Minimal Surfaces and Their Gauss Maps

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Abstract In this paper we will discuss some classical results in minimal surfaces theory, related to the Gauss map of such surfaces. In the last section we will comment on some work in progress and some open problems related to one of these results.

1 Introduction

It is generically accepted that the theory of minimal surfaces starts with the work of the Italian mathematician J. N. Lagrange who, in 1760, posed the following problem:

> consider a bounded open set $\Omega \subseteq \mathbb{R}^2$ with smooth boundary $\partial\Omega$ and a smooth function $\phi : \partial\Omega \longrightarrow \mathbb{R}$. Find a smooth function $f : \Omega \longrightarrow \mathbb{R}$ such that $f_{|\partial\Omega} = \phi$ and the graph of f has area smaller or equal to the area of the graph of any other smooth function $g : \Omega \longrightarrow \mathbb{R}$ such that $g_{|\partial\Omega} = \phi$.

Lagrange approach to the problem is the, by now, basic approach of the calculus of variations. Suppose that f is a solution of the problem. Consider a function $\eta : \Omega \longrightarrow \mathbb{R}$ such that $\eta_{|\partial\Omega} = 0$. Then the function $f_t = f + t\eta$ agrees with ϕ on $\partial\Omega$ and the area of its graph is

$$A_\eta(t) = \int_{\overline{\Omega}} \left[1 + \left(\frac{\partial f_t}{\partial x}\right)^2 + \left(\frac{\partial f_t}{\partial y}\right)^2\right]^{\frac{1}{2}} \mathrm{d}x\mathrm{d}y.$$

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Since f is supposed to be a solution of the problem, the function $A_\eta(t)$ has a minimum at $t = 0$, hence $A'_\eta(0) = 0$. A simple calculation gives

$$A'_\eta(0) = -\int_{\overline{\Omega}} \operatorname{div}\left(\frac{\nabla f}{\sqrt{1 + \|\nabla f\|^2}}\right) \eta \mathrm{d}x\mathrm{d}y = 0.$$

Since $A'_\eta(0) = 0, \ \forall\, \eta$ with $\eta|_{\partial\Omega} = 0$, it follows that solutions of the problem are solutions of the equation

$$\operatorname{div}\left(\frac{\nabla f}{\sqrt{1 + \|\nabla f\|^2}}\right) = 0. \qquad (1)$$

Equation (1) is called the *minimal surface equation* or the *Euler Lagrange equation of the problem*.

Remark 1.1 It turns out, from the regularity theory for elliptic partial differential equations, that a solution of (1) is a real analytic function (see also Lemma 2.7).

Some years later J. B. Meusnier was looking for a good concept of curvature of a regular surface M in $\mathbb{R}^3$. He considered a point $p \in M$ and a unit normal vector $N \in [T_pM]^\perp$. For a unit tangent vector $v \in T_pM$ he considered the plane determined by N and v and passing through p. Cutting M with such a plane he obtain a plane curve and he denoted by $k_p(v)$ the (oriented) curvature of this curve. It turns out that the function $k_p(v)$ has a unique minimum value, $k_m(p)$, and a unique maximum value, $k_M(p)$. These two numbers are called the *principal curvatures* of M at p and were introduced earlier by Euler.

Given the principal curvatures we can define

- the *Gaussian curvature* of the surface, $K(p) = k_m(p)k_M(p)$,
- the *mean curvature*, $H(p) = \frac{1}{2}(k_m(p) + k_M(p))$.

Meusnier showed that a function f is a solution of (1), if and only if its graph has vanishing mean curvature. This leads to the following definition.

Definition 1.2 A regular surface in $\mathbb{R}^3$ is called a *minimal surface* if its mean curvature vanishes identically.

Remark 1.3 Following Gauss, a concept should be considered only if it is "pregnant with theorems." He certainly had many important results involving the Gaussian curvature, but not so many involving the mean curvature. So he never seriously considered the latter concept.

We will take a slightly more general approach. Consider a surface M, i.e., a two-dimensional differentiable manifold, that, for simplicity, we will assume connected and *oriented by a positive atlas* of smooth charts $\{(\Omega_\alpha, \psi_\alpha)\}$.[1] Let $f : M \longrightarrow \mathbb{R}^3$ be an immersion. Then f induces a Riemannian metric on M,

$$\langle X, Y\rangle_p := \langle \mathrm{d}f(p)(X), \mathrm{d}f(p)(Y)\rangle, \quad X, Y \in T_pM,$$

which make f an isometric immersion. We can define the *Gauss map*

$$\underline{\mathbf{n}} : M \longrightarrow S^2 = \{x \in \mathbb{R}^3 : \|x\| = 1\},$$

where $\underline{\mathbf{n}}(x) \in [\mathrm{d}f(x)(T_xM)]^\perp \subseteq \mathbb{R}^3$ is the unit vector such that $(\psi_u, \psi_v, \underline{\mathbf{n}}(x))$ is a positively oriented basis of $T_{f(x)}\mathbb{R}^3$. If (Ω, ψ) is a positive chart, then

$$\underline{\mathbf{n}}(\psi(u, v)) = \frac{\psi_u \wedge \psi_v}{\|\psi_u \wedge \psi_v\|}$$

so $\underline{\mathbf{n}}$ is a well-defined smooth map.

Remark 1.4 If f is an immersion, then for all $x \in M$ there exists a neighborhood U of x such that $f(U)$ is a regular surface in $\mathbb{R}^3$. So, for local considerations, we can identify M with $f(M)$.

Consider the differential of the Gauss map

$$\mathrm{d}\underline{\mathbf{n}}(x) : T_xM \longrightarrow T_{\underline{\mathbf{n}}(x)}S^2 = T_xM$$

and the operator $A_x = -\mathrm{d}\underline{\mathbf{n}}(x)$. The operator A_x is called the *shape operator* at x or the *second fundamental form*. It turns out that A_x is a symmetric operator whose eigenvalues are exactly the principal curvatures.

Lagrange did not give any example of solutions of Eq. (1) (except for the trivial ones, i.e., the affine functions). There where many efforts to produce examples and characterize minimal surfaces with special properties. We will recall now some results in this direction.

Theorem 1.5 (Meusnier) *If f is a solution of Eq. (1) whose level curves are straight lines segments, then*

$$f(x, y) = A \arctan \frac{y - y_0}{x - x_0} + B, \quad x_0, y_0, A, B \in \mathbb{R}$$

i.e., the graph is an open part of a helicoid (figure a below).

[1] That is, an atlas such that the change of coordinates has positive Jacobian.

Theorem 1.6 (Sherk) *If f is a solution of Eq. (1) of the form $f(x, y) = g(x) + h(y)$, then*

$$f(x, y) = a^{-1} \log\left(\frac{\cos ax}{\cos ay}\right), \quad a \in \mathbb{R}$$

(figure b below).

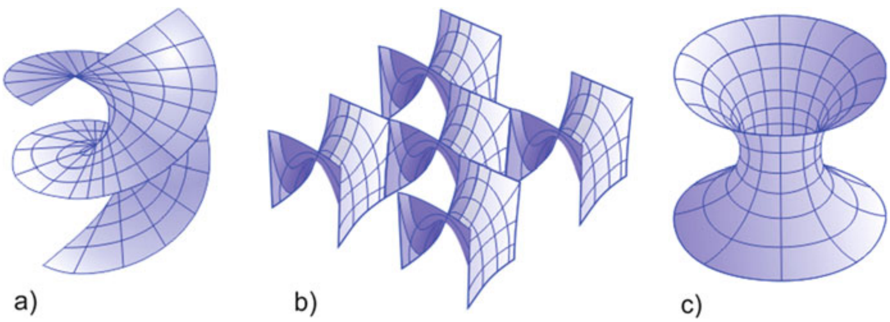
Theorem 1.7 (Euler) *A minimal surface of revolution is an open part of the catenoid*

$$(\cos v \sinh u, \sin v \cosh u, u),$$

up to rigid motions, or of a plane (figure c below).

Theorem 1.8 (Catalan) *A ruled minimal surface is an open part of a helicoid or of a plane.*

We refer to [2] for proofs and further information.



