Michele Emmer
Marco Abate
Editors

Imagine Math 6

Between Culture and Mathematics

Springer

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To
Maria Pia Cavaliere
Giorgio Israel
Reza Sarhangi

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About the Editors

Michele Emmer is a member of the Istituto Veneto di Lettere, Scienze e Arti in Venice, founded by Napoleon. Former Professor of Mathematics at La Sapienza University in Rome (until 2015), since 1997 he has organized the Mathematics and Culture conferences in Venice. He has organized several exhibitions, cooperating with The Biennale of Art of Venice and the Prada Foundation among others. He is a member of the board of the Journal "Leonardo: art, science and technology", MIT Press, a filmmaker, including a film on M.C. Escher, and author of the series "Art and Math". He is also editor of the series "Mathematics and Culture" and "Imagine Math" by Springer-Verlag, as well as the series "The Visual Mind" by MIT Press.

His most recent books include "Bolle di sapone tra arte e matematica", 2010, which won the best Italian essay award at Viareggio 2010; "Numeri immaginari: cinema e matematica", Bollati Boringhieri, 2012; "Il mio Harry's bar", Archinto ed., 2012; "Imagine Math 3", Springer 2013; "Flatlandia di E. Abbott, with DVD, music by Ennio Morricone, 2008, "Imagine Maths 4" 2015, "Imagine Maths 5", 2016. "Racconto matematico: memorie impersonali", to appear 2019, Bollati Boringhieri.

Marco Abate is a Full Professor of Geometry at the University of Pisa. He has written more than one hundred scientific papers and textbooks, as well as several papers on the popularization of mathematics. His interests include holomorphic dynamics, geometric function theory, differential geometry, writing (comic books and more), photography, origami, and travelling (having already visited Antarctica his next destination is the Moon). His web site is http://pagine.dm.unipi.it/abate/.

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Introduction

Michele Emmer

Image all the people
Sharing all the world…
John Lennon

Imagine mathematics, imagine with the help of mathematics, imagine new worlds, new geometries, new forms. Imagine building mathematical models that make it possible to manage our world better, imagine solving great problems, imagine new problems never before thought of, imagine combining music, art, poetry, literature, architecture, theatre and cinema with mathematics. Imagine the unpredictable and sometimes counterintuitive applications of mathematics in all areas of human endeavour.

Imagination and mathematics, imagination and culture, culture and mathematics. For some years now the world of mathematics has penetrated deeply into human culture, perhaps more deeply than ever before, even more than in the Renaissance. In theatre, stories of mathematicians are staged; in cinema Oscars are won by films about mathematicians; all over the world museums and science centres dedicated to mathematics are multiplying. Journals have been founded for relationships between mathematics and contemporary art and architecture. Exhibitions are mounted to present mathematics, art and mathematics, and images related to the world of mathematics.

The volumes in the series "Imagine Math" are intended to contribute to grasping how much that is interesting and new is happening in the relationships between mathematics, imagination and culture.

This sixth volume of the series begins with a homage to the architect Zaha Hadid. She died on March 31st, 2016, a few weeks before the opening on May 25th, 2016,

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of a large exhibition of her works in Palazzo Franchetti in Venice, the location of the Istituto Veneto di Scienze, Lettere ed Arti where all the conferences on Mathematics and Culture have taken place in the last years.

A large section is dedicated to literature, narrative and mathematics including a contribution by Simon Singh and a paper on "Eloge des mathématiques" by Alain Badiou. The role of media in mathematics, including museums of science, journals and movies is closely examined. Moreno Andreatta in the section on mathematics and music discusses musically driven mathematical practice. A further extensive section is dedicated to mathematics and applications and includes in particular contributions on blood circulation by Alfio Quarteroni and on preventing crimes using earthquakes by Marco Abate. Another section is dedicated to mathematics and art, including the role of math in design. Finally, a large section is dedicated to the photos of mathematicians and mathematical objects by Vincent Moncorge.

The topics are treated in a way that is rigorous but captivating, detailed but full of evocations. An all-embracing look at the world of mathematics and culture.

Part I
Homage to Zaha Hadid

ZHA, Zaha Hadid Architects

Gianluca Racana

Mathematics and its tools have always played a central role in the evolution of the human understanding of nature and the constructed world. Sir Isaac Newton's methods to derive the laws of gravitation, Henri Poincaré's extension of the Cartesian geometries to the planetary system and Lord Kelvin's use of the mathematical technique of curve-fitting to predict the tides, are just a few examples.

Mathematics provides the foundations of computing and of scientific methods of research within architectural practices. It has had a profound influence on architectural morphology and its origins, basing them on sound structural principles. The enhancement of the performative aspects of design with respect to the built environment, its manufacture and ultimately the comfortable navigation by people within these environments, forms an integral part of building on these foundations.

With historical training in geometric methods to understand morphology, architects are well positioned to contribute to this collaborative endeavour of delivering information-rich settings that support the complex needs of humans within the built environment.

A large proportion of our own work at ZHA emerges from our fascination with mathematical logic and geometry, with advances in design technology enabling us to rethink form and space. The fluid surfaces and structures of each project thus generated are defined by scientific innovations.

Our design for the *Mathematics Gallery* (2016) at the *Science Museum* in London represents a pertinent case. The successful flight of the *Handley Page* aeroplane (see Fig. 1) in the 1929 *Guggenheim competition*, with its short take-off and landing distances, represents a triumphal moment in the accessibility of aviation to ordinary men and women [1].

The spatial arrangement of the Gallery places a central emphasis on this important product of British aviation, and the transformational capacity of mathematics

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Fig. 1 Handley Page H.P.39 (1929)

and science, by taking inspiration from one of the key moments in the flight of the plane and the concepts of aerodynamics embodied within.

While mathematical logic and geometry can provide an intuitive model to understand, the natural world, computational tools allow us to examine scenarios that enable a nuanced understanding of the mechanisms of nature. Using the principles of a mathematical approach known as *computational fluid dynamics* that acts as an organizational guide, the layout of the *Gallery* allows for the virtual lines of airflow to be manifested physically. The positioning of the more than 100 historical objects and the production of robust arch-like benches using robotic manufacture, all embody the mathematical spirit of the brief. The resulting spatial experience created by these components within the *Gallery* enables visitors to see some of the many perceivable ways in which mathematics touches our lives.

However, these principles do not apply to the aforementioned project only. In facts, ZHA maintains a fully digital practice: all of our buildings are designed in 3D from the beginning, rather than being translated into 3D models at a later stage in the design process. Building on our expertise in 3D design, ZHA is using Building Information Modelling on a growing number of our projects such as the *Investcorp Building* (2015) at *Oxford University's Middle East Centre* and the *Serpentine Sackler Gallery* (2013) in *Kensington Gardens*, London.

Given the small complex site of the *Investcorp Building*, it was critical that the building was efficient and that all building systems where coordinated with architectural elements. ZHA modelled the entire building in 3D to analyse and highlight problems within the design stage. The design team frequently exchanged 3D information to overlay the outputs of each discipline resulting in a highly coordinated building. Many of the design packages included 3D information to allow subcontractors to understand working zones, tolerances and to work with the design team as the design developed.

The irregular geometry of the *Serpentine Sackler Gallery* required close collaboration between us and the engineering teams to resolve any potential issue. The sinuous, fluid form of the *tensile* roof was achieved through specific 3D software. The roof curvature was then optimised to control and minimise the stresses in the fabric and avoid the risk of ponding. The structural and steel elements, such as the columns and the perimeter truss, were modelled in 3D in close collaboration with the structural engineer.

The late Zaha Hadid first became interested in geometry while studying mathematics at university. Mathematics and geometry have a strong connection with architecture and she continued to examine these relationships throughout each of her projects; with mathematics always central to her work. As Zaha said:

> When I was growing up in Iraq, math was an everyday part of life. My parents instilled in me a passion for discovery, and they never made a distinction between science and creativity. We would play with math problems just as we would play with pens and paper to draw—math was like sketching [2].

References

1. The Handley Page H.P.39 was a wooden biplane to compete in the competition proposed by the Daniel Guggenheim Fund for the promotion of Aeronautics
2. Z. Hadid, written for CNN, 17 November 2015

Mathematics: The Winton Gallery, Science Museum, London (2016)


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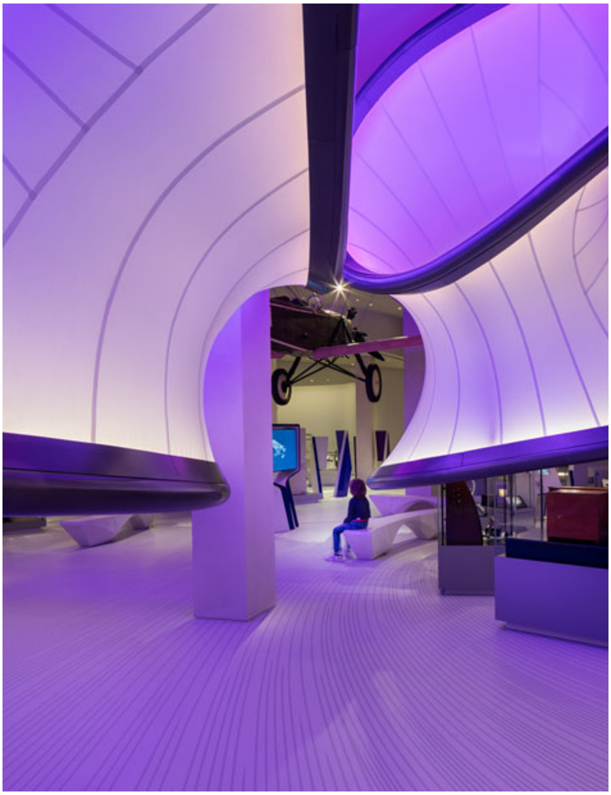

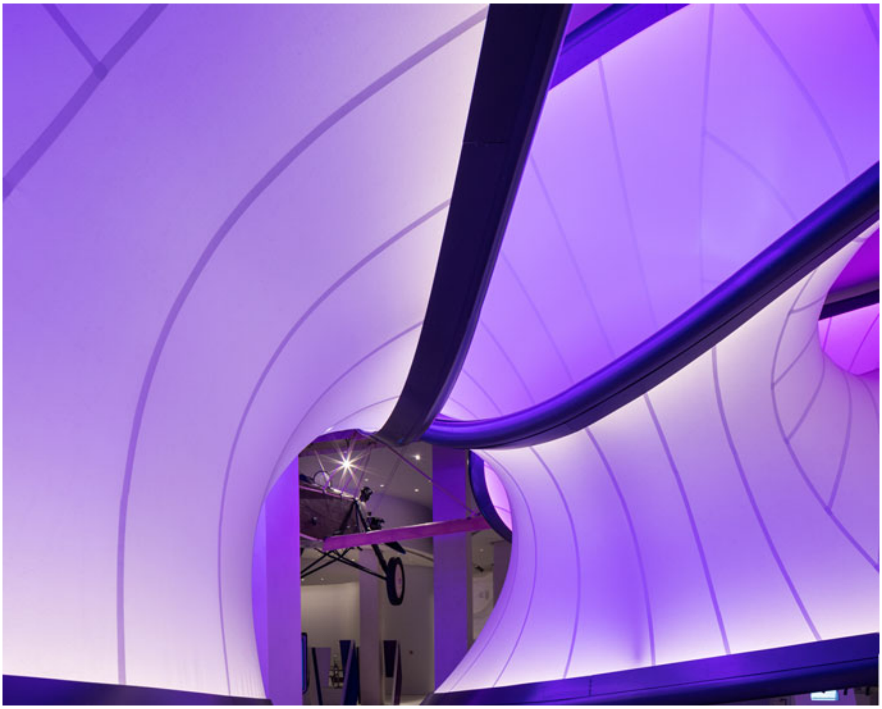

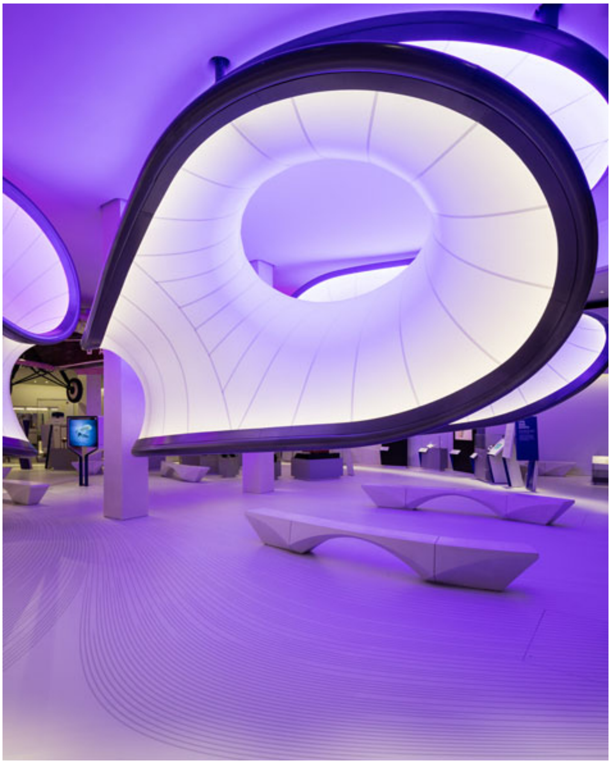


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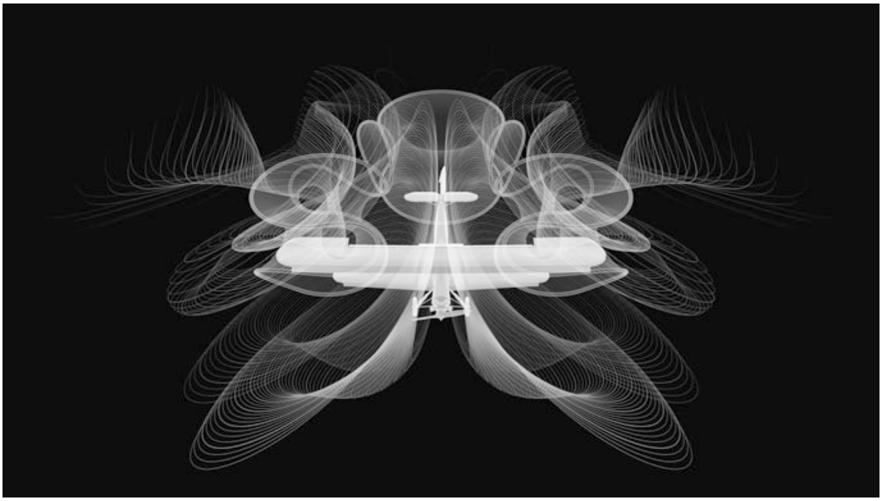

Serpentine Sackler Gallery, Royal Park of Kensington Gardens, London (2013)

All pictures © Photography by Luke Hayes, Courtesy of Zaha Hadid Architects.

Investcorp Building for Oxford University's Middle East Centre at St Antony's College (2015)

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Zaha Hadid: Fluid and Topological Architecture

Michele Emmer

New Geometries, New Spaces: Topology and Architecture

Fluidity is a key concept in Zaha Hadid's architecture. The notion has its roots in mathematics and in particular in topology. Since the end of the nineteenth century topology and its approaches to space has influenced art and more recently architecture. Hadid's architecture epitomizes this process.

Let's start with the opinion of two mathematician on the question 'What is Topology?': Richard Courant and Herbert Robbins write in the famous book *What is Mathematics?* (1941) [1]:

> The new subject, called *analysis situs* or *topology*, has as its object the study of the properties of geometrical figures that persist even when the figures are subjected to deformations so drastic that all their metric and projective properties are lost.... At first, the novelty of the methods in the new fields left no time to present their results in the traditional postulational form of elementary geometry. Instead, the pioneers, such as Poincaré, were forced to rely largely upon geometrical intuition. Even today a student of topology will find that by too much insistence on a rigorous form of presentation he may easily lose sight of the essential geometrical content in a mass of formal detail.
> The key word is geometrical intuition. Obviously over the years mathematicians have tried to bring topology into the realm of more rigorous mathematics, but there is still a strong sense of intuition involved. These two aspects, the distortions which maintain some of the geometrical properties of the figure, and intuition play an important role in the idea of space and shape from the nineteenth century to today.

It is worth noting that examples of topological surfaces, in particular that of the *Möbius strip*, have been found in places such as the harness for the horses of the *Guard of the Tsar* of Russia (seventeenth century), objects that are on display at

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Fig. 1 *Calima Culture*, gold, Colombia, Photo by M. Villarreal

the museum of the Kremlin, and in some pre-Columbian civilizations of southern Colombia, in particular the *Calima culture*, probably made for religious reasons (see Fig. 1).

This suggests that this surface is a kind of archetypal form that is rediscovered over the centuries. It is one of the reasons why the psychoanalyst Jacques Lacan took the *Möbius strip* as a symbol of his magazine *Scilicet* in the 1960s.

Even more interesting is the discovery of the strip by one of the greatest artists and architects of the twentieth century, Max Bill (for a detailed description on how he discovered Mobius' surfaces see [2]).

In 2008 a major exhibition dedicated to the works of Max Bill was held at the *Palazzo Reale* of Milan. In one of the rooms, entitled *Topology*, there were a number of sculptures, but not the series *Endless Ribbon*, a real marble *Möbius band* (see Fig. 2). In the catalog, just for the section of the topological sculptures, Karl Gerstner [3] noted, after pointing out that the new spark "began 150 years ago by those who call in question a lot of things as far away as geometry" (and explicitly mentions Bolyai, Lobachevskij, Riemann):

> Whoever keeps the whole artistic production of Max Bill, not only sculpture, will find concrete models, sensitive equivalent of thought patterns of modern abstract science, but in order not to create a misunderstanding: his works are not models for physics or any other teaching. They are autonomous works of art, but as with all great art, they are also reflection of what their time embodies at the core. The spiritual underpinning of these works is in essence art and science.

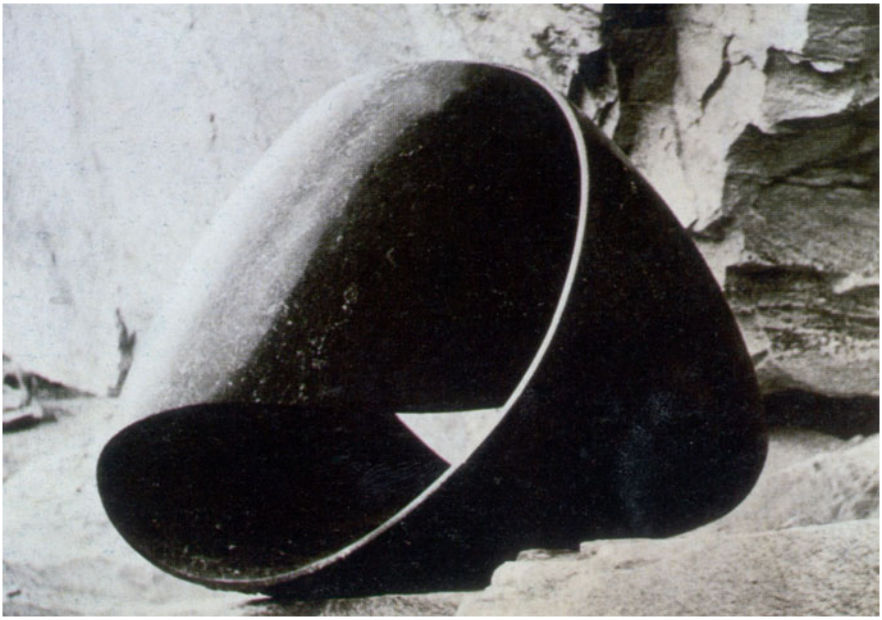

Fig. 2 From the film *The Möbius Band* by M. Emmer, with Max Bill (1984)

Stephen Perrella, one of the most interesting virtual architects describes *Architectural Topology* as follows in 2001 (Perrella died in 2008) [4]:

> *Architectural topology* is the mutation of form, structure, context and programme into interwoven patterns and complex dynamics. Over the past several years, a design sensibility has unfolded whereby architectural surfaces and the topologising of form are being systematically explored and unfolded into various architectural programmes.
> Influenced by the inherent temporalities of animation software, augmented reality, computer-aided manufactured and informatics in general, *topological space* differs from Cartesian space in that it imbricates temporal events-within form. Space then, is no longer a vacuum within which subjects and objects are contained, space is instead transformed into an interconnected, dense web of particularities and singularities better understood as substance or filled space.
> This nexus also entails more specifically the pervasive deployment of teletechnology within praxis, leading to an usurping of the real (material) and an unintentional dependency on simulation.

Other examples a few years after the paper of Perrella. The layout of the 2004 pavilion of the *Venice Biennale di Architettura* was assigned to two famous architects: Hani Rashid and Lise Anne Couture. In an article for the catalog entitled *Asymptote, the Architecture of Metamorph*, they summarized their project as follows:

> *Asymptote*'s transformation of the *Corderie* in the *Arsenale* emerged from computer generated morphing animation sequences derived from utilizing rules of perspective geometry with the actions and dynamics of torquing and stringing the space of the *Corderie*.

> The experience of *Metamorph* is spatial in that it is itself an architectural terrain of movement and flow. The exhibition architecture—from installation and exhibition design to graphic identity and catalog design—provides for a seamless experience that fuses the Arsenale, Giardini and Venice, making explicit a contemporary reading of architecture where affinities and disparities co-mingle to produce the effects of flux and metamorphoses of form and thinking [5].

One of the studies of the layout was described quite significantly as follows:

> Study of the *topological surface* that develops in the space of the Corderie and determines the movements and the curvatures used in designing levels.

Also interesting is what Hana Rashid writes in the catalogue of the *Biennale* [6]:

> With the help of computers in all its forms developments of a new architecture, an architecture influenced and modulated by the infinite and provocative possibilities offered by these technological tools, beyond the simple promise of greater efficiency and production capacity, are emerging. These new processes and methodologies associated with history, theory, conceptual thinking, experimentation and production are radically changing not only the way we see and think about space, but also the means by which we can occupy and inhabit the territory. In one form or another, it is now within the reach of artists and architects to discover and evoke digitally induced *spatial deliria* in which the merging simulation and effect with physical reality creates the possibility of a *sublime digital metamorphosis* from thought to its realization.

At the same *Biennale* in 2004:

> Many of the great creative acts in art and science can be seen as fundamentally metamorphic, in the sense that they involve the conceptual re-shaping of ordering principles from one realm of human activity to another visual analogy. Seeing something as essentially similar to something else has served as a key tool in the fluid transformation of mental frameworks in every field of human endeavour. I used the expression *structural intuitions* to try to capture what I felt about the way in which such conceptual metamorphoses operate in the visual arts and the sciences. Is there anything that creators of artefacts and scientists share in their impulses, in their curiosity, in their desire to make communicative and functional images of what they see and strive to understand?
> The expression *structural intuitions* attempts to capture what I tried to say in one phrase, namely that sculptors, architects, engineers, designers and scientists often share a profound sense of involvement with the beguiling structures apparent in the configurations and processes of nature—both complex and simple. I think we gain a deep satisfaction from the perception of order within apparent chaos, a satisfaction that depends on the way that our brains have evolved mechanisms for the intuitive extraction of the underlying patterns, static and dynamic.

These are the words of Martin Kemp, an art historian specialized in the relationship between art and science in the article *Intuizioni strutturali e pensiero metamorfico nell'arte, architettura e scienze*, in *Focus*, one of the volumes that make up the catalog of the 2004 *Biennale Internazionale di Architettura di Venezia* [7].

At the 2008 *Mostra Internazionale di Architettura* of the *Biennale of Venice*, in two separate locations, a Zaha Hadid and Patrick Schumacher project was presented. One called *Lotus* was shown in a hall of the *Arsenale* and the other by the name of *Aura*, at the *Villa Malcontenta*, one of the most famous buildings by the Renaissance architect Palladio on the Brenta River, some 50 km away from Venice.

Fig. 3 Zaha Hadid & Patrick Schumacher, *Aura*, 2008. © Courtesy of the ASAC, La Biennale, Venezia & Zaha Hadid Architects

The *Aura* installation (see Fig. 3) for the 2008 *Biennale* represents a dialogue between the *fluid* contemporary language of the *Zaha Hadid studio* and the mathematical principles of harmonious architectural composition of Andrea Palladio, on the 500th anniversary of his birth. The work focuses on the *piano nobile* of Palladio's *Villa Foscari La Malcontenta*, which encapsulates his theory of perfect form.

Accordingly, the proportions of the sequence of spaces provided the starting point for Zaha Hadid and Patrik Schumacher's study changing, transforming the rules, "instead of representing a system already domesticated through internal rules, the *Lotus* room seduces through the folds of undulating rhythm, its exclusions, its reconfigurability and its ability to remain outside of categories" [8].

In November 2009, a new space for contemporary art and architecture in Rome, MAXXI, was inaugurated (see Fig. 4).

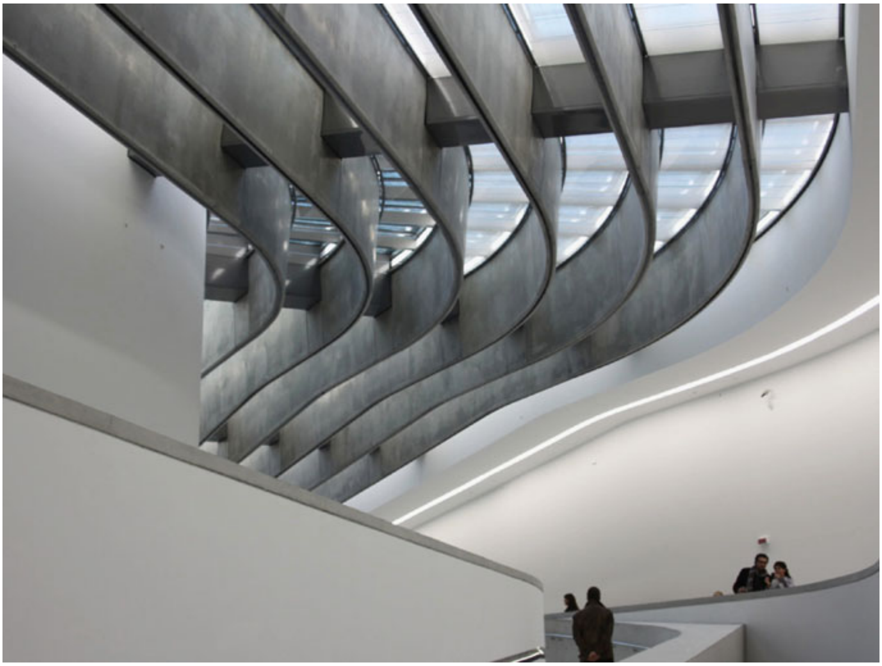

Fig. 4 Zaha Hadid, *MAXXI Museo nazionale delle arti del XXI secolo*, Roma, 2009. Photo Francesco Bolis, © Courtesy Fondazione MAXXI & Zaha Hadid Architects

This how the project is presented at the site of the study of Zaha Hadid [9]:

> MAXXI supercedes the notion of museum as *object* or fixed entity, presenting instead a field of buildings accessible to all, with no firm boundary between what is inside and what outside. Central to this new reality—its primary force—is a confluence of lines—walls that constantly intersect and separate to create indoor and outdoor spaces.
> MAXXI integrates itself with its surroundings, re-interpreting urban grids to generate its own geometric complexity. Through the flow of its walls it defines major streams—the galleries—and minor streams—interconnections and bridges, delighting in a peculiar shape footprint which in this context becomes 'liberation'—a freedom to bundle, twist and turn through existing buildings. In this very meandering MAXXI both draws on and feeds the cultural vitality of its mother city… MAXXI expresses itself through glass, steel and cement—delighting in neutrality, achieving great curatorial flexibility and variety.
> To wander through, to experience this place—these spaces—is to encounter constantly changing vistas and surprises.

At the inauguration of MAXXI Zaha Hadid said that first of all she had to decide whether or not to keep all existing buildings. Once she made the decision, she began to study the geometries that would replace them: orthogonal, parallel or diagonal.

> What appeared was a confluence of lines of different geometries present on the site. This way it started and a *fluid interpretation* of the space emerged [10].

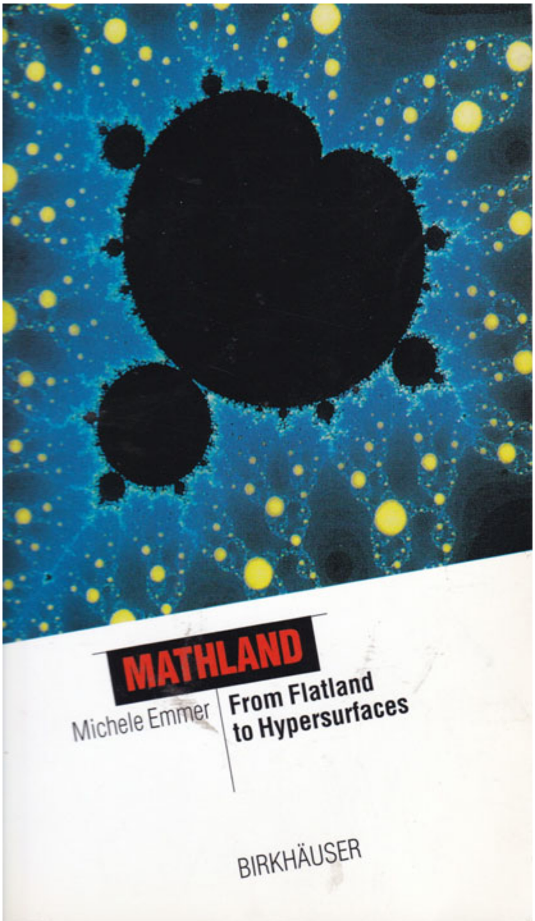

Fig. 5 Cover of [13]. Courtesy of Birkhaüser verlag, Basel

Fluidity is now one of the keywords of contemporary architecture. Among other things, Zaha Hadid has a degree in mathematics and fluidity is naturally related to *Topology*, the mathematics of transformations and isomorphisms.

So we can say that some of the topological ideas were *sensed* by artists and architects in the past decades, first by artists, then much later by architects.

This is one of the main reason why Mark Burry, who is on charge of completing Gaudi's *Sagrada Familia* in Barcelona dedicated a chapter to topology in his recent book [11] *The New Mathematics in Architecture* (2010).

He wrote [12]:

> The freedom that topology affords in architecture as a more generalized framework to geometry has received greater appreciation in the post-digital age.... The essence of architectural and urban planning is also captive in such non-geometrical diagrams, as are the relationships between component spaces or activities of building. This is regardless static, unchanging form that is also subject to detailed geometrical description. It is possible that the organization of the early development world of our childhood is a similar network of

Fig. 6 Cover of [14]. Courtesy of Birkhaüser verlag, Basel

connections between significant places and things, and it is only later and gradually that the absolute reference of metrical Cartesian space is superimposed on our established perception of proximities and relationships....

What is it about topology and its freedom of description that has seized modern architectural production, long after the underlying ideas were in common domain? One possible answer is the confluence of unimaged new levels of computer graphical representations with the transition of non-rational basis splines, or NURBS, from the automotive industry into other computer-aided design software.... The dynamism of systems could not only be represented in truly dynamic models, but their manifestations could now be understood visually. Truly visual feedback changed everything. It became possible to model surfaces that could change, stretch, adopt free from curvature, or conform to a geometrical rationale without losing their integrity—wonderful surfaces that, plastically and geometrically at least, exceeded the behaviour of any known material and could be given visual material qualities at a whim.... Topological description is being adopted as the means of mapping architectural intention, and with it arrives the progressive discovery of how to map this onto the frozen Euclidean moment in the physical world.

Fig. 7 *Exhibition Zaha Hadid*, Palazzo Franchetti, Venice, 2016. Photo credit Francesco Allegretto, courtesy Fondazione Berengo & Zaha Hadid Architects

In his chapter on *Topology* Mark Burry has a list of books as references to the topic including my volume (see Fig. 5) by the title *Mathland, from Flatland to Hypersurfaces*, on the new tendency in architecture to include *Topology* [13].

The book was published in the series *The Information Technology Revolution in Architecture* in 2004, the same year, in the same series, edited by the Italian architect Antonino Saggio, Patrick Schumacher published (see Fig. 6) the book *Digital Hadid: Landscapes in Motion* [14].

Zaha Hadid in Venice

In March 2017 the twentieth meeting of *Mathematics and Culture* was held at the *Istituto Veneto di Scienze, Lettere ed Arti*, Palazzo Franchetti, on the *Canal Grande* in Venice. The meetings of *Mathematics and Culture* are held there since 2013. On the upper floors of the building there are usually art and architecture exhibitions, some framed within the *Venice Bienniale*. From 27 May to 27 November 2016 there was a large exhibition dedicated to Zaha Hadid (see Figs. 7, 12–16), who herself designed the exhibition by choosing works, paintings and projects to be included in it's itinerary. She died on March 31, 2016.

At the Venice conference a special session was held dedicated to Zaha Hadid with the intervention of Gianluca Racana, one of the directors of Zaha Hadid Studios. On the occasion of the exhibition in Venice, a catalog was published divided into three small-format books: *Zaha Hadid Architects, Zaha Hadid Selected Works, Zaha Hadid CODE* [15].

In the introduction to the exhibition, Patrick Schumacher, partner of Zaha Hadid, stressed that the publication of the book *Digital Hadid* in 2004 remained an important milestone in his attempt to reflect on the genealogy of the digitally generated style he called *Parametricism* at the *Venice Biennale of Architecture* in 2008. He felt the need to give a name to the new language or style of architecture that had been formed by the strong convergence of an entire generation of young architects since the nineties.

> My 2004 thesis focused on the pre-digital desire for complexity and fluidity as a motivating force for the introduction of certain digital tools drawn into architecture from the realms of computer graphics, movie animation and scientific simulation.... These tools are the ever expanding set of algorithms that shape, discipline, and rationalize our design in unexpected and sometimes even counter-intuitive ways. These tools have become truly generative and intelligent, augmenting our design capacity in profound ways [16].

Always in 2004, Schumacher wrote [17]:

> There is an unmistakable new style *manifesto* within avant-garde architecture today. Its most striking characteristic is its complex and dynamic curve—linearity. Beyond this obvious surface feature, one can identify a series of new concepts and methods that are so different from the repertoire of both traditional and modern architecture that one might speak of the emergence of a new paradigm for architecture.

Schumacher reports some of the answers that Zaha Hadid gave about the role of design media in general and digital media in particular in an interview with the Chairman of the Architectural Association of Schools of Architecture Mohsen Mostafavi [18]:

> I still think that even in our later projects, where the computer was already involved, the 2-dimensional plan drawings are still seminal. I still think the plan is critical. The computer shows what you might see from various selected viewpoints. But I think this doesn't give you enough transparency, it's much too opaque. Also, I think it is much nicer on the screen that when it is printed onto paper, because the screen gives you luminosity and the paper does not, unless you do it through a painting. Further, I think if you compare computer renderings with rendering by hand, I must say that you can improvise much more with hand drawing and painting.... Only 1920s *Modernism* really discovered the full power and potential of drawing as a highly economic trial-error mechanism and an effortless plane of invention—in fact inspired by the compositional liberation achieved by *abstract art* in the first decade of the 20th century. Drawing accelerates the evolution of architecture. Modern architecture depends upon the revolution within the visual arts that finally shook off the burden of representation. Modern architecture was able to build upon the legacy of modern abstract art as the conquest of a previously unimaginable realm of constructive freedom.... Abstraction meant the possibility and challenge of creation. Through figures such as Malevich and vanguard groups such as *De Stijl* movement, this exhilarating historical moment was captured and exploited for the world of experimental architecture.

Schumacher comments: "Abstraction implies the avoidance of familiar ready-made typologies. Instead of taking for granted things like houses, rooms, windows, roofs.... Hadid reconstitutes the functions of territorialization, enclosing and interfacing by means of boundaries, fields, planes, volumes, cuts, ribbons.... One of Hadid's most audacious moves was to translate the dynamism and fluidity of her calligraphic hand directly into equally fluid tectonic systems. Another incredible move was from isometric and perspective projection to literal distortions of space and from the exploded axonometry to the literal explosion of space into fragments, from the superimposition of various fisheye perspectives to the literal bending and

Fig. 8 Zaha Hadid, *The Peak, Hong Kong*, painting (1982–1983) © Courtesy Zaha Hadid Architects

meltdown of space. All these moves initially appear as plainly illogical, akin to the operation of the surrealists." (see Fig. 8).

In the introduction *Realising Architecture's Disruptive Potential* to the volume *Zaha Hadi CODE* Shajay Bhooshan, Associate to *Zaha Hadid Computation and Design Group* (*ZH_CODE*) wrote [19]:

> The 19th century architect Antonio Gaudì could draw inspiration from the biological ideas and drawings of Ernst Haeckel, or develop an artistic repertoire influenced by the formal appearance of new mathematics of the time. Contemporary computational designers on the other hand, can use the very biological models that generate our physiology to produce geometry of architecture. They could, in equal part utilize the code of complex mathematics to generate the structural systems as in the *Bejing Water Cube* stadium. However in the final execution of the projects, hard distinctions are productive and necessary. Thus, digital

> technologies can allow for fluid transition from a co-authoring early stages to collaborative, specialised later stages of design and execution.... The geometry of the so called *minimal surfaces*... and the latest manifestation of such structures in the *Mathematics Gallery* is a result of a long history of prior experience and historically assimilated and transferred research.

I refer to the models of *minimal surfaces* and soap films used in the *Olympic Pool* of Bejing, and to Haeckel and Gaudì, in my book *Bolle di sapone: Arte e matematica* [20]. Two of the architects who participated in the design and construction of the *Olympic Pool* participated on two different occasions in the *Venice conferences* on *Mathematics and Culture* [21–23].

It is interesting that the first project presented in the volume *CODE* is that of the *Gallery of Mathematics and Computing* at the *Science Museum* in London, inaugurated in December 2016. It was Zaha Hadid's last project (it had started in 2014). A part of the aforementioned introduction is dedicated to the project: *Exemplar Project—The Gallery of Mathematics and Computing* [24].

