Eynard-Orantin B-Model and Its Application in Mirror Symmetry

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Abstract We describe the Eynard-Orantin recursive algorithm on a spectral curve, and give a biased survey on its roles as B-models which predict various higher genus A-model invariants via mirror symmetry.

1 Introduction

The Eynard-Orantin topological recursion is a recursive algorithm from the matrix model theory [24]. Mathematically speaking, it starts with an affine plane curve Σ with a choice of a fundamental normalized differential of the second kind, and then the algorithm recursively produces a series of symmetric meromorphic forms $\omega_{g,n}$ on the product of n copies of Σ. These $\omega_{g,n}$ are called B-model higher genus invariants. They are genus g correlators with n boundary components. We will survey two aspects of this recursive algorithm—its relation to a quantization of a semisimple Frobenius manifold, and its role in mirror symmetry.

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1.1 Relation to Givental's Quantization and Abstract Frobenius Structure

The Eynard-Orantin recursion is a recursive algorithm in computing higher genus invariants of a Frobenius manifold, as shown in [19]. They show that we can define the recursion formally around each ramification point using the data from a calibrated Frobenius manifold, the recursion is equivalent to Givental's quantization [39, 40]. Another important theorem of [19] is that they express Eynard-Orantin higher genus invariants as graph sums. This allows us to compare with the graph sum formula of Gromov-Witten invariants, which is essential in the proof of mirror symmetry involving Eynard-Orantin recursion.

1.2 Eynard-Orantin Recursion as B-Model in Mirror Symmetry

Mirror symmetry is the equivalence between the A-model (about the Kähler structure of a manifold) and the B-model of its mirror (about the complex structure). When the spectral curve is the B-model mirror to some A-model, like topological strings on a toric Calabi-Yau threefold, the Eynard-Orantin B-model invariants predict the A-model strings correctly. This is the Bouchard-Klemm-Mariño-Pasquetti (BKMP, [11, 12, 55]) remodeling conjecture, proved recently in [25, 31, 32]. There are also various mirror symmetry statements along this line, as long as one can have a mirror curve as the B-model, e.g. the case of $\mathbb{P}^1$ (Norbury-Scott conjecture [19, 30, 59]), Bouchard-Mariño conjecture in various settings [9, 10, 13].

These Eynard-Orantin higher genus invariants $\omega_{g,n}$ enjoy many nice properties. In [24], the authors discuss the variation of $\omega_{g,n}$ with respect to the moduli of the spectral curves. Also, the fundamental normalized differential of the second kind depends on the choice of a Torelli marking. It changes under a modular transformation. The modularity property of Gromov-Witten invariants for toric Calabi-Yau threefolds follows from the BKMP conjecture and the modularity property of the Eynard-Orantin B-model invariants $\omega_{g,n}$.

1.3 Structure of This Paper

We first review the definition of Eynard-Orantin topological recursion in Sect. 2. In Sect. 3 we will state the equivalence between the Eynard-Orantin topological recursion on a formal spectral curve and Givental's quantization on a Frobenius manifold. In Sect. 4 we will review the applications of Eynard-Orantin recursions to all genera mirror symmetry. The last section is about the modularity property of Gromov-Witten invariants from the modularity of Eynard-Orantin invariants, via the mirror symmetry statement introduced in Sect. 4.

This survey is far from covering the vast scope of the Eynard-Orantin topological recursion, which is a very active field of research of late. We do not systematically cover the fundamental properties of the Eynard-Orantin recursion, like the variations of $\omega_{g,n}$ with respect to the moduli of spectral curves [24]. There are many other fantastic applications of the recursion, not necessarily along the line of mirror symmetry for toric Calabi-Yau threefolds, like Weil-Pertersson volume [22, 23, 65]. The recent progress on “quantum curves”, and the application of Eynard-Orantin recursion to non-semisimple situations by taking non-semisimple limits, are also beyond the reach of this survey.

2 Spectral Curve and Eynard-Orantin Recursion

2.1 Spectral Curves

Let Σ be a smooth affine algebraic curve in $(\mathbb{C}^*)^2$. The coordinate Y maps Σ into the second component of $(\mathbb{C}^*)^2$. It is a holomorphic function on Σ. Let $Y = e^{-y}$. We denote the covering map π_Y

$$\pi_Y : \mathbb{C}^* \times \mathbb{C} \to (\mathbb{C}^*)^2,$$
$$(a, y) \mapsto (a, e^{-y}).$$

Let $\widetilde{\Sigma}$ be the lift of Σ under this map, and let $\overline{\Sigma}$ be a choice of smooth compactification of Σ, which is a compact Riemann surface.[1]

Recall that a Torelli marking on $\overline{\Sigma}$ is a choice of cycles $A_1, \ldots, A_{\mathfrak{g}}, B_1, \ldots, B_{\mathfrak{g}}$ in $H_1(\overline{\Sigma}; \mathbb{C})$, such that $A_i \cap B_j = \delta_{i,j}$ and $A_i \cap A_j = B_i \cap B_j = 0$, where $\mathfrak{g}$ is the genus of $\overline{\Sigma}$.[2] Given such a marking, following the notions [34], we define the fundamental normalized differential of the second kind (a.k.a. Bergman kernel in Eynard-Orantin [24]).

Definition 1 The *fundamental normalized differential of the second kind* (abbreviated as *fundamental differential* in this paper) associated to a Torelli marking on $\overline{\Sigma}$ is the symmetric meromorphic form on $(\overline{\Sigma})^2$ satisfying the following conditions.

- The only pole is the double pole along the diagonal, i.e. given any local coordinate ζ near a point $p \in \overline{\Sigma}$, the differential B has the following form near $(p, p) \in (\overline{\Sigma})^2$

$$B = \frac{d\zeta_1 d\zeta_2}{(\zeta_1 - \zeta_2)^2} + \text{holomorphic part}.$$

[1] We reserve the variable $X = e^{-x}$ for other purposes. In many but not all examples, it will be the first coordinate.

[2] We allow such cycles to be non-geometric, i.e. elements in $H_1(\overline{\Sigma}; \mathbb{C})$.

- It is normalized by the choice of A-cycles

$$\int_{q\in A_i} B(p,q) = 0, \text{ for } i = 1,\dots,\mathfrak{g}.$$

Remark 2 The pairing of cycles in $H_1(\overline{\Sigma};\mathbb{C})$ turns it into a symplectic vector space. The subspace spanned by A-cycles is a Lagrangian subspace. The fundamental differential B only depends on the choice of this Lagrangian subspace.

Definition 3 A spectral curve $\boldsymbol{\Sigma} = (\Sigma, x, B)$ consists of the following data:

- a smooth affine algebraic curve Σ in $(\mathbb{C}^*)^2$ together with a Torelli marking on $\overline{\Sigma}$;
- a holomorphic Morse function (superpotential) x from the universal cover of Σ to $\mathbb{C}^*$, such that dx descends to a meromorphic form on $\overline{\Sigma}$ with poles in $\overline{\Sigma}\setminus\Sigma$;
- a fundamental normalized differential of the second kind B on $\overline{\Sigma}$ with respect to such choice of A-cycles.

Remark 4 In the applications of the Eynard-Orantin recursion, very often $X = e^{-x}$ is the first coordinate of the affine curve Σ.

Fix a spectral curve $\boldsymbol{\Sigma}$. Since x is Morse, the critical points where $dx = 0$ form a finite set $\{p_\alpha : \alpha \in I_{\boldsymbol{\Sigma}}\}$. Define the Liouville form $\Phi = ydx = -\log Y dx$. It is a well-defined holomorphic form on $\widetilde{\Sigma}$, and is meromorphic on the smooth completion of $\widetilde{\Sigma}$.

At each critical point p_α, we define the local coordinate ζ_α by

$$x = \zeta_\alpha^2 + x_{0,\alpha},$$

where $x_{0,\alpha}$ is the critical value of x at p_α. For any p near p_α, let $\bar{p}$ be the point on Σ such that $\zeta_\alpha(\bar{p}) = -\zeta_\alpha(p)$.

The central topic of this survey, Eynard-Orantin's topological recursion, is essentially defined around each critical point of x on the spectral curve. Following [19], we define formal spectral curves below.[3]

Definition 5 A formal spectral curve $\mathfrak{C}$ is a disjoint union of $\{C_\alpha\}_{\alpha\in I_{\mathfrak{C}}}$ where each $C_\alpha = \mathrm{Spec}\mathbb{C}[[\zeta_\alpha]]$, together with following information.

- A function $y_\alpha = \sum_{i=0}^{\infty} h_i^\alpha (\zeta_\alpha)^i$ on C_α where $h_1^\alpha \neq 0$.
- A holomorphic Morse function $x_\alpha = x_{0,\alpha} + \zeta_\alpha^2$ on C_α.

[3]In [19], Eynard-Orantin recursions on such formal spectral curves are called local topological recursions.

- The "fundamental normalized differential of the second kind" $B^{\alpha,\beta} \in \Gamma(T^*(C_\alpha \times C_\beta \setminus C_{\alpha,\beta}))$

$$B^{\alpha,\beta}(\zeta_\alpha, \zeta_\beta) = \delta_{\alpha,\beta}\frac{d\zeta_\alpha \otimes d\zeta_\beta}{(\zeta_\alpha - \zeta_\beta)^2} + \sum_{i,j\geq 0} B^{\alpha,\beta}_{i,j}(\zeta_\alpha)^i(\zeta_\beta)^j d\zeta_\alpha \otimes d\zeta_\beta,$$

where $C_{\alpha,\beta} \cong \mathrm{Spec}\mathbb{C}[[\zeta]]$ is the diagonal. We require $B^{\alpha,\beta}_{i,j} = B^{\beta,\alpha}_{i,j}$.

Any spectral curve induces a formal spectral curve. We will consider the recursions for both actual and formal spectral curves in the next subsection.

2.2 Eynard-Orantin's Topological Recursion

Definition 6 The Eynard-Orantin recursive algorithm defines a sequence of symmetric meromorphic forms $\omega_{g,n}$ on $(\overline{\Sigma})^n$ for $g \in \mathbb{Z}_{\geq 0}, n \in \mathbb{Z}_{>0}$ as follows.

- Initial conditions:

$$\omega_{0,1} = 0, \quad \omega_{0,2} = B.$$

- Recursive algorithm:

$$\omega_{g,n}(p_1, \ldots, p_n)= \sum_{p'\in I_\Sigma} \mathrm{Res}_{p=p'} \frac{\int_{\xi=p}^{\bar{p}} B(p_n, \xi)}{2(\Phi(p) - \Phi(\bar{p}))}\Big(\omega_{g-1,n+1}(p, \bar{p}, p_1, \ldots, p_{n-1}) + \sum_{g_1+g_2=g}\ \sum_{I\sqcup J=\{1,\ldots,n-1\}} \omega_{g_1,|I|+1}(p, p_I)\omega_{g_2,|J|+1}(\bar{p}, p_J)\Big).$$

Proposition 7 *When $2g - 2 + n > 0$, the poles of $\omega_{g,n}$ are at $dx_i = 0$ (critical points), where dx_i is the differential of the superpotential on i-th copy of $(\Sigma)^n$.*

Proof The proof is in Appendix A of [24]. We repeat it here. Assume the statement is correct for all (g, n) such that $g < g_0$ or $g = g_0, n < n_0$ where $2g_0 - 2 + n_0 > 0$. Then by the recursion, if $p_1, \ldots, p_n$ are not any critical point, all ω_{g_0-1,n_0+1}, $\omega_{g_1,|I|+1}$ and $\omega_{g_2,|J|+1}$ on the RHS are not at their poles, and the residues are well-defined. Therefore ω_{g_0,n_0} is finite at $(p_1, \ldots, p_n)$. Notice that we can always make the contour around p' small enough to avoid $p_1, \ldots, p_{n-1}$ such that we would not encounter the diagonal pole of $\omega_{0,2}$.

Definition 6 also applies to any formal spectral curve $\mathfrak{C}=(\{C_\alpha\}, \{x_\alpha\}, \{y_\alpha\}, B^{\alpha,\beta})$, and produces a sequence of meromorphic symmetric differential n-forms $\omega_{g,n}$ on $(\sqcup_{\alpha\in I_{\mathfrak{C}}} C_\alpha)^n$.

2.3 Differential Forms on Spectral Curves

For any spectral curve $\mathbf{\Sigma}$, we define *the preferred basis of differentials of the second kind* as below

$$\theta_d^\alpha(p) := (2d-1)!!2^{-d}\mathrm{Res}_{p'\to p_\alpha} B(p,p')\zeta_\alpha(p')^{-2d-1}.$$

The form θ_d^α satisfies the following properties.

- θ_d^α is a meromorphic 1-form on $\overline{\Sigma}$ with a single pole of order $2d+2$ at p_α.
- In local coordinate

$$\theta_d^\alpha = (-\frac{(2d+1)!!}{2^d\zeta_\alpha^{2d+2}} + \text{holomorphic part})d\zeta_\alpha.$$

- $\int_{A_i}\theta_d^\alpha = 0$ for $i = 1, \dots \mathfrak{g}$.

For $k > 0$, we define

$$\hat{\xi}_{\alpha,k} = (-1)^k(\frac{d}{dx})^{k-1}(\frac{\theta_0^\alpha}{dx}),$$

which is a meromorphic function on $\overline{\Sigma}_q$. As a convention, we write $d\hat{\xi}_{\alpha,0} = \theta_0^\alpha$, although $\hat{\xi}_{\alpha,0}$ is not a well defined global meromorphic function on $\overline{\Sigma}_q$.

Similarly, for any formal spectral curve $\mathfrak{C} = \{C_\beta\}_{\beta\in I_{\mathfrak{C}}}$, we define these meromorphic forms $\theta_{d,\beta}^\alpha(\zeta_\beta)$ on C_β

$$\theta_{d,\beta}^\alpha(\zeta_\beta) := (2d-1)!!2^{-d}\mathrm{Res}_{\zeta_\alpha\to 0} B^{\alpha,\beta}(\zeta_\alpha,\zeta_\beta)(\zeta_\alpha)^{-2d-1}.$$

We have

$$\theta_{d,\beta}^\alpha = (-\frac{\delta_{\alpha,\beta}(2d+1)!!}{2^d\zeta_\beta^{2d+2}} + \text{holomorphic part})d\zeta_\beta.$$

We also define

$$\hat{\xi}_{\alpha,\beta,k} = (-1)^k(\frac{d}{dx})^{k-1}(\frac{\theta_{0,\beta}^\alpha}{dx}), \quad d\hat{\xi}_{\alpha,\beta,0} = \theta_{0,\beta}^\alpha.$$

3 Identification of Eynard-Orantin's Recursion with Givental's Quantization

3.1 Frobenius Manifold

In this section, we explain the equivalence of Givental's quantization of a semisimple Frobenius manifold and the corresponding Eynard-Orantin recursion.

Definition 8 A Frobenius algebra $(V, \star)$ is a finite-dimensional associative algebra V over a field k with unit $\mathbf{1}$ equipped with a non-degenerate bilinear pairing $(,) : V \times V \to k$ such that $(a \star b, c) = (a, b \star c)$.

We fix the dimension of the Frobenius algebra (or later, manifold) in discussion to be N. A Frobenius algebra is *semisimple* if it has a basis $\{\phi_\alpha\}_{\alpha=1,\ldots,N}$ such that $\phi_\alpha \star \phi_\beta = \delta_{\alpha,\beta}\phi_\alpha$. Such basis is unique up to a permutation.

Definition 9 A Frobenius manifold V is a k-manifold with a flat pseudo-Riemannian metric $(,)$ with the following properties.

- Locally there is a function F whose third covariant derivative F_{abc} at q defines a product $\star_q$ on the tangent by $F_{abc}|_q = (\partial_a \star_q \partial_b, \partial_c)$, such that each tangent space at a point q is a Frobenius algebra with the product $\star_q$ and the pairing from the Riemannian metric.
- The vector field the of unit $\mathbf{1}$ is covariantly constant and preserves the multiplication.

