with $r = 3$.

There are two types of quadratic summands: $c_1, c_2 \in \{\chi_1, \chi_3\}$ and $c_1, c_2 \in \{\chi_2, \chi_3\}$. They add up respectively to

$$\frac{\Omega(\phi_a z^m, \mathbf{f}^-_{\chi_1-\chi_3}(s,\lambda;z))}{y_{\chi_1-\chi_3}(s,\lambda)}\, \Omega^{(g)}_{\chi_1,\chi_3}(s,\lambda;\mathbf{t})\, d\lambda \tag{31}$$

and

$$\frac{\Omega(\phi_a z^m, \mathbf{f}^-_{\chi_2-\chi_3}(s,\lambda;z))}{y_{\chi_2-\chi_3}(s,\lambda)}\, \Omega^{(g)}_{\chi_2,\chi_3}(s,\lambda;\mathbf{t})\, d\lambda. \tag{32}$$

By definition

$$\sum_{g=0}^{\infty} \hbar^{g-3/2}\, \Omega^{(g)}_{\chi_i,\chi_3}(s,\lambda;\mathbf{t})\mathcal{A}_s = \hbar^{-1/2}\Big(:\widehat{\phi}_{\chi_i}(s,\lambda)\widehat{\phi}_{\chi_3}(s,\lambda): + P^{(0)}_{\chi_i,\chi_3}(s,\lambda)\Big)\, \mathcal{A}_s. \tag{33}$$

We claim that the propagators $P^{(0)}_{\chi_i,\chi_3}(s,\lambda)$ in (33) do not contribute to the residue at $\lambda = u_{N-1}$. Indeed, their contribution is given by the residue at $\lambda = u_{N-1}$ of the following function

$$\frac{\Omega(\phi_a z^m, \mathbf{f}^-_{\chi_1-\chi_3}(s,\lambda;z))}{y_{\chi_1-\chi_3}(s,\lambda)}\, P^{(0)}_{\chi_1,\chi_3}(s,\lambda) + \frac{\Omega(\phi_a z^m, \mathbf{f}^-_{\chi_2-\chi_3}(s,\lambda;z))}{y_{\chi_2-\chi_3}(s,\lambda)}\, P^{(0)}_{\chi_2,\chi_3}(s,\lambda).$$

Note that the above expression is invariant with respect to the local monodromy around $\lambda = u_{N-1}$ and that the coefficients in front of $P^{(0)}_{\chi_i,\chi_3}(s,\lambda)$ do not have a pole at $\lambda = u_{N-1}$. Recalling formula (23) we get that $P^{(0)}_{\chi_i,\chi_3}(s,\lambda)$ has a pole of order at mots $1/2$, which implies that the entire expression is analytic at $\lambda = u_{N-1}$.

The normally ordered product on the RHS of (33) is by definition

$$\widehat{\phi}_{\chi_3}(s,\lambda)\,\widehat{\phi}^{+}_{\chi_i}(s,\lambda) + \widehat{\phi}^{-}_{\chi_i}(s,\lambda)\widehat{\phi}_{\chi_3}(s,\lambda). \tag{34}$$

Since $(\chi_3|\alpha) = 0$ the field $\widehat{\phi}_{\chi_3}(t,\lambda)$ is analytic at $\lambda = u_{N-1}$. In addition $\widehat{\phi}^{-}_{\chi_i}(t,\lambda)$ has a pole of order at most $\frac{1}{2}$ at $\lambda = u_{N-1}$. It follows that the second summand in (34) does not contribute to the residue and therefore it can be ignored. For the RHS of (33) we get

$$\sum_{g=0}^{\infty} \hbar^{g-1}\,\widehat{\phi}_{\chi_3}(s,\lambda)\,\langle \phi^{+}_{\chi_i}(s,\lambda;\psi)\rangle_{g,1}(t;\mathbf{t})\,\mathcal{A}_s,$$

which after recalling the local recursion (28) becomes

$$-\frac{1}{4}\sum_{j=1}^{N} \mathrm{Res}_{\mu=u_j} \frac{\Omega(\phi^{+}_{\chi_i}(s,\lambda;z),\mathbf{f}^{-}_{\beta_j}(s,\mu;z))}{y_{\beta_j}(s,\mu)}\,\widehat{\phi}_{\chi_3}(s,\lambda)\,X^{u_j}_{s,\mu}(\beta_j^2)\,d\mu\,\mathcal{A}_s,$$