Around 1866 A. Enneper and K. Weierstrass gave a special parametrization for minimal surface, today known as the Weierstrass representation formula, which turns out to be a basic tool for producing examples of minimal surfaces. We will discuss this parametrization in Sect. 3.

We point out that the above results are, essentially, locally in nature. Probably the first global result is due to Bernstein who, around 1915, proved the following

Theorem 1.9 (Bernstein) *Let $f : \mathbb{R}^2 \longrightarrow \mathbb{R}$ be a solution of the minimal surface equation. Then f is an affine function, i.e., its graph is an affine plane.*

Remark 1.10 Bernstein's theorem should be compared with Liouville's theorem which states that a bounded harmonic function $f : \mathbb{R}^2 \longrightarrow \mathbb{R}$ is constant. However, in the first one, there are no conditions on the behavior of the function at infinity.

We will discuss in the next sections a couple of results that generalize Bernstein's theorem (see Remarks 2.35 and 4.2).

Remark 1.11 A natural question is if a similar theorem holds for functions $f : \mathbb{R}^n \longrightarrow \mathbb{R}$ which verify Eq. (1). Surprising enough the answer is positive if $n \leq 7$, but there are counterexamples for $n \geq 8$.

A basic question in the theory is the existence of a minimal surface whose boundary is a given simple closed curve Γ. This question has a long and rich history dating from the experiments of the Belgian physicist J. Plateau in 1847. He showed that dipping a wire on a soap solution we obtain a soup film which is a minimal surface and it is stable for small perturbations, i.e., is a *local minimum* for the area functional. So we have an "experimental proof" of the existence of minimal solution spanning a given boundary. But a "mathematical proof" proved to be a much more difficult task. It was only in 1930 that we have a first general answer to the question when, independently, Douglas (see [6]) and Radó (see [18]) proved the existence of a "minimal map" from a disk to $\mathbb{R}^3$, mapping the boundary of the disk onto a given rectifiable Jordan curve. A proof that the map is in fact an immersion, i.e., a minimal surface, appeared around 1960. There are still various problems under investigation, for example, the problem of uniqueness. If we consider the set of Jordan curves in $\mathbb{R}^3$, with a suitable topology, the subset for which the solution of the Plateau problem is unique is dense. Strangely enough the complement of this subset is also dense, and in fact there exist curves that bound an uncountable number of solutions. Naturally we are talking of *geometrically distinct* solutions, i.e., solutions up to a reparametrization. Also the existence of solutions with more complicate topology is an interesting field of investigation. We will not treat these questions here and refer to [4, 12] and the references therein for an introduction to these problems.

2 Stability

Let M be a surface, that, for simplicity we will assume connected and oriented, and let $f : M \longrightarrow \mathbb{R}^3$ be an immersion. A *domain* $D \subseteq M$ will be a connected, *relatively compact* open set such that the boundary is a finite union of disjoint piecewise smooth curves.

Definition 2.1 Let $D \subseteq M$ de a domain. A *(proper) variation* of f, supported on D, is a smooth function $F : (-\epsilon, \epsilon) \times M \longrightarrow \mathbb{R}^3$ such that:

(1) $F(0, x) = f(x)$,
(2) the restriction of F to $\{t_0\} \times M$ is an immersion,
(3) $F(t, x) = x$ if $x \notin D$.

When clear from the context we will simply say that F is a variation of f.

Given a variation F, the *variational vector field* is the vector field

$$V_F(x) := \mathrm{d}F(0, x)\left(\frac{\partial}{\partial t}\right).$$

Clearly V is a vector field along f[2] vanishing outside D. Set

$$F_t : M \longrightarrow \mathbb{R}^3, \quad F_t(x) = F(t, x).$$

Since F_t is an immersion, we can consider in M the induced metric and we will denote by $A_F(t)$ the area of D with respect to the induced metric. Then

Lemma 2.2 (First Variational Formula)

$$\frac{\mathrm{d}A_F(t)}{\mathrm{d}t}(0) = -2 \int_M \langle H\underline{\mathbf{n}}, V_F \rangle \mathrm{d}M,$$

where H is the mean curvature, and V_F is the variational vector field.

In particular if D has minimal area for all variations, $f_{|D}$ is a *minimal surface*. But in general a minimal surface is just a critical point of the area functional, not necessarily a minimum, not even a relative minimum.[3] In order to decide if a minimal surface is a relative minimum of the area functional we have to look at the second derivative of the area functional. To compute this derivative we need some preliminaries.

Let M be a Riemannian surface. If $u : M \longrightarrow \mathbb{R}$ is a smooth map, the *Laplacian* of u, Δu, is defined as

$$\Delta u = \operatorname{div} \nabla u,$$

where the gradient ∇u and the divergence are taken in relation to the metric of M.

It is useful, sometimes, to work in special local coordinates. Let $U \subseteq \mathbb{R}^2$ be an open set with coordinates (u, v).

Definition 2.3 A local parametrization $\psi : U \longrightarrow M$ of a Riemannian surface M is *isothermal* if

$$\|\psi_u\| = \|\psi_v\| = \lambda, \quad \langle \psi_u, \psi_v \rangle = 0,$$

where $\lambda : U \longrightarrow \mathbb{R}$ is a positive smooth function and subscripts denoted derivatives.

The coordinates (u, v) will be called *isothermal coordinates* (or *isothermal parameters*).

The following is well known

Theorem 2.4 *Let M be a Riemannian surface. Then $\forall\ p \in M$ there are isothermal coordinates in a neighborhood of p.*

Remark 2.5 In the case of minimal surfaces a simpler proof can be found in [16].

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[2] That is, a map $V : M \longrightarrow T\mathbb{R}^3$ such that $V(x) \in T_{f(x)}\mathbb{R}^3$.

[3] That is, a minimum with respect to nearby surfaces bounding the same curve.

Remark 2.6 If M is a connected oriented Riemannian surface, we can choose a *positive* atlas of isothermal coordinates. Once we do this, the changes of coordinates are conformal, hence holomorphic (where defined). A differentiable surface with such an atlas is called a *Riemann surface*.

In terms of isothermal coordinates the Laplacian is given by

$$\Delta = \lambda^{-2}\left(\frac{\partial^2}{\partial^2 u} + \frac{\partial^2}{\partial^2 v}\right),$$

and the Gaussian curvature is given by

$$K = -\Delta \log \lambda.$$

The following lemma is easy to prove

Lemma 2.7 *If* $f : M \longrightarrow \mathbb{R}^3$ *is an isometric immersion,* $f(u, v) = (f_1(u, v), f_2(u, v) f_3(u, v))$, *then*

$$\Delta f = (\Delta f_1, \Delta f_2, \Delta f_3) = 2H\underline{\mathbf{n}},$$

hence f is minimal if and only if the coordinate functions are harmonic. *In particular the coordinate functions of a minimal immersion are real analytic functions, and so are the Gaussian and mean curvature functions.*

Let $D \subseteq M$ be a domain. We will assume, for simplicity, that M is oriented and let $\underline{\mathbf{n}} : D \longrightarrow S^2$ be the Gauss map associated with the (fixed) orientation. We will denote by $H = H(D)$ the space of continuous functions on M, vanishing outside D, whose gradient exists almost everywhere and its norm is square integrable.

Remark 2.8 The space H has a natural norm given by

$$\|u\|^2 = \int_D u^2 \mathrm{d}M + \int_D \|\nabla u\|^2 \mathrm{d}M.$$

The subspace of smooth functions is dense in H, so, in many cases, we can assume that a function in H is smooth.

If $u \in H$ we consider the *normal variation*

$$F_u(t, x) = f(x) + tu(x)\underline{\mathbf{n}}(x),$$

whose variational vector field is

$$V = V_F = u\underline{\mathbf{n}}.$$

Theorem 2.9 (Second Variational Formula) *The second derivative of the area functional in the V direction is*

$$I(V, V) = \int_D u(-\Delta u + 2Ku)\mathrm{d}M.$$

We refer to [12] for a proof.