A Frobenius manifold V is *generically semisimple* if for generic $q \in V$, T_qV is semisimple. We sometimes write $\star$ instead of $\star_q$ when the context is clear.

Let τ^a, $a = 1, \ldots, N$ be flat coordinates on a Frobenius manifold, and let $H_a = \frac{\partial}{\partial \tau^a}$ be the corresponding frames in the tangent bundle. The *quantum connection* ∇ is given as follows

$$\nabla_a = z\partial_a - H_a\star$$

The *quantum differential equation (QDE)* is

$$\nabla\eta = 0$$

The QDE is a system of first-order differential equations, and a choice of fundamental solutions $S_\tau = (\eta_1(\tau), \ldots, \eta_N(\tau))$ is called an *S-calibration*.

Definition 10 Around a semisimple point $p \in V$ (we assume $\tau(p) = \tau_0$), we define the following notions.

- Canonical basis $\phi_\alpha(\tau)$ such that $\phi_\alpha(\tau) \star \phi_\beta(\tau) = \delta_{\alpha,\beta}$. We have

$$(\phi_\alpha(\tau), \phi_\beta(\tau)) = \frac{\delta_{\alpha,\beta}}{\Delta^\alpha(\tau)}.$$

- Canonical coordinates $u^\alpha(\tau)$ such that $\frac{\partial}{\partial u^\alpha(\tau)} = \phi_\alpha(\tau)$. They are fixed up to constants.
- Flat basis ϕ_α which is the parallel transport (according to the Levi-Civita connection of the Riemannian metric on V) of $\phi_\alpha(\tau_0)$ at $\tau = 0$. We also denote $\Delta^\alpha = \Delta^\alpha(\tau_0)$.
- Normalized basis $\hat{\phi}_\alpha(\tau) = \phi_\alpha(\tau)\sqrt{\Delta^\alpha(\tau)}$; $\hat{\phi}_\alpha = \phi_\alpha\sqrt{\Delta^\alpha}$.
- The dual basis $\{\phi^\alpha\}$ to $\{\phi_\alpha\}$, and the dual basis $\{\phi^\alpha(\tau)\}$ to $\{\phi_\alpha(\tau)\}$. The normalized basis are self-dual.

Theorem 11 *Around a semisimple point $p \in V$, there exists an S-calibration $S_\tau = (\eta_1(\tau), \ldots, \eta_N(\tau))$. Each $\eta_\alpha(\tau) = \sum_{a=1}^N (S_\tau)_a^{\hat{\alpha}} H^a$ where H^a is the dual basis to H_a. One can decompose S_τ as following*

$$(S_\tau)_a^{\hat{\alpha}} = (\Psi_\tau)_a^\beta R_\tau(z)_\beta^\alpha e^{\frac{u^\alpha(\tau)}{z}}.$$

Here Ψ_τ is the transition matrix from $\hat{\phi}_\alpha(\tau)$ to H_a

$$H_a = \sum_{a=1}^N (\Psi_\tau)_a^\alpha \hat{\phi}_\alpha(\tau),$$

and

$$(R_\tau)_\beta^\alpha(z) = \delta_\beta^\alpha + O(z)$$

is a formal power series in z, and it is unitary

$$(R_\tau)_\alpha^\gamma(z)(R_\tau)_\beta^\gamma(-z) = \delta_{\alpha\beta}.$$

Furthermore, R_τ it is uniquely determined by up to a right multiplication of $\exp(\sum_{i=1}^\infty a_{2i-1}z^{2i-1})$ *where a_{2i-1} are constant diagonal matrices.*

Let S_τ be an S-calibration. Define an operator $\mathcal{S}_\tau : T_\tau V \to T_\tau V$ by

$$(S_\tau)_a^{\hat{\alpha}} = (H_a, \mathcal{S}_\tau(\hat{\phi}^\alpha)).$$

Define the matrix

$$(S_\tau)_{\overline{\alpha}}^{\hat{\beta}} = (\hat{\phi}_\beta(\tau), S_\tau(\phi_\alpha)).$$

3.2 Quantizations of a Generically Semi-Simple Frobenius Manifold

We will introduce Givental's quantization for semi-simple Frobenius manifolds. When the Frobenius manifold comes from genus 0 Gromov-Witten theory of a toric manifold, this quantization matches higher genus Gromov-Witten invariants. First we introduce the following notations

$$\mathbf{u}(z) = (\mathbf{u}_1(z), \mathbf{u}_2(z), \dots), \quad \mathbf{u}_j(z) = \sum_{a=1}^{N} \mathbf{u}_j^a(z) H_a,$$
$$\mathbf{u}_j^a(z) = (u_j^a)_0 + (u_j^a)_1 z + (u_j^a)_2 z^2 + \dots,$$
$$\mathbf{t}(z) = \sum_{a=1}^{N} \mathbf{t}^a(z) H_a, \quad \mathbf{t}^a(z) = (t^a)_0 + (t^a)_1 z + (t_2^a) z^2 + \dots.$$

Example 12 Let X be a smooth toric manifold over $\mathbb{C}$, and $\mathbb{T} \subset X$ be the dense open torus in X. The equivariant quantum cohomology $QH^*_{\mathbb{T}}(X;\mathcal{Q})$ is a Frobenius algebra over the fractional field $\mathcal{Q}$ of $H^*_{\mathbb{T}}(\mathrm{pt};\mathbb{C})$, and it is semisimple around the origin. When X is compact, the non-equivariant quantum cohomology $QH^*(X;\mathbb{C})$ is semisimple generically. It is not necessarily semisimple when the Kähler parameter is zero, i.e. the ordinary non-equivariant cohomology algebra is not necessarily semisimple. We recall the definition of equivariant Gromov-Witten invariants for X below. We do not assume X is compact in the equivariant setting.

Let $\overline{\mathcal{M}}_{g,n}(X;\beta)$ be the moduli of the stable maps from a genus g, n-marked curve to X in class $\beta \in H_2(X;\mathbb{Z})$. Recall that ψ-class $\psi_i = c_1(\mathbb{L}_i)$ where $\mathbb{L}_i$ is formed by cotangent lines at i-th marked point on $\overline{\mathcal{M}}_{g,n}(X;\beta)$. Let $\bar{\psi}_i = \pi^*\psi_{\mathrm{pt},i}$, where $\psi_{\mathrm{pt},i}$ is the i-th ψ-class on the moduli space of curves $\overline{\mathcal{M}}_{g,n}$, and $\pi : \overline{\mathcal{M}}_{g,n}(X;\beta) \to \overline{\mathcal{M}}_{g,n}$ is the forgetful map.

The $\mathbb{T}$-equivariant genus g degree d Gromov-Witten invariants of X are defined by

$$\langle \gamma_1 \hat{\psi}_1^{a_1}, \cdots, \gamma_n \hat{\psi}_n^{a_n} \rangle_{g,n,\beta}^{X,\mathbb{T}} = \int_{[\overline{\mathcal{M}}_{g,n}(X,d)^{\mathbb{T}}]^{w,\mathbb{T}}} \frac{\iota^*\left(\prod_{j=1}^n \mathrm{ev}_j^*(\gamma_j)(\hat{\psi}_j^{\mathbb{T}})^{a_j}\right)}{e_{\mathbb{T}}(N^{\mathrm{vir}})} \in \mathcal{Q}.$$

where the weighted virtual fundamental class $[\overline{\mathcal{M}}_{g,n}(X,d)^{\mathbb{T}}]^{w,\mathbb{T}}$ [1, 2] (resp. the virtual normal bundle N^{vir} of $\overline{\mathcal{M}}_{g,n}(X,d)^{\mathbb{T}}$ in $\overline{\mathcal{M}}_{g,n}(X,d)$) is defined by the fixed (resp. moving) part of the restriction to $\overline{\mathcal{M}}_{g,n}(X,d)^{\mathbb{T}}$ of the $\mathbb{T}$-equivariant perfect obstruction theory on $\overline{\mathcal{M}}_{g,n}(X,d)$ [41], and $\iota^* : H^*_{\mathbb{T}}(\overline{\mathcal{M}}_{g,n}(X,d);\mathbb{Q}) \to H^*_{\mathbb{T}}(\overline{\mathcal{M}}_{g,n}(X,d)^{\mathbb{T}};\mathbb{Q})$ is induced by the inclusion map $\iota : \overline{\mathcal{M}}_{g,n}(X,d)^{\mathbb{T}} \hookrightarrow \overline{\mathcal{M}}_{g,n}(X,d)$. Here $\hat{\psi} = \psi$ or $\bar{\psi}$, and these invariants are called ancestors or descendants, respectively.

Let $\tau \in H^*_{\mathbb{T}}(X; \mathcal{Q})$. Define the ancestor/descendant ($\hat{\psi} = \bar{\psi}$ or ψ) potential with primary insertions (we suppress the torus symbol $\mathbb{T}$ from here in the Gromov-Witten notations for closed Gromov-Witten invariants)

$$\langle\langle \mathbf{u}_1(\hat{\psi}_1), \ldots, \mathbf{u}_n(\hat{\psi}_n) \rangle\rangle^X_{g,n} = \sum_{m=0}^{\infty} \sum_{\beta \geq 0} \frac{\langle \mathbf{u}_1(\hat{\psi}_1), \cdots, \mathbf{u}_n(\hat{\psi}_n), \tau^m \rangle^{X,\mathbb{T}}_{g,n+m,\beta}}{m!}.$$

We always assume this sum converges for a suitable domain of τ.[4]

The quantum cohomology is defined by

$$(a *_\tau b, c) = \langle\langle a, b, c \rangle\rangle^X_{0,3}, \quad a, b, c \in H^*_{\mathbb{T}}(X; \mathcal{Q}).$$

Let $\mathbf{t} = \mathbf{u}_1 = \mathbf{u}_2 = \mathbf{u}_3 = \ldots$. We define

$$F^X_{g,n}(\mathbf{t}) = \langle\langle \mathbf{t}, \ldots, \mathbf{t} \rangle\rangle^X_{g,n}, \quad F^X_g = \langle\langle\rangle\rangle^X_{g,0}.$$

Here $F^X_g = F^X_g(\tau)$ is a function of τ.

Fix a generically semisimple Frobenius manifold V with $\dim V = N$. Given two S-calibration S_τ and $\widetilde{S}_\tau$ where $\widetilde{S}_\tau$ allows such a decomposition

$$(\widetilde{S}_\tau)^\alpha_a = (\Psi_\tau)^\beta_a (R_\tau)^\alpha_\beta e^{\frac{u^\alpha}{z}},$$

we will describe the graph sum formula for higher genus descendant and ancestor potentials with these choices of S-calibrations. Let Γ be a connected graph. We introduce the following notations.

1. $V(\Gamma)$ is the set of vertices in Γ.
2. $E(\Gamma)$ is the set of edges in Γ.
3. $H(\Gamma)$ is the set of half edges in Γ.
4. $L^o(\Gamma)$ is the set of ordinary leaves in Γ. The ordinary leaves are ordered: $L^o(\Gamma) = \{l_1, \ldots, l_n\}$ where n is the number of ordinary leaves.
5. $L^1(\Gamma)$ is the set of dilaton leaves in Γ. The dilaton leaves are unordered.

We also introduce the following labels:

1. (genus) $g : V(\Gamma) \to \mathbb{Z}_{\geq 0}$.
2. (marking) $\alpha : V(\Gamma) \to \{1, \ldots, N\}$. This induces $\alpha : L(\Gamma) = L^o(\Gamma) \cup L^1(\Gamma) \to \{1, \ldots, N\}$, as follows: if $l \in L(\Gamma)$ is a leaf attached to a vertex $v \in V(\Gamma)$, define $\alpha(l) = \alpha(v)$.
3. (height) $k : H(\Gamma) \to \mathbb{Z}_{\geq 0}$.

[4]It should converge near "large radius limit" τ_0. We decompose $\tau = \tau' + \tau''$, $\tau' \in H^2_{\mathbb{T}}(X; \mathcal{Q})$ and $\tau'' \in H^{\neq 2}_{\mathbb{T}}(X; \mathcal{Q})$. Here $\tau'_0 = -\infty$ and $\tau''_0 = 0$. This fact allows us to avoid Novikov variables. It is a highly non-trivial statement (see [45]). A common practice is to utilize Novikov variables first, and the convergence follows from the B-model *after* establishing a mirror symmetry statement.

Given an edge e, let $h_1(e), h_2(e)$ be the two half edges associated to e. The order of the two half edges does not affect the graph sum formula in this paper. Given a vertex $v \in V(\Gamma)$, let $H(v)$ denote the set of half edges emanating from v. The valency of the vertex v is equal to the cardinality of the set $H(v)$: $\mathrm{val}(v) = |H(v)|$. A labeled graph $\vec{\Gamma} = (\Gamma, g, \alpha, k)$ is *stable* if

$$2g(v) - 2 + \mathrm{val}(v) > 0$$

for all $v \in V(\Gamma)$.

Let $\mathbf{\Gamma}(V)$ denote the set of all stable labeled graphs $\vec{\Gamma} = (\Gamma, g, \alpha, k)$, associated to the Frobenius manifold V. The genus of a stable labeled graph $\vec{\Gamma}$ is defined to be

$$g(\vec{\Gamma}) := \sum_{v\in V(\Gamma)} g(v) + |E(\Gamma)| - |V(\Gamma)| + 1 = \sum_{v\in V(\Gamma)} (g(v) - 1) + (\sum_{e\in E(\Gamma)} 1) + 1.$$

Define

$$\mathbf{\Gamma}_{g,n}(V) = \{\vec{\Gamma} = (\Gamma, g, \alpha, k) \in \mathbf{\Gamma}(V) : g(\vec{\Gamma}) = g, |L^o(\Gamma)| = n\}.$$

We assign weights to leaves, edges, and vertices of a labeled graph $\vec{\Gamma} \in \mathbf{\Gamma}(V)$ as follows.

1. *Ordinary leaves.* To each ordinary leaf $l_j \in L^o(\Gamma)$ with $\alpha(l_j) = \alpha \in \{1, \dots, N\}$ and $k(l) = k \in \mathbb{Z}_{\geq 0}$, we define two kinds of weight:

$$(\mathcal{L}^{\mathbf{u}}_d)^\alpha_k(l_j) = [z^k](\sum_{\alpha,a=1}^N \left(\mathbf{u}^a_j(z) S^{\hat{\beta}}{}_a(z)\right)_+ R(-z)_\beta{}^\alpha),$$

$$(\mathcal{L}^{\mathbf{u}}_a)^\alpha_k(l_j) = [z^k](\sum_{\alpha,a=1}^N \left(\mathbf{u}^a_j(z) \Psi_a{}^\beta\right) R(-z)_\beta{}^\alpha).$$

The notion $()_+$ discards negative powers of z, i.e. $(\sum_{n\in\mathbb{Z}} a_n z^n)_+ = \sum_{n\geq 0} a_n z^n$.