The description opens with a quote from Ada King Lovelace:

> The Analytical Engine weaves algebraic patterns, just as the Jacquard loom weaves flowers and leaves.

Ada Lovelace (1815–1852; see Fig. 9) was an English mathematician daughter of Lord Byron and of mathematician Anne Isabella Milbanke. She had worked with Charles Babbage on the first analytical machine and she is considered the author of the first ever algorithm designed to be processed by a machine. The programming language *Ada* is named in her honor.

Bhooshan adds that "by a wonderful coincidence, we—the CODE team—started work on the *Gallery* in the bicentennial year of birth of Ada Lovelace, a pioneering woman in the history of computers and of *poetic science*—a resonant desire for a synergetic union of man and machine, articulated more than two centuries ago. The project is a testament to the aforementioned critical aspects of innovation, collaborative design processes and the fluid exchange of means, methods and models across disciplines."

These are the models of *minimal surfaces* that were first studied in architecture in the 1960s by Frei Otto at the *University of Stuttgart*. The two German architects who participated in the *Olympic Pool* project in Bejing, Chriss Bosse and Tobias Walliser, were students of Frei Otto. Inside the *Gallery* there are rest areas, which have been created using a long line of research in mathematics and engineering, the materialization of classes of models of Surfaces which the sculptor Henry Moore had gone to see in the thirties at the *Science Museum* in London [25] Adds Bhooshan "this story of mathematics, physics and materials science was inherited from manufactured artifacts."

The *Mathematics Gallery* at *London's Science Museum* (see Fig. 10), is described as a pioneering new gallery that explores how mathematicians, their tools and ideas have helped to shape the modern world. Its design and layout is defined by mathematical equations that are used to create the 3D curved surfaces representing the airflow patterns that would have streamed around the historic 1929 aircraft at the centre of the exhibition.

Fig. 9 Alfred Edward Chalon, *Portrait of Ada Loverlace*, 1840, Courtesy of Science Museum, London, Picture Library

> CODE's research applications determined the airflow patterns that define the shapes and spaces of the *Mathematics Gallery* at the *Science Museum*, which were a result of a fluid exchange of means, methods and models across disciplines and the lineage of innovative, tensile fabric structures that the office has undertaken in the past. These galleries outline the development and continued advancements of this research and their application throughout *Zaha Hadid Architects'* body of work.

In the center of the large room around the small airplane, the air movements around the wings (see Fig. 11) are simulated using the Navier-Stokes equations of dynamic fluid of the years around 1840, today simulated by computers in all projects of aircrafts, ships, cars, and many other things.

I do not know if Zaha Hadid herself wrote the words that present the *Gallery*:

> The enactment of the performance aspects of design with respect to environment and energy, its efficient manufacture and ultimately the comfortable and harmonious forms are an integral part of building on those foundations. Our design of the *Gallery for Mathematics and*

Fig. 10 Zaha Hadid Architects, *Gallery for Mathematics and Computing*, Science Museum, London. Photo by Luke Hayes, courtesy of *Zaha Hadid Architects*

> *Computing* realizes such efforts. Mathematics and computing are intrinsically entangled in the every day production of the project—be it the design, engineering and manufacture of the fabric surfaces of the central pod, the efficacious production of robust archlike benches using the latest advances in robotic manufacture, the enumeration and harmonious positioning of the hundred odd objects in the approximately eighty show-cases, or the coordination of lighting instruments and quality to augment the *subliminal navigation of the space* by its users. In essence, the designers, as the curators, engineers and builders of the project, embody the spirit of the brief of imbuing the quotidian aspects of mathematics.
> We hope that the visitors to the *Gallery* have a chance to enjoy the spatial experience at the new *Mathematics and Computing Gallery* and that the architecture enables them to see the myriad, tangible and perceptible ways in which Mathematics touches our lives. We hope that they can indeed see the weaves of algebra as natural as the weaves of the loom.

There is one problem in the *Gallery:* the content of the room, the mathematical objects. They are for the most part old objects, obsolete systems of little interest.

For example, to show that computers and algorithms make big ships move today, there is just a nice model of a ship. In short, there are just a few ideas on modern and contemporary mathematics and this obviously does not depend on the *Zaha Hadid Architects*. It is true that English mathematics has historically been much more interested in applications (with numerous exceptions such as G.H. Hardy), but

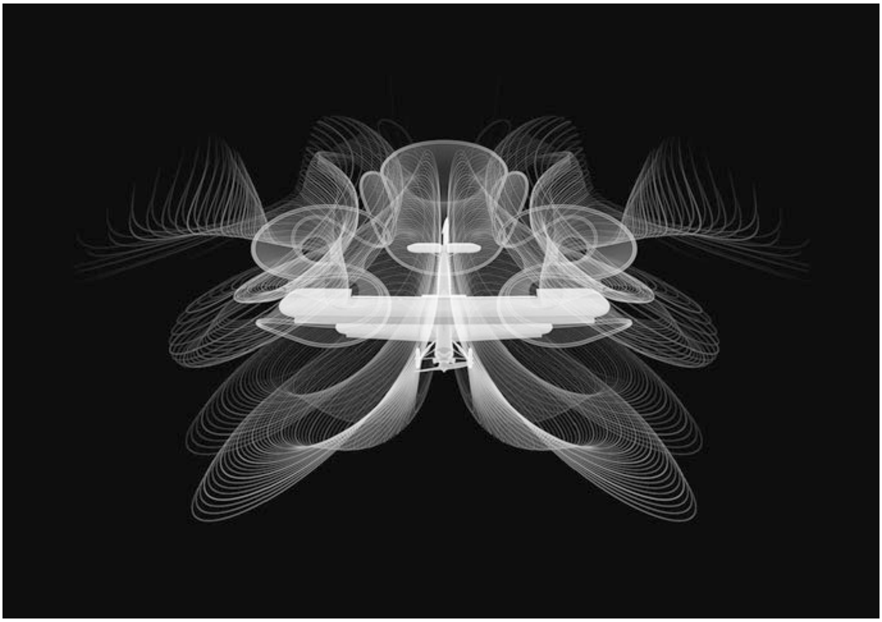

Fig. 11 Zaha Hadid Architects, *Gallery for Mathematics and Computing*, Science Museum, Londra, project, 2014–2016. Courtesy of *Zaha Hadid Architects*

in view of the complete transformation of the room perhaps it would have also been necessary to rethink what to exhibit.

Conclusions

There was no doubt that it was necessary to modify our formal language in order to use geometric instruments that were up to graphic and pictorial intuitions.

> Computer technology, i.e. the new digital design tools, have had an important and increasing influence on the work of *Zaha Hadid Architects* over the last years. This concerns primarily the handling of increasingly complex geometries within the designs. However, the desire for such tools to be imported from the animation industry originated in the fact that the tendency towards complexity and fluidity was already manifest in the work before those tools were available.

All this leads to a new conception of space. Schumacher clarifies:

> These techniques lead to a new concept of space which suggests a new orientation, navigation and inhabitation of space.... The significance and the ambition of these projects is that they might be seen as manifestos of a new type of space. As such, their defining context is the historical progression of such manifestos rather than their concrete spatial and institutional location... including the legacy of modern architecture and abstract arts as the conquest of a previously unimaginable realm of constructive freedom.

Fig. 12 *Exhibition Zaha Hadid*, Palazzo Franchetti, Venice, 2016. Photo credit Francesco Allegretto, courtesy Fondazione Berengo & Zaha Hadid Architects

Fig. 13 *Exhibition Zaha Hadid*, Palazzo Franchetti, Venice, 2016. Courtesy of Berengo Foundation & Zaha Hadid Architects

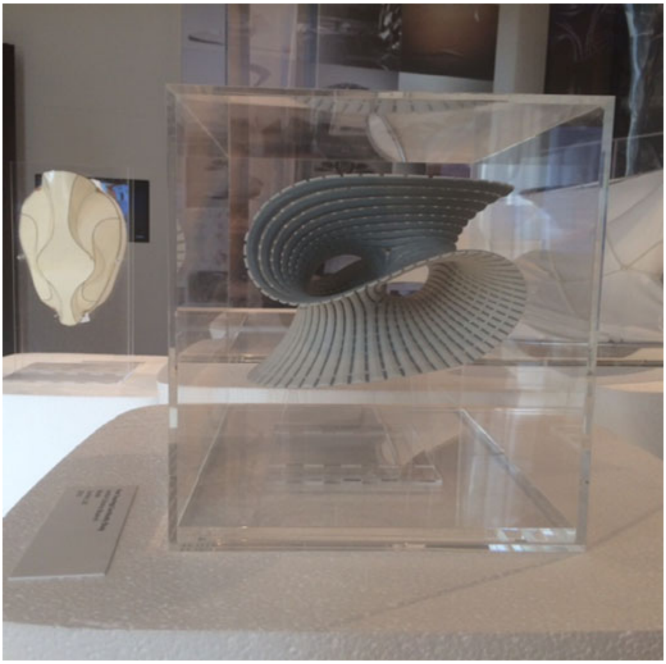

Fig. 14 *Exhibition Zaha Hadid*, Palazzo Franchetti, Venice, 2016, Photo by M. Villarreal. Courtesy of Zaha Hadid Architects

Fig. 15 *Exhibition Zaha Hadid*, Palazzo Franchetti, Venice, 2016. Photo credit Francesco Allegretto, courtesy Fondazione Berengo & Zaha Hadid Architects

Fig. 16 *Exhibition Zaha Hadid*, Palazzo Franchetti, Venice, 2016. Photo credit Francesco Allegretto, courtesy Fondazione Berengo & Zaha Hadid Architects

And that is why all these reflections and projects are interesting for a mathematician who has always been fascinated by how mathematical, geometric and spatial ideas interacted with artistic, architectural, visual culture.

Without topology it would have been very difficult to imagine all these forms, meaning with the word *topology* not only its strictly mathematical meaning. The study of topological forms and sculptures is best understood, as was witnessed at the *Venice exhibition* by the many declared topological objects (see Figs. 12–16), in particular of the *CODE team*. There were many forms on display from the prototype sculpture *Topology Optimized Concrete Shell* up to chairs and many other objects.

Schumacher concludes [26]:

> All compositions are seen as tasks for creative organic interarticulation. A refined organic architecture resists easy decomposition, a measure of its complexity.

References

1. R. Courant, H. Robbins, *What Is Mathematics? An Elementary Approach to Ideas and Methods* (Oxford University Press, New York, 1941)
2. M. Bill, Come cominciai a fare le superfici a faccia unica, in *Max Bill, Catalogue of the Exhibition*, ed. by A.C. Quintavalle, Parma (1977)
3. K. Gestner, L'estetica che nasce dallo spirito della geometria, in *Bill, Catalogo della Mostra 29 Marzo–25 Giugno 2006*, ed. by T. Buchsteiner, O. Letze (Electa Mondadori, Milano, 2006), pp. 124–127

4. S. Perrella, Electronic baroque, in *Architecture and Science*, ed. by G. Di Cristina (Wiley–Academy, Chichester, 2001), pp. 149–150
5. L.A. Couture, H. Rashid, Asymptote, L'architettura di Metamorph, in *Metamorph, 9 Mostra Internazionale di Architettura*, ed. by K.W. Forster (Marsilio Editori, Venezia, 2004), pp. 8–13.
6. H. Rashid, Morphing il sublime, in *Focus, 9 Mostra Internazionale di Architettura*, ed. by K. Forster (Marsilio Editori, Venezia, 2004), pp. 70–81
7. M. Kemp, Intuizioni strutturali e pensiero metamorfico nell'arte, architettura e scienze, in *Metamorph, Catalogue of the Mostra Internazionale di Architettura, Focus*, ed. by K.W. Forster. (Marsilio Editori, Venezia,2004), pp. 30–42
8. Z. Hadid, P. Schumacher, Lotus, in *Out There, Architecture Beyond Building, Istallazioni, Mostra Internazionale di Architettura, La Biennale di Venezia, Catalogue*, ed. by P. Jodidio (Marsilio Editori, Venezia, 2008), pp. 90–95
9. http://www.zaha-hadid.com/architecture/maxxi/
10. Z. Hadid, http://www.maxxi.beniculturali.it/progettoarchitettonico.htm
11. M. Burry, Gaudì innovator, in *Mathematics and Culture IV*, ed. by M. Emmer (Springer, Berlin, 2004)
12. J. Burry, M. Burry, in *The New Mathematics of Architecture* (Thames and Hudson, Londra, 2010)
13. M. Emmer, *Mathland, from Flatland to Hypersurfaces* (Birkhäuser, Basel, 2004)
14. P. Schumacher, *Digital Hadid, Landscapes in Motion* (Birkhäuser, Basel, 2004)
15. Z. Hadid, Catalogue of the Exhibition 27 May–27 November 2016, *Zaha Hadid Architects, Zaha Hadid Selected Works, Zaha Hadid CODE*, Palazzo Franchetti, Venice, Zaha Hadid Architects & Fondazione Berengo, Venice (2016)
16. In [15], *Zaha Hadid Architects*, pp. 15–16
17. In [14], pp. 5–7
18. In [14], pp. 10–17
19. S. Bhooshan, Realising architecture's disruptive potential, in *Zaha Hadid CODE* [15], pp. 9–15
20. M. Emmer, *Bolle di Sapone: Arte e Matematica* (Bollati Boringhieri, Torino, 2009), Premio letterario Viareggio saggistica (2010)
21. C. Bosse, L'architettura delle bolle di sapone, in *Matematica e Cultura 2007*, ed. by M. Emmer (Springer, Milano, 2007), pp. 43–55
22. T. Walliser, Other geometries in architecture: bubbles, knots and minimal surfaces, in *Mathknow, Mathematics Applied Sciences and Real Life*, ed. by M. Emmer, A. Quarteroni (Springer, Milano, 2009), pp. 91–111.
23. M. Emmer, Soap films and soap bubbles: from plateau to the Olympic swimming pool in Beijing, in *Mathknow, Mathematics Applied Sciences and Real Life*, ed. by M. Emmer, A. Quarteroni (Springer, Milano, 2009), pp. 119–129
24. In [19], p. 16
25. M. Emmer, Visible harmonies: mathematical models, in *Imagine Math 3*, ed. by M. Emmer (Springer, Heidelberg, 2015), pp. 43–65
26. In [14], p. 32

Part II
Mathematics & Media

A Mathematician at MUSE, the Science Museum of Trento

Marco Andreatta

MUSE, the Science Museum of Trento

At the beginning of the 21-st century the local government of the Provincia Autonoma di Trento decided to create a new museum of science; this is a part of the larger program of investments into research, innovation and culture, for which, since 50 years, it has been putting a large part of Gross Domestic Product. In 2011 I was nominated chairman of MUSE, to cooperate with the director, Michele Lanzinger, in order to set-up this new and innovative museum. In this paper I briefly describe my experience, of a mathematician, professor of Geometry, in this unusual role and, at the same time, I try to give an idea of what MUSE is.

The building has been conceived and realized by the architect Renzo Piano and his Italian team. It has been designed via projective geometry, following the Italian tradition, its skyline representing the dolomites (see Fig. 1). The museum is a part of a larger architectural project which redeveloped an abandoned industrial area, in accordance with Piano's vision of residential areas.

The building displays the architect point of view as a flight towards knowledge. Let me quote Piano's words, taken from an interview in Metropolis Magazine titled "How to design the perfect Museum", see [3]. *Buildings like these allow people to share experiences together, to enjoy and share life.... They fly, they are rooted, but they lift up, above the ground and that lets light to come under and inside and allow the ritual of the city life to merge with the ritual of the building life. By lifting the building, the ground floor becomes almost a continuation of the public realm. You leave space beneath it for life to happen.*

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Fig. 1 MUSE's skyline; reproduced with permission from Archivio MUSE, Museo delle Scienze-Trento

Fig. 2 The big Void; reproduced with permission from Archivio MUSE, Museo delle Scienze-Trento

Completed on July 2012, within one year it has been furnished and prepared for the inauguration in July 2013, saving money compared to the initial budget plan. For the first time the Renzo Piano Building Workshop took also care of the furnishing. It has been an intensive year of debates, sometime harsh, between architects, scientists and furniture workers which produced a lot of creative, but scientifically rigorous, solutions. Among all, the Piano's idea of the Big Void in the central building, with many taxidermied animals suspended in the void and displaced appropriately according to their area of living (see Fig. 2).

MUSE focuses mainly on natural science; it is a modern version of the nineteenth-century science museum of Trento, depicting the beautiful nature of our region, the Alps and the Dolomites (World Heritage Site of UNESCO), with the eyes, the instruments and the method of scientific research.

MUSE promotes the public understanding of science, focusing in particular on the future perspective of the world we live in, on the significant global impact that human activities have on the ecosystems, based on objective data and on theories debated by the large international scientific community.

MUSE is in between a museum and a science center; many multimedia and interactive features are installed near taxidermied animals, dinosaurs, glaciers, aquariums, green house. We introduce new technologies, like augmented reality, tomographies, interactive exhibits or boards. We keep searching a good equilibrium, beyond a certain limit abstraction (and mathematics) can annoy and prevent comprehension.

Beside ruling the administrative board, the obvious question was what could be my role, as a mathematician. I had to balance my idealist view (from old Plato to the present of Alain Connes), according to which everything in nature is written in a mathematical language, with the need of being comprehensible and tasty for a large public. This is actually a core problem for a science museum. The role of mathematical models in natural science, concretely discussed by Galileo, is more and more essential and unavoidable nowadays. To organize and understand huge quantities of data, to foresee evolutions, to test human influence on the natural environment. Moreover, a modern museum of science should support the slogan raised recently by life sciences: *converging science*, which has mathematics and informatics in its core. Therefore my main concern was to combine the natural exhibits with quantitative models, with numbers and statistics, to give meaning and support to scientific interpretations.

Different exhibits point in this direction, as an example let me take *Science on a Sphere (SOS)*, a global display system, made by the federal US agency NOAA (National Oceanic and Atmospheric Administration), that uses computers and video projectors to display planetary data onto a three meters diameter sphere; a giant animated globe of atmospheric storms, climate change, ocean temperature, volcano's activities used to explain complex environmental processes via true data (see Fig. 3). At the beginning architects strongly opposed the idea to put a *curved shape, spherical object* in a building in which everything is linear. Eventually, the captivating communicative quality of this all-round and steady representation of data won their opposition. We also use SOS *to play* with geometry: we presented some visualizations of platonic solid on the sphere at the *Imaginary - Open Math* conference in 2016.

Some Important Features of MUSE

MUSE supports exhibits and expositions with a research activity in the fields of communication, of natural science and of education, also via collaboration with universities and research centers all over the world. It has a crucial role in the European networks of Museum and Science Centers (ECSITE). It is part of my job to enlarge the net of collaborations, to convince political representatives that a good scientific communication as well as a good educational program can be achieved only together with a solid international research activity.

Fig. 3 Science On a Sphere and Fabrication Laboratory; reproduced with permission from Archivio MUSE, Museo delle Scienze-Trento

Another crucial goal of the museum is to show that scientific innovation is a key factor for our cultural, social and economic development. Let me mention in this respect our Fabrication Laboratory (Fab Lab), with 3D printers, laser cutters, arduinos, etc. (see Fig. 3) General public as well as private costumers can find experts to help in *building their ideas*; it is turning out to be a central tool for innovation all over the world. Here, of course, Mathematics plays a fundamental role. With the help of many math students on *stage* we have constructed, using different digital techniques and materials, classical mathematical objects like regular solids, special curves, algebraic surfaces (in the old spirit of Klein-Castelnuovo), Klein bottles and so on. A perfect and modern way of *touching the abstract*, using Enrico Giusti title of his talk in the last conference in Venice.

MUSE promotes also public debates on hot themes, dialoguing with leading explorers and scientists, from the astronaut Samantha Cristoforetti to the director of CERN Fabiola Gianotti. Some mathematicians contributed to the debate, from M. Emmer to P. Odifreddi, from A. Quarteroni to G. Todesco.

The Activity of the Museum Measured by Numbers

Up to now we count more then half million visitors every year; MUSE is among the ten most visited museum in Italy, the first among science museums. The *Giornale dell'Arte* recently put it in the first position, together with Roma's Maxxi, in its ranking which measures parameters like the innovative expositive structure, with interactive and multimedia tools, the young age of visitors, the excellence of presentations and services, due in particular to the so-called pilots, young scientists which help visitors at every floor, nice cafeteria and bookstore.

It employs about 120 persons and 90 pilots (temporary job). MUSE coordinates a net of other six local museum of natural science, including one in the Udzunwga mountains in Tanzania. Its annual budget is covered half by self-revenue (tickets, research grants, European projects, . . .) and half by the local government. We ran an economic impact analysis (EIA), published by the Italian newspaper Sole24ore, which says that MUSE generates yearly an economic impact in our town which amounts to more then 50 million euros.

Life at MUSE is rich of events: special openings in the evening, cultural entertainments, conferences and debates, opening session for conferences or business activities. It is a center promoting the cultural life of the town, very active on media and social networks.

Temporary Exhibition; an Example with Math

MUSE organizes also temporary exhibitions. As an example, let me present the one held February-June 2016, titled *MadeInMath, discover the mathematics of the world*. It was a revised version of *MateInItaly*, an exhibition proposed in 2014 at Milano's Triennale; the curators were some colleagues of Milano, Renato Betti, Gilberto Bini, Maria Dedò, Simonetta Di Sieno, Angelo Guerraggio, plus myself. In the MUSE style, it has been an exhibition in between a museum and a science center, with multi-media and interactive aspects which provided direct experiences. In many respects, it continues the important Italian tradition of exhibitions of mathematics, started by Castelnuovos (father and daughter) and properly described by M. Emmer in the last edition of the Venice conferences and summarized in its philosophy by E. Giusti in the same occasion (see [1] and [2]). In the beautiful stage offered by MUSE we mixed art with advertisement and effective communication. The storyboard, which keeps in mind the historical development, describes how mathematicians, through arithmetic, geometry or calculus, organize objects and principles into logical structures and models. It shows how this gives rise to innovation and how it influences human progress (see Fig. 4). Many intriguing questions were posed like neuroscience theory on mathematics, representations of the sphere on a map, geometry in higher dimension, description of powerful mathematical models in science and in everyday life. A short video of the exhibit can be found at: http://www.science.unitn.it/~andreatt/MadeinMathPROMOver2.mov

Popularization of an abstract (and difficult) science like mathematics is much harder than, for instance, natural science. Its abstract character in particular requires to make it alive through concrete examples, connected to everyday life and to technological applications. To achieve this, the exhibition had background questions which connected different themes: *what mathematicians do, what are the result of their efforts, how have they been used?* We proposed several answers at these questions.

I should point out the fact that a visitor not familiar with mathematics needed explanations which were provided by our pilots; on the other hand this confirms the idea that the best way to learn mathematics is via *direct talking*. We collected interesting feedbacks from the visitors: many were astonished not to find abstract and obscure formula or theorems, but appealing subjects and questions, everyday

Fig. 4 Made in Math; reproduced with permission from Archivio MUSE, Museo delle Scienze-Trento

life applications, amusing interactive and stimulating exhibits, an innovative and fascinating way of using multimedia presentation.

Matched to the exhibition many side activities were performed, which gave rise to a period of debates and discussion with a miscellaneous public coming from Italy or abroad. Among others: a series of conferences on specific themes of the exhibition, a round table on the *role of mathematics in the society*, some evening entertainments on the connections between math, art, sport, ... Moreover we had many educational and recreational laboratories on Math and a *crowd project*, called Musemenger, which realized a three-dimensional fractal of paper (origami), ran for 150 days with the participation of over 5,000 people (see Fig. 5).

Conclusion

MUSE is able to collect and to interpret the need of scientific knowledge and method in our society. It helps to convince the general public that strategical choices, such as how to use natural resources or how to respect other lives, need a better scientific understanding, possibly based on numerical data and mathematical models, approved by the scientific community and used appropriately. We were able to tackle complex and deep questions, for instance: the space in 4 or more dimensions (both via physics and math), the extinctions of many species in this anthropocene period, which is probably not only unavoidable but essential in the evolution of the life (the temporary exhibition of Spring 2017).

Fig. 5 Crowd project MuseMenger; reproduced with permission from Archivio MUSE, Museo delle Scienze-Trento

Last but not least, a visit to MUSE is an enjoyable experience for a miscellaneous variety of people, from students to retired persons, from families to professional scientists.

Acknowledgements I would like to thank Michele Emmer for the invitation at the conference Imagine Math 6 in Venice 2017 and also for providing remarks and useful suggestions during his visit at MUSE.

References

1. M. Emmer, Are math exhibition useful? in *Imagine Maths 5*, ed. by M. Emmer, M. Abate, M. Falcone, M. Villarreal (IVSLA-UMI, Bologna, 2016), pp. 229–250
2. E. Giusti, Touching the Abstract: Mathematics at the Museum, in *Imagine Maths 5*, ed. by M. Emmer, M. Abate, M. Falcone, M. Villarreal (IVSLA-UMI, Bologna, 2016), pp. 251–262
3. R. Piano, How to design the perfect Museum. An interview of Paul Clemence to Renzo Piano, originally published on Metropolis Magazine, available on http://www.archdaily.com/534172/renzo-piano-reveals-how-to-design-the-perfect-museum, Accessed August, 22nd, 2017.

To Bring Math to the World, Start with Mathematicians

Frank Morgan

As Editor of *Notices of the American Math Society* (AMS), I'm finding that to share math with the world, mathematicians have to get better at sharing math with each other. *Notices* is the largest publication in higher mathematics, going to at least the 30,000 AMS members, with the goal of promoting mathematics and the society. The trouble is that many articles and even conference talks are too technical, try to do too much, and have too few pictures. The math needs to be humanized.

For my first issue last January 2016 we featured the *January Joint Math Meetings* invited speakers, asked them to write alluring invitations to their talks, edited them as extensively as needed, and put their pictures on the cover.

Notices is the official publication of record for the *American Mathematical Society*. Frank Morgan, as chief editor, worked closely with managing editor Rachel Rossi and deputy editor Allyn Jackson.

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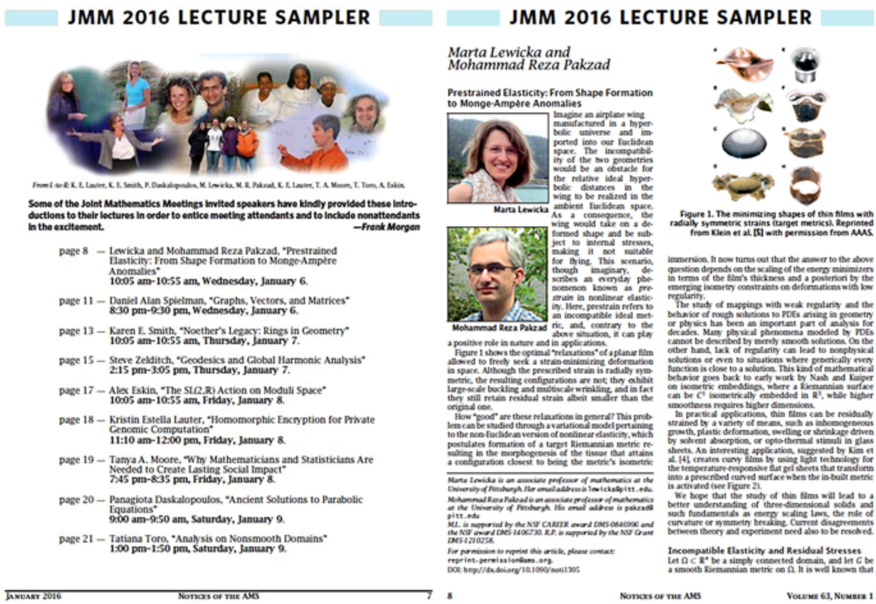
This made for an issue that was timely as well as lively and readable, and we asked for lots of illustrations in addition to pictures of the authors.



JMM 2016 LECTURE SAMPLER

From L-to-R: K. E. Lauter, K. E. Smith, P. Daskalopoulos, M. Lewicka, M. R. Pakzad, K. E. Lauter, T. A. Moore, T. Toro, A. Eskin.

Some of the Joint Mathematics Meetings invited speakers have kindly provided these introductions to their lectures in order to entice meeting attendants and to include nonattendants in the excitement.
—Frank Morgan

page 8 — Lewicka and Mohammad Reza Pakzad, "Prestrained Elasticity: From Shape Formation to Monge-Ampère Anomalies"
10:05 am–10:55 am, Wednesday, January 6.

page 11 — Daniel Alan Spielman, "Graphs, Vectors, and Matrices"
8:30 pm–9:30 pm, Wednesday, January 6.

page 13 — Karen E. Smith, "Noether's Legacy: Rings in Geometry"
10:05 am–10:55 am, Thursday, January 7.

page 15 — Steve Zelditch, "Geodesics and Global Harmonic Analysis"
2:15 pm–3:05 pm, Thursday, January 7.

page 17 — Alex Eskin, "The SL(2,R) Action on Moduli Space"
10:05 am–10:55 am, Friday, January 8.

page 18 — Kristin Estella Lauter, "Homomorphic Encryption for Private Genomic Computation"
11:10 am–12:00 pm, Friday, January 8.

page 19 — Tanya A. Moore, "Why Mathematicians and Statisticians Are Needed to Create Lasting Social Impact"
7:45 pm–8:35 pm, Friday, January 8.

page 20 — Panagiota Daskalopoulos, "Ancient Solutions to Parabolic Equations"
9:00 am–9:50 am, Saturday, January 9.

page 21 — Tatiana Toro, "Analysis on Nonsmooth Domains"
1:00 pm–1:50 pm, Saturday, January 9.

JANUARY 2016 NOTICES OF THE AMS 7

JMM 2016 LECTURE SAMPLER

Marta Lewicka and Mohammad Reza Pakzad

Prestrained Elasticity: From Shape Formation to Monge-Ampère Anomalies

Marta Lewicka

Mohammad Reza Pakzad

Imagine an airplane wing manufactured in a hyperbolic universe and imported into our Euclidean space. The incompatibility of the two geometries would be an obstacle for the relative ideal hyperbolic distances in the wing to be realized in the ambient Euclidean space. As a consequence, the wing would take on a deformed shape and be subject to internal stresses, making it not suitable for flying. This scenario, though imaginary, describes an everyday phenomenon known as *prestrain* in nonlinear elasticity. Here, prestrain refers to an incompatible ideal metric, and, contrary to the above situation, it can play a positive role in nature and in applications.

Figure 1 shows the optimal "relaxations" of a planar film allowed to freely seek a strain-minimizing deformation in space. Although the prescribed strain is radially symmetric, the resulting configurations are not; they exhibit large-scale buckling and multiscale wrinkling, and in fact they still retain residual strain albeit smaller than the original one.

How "good" are these relaxations in general? This problem can be studied through a variational model pertaining to the non-Euclidean version of nonlinear elasticity, which postulates formation of a target Riemannian metric resulting in the morphogenesis of the tissue that attains a configuration closest to being the metric's isometric immersion. It now turns out that the answer to the above question depends on the scaling of the energy minimizers in terms of the film's thickness and a posteriori by the emerging isometry constraints on deformations with low regularity.

The study of mappings with weak regularity and the behavior of rough solutions to PDEs arising in geometry or physics has been an important part of analysis for decades. Many physical phenomena modeled by PDEs cannot be described by merely smooth solutions. On the other hand, lack of regularity can lead to nonphysical solutions or even to situations where generically every function is close to a solution. This kind of mathematical behavior goes back to early work by Nash and Kuiper on isometric embeddings, where a Riemannian surface can be C^1 isometrically embedded in $\mathbb{R}^3$, while higher smoothness requires higher dimensions.

In practical applications, thin films can be residually strained by a variety of means, such as inhomogeneous growth, plastic deformation, swelling or shrinkage driven by solvent absorption, or opto-thermal stimuli in glass sheets. An interesting application, suggested by Kim et al. [4], creates curvy films by using light technology for the temperature-responsive flat gel sheets that transform into a prescribed curved surface when the in-built metric is activated (see Figure 2).

We hope that the study of thin films will lead to a better understanding of three-dimensional solids and such fundamentals as energy scaling laws, the role of curvature or symmetry breaking. Current disagreements between theory and experiment need also to be resolved.

Incompatible Elasticity and Residual Stresses

Let $\Omega \subset \mathbb{R}^n$ be a simply connected domain, and let G be a smooth Riemannian metric on Ω. It is well known that

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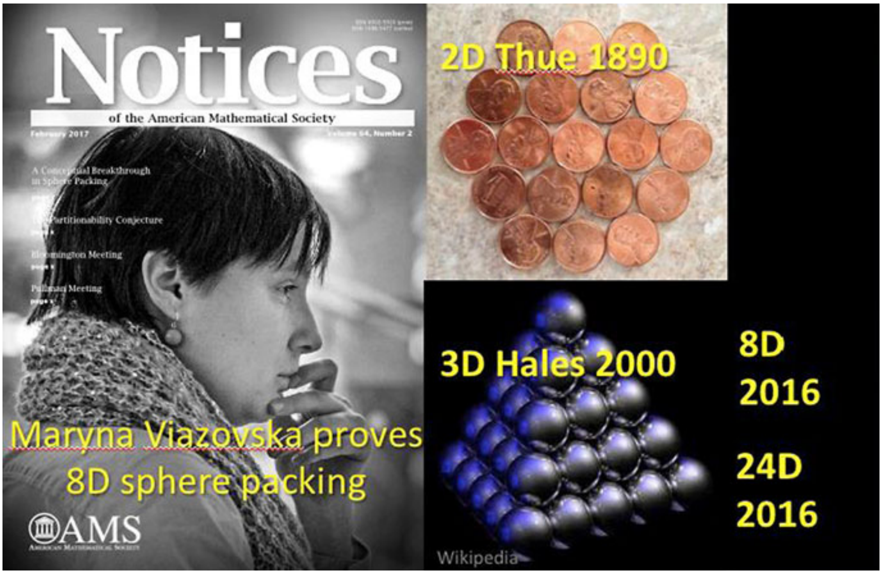
Figure 1. The minimizing shapes of thin films with radially symmetric strains (target metrics). Reprinted from Klein et al. [5] with permission from AAAS.

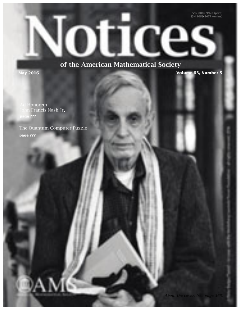
8 NOTICES OF THE AMS VOLUME 63, NUMBER 1

It was so popular that we repeated it in January 2017, along with a statement about diversity in mathematics. Everyone should feel welcome in mathematics. We're very aware of the continuing momentous need for diversity, but as we looked back over the past year, we found lots of positive signs, which we showcased in an opening editorial on *Diversity in Mathematics*.

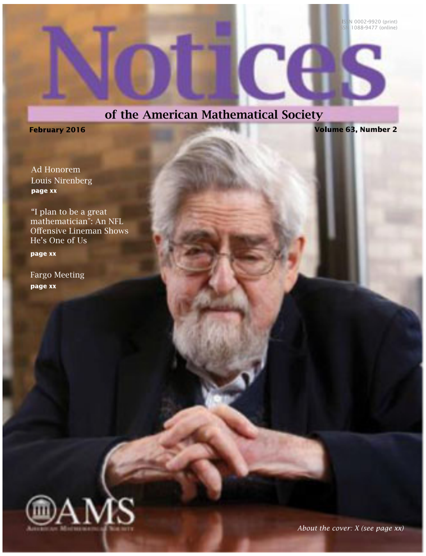
The February issue featured Maryna Viazovska's proof of optimal sphere packing in 8D. If you'd ever fiddled with coins on a tabletop, you'd have discovered that six fit perfectly around one in the center, and it's not too hard to prove that this is the densest packing, as was first proved by Axel Thue in 1890. 3D is much harder, because you can fit 12 spheres around a central sphere, it's not a perfect, tight fit, and worse, you can't continue putting 12 spheres around each of those 12 spheres. The long, complicated, most impressive proof was accomplished by Hales in 2000. In 8D the spheres fit perfectly again, so the proof is easier and shorter than in 3D. The next easy case is 24D, also proved by Viazovska with collaborators. February was the first-ever woman cover, but she wasn't there because she was a woman, rather because she proved 8D sphere packing.



Obituaries provide a popular way to present math from a more human perspective, especially for someone as famous and influential as John Nash, Nobel laureate in economics, victor over 20 years of schizophrenia, star of the book and movie *A Beautiful Mind*, and recent winner of the Abel prize, Norway's response in mathematics to Sweden's Nobel prize. Here he's receiving the prize from King Harald and talking to me at the dinner in the castle.

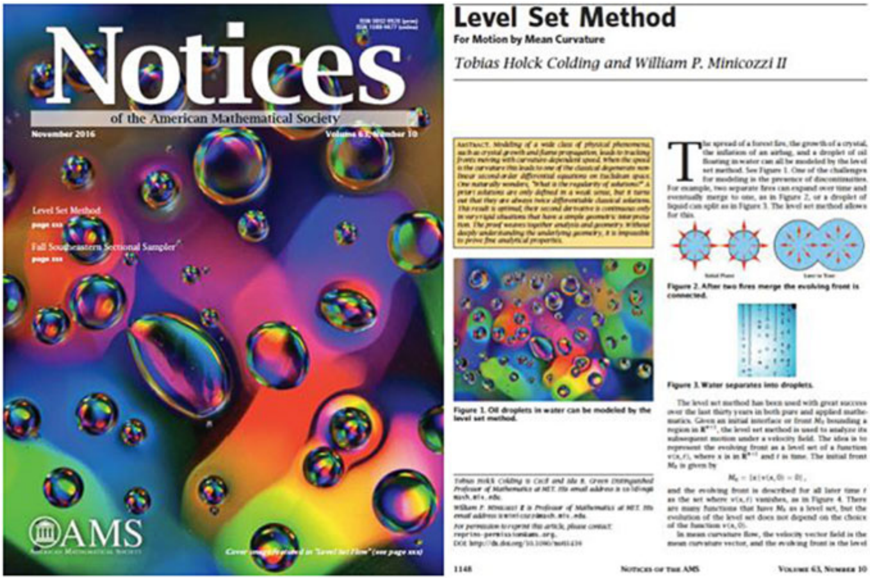



But why wait for someone to die? Louis Nirenberg, here in the wheelchair, also received the Abel prize. So we also featured him and his work on partial differential equations describing how things dependent on several variables change. You can rarely get exact solutions, and Nirenberg was famous for estimates and inequalities, which inspired an original cartoon on our new BackPage, where we also have quotes, stats, humor, and a caption contest.