The quadratic form I is called the *index form* for D. If the index form is not positive semi-definite, the domain is *not* a relative minimum of the area functional. It turns out that, for local minimality questions, it is not restrictive to consider only normal variations. In particular if the index form of D is positive definite, then D is a local minimum of the area functional.

Definition 2.10 Let $f : M \longrightarrow \mathbb{R}^3$ be a minimal immersion and let $D \subseteq M$ be a domain. We will say that D is *stable* (resp. *strongly stable*) if the index form of D is positive semi-definite (res. positive definite).

An important concept related to stability is the following:

Definition 2.11 A normal field $J = u\underline{\mathbf{n}}$ is called a *Jacobi field* if

$$-\Delta u + 2uK = 0. \tag{2}$$

Definition 2.12 Let $f : M \longrightarrow \mathbb{R}^3$ be an immersion and let $D \subseteq M$ be a domain.

1. ∂D is said to be a *conjugate boundary* if there exists a Jacobi field $u\underline{\mathbf{n}}$ with $u \in H \setminus \{0\}$.
2. The *multiplicity* or *nullity* of a conjugate boundary, denoted by $\nu(D)$, is the dimension of the space of Jacoby fields $u\underline{\mathbf{n}}$, $u \in H$.
3. ∂D is a *first conjugate boundary* if it is a conjugate boundary and for all domains $D' \subseteq D$, $\partial D'$ is a conjugate boundary if and only if $D = D'$.

Remark 2.13 In the theory of geodesics we have analogous concepts. If M is a Riemannian manifold and $\gamma : [0, a] \longrightarrow M$ is a geodesic, a vector field J along γ is a Jacobi field if $\ddot{J} + R(\dot{\gamma}, J)\dot{\gamma} = 0$ where $\ddot{J}$ is the second covariant derivative of J along γ and R is the curvature tensor. A point $t_0 \in [0, a]$ is conjugate to 0 if there exists a non-trivial Jacoby field vanishing at 0 and t_0. Then the geodesic γ is a local minimum for the energy functional acting on curves joining $\gamma(0)$ and $\gamma(a)$ if there are no conjugate values in $[0, a)$. The corresponding assertion is true in our context, i.e., if for all domains $D' \subsetneq D$ $\partial D'$ is not a conjugate boundary, then D is stable.

We will give now a short proof of the following fact:

Theorem 2.14 *If $D \subseteq M$ is a domain and ∂D is a first conjugate boundary, then* $\nu(D) = 1$.

Proof Set $\mathbb{J} = \{u \in H : -\Delta u + 2uK = 0\}$. Since ∂D is conjugate, $\mathbb{J} \neq \{0\}$. Fix a point $p \in D$. Define the linear map

$$L : \mathbb{J} \longrightarrow \mathbb{R}, \quad L(u) = u(p).$$

Consider $u \in \mathbb{J} \setminus \{0\}$. It is known that either $u(p) \neq 0$ or u change sign in D. In the latter case consider a connected component D' of the complement of the zero set of u. This is a domain properly contained in D, and u vanishes on $\partial D'$. Hence $\partial D'$ is a conjugate boundary, contradicting the fact that ∂D is a first conjugate boundary. Hence $u(p) \neq 0$ and so L is injective and surjective, hence an isomorphism. It follows that $\dim \mathbb{J} = 1 = \nu(D)$. □

Definition 2.15 Let $\mathbb{E}$ be a real vector space and let $I : \mathbb{E} \longrightarrow \mathbb{R}$ be a quadratic form. The *index* of I is the superior of the dimensions of subspaces of $\mathbb{E}$ on which I is negative definite.

Definition 2.16 If $f : M \longrightarrow \mathbb{R}^3$ is a minimal immersion and $D \subseteq M$ is a domain, we define the *index* of D, $i(D)$, as the index of the index form of D.

In the theory of geodesics, the celebrated theorem of Morse states that if M is a Riemannian manifold and $\gamma : [0, a] \subseteq \mathbb{R} \longrightarrow M$ is a geodesic, the index of γ, i.e., the index of the second derivative of the energy functional, is the number of instants $t \in [0, a)$ conjugate to 0, counted with multiplicity. This result has been generalized for minimal surfaces by Smale. He considered a domain D in a minimal surface M and *a flow of contractions*, i.e., a family of diffeomorphisms $\phi_t : M \longrightarrow M,\ t \geq 0$, such that

- $\phi_0 = \mathbb{1}_M$,
- $\phi_t(D) \subset \phi_s(D)$ if $t > s$,
- $\lim_{t \longrightarrow \infty} A(\phi_t(D)) = 0$.

Theorem 2.17 (Morse-Smale) *Let D and ϕ_t be as above and set $D_t = \phi_t(D)$. Then*

$$i(D) = \sum_{t>0} \nu(D_t).$$

For $\lambda \in \mathbb{R}$ we consider the space

$$\Sigma_\lambda = \{u \in H : \Delta u + \lambda u = 0\}.$$

If $\dim \Sigma_\lambda = n_\lambda > 0$ then λ is an eigenvalue of the operator $-\Delta$. It follows from the spectral theory of such an operator, that the eigenvalues of $-\Delta$ form a countable set of positive numbers and we can order them in such a way that

$$0 < \lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_n \leq \cdots .$$

Remark 2.18 When dealing with more than one domain we will write $\lambda_i(D)$ in order to avoid confusions.

We state now some well-known basic properties of the eigenvalues of $-\Delta$.

Theorem 2.19

(1) *If* $u \in \Sigma_\lambda$ *the* u *is analytic.*
(2) *If* $u \in H \setminus \{0\}$,

$$\lambda_1 \leq \frac{\int_M \|\nabla u\|^2 \mathrm{d}M}{\int_M u^2 \mathrm{d}M}.$$

and equality holds if and only if $u \in \Sigma_{\lambda_1}$.
(3) *If* $u \in \Sigma_{\lambda_1}$ *then* $u(x) \neq 0,\ \forall\, x \in D$.
(4) *If* $u \in \Sigma_{\lambda_i}$, $i > 1$, *then* u *changes sign in* D.
(5) *If* $D' \subseteq D$ *then* $\lambda_1(D') \geq \lambda_1(D)$, *and equality holds if and only if* $D' = D$.

Example 2.20 Consider the unit sphere $S^2 \subseteq \mathbb{R}^3$. Let $u : S^2 \longrightarrow \mathbb{R},\ u(x, y, z) = z$ (a spherical harmonic of the first kind) and consider the restriction $\tilde{u}$ of u to the half sphere $S^2_+ = \{(x, y.z) \in S^2 : z > 0\}$. It is easily seen that $\Delta\tilde{u} + 2\tilde{u} = 0$. The function $\tilde{u}$ vanishes on ∂S^2_+ and is positive in the interior of S^2_+, hence, by items (3) and (4) of Theorem 2.19, $\lambda(S^2_+) = 2$. Also, for any proper subdomain $D \subset S^2_+$, $\lambda_1(D) > 2$, by item (5) of the same theorem.

The following fact, of interest in itself, will be useful later.

Theorem 2.21 *The spherical caps of* S^2 *minimize the first eigenvalue of* $-\Delta$ *among all domains with the same area.*

Proof See [17] □

Let $f : M \longrightarrow \mathbb{R}^3$ be a minimal immersion. Since the Gauss curvature is analytic (see Lemma 2.7) either the Gauss curvature vanishes identically, in which case $f(M)$ is an open subspace of an affine plane, or the zeros of K are isolated, hence finite in number on every domain in M.[4] We will assume that K is not identically zero, and will set

$$M_0 = M \setminus \{x \in M : K(x) = 0\}.$$

If $x \in M_0$, $\mathrm{d}\underline{\mathbf{n}}(x)$ is an isomorphism and we can define a new metric in M_0 setting

$$\langle X, Y\rangle_0 = \langle \mathrm{d}\underline{\mathbf{n}}(X), \mathrm{d}\underline{\mathbf{n}}(Y)\rangle, \quad \forall\, X, Y \in TM_0.$$

[4] Since a domain is relatively compact according to our definition.