2. *Dilaton leaves.* To each dilaton leaf $l \in L^1(\Gamma)$ with $\alpha(l) = \alpha \in I_\Sigma$ and $2 \leq k(l) = k \in \mathbb{Z}_{\geq 0}$, we assign

$$(\mathcal{L}^1)^\alpha_k(l) = [z^{k-1}](-\sum_{\beta=1}^N \frac{1}{\sqrt{\Delta^\beta(\tau)}} R_\beta{}^\alpha(-z)).$$

3. *Edges.* To an edge connecting two vertices marked by $\alpha, \beta \in \{1, \dots, N\}$ and with heights k and l at its two half-edges, we assign

$$\mathcal{E}^{\alpha,\beta}_{k,l}(e) = [z^k w^l]\Big(\frac{1}{z+w}(\delta_{\alpha,\beta} - \sum_{\gamma=1}^N R_\gamma{}^\alpha(-z) R_\gamma{}^\beta(-w)\Big).$$

4. *Vertices.* To a vertex v with genus $g(v) = g \in \mathbb{Z}_{\geq 0}$ and with marking $\alpha(v) = \alpha$, with n ordinary leaves and half-edges attached to it with heights $k_1, \ldots, k_n \in \mathbb{Z}_{\geq 0}$ and m more dilaton leaves with heights $k_{n+1}, \ldots, k_{n+m} \in \mathbb{Z}_{\geq 0}$, we assign

$$(\sqrt{\Delta^{\alpha(v)}(\tau)})^{2g(v)-2+\mathrm{val}(v)} \langle \prod_{h \in H(v)} \tau_{k(h)} \rangle_{g(v)} = \left(\sqrt{\Delta^{\alpha}(\tau)}\right)^{2g(v)-2+\mathrm{val}(v)} \int_{\overline{\mathcal{M}}_{g,n+m}} \psi_1^{k_1} \cdots \psi_{n+m}^{k_{n+m}}.$$

Define the weight of a labeled graph $\vec{\Gamma} \in \mathbf{\Gamma}(V)$ to be (the letter F means "Frobenius")

$$w^{\mathbf{u}}_{F,\bullet}(\vec{\Gamma}) = \prod_{v \in V(\Gamma)} (\sqrt{\Delta^{\alpha(v)}(\tau)})^{2g(v)-2+\mathrm{val}(v)} \langle \prod_{h \in H(v)} \tau_{k(h)} \rangle_{g(v)} \prod_{e \in E(\Gamma)} \mathcal{E}^{\alpha(v_1(e)),\alpha(v_2(e))}_{k(h_1(e)),k(h_2(e))}(e) \tag{1}$$
$$\cdot \prod_{j=1}^{n} (\mathcal{L}^{\mathbf{u}}_{\bullet})^{\alpha(l_n)}_{k(l_j)}(l_j) \prod_{l \in L^1(\Gamma)} (\mathcal{L}^1)^{\alpha(l)}_{k(l)}(l),$$

where $\bullet = a$ or d.

Definition 13 Suppose that $2g - 2 + n > 0$. Define the ancestor potential

$$\langle\langle \mathbf{u}_1(\bar{\psi}_1), \ldots, \mathbf{u}_n(\bar{\psi}_n) \rangle\rangle^V_{g,n} = \sum_{\vec{\Gamma} \in \mathbf{\Gamma}_{g,n}(V)} \frac{w^{\mathbf{u}}_{F,a}(\vec{\Gamma})}{|\mathrm{Aut}(\vec{\Gamma})|}.$$

and the descendant potential

$$\langle\langle \mathbf{u}_1(\psi_1), \ldots, \mathbf{u}_n(\psi_n) \rangle\rangle^V_{g,n} = \sum_{\vec{\Gamma} \in \mathbf{\Gamma}_{g,n}(V)} \frac{w^{\mathbf{u}}_{F,d}(\vec{\Gamma})}{|\mathrm{Aut}(\vec{\Gamma})|}.$$

Remark 14 The ψ-classes and $\bar{\psi}$-classes here are just notations. If X is a toric manifold, and $V = QH^*_{\mathbb{T}}(X; \mathcal{Q})$, one may choose

$$(S_\tau)_a^{\hat{\alpha}} = (H_a, \hat{\phi}_\alpha) + \langle\langle H_a, \frac{\hat{\phi}_\alpha}{z - \psi} \rangle\rangle^X_{0,2},$$

and

$$(\widetilde{S}_\tau)_a^{\alpha} = (S_\tau)_a^{\alpha} \cdot \exp(-\sum_{n=1}^{\infty} \frac{B_{2n}}{2n(2n-1)} (\frac{z}{\chi^\alpha})^{2n-1}).$$

Givental [39, 40] shows the following (when $2g-2+n>0$)

$$\langle\langle \mathbf{u}_1(\bar{\psi}_1),\dots,\mathbf{u}_n(\bar{\psi}_n)\rangle\rangle^V_{g,n} = \langle\langle \mathbf{u}_1(\bar{\psi}_1),\dots,\mathbf{u}_n(\bar{\psi}_n)\rangle\rangle^X_{g,n},$$
$$\langle\langle \mathbf{u}_1(\psi_1),\dots,\mathbf{u}_n(\psi_n)\rangle\rangle^V_{g,n} = \langle\langle \mathbf{u}_1(\psi_1),\dots,\mathbf{u}_n(\psi_n)\rangle\rangle^X_{g,n}.$$

Let $\mathbf{u}_1=\mathbf{u}_2=\cdots=\mathbf{t}$. One can define the total ancestor potential

$$\mathcal{A}_\tau(\mathbf{t}) = \exp(\sum_{g,n=0}^{\infty}\frac{\hbar^{g-1}}{n!}\langle\langle \mathbf{t}(\bar{\psi}_1),\dots,\mathbf{t}(\bar{\psi}_n)\rangle\rangle^V_{g,n}).$$

The graph sum formula for the ancestor potentials is another form of the following Givental's quantization process [39] (without $(g,n)=(1,0)$ information,which is captured by $C(\tau)$ and not defined here)

$$e^{F_1(\tau)}\mathcal{A}_\tau(\mathbf{t}) = e^{C(\tau)}\widehat{\Psi}\widehat{R}(z)e^{\frac{\widehat{U}}{z}}\prod_{a=1}^{N}\mathcal{T}(\mathbf{t}^a).$$

For $a=1,\dots,N$, $\mathcal{T}(\mathbf{t}^a)$ is the Kontsevich tau-function

$$\mathcal{T}(\mathbf{t}^a) = \exp(\sum_{g,n=0}^{\infty}\frac{\hbar^{g-1}}{n!}\langle \mathbf{t}^a(\psi_{\mathrm{pt},1}),\dots,\mathbf{t}^a(\psi_{\mathrm{pt},n})\rangle^{\mathrm{pt}}_{g,n}),$$

Let

$$\mathcal{D}(\mathbf{t}) = \exp(\sum_{g,n=0}^{\infty}\frac{\hbar^{g-1}}{n!}\langle \mathbf{t}(\psi_1),\dots,\mathbf{t}(\psi_n)\rangle^X_{g,n}).$$

It does not depend on τ. Givental's quantization formula says

$$\mathcal{D}(\mathbf{t}) = e^{C(\tau)}\widehat{S}_\tau^{-1}(z)\mathcal{A}_\tau(\mathbf{t}).$$

It is a consequence of the graph sum formula for the descendant potential.

3.3 A Graph Sum Formula for Eynard-Orantin Recursions

Dunin-Barkowski–Orantin–Shadrin–Spitz express the Eynard-Orantin higher genus differential forms in terms of a graph sum in [19], and then compare with Givental's quantized descendant potentials.

We expand the fundamental normalized differential around (p_α, p_β) where p_α, p_β are critical points of x.

$$B(\zeta_\alpha, \zeta_\beta) = (\frac{\delta_{\alpha,\beta}}{(\zeta_\alpha - \zeta_\beta)^2} + \sum_{k,l\in\mathbb{Z}_{\geq 0}} B^{\alpha,\beta}_{k,l}(\zeta_\alpha)^k(\zeta_\beta)^l)d\zeta_\alpha d\zeta_\beta.$$

In case of a formal spectral curve, the fundamental normalized differential is defined by these coefficients $B^{\alpha,\beta}_{i,j}$. Define the propagators

$$\check{B}^{\alpha,\beta}_{k,l} = \frac{(2k-1)!!(2l-1)!!}{2^{k+l+1}} B^{\alpha,\beta}_{2k,2l},$$

and

$$\check{h}^\alpha_k = 2(2k-1)!!h^\alpha_{2k-1}.$$

Here we quote a lemma [21, Equation (D.4)] on the relation between $\hat{\xi}_{\alpha,k}$ and θ^α_k.

Lemma 15

$$\theta^\alpha_k = d\hat{\xi}_{\alpha,k} - \sum_{i=0}^{k-1}\sum_\beta \check{B}^{\alpha,\beta}_{k-1-i,0} d\hat{\xi}_{\beta,i}.$$

Here β sums over $I_{\mathbf{\Sigma}}$ or $I_{\mathfrak{C}}$ for any spectral curve $\mathbf{\Sigma}$ or formal spectral curve $\mathfrak{C}$.

Similarly to $\mathbf{\Gamma}(V)$ we define the set of decorated stable graph $\mathbf{\Gamma}(\mathbf{\Sigma})$ (or $\mathbf{\Gamma}(\mathfrak{C})$ if we are working with a formal spectral curve)—the only difference is the marking as below.

(2)' (marking) $\alpha : V(\Gamma) \to I_{\mathbf{\Sigma}}$ (or $I_{\mathfrak{C}}$). We also define the marking of leaf $\alpha(l)$ to be the marking of the vertex it attaches to.

Given a labeled graph $\vec{\Gamma} \in \mathbf{\Gamma}_{g,n}(\mathbf{\Sigma})$ with $L^o(\Gamma) = \{l_1, \ldots, l_n\}$, we define its weight to be (the letter S means "spectral curves")

$$w^{\mathbf{p}}_S(\vec{\Gamma}) = (-1)^{g(\vec{\Gamma})-1} \prod_{v\in V(\Gamma)} \left(\frac{h_1^{\alpha(v)}}{\sqrt{-2}}\right)^{2-2g-\mathrm{val}(v)} \langle \prod_{h\in H(v)} \tau_{k(h)}\rangle_{g(v)} \prod_{e\in E(\Gamma)} \check{B}^{\alpha(v_1(e)),\alpha(v_2(e))}_{k(e),l(e)} \tag{2}$$

$$\cdot \prod_{j=1}^n (\check{\mathcal{L}}^{\mathbf{p}})^{\alpha(l_j)}_{k(l_j)}(l_j) \prod_{l\in L^1(\Gamma)} (-\frac{1}{\sqrt{-2}})\check{h}^{\alpha(l)}_{k(l)}.$$

Here $\mathbf{p} = (p_1, \ldots, p_n) \in (\overline{\Sigma})^n$ in case of an actual spectral curve, and the ordinary leaf is

$$(\check{\mathcal{L}}^{\mathbf{p}})^\alpha_k(l_j) = -\frac{1}{\sqrt{-2}}\theta^\alpha_k(p_j).$$

When we are working with a formal curve, $\mathbf{p} = (\zeta_{\beta_1}, \ldots, \zeta_{\beta_n}) \in C_{\beta_1} \times \cdots \times C_{\beta_n}$. The ordinary leaf is

$$(\check{\mathcal{L}}^{\mathbf{p}})^\alpha_k(l_j) = -\frac{1}{\sqrt{-2}}\theta^\alpha_{k,\beta_j}(\zeta_{\beta_j}).$$

The graph sum formula of $\omega_{g,n}$ is the following.

Theorem 16 (Dunin-Barkowski–Orantin–Shadrin–Spitz [19])

$$\omega_{g,n}(\mathbf{p}) = \sum_{\vec{\Gamma}\in\mathbf{\Gamma}(\mathbf{\Sigma})} \frac{w_S^{\mathbf{p}}(\vec{\Gamma})}{\mathrm{Aut}(\vec{\Gamma})}.$$

Starting from a generically semisimple Frobenius manifold V with an S and $\widetilde{S}$-calibration, around a semisimple point $p \in V$, we define a family of formal spectral curves $\mathfrak{C}_V(\tau) = \{C_{V,\beta} = \mathrm{Spec}\mathbb{C}[[z_\beta]]\}_{\beta=1}^N$, together with the following information

$$h^\alpha_{2k-1}(\tau) = [z^{k-1}]\left(\sum_{\beta=1}^N \frac{\sqrt{-2}}{(2k-1)!!2^{k-1}\sqrt{\Delta^\alpha(\tau)}}(R_\tau)_{\alpha'}^{\ \alpha}(-z)\right),\ k \geq 0.$$

$$B^{\alpha,\beta}_{2k,2l}(\tau) = \frac{2^{k+l+1}}{(2k-1)!!(2l-1)!!}[z^k w^l]\left(\frac{1}{z+w}(\delta_{\alpha,\beta} - \sum_{\gamma=1}^N (R_\tau)_\gamma^{\ \alpha}(-z)(R_\tau)_\gamma^{\ \beta}(-w)\right),\ k,l \geq 0.$$

Notice that they only depend on R_τ, which comes from factorizing $\widetilde{S}$. Even coefficients of h^α_k and odd coefficients of $B^{\alpha,\beta}_{k,l}$ could be arbitrarily chosen.

Define

$$\overline{\boldsymbol{u}}^\alpha_j(z) = \sum_{z=0}^\infty (\overline{u}^\alpha_j)_k z^k = \sum_{b=1}^N \mathbf{u}^a_j(z)\Psi_a^{\ \alpha},$$

$$\widetilde{\mathbf{u}}^\alpha_j(z) = \sum_{z=0}^\infty (\widetilde{u}^\alpha_j)_k z^k = \sum_{b=1}^N \left(S^{\hat{\alpha}}_{\ b}(z)\mathbf{u}^b_j(z)\right)_+,$$

where

$$S^{\hat{\alpha}}_{\ b}(z) = (\hat{\phi}^\alpha(\tau), \mathcal{S}(H_b)).$$

Theorem 17 ([19]) *When* $2g-2+n > 0$,

$$\omega^{\mathfrak{C}_V(\tau)}_{g,n}(z^1_{\beta_1}, \ldots, z^n_{\beta_n}) = (-1)^{g-1+n}\langle\!\langle \mathbf{u}_1(\bar{\psi}), \ldots, \mathbf{u}_n(\bar{\psi})\rangle\!\rangle^V_{g,n}|_{(\overline{u}^\alpha_j)_k = d\hat{\xi}^\alpha_{k,\beta_j}(z^j_{\beta_j})},$$

$$\omega^{\mathfrak{C}_V(\tau)}_{g,n}(z^1_{\beta_1}, \ldots, z^n_{\beta_n}) = (-1)^{g-1+n}\langle\!\langle \mathbf{u}_1(\psi), \ldots, \mathbf{u}_n(\psi)\rangle\!\rangle^V_{g,n}|_{(\widetilde{u}^\alpha_j)_k = d\hat{\xi}^\alpha_{k,\beta_j}(z^j_{\beta_j})}.$$

3.4 Oscillatory Integrals on the Spectral Curves

Let $\boldsymbol{\Sigma}$ be a spectral curve, and γ_α be the Lefschetz thimble in Σ with respect to x such that p_α is the only critical point in γ_α and

$$x(\gamma_\alpha) = [x_{0,\alpha}, +\infty).$$

We define $\check{R}(z)$ as the power series in the following asymptotic expansion.

$$\int_{\gamma_\alpha} e^{-\frac{x}{z}} \theta_0^\beta \sim 2\sqrt{\frac{\pi}{z}} e^{-\frac{x_{0,\alpha}}{z}} \check{R}_\beta^{\ \alpha}(z).$$

Notice that this definition is also well-defined for formal spectral curves. We have the following property for $\check{B}_{k,l}^{\alpha,\beta}$ [22, Equation (B.9)]:

$$\check{B}_{k,l}^{\alpha,\beta} = [z^k w^l] \left(\frac{1}{z+w} (\delta_{\alpha,\beta} - \sum_{\gamma \in I_{\boldsymbol{\Sigma}}} \check{R}_\gamma^{\ \alpha}(z) \check{R}_\gamma^{\ \beta}(w)) \right).$$

We consider the space of differential forms spanned by θ_0^α, denoted by $\check{V}_\tau$. It is isomorphic to $T_\tau V$ by $\theta_0^\alpha \mapsto \frac{\hat{\phi}^\alpha(\tau)}{\sqrt{-2}}$. Denote this isomorphism by $\mathfrak{r}$. By [25, Appendix D], the differential form

$$d(\frac{dy}{dx}) = \frac{1}{2} \sum_{\beta=1}^{N} h_1^\beta(\tau) \theta_0^\beta.$$

We have the following correspondence table between the Frobenius manifold V and the family of formal spectral curves $\mathfrak{C}_V(\tau)$.