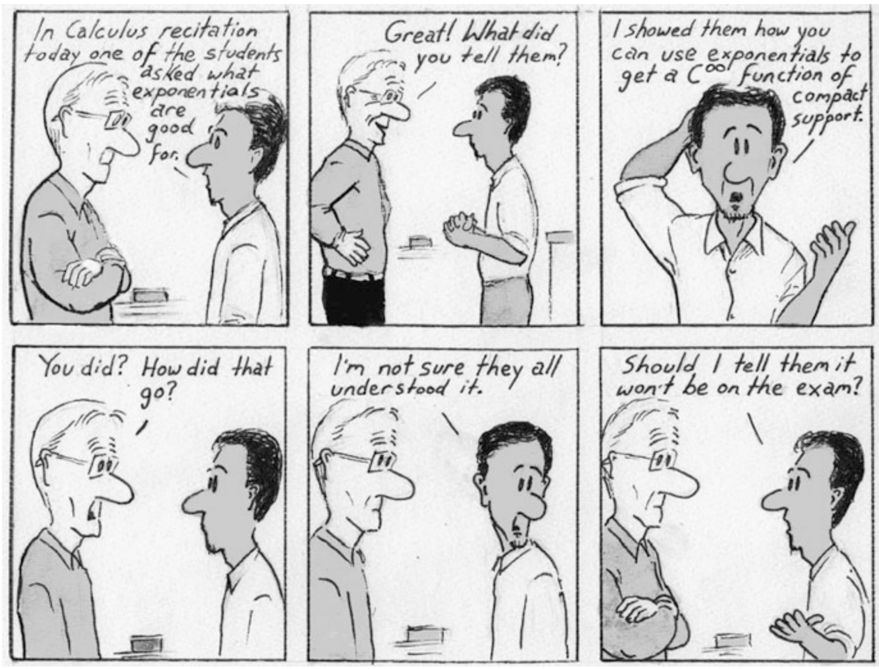
A major difficulty is how to prepare authors for the extensive revision and cutting often needed. I decided we had to tell them from the outset and added it to the posted information. Fortunately it doesn't seem to bother them at the start. Perhaps they think it's just irrelevant legal jargon. Perhaps they think it won't apply to them. Later on there's often some pushback, usually we can come to agreement, sometimes we just have to part.

Good pictures are alluring, we rarely get the quantity and quality we desire, but some articles come with beauties, like an article by Colding and Minicozzi about a general method for studying the evolution of surfaces at a rate proportional to curvature, similar to an evolution used by Perelman to prove the Thurston conjectures. We made it the cover story.



The January 2017 lecture sampler included a beautifully illustrated article by Ingrid Daubechies on the mathematics of restoring artwork by using wavelets to extract key features.

Since about a third of the AMS members are graduate students, we introduced a new graduate student section, with this major side benefit: authors writing for the graduate student section are more likely to make their articles readable by the average mathematician. We inducted the first graduate student member of any AMS editorial board, Alexander Diaz-Lopez, who has been doing an email interview for every issue. Also regularly in the *Graduate Student Section* is the ever popular *WHAT IS* feature. And we're learning how to extract what's most interesting for our readers. We used to have a long list in fine print of recent books about mathematics, but now we just feature three of the most interesting on our new BookShelf and put the rest online. Similarly we've made the *Mathematics People* and *Mathematics Opportunities* entries shorter and better illustrated. We have an original new comic strip entitled *My TA*, roughly modeled on our graduate student board member Alexander Diaz-Lopez and me.



We want everyone to find mathematics interesting and fun, and we're starting with the mathematicians.

My Life in the Shape of Maupertuis

Osmo Pekonen

It all started in 1998. I had noticed that the French mathematician Pierre Louis Moreau de Maupertuis (1698–1759) would have an anniversary year. A major international conference was in preparation in Berlin [8] where Maupertuis had once assumed the presidency of the Prussian Academy of Sciences. Probably something should be done in Finland, as well, I thought, given that Maupertuis had led in 1736–1737 a team of French academicians to Finnish Lapland—then part of the Kingdom of Sweden—to measure the shape of the Earth. I started putting up a small-scale Finnish-Swedish-French conference but little did I imagine that Maupertuis would swallow up the next twenty years of my life—and that I would progressively, on screen and stage, be transformed into Maupertuis myself.

Let us outline the story of this remarkable man. Pierre Louis Moreau de Maupertuis (see Fig. 1) was born in Saint-Malo in 1698. He was a son of René Moreau de Maupertuis, a privateer captain ennobled by the king of France. He started a military career as a Royal Musketeer but was soon won over to mathematics. He became a member of the Royal Academy of Sciences where he was an early supporter of Newtonian science in France. On a study trip to England he became convinced of the flattening of the Earth near the poles as predicted by Newton.

In early 18th century, cartographers of the leading naval powers, England and France, had begun suspecting that there was some systematical error in the atlases based on the hypothesis of a perfectly spherical Earth. Indeed, Isaac Newton had predicted in his foundational work *Principia* (1687) that the Earth is slightly flattened at poles, and bulges at the Equator, due to its rotational movement which diminishes the effect of gravitation as a function of the distance from the Equator. A flattened Earth has an oblate shape which in everyday language can be compared with a mandarin orange. In France, on the other hand, the leading geometers of the time, father and son Cassini, had come to a different conclusion based on their extensive field work done within the boundaries of France: the Earth should rather

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https://doi.org/10.1007/978-3-319-93949-0_6

Fig. 1 Pierre Louis Moreau de Maupertuis, the man who flattened the Earth. An engraving by Jean Daullé based on an oil painting (1739) by Robert Levrac-Tournières. The background scenery is from the village of Pello on the Arctic Circle

have a prolate shape like a lemon. Dubious arguments supporting this view were conjured up by appealing to the obsolete "vortex theory" due to René Descartes. As for the "fruity" images of 'mandarin orange' and 'lemon', they were actually in use in contemporary debate which opposed England to France and Newton to Descartes.

The French Royal Academy of Sciences endeavored to settle the issue once and for all by practical measurements on a global scale. For this purpose, two expeditions were launched. One expedition, under the leadership of Charles Marie de La Condamine, should sail to the Equator in South America, the other one, that of Maupertuis, as far North as possible, to perform the same geodesic task which was the precise measuring of the length of a one-degree arc of meridian. By a comparison of the results obtained, the shape of the Earth could be deduced. For an oblate spheroid, the arc lengths should increase when approaching the poles; for a prolate spheroid, they should decrease.

Maupertuis first thought of traveling to Iceland but was soon convinced that the conditions there would be overwhelmingly harsh. No scientific expedition had spent a winter in Arctic conditions before! Happily enough, the Swedish scientist Anders Celsius was sojourning on a study trip in Paris, and he could recommend the Tornio river valley in Northern Sweden as a promising venue for the measurements. The Tornio river could serve as a means of transportation through the wilderness while its frozen surface might provide a perfect Euclidean plane for the measurement of the base line. There was also the small town of Tornio, founded by Gustavus Adolphus in 1621, at the mouth of the river to guarantee hospitality and infrastructure.

Today, the Tornio river is the frontier between Sweden and Finland, and the city of Tornio lies on the Finnish side. So Maupertuis came to what is today Finnish Lapland with a team of four other members of the Royal Academy of Sciences (Charles Étienne Louis Camus, Alexis Claude Clairaut, Pierre Charles Le Monnier, abbé Réginald Outhier). Celsius joined in as a representative of Swedish science. The Venetian polymath Francesco Algarotti also dreamed of participation but his constitution was judged to be too frail for an Arctic adventure [10].

Celsius's original idea was to execute the measurements on the islands in the Gulf of Bothnia in front of Tornio but these were too low and too poorly visible from afar to be of any use. However, the expedition was able to take advantage of the orientation of the Tornio river, almost north-south, and measure the meridian arc along the river valley itself. A triangulation of the Tornio valley was conceived, using nine nodes. The southernmost starting point of the chain of triangles to be measured was the spire of the Suensaari church of Tornio; the eight other ones were the following fells from south to north: Niva, Kaakama, Huitaperi, Aavasaksa, Horila, Niemi, Pullinki, and Kittis (see Fig. 3). Today all of these mountain tops, except for Pullinki, are on the Finnish side of the border. The king of Sweden provided the expedition with a regiment of Finnish soldiers who were used to clear the tops of the fells and build markers on them.

The fog and the mosquitoes were a constant nuisance, and at one point the land surveyors accidentally ignited a forest fire. Nonetheless, at the end of the summer 1736, the suite of triangles was ready. Field work was to be combined with mathematics and astronomy. The next step was to measure the zenith distance of certain stars (the δ star of Dragon's constellation was mainly used) relative to the beginning and end of the triangulation, in order to calculate the latitude difference between those points. A one degree arc approximately corresponds to the distance between Tornio and the village of Pello at the northern end of the triangulation where the Frenchmen were housed in the Korteniemi manor.

The last practical task was to measure a base line on the frozen river, a task that was hampered by the intense cold (see Fig. 2). To warm up, the academicians consumed spirits brought from France as they were the only beverage that would not freeze. Even then, Le Monnier, the expedition's chief astronomer, got his tongue stuck to the silver flask that held his drink, and it was impossible to remove it without leaving part of its skin on it! By Christmas the field work was done. The French academicians stayed in Tornio, executing their calculations and waiting for the spring

Fig. 2 The five French academicians measuring the shape of the Earth in deep snow under the Northern lights. A 19th century fantasy picture by J. Ansseau. Source: Louis Figuier: *Vies de savants illustres* (1882)

to turn the Gulf of Bothnia navigable again, so that they could return home. The main mathematician responsible of the calculations was Clairaut. He later deduced a general formula for the dependence of the flattening on the latitude.

Maupertuis, and one of his companions, seem nonetheless to have had time to seduce two Lappish girls, Christine and Elisabeth Planström, who, to their surprise, later followed them to Paris causing a lot of embarrassment [2]. Once back in France, Maupertuis was immediately received by Louis XV at Versailles and acclaimed at the Academy as "the man who flattened the Earth". Voltaire wryly added that Maupertuis had flattened the Cassinis, as well. His achievement also provided a first major experimental proof of the Newtonian theory of gravity at large. Even so, doubts lingered on among his adversaries, led by the Cassini dynasty of geometers, until renewed measurements within France—with the participation of Cassini de Thury, a third generation geometer—finally settled the issue.

Maupertuis wrote a book on the expedition entitled *La Figure de la Terre* (1738); a great success, soon translated into English and with numerous reprints. Maupertuis became one of the first scientists to have his portrait painted like a nobleman (see Fig. 1). He appears in it wearing a Lappish costume, flattening the Earth with one hand. He ordered engravings out of the portrait, which he distributed throughout Europe. This was a media coup that not all of his colleagues appreciated. In any case, Maupertuis reached the zenith of his career when Frederick the Great invited

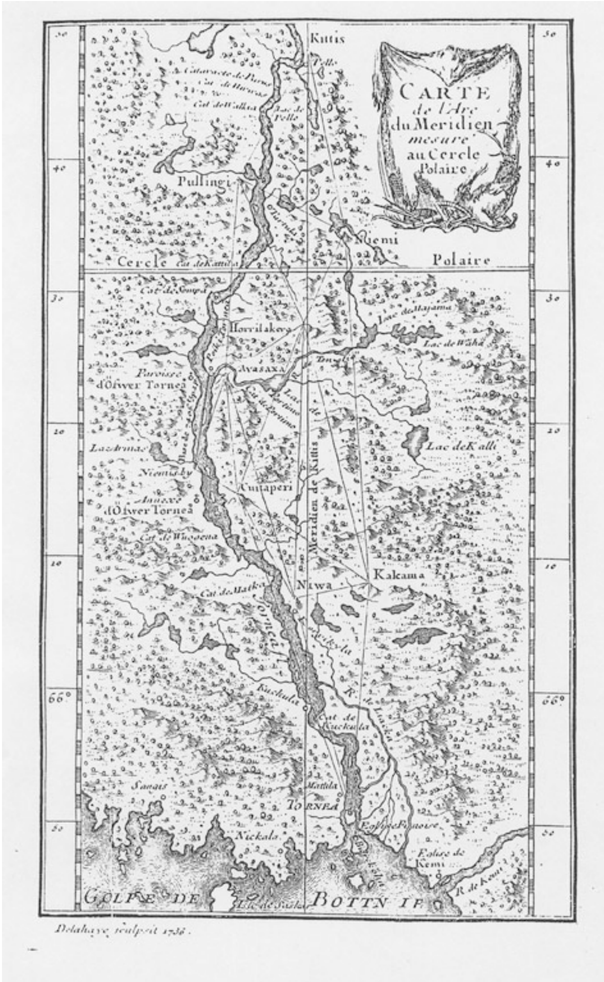

Fig. 3 The geodesic triangulation of the Tornio river valley as drawn by abbé Réginald Outhier in his *Journal d'un voyage au Nord* (1744)

him to be the president of the Prussian Academy of Sciences, founded in Berlin in 1700 by Leibniz. This was a position that aroused the jealousy of Voltaire who also had moved to Frederick's court. By the way, so had Algarotti whom Frederick made into a Prussian count.

Maupertuis and Voltaire were first friends as both championed for the Newtonian theory of gravity but their relationship soured in Berlin. Voltaire made fun of "the native of Saint-Malo" in a number of satirical texts, including his science fiction novel *Micromégas* (1752) where Maupertuis is being observed by two extraterrestrials from Saturn and Sirius who surprise him making love with the Lappish girls. Voltaire parodied Maupertuis as a greedy mad scientist named "Dr. Akakia". On the first day of Christmas 1752 Frederick ordered his executioner to burn Voltaire's two pamphlets and bring the ashes to Maupertuis but Voltaire relentlessly continued his attacks. Also the ridiculous figure of Pangloss in Voltaire's *Candide* (1759) may have been inspired by Maupertuis rather than by Leibniz. Maupertuis's posthumous fame suffered greatly. He passed away in Engelhof, the house of the Bernoullis in Basel. He could not be buried in Protestant Basel but found a very modest resting place in the Catholic church of Dornach, in the nearby canton of Solothurn.

Resurrecting Maupertuis

Maupertuis really was one of the great figures of the French Enlightenment but, until recently, he has remained somewhat forgotten even in his home country. Foreign scientists have contributed to improve Maupertuis's tarnished image. A comprehensive biography by Mary Terrall, professor of history of science at UCLA, rehabilitated Maupertuis's fame as a versatile major scientist [16]. Reviewing it spurred me to cast a closer look into the existing research literature.

Maupertuis was not a modest man. In his portrait, his travel companions are not seen at all—as if he had made the trip to Lapland, and flattened the Earth, all alone. I soon realized that one of Maupertuis's co-travelers in particular had been forgotten altogether, the Catholic priest abbé Réginald Outhier (1694–1774), a corresponding member of the Académie royale des sciences, who was a trained astronomer and an established cartographer but who also served as the chronicler of the expedition. His richly illustrated diary, published as *Journal d'un voyage au Nord* (1744), is a vivid account of the everyday business of the undertaking. It is also an important source as an acute ethnographical description of life in the Tornio river valley in the 1730s, indeed the earliest in its kind from that particular corner of the world.

So, there was a gap in the literature, and I embarked upon filling it with a study that soon grew into the proportions of a fully-fledged doctoral thesis. However, I decided to skip the mathematics of the arc measurement and to focus instead on the human side of the story that emerges so well from abbé Outhier's diary. Indeed, the calculations of the expedition have been perused over the centuries, and the entire measurement in the Tornio valley has been remade several times—for instance in 1846–1851 in conjunction with the measurement of the Struve Geodesic Arc that stretched from the Black Sea all the way to the Arctic Ocean. I chose as my topic the encounter, if not the clash, of cultures—and religions—between the elegant academicians from Paris and the Lappish natives as observed by a Catholic priest who had a keen eye on manifestations of Lappish lore and Lutheran religion. The resulting thesis that I defended in Rovaniemi, the capital of Finnish Lapland located on the Arctic Circle, was entitled *La rencontre des religions autour du voyage de l'abbé Reginald Outhier en Suède en 1736–1737* [11], and it earned me the somewhat surprising title of Doctor of Social Sciences of the University of Lapland. Mary Terrall acted as the president of the jury. I was already a Doctor of Philosophy, having written back in 1988 another thesis on Differential Geometry under the guidance of Jean-Pierre Bourguignon and Max Karoubi during my studies in Paris.

My thesis on abbé Outhier, written in French, earned me the Chaix d'Est Ange prize of the Académie des sciences morales et politiques of the Institut de France. A monument for abbé Outhier was unveiled in his home village La Marre in the Jura. I have also published, together with Anouchka Vasak, a commented critical edition of Maupertuis's writings from Lapland [13]. These texts, that include letters and scientific reports, had never been compiled in a single volume before. Besides my publications in French, I have edited a number of books on Maupertuis in Finnish. These include commented translations of the travel accounts by Outhier [9] and Maupertuis [3] and also a Finnish version of Mary Terrall's Maupertuis biography [16]. In total, I have written more than 2000 pages on Maupertuis by now.

Maupertuis's Other Achievements

Besides his heroic trip to Lapland, Maupertuis did many other things. Having become the president of the Berlin Academy he aimed high and started dabbling with Metaphysics and even Theology. He has a role in the history of the Calculus of Variations as the father of the Principle of Least Action (1744) which he believed would explain the ultimate origin of the Laws of Nature. He even ventured to propose a proof of the existence of God based on his principle—which provided additional laughing matter for Voltaire.

The Principle of Least Action is still relevant in today's Mathematical Physics even if nobody has been able to formulate the right kind of Action Principle for a Grand Unified Theory (sometimes called M-theory) of fundamental physics to explain the workings of the Universe. The problem set by Maupertuis in his metaphysical writings thus remains open. Finding an overall Action Principle would be tantamount to discovering the "Holy Grail" of fundamental science. Intriguingly, in a speech delivered to the Berlin Academy in 1747, Maupertuis claims to have seen in Lapland the ultimate knowledge about our Universe appearing in a faint writing on the mysterious Stone of Käymäjärvi that was sacred for the Lapps [5]! The story of his visit to that stone, together with Celsius who was known as an expert of runic writing, sounds like an episode of Indiana Jones. The Käymäjärvi Stone still exists, and it is still referred to as a sacred stone, or a *seita*, by the Lapps. We visited it back in 2003 with Juha Pentikäinen, professor of Comparative Religion and a renowned expert of Lappish lore, who was one of the advisors of my second doctoral thesis, but we saw no writing whatsoever appearing on the stone.

Maupertuis is sometimes regarded also as Darwin's forerunner as he formulated something like a principle of evolution and survival of the fittest in a biological treatise named *Vénus physique*, an intriguing text that I have edited in Finnish translation [4]. This astonishing text is half scientific speculation, half Maupertuis's personal erotic yearning. He was known as a restless womanizer. My recent translation frenzy has also included a philosophical treatise *Discours sur le bonheur* [6] written by one of his many girlfriends, Émilie du Châtelet (1706–1749), a major scientist in her own right. After a brief romantic involvement with Maupertuis, "la divine Émilie" spent fifteen happy years together with Voltaire in her castle of Cirey where she also received Algarotti. As a digression, let us mention that Madame du Châtelet, as well, has a connection with Finnish culture: The foremost contemporary Finnish female composer Kaija Saariaho has created an opera *Émilie*, to a libretto by the French academician Amin Maalouf, on her colorful life story, scientific achievement and dramatic death [12].

On Screen and Stage

Maupertuis's trip to Lapland—together with La Condamine's trip to the Equator in Peru—was the first time in the history of science when a major scientific undertaking was jointly funded by several governments. French and Swedish crowns both spent for Maupertuis's voyage to the Great North whereas France and Spain collaborated to organize the South American expedition. It was an early example of "Big Science", so to speak, and a landmark in the history of the exploration of our planet at large. Both expeditions ventured into difficult terrains, suffering great hardship both in the snowy plains of Lapland and in the green hell of the Amazonian rain forest that La Condamine's expedition had to cross on its way back home [7]. Both expeditions obviously would offer rich material for a cinematographic interpretation. To produce an adventure movie on La Condamine's tribulations in the Amazonian jungle would probably require the talent of another Werner Herzog whereas several other film directors have already tried their hand to make a movie on Maupertuis.

A well-known French documentarist, Yves de Peretti, began shooting a movie on Maupertuis in the summer of 1998 in conjunction with the tercentenary of our hero. I first served as a scientific consultant but soon was called to play the role of Maupertuis, as well. We couldn't afford stuntmen, so mathematicians were used instead, for instance to shoot the roaring rapids of the Tornio river! De Peretti with his team was supposed to return for winter scenes as well, but they may have lost heart because the winter of 1998–1999 was extremely cold in Lapland, with temperatures falling below -50 Centigrade. His movie was never accomplished. In 2009 another French movie team, directed by Joël Foulon and Alain-Michel Blanc, similarly hired me to play the role of Maupertuis but left their movie unfinished.

The movie idea resurfaced once again when I was contacted in 2013 by Axel Straschnoy, a visual artist born in 1978 in Buenos Aires who is now based in Helsinki. After careful planning, he shot in August 2014 in the Tornio river valley a small-budget movie on Maupertuis with two actors only. I starred as Maupertuis (in French; see Fig. 4) with my then doctoral student Johan Stén playing the role of Celsius (in Swedish; see Fig. 6). The outcome is a 24-minute film entitled *La figure de la Terre* which hovers somewhere between drama and documentary [15]. The approach is cheek-in-tongue at places because we didn't pretend to be entirely immersed in the 18th century even if we wore costumes and wigs of that era. And once again—in the absence of any stuntmen—we ended up shooting the grand rapids of Kukkolankoski in the Tornio river.

A projection of Axel Straschnoy's movie was my contribution in the Imagine Maths 6. *La figure de la Terre* (see Fig. 6) has also been screened in Helsinki, in

Fig. 4 Osmo Pekonen starring as Maupertuis. The venue is the Suensaari church of Tornio (1686) which served as the starting point of the measurement. The fruit in the vase serve to illustrate the shape of the Earth. Photo: Osmo Pekonen

Paris, in Seoul, and in Buenos Aires, so that for us with Johan Stén a career as internationally known movie stars is dawning as a promising alternative for our bread-and-butter work as historians of 18th century science. Meanwhile, Stén has defended a thesis on Anders Johan Lexell (1740–1784), a Finnish mathematician who became Euler's successor at the Saint Petersburg Academy [14].

The two of us have acquired costumes suiting our roles as 18th century scholars. Mine is from a specialized store in Paris whereas Johan's costume has been tailored in the Royal Swedish Opera. Beyond our brief stint as international movie stars, we have made good use of our expensive costumes and wigs as stage actors of the Finnish Lumières festival. It is an annual event mixing science and art taking place in Sveaborg (also called Suomenlinna), an 18th century fortress island in front of Helsinki. The Lumières festival which is always organized during the white nights of June, on the weekend before the feast of St. John the Baptist, is a high quality event of international standard where ancient music and dance alternate with reconstitutions of scenes of 18th century science. We have appeared on stage four times, incarnating figures of history of science such as Maupertuis (played by myself) and Celsius, Linné, or Lexell (played by Stén). Are we real actors—or do we only act so well that people are ready to believe so? Real actors or fake actors: the question is almost philosophical! In any case, we have sometimes appeared on stage together with high-level professional actors. Putting on the costumes tailored for us is a part of the "willing suspension of disbelief" so necessary to achieve the metamorphosis of two scholars into somewhat credible actors.

Fig. 5 The Finnish historians of science D.Phil., D.Soc.Sci. Osmo Pekonen and D.Phil., D.Tech. Johan Stén making career as movie stars. A promotional picture for the movie *La Figure de la Terre*. Reproduced with permission from Axel Straschnoy

Tempus fugit. Without hardly noticing I have spent twenty years of my life immersed in the elegant world of Maupertuis and his friends. Having played his role both on screen and stage, I have almost identified myself with a French colleague who lived three centuries ago. Indeed, besides our common profession as mathematicians, there are some common elements in our life stories. A long time ago, I did part of my military service as an artillery measurement officer of the Finnish armed forces in Lapland. In those days, we used very traditional methods of measurement, actually determining our position from the stars and using triangulation—just like Maupertuis! So I have concretely done essentially the same thing that he did in Lapland. Moreover, when I studied in France, I was a guest scholar in mathematics at the École Polytechnique which is a military academy, so I feel a little bit like a French musketeer, just like Maupertuis. By now, I have played his role so many times that Google Image Search gets crowded with pictures of me in his role.

Even if I am so favorably disposed, I haven't yet met with the ghost of Maupertuis. Even that would be possible if we are to believe a very serious report about Johann Gottlieb Gleditsch, a member of the Berlin Academy who encountered the departed soul of Maupertuis as a phantom appearing in the academy meeting rooms. Jean-Henri-Samuel Formey, the secretary of the academy, reported on this strange case on 12 February 1760 in a letter to his Venetian colleague Francesco Algarotti

Fig. 6 The poster for the Finnish premier of the movie *La Figure de la Terre* (2014). Reproduced with permission from Axel Straschnoy

(Algarotti: *Opere* 16:317) [1]. As a matter of fact, Sir Walter Scott reproduces this anecdote in his treatise *Letters on Demonology and Witchcraft* (1830). I hope that Maupertuis in the realm of shadows feels flattered—not flattened—by the fact that someone after three centuries keeps incarnating him on screen and stage.

References

1. F. Algarotti, *Opere del Conte Algarotti. 17 vols.* (Carlo Palese, Venice, 1791–1794)
2. O. Courcelle, Des fêtes du Pôle aux geôles et couvents de France: itinéraire de deux ‘Lapones’. *Cah. Voltaire* **8**, 47–62 (2009)
3. P.L. M. de Maupertuis, *La Figure de la Terre*. Paris (Väylä, Tornio, 1738/2018). Edited in Finnish by Osmo Pekonen (2018) as *Maan muoto ja muita kirjoituksia Lapista*
4. P.L. M. de Maupertuis, *Vénus Physique*. Paris (Väylä, Tornio, 1745/2017). Edited in Finnish by Osmo Pekonen (2017) as *Fyysinen Venus*
5. P.L. M. de Maupertuis, Relation d’un voyage fait dans la Lapponie septentrionale, pour trouver un ancien monument, in *Histoire de l’Académie Royale des Sciences et des Belles-Lettres 1747* (Ambroise Haude & J.C. Spener, Berlin, 1747/1749), pp. 432–445

6. É. du Châtelet, *Discours sur le Bonheur* (Hai, Kuopio, 2016), Edited in Finnish by Osmo Pekonen (2016) as *Tutkielma onnesta*
7. L.D. Ferreiro, *Measure of the Earth* (Basic Books, New York, 2011)
8. H. Hecht (ed.), *Pierre Louis Moreau de Maupertuis: Eine Bilanz nach 300 Jahren* (Verlag A. Spitz, Berlin, 1999)
9. R. Outhier, *Journal d'un Voyage au Nord*. Paris (Väylä, Tornio, 1744/2011). Edited in Finnish by Osmo Pekonen (2011) as *Matka Pohjan perille*
10. M. Pastore Stocchi, G. Pizzamiglio (eds.), *Nel Terzo Centenario della Nascita di Francesco Algarotti (1712–1764)* (Istituto Veneto di Scienze, Lettere ed Arti, Venice, 2014)
11. O. Pekonen, *La rencontre des religions autour du voyage de l'abbé Réginald Outhier en Suède en 1736–1737* (Lapland University Press, Rovaniemi, 2010)
12. O. Pekonen, L. Hautsalo, Émilie, an opera about love, death, and mathematics, in *Proceedings of Bridges 2017: Mathematics, Art, Music, Architecture, Education, Culture*, ed. by D. Swart, C.H. Séquin, K. Fenyvesi (Tessellations Publishing, Phoenix, 2017), pp. 387–390.
13. O. Pekonen, A. Vasak, *Maupertuis en Laponie. À la recherche de la figure de la Terre* (Hermann, Paris, 2014)
14. J.C.-E. Stén, *A Comet of the Enlightenment. Anders Johan Lexell's Life and Discoveries*. Vita Mathematica, vol. 17 (Birkhäuser, Basel, 2015)
15. A. Straschnoy, *La Figure de la Terre*. A movie. Duration: 00:23:50. Picture ratio: 16:9. Sound format: Dolby Surround 5.1. Producer (Kolme Perunaa, Helsinki, 2014)
16. M. Terrall, *The Man Who Flattened the Earth. Maupertuis and the Sciences in the Enlightenment*. (Chicago University Press, Chicago, 2002/2016). Translated into Finnish by Osmo Pekonen (2016) as *Maupertuis, maapallon muodon mittaaja* (Väylä, Tornio)

Part III
Mathematics & Music

From Music to Mathematics and Backwards: Introducing Algebra, Topology and Category Theory into Computational Musicology

Moreno Andreatta

The “Mathemusical” Dynamics Between Music and Mathematics (via Computer Science)

In western tradition, mathematics and music have been connected for more than 2000 years. Despite this long history of relations between the two fields, the interest of “working mathematicians” in this research domain is a rather new phenomenon [9]. Whilst the power of applying mathematics to music has been acknowledged for a long time, it is only thanks to more recent developments that also music begins to occupy an important place in the development of mathematics. In fact, music has shown to provide a number of difficult theoretical problems, in particular for what concerns their constructive formalization and algorithmic solution. This asks for a permanent feedback between musical thought, mathematical formalisation and computational modelling, a dynamic movement between different disciplines that I suggested to call “mathemusical” [3]. This feedback between music and mathematics via computer science is illustrated in the diagram of Fig. 1.

This paper provides an overview of the ongoing SMIR (Structural Music Information Research) Project, supported by the University of Strasbourg Institute for Advanced Study and carried on at the Institut de Recherche Mathématique Avancée (IRMA) in collaboration with the GREAM (Groupe de Recherche Expérimentale sur l’Acte Musical) and the Institut de Recherche et Coordination Acoustique/Musique (IRCAM). For a description of the institutional aspects of the project, including the list of participants and past and future events, see the official webpage: http://repmus.ircam.fr/moreno/smir.

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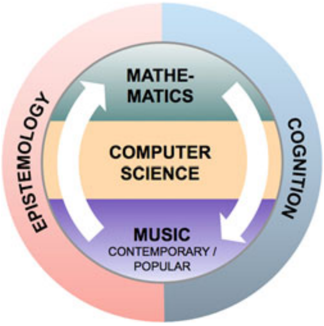

Fig. 1 A diagram showing the "mathemusical" dynamics between music and mathematics via computer-science

Among the music-theoretical problems showing a remarkable link with interesting mathematical constructions and open conjectures, one may quote the following ones that have provided the content of a considerable number of Master and PhD dissertations:[1]

- Tiling rhythmic canons and their spectral dimensions (via the Fuglede Conjecture)
- Z-relation in music theory and the study of homometric structures in crystallography
- Transformational music theory and the categorical classification of direct musical graphs
- Neo-Riemannian music analysis, spatial computing and Formal Concept Analysis (FCA)
- Diatonic Theory, Maximally Even Sets and the Discrete Fourier Transform
- Periodic sequences and finite difference calculus
- Chord classification and combinatorial block-designs in music composition

Interestingly, most of these problems are deeply interrelated showing the existence of a remarkable interplay between algebraic formalization and geometric representations of musical structures and processes.[2] Moreover, the mathemusical dynamics (from music to mathematics to music via computer science and the possible epistemological and cognitive implications) constitutes a radical change of perspective with respect to the traditional application of mathematics in the musical domain.

[1] See http://repmus.ircam.fr/moreno/production for the complete list of students' work focusing on these aspects of the relations between music and mathematics.

[2] This interplay also provides a further example of the duality between temporal and spatial constructions which are the two fundamental ingredients of music according to the field medalist Alain Connes. As he suggested in a conversation with composer Pierre Boulez on the analogy and the difference between the creative processes in mathematics and music: "Concerning music, it takes place in time, like algebra. In mathematics, there is this fundamental duality between, on the one hand, geometry—which corresponds to the visual arts, an immediate intuition—and on the other hand algebra. This is not visual, it has a temporality. This fits in time, it is a computation, something that is very close to the language, and which has its diabolical precision. [...] And one only perceives the development of algebra through music" [15].

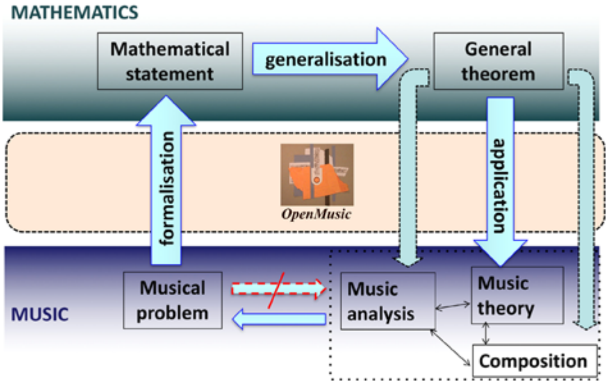

Fig. 2 A more detailed perspective on the "mathemusical" research diagram of Fig. 1, with the indication of the three main ingredients of the dynamics (formalisation, generalisation and application). The computer science is represented with the icon of *OpenMusic*, a visual programming language for computer-aided music theory, analysis and composition currently developed at IRCAM and integrating—in its *MathTools* environment—most of the computational constructions derived from the "mathemusical" research

Mathemusical problems are characterised by the fact that settling them in an appropriate mathematical framework not only gives rise to new musical applications, but also paves the way to new mathematical constructions. By analysing more carefully the different steps of this mathemusical dynamics, one observes that it can be decomposed into the following three stages:

– *Formalization*: the initial music-theoretical problem is approached by means of a combination of mathematical tools enabling to formalize it and revealing its computational character
– *Generalization*: the formalized problem is generalized by using a panoply of mathematical constructions, ranging from abstract algebra to topology and category theory and leading to general statements (or theorems)
– *Application*: once a generalized result has been obtained, it can be applied to music by focusing on one of the three main aspects, i.e. the theoretical, the analytical and the compositional one.[3]

This decomposition of the mathemusical dynamics, together with the triple perspective of the possible musical applications of a general result is shown in Fig. 2.

It is this fruitful double movement, from music to mathematics and backwards, which is at the heart of a growing international research activity where computer science is positioned in the middle of this feedback, as an interface for connecting the musical and mathematical domains. We simplify the picture by considering a homogeneous intermediate level corresponding to the place occupied by computer

[3] As largely documented, these three aspects are deeply interconnected, particularly in Twentieth-Century music and musicology. See [3] for a detailed account of music-theoretical, analytical and compositional applications of the algebraic methods in contemporary music research.

science with respect to music theoretical and mathematical research. By analyzing more carefully the different music theoretical problems, one may nevertheless distinguish the cases in which the computer-aided models are directly built in the formalization process as in the case of problems asking for a computational exploration of the solution space, more than a search of a general underlying mathematical theory. To this family belong, for example, typical enumeration problems such as the classification of all possible Hamiltonian paths and cycles within music-theoretical geometric spaces, such as the *Tonnetz* and their multi-dimensional extensions [8]. Conversely, there are cases in which the computational models are built starting from some general algebraic results, as in the case of the construction of tiling rhythmic canons corresponding to the decomposition of a cyclic group of order n into a direct sum of two non-periodic factors (i.e. two subsets of periodicity equal to n).[4] The ubiquitous role of computational modeling in the mathemusical dynamics clearly show that one may balance the usual Leibnizian perspective of music as an *exercitium arithmeticae* by proposing that also the reverse hypothesis holds, according to which mathematics can be considered, in some special cases, as an *exercitium musicae* [4].

The new interplay which I propose to establish among algebra, topology and category theory in the service of computational musicology is also necessary to successfully tackle difficult "mathemusical" problems which are linked to open conjectures in mathematics. This is the case of two major problems that have been the object of study in the last fifteen years and which can be approached in a new way: the construction of *tiling rhythmic canons* and the classification of *homometric musical structures*. Tiling canons are special rhythmic canons having the property of tiling the time axis by temporal translation of a given rhythmic pattern [5]. This compositional process is deeply connected with an open conjecture in mathematics dating from the 1970s, i.e. Fuglede or "Spectral Conjecture" [7]. This conjecture turns out to be also linked to homometry theory, a field in mathematical combinatorics that originates in crystallography, where one may find crystals having the same X-ray spectrum without being isometric. Analogously, composition naturally provides examples of musical homometric structures having the same distributions of intervals but being not equivalent up to elementary musical transformations such as transpositions or inversions. The deep connection between *tiling* and *homometry* comes from the observation that if a rhythmic pattern tiles the musical line by translation (i.e. it generates a tiling rhythmic canon), so does any rhythmic pattern that is homometric to the initial one. Moreover, in this tiling process one only has to consider tiling canons associated to factorizations of a cyclic group as a direct sum of two *non-periodic* subsets, since all the other canons verify Fuglede's Conjecture [2].

This new perspective on the mathematical relevance of many music-theoretical constructions, with an emphasis on their computational character, probably played an important role in the change of perspective by mathematicians on music and mathematics as a research field. This led in 2007 to the constitution of an interna-

[4] See [7] for a description of Tiling Canons as a key to approach open Mathematical Conjectures.

tional society (the "Society for Mathematics and Computation in Music")[5] and the launching of the first mathematical Journal devoted to "mathemusical" research (the "Journal of Mathematics and Music", edited by Taylor and Francis).[6] The recognition of the mathematical dimension of the research carried on in this domain enabled the inscription in 2010 of "Mathematics and Music" as an official research field within the Mathematics Subject Classification of the American Mathematical Society (under the code 00A66). Although the research conducted by the members of this community mainly focused on classical or contemporary art music, there is a growing interest on *popular music*, whose theoretical problems are often as deep as those belonging to contemporary music. The SMIR project clearly suggests the necessity to push the boundaries between music *genres* and to look at new possibilities of interaction between contemporary art music and popular music, both providing rich music-theoretical constructions for possible collaborations between mathematicians, computer scientists, musicologists and composers.