where $X^{u_j}_{s,\mu}(a)$ is the Laurent series expansion of $X_{s,\mu}(a)$ in $(\mu - u_j)$. Therefore we need to compute the residues $\mathrm{Res}_{\lambda=u_{N-1}}\,\mathrm{Res}_{\mu=u_j}$ of the following expressions

$$-\frac{1}{4}\sum_{i=1,2} \frac{\Omega(\phi_a z^m,\mathbf{f}^{-}_{\chi_i-\chi_3}(s,\lambda;z))}{y_{\chi_i-\chi_3}(s,\lambda)}\,\frac{\Omega(\phi^{+}_{\chi_i}(s,\lambda;z),\mathbf{f}^{-}_{\beta_j}(s,\mu;z))}{y_{\beta_j}(s,\mu)}\,\widehat{\phi}_{\chi_3}(s,\lambda)\,X^{u_j}_{s,\mu}(\beta_j^2)\,d\mu\,\mathcal{A}_s.$$

The operator $\widehat{\phi}_{\chi_3}(s,\lambda)\,X^{u_j}_{s,\mu}(\beta_j^2)$ can be written as

$$:\widehat{\phi}_{\beta_j}(s,\mu)^2\,\widehat{\phi}_{\chi_3}(s,\lambda): +2[\widehat{\phi}^{+}_{\chi_3}(s,\lambda),\widehat{\phi}^{-}_{\beta_j}(s,\mu)]\,\widehat{\phi}_{\beta_j}(s,\mu) + P^{(0)}_{\beta_j,\beta_j}(s,\mu)\widehat{\phi}_{\chi_3}(s,\lambda). \tag{35}$$

Since $(\chi_3|\alpha) = 0$ the operator $\widehat{\phi}^{+}_{\chi_3}(s,\lambda)$ is regular at $\lambda = u_{N-1}$. It follows that the commutator

$$[\widehat{\phi}^{+}_{\chi_3}(s,\lambda),\widehat{\phi}^{-}_{\beta_j}(s,\mu)] \in \mathbb{C}(\!(\lambda - u_{N-1},\mu - u_j)\!)$$

and therefore we may recall Lemma 18. The above residue is non-zero only if $j = N-1$. In the latter case we get

$$-\frac{1}{4}\operatorname{Res}_{\lambda=u_{N-1}} \sum_{i=1,2} (\chi_i|\alpha)\, \frac{\Omega(\phi_a z^m, \mathbf{f}^-_{\chi_i-\chi_3}(s,\lambda;z))}{y_{\chi_i-\chi_3}(s,\lambda)\, y_\alpha(s,\lambda)}\, \widehat{\phi}_{\chi_3}(s,\lambda)\, X^{u_{N-1}}_{s,\lambda}(\alpha^2)\, d\lambda\, \mathcal{A}_s. \tag{36}$$

Note that

$$[\widehat{\phi}^+_{\chi_3}(s,\lambda), \widehat{\phi}^-_{\beta_j}(s,\mu)] = \iota_{\lambda-u_{N-1}}\, \iota_{\mu-u_{N-1}}\; P_{\chi_3,\beta_j}(s,\lambda,\mu),$$

where $\iota_{\lambda-u_{N-1}}$ is the Laurent series expansion at $\lambda = u_{N-1}$. Hence

$$\widehat{\phi}_{\chi_3}(s,\lambda)\, X^{u_{N-1}}_{s,\lambda}(\alpha^2) = \iota_{\lambda-u_{N-1}} X_{s,\lambda}(\chi_3\alpha^2).$$

By definition

$$-\frac{1}{4}\alpha^2 = \chi_1\,\chi_2 - \frac{1}{4}\,\chi_3^2$$

and since χ_3 is invariant with respect to the local monodromy around $\lambda = u_{N-1}$, the field $X_{s,\lambda}(\chi_3^3)$ does not contribute to the residue. We get the following formula for the residue (36):

$$\operatorname{Res}_{\lambda=u_{N-1}} \sum_{i=1,2} (\chi_i|\alpha)\, \frac{\Omega(\phi_a z^m, \mathbf{f}^-_{\chi_i-\chi_3}(s,\lambda;z))}{y_{\chi_i-\chi_3}(s,\lambda)\, y_\alpha(s,\lambda)}\, X^{u_{N-1}}_{s,\lambda}(\chi_1\chi_2\chi_3)\, d\lambda\, \mathcal{A}_s.$$