Frobenius manifold V	Correspondence	Family of formal spectral curves $\mathfrak{C}_V(\tau)$
Dimension N	$=$	# of formal disks
R-matrix $R_\alpha^{\ \beta}(z)$	$=$	$\check{R}_\alpha^{\ \beta}(-z)$
Propagator $\mathcal{E}_{i,j}^{\alpha,\beta}$	$=$	Propagator $\check{B}_{i,j}^{\alpha,\beta}$
Canonical coordinate u^α	$=$	Critical value $x_{0,\alpha}$
S-matrix $\widetilde{S}_{\underline{\hat{\beta}}}^{\ \hat{\alpha}} = (\hat{\phi}_\beta(\tau), S_\tau(\hat{\phi}^\alpha))$	$\sim$	Oscillatory integral $\frac{1}{2\sqrt{\pi z}} \int_{\gamma_\alpha} e^{-\frac{x}{z}} \theta_0^\beta$
Meromorphic form $\frac{\theta_0^\alpha}{\sqrt{-2}}$	$\overset{\mathfrak{r}}{\mapsto}$	Canonical basis $\hat{\phi}^\alpha(\tau)$
$d(\frac{dy}{dx})$	$\overset{\mathfrak{r}}{\mapsto}$	Identity $\mathbf{1}$
$\sqrt{\Delta^\alpha(\tau)}$	$=$	$\frac{-\sqrt{-2}}{h_1^\alpha(\tau)}$

4 Applications of Eynard-Orantin Recursion: Mirror Symmetry

Mirror symmetry relates the A-model theory on a target space to the B-model theory on its mirror space. Gromov-Witten invariants are a typical type of A-model invariants. In order to apply the recursion algorithm and to use Eynard-Orantin higher genus invariants $\omega_{g,n}$ to predict Gromov-Witten invariants, we need a mirror B-model in the form of a spectral curve. When the target space is a 1-dimensional toric variety, like $\mathbb{P}^1$, the mirror Landau-Ginzburg model is a superpotential on $\mathbb{C}^*$. After suitable compactification, one may directly regard this as a spectral curve. Another (much bigger) class of examples is toric Calabi-Yau 3-orbifolds. Their mirrors, although 3-dimensional, could be reduced to mirror curves by dimensional reduction. Lying at the intersection of these two classes is the Lambert curve, which could be regarded as $\mathbb{P}^1$ in the large radius limit, or as $\mathbb{C}^3$ with limiting equivariant data (large framing limit). The relations among these examples is summarized in the following diagram.

$$\begin{tikzcd}
 & & \text{toric CY 3-folds} \arrow[d, "\text{large radius limit}"] \\
 & \text{equivariant } \mathbb{P}^1 \arrow[dl, "\text{non-equivariant limit}"'] \arrow[d, "\text{large radius limit}"] & \mathbb{C}^3 \arrow[dl, "\text{large equivariant (framing) limit}"] \\
\text{Norbury-Scott conjecture} & \text{Lambert curve (Bouchard-Mariño conjecture)} &
\end{tikzcd}$$

4.1 Airy Curve

Let's look at the easiest case, which is roughly "mirror symmetry of a point". The Airy curve, in our notation, is a formal curve $\mathfrak{C} = (C, y, x, B)$ where

$$C = \mathrm{Spec}[[\zeta]],$$

$$y = \zeta,\ x = \zeta^2,$$

$$B = \frac{d\zeta_1 d\zeta_2}{(\zeta_1 - \zeta_2)^2}.$$

Remark 18 We may regard this curve as the parabola $x = y^2$ in $\mathbb{C}^2$, which is the Airy curve in the usual sense. The fundamental normalized differential B is the unique one on its compactification $\mathbb{P}^1$.

Once we run the Eynard-Orantin recursion for the spectral curve $\mathfrak{C}$, we have the following theorem.

Theorem 19

$$\omega_{g,n}(\zeta_1,\ldots,\zeta_n) = (-2)^{2-2g-n} \sum_{d_1+\cdots+d_n=d_{g,n}} \prod_{i=1}^{n} \frac{(2d_i+1)!!d\zeta_i}{\zeta_i^{2d_i+2}} \langle \prod_{i=1}^{n} \psi_i^{d_i} \rangle.$$

This theorem is a direct consequence of the graph sum formula for Eynard-Orantin recursions (Theorem 16). There is only one critical point of x, labeled by 1. It is straightforward to check that all propagators $\check{B}^{1,1}_{i,j} = 0$ for all i, j. The differential forms

$$\theta^1_k(\zeta) = -\frac{(2k+1)!!d\zeta}{2^k \zeta^{2k+2}},$$

and $h^1_1 = 1$. There are no dilaton leafs since all of them are zero.

4.2 Lambert Curve

Lambert curve is given by $\mathbf{\Sigma} = (\Sigma, x, B)$ where

$$\Sigma = \{0\} \times \mathbb{C}^* \in (\mathbb{C}^*)^2, \ \overline{\Sigma} \cong \mathbb{P}^1$$
$$x = e^{-y} + y,$$
$$B = \frac{dY_1 dY_2}{(Y_1 - Y_2)^2}.$$

Here $X = e^{-x}$, $Y = e^{-y}$. The only branch point is at $Y = 1$. At $Y = 0$, the value of X is well defined. Using the Eynard-Orantin recursion, we construct $\omega_{g,n}$ as symmetric meromorphic n-form on $(\mathbb{P}^1)^n$. Notice that $\omega_{g,n}$ is smooth at $Y = 0$, and can be expanded in series by X.

Lambert curve predicts *Hurwitz numbers* on the A-side. Consider ramified covers of $\mathbb{P}^1$ by a genus g curve with a specified ramification profile at a special point on $\mathbb{P}^1$. All other branch points in $\mathbb{P}^1$ are simple and fixed. The ramification profile is given by a partition μ of length $n := \ell(\mu)$. The number of such covers is denoted by $H_{g,\mu}$.

We collect all Hurwitz numbers at fixed genus g for all ramification profiles μ of the same length $n = \ell(\mu)$ into a generating function

$$H_g(X_1,\ldots,X_n) = \sum_{\ell(\mu)=n} \frac{m_\mu(X)|\mathrm{Aut}(\mu)| \prod_{i=1}^n \mu_i H_{g,\mu}}{(2g-2+n+|\mu|)!},$$

where $m_\mu(X)$ is a monomial symmetric function in $X_1, \ldots, X_n$ defined by

$$m_\mu(X) = \frac{1}{|\mathrm{Aut}(\mu)|} \sum_{\sigma \in S_n} \prod_{i=1}^{n} (X_{\sigma(i)})^{\mu_i - 1}$$

Here S_n is the permutation group.

The Bouchard-Mariño conjecture says the following [10].

Theorem 20 (Bouchard-Mariño Conjecture) *When* $2g - 2 + n > 0$*,*

$$H_g(X_1, \ldots, X_n) dX_1 \ldots dX_n = \omega_{g,n}.$$

The right side should be understood as an power series expansion at $X = 0$*.*

We omit the unstable cases $(g, n) = (0, 1), (0, 2)$ for simplicity here. This theorem is proved in [9, 27, 58]. Here we introduce the ELSV formula [20, 42] for later use. This relates Hurwitz numbers to Hodge integrals, which are more relevant to A-model GW theory in mirror symmetry.

Theorem 21 (ELSV Formula)

$$H_{g,\mu} = \prod_{i=1}^{n} \frac{\mu_i^{\mu_i}}{\mu_i!} \int_{\overline{\mathcal{M}}_{g,n}} \frac{\Lambda_g^\bullet(1)}{(1 - \mu_1 \psi_1) \ldots (1 - \mu_n \psi_n)}$$

Here $\Lambda_g^\bullet(u) = u^g - \lambda_1 u^{g-1} + \cdots + (-1)^g \lambda_g$, and $\lambda_i = c_i(E)$ where E is the Hodge bundle. We will see that from this formula, the Bouchard-Mariño conjecture is a consequence of all genera equivariant mirror symmetry for $\mathbb{P}^1$ (Sect. 4.3), and also a consequence of the BKMP conjecture for $\mathbb{C}^3$ (Sect. 4.4).

4.3 Projective Line

Let $X = \mathbb{P}^1$. Its mirror is a Landau-Ginzburg model

$$W(Y) = t_0 + Y + \frac{e^{t_1}}{Y}.$$

To capture the equivariant data of $\mathbb{P}^1$, we use a modified *equivariant* superpotential

$$\widetilde{W} = W + \mathsf{w}_1 \log Y + \mathsf{w}_2 \log \frac{e^{t_1}}{Y}.$$

The 2-torus $\mathbb{T}$ acts by turning homogeneous coordinates $(s_1, s_2) \cdot (z_1 : z_2) = (s_1 z_1 : s_2 z_2)$. The characters w_i are basis in the character lattice $\mathsf{w}_i : (s_1, s_2) \mapsto s_i \in$

$\mathbb{C}^*$. Let $x = \widetilde{W}$, and $\overline{\Sigma} = \mathbb{P}^1$ where $(1 : Y)$ is its coordinate. There is only one fundamental differential

$$B = \frac{dY_1 dY_2}{(Y_1 - Y_2)^2}.$$

since there is no choice of A-cycles. The spectral curve is $\mathbf{\Sigma} = (\Sigma, x, B)$. The all genera mirror symmetry for $\mathbb{P}^1$ is the following theorem [30].

Theorem 22 (Fang-Liu-Zong) *Let $\mathbf{t} = t_0\mathbf{1} + t_1 H$ where $\mathbf{1}$ is the unit in $H^0_{\mathbb{T}}(\mathbb{P}^1; \mathbb{C})$ and H is the equivariant lift of the hyperplane class whose restriction at two $\mathbb{T}$-fixed points gives w_1 and w_2 respectively.*

$$\omega_{g,n}|_{\frac{1}{\sqrt{-2}}d\xi_{\alpha,k}(Y_j)=(\tilde{u})^\alpha_k} = (-1)^{g-1+n} F^{\mathbb{P}^1,\mathbb{T}}_{g,n}(\mathbf{u}_1, \ldots, \mathbf{u}_n, \mathbf{t}).$$

Since the proof utilizes the same idea as in the proof of the BKMP conjecture which will be discussed in more details (see Sect. 4.5), we only briefly remark a few words here. Notice the similarity between this theorem and Theorem 17—the right side is the actual Gromov-Witten potential, while the one in Theorem 17 comes from the quantization for the Frobenius manifold. They agree as shown in [39, 40].

We mention that taking the non-equivariant limit $\mathsf{w} \to 0$ and when there is no primary insertions, this theorem leads to the Norbury-Scott conjecture [19, 59].

Theorem 23 (Norbury-Scott) *Near $Y = 0$, in the non-equivariant limit ($\mathsf{w}_1 = \mathsf{w}_2 = 0$, $t_0 = 0$), $x^{-1} = (Y + \frac{e^{t_1}}{Y})^{-1}$ is a coordinate such that one can expand $\omega_{g,n}$ in power series*

$$\omega_{g,n} = (-1)^{g-1+n} \sum_{a_1,\ldots,a_n \in \mathbb{Z}_{\geq 0}} \langle\!\langle \tau_{a_1}(H) \ldots \tau_{a_n}(H) \rangle\!\rangle^{\mathbb{P}^1}_{g,n} \prod_{j=1}^n \frac{(a_j+1)!}{x^{a_j+2}} dx_j.$$

Remark 24 The divisor equation says ($q = e^{t_1}$)

$$\langle\!\langle \tau_{a_1}(H) \ldots \tau_{a_n}(H) \rangle\!\rangle^{\mathbb{P}^1}_{g,n} = q^{\frac{1}{2}\sum_{i=1}^n a_i + 1 - g} \langle \tau_{a_1}(H) \ldots \tau_{a_n}(H) \rangle^{\mathbb{P}^1}_{g,n}.$$

The Norbury-Scott conjecture corresponds to setting $q = 1$, i.e. $t^1 = 0$.

Taking the large radius limit of the equivariant mirror theorem (Theorem 22) for $\mathbb{P}^1$, one could recover the Bouchard-Mariño conjecture. The superpotential becomes $x = Y + \mathsf{w}_1 \log Y$ by setting $q = 0$. Setting $\mathsf{w}_1 = -1$ turns this into a Lambert curve. The localization calculation of $F^{\mathbb{P}^1,\mathbb{T}}$ in the limit produces the Hodge integrals, and ELSV formula turns it into the desired generating function involving Hurwitz numbers, as shown in [30, Section 5].

4.4 Mirror of $\mathbb{C}^3$ (Topological Vertex)

Let's switch gears and proceed to toric Calabi-Yau threefolds. The mirror of a toric 3-(orbi)fold, by the construction of Givental [38], is a Landau-Ginzburg model $W : (\mathbb{C}^*)^3 \to \mathbb{C}$. A Calabi-Yau should have a Calabi-Yau mirror. A special feature of a toric Calabi-Yau variety is that its mirror's superpotential $W = H(X, Y)Z,\ (X, Y, Z) \in (\mathbb{C}^*)^3$. As pointed out in [44], the Calabi-Yau mirror is $\{H(X, Y) = uv,\ u, v \in \mathbb{C},\ X, Y \in \mathbb{C}^*\} \subset (\mathbb{C}^*)^2 \times \mathbb{C}^2$. Furthermore, this Calabi-Yau mirror can be reduced to a mirror curve $\{H(X, Y) = 0\} \subset (\mathbb{C}^*)^2$. All these different mirrors should be equivalent, carrying the same B-model information.

LG-model $((\mathbb{C}^*)^3, W)$ —— CY 3-fold mirror $\check{\mathcal{X}}$

The mirror curve $H(X, Y) = 0$ (higher genus invariants defined by the E-O recursion)

The simplest toric Calabi-Yau threefold is $\mathbb{C}^3$, equipped with the Calabi-Yau form $dZ_1 \wedge dZ_2 \wedge dZ_3$ where $(Z_1, Z_2, Z_3) \in \mathbb{C}^3$ are the coordinates. Its mirror is a 3-dimensional Landau-Ginzburg model $(\mathbb{C}^*)^3$, with a superpotential $W = XZ + YZ + Z = H(X, Y)Z$, where $X, Y, Z \in \mathbb{C}^*$ [35]. The mirror curve is $\{H(X, Y) = 0\}$ as an affine plane curve in $(\mathbb{C}^*)^2$.

We want to consider *open* Gromov-Witten invariants of $\mathbb{C}^3$, which count holomorphic maps from bordered Riemann surfaces to $\mathbb{C}^3$ mapping boundaries to a Lagrangian submanifold L (an A-brane). The construction of such invariants is very complicated. Here we require that L is a so-called *Aganagic-Vafa* brane. This gives a very important class of open Gromov-Witten invariants. They play central roles in many interesting topics involving mirror symmetry and the theory of topological vertex [3–5].

In this particular example $\mathbb{C}^3$, an Aganagic-Vafa brane L is given by

$$L = \{(Z_1, Z_2, Z_3) \in \mathbb{C}^3 : |Z_1|^2 - |Z_2|^2 = c, |Z_2|^2 - |Z_3|^2 = 0, \mathrm{Arg}(Z_1 Z_2 Z_3) = \mathrm{const}\} \cong S^1 \times \mathbb{R}^2.$$

It is a Harvey-Lawson special Lagrangian [43], and c is its "open Kähler parameter". Let $\mu = \{\mu_1, \ldots, \mu_n\}$ be a partition of length $\ell(\mu) = n$. Naïvely, we denote the number $N^{\mathbb{C},L}_{g,n,\mu}$ by the counting of the holomorphic maps described below.

$$\left\{\begin{array}{l}(C, \partial C), \text{ where the genus of} \\ C \text{ is } g, \text{ and } \partial C \text{ has } n \text{ com-} \\ \text{ponents}\end{array}\right\} \xrightarrow{f} (\mathbb{C}^3, L),$$

The winding number of each boundary component is given by μ_i, $i = 1, \ldots, n$. The definition and computation of such maps in symplectic and algebraic settings can be found in [46, 48, 52–54].