We will denote by $\mathrm{d}M_0$ the volume form with respect to this metric and by Δ_0 the Laplace operator of this metric. Then, as it is easily seen,

$$\Delta = -K\Delta_0, \tag{3}$$

$$\mathrm{d}M_0 = -K\mathrm{d}M. \tag{4}$$

Let $D \subseteq M$ be a domain and $u \in H(D)$. We will denote by $\tilde{u}$ the restriction of u to $M_0 \cap \overline{D}$.

Lemma 2.22 *Let $X = u\underline{\mathbf{n}}$. Then*

(1) *The index form is given by*

$$I(X, X) = -\int_{\overline{D}\cap M_0} (\tilde{u}\Delta_0\tilde{u} + 2\tilde{u}^2)\mathrm{d}M_0,$$

(2) *X is a Jacobi field if and only if*

$$\Delta_0\tilde{u} + 2\tilde{u} = 0.$$

Proof The first assertion follows from $\mathrm{d}M_0 = -K\mathrm{d}M$ and the fact that $M \setminus M_0$ has measure zero. The second one follows from $\Delta = -K\Delta_0$ and continuity.

Next we will prove a first result relating stability and eigenvalues of the Laplacian.

Theorem 2.23 (Schwarz) *Let $f : M \longrightarrow \mathbb{R}^3$ be a minimal immersion and let $D \subseteq M$ be a domain. Assume that $\underline{\mathbf{n}}(D)$ is a domain in S^2 with first eigenvalue of $-\Delta_0$ smaller than 2. Then D is not stable.*

Proof Since the zero curvature points in $\overline{D}$ are finite in number, $\underline{\mathbf{n}} : \overline{D} \longrightarrow \underline{\mathbf{n}}(\overline{D})$ is a branched covering map and $\underline{\mathbf{n}}(\partial D) \subseteq \partial\underline{\mathbf{n}}(\overline{D})$. Let v be a function in the first eigenspace of $\underline{\mathbf{n}}(D)$. Consider $u = v \circ \underline{\mathbf{n}}$. Then u is a function in $H(D)$ and

$$\Delta_0 u + \lambda_1 u = 0.$$

Consider the vector field along D, $X = u\underline{\mathbf{n}}$ and denote, as before, by $\tilde{u}$ the restriction to $D \cap M_0$. Then, by Lemma 2.22,

$$I(X, X) = -\int_{\overline{D}\cap M_0} [\tilde{u}\Delta_0\tilde{u} + 2\tilde{u}^2]\mathrm{d}M_0 = (\lambda_1 - 2)\int_{\overline{D}\cap M_0} \tilde{u}^2\mathrm{d}M_0 \quad < 0.$$

The operator Δ_0 is, essentially, the Laplacian on the sphere S^2 and we will use the same symbol for the two operators. It follows from Theorem 2.23, that a domain whose image by the Gauss map contains properly an hemisphere is not stable.

The next result relates stability with the image of the Gauss map of a domain.

Theorem 2.24 (Barbosa-do Carmo) *If $f : M \longrightarrow \mathbb{R}^3$ is a minimal immersion and $D \subseteq M$ is a domain such that $A(\underline{\mathbf{n}}(D)) < 2\pi$, then D is strongly stable.*

Proof (Sketch) The proof is by contradiction. Suppose D is not stable. Then, by Theorem 2.17, there is a domain $D' \subseteq D$ such that $\partial D'$ is a first conjugate boundary. The heart of the proof is the existence of a function $v \in H(\underline{\mathbf{n}}(D'))$ such that

$$\int_{\underline{\mathbf{n}}(D')} \|\nabla v\|^2 \leq 2 \int_{\underline{\mathbf{n}}(D')} v^2. \tag{5}$$

Once we have such a function we proceed as follows. From Theorem 2.19 (2), we have $\lambda_1(\underline{\mathbf{n}}(D')) \leq 2$. Consider a spherical cup C such that $A(C) = A(\underline{\mathbf{n}}(D')) < 2\pi$. Then $\lambda_1(C) > 2$ (see Example 2.20) and, by Theorem 2.21,

$$\lambda_1(\underline{\mathbf{n}}(D')) \geq \lambda_1(C) > 2,$$

a contradiction.

We will sketch the construction of the function v.

Since $\partial D'$ is a first conjugate boundary, we have a Jacobi field $u\underline{\mathbf{n}},\ u \in H(D'),\ u$ positive in D'. If $p \in \overline{D'}$, the Gauss map in a suitable neighborhood of p looks like $\underline{\mathbf{n}}(z) = z^n$, with respect to a complex local coordinate z (see also Sect. 5). We set $\nu(p) = n$. Observe that $\nu(p) = 1$ if $K(p) \neq 0$. Given $q \in \underline{\mathbf{n}}(\overline{D'})$ the set $\underline{\mathbf{n}}^{-1}(q) \cap \overline{D'}$ is finite and we define

$$v(q) = \sum_{p \in \underline{\mathbf{n}}^{-1}(q) \cap \overline{D'}} \nu(p) u(p).$$

Then v vanishes on $\overline{\partial \underline{\mathbf{n}}(D')}$ and is positive in the interior. The proof that v verify Eq. (5) is not simple and we refer to [1] for it.

Essentially the same proof shows the following "companion" if Theorem 2.23.

Theorem 2.25 *Let $D \subseteq M$ be a domain such that $\lambda_1(\underline{\mathbf{n}}(D)) > 2$. Then D is strongly stable.*

Example 2.26 A good example to keep in mind is the catenoid. Consider the domain of the catenoid between the panes $z = -\epsilon,\ z = N,\ 0 \leq \epsilon \leq N$. Then the image of this domain is a spherical ring shaped domain bounded by two parallels. If $\epsilon = 0$ then the area of the spherical image is less than 2π, so the domain is stable. If $\epsilon > 0$, then, by a limiting argument, if N is sufficiently large, the first eigenvalue is smaller than 2. Hence the domain is not stable. In particular the estimates in the above results are sharp.

Next we will consider stability from a global point of view.

Definition 2.27 Let $f : M \longrightarrow \mathbb{R}^3$ be a minimal immersion. We will say that M is *stable* if every domain in M is stable.

Before describing the next result, which characterizes complete stable minimal surfaces, we will recall a basic fact in Riemann surfaces theory, the *uniformization theorem*.

Theorem 2.28 (Uniformization Theorem) *Let M be a Riemann surface. Then there is a conformal covering map $\pi : \tilde{M}_c \longrightarrow M$, where $\tilde{M}_c$ is either the complex plane $\mathbb{C}$, the unit disk $\mathbb{D} = \{z \in \mathbb{C}; |z| < 1\}$ or the sphere S^2.*

Definition 2.29 The space $\tilde{M}_c$ in the theorem above is called the conformal universal covering of M.

Remark 2.30 The disk and the plane are obviously diffeomorphic but they are not conformally equivalent since there are no non-constant conformal maps from the plane to the disk, by Liouville's theorem.

Remark 2.31 If $f : M \longrightarrow \mathbb{R}^3$ is an isometric immersion, with M compact and without boundary, then there exists a point in M where the Gaussian curvature is positive (for example, a point $p \in M$ such that $f(p)$ has maximal distance from the origin). If f is a minimal immersion, its Gaussian curvature is non-positive and so M cannot be compact. Since $\pi : \tilde{M}_c \longrightarrow M$ is conformal and the coordinate functions of f are harmonic, so are the ones of $f \circ \pi$, hence $f \circ \pi$ is a minimal immersion and $\tilde{M}_c$ cannot be compact. In particular its conformal universal covering of a minimal surface is either the complex plane $\mathbb{C}$ or the unit disc $\mathbb{D}$.

The following result is due to do Carmo and Peng (see [5]) and, independently, to Fisher Colbrie and Shoen (see [8]).