The Originality of a Structural Approach in MIR (Music Information Research)[7]

More specifically, the SMIR project proposes to approach "mathemusical" research by pushing further the fruitful interplay between algebraic computations, topological representations and categorical formalisations. It has two interdisciplinary levels. The first interdisciplinary degree arises within the field of music itself by treating simultaneously, as we have seen, a wide spectrum of different musical genres, ranging from contemporary art music to popular music, including rock, pop, jazz and *chanson*. Surprisingly, far from being easy to formalize, the compositional process in popular music can be a very rich research domain, particularly once the harmonic structures are represented in a topological way through the formalization of the Tonnetz and their generalized versions as simplicial complexes [13, 14]. Figure 3 shows the construction of the Tonnetz as a simplicial complex starting from the topological representation of notes (0-cells), intervals (1-cells), 3-note chords (2-cells) and their self-assembly by identification of a common edge.

The second degree of interdisciplinarity of the project relies in the use of different mathematical concepts in a *structural approach* which is in contrast to the current state of Music Information Research (MIR), mainly relying on the application of *statistical methods* in signal processing. Instead of focusing on the signal content, our approach emphasizes the symbolic aspects of musical representations and their structural algebraic, topological and categorical formalisations. For example, a musical chord progression can be seen as a family of subsets of a cyclic or

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[5] See http://www.smcm-net.info/.

[6] See http://www.tandfonline.com/toc/tmam20/current.

[7] Following the Roadmap described in [41], we prefer to consider MIR as the field of Music Information Research instead of limiting the scope of purely Music Information Retrieval.

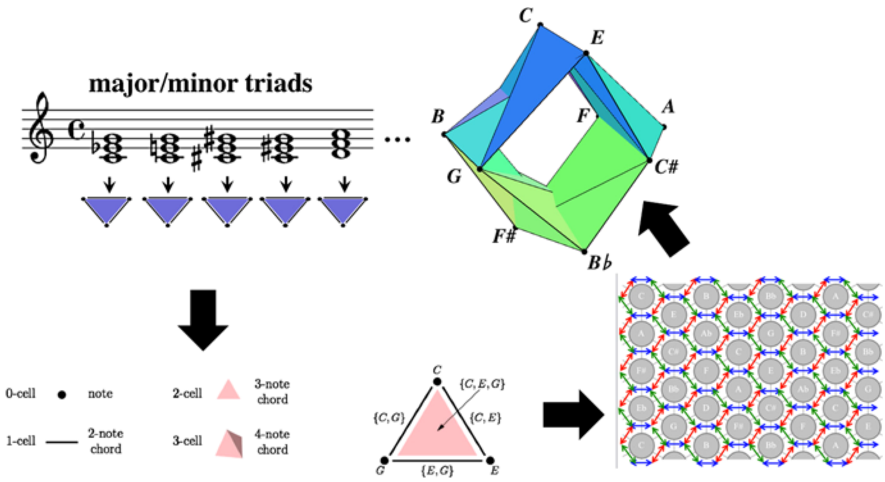

Fig. 3 The Tonnetz as a simplicial complex obtained by self-assembly of major and minor chords viewed as 2-cells. See [13]

dihedral group, as a path in a simplicial complex space or, in a more abstract way, as a collection of functors and natural transformations. As an example, let us take the generating hexagonal shape of the Tonnetz structure (see Fig. 4). The simplest way to describe this shape is to consider it as a collection of symmetries relying major and minor chords. More precisely, according to the neo-Riemannian music analysis,[8] there are three ways of transforming a major chord into a minor chord by preserving two common notes and these three symmetries correspond to the **R**, **P** and **L** transformations. The **R** transformation (as "relative"), changes for example the C major chord into the A minor chord, whereas the **P** transformation (as "parallel") and the **L** transformation (as "Leading-Tone operator") change respectively the C major chord into the C minor and the E minor chord. In a categorical framework, generalizing the K-net theory (or Klumpenhouwer Networks), the neo-Riemannian transformations are viewed as natural transformations between major and minor chords represented as labelled graphs with vertices corresponding to the notes and arrows corresponding to transposition and inversion operations [35]. By definition a transposition by $h$semi-tons is an operation indicated by T_h that sends a generic element $x$of the cyclic group of order 12 (i.e. a pitch-class in the musical set-theoretical terminology) into $x + h$ (modulo 12). Similarly, one may define the generic inversion operation indicated by I_k that sends a pitch-class x into $k - x$ (always modulo 12).

[8]This approach takes origin in the writings of the German musicologist Hugo Riemann who proposed a "dualistic" perspective of Euler's Tone System [23] based on inversional relations between major and minor chords. After a first algebraic formalisation by David Lewin through the concept of GIS or Generalized Interval System [26], neo-Riemannian theory and analysis has progressively integrated mathematical concepts belonging to topology and algebraic geometry [11] and shown its relevance to the analysis of a popular music repertoire [13].

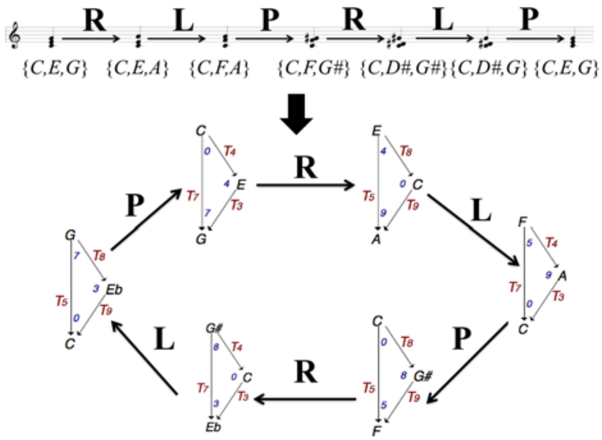

Fig. 4 The hexagonal cycle generating the Tonnetz and represented in a categorical framework displaying the transformations between major and minor chords viewed as labelled graphs with vertices corresponding to the notes and arrows corresponding to transposition operations. See [36]

One of the interests of using a categorical approach instead of a group-action based approach in the representation of musical structures such as chords or melodic patterns relies in the possibility to provide a flexible notion of equivalence relation, not necessarily linked to an underlying group action. More precisely, in the usual enumeration and classification of harmonic structures, two chords A and B are considered as equivalent if there exists a musically-relevant[9] group Gacting on the family of all possible chord structures (i.e. the family of subsets of the cyclic group of order 12 seen as the space of pitch-classes where the algebraic structure has been forgotten) and such that $A$and $B$are related via a transformation f belonging to G. Music offers many example where one would feel the need to establish a formal equivalence relation between chords which are not orbits with respect to an underlying group action. The categorical approach[10] enables precisely to overcome

[9]To the class of "musically-relevant" groups acting on the family of all possible chords belong groups such as the *cyclic group* of order 12 (or group of transpositions), the *dihedral group* of order 24 (or group of transposition and inversions) and the *affine group* of order 48 (or group of "augmentation", i.e. applications f of the form $f(x) = ax + b$, where a belongs to the set of invertible elements of $\mathbf{Z_{12}}$ and b is any possible transposition factor). By using a term which has a strong philosophical meaning [25], we suggested to call "paradigmatic" a classification approach of musical structures based on an underlying group action [3]. This provides an elegant formalization of the most common chord catalogues, from Anatol Vieru's catalogue of transposition classes of chords [42] to Mazzola/Morris catalogue of affine orbits [27, 31], including Julio Estrada's catalogue of "identities" [17].

[10]Category theory was originally introduced in music theory by Guerino Mazzola in his dissertation *Gruppen und Kategorien in der Musik* [27] and further extended in *Geometrie der Töne* [28] and *The Topos of Music* [29]. For an alternative approach to the categorical formalization of music theory, see Fiore and Noll [18] and our series of papers dealing with the categorical interpretation of Klumpenhouwer Networks, initially within the framework of *Topos of Music* [30] and,

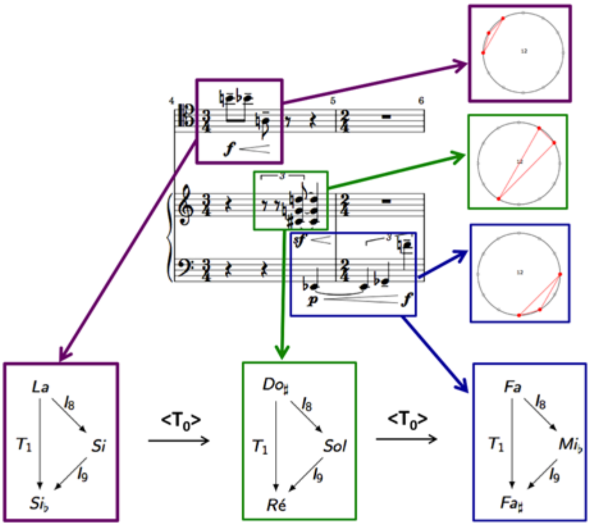

Fig. 5 Three chords belonging to Anton Webern's *Drei Kleine Stücke*, Op. 11/2 which literally share the same type of transpositions and inversions without being in the same orbit under the action of the cyclic, dihedral or affine groups. See [36]

some limitations of the paradigmatic approach by establishing isomorphic relations between configurations of elements (i.e. objects and morphisms between objects) instead of pre-existing "types" of chords. An example is shown in Fig. 5 where three chords are "strongly isographic" (i.e. they have the same configuration of arrows) despite the fact that they do not belong to the same orbit under the action of any the three "musically-relevant" groups we mentioned before (cyclic, dihedral and affine groups).

A (Very) Short Journey Through Some Tools and Research Axes of the SMIR Project

As we have seen by briefly describing some examples of the interplay between algebraic, topological and categorical approaches, the SMIR project proposes to use structural approaches in Music Information Researches based on advanced mathematics. These approaches include the search of algebraic invariants to classify musical structures, the use of simplicial complexes to represent musical spaces, the use of Galois lattices and ordered structures, together with persistent homological tools and functorial approaches to describe graph-theoretical musical constructions. These theoretical concepts are systematically accompanied by computational modelling including *spatial computing*, a non-conventional paradigm in computer science aiming to reformulate in spatial terms the data structures and their formal manipulations, as in the case of the simplicial representation of the Tonnetz and its possible generalisations. The panoply of tools and theoretical constructions which

successively, through the set-up of generalized K-nets called "Poly-Klumpenhouwer Networks" [35–37].

are potentially useful within a structural approach to music information research can be classified into the three main research axes that will be shortly described in the following sections.

The Interplay Between Mathematical Morphology and Formal Concept Analysis in Computational Musicology

By using some recent results that we obtained concerning the connections between topology, music and Formal Concept Analysis [19, 39], on the one hand, and some existing relations between Mathematical Morphology [40] and Formal Concept Analysis [20] on the other hand [10], the project will investigate how to combine Formal Concept Analysis and Mathematical Morphology in order to approach in a new way some classical problems of Music Information Research. One of the most prominent examples is the automatic retrieval of musical structure. Some promising results have been obtained by Pierre Relaño in his Master dissertation devoted to the application of techniques developed within the field of mathematical morphology to the lattice representation of musical structures, with a special focus on harmony [38].

Generalized Tonnetze, Persistent Homology and Automatic Classification of Musical Styles

The SMIR project will build on some recent findings concerning the simplicial complexes and neo-Riemannian music analysis [13, 14] in order to study the problem of automatic classification of musical styles via a purely topological approach based on techniques such as *persistent homology*, which appears as a fundamental tool in the field of *Topological Data Analysis*. As in the case of this new research area that emerged from the application of computational topology in data analysis, by computing persistent homology from musical data sets it is possible to characterise the underlying musical space with a “topological signature” that reveals its structural properties. Some interesting results have been obtained by Mattia Bergomi in his PhD dissertation that is the first doctoral thesis devoted to the application of persistent homology to automatic style classification [11]. This approach has been successively applied to a structural computational analysis of popular music [12].

Category Theory and Transformational (Computer-Aided) Music Analysis

Several other “mathemusical” problems emerged in the recent years, which ask for extending the mathematical framework by also including tools and constructions belonging to other fields of mathematics, in particular category theory. As a theory of abstract mathematical structures, category theory is in fact particularly suitable for unifying different music-theoretical constructions. As we have shown, it constitutes the natural mathematical framework for the so-called "transformational music theory" [35]. Following our first attempt at developing a category-based approach to creativity [6], we will explore in this project different categorical constructions at the basis of the creative process in music analysis and composition. As we suggested elsewhere [1], category theory also provides very powerful conceptual tools that can have crucial theoretical implications for cognitive sciences and mathematical psychology. We strongly believe that the fact of coupling an algebraically formalized geometrical approach, such as the transformational one, with a computational perspective has some crucial theoretical implications for cognitive sciences and mathematical psychology. One simple way to have the intuition of this change is to compare the transformational approach in music analysis with some mathematically-oriented directions in developmental psychology and cognition, such as Halford and Wilson's neostructuralistic approach [22], Ehresmann and Vanbremeersch's Model of Memory Evolutive Systems [16] and Phillips and Wilson's Categorical Compositionality [32]. From an epistemological point of view, transformational analysis provides an instantiation, in the music domain, of Gilles-Gaston Granger's articulation between the "objectal" and the "operational" dimensions [21]. This duality was considered by the French epistemologist as the foundational basis for the very notion of "concept" in philosophy.[11] From the perspective of developmental psychology, among the three problematics which—according to the psychologist Olivier Houdé—mark the renewal of Piaget's genetic epistemology, category theory occupies a central place [24]. Differently from the structural approach which Piaget developed starting from his logical treatise [33] and which also constitutes the conceptual framework of his researches on the "*abstraction réfléchissante*", category theory introduces, according to Houdé, a new element in the operational thinking. Morphisms enable to take into account an "aspect of logical-mathematical cognition which does not proceed from the transformation of the reality (operations and grouping of operations) but which takes into account the simple relational activity [*mise en relation*]".[12] Being capable of integrating these epistemological and cognitive aspects within a theoretical research will be one of the major challenges of the SMIR project. Starting from the reflections of mathematicians on the phenomenological account of contemporary mathematics, and comparing these authors with some more epistemological orientations on the cognitive aspects of the phenomenological method, the structural Music Information Researcher might find the way to constitute a new conceptual space within which some mathematical problems raised by music open new perspectives enabling to enrich the philosophical quest. This would surely lead to a better understanding of the interplay between algebraic formalizations, topological descriptions and categorical representations of musical structures and processes.

[11] See, in particular, the article "Contenus formels et dualité", reprinted in [21].

[12] See [24] as well as Piaget's posthumous *Morphismes et catégories* [34].

References

1. E. Acotto, M. Andreatta, Between mind and mathematics. Different kinds of computational representations of music. Math. Soc. Sci. **199**, 9–26 (2012)
2. E. Amiot, *Music Through Fourier Space: Discrete Fourier Transform in Music Theory*. Computational Music Science (Springer, Heidelberg, 2016)
3. M. Andreatta, Méthodes algébriques dans la musique et musicologie du XX^{e} siècle: aspects théoriques, analytiques et compositionnels. PhD thesis in computational musicology, Ecole des hautes études en sciences sociales/IRCAM, Paris (2003)
4. M. Andreatta, Mathematica est exercitium musicae. La recherche 'mathémusicale' et ses interactions avec les autres disciplines. Habilitation Thesis, IRMA/University of Strasbourg (2010)
5. M. Andreatta, C. Agon (eds.), Special Issue "Tiling Problems in Music". J. Math. Music **3**(2) (2009).
6. M. Andreatta, A. Ehresmann, R. Guitart, G. Mazzola, Towards a categorical theory of creativity, in *Mathematics and Computation in Music 2013*, ed. by J. Yust, J. Wild, J.A. Burgoyne. LNCS/LNAI, vol. 7937 (Springer, Heidelberg, 2013), pp. 19–37
7. M. Andreatta, Tiling Canons as a key to approach open mathematical conjectures? in *Mathemusical Conversations*, ed. by E. Chew et al.(Wiley, New York, 2016), pp. 86–104
8. M. Andreatta, G. Baroin, Formal and computational models in popular music, in *Aesthetics & Neurosciences: Scientific and Artistic Perspectives*, ed. by Z. Kapoula et al.(Springer, Berlin, 2016), pp. 257–269
9. G. Assayag, J.F. Rodrigues, H.G. Feichtinger (eds.), *Mathematics and Music: A Diderot Mathematical Forum* (Springer, Berlin, 2002)
10. J. Atif, I. Bloch, F. Distel, C. Hudelot, Mathematical morphology operators over concept lattices, in *Formal Concept Analysis. ICFCA 2013*, ed. by P. Cellier, F. Distel, B. Ganter. Lecture Notes in Computer Science, vol. 7880 (Springer, Berlin, 2013)
11. M. Bergomi, Dynamical and topological tools for (modern) music analysis. PhD thesis, UPMC/LIM, Milan (2015)
12. M. Bergomi, F. Fabbri, M. Andreatta, Hey Maths! Modèles formels et computationnels au service des Beatles. Volume! La revue des musiques populaires, ed. by G. Tosser, O. Julien (2015), pp. 161–177
13. L. Bigo, M. Andreatta, Topological structures in computer-aided music analysis, in *Computational Music Analysis*, ed. by D. Meredith (Springer, Berlin, 2015), pp. 57–80
14. L. Bigo, D. Ghisi, A. Spicher, M. Andreatta, Spatial transformations in simplicial chord spaces, in *Proceedings ICMC/SMC*, ed. by A. Georgaki, G. Kouroupetroglou, Athens, Greece (2014), pp. 1112–1119
15. P. Boulez, A. Connes, Creativity in music and mathematics. A meeting organized within the Mathematics and Computation in Music Conference at IRCAM, 15 June 2011. Video available at http://agora2011.ircam.fr
16. A.C. Ehresmann, J.-P. Vanbremeersch, *Memory Evolutive Systems: Hierarchy, Emergence, Cognition* (Elsevier, Amsterdam, 2007)
17. J. Estrada, Théorie de la composition: discontinuum-continuum. PhD thesis, Université de Strasbourg (1994)
18. T. Fiore, T. Noll, Commuting groups and the topos of triads, in *Mathematics and Computation in Music 2011*, ed. by C. Agon, E. Amiot, M. Andreatta, G. Assayag, J. Bresson, J. Mandereau. LNCS/LNAI, vol. 6726 (Springer, Heidelberg, 2011), pp. 69–83
19. A. Freund, M. Andreatta, J.-L. Giavitto, Lattice-based and topological representations of binary relations with an application to music. Ann. Math. Artif. Intell. **73**(3–4), 311–334 (2015)
20. B. Ganter, R. Wille, *Formal Concept Analysis: Mathematical Foundations* (Springer, Berlin, 1998)
21. G.-G. Granger, *Formes, Opérations, Objets* (Librairie Philosophique J. Vrin, Paris, 1994)
22. G.S. Halford, W.H. Wilson, A category-theory approach to cognitive development. Cogn. Psychol. **12**, 356–411 (1980)

23. H. Hascher, A. Papadopoulos (eds.), *Leonhard Euler. Mathématicien, Physicien et Théoricien de la Musique* (Editions du CNRS, Paris, 2015)
24. O. Houdé, La référence logico-mathématique en psychologie: entre méthode universelle et rationalité arrogante, in *Pensée Logico-Mathématique, Nouveaux Objets Interdisciplinaires*, ed. by O. Houdé (Presses Universitaires de France, Paris, 1993), pp. 47–119
25. T. Kuhn, *The Structure of Scientific Revolutions* (University of Chicago Press, Chicago, 1962)
26. D. Lewin, *Generalized Musical Intervals and Transformations* (Yale University Press, New Haven, 1987)
27. G. Mazzola, *Gruppen und Kategorien in der Musik* (Hermann, Paris, 1985)
28. G. Mazzola, *Geometrie der Toene* (Birkhäuser, Basel, 1990)
29. G. Mazzola, *The Topos of Music* (Birkhäuser, Basel, 2002)
30. G. Mazzola, M. Andreatta, From a categorical point of view: K-nets as limit denotators. Perspect. New Music **44**(2), 88–113 (2006)
31. R. Morris, *Composition with Pitch-Classes: A Theory of Compositional Design* (Yale University Press, New Haven, 1988)
32. S. Phillips, W.H. Wilson, Categorical compositionality: a category theory explanation for the systematicity of human cognition. PLoS Comput. Biol. **6**(7), 1–14 (2010)
33. J. Piaget, in *Traité de Logique, Essai de Logistique Opératoire*, (A. Colin, Paris, 1949)
34. J. Piaget, G. Henriques, E. Ascher, *Morphismes et Catégories: Comparer et Transformer* (Delachaux et Niestlé, Lausanne, 1990)
35. A. Popoff, M. Andreatta, A. Ehresmann, A categorical generalization of Klumpenhouwer networks, in *Mathematics and Computation in Music 2015*, ed. by T. Collins, D. Meredith, A. Volk. LNCS/LNAI, vol. 9110 (Springer, Heidelberg, 2015), pp. 303–314
36. A. Popoff, C. Agon, M. Andreatta, A. Ehresmann, From K-nets to PK-nets: a categorical approach. Perspect. New Music **54**(2), 5–63 (2017)
37. A. Popoff, M. Andreatta, A. Ehresmann, Relational PK-Nets for transformational music analysis. J. Math. Music **12**(1) (2018, in press). arXiv:1611.02249
38. P. Relaño, Morphologie mathématique, FCA et musicologie computationnelle. Master thesis, ENS-Lyon/LTCI/Télécom ParisTech/LAMSADE, Université Paris Dauphine/IRCAM-CNRS-UPMC (2017)
39. T. Schlemmer, M. Andreatta, Using formal concept analysis to represent chroma systems, in *Mathematics and Computation in Music 2013*, ed. by J. Yust, J. Wild, J.A. Burgoyne. LNCS/LNAI, vol. 7937 (Springer, Heidelberg, 2013), pp. 189–200
40. J. Serra, *Image Analysis and Mathematical Morphology* (Academic Press, London, 1982)
41. X. Serra, M. Magas, E. Benetos, M. Chudy, S. Dixon, A. Flexer, E. Gómez, F. Gouyon, P. Herrera, S. Jorda, O. Paytuvi, G. Peeters, J. Schlüter, H. Vinet, G. Widmer, Roadmap for music information ReSearch, in *Creative Commons BY-NC-ND 3.0 License*, ed. by G. Peeters (2013), 978-2-9540351-1-6
42. A. Vieru, *Cartea Modurilor, 1 (Le Livre des Modes, 1)* (Muzicala, Bucarest, 1980)

Mathematics and Labyrinths in Twentieth Century's Music

Susanna Pagano

Introduction

We often consider mathematics and music as two subjects very far one from the other, the first led by calculation and accuracy, the second by fantasy and inspiration. Actually, there is a lot of fantasy in maths and a lot of calculation in music, more than can be imagined. Maths and music have always been connected: for example, in ancient times music formed, together with arithmetic, astronomy and geometry the quadrivium, the group of scientific subjects. Of course, in history there have been periods when this link was hidden and others when it was more evident and clear. Perhaps, in the period of tonality (XVIII–XIX centuries) this relationship was more hidden, and musical structures respected—above all—musical rules and logic following the principles of tonality. In the Renaissance and then again in the XX century this relationship became more evident, even if in a different way. In particular, in the last century tonality was gradually abandoned and each composer chose his personal language, personal order and rules. Frequently, rules and inspiration came from the world of mathematics.

One of the most discussed points of the mathematical analysis of music is the awareness of the composer of the intricate procedures and proportions, and the balance between awareness and instinct, calculation and inspiration. However, this is not so important, in my opinion. Some of the structures we will analyse are really too complex to doubt the composer's consciousness. Then, instinct and calculation must always integrate themselves to create an artistic structure, the architecture of a piece of music, even if in the second case it is not always so evident. Even if the structure is complex and even if it is not easy for a listener to appreciate a musical form, it is just this form to create the climax and the dramatic effect of the piece itself, and for such a reason its effectiveness is directly perceivable.

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Claude Debussy and the Golden Section

Debussy's music often seems free and unpredictable, very different from any previous musical form. Somebody defined his style as Impressionist, even if this bothered him. Debussy wrote that sometimes trying to explain things could kill their mystery. This does not mean that his works have no structure, even if he often hid this structure accurately. Many of his works are characterized by complex symmetries, transpositions and proportions, and these features can seem to be in contrast with the free and instinctive impression that his works give to the listener. Jules Laforgue, one of Debussy's favourite authors, wrote that in art there will always be, as there always was, instinct and reflection, inspirational or divining instinct and knowledge or science. Debussy in fact admitted that his works had columns, had a structure but that he enjoyed hiding them.

The principles Debussy used for the organization of his music are symmetry and golden section: some of his more important works have been deeply analysed from such a perspective, as *La Mer, La Cathedrale engloutie, Isle Joyeuse* and others [6]. If the use of the golden section often is easily deducible in space, it is not in music. Part of the problem is that in looking at an architecture, a picture, a sculpture you can immediately perceive and understand the presence of a proportion. Music develops in time, so it not so easy and rapid to perceive its form, especially when listening to a long piece. In music the golden section concerns tension, climax and balance between the parts and heavily influences the effectiveness of the musical speech, even if we must listen to the entire piece more than one time, and perhaps we must analyse it to be intellectually conscious of these features.

We will see the proportions of the golden section applied to one of his twenty-four *Preludes*, as David Lewin did in his book *Musical Form and Trasformation* [7], to show these features. The *Preludes* were written between 1914 and 1915, and each of them is followed (not preceded) by a "title" that just for its position cannot be defined as a title. It is rather an image, suggested when the last notes disappear. Many of the images are very suggestive: Walks in the snow, Winds, Fairs. . . . The "title" of the last one is *Feux d'artifice*, and we attend a fireworks show, probably during a 14th July evening, a French National holiday, the Bastille Day. During the celebrations there have been wonderful fireworks, described in all the piece through different pianistic effects. In the end, people go home while fragments of the French Hymn come from a far away band. Disclosed by this climax and accompanied by a tremolo, this quote is a signature for the entire series of the Preludes. Usually Debussy signed as *"C.D., French musician"*; the initials C and D are represented by the final notes of the French Hymn (C and D in fact), while the epithet *"French musician"* is represented by the Hymn itself. This signature hides a passionate nationalism, since the composer underlines that these are not Bach or Chopin's Preludes, but (finally) a French composer's Preludes. Such a signature hides an aggressive nationalism, in fact in the Hymn we do not listen to *"Allons enfants de la patrie. . . "*, but to *"Aux armes, citoyens! Formez vos bataillons! Marchons! Marchons!"*. France would have actually entered a war one year later, and in this piece war premonitions can be recognised in the sudden explosions of fireworks and in the violent dynamic contrasts.

The opening of the piece introduces the total chromatic, the twelfth note of which arrives with the climatic E of measure 25. Just after, the theme is introduced, and it is contained between the notes C and D. Just at the beginning of the piece we listen to two alternate triplets, the first on white notes, the second on black ones. The contrast between white and black will be laid all through the entire piece, from the beginning to the end, symbolically, since the contrast white/black represents the contrast between light and darkness. One triplet retrogrades the other, the first ascends, the second descends, and so on. Sparkling octaves appear on the right hand side: they are a D and a flat A, and they are the two bisectors of the black triplet, one at the opposite of the other in whole-tone scales. The two "sign-notes" C and D are introduced as a dyad with the white triplet to form a pentatonic scale, and with the black one to form a whole-tone scale, so just in a short time we have a chromatic, whole-tone and pentatonic material. The last note to form the entire whole-tone scale is just the E, so while the pressure grows, the register ascends, the D and the flat A sparks shoot off at twice their earlier rate, there is a crescendo and an acceleration. There is only a temporary stop in ascension, a descending glissando on black notes (which correspond to the white climb we have just listened to), then the rising starts again and reaches, at last, the E note, which completes not only the whole tone scale, but also the total chromatic. Just here, the theme of the piece appears for the first time, and of course C and D are its most important pitches, because C is the first and D the last one. The theme is then variated. Its opening interval is a rising fifth but in first variation it becomes a tritone, a fourth in the second variation, a sixth (wider) in third variation, more intense.

Measures 61–64 are also very peculiar, as shown in Fig. 1. Again we find a contrast between black and white: a theme is repeated twice and the first time its first chord and the glissando that follows is totally white, the second time totally black. The theme is comprised in a tritone, and the centre of symmetry of this interval is the first note of the previous/next phrase: the flat E in m. 63 is a bisector for the (C, sharp F) of m. 61 and the C in m. 61 is a bisector for the (flat E, A) of m. 63. It is not accident that right here there is the golden section of the entire Prelude, composed by 98 measures. After this magic suspension, the second part of the piece begins, and it is a sort of reprise where the theme is variated and transposed.

But the game of proportions goes on, because the first 64 measures have their golden section at measure 25, where for the first time the E note appears, completing the total chromatic, as evidenced before.

At the end of the Prelude the principal theme returns, but it is transposed. We have two reprises. The first begins on sharp C (or flat D), the second on C (like at the beginning), and leads to a strong climax and a long descending glissando of white and black keys together, and right here there is the golden section of the last 38 measures! At this point, our ear considers flat D as the root, because after the glissando a low tremolo begins, just on the notes flat D and flat A. And right here, pianissimo, far away, we can hear both the Marseillaise and, again, the incipit of the theme, this time beginning on C. A contrast between black and white, between the first and the second reprise appears again. At this point we recognize the flat D as the final root of the piece and the true reprise as a false one. This is a magical

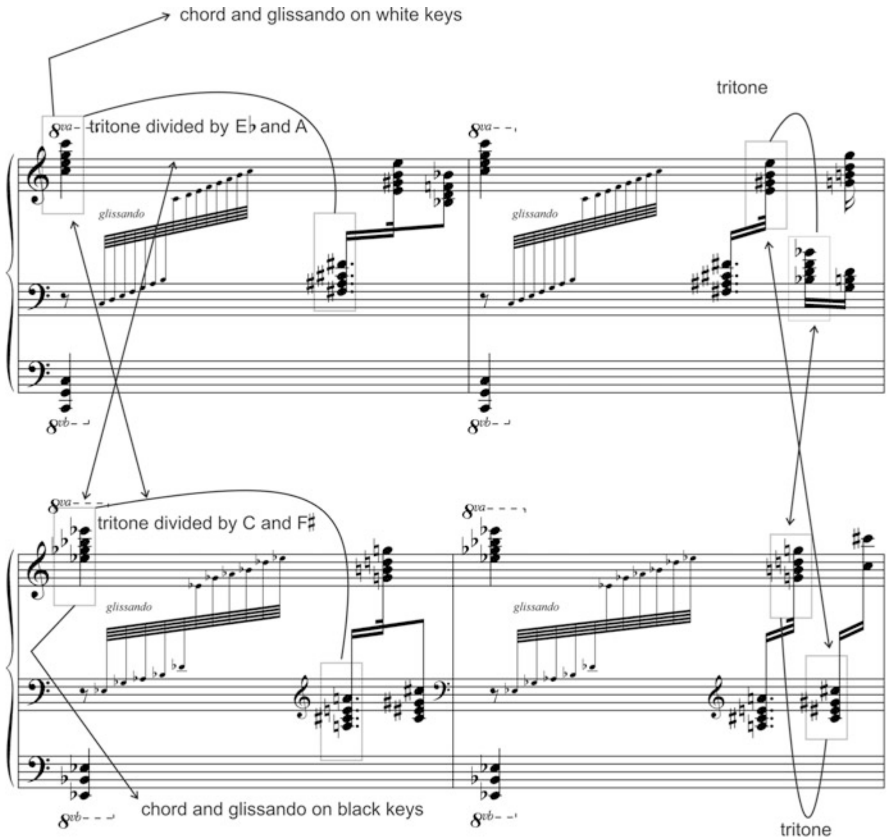

Fig. 1 Symmetry's games in measures 61–64

effect: the quote from the Marseillaise would be rather insignificant if the piece ended unequivocally in C, and the final measures would lose their hidden menace if the Prelude ended clearly in flat D. Debussy teased us in a sort of sound illusion. That is why I spoke about "Labyrinths" in the book I wrote on the subject [9].

Olivier Messiaen, Palindromes and Prime Numbers

The next composer I will deal with was French as Debussy, but his language was completely different. His works are very original. He did not use the scales of the western tradition anymore. One of the main features of his music, in fact, was the search of symmetry. For such a reason he divided the octave into equal interval groups, where the last note was the first of the following group. He created some scales, which he defined *"Modes of limited transposition"*. The first of these nodes was the whole-tone scale, used before by Debussy, the second the octatonic scale, used by Bartok. The other five modes he used in his music are more complex, but each of them returns at its original form after a limited number of transpositions, justifying their name. These scales confer a distinctive sound to his music, unusual but very characteristic.

The rhythm of Messiaen's compositions is also very peculiar. Often there is no time indication at the beginning of his works, so every measure can be very different by the others. Furthermore, he often used Indian rhythms, called "*Tala*", that sound exotic and asymmetrical because there is not a common pulsation and short values create the unity of the rhythmic measure of the piece. For this reason and for the use of prime numbers, because many of his melodies were composed by a prime number of sounds or rhythmic pulsations (that is a very unusual feature), he seems to refuse the typical symmetry of Western music. However, he created a new one, above all using rhythmic palindromes, that are non retro-gradable. This is one of the most interesting features in Messiaen's music: the balance between symmetry and asymmetry fused in an original and personal language. Since he was deeply influenced by Debussy's music, and according to the French taste, his music is very fine and innovative using colours and all the possibilities and registers of the instrument. Often in his pianistic scores there are several written references to other instruments, sometimes exotic, as gamelan, tam-tams and bells.

Messiaen loved nature and was a deeply religious man. His knowledge of ornithology was a constant source of inspiration and he dedicated part of his works to describe bird songs. He also composed many religious cycles, and the two pieces we are going to analyse are part of one of them. Its title is *Vint Regards sur l'enfant Jesus*, and—in fact—each of the twenty pieces forming the cycle describes a scene about the birth of Christ, as the composer himself wrote in the short comments at the beginning of them. *"Christmas"*, *"Contemplation of the Prophets, the Sheperds and the Magi"*, *"The all-powerful word"*, *"The Virgin's first Communion"*, *"Contemplation of the joyful Spirit"*: these are only some of the titles and of the scenes evoked by this music.

The first piece I will analyse is *La parole tuit-puissante*. It is based on two opposing elements, a melodic line doubled in octaves in the high part and a rhythmic tam-tam in the lower part of the keyboard, a three note cluster (A-sharp A–H), always in *fortissimo*.

The cluster is composed by 3, 5, 8 and then again 5 and 3 semiquavers, so its values, as F.J.C. Ciscar writes in his analyses [3], are the first elements of Fibonacci's series. It appears twenty-one times and each repetition is separated by the other by a rest of 7 semiquavers (7 is a prime number). The rhythm of the monody is completely different, because it derives from Indian rhythms, deeply asymmetric, thus creating a continuous polyrhythmic game cleared by the separation of the staves. Many of the measures of this piece are composed by a prime number of notes making it very asymmetric. The melody is composed by 12 notes, always the same pitches like in a set, except the D note which is the first and the last sound in the piece and in each of its parts. The various musical phrases are obtained by a continuous permutation of the twelve notes. On the contrary, the rhythmic tam-tam is symmetrical and it is a palindrome (as shown in Fig. 2) and Messiaen's scales are

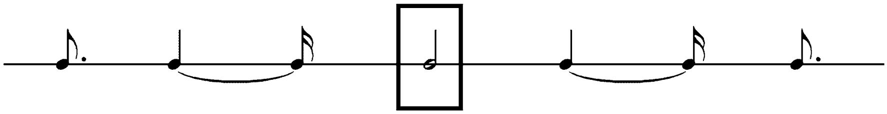

Fig. 2 Rhythmic palindrome and its centre of symmetry

very symmetrical because they are obtained through a symmetrical division of the octave. The result is a strong and continuous contrast between the symmetry of the palindrome and of the arch form of the piece and the asymmetry of the melodic line and of prime numbers. The end is sharp and sudden, and the tam-tam is for the first time interrupted before its ending. The note of the beginning and the end is the same: this and the continuous permutation of the same sounds let me speak about a labyrinth this time too.

The second piece we will analyse is *Regard des prophètes, des bergers et des mages*, an arch-shaped piece, because the first section is repeated at the end. At the beginning, the right hand plays a four chord ostinato, while the left hand plays a deep four quarters chord, repeated many times, each time shortened by a semiquaver. In this way, at the end it is reduced to a semiquaver, repeated 33 times. Then, we can hear a distinguishing theme, composed by phrases of a prime number of notes (11, 7 and 17), monotonous and insistent like a bird's tweet. The central section is deeply counterpointistic, like a canon. In the end, we hear the first section again, but this time it is developed in the opposite direction. So, the ostinato on the right hand side is the same, but the tam-tam on the left hand begins from the 33 semiquavers and becomes longer and longer to reach a four quarters chord, when the arch path is concluded and just there, after a short Coda, the piece ends. Here we find again a palindrome, but this time it is formed by the opposite sections of the *Regard* reflecting each other.

Georgy Ligeti: Fractal Geometry and Chaos Theory

Georgy Ligeti was Hungarian and all his family, except his mother, died in concentration camps during the Second World War. He lived in Hungary until 1956, and in spite of his great interest for contemporary music and its languages, he could not access Western developments and works for the restrictions of the communist government. However, in 1956 he left Hungary and reached Western Europe, where he knew other important and famous composers such as Stockhausen, Kagel and Boulez, their compositional languages and electronic music. In this period he wrote his most famous works and became one of the most important and original composers of the century.

Ligeti's style is very different from the composers we have spoken about. When he was a child, his father wanted him to become a scientist, so he could not study music until his teenage years. His scientific education opened his mind towards many different interests and inspirations that influenced his music, as the Benoit

Mandelbrot's fractal geometry, the theory of chaos and the books of Douglas Hofstadter and Jorge Luis Borges.