A common phenomenon in open string counting is that the moduli space of such maps has codimension 1 boundaries (walls). The counting changes across the wall, thus depends on a particular choice of chamber in the moduli space. In case that L is an Aganagic-Vafa brane in a toric Calabi-Yau threefold, the result depends on a *framing* parameter, an integer $f \in \mathbb{Z}$. We denote this number by $N^{\mathbb{C}^3,L,f}_{g,n,\mu}$.

A simple way to understand this $N^{\mathbb{C}^3,L,f}_{g,n,\mu}$ is to write down the localization formula—assuming one can actually do the localization (see e.g. [47]). It turns out that the answer we get depends on the torus we choose to localize, unlike the case of closed Gromov-Witten invariants. Denote the Calabi-Yau torus by $\mathbb{T}' = \{(Z_1, Z_2, Z_3) \in (\mathbb{C}^*)^3, Z_1 Z_2 Z_3 = 1\}$, which preserves the Calabi-Yau form. Let w_1 and w_2 be the following character in $\mathrm{Hom}(\mathbb{T}', \mathbb{C}^*)$

$$\mathsf{w}_1(Z_1, Z_2, \frac{1}{Z_1 Z_2}) = Z_1, \quad \mathsf{w}_2(Z_1, Z_2, \frac{1}{Z_1 Z_2}) = Z_2.$$

If we choose $\mathbb{T}'_f = \mathrm{Ker}(\mathsf{w}_2 - f\mathsf{w}_1) \subset \mathbb{T}'$, we will get $N^{\mathbb{C}^3,L,f}_{g,n,\mu}$ by the localization formula.[5] We can assemble these $N^{\mathbb{C}^3,L,f}_{g,n,\mu}$ into a generating function

$$F^{\mathbb{C}^3,L,f}_{g,n} = \sum_{\mu_1,\ldots,\mu_n=1}^{\infty} N^{\mathbb{C}^3,L,f}_{g,n,\mu} X_1^{\mu_1} \ldots X_n^{\mu_n}.$$

The mirror B-model starts from the reparametrized mirror curve

$$H_f(X, Y) = X^{-f} Y + Y + 1.$$

This defines an affine plane curve $\Sigma \subset (\mathbb{C}^*)^2$ whose compactification $\overline{\Sigma} \cong \mathbb{P}^1$. Define the superpotential and the fundamental differential

$$W = x, \quad B(Y_1, Y_2) = \frac{dY_1 dY_2}{(Y_1 - Y_2)^2}.$$

Here $X = e^{-x}$ and $Y = e^{-y}$. The moment map and the mirror curve is illustrated in Fig. 1.

After running the Eynard-Orantin recursion, we get a sequence of $\omega_{g,n}$. The famous BKMP remodeling conjecture [11, 55] asserts the following.

[5] If one insists on algebraic geometry, we can use relative Gromov-Witten invariants as the definition. This involves partially compactifying $\mathbb{C}^3$ into the total space of $\mathcal{O}_{\mathbb{P}^1}(-1-f) \oplus \mathcal{O}_{\mathbb{P}^1}(f)$, and define $N^{\mathbb{C}^3,L,f}_{g,n,\mu}$ as the relative Gromov-Witten invariants on this space relative to the fiber divisor at the infinity in $\mathbb{P}^1$. The tangency condition at the divisor is given by μ. (See [48, 53, 54].)

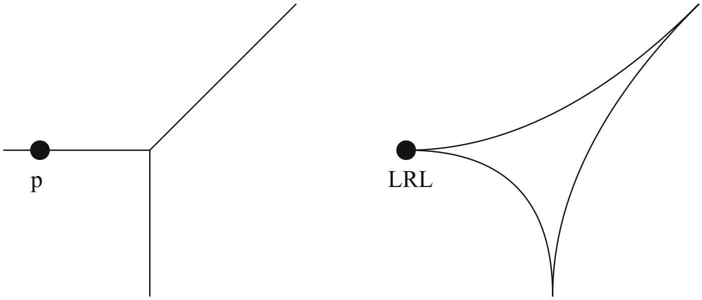

Fig. 1 Under the moment map of the real Calabi-Yau torus $\mathbb{T}'_{\mathbb{R}}$, the toric graph is the image of $\mathbb{T}'$-invariant 1-dimensional subvariety (left). The image of the Aganagic-Vafa brane is a point on the toric graph. The mirror curve (right) is $\mathbb{P}^1$ with three punctures, and its compactification is $\mathbb{P}^1$. The large radius point is where $X = 0$

Theorem 25 ([14, 67, 68]) *For $g \geq 0$, $n \geq 1$, $2g - 2 + n > 0$,*

$$F_{g,n}^{\mathbb{C}^3,L,f} = \int^{X_1} \cdots \int^{X_n} \omega_{g,n}.$$

Remark 26 The cases $(g,n) = (0,1), (0,2)$ have special forms which are omitted here for simplicity. There are also predictions on the free energies F_g which are the generating functions of closed GW invariants based on $\omega_{g,n}$. We will discuss F_g in general in the next subsection.

We will postpone the discussion of the proof to the next subsection. The following theorem relates Hurwitz numbers to $F_{g,n}^{\mathbb{C}^3,L,f}$ in the large framing limit.

Theorem 27 ([13])

$$\lim_{f \to \infty} (-1)^n f^{2-2g+n} F_{g,n}^{\mathbb{C}^3,L,f}\left(\frac{X_1}{f}, \ldots, \frac{X_n}{f}\right) = H_g(X_1, \ldots, X_n).$$

We have a localization formula for $F_{g,n}^{\mathbb{C}^3,L,f}$. One can write it as a triple Hodge integral with some disk factors (elementary functions). The proof of this theorem is a direct calculation, in which one takes the limit $f \to \infty$ in the triple Hodge integral (cf. ELSV formula 21)

$$\Lambda_g^\bullet(1)\Lambda_g^\bullet(f)\Lambda_g^\bullet(-1-f) = \Lambda_g^\bullet(1)(-1)^{g-1} f^{2g-2}\left(1 + O\left(\frac{1}{f}\right)\right).$$

On the other hand, the mirror curve

$$XY^{-f} + Y + 1 = 0$$

reduces to the Lambert curve $X' = Y'e^{-Y'}$ under the change of variable

$$X = -(-1)^f \frac{X'}{f}, \quad Y = -1 + \frac{Y'}{f}.$$

and taking limit $f \to \infty$. Theorem 20 is a consequence of Theorem 25 after one takes limit $f \to \infty$, and rewrites the open GW potential in the Hurwitz potential by Theorem 27.

4.5 *Mirror of a Semi-Projective Toric Calabi-Yau Threefold*

A toric Calabi-Yau threefold X is a Calabi-Yau 3-dimensional manifold (or more generally, an orbifold) with a Zariski open and dense algebraic torus $\mathbb{T} \cong (\mathbb{C}^*)^3$. The action of $\mathbb{T}$ on itself extends to X. For simplicity, we require X is a smooth manifold, and will remark briefly on orbifolds in Sect. 4.6. We also require that X is semi-projective, i.e. it is projective over its affinization. The last condition is equivalent to that the union of all cones defining X is convex in $\mathbb{R}^3$. Let $\mathbb{T}'$ be the 2-dimensional subtorus preserving the Calabi-Yau form.

Let $N = \mathrm{Hom}(\mathbb{C}^*, \mathbb{T})$ and $M = \mathrm{Hom}(\mathbb{T}, \mathbb{C}^*) = N^\vee$. The Calabi-Yau torus $\mathbb{T}' = \mathrm{Ker}(\mathsf{w}_3)$ for some $\mathsf{w}_3 \in M$. Being a Calabi-Yau threefold, the fan data to define X is the cone with vertex at the origin over a triangulated integral convex polytope Δ_X on $\{\mathsf{w}_3(y) = 1 | y \in N_{\mathbb{R}}\}$. If this triangulation cannot be further refined, i.e. each triangle has area $\frac{1}{2}$, the resulting X is a smooth manifold (see Fig. 2).

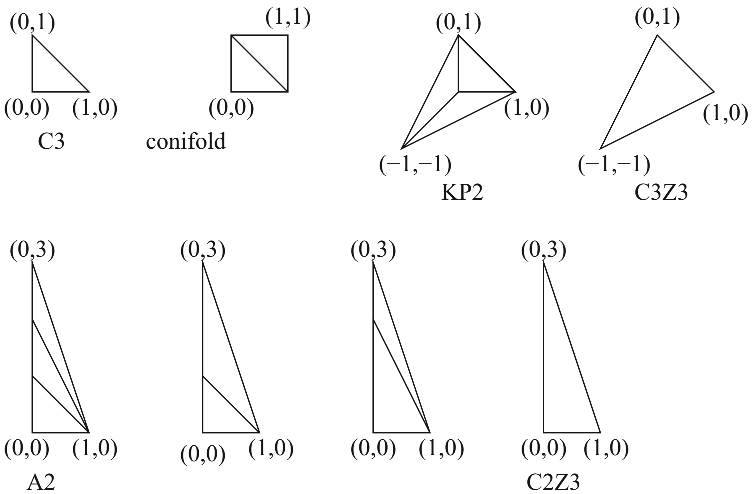

Fig. 2 Defining polytopes of some toric CY 3-(orbi)folds

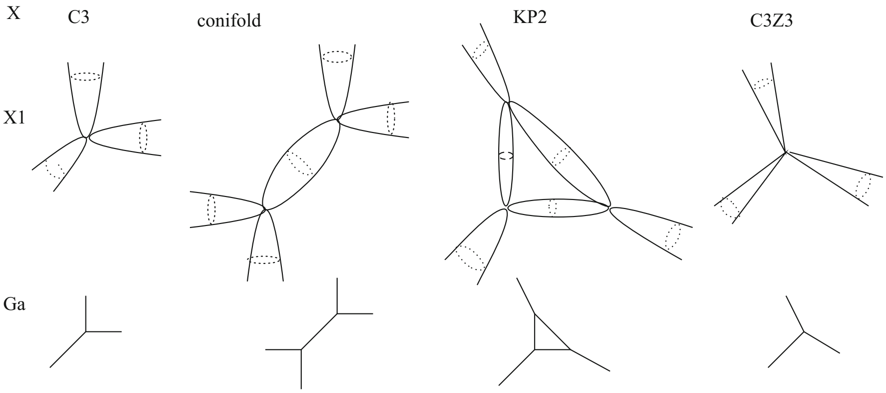

Fig. 3 1-Dimensional $\mathbb{T}'$-invariant subvarieties and toric graphs. We use $\mathcal{X}$ since some are orbifolds

The action of the real torus $\mathbb{T}'_{\mathbb{R}} \subset \mathbb{T}'$ is Hamiltonian, and we can consider the image of all 1-dimensional $\mathbb{T}'$-invariant subvarieties in X under the moment map μ'. Such image is called the toric graph of X (Fig. 3).

An Aganagic-Vafa brane is a Lagrangian 3-dimensional submanifold in X, given by the following condition

$$L \subset \mu'^{-1}(p), \quad \mathrm{Arg}(Z_1 \dots Z_{p+3}) = \text{const on } L.$$

where $(Z_1, \dots, Z_{p+3}) \in \mathbb{C}^{p+3}$ are homogeneous coordinates, and p is a non-vertex point on the toric graph. When p is on a finite segment, L is called an *inner braner*; when p is on a ray, L is called an *outer brane*. We restrict to the case of an outer brane L for simplicity. By suitable arrangement (by some $SL(2;\mathbb{Z})$-action and translation of the defining polytope), we always assume that vertex containing p consists of half-edges in the direction $(-1, 0)$, $(0, -1)$ and $(1, 1)$, and p is on the half-edge in the direction $(-1, 0)$. This half-edge is a ray since L is outer.

Similar to the case of $\mathbb{C}^3$, we consider the open Gromov-Witten invariant $N^{X,L,f}_{g,n,\beta,\mu}$ which counts the maps of the bordered Riemann surface in the topological type (g, n) to the target $(X, (L, f))$ in the curve class β. They form a generating function

$$F^{X,L,f}_{g,n} = \sum_{\mu_1,\dots,\mu_n=1}^{\infty} N^{X,L,f}_{g,n,\beta,\mu} \hat{X}_1{}^{\mu_1} \dots \hat{X}_n^{\mu_n} Q^{\beta}.$$

Here we use $\hat{X}$ as open variables since they might differ from X in B-model by an open mirror map. The Kähler parameter $Q^{\beta} = \prod_{a=1}^{p} Q_a{}^{\langle p_a, \beta\rangle}$, where p_a form an integral basis in the Kähler cone, and we let $Q_a = e^{-\tau_a}$. The B-model is a mirror curve Σ_q

$$H(X, Y) = XY^{-f} + Y + 1 + \sum_{a=1}^{N-3} q_a X^{m_a} Y^{n_a - f m_a} = 0.$$

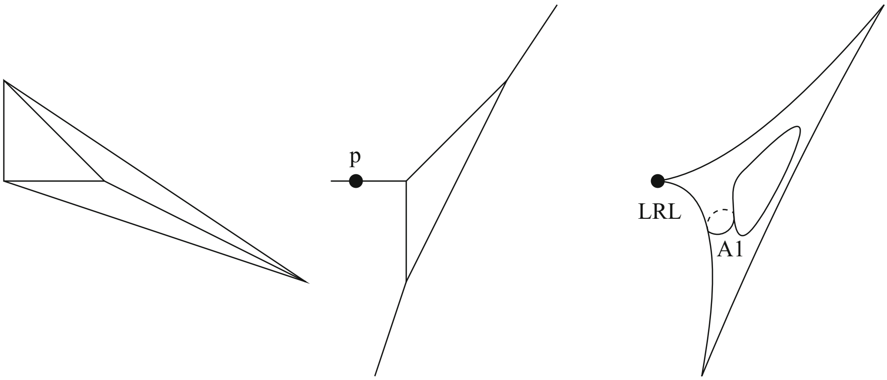

Fig. 4 Defining polytope, toric graph and mirror curve of $\mathcal{O}_{\mathbb{P}^2}(-3)$. Notice that we've arranged the defining polytope and the toric graph in the desired form such that the half edges of the vertex adjacent to p are in the desired direction. The point LRL on the mirror curve is the B-model large radius point, and the period integral around cycle A_1 gives the mirror map

In the equation, $q_1, \ldots, q_{N-3}$ are complex parameters, and $q \to 0$ at the large radius point. The number of complex parameter is $N-3$, where N is the number of integer points inside the defining polytope of X. Under mirror symmetry, N is also the dimension of the equivariant (quantum) cohomology. The integer points inside the defining polytope are denoted by (m_a, n_a). The tropicalization of this curve reduces back to the toric graph (see Fig. 4 as an example). It is also an SYZ dual to the union of 1-dimensional $\mathbb{T}'$-invariant subvarieties. Depending on the choice of the Aganagic-Vafa brane, there is a large radius limit point (the LRL-point in Fig. 4) on the mirror curve where $X = 0$. We specify $e^{-x} = X$, $e^{-y} = Y$ (so at LRL, $x = \infty$ and thus the name large radius). The Landau-Ginzburg superpotential W and its equivariant version $\widetilde{W}$ are

$$W = H(X, Y)Z, \quad \widetilde{W} = W - \log X.$$

The open-closed mirror map is the following

$$\begin{aligned} \tau_a &= \tau_a(q) = \log q_a + h_a(q),\ a = 1, \ldots, N-3 \qquad (3) \\ \log \hat{X}_i &= \log X_i + h_0(q), \end{aligned}$$

where $h_a(q) = O(q)$ for $a = 1, \ldots, N-3$. In particular, there are certain choices of geometric cycles $A_a \in H_1(\Sigma_q; \mathbb{Z}), a = 1, \ldots, N-3$, which can be lifted to cycles in $H_1(\widetilde{\Sigma}_q; \mathbb{Z})$, such that

$$\tau_a = \int_{A_a} y dx.$$

The closed part of this map (first line of Eq. (3)) maps q to a Kähler class $\tau \in H^2(X)$. Furthermore there is another cycle $A_0(X)$ which lifts to a path $\widetilde{A}_0(X)$ in the universal cover of Σ_q, such that

$$\log \hat{X} = \int_{\widetilde{A}_0(X)} ydx = \log X + h_0(q).$$

The genus 0 mirror symmetry for closed descendant Gromov-Witten theory is the celebrated toric mirror theorem of Givental and Lian-Liu-Yau [36, 38, 50, 51], in which closed mirror maps are also explicitly given. The open-closed mirror maps are explicitly computed in [49, 57], and the mirror symmetry for disk invariants is conjectured in [3, 4] and proved in [28, 33] under these mirror maps.