Theorem 2.32 *Let $f : M \longrightarrow \mathbb{R}^3$ be a complete minimal immersion. Then, if M is stable, $f(M)$ is a plane.*

Proof (Sketch) We will start with the following:

Lemma 2.33 *Let $\pi : \tilde{M} \longrightarrow M$ be a conformal covering map. Then $f \circ \pi : \tilde{M} \longrightarrow \mathbb{R}^3$ is a complete stable minimal surface.*

Proof By Remark 2.31 $f \circ \pi$ is minimal. Also it is well known that $\tilde{M}$, with the covering metric, is complete. It remains to show that it is stable. Suppose that $\tilde{D}$ is a domain which is not stable. Then there exists a domain $\tilde{D'} \subseteq \tilde{D}$ such that $\partial \tilde{D'}$ is a first conjugate boundary. Hence we have a function $\tilde{u} \in H(\tilde{D'})$, positive in $\tilde{D}'$, such that $\Delta \tilde{u} - 2K\tilde{u} = 0$. Consider $q \in \pi(\tilde{D'}) := D'$. Since $\tilde{D'}$ is relatively compact, $\pi^{-1}(q) \cap \tilde{D'}$ is a finite set of points say $\{p_1, \ldots, p_k\}$. Set

$$u(q) = \sum_1^k \tilde{u}(p_i).$$

Then $u \in H(D')$ and is positive in D'. As in Theorem 2.24, it is possible to show that

$$\int_{\overline{D'}} \|\nabla u\|^2 \mathrm{d}M \;\leq\; 2\int_{\overline{D'}} -Ku^2 \mathrm{d}M$$

and the same argument as in Theorem 2.24 shows that D' is not stable, a contradiction.

By the lemma, we can assume that M is simply connected. Then, by the uniformization theorem, M is conformally equivalent to either the complex plane $\mathbb{C}$ or the unit disk $\mathbb{D}$. Assume the latter and let $\mathrm{d}s^2 = \lambda^2|\mathrm{d}z|^2$ be the induced metric.

We will proceed by contradiction supposing that the Gaussian curvature is not identically zero.

Set $\phi = \lambda^{-1}$. Then we have

$$K = -\phi^2 \Delta_0 \phi^- 1, \quad \nabla = \phi \nabla_0, \quad \Delta = \phi^2 \Delta_0, \quad \mathrm{d}M = \phi^2 \mathrm{d}A, \tag{6}$$

where $\nabla_0,\ \Delta_0, \mathrm{d}A$ are the gradient, the Laplacian, and the area form with respect to the flat metric.

Since M is stable, we have, for every piecewise smooth compactly supported function $u : M \longrightarrow \mathbb{R}$,

$$\int_M (u\Delta u - 2u^2 K)\mathrm{d}M \leq 0. \tag{7}$$

Using (6) we have that (7) can be written as

$$\int_D (u\Delta_0 u + u^2 \Delta_0 \log \lambda^2)\mathrm{d}A \leq 0. \tag{8}$$

Replacing u by ϕu in Eq. (6) we have

$$\int_D (\phi u \Delta_0(\phi u) + u^2\phi^2 \Delta_0 \log \phi^{-2})\mathrm{d}A \leq 0. \tag{9}$$

Since $u\phi$ has compact support, integration by parts give

$$\int_D \phi u \Delta_0 \phi u \mathrm{d}A = -\int_D (u^2|\nabla_0\phi|^2 + \phi^2|\nabla_0 u|^2 + 2\langle \nabla_0 u, \nabla_0 \phi\rangle_0)\mathrm{d}A, \tag{10}$$

$$\int_D u^2\phi^2\Delta_0 \log \phi^{-2}\mathrm{d}A = 4\int_D (\phi u\langle \nabla_0 u, \nabla_0\phi\rangle_0 + u^2|\nabla_0\phi|^2)\mathrm{d}A. \tag{11}$$

where $|\cdot|$ and $\langle \cdot, \cdot\rangle_0$ are the norm and scalar product of the flat metric. Adding (10) and (11), we get

$$3\int_D |\nabla_0\phi|^2\mathrm{d}A \leq \int_D \phi^2|\nabla_0 u|^2\mathrm{d}A - 2\int_D \phi u\langle \nabla_0 u, \nabla_0\phi\rangle_0\mathrm{d}A. \tag{12}$$

Using the inequality

$$|\phi u \langle \nabla_0 u, \nabla_0 \phi \rangle_0| \leq \epsilon |\nabla_0 \phi|^2 u^2 + \epsilon^{-1} |\nabla_0 u|^2 \phi^2, \quad \forall \ \epsilon > 0,$$

we obtain

Lemma 2.34 *There exists a positive constant b such that*

$$\int_M |\nabla \phi|^2 u^2 \mathrm{d}M \leq b \int_M \phi^2 |\nabla u|^2 \mathrm{d}M.$$

Let B_r be the geodesic ball of radius r, and $\theta \in (0, 1)$. Let $u : M \longrightarrow \mathbb{R}$ be a function which vanishes outside B_r, it is 1 in $B_{\theta r}$ and is linear in $B_r \setminus B_{\theta r}$. From Lemma 2.34 we have

$$\int_{B_r} |\nabla \phi|^2 \mathrm{d}M \leq \frac{b}{(1-\theta)^2 r^2} \int_M \phi^2 \mathrm{d}M = \frac{b}{(1-\theta)^2 r^2} \int_{B_r} \mathrm{d}A = \frac{\pi b}{(1-\theta)^2 r^2}.$$

Letting $r \longrightarrow \infty$, we get $|\nabla \phi| = 0$, hence $\phi = constant$ and the metric $\mathrm{d}s^2$ is not complete, a contradiction.

With similar techniques we can treat the case in which M is conformally equivalent to $\mathbb{C}$.

Remark 2.35 Let $f : \mathbb{R}^2 \longrightarrow \mathbb{R}$ be a solution of the minimal surfaces equations. Then the graph of f is a complete minimal surface. Moreover, given a domain in the graph of f, the image of this domain by $\underline{\mathbf{n}}$ is strictly contained in a hemisphere and has area smaller than 2π. Hence, by Theorem 2.25, the domain is stable. Hence, as a corollary of Theorem 2.32, we have that f is affine, i.e., the Bernstein's Theorem 1.9.

3 The Weierstrass Representation Formula

The Weierstrass representation formula is a basic tool in the study of minimal surfaces in $\mathbb{R}^3$ because, on one hand, it is a "machine" to produce examples of minimal surfaces and, on the other hand, it allows to use the powerful theory of holomorphic functions to treat theoretical problems. We will start with the local version.

Consider $\mathbb{C} \cong \mathbb{R}^2$ with the complex coordinate $z = u + iv, i = \sqrt{-1}$ and the differential operators

$$\frac{\partial}{\partial z} = \frac{1}{2}\left(\frac{\partial}{\partial u} - i\frac{\partial}{\partial v}\right), \quad \frac{\partial}{\partial \bar{z}} = \frac{1}{2}\left(\frac{\partial}{\partial u} + i\frac{\partial}{\partial v}\right)$$

In particular, if $U \subseteq \mathbb{C}$ is an open set, a function $g : U \longrightarrow \mathbb{C}$ is *holomorphic* if and only if

$$\frac{\partial g}{\partial \bar{z}} = 0.$$

The local version of the Weierstrass representation formula can be stated as follows.

Theorem 3.1 (Weierstrass Representation) *Let $\Omega \subseteq \mathbb{C}$ be an open set and let $f : \Omega \longrightarrow \mathbb{R}^3$ be a* conformal *minimal immersion. Consider the "complex tangent vector"*

$$\frac{\partial f}{\partial z} = \sum \phi_i e_i,$$

where e_i is the standard basis of $\mathbb{R}^3$ and the ϕ_i's are complex valued functions. Then

(1) $\sum |\phi_i|^2 \neq 0,$
(2) $\sum \phi_i^2 = 0,$
(3) $\dfrac{\partial \phi_i}{\partial \bar{z}} = 0.$

Conversely, given functions ϕ_i verifying the condition above, if Ω is simply connected, the function $f : \Omega \longrightarrow \mathbb{R}^3$, given by

$$f_i(z) = 2\mathfrak{Real} \int_\gamma \phi_i \mathrm{d}z, \quad i = 1, 2, 3,$$

is a well-defined conformal minimal immersion. Here $\mathfrak{Real}$ stands for the real part and the integral is taken along a curve γ in Ω joining a fixed point z_0 to z.