Using that $\alpha = \chi_1 - \chi_2$, $(\chi_1|\alpha) = 1$, and $(\chi_2|\alpha) = -1$ we get

$$\operatorname{Res}_{\lambda=u_{N-1}} \Bigg(\frac{\Omega(\phi_a z^m, \mathbf{f}^-_{\chi_1}(s,\lambda;z))}{y_{\chi_2-\chi_1}(s,\lambda)\, y_{\chi_3-\chi_1}(s,\lambda)} + \frac{\Omega(\phi_a z^m, \mathbf{f}^-_{\chi_2}(s,\lambda;z))}{y_{\chi_1-\chi_2}(s,\lambda)\, y_{\chi_3-\chi_2}(s,\lambda)}$$
$$+ \frac{\Omega(\phi_a z^m, \mathbf{f}^-_{\chi_3}(s,\lambda;z))}{y_{\chi_1-\chi_3}(s,\lambda)\, y_{\chi_2-\chi_3}(s,\lambda)} \Bigg) \times \sum_{g=0}^{\infty} \hbar^{g-3/2}\Omega^{(g)}_{\chi_1\chi_2\chi_3}(s,\lambda;\mathbf{t})\, d\lambda\, \mathcal{A}_s.$$

This sum cancels out the contribution to the residue at $\lambda = u_{N-1}$ of the cubic terms (i.e. the terms with $r = 3$) of the integral (29). □

Note that in the integral (29) we may choose the integration contour to be the boundary of the disk D_{N-1}. Since the integrand in (29) has singularities only at the critical values $u_{N-1}(s)$ and $u_N(s)$, which are inside the disk D_{N-1} for all $s \in S_0$, we get that the integral (29) depends analytically on $s \in S_0$. Using Theorem 17 we can set up a recursion that produces functions analytic in a neighborhood of any generic point $b_0 \in B$. For example, let us write the recursion for $\langle \phi_a \rangle_{1,1}(s;0)$. Since

$$\Omega^{(1)}_{c_1,c_2}(s,\lambda;0) = P^{(0)}_{c_i,c_j}(s,\lambda), \quad \Omega^{(1)}_{c_1,c_2,c_3}(s,\lambda;0) = 0,$$

we have

$$\langle \phi_a \rangle_{1,1}(s;0) = \frac{1}{4}\sum_{i=1}^{N-2} \frac{1}{2\pi\sqrt{-1}} \oint_{C_i} \frac{(I^{(-1)}_{\beta_i}(s,\lambda),\phi_a)}{(I^{(-1)}_{\beta_i}(s,\lambda),\mathbf{1})} P^{(0)}_{\beta_i,\beta_i}(s,\lambda)d\lambda$$
$$-\frac{1}{2\pi\sqrt{-1}} \oint_{C_{N-1}} \sum_{1\le i<j\le 3} \frac{(I^{(-1)}_{\chi_i-\chi_j}(s,\lambda),\phi_a)}{(I^{(-1)}_{\chi_i-\chi_j}(s,\lambda),\mathbf{1})} P^{(0)}_{\chi_i,\chi_j}(s,\lambda)d\lambda,$$

where C_i is the boundary of the disk D_i, $1 \le i \le N-1$. By definition if $(s,\lambda) \in S_0 \times C_i$, then (s,λ) is not a point on the discriminant. Note that $I^{(-1)}_{\varphi}(s,\lambda) \neq 0$ if φ is a vanishing cycle and (s,λ) is not on the discriminant, because according to Lemma 14

$$2 = (\varphi|\varphi) = (I^{(0)}_{\varphi}(s,\lambda), (\theta+1/2)I^{(-1)}_{\varphi}(s,\lambda)).$$

All integrals depend analytically on $s \in S_0$, so the correlator $\langle \phi_a \rangle_{1,1}(s;0)$ is analytic in the entire neighborhood S_0 of $b_0 \in B$. Using induction on the lexicographical order of the pairs (g,n), where g is the genus and n is the number of insertions, we can prove by induction that all ancestor correlators are analytic in the neighborhood S_0. Theorem 1 follows from the Hartogues extension theorem.

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