The cycles $A_a, a = 1, \ldots, N-3$ induce a Lagrangian subspace of $H_1(\overline{\Sigma}_q; \mathbb{C})$, and thus defines a fundamental differential form B. Define the spectral curve $\boldsymbol{\Sigma}_q = (\Sigma_q, x, B)$. The Eynard-Orantin recursion gives a sequence of higher genus B-model invariants $\omega_{g,n}$. The BKMP remodeling conjecture says

Theorem 28 (Fang-Liu-Zong, [31, 32]) *When $2g-2+n > 0$, $g \geq 0$, $n \geq 1$,*

$$\int^{X_1} \cdots \int^{X_n} \omega_{g,n} = F_{g,n}^{X,L,f}(\hat{X}_1, \ldots, \hat{X}_n).$$

In this theorem, we understand that $\omega_{g,n}$ as power series in X around the large radius limit point. When $2g-2 > 0$, closed free energy is predicted by the following formula

$$F_g^X = \sum_{p_0 \in I_{\boldsymbol{\Sigma}_q}} \mathrm{Res}_{p=p_0} \omega_{g,1}(p) \widetilde{\Phi}(p),$$

where $I_{\boldsymbol{\Sigma}_q}$ is the set of ramification points, and $d\widetilde{\Phi}(p) = \Phi$ is a function locally defined around each ramification point.

4.5.1 Sketch of the BKMP Remodeling Conjecture: Graph Sums

We illustrate the idea of using graph sums to give a proof of this conjecture. As we discussed in Sect. 3, the B-model side could be written as the following graph sums

$$\int^{X_1} \cdots \int^{X_n} \omega_{g,n} = \sum_{\vec{\Gamma} \in \boldsymbol{\Gamma}_{g,n}(\Sigma_q)} \frac{w_{B,O}^X(\vec{\Gamma})}{\mathrm{Aut}(\vec{\Gamma})}. \tag{4}$$

The only difference between $w_{B,O}^X(\vec{\Gamma})$ and $w_S^{\mathbf{p}}(\vec{\Gamma})$ in Eq. (2) is that the ordinary leaf term $(\check{\mathcal{L}}^{\mathbf{p}})_k^\alpha(l_j) = -\frac{1}{\sqrt{2}}\theta_k^\alpha(p_j)$ is replaced by its integral

$$(\check{\mathcal{L}}^O)_k^\alpha(l_j) = -\frac{1}{\sqrt{2}} \int^{X_j} \theta_k^\alpha \tag{5}$$

The first step to deal with the A-model side in Theorem 28 is to reduce it to closed descendant Gromov-Witten invariants by the technique of localization, as done in [28, Proposition 3.4] and [33, Proposition 3.1, 3.2].

By this localization formula, we have the graph sum formula

$$F_{g,n}^{X,L,f} = \sum_{\vec{\Gamma} \in \mathbf{\Gamma}(V(X))} \frac{w_{A,O}^{\hat{X}}(\vec{\Gamma})}{\mathrm{Aut}(\vec{\Gamma})}. \tag{6}$$

The only difference between $w_{A,O}^{\hat{X}}(\vec{\Gamma})$ and $w_{F,\bullet}^{\mathbf{u}}(\vec{\Gamma})$ in Eq. (1) is that the ordinary leaf term $(\mathcal{L}_d^{\mathbf{u}})_k^{\alpha}(l_j)$ is replaced by

$$(\mathcal{L}^O)_k^{\alpha}(l_j) = [z^k](\sum_{\beta,\gamma=1}^{N} \left(\widetilde{\xi}^{\beta}(z, X_j) S_{\hat{\beta}}^{\hat{\gamma}}(z) \right)_+ R(-z)_{\gamma}^{\beta}). \tag{7}$$

Roughly speaking, $\widetilde{\xi}^{\beta}(z, X)$ is the generating function counting 1 interior-pointed holomorphic disks mapped to (X, L) with no curve class but all winding numbers. The class ϕ^{β} is inserted in the interior. In order to compare the graph sum formulae (4) and (6), we need to identify $\mathbf{\Gamma}(V(X))$ and $\mathbf{\Gamma}(\mathbf{\Sigma}_q)$ first, and then we will identify the contribution from each graph $w_{A,O}^{\hat{X}}(\vec{\Gamma})$ and $w_{B,O}^{X}(\vec{\Gamma})$. The sets $\mathbf{\Gamma}(V(X))$ and $\mathbf{\Gamma}(\mathbf{\Sigma}_q)$ are just sets of stable decorated graphs, and the part of the decoration that depends on $V(X)$ or $\mathbf{\Sigma}_q$ is the labeling of a vertex by a canonical basis of the Frobenius algebra $V(X)$ for $\mathbf{\Gamma}(V(X))$, or a ramification point of x in the case of $\mathbf{\Gamma}(\mathbf{\Sigma}_q)$. The mirror theorem of semi-positive toric manifolds [36, 38, 50, 51], or later of semi-positive toric orbifolds [17], says the following.

Theorem 29

$$\mathrm{Jac}(\widetilde{W}) \cong QH^*_{\mathbb{T}'_f}(X)$$

in the small phase space $\tau \in H^2(X)$ and under the closed mirror map (3).

The Jacobian ring is

$$\mathrm{Jac}(\widetilde{W}) = \frac{\mathbb{C}[X^{\pm}, Y^{\pm}, Z^{\pm}]}{\langle \frac{\partial \widetilde{W}}{\partial X}, \frac{\partial \widetilde{W}}{\partial Y}, \frac{\partial \widetilde{W}}{\partial Z} \rangle}.$$

It is a Frobenius algebra for given q. Here we refrain from saying that it is a Frobenius *manifold*, i.e. we do not give a metric and discuss the flatness condition. This theorem already identifies the canonical basis of both sides.

The canonical basis of left hand side $\mathrm{Jac}(\widetilde{W})$ consists of functions taking value 1 on one critical point of $\widetilde{W}_{\mathbb{T}'}$ and vanish on other critical points. A critical point (X_0, Y_0, Z_0), by direct calculation in [32], is the solution to the following equation

$$H(X, Y) = 0, \quad \frac{\partial H}{\partial Y} = 0, \quad ZX\frac{\partial H}{\partial X} = 1.$$

We see the critical points of $\widetilde{W}$ has a 1-to-1 correspondence to the ramification points of $x : \Sigma_q \to \mathbb{C}^*$.

Once we identify the set of stable decorated graphs, after looking at the weights (1) and (2) (the ordinary leaf terms should be replaced by (7) and (5), as discussed before), we need to show the following:

- Show that $R_\beta^{\ \alpha}(-z) = \check{R}_\beta^{\ \alpha}(z)$. Notice that this matches vertices, edges and dilaton leafs. Both R and $\check{R}$ are obtained through the decomposition of S-matrices. Dubrovin and Givental's results [18, 37, 40] on this decomposition ensures the uniqueness of R-matrices up to a constant matrix, which can be fixed at the large radius limit ($q = 0$). When $q = 0$, A-side R is computed by the quantum Riemann-Roch [16, 64] and B-side $\check{R}$ is computed by direct integral which produces triple Gamma functions [31], in which we show that they match.
- Show that open leafs (7) and (5) match. By localization, $(\widetilde{\xi}^\beta(z, X_j) S^{\hat{\gamma}}_{\ \beta}(z))_+$ in Eq. (7) is the generating functions of $\hat{\phi}_\gamma(\tau)$ interiorly inserted disk invariants with all winding numbers and in all possible curve classes. It is shown in [28, 33] that this corresponds to $\hat{\xi}^\gamma(z, X)$, which is the B-model counting of disk invariants with canonical basis inserted.

4.6 Remarks on Orbifolds

Once we adopt the orbifold Gromov-Witten invariants [1, 2, 15], there is no essential difference to state the BKMP conjecture when the toric Calabi-Yau threefold is a toric orbifold in the sense of [8]. When the defining polytope contains a triangle with areas larger than $\frac{1}{2}$, it defines a toric orbifold (with non-trivial orbifold structure). Some examples in Fig. 2 are orbifolds.

In this paper, all quotients of a smooth variety by a finite group are stacky. Let $\mathcal{X}_1 = \mathbb{C}^3/\mathbb{Z}_3$ and $\mathcal{X}_2 = \mathbb{C}^2/\mathbb{Z}_3 \times \mathbb{C}$. For $\mathcal{X}_1$, the generator of $\mathbb{Z}_3$ acts diagonally; while for $\mathcal{X}_2$ the generator of $\mathbb{Z}_3$ acts on each piece of $\mathbb{C}^2$ with opposite non-trivial weights. The Aganagic-Vafa Lagrangian brane L is stacky for $\mathcal{X}_2$ (Fig. 5).

The mirror curves are

$$\mathcal{X}_1 : \quad X^3 Y^{-1-3f} + Y + 1 + qXY^{-f} = 0,$$
$$\mathcal{X}_2 : \quad XY^{-f} + Y^3 + 1 + q_1 Y + q_2 Y^2 = 0.$$

When the Aganagic-Vafa brane L is not stacky, the orbifold BKMP conjecture is conjectured by [12]. It takes the same form as in Theorem 28. One needs to make some adjustment for gerby legs (stacky Lagrangian), as in [31, 32].

The topological vertex algorithm works efficiently for smooth toric Calabi-Yau threefolds [5, 53, 54, 56] and is extended to hard Lefschetz orbifolds [60–62, 69]. However, this algorithm fails when the orbifold $\mathcal{X}$ is non-hard Lefschetz. The affine orbifold $\mathcal{X}_1 = \mathbb{C}^3/\mathbb{Z}_3$ is non-hard Lefschetz—the only vertex in the toric

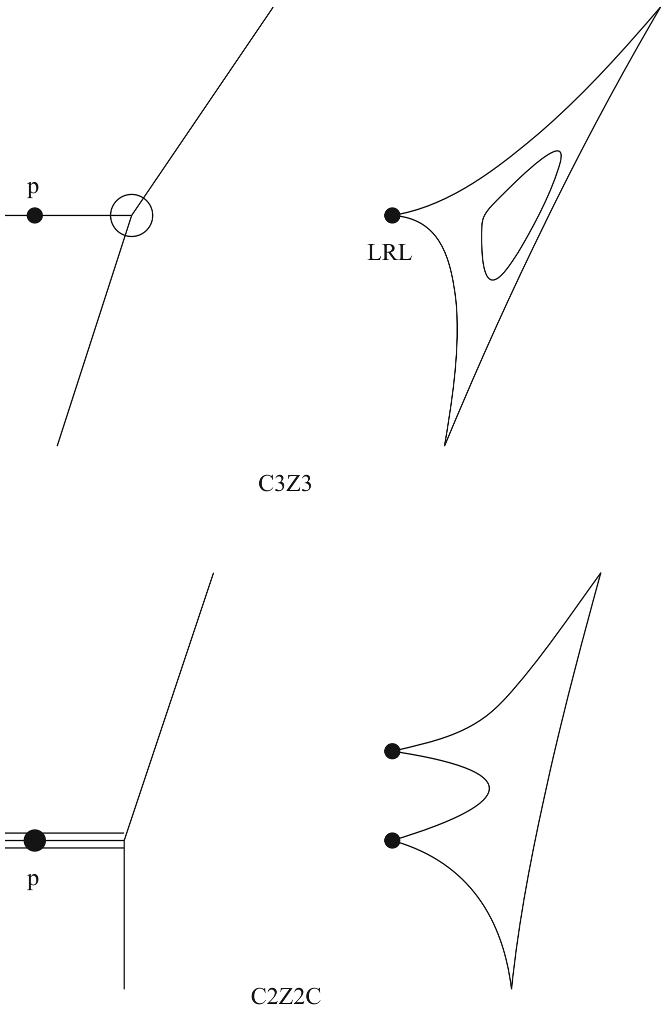

Fig. 5 Toric graph and mirror curves of $\mathbb{C}^3/\mathbb{Z}_3$ and $\mathbb{C}^2/\mathbb{Z}_3 \times \mathbb{C}$

graph corresponds to a higher genus part of the mirror curve. Thus orbifold BKMP conjecture is powerful in the sense that it first provides an effective algorithm.

The case of affine toric Calabi-Yau threefolds ($\mathbb{C}^3/G$ for a Calabi-Yau action by the abelian group G) is proved in [31], and the general toric Calabi-Yau 3-orbifolds is proved in [32]. For affine cases, a particular complication compared to smooth cases is the A-side orbifold Riemann-Roch calculation [29, 64]. For a general toric Calabi-Yau 3-orbifold, one cannot rely on Givental's quantization [39, 40] since his technique is restricted to the smooth situation if not modified extensively. The paper [32] uses Zong's thesis [70], which relies on Teleman's groundbreaking work [63].

There is also an orbifold version of Bouchard-Mariño formula [13]. As shown in [13], it is the large framing limit of the BKMP conjecture for an affine toric Calabi-Yau 3-orbifold. It should also be the large radius limit of an all genera mirror symmetry statement about an orbifold $\mathbb{P}^1$. However this is not addressed in any literature as far as the author knows.

5 Modularity of the Topological Recursion and Its Application

In this section, we will briefly review the modular invariance of $\omega_{g,n}$, and then as an application, illustrate how BKMP remodeling conjecture implies the modularity of Gromov-Witten invariants through an example. The modular invariance is a property emanating from the modular transformation of the fundamental differential $B(p, q)$ on an actual Riemann surface, and thus it is not a feature of formal spectral curves.

5.1 Modular Invariance of Fundamental Normalized Differentials of the Second Kind

Let (Σ, x, B) be a spectral curve, and $\overline{\Sigma}$ be its compactification. We fix two sets of Torelli markings

$$(A_k, B_k), (A'_k, B'_k),\ k = 1, \ldots, \mathfrak{g}$$

on $\overline{\Sigma}$. They differ by an $\mathrm{Sp}(2\mathfrak{g}; \mathbb{Z})$ transformation

$$\begin{pmatrix} A \\ B \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} A' \\ B' \end{pmatrix},$$

where a, b, c, d are $\mathfrak{g} \times \mathfrak{g}$ matrices, and $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{Sp}(2\mathfrak{g}; \mathbb{Z})$. Let $\theta_k,\ k = 1, \ldots, \mathfrak{g}$ be linearly independent holomorphic forms on $\overline{\Sigma}$ given by the Torelli marking (A_k, B_k), i.e.

$$\int_{A_i} \theta_j = \delta_{ij}.$$

The period matrix τ_{ij} is given by

$$\tau_{ij} = \int_{B_j} \theta_i.$$

We know $\mathrm{Im}(\tau) > 0$ (positive definite), and $\tau_{ij} = \tau_{ji}$.