Remark 3.2 In Theorem 3.1, the first condition tells us that f is an immersion, the second one that f is conformal and the last one that f is minimal.

The second condition in Theorem 3.1 says that, essentially, one of the three function depends only on the other two. Set

$$\omega = \phi_1 - i\phi_2, \quad g = \frac{\phi_2}{\omega}.$$

Then ω is a holomorphic function and g a meromorphic one. Moreover, given ω and g, we can recover the ϕ_i's:

$$2\phi_1 = (1 - g^2)\omega, \ \ 2\phi_2 = (1 + g^2)\omega, \ \ \phi_3 = g\omega. \tag{13}$$

Remark 3.3 To make sense to the above procedure we have to ask that g has a pole of order m at z if and only if ω has a zero order $2m$ at z.

Definition 3.4 The pair (g, ω) is called the Weierstrass data of f.

The geometry of the immersion can be described in terms of Weierstrass data. For example, the metric is given by

$$ds^2 = |\omega|^2(1 + |g|^2)^2|dz|^2,$$

and the Gaussian curvature by

$$k = -\left(\frac{2|g'|^2}{|\omega|(1 + |g|^2)^2}\right)^2.$$

The function g has a very interesting geometric interpretation. Let $\sigma : S^2 \longrightarrow \mathbb{C} \cup \{\infty\}$ be the stereographic projection from the north pole. The following is easy to prove:

Lemma 3.5 $g = \sigma \circ \underline{\mathbf{n}}$.

Let M be a Riemann surface, with an atlas of isothermal coordinates, and let $f : M \longrightarrow \mathbb{R}^3$ be a conformal minimal immersion. The function g is a well-defined meromorphic function from M to $\mathbb{C}$, by Lemma 3.5. It turns out that the locally defined holomorphic 1-forms ωdz coincide in the intersection of the domains and so they define a global holomorphic 1-form that we still denote by ω. Now, if we suppose that the zeros of ω are related to the poles of g as in Remark 3.3 and that the forms below in (14) have no real periods,[5] we can recover f from the Weierstrass data by integration:

$$f(z) = \mathfrak{Real} \int_{z_0}^{z} ((1 - g^2)\omega,\ (1 + g^2)\omega,\ 2g\omega), \tag{14}$$

where the integral is along a smooth curve joining a fixed point z_0 to z.

Remark 3.6 In the construction of examples using the formula above, the hard point is, in general, the proof that the forms in question have no real periods.

Example 3.7 Consider $M = \mathbb{C},\ g(z) = -ie^z,\ \omega(z) = e^{-z}dz$. Then g has no poles and ω has no zeros. Since the domain is simply connected, the forms in (14) have no periods and we have

$$f(u, v) = (\cos(v)\sinh(u), \sin(v)\sinh(u), v), \qquad z = u + iv$$

i.e., a helicoid.

[5] A *period* of a 1-form is the value of the integral of the form along a closed curve.

Example 3.8 Let $M = D$ be the unit disk, $g(z) = z,\ \omega = 4\mathrm{d}z(1 - z^4)^{-1}$. Again g has no poles, ω has no zeros, and the domain is simply connected. After integration, and some calculations, we have

$$f(u, v) = \left(u, v, \log \frac{\cos v}{\sin u}\right),$$

i.e., the Scherk's surface.

Example 3.9 Consider $M = \mathbb{C},\ g(z) = -e^x,\ \omega = -e^{-z}\mathrm{d}z$. Again we are in a "good situation" and integration gives

$$f(u, v) = (\cos v \sinh u, \sin v \cosh u, u) - (0, 0, 1),$$

i.e., a catenoid (up to a translation). Such a parametrization wraps the plane around the (geometric) catenoid infinitely many times.

An alternative way to obtain a catenoid is the following: take $M = \mathbb{C} \setminus \{0\},\ g(z) = z,\ \omega(z) = z^{-2}\mathrm{d}z$. Since M is not simply connected we have to check that the forms have no real periods. Since $\pi_1(M) \cong \mathbb{Z}$, and the forms are closed, it is sufficient to consider the integrals of those forms on the unit circle $\gamma(t) = (\cos t, \sin t),\ t \in [0, 2\pi]$. A simple calculation shows that the first two forms have no periods and the third one has purely imaginary periods. After integration we get

$$f(u, v) = \left(-u\left(1 + \frac{1}{u^2 + v^2}\right) + 1, -v\left(1 + \frac{1}{u^2 + v^2}\right),\ \log(u^2 + v^2)\right),$$

which is a parametrization of the catenoid, up to a translation.

Remark 3.10 The Weierstrass representation formula holds, mutate mutandis, for minimal surfaces in $\mathbb{R}^n,\ n \geq 3$. The important point is that the domain is two-dimensional.

Remark 3.11 The Weierstrass representation formula asked, for a long time, for a generalization to the case of minimal surfaces in more general spaces. The first two conditions in Theorem 3.1 have an obvious extension, while the third one is replaced by an integral differential equation involving the Riemannian connection of the ambient manifold. This equation is, in general, very difficult to solve explicitly, but, depending on the ambient manifold, arguments ad hoc can be used to produce explicit solutions hence examples and general results. This is still an active field of investigation.

4 On the Image of the Gauss Map: The General Case

As we have seen in Sect. 3 the Gauss map of a minimal surface is a meromorphic map. A classical problem in minimal surface theory is to know which results from the classical complex function theory remain true for the Gauss map. For example, there is a Picard type theorem for the Gauss map of a complete minimal surface? Questions like this were asked since the middle of the last century and still puzzle researchers in the field. L. Nirember conjectured, around 1950, that the image of the Gauss map of a complete non-flat minimal surfaces in $\mathbb{R}^3$ is dense. A positive answer to this conjecture was given by Osserman (see [15] and [14]). We will sketch now Osserman's proof.

Theorem 4.1 (Osserman) *Let $f : M \longrightarrow \mathbb{R}^3$ be a complete, non-flat, minimal surface. Then the image of the Gauss map is dense.*

Proof (Sketch) The map $f \circ \pi : \tilde{M}_c \longrightarrow \mathbb{R}^3$ is a complete minimal immersion, if we consider in $\tilde{M}_c$ the covering metric. By Remark 2.31 $\tilde{M}_c$ is either the plane or the disk. The Gauss maps of $f \circ \pi$ and of f have the same image so we may consider the Gauss map as a meromorphic map defined on $\tilde{M}_c$. If $\tilde{M}_c = \mathbb{C}$ the theorem follows from the classical Liouville's (or Picard) theorem. So we suppose that $\tilde{M}_c$ is the unit disk. Let us suppose that the Gauss map misses a neighborhood of a point, that, without loss of generality, we can assume to be $e_3 = (0, 0, 1)$. Then, for the Weierstrass data we have:

- there exists a constant $A < \infty$ such that $|g(z)| < A \ \forall\, z \in \tilde{M}_c$,
- $\omega(z) \neq 0, \ \forall\, z \in \tilde{M}_c$, since g has no poles (see Remark 3.3).

  Consider the map $F : \tilde{M}_c \longrightarrow \mathbb{C}$ defined by

$$F(z) = \int_0^z \omega(\xi) \mathrm{d}\xi.$$

Since $\tilde{M}_c$ is simply connected and ω is holomorphic, F is well defined, $F(0) = 0$, and F is locally invertible since $F'(z) = \omega(z) \neq 0$. Let H be a local inverse defined in a neighborhood of 0. Observe that H cannot be defined in the all of $\mathbb{C}$ otherwise it would be a bounded entire holomorphic function, hence constant. So

$$R = \sup\{r \in \mathbb{R} : H \ \text{ is defined for all } \ z \ \text{ with } \ |z| < r\} < \infty.$$

In particular there is a $v \in \mathbb{C}$ with $|v| = R$ such that H cannot be defined in a neighborhood of v. Consider the curve $\sigma(t) = tv, \ t \in [0, 1)$ and let $\gamma(t) = H(\sigma(t)$. Then is not difficult to prove that

- γ is a divergent curve, i.e., for every compact set $K \subseteq \tilde{M}_c$ there exist t_0 with $\gamma(t_0) \notin K$,
- the length of γ is $R < \infty$.