The piece I will analyse is one of the Studies for piano, written between 1985 and 2001 and divided in three books. These studies are very innovative, for the use of African polyrhythms and of many different and distant influences, from Bartok to Conlon Nancarrow, who wrote works for player piano (an auto-playing instrument). Such works often recall geometrical plays and acoustical illusions. This is the case of the last three studies of the Second Book: *Vertigo*, *L'Escalier du Diable* and *Columna Infinita*. Each of them describes, in a different way, an infinite scale which creates itself again every time we think it has come to an end.

L'Escalier du Diable was written in 1991, and creates the same illusion of the Shepard's tone on the piano. It is different from everything else you can usually hear at a Concert Hall. You have to imagine a staircase, a long, infinite staircase, which emerges from the black depth of the ground and climbs to the sky. You have to imagine to arrive to the top and to fall again back to the starting point to begin the ascent again and again, reaching nothing, like a blame, like Sisifo with his stone in a sort of diabolic uninterrupted cycle. You have to imagine an acoustical play which entraps us in a labyrinth, similar to Escher's world, where a staircase can be at the same time endlessly descended or climbed, where a staircase can lead someone at the beginning again, or can be passed by mysterious characters which belong to irreconcilable dimensions, or can fill and completely saturate a whole house and extend ad infinitum in a foolish senseless structure without exit. You can imagine a senseless staircase, as in *The immortal*, one of Borges' tales, where the protagonist searches a mysterious city and its mysterious people, the immortals, for a long time. When he finally arrives he discovers that Immortals were all mad and left the city, now abandoned and awful. He describes its terrible, disturbing and senseless structure, with its devilish ladders that carry nowhere:

> More than any other feature of that incredible monument, I was arrested by the great antiquity of its construction. I felt that it had existed before humankind, before the world itself. Its patent antiquity (though somehow terrible to the eyes) seemed to accord with the labor of immortal artificers. Cautiously at first, with indifference as time went on, desperately toward the end, I wandered the staircases and inlaid floors of that labyrinthine palace. (I discovered afterward that the width and height of the treads on the staircases were not constant; it was this that explained the extraordinary weariness I felt.) This palace is the work of the gods, was my first thought. I explored the uninhabited spaces, and I corrected myself: The gods that built this place have died. Then I reflected upon it peculiarities, and told myself: The gods that built this place were mad. I said this, I know, in a tone of incomprehensible reproof that verged upon remorse—with more intellectual horror than sensory fear. The impression of great antiquity was joined by others: the impression of endlessness, the sensation of oppressiveness and horror, the sensation of complex irrationality. I had made my way through a dark maze, but it was the bright City of the Immortals that terrified and repelled me. A maze is a house built purposely to confuse men; its architecture, prodigal in symmetries, is made to serve that purpose. In the palace that I imperfectly explored, the architecture had no purpose. There were corridors that led nowhere, unreachably high windows, grandly dramatic doors that opened onto monklike cells or empty shafts, incredible upside-down staircases with upside-down treads and balustrades. Other staircases, clinging airily to the side of a monumental wall, petered out after two or three landings, in the high gloom of the cupolas, arriving nowhere. I cannot say whether these are literal examples I have given; I do

> know that for many years they plagued my troubled dreams I can no longer know whether any given feature is a faithful transcription of reality or one of the shapes unleashed by my nights. This City, I thought, is so horrific that its mere existence, the mere fact of its having endured—even in the middle of a secret desert—pollutes the past and the future and somehow compromises the stars. So long as this City endures, no one in the world can ever be happy or courageous. I do not want to describe it; a chaos of heterogeneous worlds, the body of a tiger or a bull pullulating with teeth, organs, and heads monstrously yoked together yet hating each other—those might, perhaps, be approximate images [2, pp. 9–10].

What I did not know when I began to study this piece is that the *Devil's Staircase* is a mathematical function and the piece effectively describes it. Ligeti said that somewhere underneath, very deeply, there's a common place in our spirit where the beauty of mathematics and the beauty of music meet. He taught that they don't meet on the level of algorithm or making music by calculation; it's much lower, much deeper—or much higher, if you prefer. In fact, the concepts of modern mathematics and physics have provided him with models—processes and patterns which have stimulated comparable but different processes in his music. Only the inspiration he has drawn from recent research is poetic and procedural more than precisely computed. Calculation of proportion, Fibonacci numbers and set theory are used frequently by many contemporary composers. However, according to Ligeti, the inner ear must govern every compositional decision, so his interest is directed towards mathematical ideas more than towards calculations. In particular, it was directed to fractal geometry and to the theory of chaos, because the balance between chaos and order and recursive procedures had similarities with his compositional language. In fact, the degeneration of simple premises into chaotic outcomes has a direct parallel in Ligeti's music, where the progressive deformation of apparently innocent material can lead to spectacularly anarchic result. The last three Studies for piano, and in particular *L'escalier du diable*, create an infinite staircase on the keyboard, as in the electroacoustic illusion by the psychologist Shepard. Shepard's tone is constructed through superimposing chromatic scales at different octaves with varying intensity. Each note belongs to different octaves at the same time. The scales increase their volume at the beginning and then, becoming higher and higher (or lower and lower), they fade away, softer and softer. Thus it is impossible for the listener to understand the exact frequency of the notes. This creates a sound illusion evocating an infinite scale, ascending or descending, always moving remaining the same. The effect is very suggestive, as it is demonstrated in the soundtrack of the recent film *Dunkirk* by C. Nolan.

This Study is built on numerous ascending chromatic scales and it makes a direct reference to mathematics in its title, since a *Devil's staircase* is also a particular instance of self-similarity based on Cantor's sets. Take a straight line, then remove the middle third and repeat the process to the two fragments remained, *ad infinitum*. The result is Cantor's dust, an infinite number of points arranged in groups, infinitely sparse, whose total length is 0. If we apply the process to an inclined line and substitute with straight lines the middle fragments we remove each time we repeat the process, we obtain the Devil's staircase. This is one of the simplest examples of mathematical constructions called fractals, characterized by recursion: their structure remains the same if we observe it enlarged or reduced to infinity.

As R. Steinetz notates in his interesting articles [10, 11], Ligeti builds his musical staircase using rhythmic cells of two and three notes, and this clearly recalls the binary-ternary geometry of the Devil's staircase. After a bar and a half's false start, we finally hear the real metrical model, and it contains subgroups of 7, 9, 11 and 9 quavers, divided in this manner: $2+2+3$, $2+2+2+3$, $2+2+2+2+3$ etc. The larger step of three quavers makes a small plateau, and the irregular progression of the subgroups recalls the irregular staircase of the graphic image.

Also the study has a "pitch model", which starts just at the beginning of the piece; so, pitch and rhythmic models are immediately out of synchronization. The melody is divided into many cells, and their first note follows a chromatic scale, from B to sharp A, higher and higher just as in a staircase. The remaining pitches of the cells belong to the whole-tone scale commencing on flat G.

According to the theory of chaos, for which minute inaccuracies in initial conditions can quickly lead to vast differences, this systematic and ordered musical structure begins to reveal some divergence. For instance, the pitch model recurs every 12 cells and three rhythmic ones every 16. Then, there are several musical bifurcations, each line can descend for a few notes or can divide itself and originate two lines, or more.

The scale climbs up in an endless staircase, octave by octave, but every time we think it has come to an end the pattern restarts in low register, obsessive and frightening, and begins rising again. Its pattern seems the same, even if we perceive its internal disorder rising more and more becoming foolish and nonsensical. Then, the climb is arrested by a deep B flat minor triad. Here begins a very slow section, perhaps even more anguished with its wide and low chords. These chords climb too, one note at a time, majestically and inexorably, slowly becoming louder and louder, together with three different bell-like ostinatos in different rhythms to create a percussion effect of gongs and tam-tams, as it is written in the score. This symbolism recalls Messiaen, as we saw in the previous paragraph. Moreover, this slow, central section recalls the wide plateau in the Devil's Staircase mathematical function.

Then, the climbing begins again, but we are not at $2/3$ of the piece as in the mathematical function. I could not believe it when I discovered it, but the quavers start again exactly in the golden section of the piece. The point is exactly calculated. In fact, if you count the number of the measures in all the Study (all the measures are equal, even if they only have a graphic function, because rhythmic patterns are very irregular and different between left and right hand) and you find its golden section, you will not obtain an entire number because it falls inside a measure. On the other hand, if you count all the quavers and then find the golden section you can find exactly where the note falls. This cannot be accidental!!!

When the movement starts again, the disorder appears rather immediately, and its volume increases more and more. Since we are listening to an infinite scale, the piece cannot end; it is in fact roughly interrupted, at the end of the last flight, on a wide, harsh chord. The effect is suggestive, because the extreme disorder and loudness of the closure of the piece dies in an endless resonance, until the absolute silence.

Conclusions

The last piece I will write about is Bach's *Ciaccona*, written for violin only and transcripted for piano by Brahms for only left hand. The term *Ciaccona* indicates a specific musical form, characterized by the return of a theme always varied and with a repetitive bass-line, so at the end of the piece we hear the same theme that we heard at the beginning, like in the other pieces I described before. We are talking about a different period and a different musical language, this is clear, but if we analyze this work we can verify that it is composed by three sections: two in D moll and the middle one in D major. The first part is the golden section of the sum of the first two sections, and the second part is the golden section of the second and third parts. As it can be observed, Bach very often used mathematic proportions intentionally, because for example in his *Ciaccona* each musical phrase is composed by four or eight measures. As notated by P. Marconi [8], Bach wanted to use only multiples of four and the golden section was applied in the only compatible way. However, even if the use of mathematic principles in music was not new, hearing Bach's music is like a perfect circle, perhaps because the presence of the tonality gives the listener a centre, differently from the other works we analyzed before, which leave the listener with a subtle idea that something is unfinished.

I am a pianist, and these pieces were in the program I played for my second level degree in piano, so I analyzed them while I was studying. I chose the authors because of their different musical languages, even if, except Bach, they lived in the same century. In fact, XX century is a variegated period, full of composers, and each of them developed his own style, with own features and rules, lacking at this point the tonal system which had been able to unify the musical world so far. I had been searching the differences between the authors for a long time, but—in the end—I began to think that similarities were more interesting. I bumped into fractal-like structures, because each piece had similar elements which appeared and disappeared depending on the detail I observed.. Was it a coincidence that every piece closed with the same elements of its beginning even if they were hided? Was it a coincidence that every illusory reprise was actually different, that different elements were actually similar, that every piece seemed vaguely unfinished? Was it a coincidence that after an illusory linear path each composer brought us at the starting point, like in a Moebius strip?

Then I discovered the wonderful Hofstadter's book *Godel, Escher, Bach*. In this book the author writes about strange loops, where a strange loop is a hierarchy of linked levels, where moving through the levels one returns to the starting point. He writes in "*I am a strange loop*" that a strange loop is

> Not a physical circuit but an abstract loop in which, in the series of stages that constitute the cycling-around, there is a shift from one level of abstraction (or structure) to another, which feels like an upwards movement in a hierarchy, and yet somehow the successive "upward" shifts turn out to give rise to a closed cycle. That is, despite one's sense of departing ever further from one's origin, one winds up, to one's shock, exactly where one had started out. In short, a strange loop is a paradoxical level-crossing feedback loop" [5, pp. 101–102].

Hofstadter's examples of strange loops are Escher's works and self-referential Godelian statements in formal systems. Thus, I finally understood. I found strange loops in the music I played, and that is the reason why I have written about *Labyrinths* in this article and in the book I wrote on the subject. Furthermore, I found that labyrinths are a recurrent image in XX Century culture. In science we can remember Gödel and Heisenberg's theories, characterized by inner circularity and self-reference. According to Heisenberg's theories, it is not possible to describe a phenomenon without interacting, because our observation changes the situation we are observing. This is a great example of circularity, like in Escher's lithography *Print Gallery*, so clearly explained in Hofstadter's book.

> What we see is a picture gallery where a young man is standing, looking at a picture of a ship in the harbor of a small town, perhaps a Maltese town, to guess from the architecture, with its little turrets, occasional cupolas, and flat stone roofs, upon one of which sits a boy, relaxing in the heat, while two floors below him a woman—perhaps his mother—gazes out of the window from her apartment which sits directly above a picture gallery where a young man is standing, looking at a picture of a ship in the harbor of a small town, perhaps a Maltese town—What!? We are back on the same level as we began, though all logic dictates that we cannot be... [4, p. 710].

In art, Escher's works are a clear image of circularity. In literature, Borges animates mathematical labyrinths, entrapping the reader. In Hofstadter's book, self-reference is the feature allowing each of us to be as each of us is, it is the source of self-consciousness and personality. So, I believe that part of the interest and fascination we feel on the subject is due to the fact that strange loops and labyrinths are part of us. As Borges writes,

> A fleeing man doesn't hide out in a labyrinth. He doesn't throw up a labyrinth on the highest point on the coast, and he doesn't throw up a crimson-colored that sailors see from miles offshore. There's no need to build a labyrinth when the entire universe is one [1, p. 260].

References

1. J.L. Borges, Ibn-Hakam Al Bokhari, murdered in his labyrinth, in *Collected Fictions* (Penguin, London, 2000), trans. by Andrew Hurley
2. J.L. Borges, *The Aleph and Other Stories* (Penguin, Baltimore, 2004)
3. F.J.C. Ciscar, *Aproximacion al Lenguaje de Olivier Messiaen, Analisis de la Obra Para Piano Vingt Regards sur L'Enfant Jesus* (Valencia University, Valencia, 2004)
4. D.R. Hofstadter, *Gödel, Escher, Bach* (Penguin, Baltimore, 2000)
5. D.R. Hofstadter, *I Am a Strange Loop* (Perseus, New York, 2007)
6. R. Howatt, *Debussy in Proportion*. (Cambridge University Press, Cambridge, 1989)
7. D. Lewin, *Musical Form and Transformation* (Oxford University Press, London, 2007)
8. P. Marconi, Prodigi matematici e simboli numerici, in *La "Ciaccona" di Bach. Saggio di Storia della Interpretazione by P. Rattalino*, (Unicopli, Milano, 1988)
9. S. Pagano, *Labirinti e Circolarità. Viaggio nella Musica del '900 e Dintorni* (Cantagalli, Siena, 2014)
10. R. Steinitz, Music, maths & chaos, in *The Musical Times* (1996)
11. R. Steinitz, Weeping and wailing, in *The Musical Times* (1996)

Part IV
Mathematics & Applications

How to Prevent Crimes Using Earthquakes

Marco Abate

Minority Report

In 2015, the Fox channel produced a TV series called *Minority Report,* created by Max Borenstein with Steven Spielberg among the executive producers. The series was sort of a sequel to the 2002 movie *Minority Report* by Steven Spielberg; the latter was in turn inspired by a short story written by Philip K. Dick in 1956 whose title was (as you can guess) *Minority Report.*

The main point of the short story, the movie and (somewhat less) the series was to explore the ethical implications of the possibility of predicting violent crimes *before* their actual occurrence. In a not-too-far future two brothers and one sister are born with the uncanny (and disturbing) ability of seeing violent crimes before they happen. The police uses their visions to identify the culprits and arrest them *before* the crimes are actually committed, thus preventing their occurrence and consequently saving lives.

Already here we have an ethical problem, because the alleged culprits are arrested and sentenced *without* having committed the crime they are accused of; but, on the other hand, the potential victims keep living their lives without being harmed, which supposedly is a good thing by itself. Moreover, often enough it was possible to find proofs that the alleged culprits were actually preparing the crimes the three siblings (the *pre-cogs*) predicted, thus justifying the arrests also from a more conventional point of view.

It turns out that the predictions of the pre-cogs are stronger and more accurate when the three siblings are kept together immersed in a futuristic milk bath (don't ask). So they are effectively segregated and imprisoned by the police, forced to stay as much as possible in the milk bath without any personal life outside. It is a small price to pay to have a crime-free city, isn't it?

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And yet, predicting the future is a tricky business, in particular when the act of prediction changes the future thus invalidating those very same predictions. It is then not surprising that in rare cases the three siblings might have different opinions. Two of them predict a particular culprit for a soon-to-happen violent crime, but the third one does not agree; there is a *minority report.* From a statistical point of view, this is a rare event; the potential error is negligible for the society considered as a whole. But from the point of view of the individuals directly involved, the potential error might be devastating; both acting and not acting can lead to the destruction of somebody's life.

These are the ethical points making the reading of the short story and the vision of the movie compelling (the TV series went in a slightly different direction, ending up cancelled after only one season), and they can be summarised in the old and fundamental conundrum underlying any democracy: up to which point we are ready to go in limiting the freedom of the individual for the sake of the safety of the whole society? Are we willing to limit the freedom of individuals because, according to tools that might be mistaken, we *think* they might commit a crime? If you believe this is just a science-fiction problem and thus it does not matter, start thinking about how religion profiling might be used to allegedly prevent terrorism.

However, this is *not* what this note is about. Or is it?

Preventing vs. Predicting

In 2011, a start-up based in California earned surprising titles in the first pages of many newspapers and online news sites. One site devoted to technological news, SingularityHub.com, summarised the hubbub quite efficiently with the following headline:

> *Pre-Cog is here—A new software stops crime before it happens* (SingularityHub.com, August 29, 2011)

The article, written by Peter Murray (see [1]), reported on a software developed by a small company called *PredPol* that was being tested in California by Santa Cruz and Los Angeles Police Departments to prevent crimes from happening.

The journalist of course quoted *Minority Report* (and the TV series *Numb3rs*), but there is a fundamental difference in the way the algorithm devised by PredPol works with respect to the way the pre-cogs worked in the movie: the aim of the algorithm is to *prevent* crimes, not to *predict* crimes. More precisely, the aim of the algorithm is to identify areas where a particular kind of crime is more likely to occur, without giving any information on who will possibly commit a crime there.

This is explained in the web page of the company [2]. After presenting itself as *The Predictive Policing Company,* the company states that "PredPol uses artificial intelligence to help you prevent crime by predicting when and where crime is most likely to occur, allowing you to optimize patrol resources and measure effectiveness." In another page of the site it is clarified what this algorithm does and what it does not:

PredPol does:

- Increase Law Enforcement's odds of stopping crime
- Predict *where* and *when* crime is most likely to occur
- Work for both large and small agencies
- Help new officers on-board quicker
- Make it easy to access predictions anywhere and anytime

PredPol does not:

- Map out past crimes or another 'hotspot' tool [we shall discuss hotspots later]
- Predict *who* will commit crimes
- Use PII (Personally identifiable information), eliminating civil liberties issues
- Replace law enforcement veterans or analysts' intuition and experience
- Require additional hiring or new hardware

More precisely, the aim of the PredPol algorithm is to suggest to the police force *where and when* to send patrols, with the idea that police presence in the right place at the right time will prevent crimes from happening. Nobody specific is targeted in advance; only the locations are indicated, and updated in (almost) real time.

Sending patrols in high-risk areas has always been standard police procedure. To explain in which sense the PredPol algorithm is different, let's first take a look to more standard approaches used by police departments in major cities.

Traditionally, the decisions about where and when to send patrols were taken by senior officers relying on their own experience and intuition. This can be very effective or completely ineffectual depending on the officer; and discontinuities are anyway created by the retirement of a senior officer in charge, with the consequent loss of his/her experience.

In the last forty years or so, using the huge amount of data collected by law enforcements, hotspot maps have been devised and have become a standard tool to help less experienced officers to decide where to send patrols, and that might provide an approach more consistent and less depending on the specific senior officer in charge.

A *hotspot map* is a map of a (given area of) a city showing (usually using colours or shades of grey; see Fig. 1) the probability of occurrence of a specific type of crime in a particular place, sometimes also taking into account the specific time of day (or night). In standard hotspot maps the probability is computed in a very simple (and naive) way: it is simply the quotient between the number of crimes of the specified type occurred in the given place at the specified time and the total number of crimes of the specified type occurred in the city at the specified time. This is a computation that can be done as soon as enough data are accumulated, and can be updated any time new data are acquired. In other words, crime hotspot maps are frequency maps based on past occurrences of a specified typology of crime; and are used by sending police patrols more often in areas with higher probability of occurrence of crimes (that is, in areas where that kind of crime has occurred more often).

As a quick Google search will show, there exist many kinds of crime hotspot maps (see [4] for hotspot maps of crime in Italy), but most of them can be reduced to two categories: long-term maps, and short-term maps.

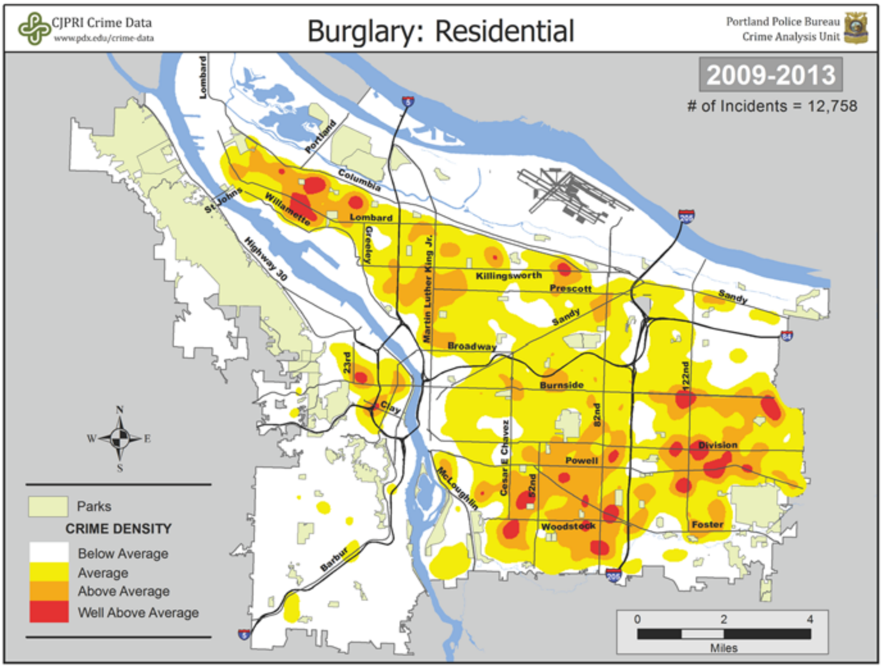

Fig. 1 Hotspot map of residential burglaries in Portland in the years 2009–2013. (Source: Henning, K., (2017, November 1). Residential burglary hotspot. Retrieved from [3])

Long-term maps are based on data collected on a span of several years, sometimes decades. Being based on large amounts of data, they are fairly stable; this is both a strength and a weakness. It is a strength because they are robust; for instance, occasional mistakes in classifying past occurrences of crimes have a very limited effect on long-time hotspot maps. By the same token, they are slow to change; one needs many new data to significantly change a long-time hotspot maps. As a consequence, they are not able to detect recent or transient phenomena. Moreover, they tend to suggest patterns of police patrolling mostly constant in time, and thus foreseeable by criminals that then adapt their behaviour to escape being spotted by the patrols. Of course such changes in behaviour will eventually be detected by long-term hotspot maps; but this needs time, and thus limits the utility of such maps.

Short-term maps are instead based only on more recent data, and are used to try and detect recent trends and behaviours. They complement the information provided by long-term maps, giving visibility to phenomena that would otherwise have remained hidden; but being based on small amounts of data they are unstable, and thus more susceptible to mistakes in classification. Furthermore, their predictions change fast, sometimes too fast to be useful.

Two examples of quickly developing crime trends can help understanding the difference between long-term and short-term hotspot maps.

The first example concerns home burglaries in suburban areas. It is unfortunately not uncommon that usually quiet suburban areas are subjected to an unusual number of home burglaries concentrated in a short amount of time. What has happened is that a burglar has discovered that the area is poorly guarded, and thus (alone or with accomplices) has burglarised as many houses he could in the shortest possible amount of time, before the inevitable increase in police patrolling that would shortly follow. Thus we have a spike (high in magnitude, short in time) of burglaries there.

The second example concerns gang-related crimes. An incident between members of rival gangs may lead to a sequence of retaliations ending only when some sort of (fragile) truce is reached—or enough gang members are injured or dead. Again, we have a spike of incidents, even though this time in an area already indicated as potentially critical by long-term hotspot maps.

Spikes in crimes tend to make headlines more than usual crimes; and this makes police departments even more interested in trying to prevent them. Short-term hotspot maps are used to try and identify spikes when they are starting to happen; but clearly they are a very rough tool.

So one needs a way to merge long-term maps and short-term maps in a single tool able to keep into account both history and transient trends; and to do so some mathematics can be useful.

Earthquakes and Crimes

This is the problem a group of researchers at UCLA started to study around the year 2009. It was quite an interdisciplinary group, comprising mathematicians (A. Bertozzi, M.B. Short and G.O. Mohler, who later moved to Santa Clara University where he refined and expanded the models), a statistician (F.P. Schoenberg), an anthropologist (P.J. Brantigham) and a criminologist (G.E. Tita), with the fundamental assistance of experienced police officers, W. J. Bratton from LAPD and Z. Friend from Santa Cruz PD.

The starting point was the idea that to provide an effective model of crime occurrences one needs to keep into account two concurrent aspects:

- a ground state of likelihood of crimes mostly due to the (social, urban and cultural) environment, slowly changing with time;
- spikes of events, short in time but possibly strong in magnitude, often sparked by single incidents.

Possibly because they were living in California, the group realised that there was another type of phenomena presenting the same kind of concurrent aspects: earthquakes.

Indeed, the likelihood of an earthquake in a particular zone is mostly due to the geological structure of the local terrain, and in particular to the characteristics and positions of faults in the area. This gives a background probability, slowly changing over (geological) time. But it is also very well-known that a single event very often causes swarms of events (aftershocks) concentrated in a (relatively) short time

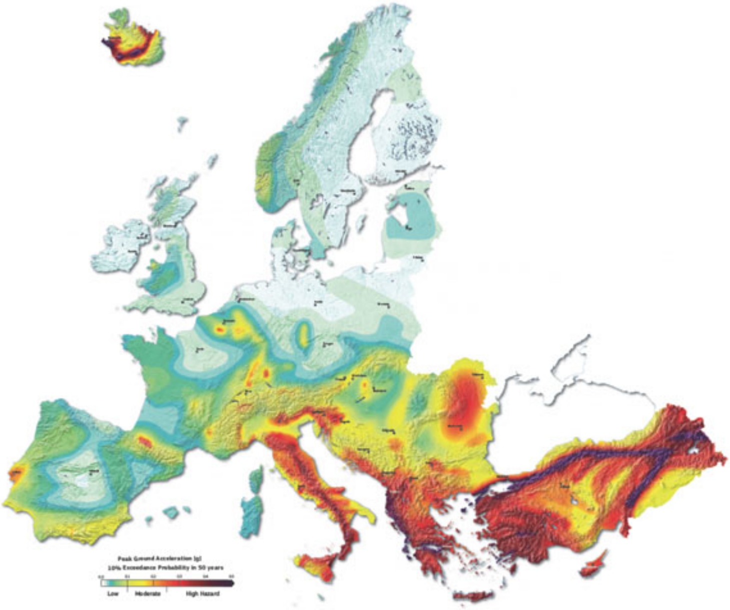

Fig. 2 European seismic hazard map. (Source: SHARE project [5])

span. Thus we have coexistence of slowly-changing and quickly-happening causes of events, in a way very similar to what happens with crime—and, not surprisingly, a typical way for visualising the likelihood of earthquakes is by hazard maps (see Fig. 2), which is the way earth scientists call hotspot maps.

Mostly led by Mohler, the group started to adapt the models used by geologists for computing the probability of earthquakes so that they could be used for computing the probability of crimes. It turned out that from this point of view crimes had a number of advantages over earthquakes: the amount of data available is much larger, the time frames to consider are much shorter and the events present much smaller variations in magnitude, all elements making the construction of probabilistic models easier.

The results of their studies were first published in the paper [6] "Self-Exciting Point Process Modeling of Crime", published online by the Journal of the American Statistical Association on January 01, 2012 (the paper was originally submitted to the journal more than two years before, on September 2009, and accepted after revision on October 2010). Many other papers followed, with different applications and/or generalizations of the original model (see, e.g., [7–10]). More to the point, the models have been applied to real data provided by the police departments of Los Angeles and Santa Cruz, and then have been used for field tests.

The tests consisted in deciding when and where to dispatch patrols using in some districts the models provided by Mohler's group, while using more traditional methods in other districts, and then comparing the outcomes. The tests gave promising results, and this convinced Mohler and his collaborators to start the spin-off company PredPol to commercialise a software based on the mathematical models they developed.

The Model

From a mathematical point of view, the model is not too complicated. The probability $p(x, y, t)$ that a crime (of a given kind) will occur at the position of coordinates (x, y) at the time t is represented by a function of the form

$$p(x, y, t) = G(x, y) + \sum_{t_j < t} g(x - x_j, y - y_j, t - t_j). \tag{1}$$

In this formula $G(x, y)$ represents the background probability of occurrence of a crime at (x, y), and corresponds to the contribute given by long-term hotspot maps. The sum takes instead into account the probability that a given crime will spark a swarm of similar crimes; indeed, $g(x - x_j, y - y_j, t - t_j)$ represents the probability that a crime occurred at (x_j, y_j) at the time t_j will cause another similar crime at (x, y) at the time $t > t_j$, and the sum ranges over all past crimes.

Formula (1) is very general; suitably choosing the functions G and g it can be adapted to model many different kinds of phenomena. It turned out that for modelling crimes (as well as for modelling earthquakes) it is not necessary to be very creative in the choice of G and g, and it is sufficient to start from one of the most used functions in probability theory, the Gaussian function

$$G_\sigma(x, y) = \frac{1}{2\pi\sigma^2} \exp\left(-\frac{x^2 + y^2}{2\sigma^2}\right),$$

where σ is a normalization factor. Figure 3 contains the graph of G_σ for two different values of σ. The smaller is σ the faster G_σ decreases away from the origin $(0, 0)$.

The function G should take into account the history of crimes in the area, and in particular the fact that a crime occurred at (x_j, y_j) might influence the later occurrence of crimes in (x, y). Clearly, the influence should decrease when (x, y) is far from (x_j, y_j). This suggested the following choice for G:

$$G(x, y) = \frac{a}{2\pi\sigma^2} \sum \exp\left(-\frac{(x - x_j)^2 + (y - y_j)^2}{2\sigma^2}\right), \tag{2}$$

where a and σ are parameters to be chosen, and the sum ranges over all past occurrences of crimes.

The function g should also take into account the time passed from the occurrence of the given crime, decreasing as the interval of time increases. Since one would like to model spikes concentrated in time without long-lasting effects, the

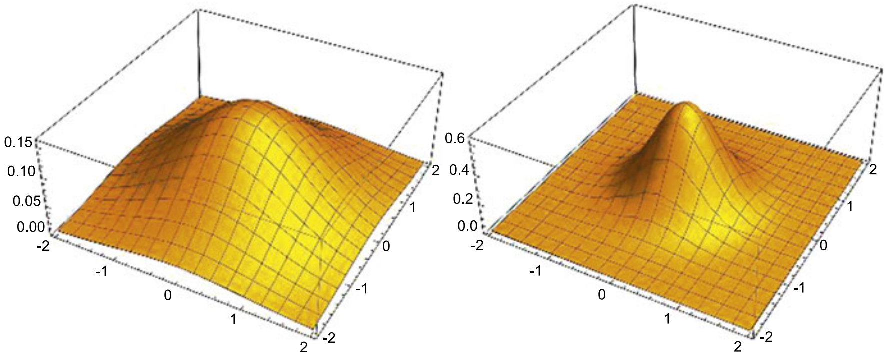

Fig. 3 Graph of G_σ for $\sigma = 1$ (left) and $\sigma = 1/2$ (right)

time dependence should be represented by a quickly decreasing function, possibly exponentially decreasing. This suggested the following form for the function g:

$$g(x - x_j, y - y_j, t - t_j) = b\gamma \exp\bigl(-\gamma(t - t_j)\bigr)\frac{1}{2\pi\tau^2} \times \exp\left(-\frac{(x - x_j)^2 + (y - y_j)^2}{2\tau^2}\right), \tag{3}$$

where b, γ and τ are parameters to be chosen.

The model then depends on five parameters: a, b, σ, τ and γ. The parameters σ and τ control the spatial influence of past crimes, the parameter γ controls the time influence of past crimes, and the parameters a and b can be used to control the relative importance of the background crime probability (corresponding to long-term maps) with respect to the effects of recent crimes (corresponding to short-term maps).

Up to here the mathematics is relatively elementary. The difficult part is the *calibration* of the model, that is the choice of the parameters so that at the time t_0 when the computation is done the probability distribution p given by (1) is as close as possible to the actual probability distribution computed by using the frequency of past crimes as we described above for hotspot maps. This is done by using non trivial techniques coming from statistics and numerical analysis.

Once the model is calibrated (that is, the parameters are chosen), $p(x, y, t)$ will give the probability of occurrence of a crime at the position (x, y) at the time $t > t_0$, thus suggesting to send patrols at time t in the places where $p(x, y, t)$ is higher. Of course, one expects the accuracy of the prediction to decrease with time; thus one needs to repeat every so often the calibration by including the new crimes occurred since the previous calibration. And this introduces another difficulty: the calibration algorithms should be fast enough to run as close as possible to real time.

Not surprisingly, the scientific papers do not describe in detail the calibration algorithms used (and as a consequence we shall not talk about them here, referring to

[6–10] for some information). Indeed, it is the efficiency of the algorithms to determine whether a model is commercially viable; and to be too public about the details will just help the competition. Furthermore, the scientific papers describe the general approach only; the actual implementation might include tweaks requested by a single customer and valid only for that specific kind of crimes in that specific kind of area. Indeed, a specific model might need to take into account, e.g., geographical features; for instance, the probability of a home burglary in the middle of a lake is zero independently of the distribution of past home burglaries along the shores of the lake.

Right now there are several companies selling predictive policing algorithms (beside PredPol, one can quote at least HunchLab [11] and CivicScape [12]). The competition is strong, and this explains why it is difficult to find details of the actual implementation of such models (there is a notable exception: CivicScape recently released publicly its software; see below). Furthermore, the use of such algorithms is becoming popular; the example set by Mohler and his collaborators is spreading fast.

Final Thoughts

A lot of research is currently going on in this area. For instance, up to now we assumed that the calibration of the model is based on the past occurrences of the *same* kind of crime one would like to prevent. But this is a limitation: for instance, a street assault with knives can lead to retaliations with guns, and possibly to murders—or, conversely, a murder can cause less severe repercussions in the same area. Thus a more accurate model should take into account several kind of crimes, but with appropriate weights and possibly using different probability functions. See, e.g., [13] for a discussion.

One natural question now is: do these models actually work? As often happens, there is not a clear cut answer. According to most of their users, they do. Here is a typical quote (taken from the PredPol web site [14]):

> The Santa Cruz, CA Police Department saw assaults drop by 9%, burglaries decrease by 11%, and robberies down 27% in its first year using the software (2011–2012). Crime overall dropped 25% in June 2013 and 29% in July 2013 compared with those same months the previous year.

On the other hand, not everybody is convinced (see, e.g., [15] and [16]). A criticism often raised is that these algorithms seem to target minorities too often, and this has led to advance doubts about racial bias included in the algorithms themselves. It is the need to dispel such doubts that has led CivicScape to the decision of making its software public [17], so that anybody interested could check whether this allegation would be true or not (of course, they claim that it is not true).

To better put these models in context there are two underlying assumptions that I think deserve to be made explicit.

The first one can be summarised as “crime begets crime”. The prediction of crimes in the future is based on the distribution of crimes in the past. Said more bluntly, these models do not believe in the possibility of reform: bad neighbourhoods will stay bad. This has nothing to do with the possibility of reforming single individuals; and indeed these algorithms will not predict the behaviour of an individual, they are not interested in who will commit a crime, but only in where and when. In reality, neighbourhoods can get better; but it is a slow process, and a process usually (but not always) started by an external influence, or anyway by an influence not represented by the distribution of past crimes. To circumvent this problem, more recent models have begun to include in the computations other factors too, ranging from community inputs to weather forecasts.

The second assumption can be summarised as "crime is boring". Being statistical models, they are geared toward predicting standard, average, repetitive (boring...) behaviours; they might miss anomalies or unprecedented incidents. In particular, they cannot be used to predict crimes of passion, or crimes due to mental illnesses—unless such crimes were more frequent in particular areas up to the point of becoming statistical meaningful.

An inevitable question in present day Europe is: can these models be used to predict terrorist attacks? At present (and hopefully forever) the answer is necessarily negative. Media exposure notwithstanding, terrorist attacks in Europe are quite rare (compared to home burglaries or car thefts, for instance); there are not enough data to meaningfully calibrate a statistical model. Terrorist attacks in Europe are isolated incidents, with a distribution closer to the one of crimes due to mental illnesses than to the one of more standard crimes, and with an even lower number of occurrences. Other models might in principle be used to prevent them, but they must be models based on different assumptions.

However, this applies to terrorist attacks in Europe. Terrorist attacks in other areas of the world, e.g., Iraq, Pakistan or the Middle East, are much more common, and they follow behaviours that are more standardised. Thus they might be amenable to being modelled with tools similar to the ones described here; if so we might rightfully say that mathematics can save lives.