Define the modified cycles

$$A_i(\tau) = A_i - \sum_j \kappa_{ij} B_j(\tau), \quad B_i(\tau) = B_i - \sum_j \tau_{ij} A_j.$$

Here

$$\kappa_{ij}(\tau, \overline{\tau}) = \frac{1}{\overline{\tau} - \tau}$$

is a $\mathfrak{g} \times \mathfrak{g}$ matrix function of τ (not holomorphic). As a convention in this section, we denote the fundamental differential associated to the A-cycles A_i by B_A, and the fundamental differential associated to the modified A-cycles $A_i(\tau)$ by $B_{A(\tau)}$.

By direct calculation, Eynard-Orantin show that in [24]

$$B_{A(\tau)} = B_A + 2\pi\sqrt{-1} \sum_{i,j=1}^{\mathfrak{g}} \theta_i \kappa(\tau, \overline{\tau}) \theta_j.$$

They also show that

$$B_{A'} = B_A + 2\pi\sqrt{-1} \sum_{i,j} \theta_i \hat{\kappa}_{ij}(\tau) \theta_j,$$

where $(\hat{\kappa}_{ij}) = bJ$ and $J = (d - \tau b)^{-1}$. Here τ' is the period matrix fixed by the Torelli marking (A'_k, B'_k).

$$\tau'_{ij} = \int_{B'_j} \theta'_i, \quad \int_{A'_j} \theta'_j = \delta_{ij}.$$

We have

$$\tau' = \frac{\tau a - c}{d - \tau b}, \quad \theta'_i = \sum_{ij} J_j \theta_j.$$

The fact that

$$J^t \kappa(\tau') J + \hat{\kappa}(\tau) = \frac{1}{\overline{\tau} - \tau}$$

implies

$$B_{A'(\tau')} = B_{A(\tau)}.$$

Proposition 30 (Eynard-Orantin) *Given any Torelli marking* (A_k, B_k) *for* $k = 1, \ldots, \mathfrak{g}$, *the modified fundamental differential* $B_{A(\tau)}$ *given by the modified Torelli marking* $(A_k(\tau), B_k(\tau))$ *is independent of the choice of* (A_k, B_k).

This property implies that given a fixed spectral curve $\mathbf{\Sigma}$, we have a preferred choice of the fundamental differential $B_{A(\tau)}$ independent of the choice of the A-cycles. We denote this by $\widetilde{B}$. Moreover, under the limit $\mathrm{Im}\tau \to \infty$, $\widetilde{B} \to B_A$.

From the explicit expression of the Eynard-Orantin recursion (Definition 6), for any spectral curve $\mathbf{\Sigma}$, we can define its modified B-model invariants $\widetilde{\omega}_{g,n}$ based on this modified fundamental differential $\widetilde{B}$.

5.2 *Modularity of* $\mathcal{O}_{\mathbb{P}^2}(-3)$*: An Example*

The modular invariance of $\widetilde{B}$ and $\widetilde{\omega}$, together with the BKMP remodeling conjecture 28, implies the modularity of the Gromov-Witten invariants of toric Calabi-Yau 3-(orbi)folds. This is a long-expected property of GW invariants. It follows naturally from the modularity of mirror curves from the view point of the remodeling conjecture. We illustrate by an example.

5.2.1 Family of Mirror Curves

Let $X = \mathcal{O}_{\mathbb{P}^2}(-3)$. Its fan is the cone over the defining polytope Δ, as shown in Fig. 4 in Sect. 4.5.

Its secondary stacky fan $\mathfrak{S}$ is a complete fan in $\mathbb{R}$, as shown in Fig. 6. The generators of is 1-cones are

$$b_1 = 1, \quad b_2 = 1, \quad b_3 = 1, \quad b_4 = -3.$$

The toric orbifold $\mathcal{M}_B \cong \mathbb{P}(1,3)$ defined by $\mathfrak{S}$ is the moduli space of the B-model, or conjecturally, is the stringly Kähler moduli space of the mirror A-model on X. Denote the stacky torus fixed point by $\mathfrak{p}_{\mathrm{orb}}$ and the non-stacky smooth torus fixed point by $\mathfrak{p}_{\mathrm{LRL}}$.

We now define the following extended secondary fan $\widetilde{\mathfrak{S}}$ as a complete fan in $\mathbb{R}^3$ as in Fig. 7. The generators of its 1-cones are

$$\widetilde{b}_1 = (0,0,1), \quad \widetilde{b}_2 = (-1,0,1), \quad \widetilde{b}_3 = (0,-1,1), \quad \widetilde{b}_4 = (-1,-1,-3),$$
$$\widetilde{b}_5 = (1,1,0), \quad \widetilde{b}_6 = (-2,1,0), \quad \widetilde{b}_7 = (1,-2,0).$$

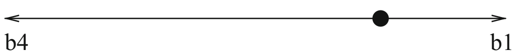

Fig. 6 The secondary fan of $\mathcal{O}_{\mathbb{P}^2}(-3)$

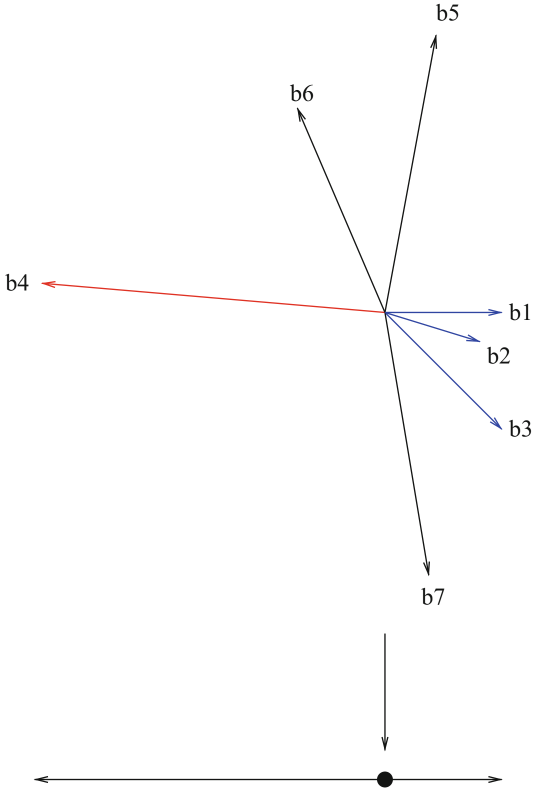

Fig. 7 The extended secondary fan of $\mathcal{O}_{\mathbb{P}^2}(-3)$. The third coordinates of the generators $\widetilde{b}_i$ are the same if they are in the same color. The rays $\widetilde{b}_5, \widetilde{b}_6, \widetilde{b}_7$ form the toric graph of $\mathbb{C}^3/\mathbb{Z}_3$. There is an obvious fan map $\widetilde{\mathfrak{S}} \to \mathfrak{S}$

The top dimensional cones are spanned by $\widetilde{b}_i$ where i ranges from the following index sets

$$\{4,5,6\},\ \{4,6,7\},\ \{4,5,7\},\ \{5,1,2\},\ \{5,1,3\},$$
$$\{6,1,2\},\ \{6,2,3\},\ \{7,2,3\},\ \{7,1,3\},\ \{1,2,3\}.$$

The 2-cones are faces of 3-cones. We denote the toric orbifold associated to the fan $\widetilde{\mathfrak{S}}$ by $\widetilde{\mathcal{M}}_B$ (Fig. 7).

There is an obvious fan map $\widetilde{\mathfrak{S}} \to \mathfrak{S}$ which forgets the first two factors. It induces a toric map $\pi : \widetilde{\mathcal{M}}_B \to \mathcal{M}_B$. The fiber $\pi^{-1}(\mathfrak{p})$ for $\mathfrak{p} \neq \mathfrak{p}_{LRL}$ is a toric orbifold defined by the stacky fan given by $\widetilde{b}_5, \widetilde{b}_6, \widetilde{b}_7$ (on $\mathbb{R}^2$). It is isomorphic to $\mathbb{P}^2/\mathbb{Z}_3$. Over the smooth torus fixed point, the fiber $\pi^{-1}(\mathfrak{p}_{\mathrm{LRL}})$ is three $\mathbb{P}^2$ intersecting along three $\mathbb{P}^1$ with normal crossing singularities (see Fig. 8). If one intersects the fan $\widetilde{\mathfrak{S}}$ by a vertical plane, at different horizontal position, we get the fan of each fiber toric surface. See Fig. 8.

We understand X, Y, q as characters in $\mathrm{Hom}(\mathbb{T}_B, \mathbb{C}^*) = M_B := N_B^\vee$, where $\mathbb{T}_B$ is the open dense 3-torus in $\widetilde{\mathcal{M}}_B$, and $N_B \cong \mathbb{Z}^3$ is the lattice that $\widetilde{b}_i$ belong to. Then

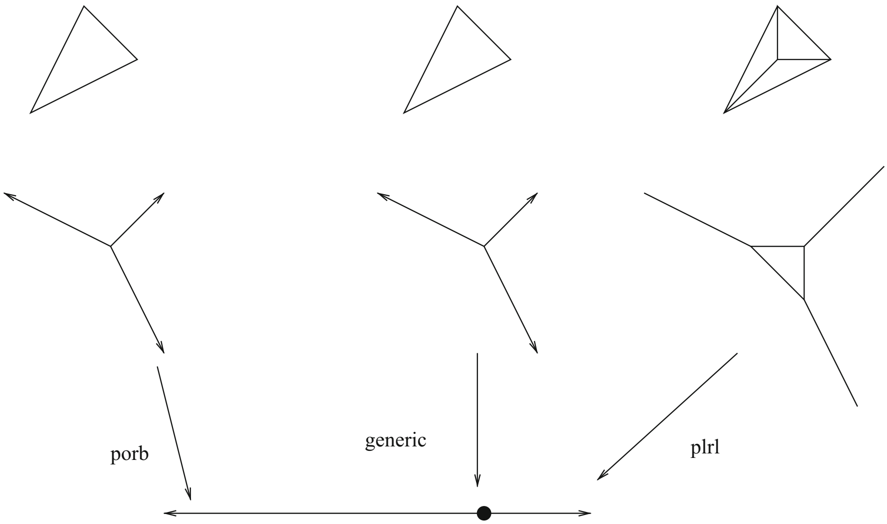

Fig. 8 Over $\mathcal{M}_B$, we have a family of toric surfaces given by π. When $\mathfrak{p} \neq p_{\mathrm{LRL}}$, the fiber $\pi^{-1}(\mathfrak{p}) \cong \mathbb{P}^2/\mathbb{Z}_3$, given by the stacky fan spanned by $\widetilde{b}_5, \widetilde{b}_6, \widetilde{b}_7$. Over p_{LRL}, the toric surface degenerates to a normal crossing of three $\mathbb{P}^2$, as shown by the "fan" and the polytope. The first rows are polytopes and the second rows are fans for fiber toric surfaces at different points in $\mathcal{M}_B$

X, Y, q corresponds to $(1, 0, 0)$, $(0, 1, 0)$ and $(0, 0, 1)$ in M_B respectively. They are sections of a line bundle $\widetilde{\mathcal{L}} = \mathcal{O}_{\widetilde{\mathcal{M}}_B}(\sum_{i=1}^{6} D_i)$. We define a section $H \in \Gamma(\widetilde{\mathcal{L}})$

$$H = X + Y + 1 + qX^{-1}Y^{-1}.$$

We define the compactified global mirror curve $\widetilde{\Sigma} = H^{-1}(0) \subset \widetilde{\mathcal{M}}_B$. It is parametrized over $\mathcal{M}_B$ by $\pi_{\widetilde{\Sigma}} = \pi|_{\widetilde{\Sigma}} : \widetilde{\Sigma} \to \mathcal{M}_B$. For any $\mathfrak{p} \in \mathcal{M}_B$, the fiber $\pi_{\widetilde{\Sigma}}^{-1}(\mathfrak{p})$ is a compact (possibly singular) curve. Let $\mathcal{M}_{B,0}$ be the part of $\mathcal{M}_B$ where the fiber curves are smooth. As shown in Fig. 9, $\mathfrak{p}_{\mathrm{LRL}} \notin \mathcal{M}_{B,0}$ and $\mathfrak{p}_{\mathrm{orb}} \in \mathcal{M}_{B,0}$. There is another point other than $\mathfrak{p}_{\mathrm{LRL}}$ not in $\mathcal{M}_{B,0}$. The fiber has one nodal singularity. This point is called the conifold point $\mathfrak{p}_{\mathrm{con}}$. Thus $\mathcal{M}_{B,0} = \mathcal{M}_B \setminus \{\mathfrak{p}_{\mathrm{LRL}}, \mathfrak{p}_{\mathrm{con}}\}$.

5.2.2 Modularity

The monodromies of the Gauss-Manin connection on the local system $H^1(\Sigma_q; \mathbb{C}) \cong H_1(\Sigma_q; \mathbb{C})$ over $\mathcal{M}_{B,0}$ around $\mathfrak{p}_{\mathrm{LRL}}$ and p_{con} (as computed in [6]) gives the *modular group* Γ of this local system. It is a normal subgroup of the symplectic group $SL(2; \mathbb{Z})$ of index 3.

Over $\mathcal{M}_{B,0}$, we have a smooth family of mirror curves, and the coordinates X, Y are well defined. So X, Y are invariant under the action of the modular group Γ. If

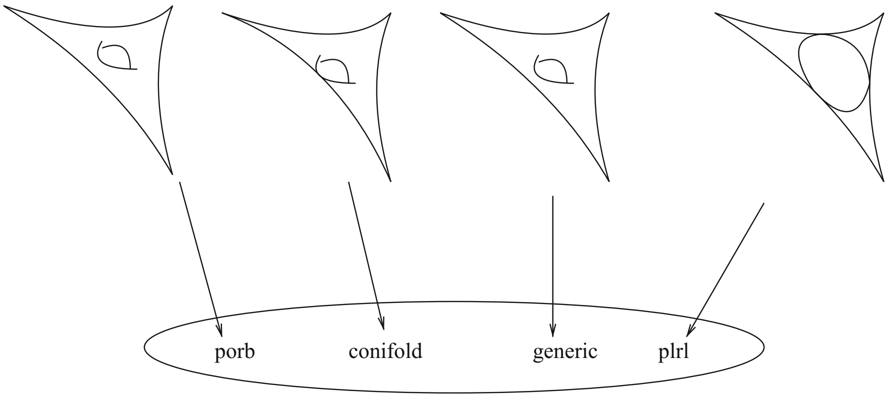

Fig. 9 Over $\mathcal{M}_B$, we have a family of compactified mirror curves $\widetilde{\Sigma}$. At p_{con} and p_{LRL} the mirror curves are singular. As before, the sharp ends in the mirror curve picture are the punctures on the mirror curve. After compactification, they become compact curves in $\pi^{-1}(\mathfrak{p})$. All puncture points are smooth

we use the modified fundamental differential $\widetilde{B}$ to define the higher genus B-model invariants $\widetilde{\omega}_{g,n}$, then they are all well-defined global invariants on $\widetilde{\Sigma}|_{\mathcal{M}_{B,0}}$. In other words, if one uses Torelli-marking-sensitive coordinates τ to express these $\widetilde{\omega}_{g,n}$, they are invariant under the action of the modular group Γ.