But these two facts contradict the completeness of the metric and the theorem follows.

Remark 4.2 We observe that, in particular, Theorem 4.1 generalizes Bernstein theorem, since an entire graph is complete and its Gauss map covers at most an hemisphere. Hence, if it is minimal, has to be flat.

Remark 4.3 Really Osserman showed a slight more general result: the complement of the image of the Gauss map of a complete non-flat minimal surface has zero logarithmic capacity.

Subsequently Xavier in [20] improves considerably Osserman result showing that the Gauss map of a complete, non-flat, minimal surface omits at most 6 points. Finally Fujimoto proved in [7] that the Gauss map of such a surface omits at most 4 points. Fujimoto result is sharp since the Gauss map of the Sherk surface misses exactly 4 points.

5 On the Image of the Gauss Map: The Finite Total Curvature Case

In [14] Osserman studies the size of the complement of the image of the Gauss map for the class of complete minimal surfaces of finite total curvature, i.e., minimal surface for which

$$\int_M k \mathrm{d}v > -\infty.$$

For this class of minimal immersions he proved the following basic properties:

Theorem 5.1 *Let $f : M \longrightarrow \mathbb{R}^3$ be a complete non-flat minimal surface of finite total curvature. Then*

(1) *M is conformally equivalent to a compact surface $\overline{M}$ minus a finite set of points $E = \{w_1, \dots, w_k\}$.*
(2) *The Gauss map extends to a branched covering map $\underline{\mathbf{n}} : \overline{M} \longrightarrow S^2$.*

Using Theorem 5.1 and the Weierstrass representation formula, he proved the following:

Theorem 5.2 *Let $f : M \longrightarrow \mathbb{R}^3$ be a complete, non-flat, minimal surface of finite total curvature. Then the Gauss map omits at most three points. Moreover, if $\chi(\overline{M}) = 2$ the Gauss map omits at most two points.*

Although not clear from Osserman's proof, the general case and the finite total curvature case are very different in nature. While the general case is a problem in value distribution theory for holomorphic functions in the disk, the finite total curvature case is of topological nature. We will try to explain the last assertion. For this we will introduce a more general class of surfaces.

Definition 5.3 Let M be a complete Riemannian surface and let $f : M \longrightarrow \mathbb{R}^3$ be an isometric immersion. We will say that f is of *finite geometric type* if

(1) M is diffeomorphic to a compact surface $\overline{M}$ minus a finite set of points $E = \{w_1, \ldots, w_k\}$,
(2) the Gauss map of f extends to a branched covering map $\underline{\mathbf{n}} : \overline{M} \longrightarrow S^2$,

Remark 5.4 Let $f : M \longrightarrow \mathbb{R}^3$ be an immersion of finite geometric type. The fact that the Gauss map extends to a branched covering of $\overline{M}$ over S^2 means that this extension is a covering map outside a finite number of points, the *branch points* of the map. The set of such branch points is finite and includes the points of zero Gaussian curvature and, possibly, some of the ends. For such a branch point v, a small punctured neighborhood is mapped onto its image as a covering map of order $\nu(v)$. The number $\beta(v) = \nu(v) - 1$ is called the *branching number* at v. Observe that if v is not a branch point then $\beta(v) = 0$. In particular *M is non-flat and the Gaussian curvature of M vanishes only at a finite set of points.*

The following fact is well known in covering space theory:

Theorem 5.5

$$-2dg(\underline{\mathbf{n}}) = \chi(\overline{M}) + \sum_{w \in \overline{M}} \beta(w) \qquad (\textit{Riemann-Hurevitz relation}), \tag{15}$$

where $dg(\underline{\mathbf{n}})$ is the degree of the Gauss map and $\chi(\overline{M})$ is the Euler characteristic of $\overline{M}$.

When clear from the context we will also say that M is a *surface of finite geometric type*.

Remark 5.6 Clearly complete, non-flat, minimal surfaces of finite total curvature are immersions of finite geometric type. The latter is a quite wider class. For example, it is stable for local small deformations near points of non-zero Gaussian curvature while minimal surfaces are not.

Remark 5.7 The concept of surface of finite geometric type can be extended to the case of hypersurfaces of $\mathbb{R}^{n+1}$. In this case condition (3) is replaced by the condition that the set of zeros of the Gauss-Kronecker curvature, i.e., the zeros of the determinant of the second fundamental form, does not disconnect M. The basic idea in introducing such hypersurfaces is that while the complex analysis methods for minimal surfaces, i.e., the Weierstrass representation formula, do not extend to the higher dimensional case, some of the topological methods do extend.

Definition 5.8 The points of E or, sometimes, punctured neighborhoods of such points, are called the *ends* of M.

The behavior of an immersion of finite geometric type near the ends is described in [10]. We will recall some basic facts. Let $w \in E$ be an end. The tangent space of $\overline{M}$ at w, as a linear subspace of $\mathbb{R}^3$, is well defined, namely $T_w\overline{M} = [\underline{\mathbf{n}}(w)]^{\perp}$. It can be shown that, for a sufficiently small neighborhood of w, the immersion, composed with the projection over $T_w\overline{M}$, is a covering map over the complement of a disk, of finite degree $I(w)$.

Definition 5.9 The number $I(w)$ is called the *geometric index* of w.

Remark 5.10 Geometrically, $I(w)$ counts the number of times that f warps a punctured neighborhood of w around the direction $\underline{\mathbf{n}}(w)$. If $I(w) = 1$ there exists a suitable punctured neighborhood W of w such that the projection of $f(W)$ over $[\underline{\mathbf{n}}(w)]^{\perp}$ is $1 - 1$. Hence $f(W)$ is a graph over the complement of a big ball $B \subseteq [\underline{\mathbf{n}}(w)]^{\perp}$. In particular $f_{|W}$ is a homeomorphism onto its image, i.e., an embedding. Conversely, if $f_{|W}$ is an embedding, $I(w) = 1$.

We will compute the Euler characteristic of $\overline{M}$ by counting the indexes of a suitable vector field on $\overline{M}$. Let ξ be a fixed unit vector in $\mathbb{R}^3$ such that ξ is a regular value of the Gauss map and $\xi \neq \pm\underline{\mathbf{n}}(w_i),\ w_i \in E$. Consider the tangent vector field $\eta(x) = P_x(\xi)$ where P_x is the projection onto T_xM. η is the gradient of the height function $h_\xi(x) = \langle f(x), \xi\rangle$, which is a Morse function since ξ is a regular value of $\underline{\mathbf{n}}$. We observe that η extends to a tangent vector field on $\overline{M}$, setting, for $w \in E$, $\eta(w) = \xi - \langle \underline{\mathbf{n}}(w), \xi\rangle \underline{\mathbf{n}}(w)$. Then the singularities of η are the points in $\underline{\mathbf{n}}^{-1}(\pm\xi)$ and, possibly, the ends.

In [3] is shown that the Gauss curvature of a surface of finite geometric type is non-positive. In particular the index of η at a point in $\underline{\mathbf{n}}^{-1}(\pm\xi)$ is -1. For the ends we have the following:

Lemma 5.11 *The index of η at an end w is* $1 + I(w)$.

Proof We will sketch the proof for the case in which the end is embedded (see Remark 5.10) and refer to [3] for the general case.

The orthogonal projection of η over the complement of $B \subseteq [\underline{\mathbf{n}}(w)]^{\perp}$ is almost constant. Hence the index on the sphere bounding B is zero. Hence we can extend the projection to a non-vanishing vector field on B. We can take the stereographic projection of $[\underline{\mathbf{n}}(w)]^{\perp}$ over the sphere S^2. The image of the projection of η, $\tilde{\eta}$ gives a vector field on the sphere with just one singularity at the south pole, hence the index of this singularity is 2. Since the composition of f with the projection into $[\underline{\mathbf{n}}(w)]^{\perp}$ and the stereographic projection is an orientation preserving diffeomorphism of a small punctured neighborhood of w onto a small neighborhood of the south pole, the conclusion follows.