References

1. http://www4.stat.ncsu.edu/~reich/st733/Journal/Logan3.pdf
2. http://www.predpol.com
3. K. Henning, Residential burglary hotspot, 1 November 2017. Retrieved from http://www.pdx.edu/crime-data/
4. http://www.transcrime.it
5. http://www.share-eu.org
6. G.O. Mohler, M.B. Short, P.J. Brantigham, F.P. Schoenberg, G.E. Tita, Self-exciting point process modeling of crime. J. Am. Stat. Assoc. **106**, 100–108 (2011)
7. G.O. Mohler, Modeling and estimation of multi-source clustering in crime and security data. Ann. Appl. Stat. **7**, 1525–1539 (2013)
8. M.B. Short, G.O. Mohler, P.J. Brantigham, G.E. Tita, Gang rivalry dynamics via coupled point process networks. Discrete Contin. Dyn. Syst., Ser. B **19**, 1459–1477 (2014)
9. J.T. Woodworth, G.O. Mohler, A.L. Bertozzi, P.J. Brantigham, Non-local crime density estimation incorporating housing information. Philos. Trans. R. Soc. A **372**, 20130403 (2014)

10. G.O. Mohler, M.B. Short, S. Malinowski, M. Johnson, G.E. Tita, A.L. Bertozzi, P.J. Brantigham, Randomized controlled field trials of predictive policing. J. Am. Stat. Assoc. **110**, 1399–1411 (2015)
11. http://www.hunchlab.com
12. http://www.civicscape.com
13. G.O. Mohler, Marked point process hotspot maps for homicide and gun crime prediction in Chicago. Int. J. Forecast. **30**, 491–497 (2014)
14. http://www.predpol.com/results/
15. http://www.nature.com/news/reform-predictive-policing-1.21338
16. http://singularityhub.com/2017/02/02/could-predictive-policing-lead-to-a-real-life-minority-report/
17. http://github.com/CivicScape/CivicScape

A Negative Number as Sum of All Natural Numbers

Marco Codegone

Introduction

A full professor of Applied Geophysics at the Politecnico di Torino wrote to me: "... these physicists are crazy, in remarkable books in Theoretical Physics and String Theory [1] an indication appears that the sum of the natural numbers is equal to minus one divided by twelve" (see also the paper [2]). Over the Internet one can find a proof which uses, in an arbitrary manner, computations with non-convergent series [3]. Starting from this suggestion we observe that the questions related to the divergent series is old. There is a paper that Leonhard Euler presented to the St. Petersburg Academy on April 25, 1727, in Latin, on "*Variae Observationes circa Series Infinitas*" [4]. In this paper the word "Infinitas" does not mean "divergent series", but rather "sum of infinite terms". More interesting for our problem is Euler's paper [5] read in 1749 to the Berlin Academy of Sciences, when he deals with the series that we call non-convergent. In this paper, Euler considers the non-convergent series $1-2+3-4+5-6+\cdots$, which has a value he claims to be $1/4$. He writes: "since the terms become increasingly large, it is quite true that we could not create a correct idea of their sum, if we understand by the sum, a value, that we all the more approach the more we add terms to the series. Thus, when it is said that the sum of this series $1-2+3-4+5-6+$ etc. is $1/4$, that must appear paradoxical. For by adding 100 terms of this series, we get -50, however, the sum of 101 terms gives $+51$, which is quite different from $1/4$ and becomes still greater when one increases the number of terms. But I have already noticed at a previous time, that it is necessary to give to the word sum a more extended meaning. We understand the sum to be the numerical value, or analytical relationship which is arrived at accord-

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ing to principles of analysis, that generate the same series for which we seek the sum. After having established this relationship, it is no more doubtful that the sum of this series $1-2+3-4+5+$etc. is $1/4$; since it arises from the expansion of the formula $1/(1+1)^2$, whose value is incontestably $1/4$. The idea becomes clearer by considering the general series: $1-2x+3x^2-4x^3+5x^4-6x^5+\cdots$ that arises while expanding the expression $1/(1+x)^2$, which this series is indeed equal to after we set $x=1$." (translation made by Pelegrí Viader Sr., Lluis Bibiloni and Pelegrí Viader Jr. [6]).

We will discuss the series

$$\sum_{n=1}^{+\infty}(-1)^{n+1}n = 1-2+3-4+5+\cdots+(-1)^n+\cdots = \frac{1}{4}, \tag{1}$$

illustrated by Euler, with the subjet of analytic continuation [7] in section "Analytical Continuation and Smoothed Asymptotics".

It is interesting to note that in 1710 the Italian mathematician Guido Grandi in his text [8] (see also [9]) discusses the following series:

$$\sum_{n=1}^{+\infty}(-1)^{n+1} = \frac{1}{2}, \tag{2}$$

finding a reason to motivate the value equal to $1/2$. He wrote: "Two brothers inherit from their father a precious stone of inestimable value, which the will forbids to sell; they established, by common agreement, that the stone is guarded alternately, one year from the first and one from the second. If it is now agreed that this rule should be between the two families of the inheritance, it turns out that the brothers will give each other the stone an infinite number of times; so each of the two brothers will eventually have half possession of the stone." This problem is discussed in section "Cesaro Summation and a Theorem by Abel" in the context of Cesaro sums (see for example in [10] p. 599) and in connection with a theorem by Abel.

A general insight comes with Hardy's book of 1949 on divergent series [11] in which he also presents Ramanujan's calculations on

$$\sum_{n=1}^{+\infty}n = 1+2+3+4+5+\cdots+n+\cdots = -\frac{1}{12}$$

which we will show in section "Ramanujan's Computation".

We also observe that it is possible (see for example Terence Tao in [12]) to shown the consistency of the result on the divergent series reasoning only with Real Analysis without the use of analytic continuation [7].

Standard Definition of Convergence

We give the definition of convergence of a numerical series $\sum_{n=1}^{+\infty} a_n$ with a_n real (or complex), in the following way. We define S_n the partial sum

$$S_n = \sum_{k=1}^{n} a_k = a_1 + a_2 + a_3 + \cdots + a_n$$

and we say that the series converges if

$$\lim_{n \to +\infty} S_n = S,$$

with S a real (or a complex) number and if this limit does not exist we say that the series is not convergent. If we have a power series $\sum_{n=1}^{+\infty} a_n x^n$ with a_n real (or complex), we define $S_n(x)$ the partial sum of the series

$$S_n(x) = \sum_{k=1}^{n} a_k(x) = a_1(x) + a_2(x) + a_3(x) + \cdots + a_n(x)$$

and we say that the series converge if

$$\lim_{n \to +\infty} S_n(x) = S(x),$$

with $S(x)$ a real (or complex) function defined in an interval (or in a disk of the complex plane), bounded or not bounded, and if this limit does not exist we say that the series is not convergent. With this definition it is a classical result that:

$$\sum_{n=1}^{+\infty} \frac{1}{2^n} = \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \cdots + \frac{1}{2^n} + \cdots = 1 \tag{3}$$

Indeed the partial sum is

$$\sum_{k=1}^{n} \frac{1}{2^k} = \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \cdots + \frac{1}{2^n} = -1 + \frac{1-(1/2)^{n+1}}{1-1/2} = \frac{(1/2)(1-(1/2)^n)}{1-1/2}$$

and by the fact that $(1/2)^n \longrightarrow 0$ for $n \longrightarrow +\infty$ we get (3).

With these definitions it is clear that the series in (1) and in (2) do not converge. We can also observe that manipulating series that do not converge in absolute value requires a lot of caution as the following Riemann-Dini theorem shows.

Let

$$\sum_{n=1}^{+\infty} a_n = a_1 + a_2 + a_3 + \cdots + a_n + \cdots$$

be a numerical series such that

$$\sum_{n=1}^{+\infty} |a_n| = |a_1| + |a_2| + |a_3| + \cdots + |a_n| + \cdots$$

does not converge, then you can give a permutation of the terms of the series such that the series has every kind of behavior.

Cesaro Summation and a Theorem by Abel

The idea of Guido Grandi, which we presented in the Introduction, has a proper justification in the Cesaro summation. If we consider the series (2) we get that the partial sums depend on the index, if the index is odd the partial sum is:

$$S_{2n+1} = \sum_{k=1}^{2n+1} (-1)^{k+1} = 1 - 1 + 1 - 1 + \cdots - 1 + 1 = 1,$$

instead, if the index is even the partial sum is:

$$S_{2n} = \sum_{k=1}^{2n} (-1)^{k+1} = 1 - 1 + 1 - 1 + \cdots + 1 - 1 = 0.$$

Then the series in not convergent, but Cesaro assumes as the sum of the series the limit of the mean value of the partial sums:

$$\lim_{n\to+\infty} \frac{S_1 + S_2 + S_3 + S_4 + S_5 + \cdots + S_n}{n}$$

and then if $n = 2k + 1$ is odd we get:

$$\lim_{k\to+\infty} \frac{1+0+1+0+1+\cdots+0+1}{2k+1} = \lim_{k\to+\infty} \frac{k+1}{2k+1} = \frac{1}{2},$$

if $n = 2k$ is even we get:

$$\lim_{k\to+\infty} \frac{1+0+1+0+1+\cdots+1+0}{2k} = \lim_{k\to+\infty} \frac{k}{2k} = \frac{1}{2},$$

we can conclude that the Cesaro summation of the Grandi series is $1/2$.

It seem that Abel's Theorem can work in this case, but is not a correct application. The Abel's theorem works in the following manner: let

$$\sum_{n=0}^{\infty} a_n = L$$

a convergent series, then we define the following function

$$f(x) = \sum_{n=0}^{\infty} a_n x^n \tag{4}$$

that converges for any $|x| < 1$ and the thesis of the theorem is:

$$\lim_{x \to 1^-} f(x) = L.$$

We present an example of application of Abel's Theorem to the convergent series

$$\sum_{n=1}^{\infty} (-1)^{n-1} 1/n.$$

We know that

$$\log(1+x) = \sum_{n=1}^{\infty} (-1)^{n-1} \frac{1}{n} x^n, \quad -1 < x \leq 1$$

then by Abel theorem we get

$$\log 2 = \sum_{n=1}^{\infty} (-1)^{n-1} \frac{1}{n}.$$

Abel's Theorem assigns to the power series for $x = 1$ the value $f(1)$ only in the case that the series $\sum_{n=0}^{\infty} a_n$ converges.

But, following the suggestion by Euler, one can abuse the theorem and, also in the case when $\sum_{n=0}^{\infty} a_n$ does not converge, assign the value $f(1)$ as in the case of the Grandi series:

$$\frac{1}{1+x} = \sum_{n=0}^{\infty} (-1)^n x^n, \quad |x| < 1 \tag{5}$$

we get

$$\begin{aligned} \frac{1}{2} &= \left[\frac{1}{1+x} \right]_{x=1} = \sum_{n=0}^{\infty} (-1)^n \\ &= 1 - 1 + 1 - 1 + 1 - 1 + 1 - 1 + \cdots + (-1)^n + \cdots . \end{aligned} \tag{6}$$

Using in this incorrect manner Abel's theorem, we can obtain the result present in the paper of Euler:

$$\frac{1}{(1+x)^2} = \sum_{n=1}^{\infty} (-1)^{n-1} n x^{n-1}, \quad |x| < 1.$$

We get

$$K = \frac{1}{4} = \left[\frac{1}{(1+x)^2}\right]_{x=1} = \sum_{n=1}^{\infty}(-1)^{n-1}n$$
$$= 1 - 2 + 3 - 4 + 5 - 6 + 7 - 8 + \cdots + (-1)^{n-1}n \cdots . \tag{7}$$

The formulas (6) and (7) have some justification through this reasoning, but they still are series which do not converge neither in absolute value nor in the usual sense nor by Abel theorem.

We observe that we can not even apply this type of reasoning for the convergence of the sum of all natural number

$$S = \sum_{n=1}^{\infty} n = 1 + 2 + 3 + 4 + 5 + 6 + 7 + \cdots . \tag{8}$$

Ramanujan's Computation

Ramanujan's computation of the sum of the natural numbers start with the following sum

$$\sum_{n=0}^{+\infty}(-1)^n = \frac{1}{2}.$$

This result is not particularly shocking because it can be justified by the Cesaro method which we considered in the preceding paragraph.

Furthermore for the series (6) and (7) we can not apply the usual algebra of convergent series and the following computation is only formal. Let S be the sum of the expression (8) and let be K the expression (7); we get

$$S - K = 1 + 2 + 3 + 4 + 5 + 6 + \cdots - (1 - 2 + 3 - 4 + 5 - 6 + \cdots)$$
$$= +4 + 8 + 12 + \cdots = 4S,$$

but $K = 1/4$ and then

$$S - \frac{1}{4} = 4S \quad \Longrightarrow \quad S = -\frac{1}{12} \tag{9}$$

that is

$$1 + 2 + 3 + 4 + 5 + 6 + 7 + \cdots = -\frac{1}{12}. \tag{10}$$

This shocking formula, which can be found in Theoretical Physics books on String Theory, can be justified, in a correct manner, only by using the theory of analytic continuation.

Analytical Continuation and Smoothed Asymptotics

To apply the principle of analytical continuation, we need to consider the complex variable $z = x + \mathrm{i}y$. The principle states that if a functions f is analytic in an open connected region A and g is analytic in a connected open region B and $A \subseteq B$ and $g = f$ in A, then g is unique and is called the analytic continuation of f.

We begin by applying this result to the formula (5), where the analytical continuation is thought in the complex variable and then it is applied to real values.

Let us put in this formula (5) $x = -2$. We obtain

$$-1 = \sum_{n=0}^{\infty} 2^n = 1 + 2 + 4 + 8 + 16 + \cdots + 2^n + \cdots \tag{11}$$

but this equality has the meaning that the function $f(x) = 1/(1 + x)$, that is equal to the series (5) only for any $|x| < 1$, is such that

$$f(-2) = \left[\frac{1}{1+x}\right]_{x=-2} = -1.$$

This value corresponds to the analytical continuation of the series.

Likewise, if we take the derivative of $-1/(1 + x)$, we get $1/(1 + x)^2$ and then

$$g(x) = \frac{\mathrm{d}}{\mathrm{d}x}\frac{-1}{1+x} = \frac{1}{(1+x)^2} = \sum_{n=1}^{+\infty}(-1)^{n-1}\, n\, x^{n-1}$$

for any $|x| < 1$. The value

$$g(1) = \frac{1}{4}$$

correspond to the analytic continuation of the series and the expression (1) must be understood in this sense.

In our case we consider the function

$$Z(x) = \sum_{n=1}^{\infty} \frac{1}{n^x} \tag{12}$$

that is a convergent series only for $x > 1$. If in the place of the variable x in the series (12) we put the complex variable $z = x + \mathrm{i}y$ the Riemann function $\zeta(z)$ is identical to the series

$$\sum_{n=1}^{\infty} \frac{1}{n^z} \tag{13}$$

only for $\mathfrak{Re}\, z > 1$. Then the correct expression is not (10) but

$$\zeta(-1) = -\frac{1}{12} \tag{14}$$

so that this value corresponds to the value of the analytic continuation of the function defined by the series (13). When in the place of formula (14) one writes

$$\sum_{n=1}^{\infty} \frac{1}{n^{-1}} = \sum_{n=1}^{\infty} n = -\frac{1}{12}$$

the result is written in an ambiguous manner.

Finally, we want to present a topic discussed by Terence Tao [12]. He suggests to study the series

$$\sum_{n=1}^{\infty} \frac{1}{n^z}\, \eta(n/N)$$

where η is a cut-off function, or more precisely a compactly supported bounded function such that $\eta(0) = 1$, for $x > 0$ $\eta(x)$ is $\mathcal{C}^{+\infty}$ and $\eta(x) = 0$ for $x > 1$. In this case, one can prove (see also [13]) that the expansion of

$$\sum_{n=1}^{\infty} n\; \eta(n/N)$$

is asymptotic to

$$-\frac{1}{12} + cN^2$$

with c a constant that depends on η, as $N \to +\infty$. The leading term $-1/12$ does not depend on η.

Conclusion

The statement that the sum of all natural numbers is $-1/12$ has given me the opportunity to discuss the subject of divergent series, which is ancient and has been studied by several authors.

Divergent series can be seen as the computation of the value of a function defined as series, but its coincidence with the series is justified only in the region of convergence of the series.

These arguments are present in many books and papers on Theoretical Physics, where are often given a proof using the renormalization methods. Terence Tao gives a mathematical justification without using analytical continuation.

We wish to point out that the result that the Riemann function ζ has value $-\frac{1}{12}$ would not have attracted the attention of more than a million and a half visitors on [3], whereas writing

$$1+2+3+4+5+6+7+\cdots\,.. = -\frac{1}{12}$$

has achieved a great success. Based on this, we can conclude that to make Mathematics more attractive, even at the cost of writing some inaccuracies, we should exhibit some shocking results.

References

1. J. Polchinski, *String Theory Vol. I: An Introduction to the Bosonic String* (Cambridge University Press, Cambridge, 1998)
2. F. Gliozzi, The infrared limit of QCD effective string. Acta Phys. Pol. B **23**(10), 971–989 (1992)
3. https://www.youtube.com/watch?v=w-I6XTVZXww
4. L. Euler, *Variae Observationes circa Series Infinitas*. Commentarii Academiae Scientiarum Imperialis Petropolitanae E72 (1727)
5. L. Euler, Remarques sur un beau rapport entre les series des puissances tant directes que reciproques. Mem. Acad. Sci. Berlin **17**, 83–106 (1768), originally published. Opera Omnia: Ser. 1 **15**, 70–90
6. http://eulerarchive.maa.org/docs/translations/E072en.pdf
7. H. Cartan, *Théorie Élémentaire des Fonctions Analytiques d'une Ou Plusieurs Variables Complexes* (Hermann, Paris, 1961)
8. G. Grandi, *De Infinitis Infinitorum, et Infinite Parvorum Ordinibus Disquisitio Geometrica* (Typographia Francisci Bindi Impress, Archiepisch, Pisa 1710)
9. S. Leonesi, in *Xenia: Suggestioni Matematiche Verso L'infinito*. Associazione Archivio Storico Bellabarba, San Severino Marche (2015)
10. G.E. Andrews, R. Askey, R. Roy, *Special Function* (Cambridge University Press, Cambridge, 2000)
11. G.H. Hardy, *Divergent Series* (Clarendon Press, Oxford, 1949)
12. https://terrytao.wordpress.com/2010/04/10/the-euler-maclaurin-formula-bernoulli-numbers-the-zeta-function-and-real-variable-analytic-continuation/
13. https://en.wikipedia.org/wiki/1_%2B_2_%2B_3_%2B_4_%2B_%E2%8B%AF

The *Arch of Titus* at the *Circus Maximus*: From Mathematical Models to Virtual Reconstruction

Corrado Falcolini

Introduction

A new accurate parametric model reconstruction of the *Arch of Titus* at the *Circus Maximus* in Rome has been presented ([6]), based on previously and newly found fragments (see Fig. 1), historical knowledge and architectural order proportions. It is a joint work, with M. Canciani and M. Saccone of the Department of Architecture of *Roma Tre University* and M. Buonfiglio and S. Pergola archeologists of the *Sovrintendenza Capitolina* (see [1]), that started in 2012 and ended up by a large excavation of the area in 2015, when many new pieces of the ancient arch were found. The initial collaboration between *Roma Tre University* and the *Sovrintendenza Capitolina* was joined by two master's students (B. Mammì and G. Romito) during the first excavation of the area in 2013 (see also [3, 7]) and the final study was conducted also with M. Pastor, A. Coletta and M.G. Granino (see [6]). The virtual reconstruction of the arch was made possible through more than one hundred surveyed fragments: with laser scanning and photogrammetric programs we got their point cloud (a list of million points coordinates) 3D representation, useful for measuring and features extraction and archived in a specific database. The model has been optimized, with original algorithms and programs, using all the processed data and has been verified and improved after each new fragment acquisition.

Here we present some simple mathematical applications and algorithms that have been applied in a few phases of the virtual reconstruction process. We will try to answer two questions: (1) is it possible to find the base diameter of a damaged, fluted column with high accuracy? and (2) given two fragments, is it possible to check their probability of matching along a contact surface?

The answers will also tell a story of search and accomplishments.

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Fig. 1 Actual position of old and new found fragments of the *Arch of Titus* inside the *Circus Maximus* in Rome

The 3D Model

The final 3D model of the arch has been obtained with the help of mathematical algorithms in several of the steps, during the full process: the development of a database (containing more than one hundred fragments of different size), using automatic feature extraction, contact surfaces detection and optimization of geometrical and parametrical description are only some of the developed applications. Similar procedures and techniques have been widely used in recent years (see for example [10, 11]).

The database collects, in a reference frame, the point cloud of every fragment; it is based on a survey done with a laser scanner and photogrammetry programs, together with a classification of several features suited for matching comparison. A general method was proposed for a classification procedure adaptable for anastylosis in similar cases. The algorithms discussed here are applied to a starting point cloud of one given fragment and deal with the best possible section curve on a plane orthogonal to the moulding extrusion path, an optimal fluting column model and contact probability of two possibly adjacent fragments. One of the most useful algorithms is an automatic procedure (see [2]) for the extraction of a piecewise regular parametric curve section directly from the point cloud: the plane of section can be detected analyzing the normal components in the standard point cloud data, followed by a parametric planar curve fitting of properly selected points; special points on this parametric curve allow an automatic geometrical moulding construction which can be translated in feature elements of the database and used in matching procedures. All measures have been optimized up to a certain error using the processed data and have been verified and improved after each new fragment addition in the database.

Fig. 2 Old ruins of the *Arch of Titus* at the *Circus Maximus* in Rome; one column base is still in situ. Right: 3D surveyed point cloud of a (damaged) fluted column drum

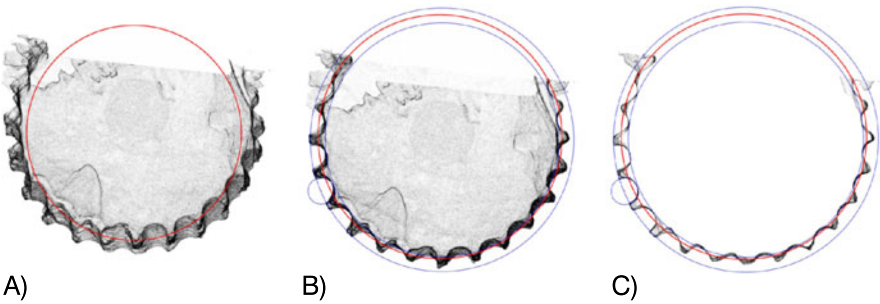


Fig. 3 Orthogonal projection of a surveyed column drum fragment along the best cylinder axis. (**a**) First automatic fitting: the column is not well oriented. (**b**) Second automatic fitting. (**c**) Drum bases removed: note the part with a very clear border

Column Radius

From the survey of a ruined column drum (see Fig. 2), the first algorithm example shows how it is possible to find an optimal fluted column model: comparing a cylindrical projection of the oriented point cloud with several section models allows an optimal measure of column and flutings radii, as well as a possible drum reposition (see [9]).

The orientation of the surveyed column can be automatically fixed by the best fitting cylinder: the sum of the distances of every point of the cloud from a parametrized surface can be minimized with a least square fitting. The resulting parameters give the cylinder axis, which can be aligned with the vertical direction. Such orientation, in general, is not exact due to the non-symmetrical shape of the fragment (see the effect of the vertical projection of the drum on an horizontal plane in Fig. 3a).

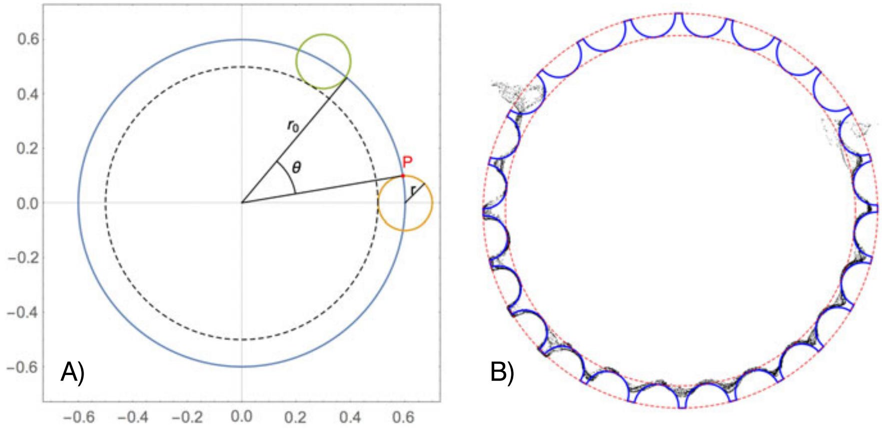


Fig. 4 Fluting column model. (**a**) Fluting centers are on the outer column circle. (**b**) Good correspondence between the model and the surveyed data

The automatic procedure can be repeated gaining a better orientation (see Fig. 3b). The final clear shape is obtained removing the drum bases which affected the object symmetry (see Fig. 3c): it is now possible to find column and flutings radii with high accuracy comparing different sections of a column model.

The horizontal section of the column is modeled (see Fig. 4a) with a circle of radius r_0 and n uniformly distributed smaller circles of radius r centered on its border. The coordinates (x, y) of the points P of intersection of the circles and the angular distance θ between two flutings are given by elementary formulas:

$$\begin{aligned} x^2 + y^2 &= r_0^2 \\ (x - r_0)^2 + y^2 &= r^2 \end{aligned} \tag{1}$$

$$x_P = r_0 - \frac{r^2}{2r_0} \tag{2}$$

$$\theta = \frac{2\pi}{n} - 2 \arccos \frac{x_P}{r_0} \tag{3}$$

and varying the parameters n, r, r_0 it is possible to find a model which best approximates the chosen section (see Fig. 4b).

Despite the very small fraction of the arch left, we are lucky to have a few moulding pieces, four letter pieces of the epigraph and at least one fragment of the final top part of a column, the *sommoscapo* (see Fig. 5a), which gave us the exact tapering ratio of radii at the base and the top of the column shaft.

Vertical planar sections of the oriented fragment (CM454 in the database) of the *sommoscapo* are shown in Fig. 5b. The sections with different planes, all containing the vertical axes of the column, are rotated and shown on a common plane to test its cylindrical symmetry and measure possible molding proportions. The column top

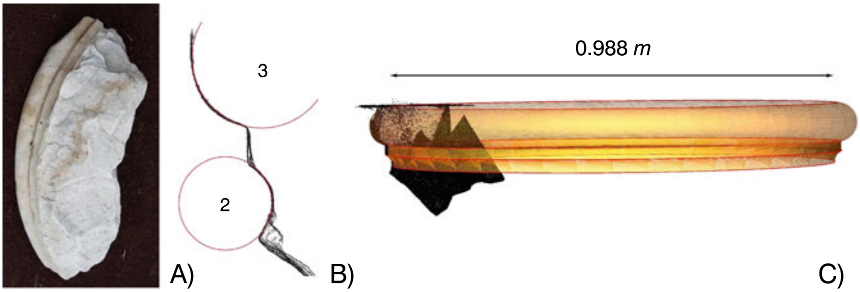


Fig. 5 (**a**) A fragment of the *sommoscapo*. (**b**) Vertical planar sections and proportions: note the ratio 2:3 of the molding arcs radii. (**c**) Diameter evaluation using the point cloud model

diameter is evaluated with the best circle fitting particular horizontal sections (see Fig. 5c).

The column model is then completed using the information on the historical architectural order of that period. Model and measure results are finally tested on several other column fragments: in some cases they allow to recognize other fragments of the same column or their vertical position, proportional to the tapering inclination of 100/3 .

The diameter of some ruined columns was evaluated with a relative error less than 1%, for the column drums with several flutings (see Fig. 3), and less than 3% for the piece of the *sommoscapo* (see Fig. 5). We can now answer the first question: the column base diameter was found very accurately close to the value of 1.186 m which is the architectural order modulus length of 4 *Roman feet* (1 foot = 29.65 cm) and the *sommoscapo* diameter is very close to the presumed value (5/6 of the base diameter) of 98.83 cm.

The Epigraph

The second example shows the use of section curves for a precise fragment orientation and alignment: the required accuracy allowed a virtual segmentation of the surface of contact of two fragments of the inscription, leading to a final perfect matching and a better understanding of its probable size and shape.

Four small pieces of the long, original epigraph were found during the last excavation: three of them are shown in Fig. 6 and all have a unique position in the virtual reconstruction model. How is that possible?

The story of this epigraph is very interesting: the arch was still standing in the IX century when an unknown visitor wrote a sort of *touristic guide* of Rome. He copied many epigraphs from a majority of the Ancient Roman monuments (see Fig. 7a) and this manuscript booklet is now stored at the Abbey Library of Einsiedeln (Ch). The accuracy of the transcription can be tested on preserved monuments, like the *Arch of Septimius Severus* or the *Arch of Constantine* (see Fig. 7a for the text). Moreover,

Fig. 6 New found fragments of the epigraph of the *Arch of Titus* at the *Circus Maximus* in Rome. Right: two fragments recomposition and other new found fragments of the arch

Fig. 7 The epigraph: (**a**) from the manuscript of an Unknown (IX century) at Einsiedeln, Stiftsbibliothek/Codex 326(1076)—Manuscript of collected works/f. 71v, 72r—Virtual Manuscript Library of Switzerland (www.e-codices.unifr.ch); (**b**) virtual reconstruction hypothesis with the fragments of Fig. 6 in place

the dimension of a rectangle containing the epigraph depends on letter size and lines length and should fit in the Roman arch proportions. It seems that there are too many uncertainties, but it turned out that this is not the case: in fact two pieces are close

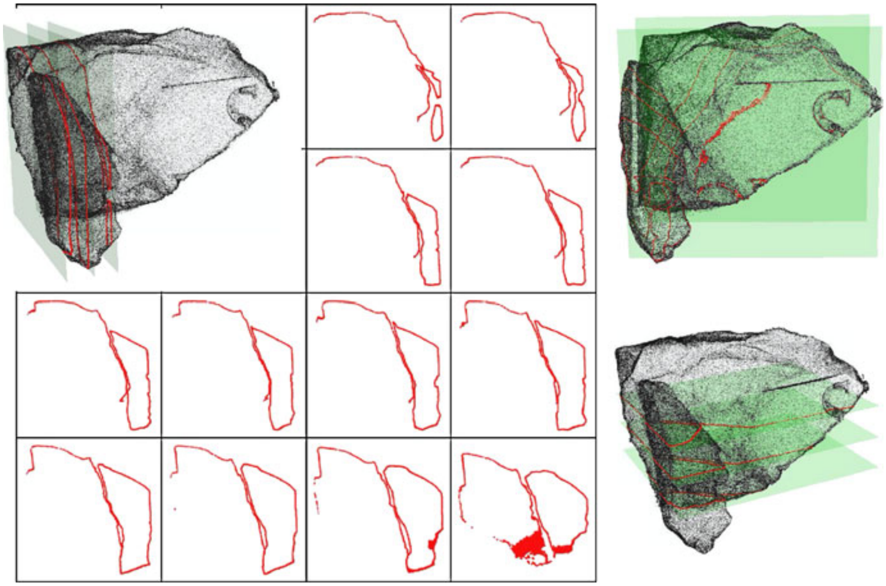


Fig. 8 Section curves along parallel planes looking for a possible contact surface between the two fragments

to the frame and a third has two uncommon partial letters. The biggest letter (see Fig. 6), part of an “S”, is on top and is easily associated to the word “SENATUS”, while the letter “R” is at the bottom but its position is determined by the following partial letter.

The intriguing part was the fragment of a letter “P” which is not near the frame and is a letter repeated many times. The solution of the puzzle came up with the least probable hypothesis: matching two of the four pieces on a small, and very irregular, surface of contact. Given the size and height of the pieces, we studied their alignment on parallel planar sections along the three principal directions (see Fig. 8) looking for similar slopes or change of the curvature allowing small movements but preventing intersections. Note the difference in the distance function between the two curves, due to the very irregular shape along the fracture. We first estimated the matching probability value along the surfaces and, finally, successfully placed the two pieces together (see Fig. 6).

Conclusions

The *Arch of Titus* area of the *Circus Maximus* in Rome has now been opened to the public, with a documented visitor’s itinerary along many of the arch fragments (see Fig. 6). The mathematical models used in the virtual reconstruction of the arch (see Fig. 9) are in a parametric form suited for improvements using features of all

Fig. 9 Virtual reconstruction: all the main fragments have found a displacement which is compatible with the fixed parameter values of the model.(Elaboration of M. Canciani)

the known fragments and adaptable to successive findings: optimization is made possible by measures for an accurate quantitative analysis.

Some of the original algorithms presented here have been also applied in different contexts: for instance several studies on the shape and tessellation of Borromini's *San Carlino* dome (see for example [2, 8]), the recomposition and restoration of the fragmented statue of *S. Andrea* at Stiffe, L'Aquila (see [5]) or the bricks detection and analysis of masonry type of *Villa dei Misteri* in Pompei (see [4]). Some of the starting procedures (surveying, best fit procedures, mathematical models and optimization) are currently part of an advanced architecture university course.

References

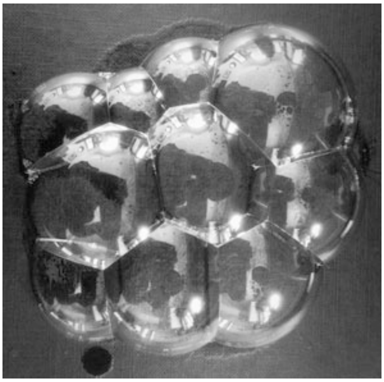
1. M. Canciani, C. Falcolini, M. Buonfiglio, S. Pergola, M. Saccone, B. Mammì, G. Romito, A method for virtual anastylosis: the case of the Arch of Titus at the Circus Maximus in Rome. ISPRS Ann. Photogramm. Remote Sens. Spatial Inf. Sci. **II-5/W1**, 61–66 (2013)
2. M. Canciani, C. Falcolini, M. Saccone, G. Spadafora, From Point Clouds to Architectural Models: algorithms for shape reconstruction. ISPRS Ann. Photogramm. Remote Sens. Spatial Inf. Sci. **XL-5/W1**, 27–34 (2013)

3. M. Canciani, C. Falcolini, M. Buonfiglio, S. Pergola, M. Saccone, B. Mammì, G. Romito, Virtual Anastylosis of the Arch of Titus at Circus Maximus in Rome. International Journal of Heritage in the Digital Era **3**(2), 393–412 (2014)
4. M. Canciani, C. Falcolini, M. Saccone, G. Spadafora, From Point Cloud to Archaeology: the case study of Villa dei Misteri in Pompeii, in *XII International Forum Le vie dei Mercanti, Best Practice in Heritage Conservation Management, from the World to Pompeii* (2014)
5. M. Canciani, G. Capriotti, L. D'Alessandro, C. Falcolini, Algoritmi rinascimentali: Il progetto di Image modeling 3D per il restauro della scultura in terracotta del Sant'Andrea di Stiffe (Aq), in *XIV Congresso Nazionale IGIIC—Lo stato dell'Arte—Accademia Di Belle Arti di L'Aquila* (2016), pp. 79–90
6. M. Canciani, C. Falcolini, M. Pastor Altaba, M. Saccone, Ricostruzione virtuale e realtà aumentata, in *Bullettino della Commissione Archeologica Comunale di Roma*, vol. 118 (2018), pp. 237–250
7. A. Coletta, S. Pergola, Circo massimo. Considerazioni sulla decorazione architettonica dell' Arco di tito, in *Bullettino della Commissione Archeologica Comunale di Roma*, vol. 115 (2016)
8. C. Falcolini, Algorithms for Geometrical Models in Borromini's San Carlino alle Quattro Fontane, in *Handbook of Research on Visual Computing and Emerging Geometrical Design Tools*, vol. 26 (2016), pp. 642–665
9. C. Falcolini, V. Talamanca, Modelli geometrici per nuvole di punti, in *Atti del VII Convegno Mathematica Italia UGM*, Napoli (2015). ISBN 978-88-96810-04-0
10. Q.X. Huang, S. Flöry, N. Gelfand, M. Hofer, H. Pottmann, Reassembling fractured objects by geometric matching. ACM Trans. Graph. **25**(3), 569–578 (2006)
11. R. Scopigno, M. Callieri, P. Cignoni, M. Corsini, M. Dellepiane, F. Ponchio, G. Ranzuglia, 3D models for cultural heritage: beyond plain visualization. IEEE Comput. **44** (2011)

The Space of Planar Soap Bubble Clusters

Frank Morgan

Soap bubbles and foams have been extensively studied by scientists, engineers, and mathematicians as models for organisms and materials, with applications ranging from extinguishing fires to mining to baking bread; see for example Cantat et al. [2], Weaire-Hutzler [15], Morgan [9, Chap. 13], and references therein. Here we provide some basic results on the space of planar clusters of n bubbles of fixed topology. We show for example that such a space of clusters with positive second variation is an n-dimensional manifold, although the larger space without the positive second variation assumption can have singularities. Earlier work of Moukarzel [12] showed how to realize a cluster as a certain generalized Voronoi partition, though not canonically.



Soap Bubble Clusters

Definition 1 A planar *cluster* consists of disjoint circular arcs/line segments meeting in threes at positive angles, enclosing n connected regions, with areas denoted

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$A_1, A_2, \ldots, A_n$. We assume that the cluster is connected and has at least two regions ($n \geq 2$). A cluster is in *quasi-equilibrium* if the arcs meet at 120 degrees. For *equilibrium* we further assume that the sum of the curvatures around a path from a region to itself is 0, or equivalently around a vertex. This condition makes *pressure* well defined as the sum of the curvatures along a path from the exterior to the point. An equivalent form of the condition is that circles/lines through a vertex meet again (each forward or backwards, at infinity in the case of three straight lines), as follows by the law of sines [9, Fig. 14.1.2]. All of these definitions are preserved by Möbius (linear fractional) transformations. A cluster is in equilibrium if and only if the cluster has vanishing first variation of length under smooth deformations of the plane which preserve the areas [13, Prop. 2.6 and Appendix A]. A cluster is *minimizing* if it minimizes length for given areas. A minimizing cluster has vanishing first variation and nonnegative second variation.

More generally and technically one might define a soap bubble cluster in $\mathbf{R}^N$ as n disjoint regions of finite volume and perimeter such that Lipschitz deformations inside small balls preserving the volumes cannot reduce the area of the union of the boundaries [9, §11.3]. In $\mathbf{R}^2$ such are equilibrium clusters as defined above [9, §13.10].

Parameterize clusters of given combinatorial type, with say v vertices and e edges, by the vertices $V_1, V_2, \ldots, V_v$, and for each edge from V_i to V_j the oriented area $a_{ij} = -a_{ji}$ between the edge and a straight line between vertices. Since $e = 1.5v$, by Euler's formula there are $2(n-1)$ vertices and $3(n-1)$ edges. On this $(7n-7)$-dimensional manifold C the areas A_i and perimeter P are smooth functions. (Using areas a_{ij} instead of curvatures avoids ambiguity of large and small circular arcs of the same curvature.) Since rigid motions infinitesimally have no fixed points (because by the assumption $n \geq 2$ we exclude a circle ($n = 1$)), there is a smooth quotient manifold Q of dimension $7n - 10$.