Using the mirror map (3) we define the open potential in the holomorphic polarization under A-model flat coordinates.

$$\widetilde{F}_{g,n}^{X,L,f}(\hat{X}_1,\ldots,\hat{X}_n,Q)=\int^{X_1}\ldots\int^{X_n}\widetilde{\omega}_{g,n}.$$

The A-model coordinate $Q=Q(\mathfrak{p})$ is well-defined around the LRL point, and is related to B-model coordiante q around the LRL point under the closed mirror map. The open potential $\widetilde{F}_{g,n}^{X,L,f}$ has non-holomorphic dependence on Q, in contrast to the name "holomorphic polarization". Under the holomorphic limit

$$\lim_{\mathrm{Im}\tau\to\infty}\widetilde{\omega}_{g,n}=\omega_{g,n}.$$

With the BKMP remodeling conjecture (Theorem 28), we have for $2g-2+n>0$ and $n\geq 1$

$$\lim_{\mathrm{Im}\tau\to\infty}\widetilde{F}_{g,n}^{X,L,f}=F_{g,n}^{X,L,f}. \tag{8}$$

If one defines

$$\widetilde{F}_g^X=\frac{1}{2-2g}\sum_{p_0\in I_{\Sigma_q}}\mathrm{Res}_{p=p_0}\widetilde{\omega}_{g,1}(p)\widetilde{\Phi}(p),$$

then for $g \geq 2$

$$\lim_{\mathrm{Im}\tau\to\infty} \widetilde{F}_g^X = F_g^X.$$

The potential $\widetilde{F}_{g,n}^{X,L,f}$ and $\widetilde{F}_g^X$ are globally defined over $\mathcal{M}_B$, although their expansions in Q are only defined around $\mathfrak{p}_{\mathrm{LRL}}$ since Q is a flat coordinate around $\mathfrak{p}_{\mathrm{LRL}}$. Their dependence on $\mathfrak{p} \in \mathcal{M}_B$ is not holomorphic.

Theorem 31 *The Gromov-Witten potential F_g^X can be completed into an analytic function $\widetilde{F}_g^X$, which under the mirror map* (3) *is globally defined on $\mathcal{M}_B$. If $\mathcal{M}_B$ is a modular curve, e.g. when $X = \mathcal{O}_{\mathbb{P}^2}(-3)$, the function $\widetilde{F}_g^X$ is a function of τ and modular invariant.*

Remark 32 The theorem also holds for unstable cases $(g, n) = (0, 0), (0, 1), (0, 2), (1, 0)$ but we need to treat these cases separately. We did not very clearly spell out what this "anti-holomorphic completion" is, as it should be stronger than (8). Indeed, $\widetilde{\omega}_{g,n}$ can be written as a polynomial in $\frac{1}{\mathrm{Im}\tau}$ with holomorphic coefficients [24, 26]. The lowest order of $\mathrm{Im}\tau$ is $2-2g$, and each coefficient in non-holomorphic terms are given by combinations of $\omega_{g',n}$, $g' < g$ in a graph sum formula. The BKMP conjecture allows us to say the same—$\widetilde{F}_g^X$ is a polynomial in $\frac{1}{\mathrm{Im}\tau}$ with the highest power term $(\frac{1}{\mathrm{Im}\tau})^{2g-2}$ and holomorphic term F_g^X. Each coefficient in the non-holomorphic terms are given by $F_{g'}^X$ in a graph sum formula where $g' < g$.

Remark 33 One could use the modularity property to compute higher genus Gromov-Witten invariants for certain toric Calabi-Yau 3-(orbi)folds, thanks to the complete structure theorem of almost holomorphic modular forms. See [6, 7, 66] for numerical calculations and closed formulae for some $\widetilde{F}_g^X$ and F_g^X.

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References

1. Abramovich, D., Graber, T., Vistoli, A.: Algebraic orbifold quantum products. In: Orbifolds in Mathematics and Physics (Madison, WI, 2001). Contemporary Mathematics, vol. 310, pp. 1–24. American Mathematical Society, Providence (2002)
2. Abramovich, D., Graber, T., Vistoli, A.: Gromov-Witten theory of Deligne-Mumford stacks. Amer. J. Math. **130**(5), 1337–1398 (2008)
3. Aganagic, M., Vafa, C.: Mirror symmetry, D-branes and counting holomorphic discs (2000). arXiv:hep-th/0012041
4. Aganagic, M., Klemm, A., Vafa, C.: Disk instantons, mirror symmetry and the duality web. Z. Naturforsch. **A 57**, 1–28 (2002)

5. Aganagic, M., Klemm, A., Mariño, M., Vafa, C.: The topological vertex. Commun. Math. Phys. **254**(2), 425–478 (2005)
6. Aganagic, M., Bouchard, V., Klemm, A.: Topological strings and (almost) modular forms. Commun. Math. Phys. **277**(3), 771–819 (2008)
7. Alim, M., Scheidegger, E., Yau, S.-T., Zhou, J.: Special polynomial rings, quasi modular forms and duality of topological strings. Adv. Theor. Math. Phys. **18**(2), 401–467 (2014)
8. Borisov, L., Chen, L., Smith, G.: The orbifold Chow ring of toric Deligne-Mumford stacks. J. Am. Math. Soc. **18**(1), 193–215 (2005)
9. Borot, G., Eynard, B., Mulase, M., Safnuk, B.: A matrix model for simple Hurwitz numbers, and topological recursion. J. Geom. Phys. **61**(2), 522–540 (2011)
10. Bouchard, V., Mariño, M.: Hurwitz numbers, matrix models and enumerative geometry. In: From Hodge Theory to Integrability and TQFT tt*-Geometry, pp. 263–283. Proceedings of Symposia in Pure Mathematics, vol. 78. American Mathematical Society, Providence (2008)
11. Bouchard, V., Klemm, A., Mariño, M., Pasquetti, S.: Remodeling the B-model. Commun. Math. Phys. **287**(1), 117–178 (2009)
12. Bouchard, V., Klemm, A., Mariño, M., Pasquetti, S.: Topological open strings on orbifolds. Commun. Math. Phys. **296**(3), 589–623 (2010)
13. Bouchard, V., Hernández Serrano, D., Liu, X., Mulase, M.: Mirror symmetry for orbifold Hurwitz numbers. J. Differ. Geom. **98**(3), 375–423 (2014)
14. Chen, L.: Bouchard-Klemm-Marino-Pasquetti conjecture for $\mathbb{C}^3$ (2009). arXiv:0910.3739
15. Chen, W., Ruan, Y.: Orbifold Gromov-Witten theory. In: Orbifolds in Mathematics and Physics. Contemporary Mathematics, vol. 310, pp. 25–85. American Mathematical Society, Providence (2002)
16. Coates, T., Givental, A.: Quantum Riemann-Roch, Lefschetz and Serre. Ann. Math. (2) **165**(1), 15–53 (2007)
17. Coates, T., Corti, A., Iritani, H., Tseng, H.-H.: A mirror theorem for toric stacks. Compos. Math. **151**(10), 1878–1912 (2015)
18. Dubrovin, B.: Geometry of 2D topological field theories. In: Integrable Systems and Quantum Groups (Montecatini Terme, 1993). Lecture Notes in Mathematics, vol. 1620, pp. 120–348. Springer, Berlin (1996)
19. Dunin-Barkowski, P., Orantin, N., Shadrin, S., Spitz, L.: Identification of the Givental formula with the special curves topological recursion procedure. Commun. Math. Phys. **328**(2), 669–700 (2014)
20. Ekedahl, T., Lando, S., Shapiro, M., Vainshtein, A.: Hurwitz numbers and intersections on moduli spaces of curves. Invent. Math. **146**(2), 297–327 (2001)
21. Eynard, B.: Invariants of spectral curves and intersection theory of moduli spaces of complex curves. Commun. Number Theory Phys. **8**(3), 541–588 (2014)
22. Eynard, B.: Intersection number of spectral curves (2011). arXiv:1104.0176
23. Eynard, B., Nicolas, O.: Weil-Petersson volume of moduli spaces, Mirzakhani's recursion and matrix models (2007). arXiv:0705.3600
24. Eynard, B., Orantin, N.: Invariants of algebraic curves and topological expansion. Commun. Number Theory Phys. **1**(2), 347–452 (2007)
25. Eynard, E., Orantin, N.: Computation of open Gromov-Witten invariants for toric Calabi-Yau 3-folds by topological recursion, a proof of the BKMP conjecture. Commun. Math. Phys. **337**(2), 483–567 (2015)
26. Eynard, B., Marño, M., Orantin, N.: Holomorphic anomaly and matrix models. J. High Energy Phys. (6) **2007**, 058, 20 (2007)
27. Eynard, B., Mulase, M., Safnuk, B.: The Laplace transform of the cut-and-join equation and the Bouchard-Mariño conjecture on Hurwitz numbers. Publ. Res. Inst. Math. Sci. **47**(2), 629–670 (2011)
28. Fang, B., Liu, C.-C.M.: Open Gromov-Witten invariants of toric Calabi-Yau 3-folds. Commun. Math. Phys. **323**(1), 285–328 (2013)
29. Fang, B., Liu, C.-C.M., Zong, Z.: Equivariant Gromov-Witten theory of affine smooth toric Deligne-Mumford stacks. Int. Math. Res. Not. IMRN **2016**(7), 2127–2144 (2016)

30. Fang, B., Liu, C.-C.M., Zong, Z.: The Eynard-Orantin recursion and equivariant mirror symmetry for the projective line. Geom. Topol. **21**(4), 2049–2092 (2017)
31. Fang, B., Liu, C.-C.M., Zong, Z.: All genus open-closed mirror symmetry for affine toric Calabi-Yau 3-orbifolds (2013). arXiv:1310.4818
32. Fang, B., Liu, C.-C.M., Zong, Z.: On the remodeling conjecture for toric Calabi-Yau 3-orbifolds (2016). arXiv:1604.07123
33. Fang, B., Liu, C.-C.M., Tseng, H.-H.: Open-closed Gromov-Witten invariants of 3-dimensional Calabi-Yau smooth toric DM stacks (2012). arXiv:1212.6073
34. Fay, J.D.: Theta Functions on Riemann Surfaces. Lecture Notes in Mathematics, vol. 352. Springer, Berlin (1973)
35. Givental, A.B.: Homological geometry and mirror symmetry. In: Proceedings of the International Congress of Mathematicians (Zrich, 1994), vol. 1, 2, pp. 472–480. Birkhuser, Basel (1995)
36. Givental, A.B.: Equivariant Gromov-Witten invariants. Int. Math. Res. Not. **1996**(13), 613–663 (1996)
37. Givental, A.: Elliptic Gromov-Witten invariants and the generalized mirror conjecture. In: Integrable Systems and Algebraic Geometry (Kobe/Kyoto, 1997), pp. 107–155. World Scientific Publishing, River Edge (1998)
38. Givental, A.B.: A mirror theorem for toric complete intersections. In: Topological Field Theory, Primitive Forms and Related Topics (Kyoto, 1996). Progress in Mathematics, vol. 160, pp. 141–175. Birkhäuser, Boston (1998)
39. Givental, A.B.: Gromov-Witten invariants and quantization of quadratic hamiltonians. Mosc. Math. J. **1**(4), 551–568 (2001)
40. Givental, A.B.: Semisimple Frobenius structures at higher genus. Int. Math. Res. Not. **2001**(23), 1265–1286 (2001)
41. Graber, T., Pandharipande, R.: Localization of virtual classes. Invent. Math. **135**(2), 487–518 (1999)
42. Graber, T., Vakil, R.: Hodge integrals and Hurwitz numbers via virtual localization. Compos. Math. **135**(1), 25–36 (2003)
43. Harvey, R., Lawson, H.B. Jr.: Calibrated geometries. Acta Math. **148**, 47–157 (1982)
44. Hori, K., Vafa, C.: Mirror symmetry (2000). arXiv:0002222
45. Iritani, H.: Convergence of quantum cohomology by quantum Lefschetz. J. Reine Angew. Math. **610**, 29–69 (2007)
46. Katz, S., Liu, C.-C.M.: Enumerative geometry of stable maps with Lagrangian boundary conditions and multiple covers of the disc. Adv. Theor. Math. Phys. **5**(1), 1–49 (2001)
47. Klemm, A., Zaslow, E.: Local mirror symmetry at higher genus. In: Winter School on Mirror Symmetry, Vector Bundles and Lagrangian Submanifolds (Cambridge, MA, 1999). AMS/IP Studies in Advanced Mathematics, vol. 23, pp. 183–207. American Mathematical Society, Providence (2001)
48. Li, J., Liu, C.-C.M., Liu, K., Zhou, J.: A mathematical theory of the topological vertex. Geom. Topol. **13**(1), 527–621 (2009)
49. Lerche, W., Mayr, P.: On N=1 mirror symmetry for open type II strings (2001). arXiv:hep-th/0111113
50. Lian, B.H., Liu, K., Yau, S.-T.: Mirror principle I. Asian J. Math. **1**(4), 729–763 (1997)
51. Lian, B.H., Liu, K., Yau, S.-T.: Mirror principle III. Asian J. Math. **3**(4), 771–800 (1999)
52. Liu, C.-C.M.: Moduli of J-holomorphic curves with Lagrangian boundary conditions and open Gromov-Witten invariants for an S^1-equivariant pair (2002). arXiv:math/0211388
53. Liu, C.-C.M., Liu, K., Zhou, J.: A proof of a conjecture of Mariño-Vafa on Hodge integrals. J. Differ. Geom. **65**(2), 289–340 (2003)
54. Liu, C.-C.M., Liu, K., Zhou, J.: A formula of two-partition Hodge integrals. J. Am. Math. Soc. **20**(1), 149–184 (2007)
55. Mariño, M.: Open string amplitudes and large order behavior in topological string theory. J. High Energy Phys. **2008**(3), 060, 34 (2008)

56. Maulik, D., Oblomkov, A., Okounkov, A., Pandaripande, R.: Gromov-Witten/Donaldson-Thomas correspondence for toric 3-folds. Invent. Math. **186**(2), 435–479 (2011)
57. Mayr, P.: $\mathcal{N} = 1$ mirror symmetry and open/closed string duality. Adv. Theor. Math. Phys. **5**(2), 213–242 (2001)
58. Mulase, M., Zhang, N.: Polynomial recursion formula for linear Hodge integrals. Commun. Number Theory Phys. **4**(2), 267–293 (2010)
59. Norbury, P., Scott, N.: Gromov-Witten invariants of $\mathbb{P}^1$ and Eynard-Orantin invariants. Geom. Topol. **18**(4), 1865–1910 (2014)
60. Ross, D.: On GW/DT and Ruan's conjecture in all genus for Calabi-Yau 3-orbifolds (2014). arXiv:1409.7015v2
61. Ross, D., Zong, Z.: The Gerby Gopakumar-Mariño-Vafa formula. Geom. Topol. **17**(5), 2935–2976 (2013)
62. Ross, D., Zong, Z.: Cyclic hodge integrals and loop schur functions. Adv. Math. **285**, 1448–1486 (2015)
63. Teleman, C.: The structure of 2D semi-simple field theories. Invent. Math. **188**(3), 525–588 (2012)
64. Tseng, H.-H.: Orbifold quantum Riemann-Roch, Lefschetz and Serre. Geom. Topol. **14**(1), 1–81 (2010)
65. Zhou, J.: Topological recursions of Eynard-Orantin type for intersection numbers on moduli spaces of curves. Lett. Math. Phys. **103**(11), 1191–1206 (2013)
66. Zhou, J.: Arithmetic properties of moduli spaces and topological string partition functions of some Calabi-Yau threefolds. Thesis (Ph.D.), Harvard University (2014), 155 pp. ISBN: 978-1321-02505-7
67. Zhou, J.: Local mirror symmetry for one-legged topological vertex (2009). arXiv:0910.4320
68. Zhou, J.: Local mirror symmetry for the topological vertex (2009). arXiv:0911.2343
69. Zong, Z.: Generalized Mariño-Vafa formula and local Gromov-Witten theory of orbi-curves. J. Differ. Geom. **100**(1), 161–190 (2015)
70. Zong, Z.: Equviariant Gromov-Witten theory of GKM orbifolds (2016). arXiv:1604.07270