Adding up the indexes of η we get

Theorem 5.12

$$\chi(\overline{M}) = 2dg(\underline{\mathbf{n}}) + \sum_{w\in E}[I(w) + 1] \qquad (\textit{Total curvature formula}). \tag{16}$$

Remark 5.13 The total curvature formula was first proved by Osserman in [14], as an inequality, using the Weierstrass representation formula. So Osserman proof works only for minimal surfaces. The equality was proved in [10] and in [3] using only the topological properties of a surface of finite geometric type.

At this point a suitable combination of the Riemann-Hurevitz relation and the total curvature formula gives

Theorem 5.14 *The Gauss map of a surface of finite geometric type misses at most 3 points.*

Proof Let $\{\xi_1, \ldots, \xi_l\}$ be the set of points omitted by the Gauss map. Set

$$A_i = \{w \in E : \underline{\mathbf{n}}(w) = \xi_i\}, \quad B = \{w \in E : \underline{\mathbf{n}}(w) \neq \xi_i \ \forall\, i\},$$
$$C = \{q \in M : \nu(q) > 1\}.$$

Let $n = -dg(\underline{\mathbf{n}})$. Then Eq. (15) becomes

$$\chi(\overline{M}) = 2n + \sum_{i=1}^{l} \sum_{p \in A_i} (1 - \nu(p)) + \sum_{p \in B} (1 - \nu(p)) + \sum_{p \in C} (1 - \nu(p)). \tag{17}$$

Observe that

$$\sum_{p \in A_i} \nu(p) = n, \quad \sum_{i=1}^{l} |A_i| + |B| = |E|.$$

Then we have

$$\chi(\overline{M}) = (2 - l)n + |E| - \sum_{p \in B} \nu(p) + \sum_{p \in C} (1 - \nu(p)). \tag{18}$$

Comparing Eq. (18) with Eq. (16), we obtain

$$0 < \sum_{w \in (\cup A_i) \cup B} I(w) = (4 - l)n - \left[\sum_{p \in B} \nu(p) - \sum_{p \in C} (1 - \nu(p)) \right]. \tag{19}$$

Therefore $l < 4$.

There are no examples of surfaces of finite geometric type, in particular complete minimal surface with finite total curvature, whose Gauss map misses three points. There have been various tentatives to prove the following conjecture that we will call the *Osserman conjecture*:

Conjecture 5.15 The Gauss map of a complete minimal surface with finite total curvature omits at most two points.

We will discuss now some results in the direction of giving a positive answer to Osserman conjecture.

Proposition 5.16 *Let $f : M \longrightarrow \mathbb{R}^3$ be a surfaces of finite geometric type. If the Gauss map omits 3 points then $\chi(\overline{M}) \leq 0$. Moreover, if $\chi(\overline{M}) = 0$ we have:*

(1) *$l = |E|$, i.e., all ends are omitted,*
(2) *$B = \emptyset = C$,*
(3) *$\sum I(p_i) = |E|$, i.e., all ends are embedded.*

Proof Just combine (19) with the total curvature formula.

A unit vector $\xi \in S^2$ is a regular value of the Gauss map if its inverse image $\underline{\mathbf{n}}^{-1}(\xi)$ does not contain flat points. In particular $\nu(p) = 1 \ \forall \ p \in \underline{\mathbf{n}}^{-1}(\xi)$. In order to extend this concept to an end $w \in E$, we have to take into account first that the curvature goes to zero approaching w and second that the end may not be embedded. The latter fact is measured by the geometric index $I(w)$. These considerations lead to the following definition:

Definition 5.17 We will say that an end $w \in E$ is *non-degenerate* if $\nu(w) \leq 1 + I(w)$.

Examples 5.18 Let $f : M \longrightarrow \mathbb{R}^3$ be a minimal surface, w an end with $\underline{\mathbf{n}}(w) = e_3$. Suppose that the end is embedded. Then the end may be parameterized as the graph of a function F, defined on the complement of a disk in the $\{e_1, e_2\}$ plane. If $z = x_1 + ix_2$ is the complex coordinate in this plane, F is of the form

$$F(z) = a \log |z| + b + \langle z_0, z\rangle |z|^{-2} + O(|z|^{-2}),$$

where z_0 is a given vector in the plane (see [19]). If $a \neq 0$ the end is of *catenoid type*. If $a = 0, \ z_0 \neq 0$ we have a *simple flat end*. In both cases the end is non-degenerate.

There are also many examples of non-degenerate ends which are not embedded. For example, for the (unique) end w of the Enneper surface, we have $I(w) = 3, \ \nu(w) = 1$, hence the end is non-degenerate, but not embedded.

Theorem 5.19 *If $f : M \longrightarrow \mathbb{R}^3$ is a surface of finite geometric type and all end are non-degenerate, then the Gauss map omits at most two points.*

Proof Suppose that the Gauss map omits the (distinct) values $\xi_i, \ i = 1, 2, 3$. Assume first that $\xi_1 = -\xi_2$. Computing $\chi(\overline{M})$ $(\leq 0$ by Corollary 5.16) and counting the singularities of η, we obtain

$$0 \geq \chi(\overline{M}) = \sum_{w \in A_1 \cup A_2} [I(w) + 1 - \nu(w)] + \sum_{w \in A_3 \cup B} [I(w) + 1],$$

which together with the condition $\nu(w) \leq 1 + I(w)$ imply $A_3 \cup B = \emptyset$, a contradiction. Assume now that no two of the ξ_i's are parallel. Then we have

$$0 \geq \chi(\overline{M}) = \sum_{w \in A_1} [I(w) + 1 - \nu(w)] - n + \sum_{w \in A_2 \cup A_2 \cup B} [I(w) + 1],$$

which again lead to the contradiction $A_3 \cup B = \emptyset$.

For minimal surfaces Y. Fang proved, in [9], the following:

Theorem 5.20 *If $f : M \longrightarrow \mathbb{R}^3$ is a complete minimal surfaces with finite total curvature and*

$$\int_M k \geq -20\pi,$$

then the Gauss map misses at most two points.

Remark 5.21 Those results should take care of the hard cases, since, intuitively, the more complicated the topology/geometry is, the "more surjective" the Gauss map should be. But it turns out that this is not the case!

6 Work in Progress and Some Problems

The main idea behind the proof of Theorem 5.14 is to consider the gradient of the function h_ξ which is the projection of the surface onto the ξ-axes. In the last few years we have tried a "dual approach," i.e., considering projection of the surface onto a plane. The general philosophy is that the singularities of such maps are strongly related to the topology of M. There are classical results due to Levine, Whitney, and others that relate the topology of a *compact* surface to the singularities of maps of these surfaces into a plane (see [11, 21] between others). We were able to extend some of these results to the case of surfaces of finite geometric type, but still we will need something finer to give a positive answer to Osserman conjecture.

We are also studying a different approach: find a locally invertible conformal map $\pi : \mathbb{C} \longrightarrow M$. If such a map exists, the composition with the Gauss map will provide a holomorphic function $\phi : \mathbb{C} \longrightarrow S^2$ that, by Picard theorem, misses at most two points. The existence of such a function may be established looking at solutions of a Beltrami type equation

$$\frac{\partial}{\partial \bar{z}} W = \mu \frac{\partial}{\partial z} W,$$

where μ is an expression in the coefficients of the metric. Curvature estimates at the ends should imply that $\sup(|\mu(z)|) < 1$, a fact that guarantees the existence of solutions.

A natural question on this line is the characterization of minimal surfaces of finite total curvature whose Gauss maps miss exactly two points. In [13] Miyaoka and Sato constructed examples of minimal spheres (or tori) punctured tree times whose Gauss maps miss two points that are not antipodal. What can we say if the two missed points are antipodal?

We can also ask the same question for surfaces as above whose Gauss maps miss just one point.

The characterizations above may be intended in terms of the type and number of the ends, the value of the total curvature, and the genus g of the surface. For example, we can ask if there are minimal surfaces of finite total curvature, with one end of Enneper type, i.e., $I(w) = 3$, two ends of catenoid type, i.e., $I(w) = 1$ and total curvature $-4\pi(g+1)$.

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