Lemma 2 *The Jacobian of the area vector* $(A_1, A_2, \ldots, A_n)$ *has full rank n* (*on C and on Q*).

Proof The area of a region sharing an edge E_{ij} with the exterior can be varied by varying a_{ij}. Adjacent regions can be similarly adjusted by varying the a_{ij} of a shared edge and restoring the area of the outer region. Work your way inward to obtain arbitrary variations. □

Corollary 3 *Level sets of fixed area vector in the spaces of clusters* (*C and Q*) *provide a smooth foliation* (*by the Inverse Function Theorem*).

Theorem 4 *The space in Q of equilibrium clusters modulo rigid motions with n regions with positive second variation for fixed areas is a smooth n-dimensional manifold, locally parametrized by the areas.*

Proof Varying areas smoothly preserves positive second variation equilibria. □

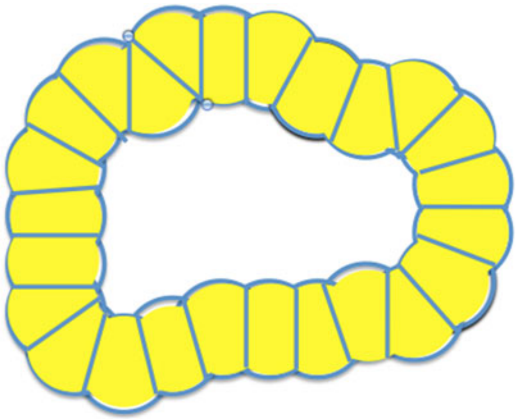


Fig. 1 A long chain or "necklace" of $n - 1$ say unit curvature bubbles surrounding a chamber with 0 pressure is floppy

Conjecture 5 *An equilibrium of positive second variation is unique for given area vector and given combinatorial type.*

Remarks 6 The theorem and conjecture do not hold with positive second variation replaced by nonnegative second variation or stability, as observed by Weaire et al. [16]. As a trivial example, consider a bubble with two small lenses in its boundary, with the extra degree of freedom of the distance between the lenses, yielding a $(3 + 1)$-dimensional manifold of stable equilibria. Lenses tend to add an extra dimension. In the symmetric case, the two middle arcs can be replaced by arcs of a different curvature, yielding quasi-equilibria, as pointed out to me by John M. Sullivan, another 4-dimensional manifold containing the symmetric case, which is thus a non-manifold point in the space of quasi-equilibria.

For a more interesting example, a long chain or "necklace" of $n - 1$ say unit curvature bubbles surrounding a chamber with 0 pressure as in Fig. 1 is floppy, with a multi-parameter family of configurations with the same areas and pressures. To compute the dimension, note that you can slide each bubble around the adjacent one, for $n - 2$ free parameters, minus 2 for them to match up and 1 for rigid rotation and 1 more to preserve the area of the central chamber, for a total of $n - 5$. These necklaces can probably be shown minimizing by the methods of Cox et al. [3]. The symmetric one also sits in a one-parameter family of varying area and pressure of the central chamber, apparently not available in the non-symmetric case, so that the space is not a manifold at that point. For smaller area the pressure goes negative and the cluster becomes unstable. For larger area the pressure goes positive, the cluster becomes strictly stable, and all of the areas can be varied, but the cluster is no longer floppy.

Is the set of equilibria of fixed combinatorial type connected? Equilibria with nonnegative second variation? For fixed area vector?

The theorem and proof hold if length is replaced by any smooth, uniformly convex norm. For equilibrium, curves meet in threes such that the unit duals of the tangent vectors and the (constant) generalized curvatures both sum to zero. See [8, 11].

It is unknown whether the manifold of Theorem 4 is locally parametrized by pressures, except for double bubbles in $\mathbf{R}^2$, where the only stable double bubble is the standard double bubble [10].

Dimension of Equilibria

An upper bound on the dimension on the space of equilibria is given by placing the vertices ($2v$) and the orientation of the curves leaving one vertex (1), which determines the orientations at adjacent vertices, etc., minus rigid motions (3), for a total of $2v - 2 = 4n - 6$. Fenyes [4] proved that if every bubble has at least three sides the conditions on each edge that the angle of the curve with the chord is the same at both ends (which he denotes by s_{ij}) are independent on vertices plus orientations, which yields a bound of $3v - 3 - 1.5v = 1.5v - 3 = 3n - 6$. Adding $v - 1$ curvature constraints would bring the estimate down to $0.5v - 2 = n - 3$, just 3 less than expected, probably due to some minor dependence among the curvature constraints. Indeed, none are needed for the triple bubble ($v = 4$), and using $v - 4$ instead of $v - 1$ would yield exactly the expected n. So probably generically the dimension is due to placing v vertices, choosing directions at each vertex, the s_{ij} edge constraints, the curvature constraints, and modding out by rigid motions:

$$2v + v - 1.5v - (v - 4) - 3 = 0.5v + 1 = n.$$

To prove Fenyes's lemma, sending the conjugate vertex to infinity, one has straight lines from points x_i, x_j, and x_k meeting x_h at 120 degrees. The result holds because for fixed x_i, x_j, x_k and directions from each of them, the point x_h is the only point where circular arcs from x_i, x_j, and x_k in those directions can meet at 120 degrees.

Question Is the space of equilibria of given combinatorial type connected? for given area vector?

Small Clusters

Any two vertices support a 1-parameter family of quasi-equilibrium double bubbles, all equilibria by trigonometry, all minimizing by the planar double bubble theorem [9]. The number of parameters equals $4 - 3$ [rigid motions] $= 1$, and $1 + 1 = 2$.

Any three vertices uniquely determine the fourth vertex and a triple quasi-equilibrium bubble (easy in the most symmetric case, in general by Möbius transformations), which in fact are equilibria. By Wichiramala [18], they are minimizing. The number of parameters is $6 - 3 = 3$ (or the previous 2 plus the 1-parameter family of decorations).

Similarly any such three vertices and an appropriate fourth on the triple bubble determine a standard 4-bubble. The number of parameters equals $6 + 1 - 3 = 4$ (or the previous 3 plus 1).

A 4D family of equilibrium 3-clusters is given by a circle decorated by two lenses. In the symmetric case, the two middle arcs can be replaced by arcs of a different curvature, as pointed out to me by John M. Sullivan, yielding another 4D family of quasi-equilibria.

The standard type 4-cluster similarly admits quasi-equilibria. Start with an equal-area double bubble, with one bubble above the other. Decorate the two vertices symmetrically to produce a 4-bubble symmetric under horizontal as well as vertical reflection. Now lengthen the horizontal line in the middle and replace the top and bottom arcs with others of smaller curvature to maintain the 120-degree angles.

The first equilibrium cluster not to arise from repeated decorations is the flower with four petals around a 4-sided central bubble as in Fig. 2 below. Is it part of a 5-dimensional family of equilibria? (Yes, next paragraphs.) Are they stable? (I think so.) Are there more quasi-equilibria? (Yes, similar to previous constructions.)

As Sullivan (email 2014) pointed out, “It’s easy to find a 4-parameter family: note that in the symmetric cluster, the size of the inner bubble can be varied; then apply Möbius transformations including scaling. What this doesn’t give is clusters of the form where the four bubbles in the ring around the center alternate large, small, large, small [because any inversion keeping one pair of opposites equal makes the other pair unequal]. But I think those could be constructed easily explicitly (with D_2 symmetry).”

An easy way to construct these 5-clusters with D_2 symmetry is to note that a quarter is part of a double bubble with a lens (which may extend outside the double bubble). So start with a double bubble, with one bubble above the other, with a centered lens (three parameters) and slide the lens until the line perpendicular to two upper circular arcs is perpendicular to the line perpendicular to the two lower circular arcs. Now reflection across the two lines yields the desired 5-cluster. At least for nearly equal petal areas, these clusters are all stable [1]. Now Möbius inversion yields a 5-parameter family of equilibrium 5-clusters (without imposed symmetry).

The space of circular arc triangles with three 120-degree angles has dimension 3 (vertices arbitrary by linear fractional transformations and uniqueness obvious from equilateral case).

3D Clusters

The space of clusters in space is harder to parameterize; we follow the methods of White [17] (see his Intro. and Thm. 3.1). We consider clusters defined as smooth surfaces meeting in threes at positive angles along smooth curves, which in turn meet in fours at positive angles, enclosing n connected regions, with volumes denoted V_i. An equilibrium cluster consists of constant-mean curvature surfaces (not necessarily spherical) meeting in threes at 120 degrees along smooth curves, which in turn meet in fours at equal angles of $\cos^{-1}(-1/3) \approx 109$ degrees [9, 11.3]. We assume that a cluster is connected and has at least two regions ($n \geq 2$). At a given equilibrium cluster C, we consider perturbations as follows. First we perturb the singular points, replacing the singular curves by homothetic copies. Next via real-analytic transverse vectorfields $\mathbf{u}, \mathbf{v}$ we perturb each singular curve by a smooth linear combination of $\mathbf{u}$ and $\mathbf{v}$ vanishing at the endpoints. We adjust each component of the surface by the harmonic function with the given change in boundary values. Finally, via a real-analytic transverse vectorfield $\mathbf{w}$, we perturb each component of the surface by a smooth multiple of $\mathbf{w}$ vanishing at the boundary. This process parameterizes a neighborhood in the space of clusters of a given cluster with a neighborhood of the origin in a space of smooth functions.

We take as our space of smooth functions the Banach space $C^{2,\alpha}$ (fixed $0 < \alpha < 1$) of twice Hölder differentiable functions under the standard $C^{2,\alpha}$ norm, making the space of clusters a smooth Banach manifold. The enclosed volume vector is a real-analytic function on this linear space. The advantage of the Hölder spaces is that harmonic functions inherit the smoothness of the boundary values up to the boundary. White [17, §1.5] explains how to modify the space to make it separable. A good reference on Banach manifolds is provided by Lang [6].

Theorem 7 *The space of clusters of n bubbles modulo rigid motions with positive second variation in* $\mathbf{R}^N$ *is a smooth n-dimensional manifold, locally parametrized by the volumes.*

Proof We consider perturbations as above of a fixed cluster with positive second variation. As in the 2D case, now by the inverse function theorem on Banach manifolds, the space is C^∞ foliated by submanifolds of fixed volume vector. The theorem follows. (Because the range of the volume vector V is finite dimensional, there is trivially a subspace complementary to ker DV, yielding the desired local isomorphism of the tangent space with ker $DV \times \mathbf{R}^n$, which by the inverse function theorem implies that the space of clusters is locally diffeomorphic to $\{V{=}0\} \times \{V\}$.) □

Remarks It is unknown whether the manifold is locally parametrized by pressures, even for double bubbles in $\mathbf{R}^3$, where it was a major advance to prove that pressure is strictly decreasing in volume for *area-minimizing* double bubbles ([5, Thm. 3.2], [9, 14.5]).

The above approach and Theorem 7 hold for piecewise smooth clusters (stratified manifolds) in $\mathbf{R}^N$. More generally and technically one could define an (equilibrium) soap bubble cluster in $\mathbf{R}^N$ as n disjoint regions of finite volume and perimeter such that Lipschitz deformations inside small balls preserving the volumes cannot reduce the area of the union of the boundaries [9, 11.3]. In $\mathbf{R}^2$ and $\mathbf{R}^3$ such are equilibrium clusters as previously defined, although in $\mathbf{R}^3$ it is not known whether every four-curve singularity satisfies the local area minimization condition [7]. In higher dimensions, it is not known whether such general equilibrium clusters are piecewise smooth (stratified manifolds).

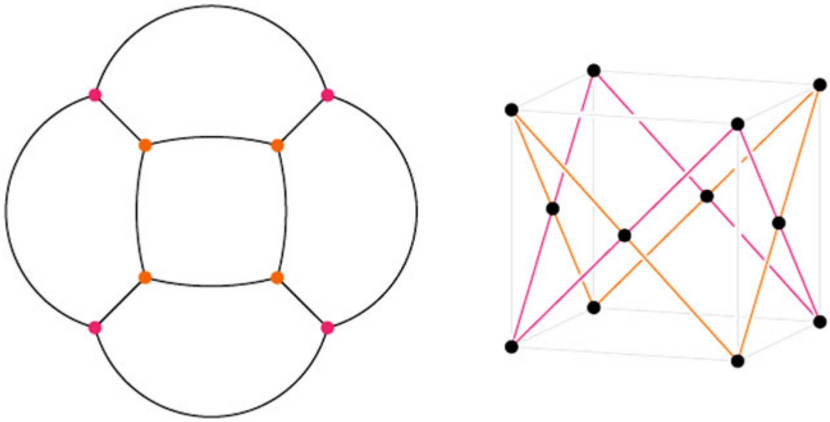


Fig. 2 A 4-flower and a schematic of its representation in de Sitter spacetime, where points (representing planar circles) occur in equally spaced triples on geodesics by the equilibrium conditions at each junction of the soap film. Aaron Fenyes

de Sitter Spacetime

The following theorem was communicated to me by Aaron Fenyes, who also provided Figs. 2 and 3.

Theorem 8 (Fenyes, 2014) *There is a natural correspondence between connected n-junction immersed equilibrium clusters in the plane with a point at infinity and the algebraic variety of n triples of points in de Sitter spacetime, each triple evenly spaced at distances* $2\pi/3$ *on an oriented line, the* $3n$ (*not necessarily distinct*) *points in antipodal pairs.*

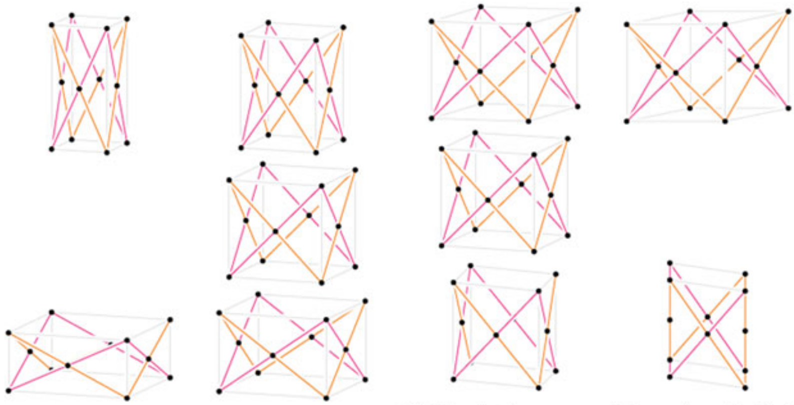
For simplicity of description, we are assuming that the junctions of the cluster are distinct and that the triples on oriented lines are distinct (although for example you could have the same triple evenly spaced on the same line with *opposite* orientations, corresponding to three films meeting at two points, as in the double bubble). The triple associated to a junction consists of the oriented circles/lines leaving the junction, the orientation of their line giving their counterclockwise order around the junction.

Figures 2 and 3 provide a schematic representation in de Sitter spacetime of the 4-flower discussed above and a two-parameter family of deformations.

Proof of Theorem 8 de Sitter spacetime S consists of the oriented spacelike lines in

$$M = (3+1)\text{-dimensional Minkowski spacetime} = \{2 \times 2 \text{ Hermitian matrices}\}$$

(see [14]). Three oriented circles/lines meet at two points at $2\pi/3$ radians if and only if they are collinear and evenly spaced at distances $2\pi/3$ in de Sitter spacetime, and



(a) Vertically squashing and stretching the cage swells and shrinks the central bubble.

(b) Deforming the square cap of the cage into a rhombus breaks the rotational symmetry of the flower, swelling one pair of opposite petals and shrinking the other pair.

Fig. 3 Two families of deformations of the representation in de Sitter spacetime of the 4-flower. Aaron Fenyes

the orientation of the line determines their cyclic order and hence which of the two points is in the cluster. The pairing of the points in antipodal pairs corresponds to the fact that each circle/line leaves one junction and enters another; hence appears with both orientations, antipodal points in S. □

Acknowledgements These discussions began with undergraduate research by Aaron Fenyes [4] and continued when Morgan and John M. Sullivan were attending the inspiring conference "Foams and Minimal Surfaces—12 Years On" newton.ac.uk/programmes/FMS/fmsw02.shtml at the Isaac Newton Institute, 24–28 February 2014, organized by Simon Cox and Denis Weaire. I am grateful to Fenyes and Sullivan for their contributions.

Appendix: Decoration

Proposition 9 *A* 3*-sided bubble in a cluster can be expanded or shrunk* (*even to a point*) *without affecting the rest of the cluster.*

Proof sketch Given a 3-sided bubble, use a linear fractional transformation to map a vertex to infinity and the other point where the two incident edges meet to the origin. The third edge is mapped to one side of an equilateral 3-sided bubble centered at the origin. Now map its third vertex to infinity. Our original bubble is now mapped to an equilateral 3-sided bubble, which can be shrunk or expanded without affecting the rest of the cluster. □

To show failure for k-sided bubbles, we need the following lemma about double bubbles. We know from double bubbles that when a circular arc splits into two others in an equilibrium cluster and they continue until they meet again, the curve that emerges is a continuation of the original circular arc.

Lemma 10 *For a double bubble consisting of bubbles of radius r_1 and r_2, increasing r_2 causes the distance d between their centers to decrease if $r_2 < 0.5r_1$ and increase if $r_2 > 0.5r_1$.*

Proof See Morgan [9, Fig. 14.1.2]. By the law of cosines,

$$d^2 = r_1^2 + r_2^2 - r_1 r_2.$$

Therefore $2d(\mathrm{d}d/\mathrm{d}r_2) = 2r_2 - r_1$, which is negative if $r_2 < 0.5r_1$ and positive if $r_2 > 0.5r_1$. □

Proposition 11 *In general, a bubble cannot vary pressure without affecting the rest of the cluster.*

Proof Consider a bubble in contact with three other surrounding, fixed, say smaller, bubbles. As its pressure decreases and it grows, it must move farther from all three—impossible. The other three are connected if you like by chains of smaller bubbles. □

Remark 12 The easier, well-known Decoration Proposition, that you can insert a three-sided bubble at a triple junction, follows from inversion to infinity of the other point where the three curves meet, which exists by double bubble trigonometry. In 3D, such an argument, based on triple bubbles, holds only if the six surfaces meeting at the tetrahedral junction are all spherical. How do you prove a general decoration theorem in 3D?

References

1. K.A. Brakke, F. Morgan, Instabilities of cylindrical bubble clusters. Eur. Phys. J. E **9**, 453–460 (2002)
2. I. Cantat, S. Cohen-Addad, F. Elias, F. Graner, R. Höhler, O. Pitois, F. Rouyer, A. Saint-Jalmes, *Foams: Structure and Dynamics* (Oxford University Press, London, 2013), translated by Ruth Flatman and edited by Simon Cox
3. C. Cox, L. Harrison, M. Hutchings, S. Kim, J. Light, A. Mauer, M. Tilton, The shortest enclosure of three connected areas in $\mathbf{R}^2$. Real Anal. Exch. **20**, 313–335 (1994/1995)
4. A. Fenyes, Geometric spaces of equilibrated two-dimensional foams (2010), uncompleted draft
5. M. Hutchings, The structure of area-minimizing double bubbles. J. Geom. Anal. **7**, 285–304 (1997)
6. S. Lang, *Differentiable Manifolds* (Addison–Wesley, Reading, 1972)
7. G. Lawlor, F. Morgan, Curvy slicing proves that triple junctions locally minimize area. J. Differ. Geom. **44**, 514–528 (1996)
8. A. Levy, Energy-minimizing networks meet only in threes. J. Undergrad. Math. **22**, 53–59 (1990)

9. F. Morgan, *Geometric Measure Theory: A Beginner's Guide*, 5th edn. (Academic Press, San Diego, 2016)
10. F. Morgan, W. Wichiramala, The standard double bubble is the unique stable double bubble in $\mathbf{R}^2$. Proc. Am. Math. Soc. **130**, 2745–2751 (2002)
11. F. Morgan, C. French, S. Greenleaf, Wulff clusters in $\mathbf{R}^2$. J. Geom. Anal. **8**, 97–115 (1998)
12. C. Moukarzel, Geometrical consequences of foam equilibrium. Phys. Rev. E **55**, 6866–6880 (1997)
13. C. Quinn, Area-minimizing partitions of the sphere. Rose-Hulman Undergrad. Math. J. **8**(2) (2007)
14. H. Schwerdtfeger, *Geometry of Complex Numbers* (Dover, New York, 1980)
15. D. Weaire, S. Hutzler, *Physics of Foams* (Oxford University Press, London, 1999)
16. D. Weaire, S. Cox, F. Graner, Uniqueness, stability and Hessian eigenvalues for two-dimensional bubble clusters. Eur. Phys. J. E **7**, 123–127 (2002)
17. B. White, The space of minimal submanifolds for varying Riemannian metrics. Indiana Univ. Math. J. **40**, 161–200 (1991)
18. W. Wichiramala, Proof of the planar triple bubble conjecture. J. Reine Angew. Math. **567**, 1–49 (2004)

Mathematics as a Social Language: A (Past and Present) Phenomenology

Paolo Rossi

Actually an alternative title of this essay might have been "Mathematics BETWEEN formal language and social language". Indeed we would just like to argue the thesis that, in the continuous exchange between the two levels of the language, the transition from formal to social is often conveyed by what is sometimes called "recreational mathematics".

As a matter of fact mathematics arises from the very beginning as a social language. Arithmetic is a necessary tool for establishing the terms of exchange for the products of hunting, and later on for crops and livestock, but it is also needed in order to measure time and space (the days requested to walk from one place to another, the lunar and seasonal cycles). Geometry (as the name itself says) has its first motivation in land distribution, and then also in the increasing complexity of building (dams, palaces and pyramids!).

The first split between the social uses and their formalization can be symbolically identified with the day when Thales "measures" the height of the Great Pyramid by the shadow of his stick. Empirical knowledge turns into a formal theorem even before Pythagoras (and the Pythagoreans), and not by chance philosophy is born at the same time, and Plato in the *Phaedo* will soon combine mathematical and philosophical knowledge, using the proof of Pythagoras' theorem as evidence for his theory of mind and knowledge.

Mathematics as a formal language was born and then developed in classical Greece, but we cannot ignore the existence of a very different tradition, that of the Bible and the Kabbalah, and later on that of early medieval Christianity. Jews and Christians turned mathematics into a sacred language, starting from the words of the *Book of Wisdom* (11:21): *Omnia in mensura et numero et pondere disposuisti*.

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This approach justified the study of mathematics, but restricted it to an elite, that of the clergy admitted to the interpretation of sacred texts, and in the early Middle Ages the main use of mathematics was the *Computus*, the complicated method that had to be used in order to fix the date of Easter. The method of *Computus was* almost definitively coded by Bede the Venerable (773-735 AD).

But already Augustine (354-430 AD) warns on the dangers of a "lay" conception of mathematics:

"*The good Christian should beware of mathematicians and all those who make empty prophecies. There is a danger that mathematicians have made a pact with the devil to darken the spirit and deliver the man to hell*" (*De Genesi ad Litteram*, Book II, xviii, 157). The bishop of Hippo probably had in mind above all the astrologers, but so be it.

In order to go back to a more social "conception" of the mathematical language it was necessary to switch to a playful attitude, which took place for the first time in the post-classical Western world, with Alcuin of York (735–804), a pupil of Archbishop Ecgbert, who in turn was a pupil of Bede.

Alcuin re-established the "public" school at the Court of Charlemagne and to this purpose he wrote the "*Propositiones ad acuendos Iuvenes*", a collection of 53 soluble problems involving (relatively) simple mathematical reasoning. The themes were not particularly original: for some a trace already exists in the Rhind Papyrus, a Egyptian text of the sixteenth century BC, and among the propositions one can also find the classic problem of "saving both ways".

But the interest of the text is mainly in the "revolutionary" (for the time) idea that mathematics could be useful in order to 'sharpen' the intellect of young people, and it could become a social language to the extent that, through the schools, its methods and content would be shared by many.

The medieval school involved two levels of education, the *Trivium* (lower lever) devoted to grammar, logic and rhetoric, and the *Quadrivium* (advanced level). All the disciplines of *Quadrivium* (arithmetic, geometry, music and astronomy) are of a mathematical nature, but this fact does not mean that the transfer of this kind of knowledge from the cloister to the public street was seen with favour by political and religious institutions: any attempt to vulgarization was fought tenaciously.

When Gerbert of Aurillac (947–1003), Pope in the year 1000 by the name of Sylvester II, attempted to introduce Arabic numerals, the abacus and the astrolabe, his work was mystified, attributing it a diabolic doctrine, and Gerbert himself was presented as a wizard, a notion gaining widespread fame thanks to imaginative legends told by William of Malmesbury, a twelfth-century chronicler. We shall not belabor on these legends (including the creation of a mechanical talking head), but they are full evidence of the hostility created by any attempt to make mathematics "popular".

Things went slightly better for Leonardo Pisano, also known as Leonardo Fibonacci (1175–1250), who with his *Liber Abaci* (1202) managed to introduce the use of the Arabic numerals to a much wider audience.

We must stress that even in this case the transition to the social dimension went through mathematical games. Let's recall the "competition" between Leonardo and users of Roman numerals (and of the related algorithms) at the Palermo Court od Emperor Frederick II. Let's also recall the many problems that Fibonacci tackled in his book, including the most famous problem of rabbit reproduction, which gave rise to the introduction of the numbers today known as "Fibonacci numbers", linked also to classic idea of the "golden section".

Nevertheless, at the end of the thirteenth century in Florence the prohibition was still enforced of using Arabic numbers in bank transactions, probably because of the supposed competitive advantage that those who were able to use them would have had against those who did not know the techniques.

A quite different fate, it is worth noting, had mathematics in the Arab and oriental world. We shall not dwell on the legend about the birth of chess or on the popularity of Mathematics in India (still true in present times, just think of Ramanujan's story), nor on the production and dissemination of astrolabes and astronomy texts (essential for the proper orientation of prayer towards Mecca, which reminds us of the sacred value of mathematics).

Part of this culture is perhaps reflected in the complex astronomical and numerological architecture of the *Comedy* (at least so Asin Palacios thinks and argues in his "*Dante and Islam*"). Numerology is already an integral part of *Vita Nova*, all played on the constant return of the number nine. But it is still a secret code, a *trobar clus* speaking to the cognoscenti, as the contemporary *Ars Magna* by Ramon Llull (later renamed *ars combinatoria* by Leibnitz), which is also clearly influenced by Islamic culture of Spain. And we see that in Dante and Llull the recreational dimension is completely lacking. But we find it again in the real social dynamics of their contemporaries. The "abacus treaties" written in the Late Middle Ages are rich in mathematical games: as Giovanni de Danti wrote in 1370 "*tractaremo de certe materie più per dare dilecto che per utilità che crediamo trare d'esse*" (we shall treat some issues more in order to give pleasure than for the uses we think they may have).

With the Renaissance (and with the revival of Platonism) we see mathematics regain a "social" role when it becomes a tool for painting, with the studies on the perspective of Piero della Francesca. We may also recall the drawings by Leonardo for the *De Divina Proportione* by Luca Pacioli on the golden section (notice that Luca wrote also a booklet on games, *De Viribus quantitatis*). Geometry is essential for architecture (and in the recreational context think of the *Ex ludis* by Leon Battista Alberti). But we see the real triumph of this "social" reinterpretation of the role of mathematics in its identification with the language of nature, proposed by Galilei and soon to become the "common sense" of modern physics.

It is also a great season for "recreational" mathematics, now in close relationship with the calculation of chances, so important in card games that are more and more common, but also in games with dice, which have very ancient history. And we must not overlook the role of "scientific" astrology, which in the sixteenth century, by Nostradamus and Cardano, got a social credibility not to be anachronistically interpreted in the light of modern knowledge (and we must remember that even Galileo did not disdain to calculate horoscopes, for a fee!).

With the successive generations, those of Descartes, Mersenne, Fermat and Pascal, even "formal" mathematics became a kind of game: number theory was developed, and it was seen by its practitioners as a pure intellectual entertainment. But in the meantime the way was paved for the extraordinary developments that soon followed, with the Newtonian and Leibnitzian "*Calculus*", and with the "social" uses of mathematics, in the eighteenth century, in the context of the new economic science associated with the industrial revolution. In this context let's recall also the paradox of St. Petersburg, which was devised by Bernoulli in the early eighteenth century and played a significant role in the development of economic theory with the introduction of the utility function.

Actually mathematics seems to accompany all the great political and social revolutions, oscillating always among the (esoteric) formal developments and (exoteric) recreational and practical applications.

We must now come to the twentieth century, and to the last pieces of our argument.

Game theory was created "officially" in 1944 by Von Neumann, and developed later by Nash although, as we have seen the birth of the theory should be backdated at least to the time of Blaise Pascal.

Certainly, the development of financial economy has contributed more than ever to the growth of the social dimension of mathematics, with an impact perhaps equaled only by the birth and the growth of computer science. Today we have on our tablets a game called Fibonacci game (along with many other math-based games); computers are now world champions in chess and go (which seems to be the hardest game ever invented); but especially we have a new mathematical mass folklore, a new real mythology.

A good example of this statement can be found in the television series *Numb3rs*, in which a young talented mathematician makes good use of its knowledge in order to solve intricate criminal cases. In this case it is proper to say that the medium is the message: if mathematics can be the subject of fiction serial, then it acquires the nature of serial fiction, and its "recreational" dimension becomes meta-recreation to the extent that it becomes a show.

Part V
Visual Mathematics

Balance in. . . Forms

The Dance of Design

Giordano Bruno and Massimo Ciafrei

We present some projects carried out by students attending the second year of the bachelor's course held at *ISIA Institute* in Rome and in the seats in Pescara and Pordenone for the joint workshop combining the course *Mathematics for Design* and the course *Theory of the Form*. The topic explored is balance in its broadest sense. The workshop "*Balance in. . . Forms*" is an experimental workshop where students have been invited to investigate the different aspects of the relationship between maths and design. By developing new, ambitious and original solutions to the problem of the formalization of space structures with analytical control, students have learned to work with languages, principles, techniques and elements of various nature. The artefacts have been created by integrating industrial design with ideas and experiments on languages, which were elaborated by young design students in the scientific, artistic, technical and speculative field using their sensibility and rationality, intuition and imagination. The projects range from complex space structures to different forms of creative investigation and try to develop an aesthetic-figurative and culturally diverse production keeping in mind their potential use in future design and scientific applications.

Mathematics and Balance

Undoubtedly, there is a strong link between these two terms. In mathematics, balance refers to the calculation of balance points, to dynamic systems, to Poincaré's and Thom's theories, to applications for financial markets and so on. But we also have mathematics of balance, that is, a science which has found a way to bring us closer to and make us understand reality in the relation between its forms: geometric shapes and algebraic and analytic formulas. How fascinating the Fibonacci sequence is to us—to know that when n increases, the ratio between the previous term and the

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next one in the sequence tends to that extraordinary number referred to as *golden ratio*, which nature has *produced* and human intelligence has tried to comprehend! And what about the balance (and therefore the elegance) of an algebraic formula such as the well-known $e^{i\pi} + 1 = 0$? The same is true of music, ballet, and artistic gymnastics, which are expressions of structural and body balance and owe it all to mathematics. That is why we wanted our students to investigate and look for old and new balance relations between forms and shapes, giving birth to a sort of dance of design.

Course in Theory of the Form

Professors: Massimo Ciafrei, Claudia Iannilli

Learning to investigate and express
the untapped potential
of biological and artificial
structures of materials
and technologies to
trans-form them in new spatial
and complete organisms
through a design approach

The course in *Theory of the Form*, characterized by a didactic methodology based on research and experimentation, aims to bring students closer to the world of design by stimulating their ability to define logic processes and to handle complex systems of relationships. The didactics of metadesign, the scientific-cultural area in which the course is rooted, informs the designer's identity and includes the morphological disciplinary aspects necessary for the investigation of and experimentation with the formal features of projects. The course content deals with the study and modelling of single structures as well as of formal sets which build the contemporary visual and spatial landscape. The course focuses on the evolutionary processes of the forms and illustrates the significant syntactic systems obtained and the maturation of functional properties in a functional organism. The course's aim is to develop cultural, aesthetic-figurative and compositional works by conceiving the right use for them in relation to future design applications.

Course in Mathematics for Design

Professor: Giordano Bruno

The course in *Mathematics for Design* is taught in the first and second year of the BA syllabus in *Industrial Design*. The course purses several goals and aims at combining mathematics and its richness with design. Firstly, we have approached the topic of "uncertainty", which is common in every design project. The approach to uncertainty follows the methodology defined by Bruno de Finetti and based on the "coherence" of probability assessments (starting with qualitative ones) and the way to properly update them with new information. Learning this way of thinking based on probabilistic logic, now regarded as primary by neuroscience, strengthens the analytical and critical abilities needed for design. Secondly, we have chosen to explore at least some of the many relations that exist and overlap between mathematics, science, art and design. Approaching these fields of creativity and human knowledge has the purpose of contributing to raise awareness about the fact that design is basically a cultural phenomenon which finds its formal expression—and other expressions—by drawing on these and other disciplines. All this has been possible thanks to Michele Emmer's films on Art and Mathematics: from platonic solids to combinatorics, from perspective to golden ratio, spirals and helices, from knot theory to the Möbius strip, from *Flatland* to the fourth dimension, from Escher and his impossible worlds to soap bubbles, from fractals to complexity and chaos. The imaginative and artistic aspect that illustrates these themes along with the mathematical concepts that accompany them—expressed in qualitative rather than quantitative terms—have allowed students to develop their own cultural production to be applied to their projects, as I have witnessed over the past thirty or more years in which I have had the opportunity and honour of teaching at *ISIA* in Rome.

The Projects

Tin tan **(Tin tan)**
Dynamic Vibration

Designer: Aleksandra Mehmetaj

"Tin tan" is a cylindrical frame containing some spring steel plates to which small metal balls are attached. When rotating on itself, the object generates a sound due to the contact between the balls and the metal plates. The experience is complemented by the vibration of the spring steel, stressed by the impact of the various components.

***Equilibro* (Balance Book)**
Small Guide to Visual Balance

Designer: Irene Caretti

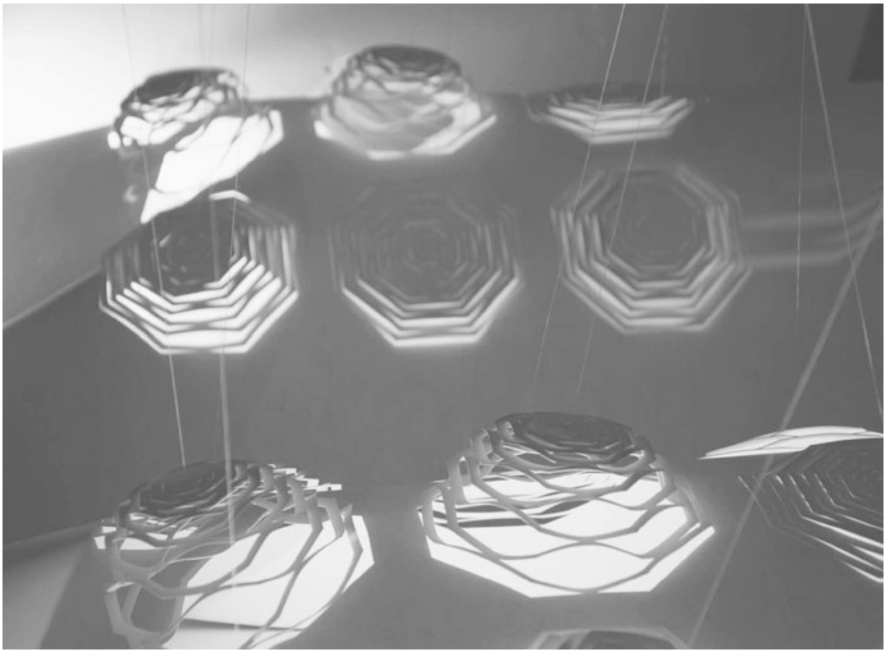
"Balance Book" stems from a research study on graphic-perceptive mechanisms. Such mechanisms are responsible for the achievement of visual balance in the composition of a work of art or, more generally, in the creation of graphic features. The purpose is to explain this dynamic to children through illustrations and very simple texts so that they can learn to consciously apply them in their drawings. As a matter of fact children draw by applying these mechanisms from a very young age, but they do it unconsciously.

Conscrict (Conscrict)
Interaction between Complex Elements

Designer: Flo Casco

"Conscrict" is a project that investigates the balance of some neoprene rings of different sizes subjected to compression and decompression. A transparent box houses these elements, presenting graphic features inside them to better observe their behaviour when compressed. The rings are forced into the box and compressed by a transparent cover. When the box is closed, it is possible to observe their behaviour under compression; when the box is open, the elements are released into space in a sudden and unpredictable manner.

***Amiens* (Amiens)**
Interplay on solids and voids



Designer: Elisa Cavezzan

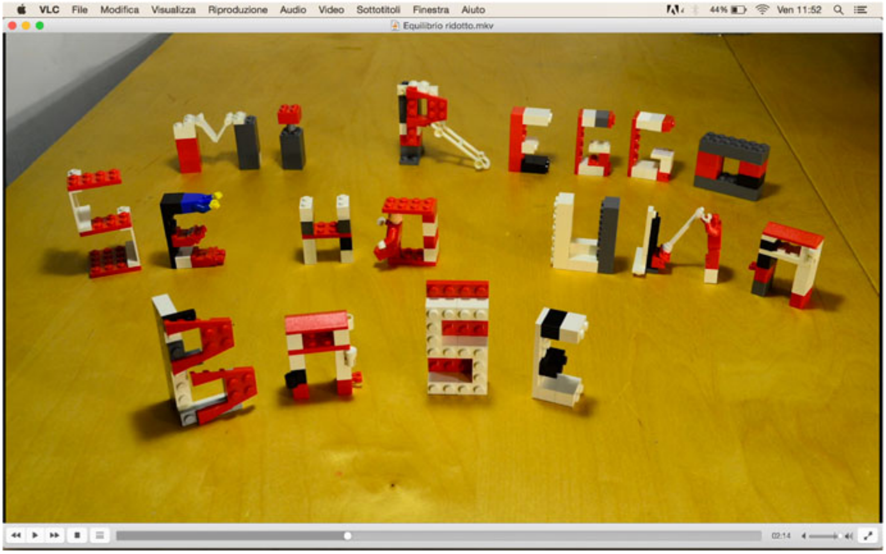
Amiens stems from the idea of finding a correlation between the solid and the void within a single element. Thanks to cuts on the surface of the objects, it is possible to move from the two-dimension regularity to a three-dimension irregularity. The transition between these two moments of the objects is the founding element of the project. It is also possible to catch the reverberation of light in the surrounding environment.

Anemoi (Anemoi)
Natural Connections

Designer: Matteo Ciafrone

Anemoi is the Greek name of the wind god. The project was born out of the need to investigate the concept of balance in the natural world, where man is not at the centre and cannot have any control. The shape of the single elements, which is inspired by the marine world, is decontextualized and set into a new environment. The project wants to create a sensory experience, an invitation to loosen up in order to find your own balance, guided by the sound and dance of the small elements moved by the wind.

Equilibrio, come giocarci **(Balance—How to Play with It)**
A short lesson on Balance (Video)



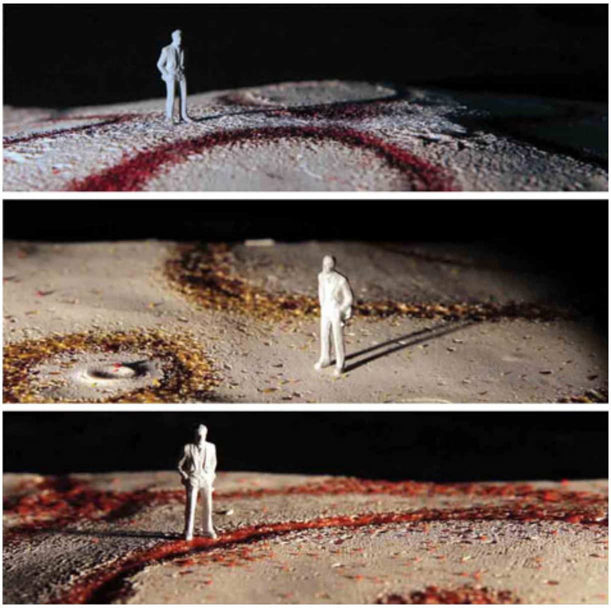
Designers: Giacomo Fabbri, Alessandro Fiorentino

"Balance- How to Play with It" was our way of designing balance. And in order to design it, you need to know it and to understand it. The challenge therefore became: let's explain it! To everyone and in the best possible way. And who can be the best judge if not children? They are impossible to deceive, always ready to get distracted and beyond any speculation attempt. Everyday objects have come to life and balance has taken over and led the way. Three points are stable, two need a push and what about a single one?

Morfologia del suono (Morphology of Sound)
Cymatics

Designers: Laura De Paolis, Sara Gentili, Valeria Gallo

The universe is an energetic whole that manifests itself in vibrations where every individual part is attracted by the resonance to similar sounds. Cymatics is the science that studies frequencies and the forms they produce: vibrations regarded as true entities. Their limit is indeed the perfect balance point; this is the generative energy in constant transformation. From this point of view cymatics can be interpreted as a dynamic holism that explores disintegration as a point of strength and rebirth.



400.000 years after the Big Bang, as space became denser and denser, it reached the optimal physical condition for sound waves to propagate. Today we can detect these ripples on the "surface of the universe" and, by analysing the traces of the initial sound waves, we can trace back the nature of the material that carried them during the early phases, thus obtaining information on how the universe was born and has evolved. Sounds from space are therefore the embodiment of previous cymatic experiments. The aim is to transform their mark into a permanent graphic element. The result is the impression of the sign through silica on the ultimate material—clay. Each shape matches the colour and the surface of a planet in our solar system, described thanks to three literary texts.

The sands of Mars by Arthur C. Clarke
A fall of moon-dust by Arthur C. Clarke
Lucky Starr and the big sun of Mercury by Isaac Asimov

***Discordante* (Discordant)**
The Hybrid of Sounds

Designer: Simona Girardi

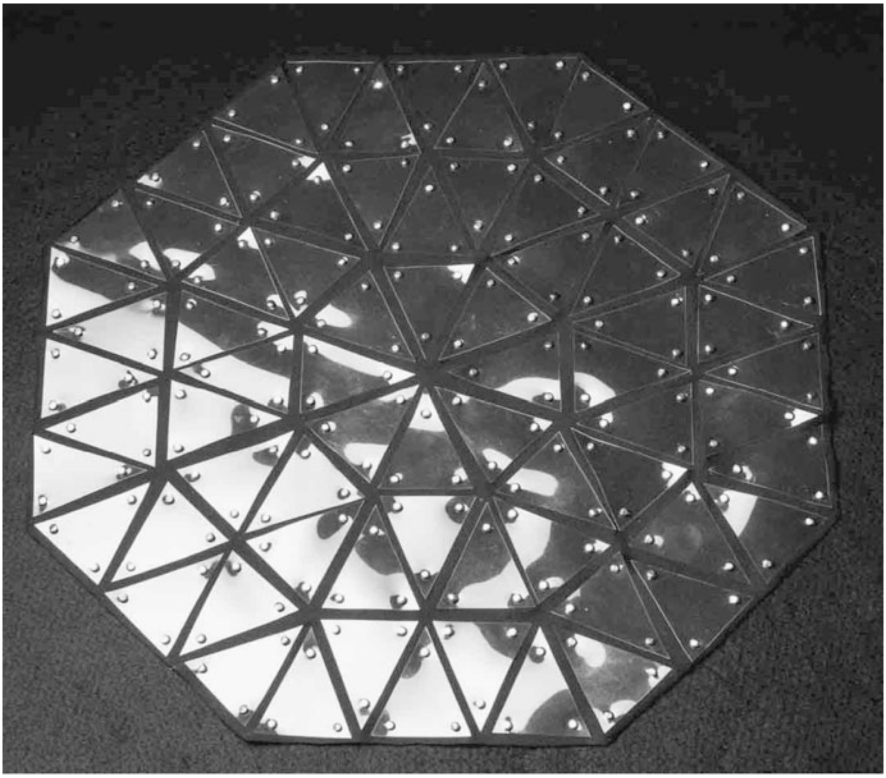
Discordant is a dynamic structure designed to produce sounds through motion. The aim is to create unexpected confusion thanks to the use of different materials, while elastic threads trigger the movement. The intensity of the sound depends on the speed released by the potential energy of the object. *Discordant* loses and finds back its stability simply through movement.

***Riccio* (Sea Urchin)**
Magnetic Balance

Designer: Jovita Kaulinyte

Riccio (Sea Urchin) exploits the characteristics of iron powder when subjected to a magnetic field. Metallic powder creates structures within the cubic space thanks to the influence of different magnets. The disposition of magnets characterizes the arrangement of powder grains within the circumscribed space of the cube, thus creating constantly new configurations.

***Umihotaru* (Umihotaru)**
Luminescent Structure



Designer: Daniela Lucci

The light blue bright stripes that give their colour to the Okayama beach rocks are not a strange special effect, but small, luminescent beings called *Umihotaru*. These crustaceans emit a powerful light beam that makes the Japanese sea a real multi-coloured sensation. "Umihotaru" consists of a modular surface made out of a series of triangular plastic elements. The surface creates fascinating games of reflection due to the action of natural light.

Il corpo in movimento (Moving Body)
Space, Time, Effort
Designers: Marzia Lupi, Gaia Stirpe

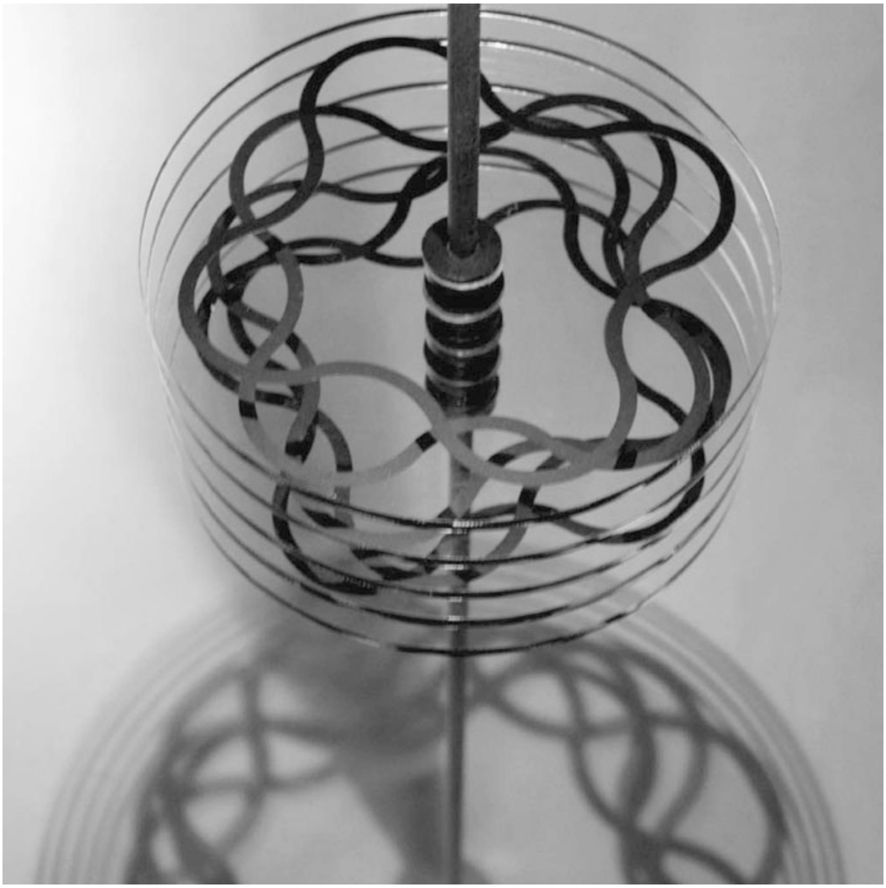
The purpose of the project is to illustrate body positions during movement through a visual code. The outlined visual code allows to correctly interpret sport gestures, which are explored in the immediate search for balance caused by muscular effort. Some body points have been taken as reference to graphically convey body movements. Among all anatomical markers considered, two were the fundamental elements chosen to analyse motion: leg and arm joints. The most spectacular and scenographic movements of sport gestures were explored through the technique of motion capture and it was possible to obtain an expressive synthesis of the movement in the space-time-effort relationship thanks to the visual code developed. The visual language created is expressed through light signs that vary according to certain differentiation criteria. These allow to identify the evolution of the athletic gesture while searching for balance. The criteria are:

Colour: the five main shades refer to physical effort, from the minimal to the maximum effort (blue, light blue, green, fuchsia and red);

Deformation: the sinuosity of curves refers to muscular intensity in a given time span;

Size: the circles around the light signs indicate effort intensity- the greater the force exerted, the greater the size of the circles.

***Aum* (Aum)**
Vibrations in Mechanical Form



Designer: Marcello Migliore

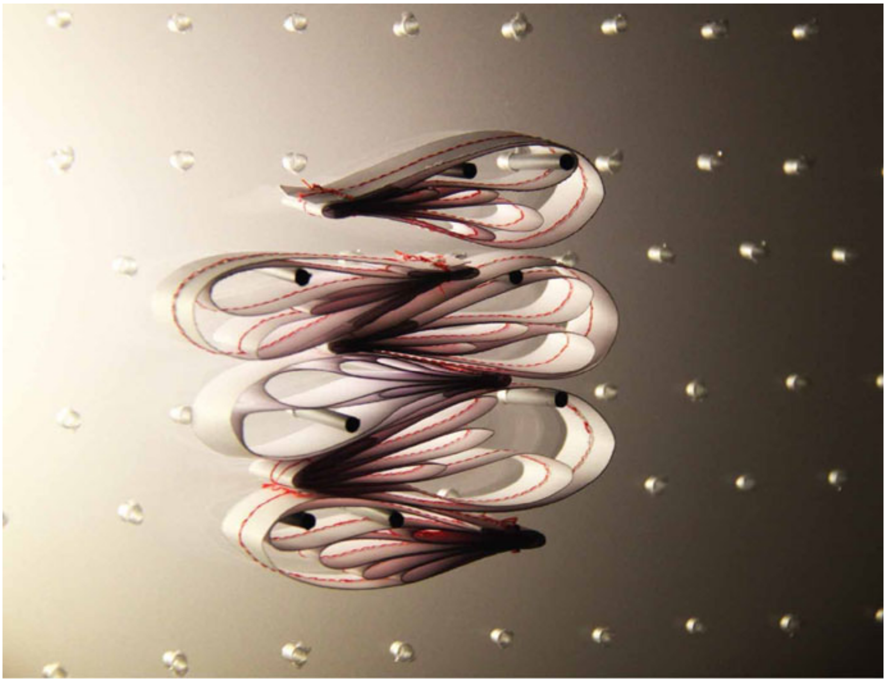
Aum is a system that combines forms, shapes, signs and structures in a precise and balanced relation to each other with the aim to capture the attention of the observer through the mechanic reproduction of marks and perceptions found in nature.

***Kujaku* (Kujaku)**
Wind is Blowing Outside

Designer: Stefania Moscatiello

Kujaku, changeable in its appearance, looks like an animal: it may be the metaphor of a peacock with a filamentous tail which creates complementary or splitting optical effects during its motion. *Kujaku* is inspired by the oriental taste and made of wood cylinders and spring steel. Its appearance is light and thin. *Kujaku* is a structure composed of threadlike elements and moves thanks to air currents. The object, free to roam around, marks its passage and traces unpredictable paths. It is a meta-design project based on experimentation on the helix and embraces the geometry of forms and the rotating and oscillating dynamics generated by the wind and random movements. It is made of spring steel and has a fixed axis and four independent screws that revolve around it. This creates a particular shape that favours the movement of rotation and oscillation depending on where it is located. *Kujaku* will react differently to wind thrusts and other external agents.

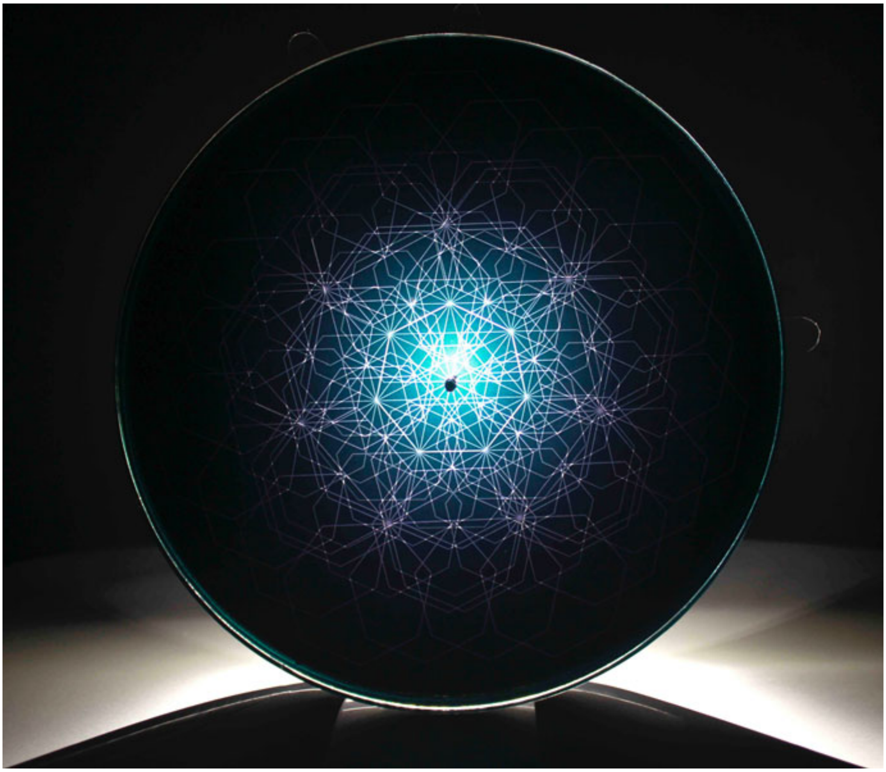
Drop (Drop)
Drop of Light



Designer: Ileana Nuzzi

The simple harmony of *Drop* and the ephemeral shape of drops have inspired not only the name but above all the origin of the project. The composition of the modules and the delicate reflections of light on the folds of the object create light and shadows on a modular matrix. Depending on the aggregation of the elements, light permeability can be intensified or decreased. The characteristics of the paper employed allow light to spread, while cotton finishing gives daintiness and colour to the elements used.

***Hepta* (Hepta)**
Geometric Chaos



Designer: Dimitri Spagnulo

The study of the morphology of this light modulator is an attempt to find a non-trivial geometric construction that may convey visual and sensory emotions. The basic element chosen is a regular heptagon: it is a complex and interesting shape due to the disparity of the sides and the width of the angles, which are not compatible with simple systems. Several cases of interlacing, overlapping, colour changes, solids and voids were analysed in order to work on different levels interacting with each other. The result is intriguing and fascinating.

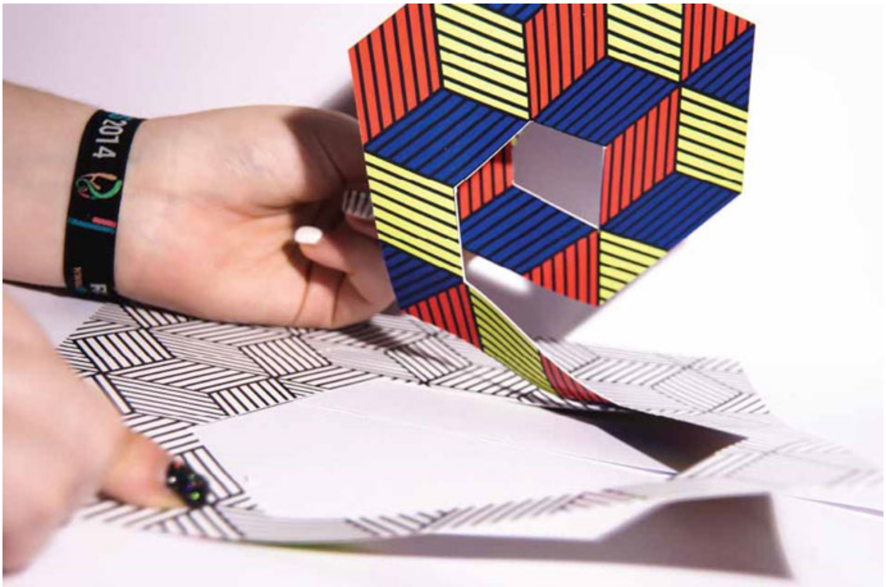
Where we see chaos, we see natural order.

***Zero waste* (Zero Waste)**
Natural Semi-Finished Products

Designers: Enrica Tartaglione, Adrienn Sasvári

The project idea is to focus on the manufacturing of industrial products that can find their way back into the decomposition cycle without any environmental impact. In line with this, natural elements deriving from everyday waste have been identified during the development of a kind of packaging that may ensure strength as well as may display remarkable aesthetic qualities. The main components are flour, pea peels and peanut discards. These have been added to binding materials, such as honey, water and corn starch. The material developed evaluates the interactions of the product with the environment, considering its entire life cycle, from pre-production, recycling to final disposal. It is about circular economy: this refers to a production system where the same resources are employed several times through re-use and re-cycling, thus resulting in remarkable efficiency gains. The system can be used in various fields depending on specific needs.

***Vertigo* (Vertigo)**
Colour and Shape



Designer: Francesca Tucci

The project was born with the aim of de-constructing the shapes in our mind in order to create totally different ones. *Vertigo*, as the name says, recalls both something that gives dizziness and something that gives a sense of rotational instability. It represents a new birth for unknowable shapes and colours. The fundamental element for this rebirth is the spiral that connects the monochromatic world, representing the scientific and rational part of the mind, and, dichotomously, the chromatic one that represents the creative and extravagant part of thought.

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References

1. M. Emmer, C. Van Vlandereen (eds.), *M.C. Escher*. Catalog of the Exhibition (Istituto Olandese, Roma, 1985)
2. M. Emmer, *Lo Spazio tra Matematica e Arte*. Catalog of the Section Spazio, XLII Biennale di Venezia (1986), pp. 37–39

3. M. Calvesi, M. Emmer (eds.), *I Frattali: La Geometria dell'Irregolare*. Catalog of the Exhibition (Ist. della Enciclopedia Italiana, Roma, 1988)
4. M. Emmer (ed.), *L'occhio di Horus: Itinerari Nell'immaginario Matematico* (Istituto della Enciclopedia Italiana, Roma, 1989), pp. 18–21
5. M. Emmer, Art and mathematics: a series of interdisciplinary movies, in *Zentralblatt fur Didaktik der Mathematik*, vol. 21, Febbraio, ed. by W. Blumm (1989), pp. 23–26, n. 1
6. M. Emmer (ed.), *Visual Mathematics, Special Issue Leonardo*, vol. 25 (Pergamon, Oxford, 1992)
7. M. Emmer, V. Marchiafava, Bulles de savon, un spectacle des mathématiques, in *Science et Technique en Spectacle*, L.I.R.E.S.T., Univ. Paris 7, ed. by A. Giordan, J.-L. Martinand, D. Reichwarg (1993), pp. 61–70
8. M. Emmer (ed.), *The Visual Mind: Art and Mathematics* (MIT Press, Cambridge, 1993)
9. M. Emmer, Soap bubble in math and architecture, in *Proceedings Nexus: Mathematics and Architecture*, ed. by K. Willliams (Edizioni dell'Erba, Firenze, 1996), pp. 53–65
10. M. Emmer (ed.), *Matematica e Cultura 1997*, Lettera Matematica, Univ. Ca' Foscari Venezia (Springer, Milano, 1998)
11. M. Emmer, M.C. Escher: design, math & cinema, in *Mathematics & Design 98*, Universidad del Pais Basco, S. Sebastian ed. by J. Barallo (1998), pp. 325–332
12. M. Emmer (ed.), *Matematica e Cultura 2*. Lettera Matematica (Springer, Milano, 1998)
13. M. Emmer, *Escher a Roma, Ancora*. Catalog of the Exhibition Homage to Escher, Museo laboratorio di Arte Contemporanea, Univ. Roma La Sapienza, 24–26 giugno (Diagonale, Namur, 1998), pp. 12–13
14. M. Emmer (ed.), *Matematica e Cultura 1998*. Supplemento a Lettera Matematica (Springer, Berlin, 1998)
15. M. Emmer, *Mathematicians: The New Artists?* Leonardo, vol. 32 (MIT Press, Cambridge, 1999), p. 220, n. 3
16. M. Emmer (ed.), *Matematica e Cultura 1999* (Springer, Berlin, 1999)
17. M. Emmer (ed.), *Matematica e Cultura 2000* (Springer, Berlin, 2000)
18. M. Emmer (ed.), *Matematica e Cultura 2001* (Springer, Berlin, 2001)
19. M. Emmer, M. Manaresi (eds.), *Matematica, Arte, Tecnologia, Cinema* (Springer, Berlin, 2002)
20. M. Emmer (ed.), *Matematica e Cultura 2002* (Springer, Berlin, 2002)
21. M. Emmer (ed.), *Matematica e Cultura 2003* (Springer, Berlin, 2003)
22. G.I. Bischi, R. Carini, L. Gardini, P. Tenti, *Sulle Orme del Caos. Comportamenti Complessi in Modelli Matematici Semplici* (Bruno Mondadori Editore, Milano, 2004)
23. M. Emmer (ed.), *Matematica e Cultura 2004* (Springer, Berlin, 2004)
24. M. Emmer (ed.), *Matematica e Cultura 2005* (Springer, Berlin, 2005)
25. M. Emmer (ed.), *Matematica e Cultura 2006* (Springer, Berlin, 2006)
26. M. Emmer (ed.), *Matematica e Cultura 2007* (Springer, Berlin, 2007)
27. M. Emmer (ed.), *Matematica e Cultura 2008* (Springer, Berlin, 2008)
28. M. du Sautoy, *Symmetry: A Journey into the Patterns of Nature* (Harper, New York, 2008)
29. G.I. Bischi, P. Nastasi, *Un 'Leonardo' del Novecento: Leonardo Sinisgalli (1908–1981)*. PRISTEM/Storia-Note di Matematica, Storia, Cultura 23/24 (2009)
30. G. Bruno, *Una Passeggiata Fra Arte e Matematica*. MATEpristem (2010)
31. M. Emmer (ed.), *Matematica e Cultura 2010* (Springer, Berlin, 2010)
32. M. Emmer (ed.), *Matematica e Cultura 2011* (Springer, Berlin, 2011)
33. M. Emmer (ed.), *Imagine Math 2012* (Springer, Berlin, 2012)
34. M. Emmer (ed.), *Imagine Math 2 2013* (Springer, Berlin, 2013)
35. M. Emmer (ed.), *Imagine Math 3 2014* (Springer, Berlin, 2014)
36. G. Bruno, Mathematical forms, in *Imagine Maths 4: Between Culture and Mathematics*, Unione Matematica Italiana & Istituto Veneto di Scienze, Lettere ed Arti, ed. by M. Emmer, M. Abate, M. Villareal (2015)
37. M. Emmer, M. Abate, M. Villareal (eds.), *Imagine Maths 4: Between Culture and Mathematics*, Unione Matematica Italiana & Istituto Veneto di Scienze, Lettere ed Arti (2015)

38. M. Emmer, M. Abate, M. Villareal (eds.), *Imagine Maths 4: Between Culture and Mathematics*, Unione Matematica Italiana & Istituto Veneto di Scienze, Lettere ed Arti (2016)
39. F. Ragazzo, *Contributi di Design Alla Sezione di Ottica*. Lettera Matematica Pristem 97, luglio 2016 (2016)
40. A. Vulpiani, *Funzioni Irregolari e Oggetti Frattali: Da Weierstrass a Mandelbrot*. Lettera Matematica Pristem 97, luglio 2016 (2016)

Mathematical and Numerical Description of the Heart Function

Alfio Quarteroni, Christian Vergara, and Mikel Landajuela

Introduction and Motivations

Mathematical and numerical modeling of the cardiac function is a research topic that has attracted a lot of interest from both the mathematical and bioengineering communities over the past 25 years. The main reason is the great impact that cardiovascular diseases have on our lives. As a matter of fact, according to [11], cardiovascular diseases are the major cause of death worldwide. In Europe, they correspond to nearly half of all deaths (47%).

Heart comprise the so-called left and right heart, each consisting of two chambers, an atrium and a ventricle. It pumps blood into the arteries of the vascular circulations and collects it after the return through the veins. To do this, it needs to contrast the resistance in the arteries where blood has a non-null pressure and to supply blood with the energy needed to reach the microvasculature and the lungs.

The main agent of blood ejection in the circulatory system is the active ventricular contraction. This is generated by an electrical impulse produced by the heart itself. This impulse consists in an electrical potential that propagates along a specialized network (the Purkinje system) and then in all the myocardium. The propagation of the electrical signal is possible owing to the excitability of the heart cells, the cardiomyocytes, which, when suitably stimulated, yield a variation in membrane voltage. This produces an inward flux of extra-cellular calcium ions just after the depolarization of the cell. Once in the intracellular space, calcium ions bind to some proteins producing a shortening of the cell and allowing the whole heart contraction.

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Blood flow in the cardiac chambers is characterized by a different behaviour with respect to the vascular flow, resulting in specific, disturbed flow patterns and an higher Reynolds number (about 4000), leading to transition to turbulence also in non-pathological conditions [10].

All these interconnected physical processes can be modeled by a complex mathematical problem that should account for the coupled and non-linear nature of the problem. Owing to its composite nature, the heart function is first modelled by means of stand-alone core components, each describing a single functionality. Each core model needs careful mathematical analysis and efficient numerical approximation. The crucial step is then the integration of these core models into a global and integrated model suitable for describing the heart function. This step requires the introduction of suitable coupling conditions and efficient numerical strategies of the stand-alone cores.

Mathematics, together with the corresponding numerical approximation, could provide quantitative answers to many clinical issues related to the heart malfunctioning. The latter could be schematically subdivided according to the nature of the involved physical process: electrical propagation, mechanical contraction, blood dynamics. As observed, these processes are highly connected. For example, ischaemic cardiopathies depend from a reduced coronary flow rate (due to atherosclerosis of coronary arteries) with consequent malnutrition of the myocardium could lead to a decreased oxygen supply in the myocardium and, possibly, to an *infarct*. Another example is given by *ventricular fibrillation*, where the cardiomyocytes are not excited in a coordinated way and do not contract correctly. This chaotic excitation could inhibit the normal heart function, which is no longer able to properly pump the blood. At the same time, cardiopathies as the infarct could yield altered electrical properties, which in turn could generate ventricular fibrillation.

Examples of applications of mathematical and numerical modeling to describe heart diseases and therapies can be found in [1] for the effect of an ischaemia on the electrical propagation, in [17] for studies on the vulnerability of the ventricles to defibrillation, in [2] for the study of ventricular arrhythmia, in [7] for the study of hypertrophic cardiomyopathy, just to mention a few.

Reconstruction of Cardiac Images

Together with the mathematical and numerical modeling, a crucial role in developing tools that are able to provide quantitative results to be used for the study of cardiac clinical problems is played by the data. In particular, the use of patient-specific data is mandatory for numerical modeling aiming to understand biophysical processes and support clinicians. Data can be schematically classified as *geometric data, boundary data, and biological data*.

Geometric data are essential for the construction of the computational domains, where the equation underlying the process at hand are solved numerically. For the heart, these represent the muscle region Ω_{mus}, delimited by the epicardium Σ_{epi} and

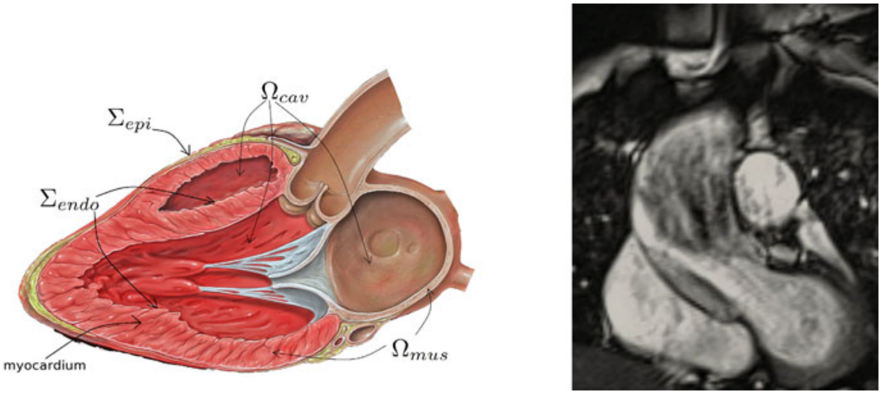

Fig. 1 Left: Longitudinal section of a complete heart domain. Right: MRI acquisition of ventricular geometry and blood velocity

by the endocardium Σ_{endo}, and the heart cavities or chambers Ω_{cav} delimited by the endocardium, see Fig. 1, left. The electrical propagation and mechanical contraction and relaxation occur in the muscle region, whereas blood fluid-dynamics occurs in the four hollow regions (chambers).

This geometric reconstruction process consists in the acquisition of clinical images, image segmentation, and generation of the computational mesh. The most typical acquisition techniques are *Magnetic Resonance Imaging* (MRI) and *Computed Tomography (CT)*, see Fig. 1, right.

In fact, the cardiac image reconstruction process relies on identifying the endocardium and epicardium surfaces. Unlike the vascular case, the external wall surface cannot be obtained by extruding the internal one under the assumption of constant wall thickness. Indeed, the thickness of myocardium changes significantly in the same heart and, obviously, from patient to patient. Moreover, not all of the slices have the same degree of complexity in the reconstruction. For example, apical and basal slices are more difficult to segment than mid-ventricular ones. Another difficulty relies on the fact that the large displacements induced by heart motion requires a dynamic acquisition procedure consisting of several (20–30) frames per heartbeat.

The most studied cardiac district is the left ventricle, due to its vital importance and its pronounced thickness, ranging between 6 and 16 mm. Its shape is often approximated by an ellipsoid but accounting for patient-specific geometries is essential to answer to clinical questions. In contrast, the right ventricle and the atria feature a wall thickness that usually is smaller than the spatial resolution of the acquisition technology, so their reconstruction is harder.

For cardiac segmentation automatic methods have been developed that are based on a strong a priori information of the shape of the ventricles. This information is included in the segmentation algorithm by using statistical models and relying on identifying an average shape of available geometries forming a *training set*, and modelling the variability within the latter. This is usually done by means of principal component analysis of positions and displacements. These strategies allow to

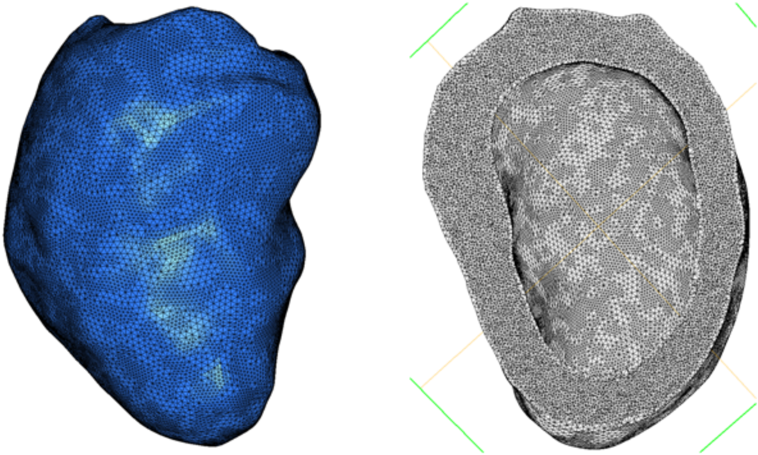

Fig. 2 Example of computational mesh composed by tetrahedra of a patient-specific left ventricle

perform an automatic segmentation minimizing the intervention of the user, provided that a training set is available. A very common strategy is the *atlas-guided segmentation*. Given an atlas, that is an integrated image obtained from reconstructions of several hearts, the mapping of the coordinates of the image under investigation to those of the atlas is performed (registration) [9]. This transformation is then applied to the atlas obtaining the segmentation of the geometry at hand. The registration process could be based on non-rigid transformations that account for elastic deformations. At the end of the segmentation process, the volume delimited by the reconstructed surfaces is subdivided in tetrahedra, obtining the computational mesh, see Fig. 2.

For a recent review of cardiac segmentation methods we refer to [13]. For a recent review on the numerical modeling of the cardiac function we refer instead to, e.g., [14, 15, 18].

An Example of Numerical Simulations

In this section, we present an example of numerical results that highlight an important characteristic of the heart function, namely the electro-mechanical coupling in the left ventricle.

We consider in particular the numerical results obtained by the coupling between the electrical and mechanical activity of a patient-specific left ventricle. For the former, we also consider the coupling with the Purkinje system, a specialized network placed on the endocardium that carries the electrical signal with an increased conduction velocity with respect to the myocardium, allowing for the almost simultaneous activation of the latter [16].

For the mathematical modeling, we considered the bidomain equations for the myocardium electrical activity [4], the monodomain equation for the activation in

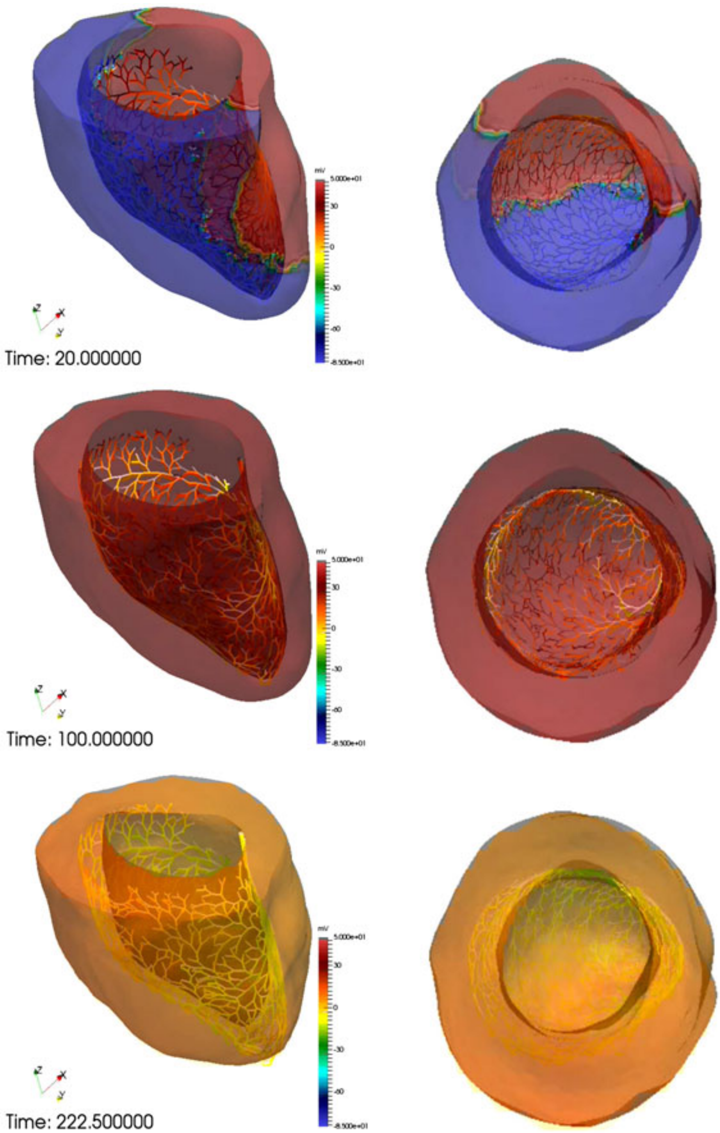

Fig. 3 Electrical activation in the current configuration of the myocardium in a patient-specific left ventricle at three different instants. The Purkinje network is kept fixed in the figures to indicate the reference configuration. CT images from the Cardiology Division at Ospedale S. Maria del Carmine, Rovereto (TN), Italy, and from the Radiology Division of Borgo-Trento (TN), Italy

the Purkinje network [3, 19, 20] and an active strain formulation [12] together with the orthotropic model proposed in [6] for the passive mechanics. The electrical activation and the active mechanics interact through the dynamics of a suitable link-ing variable that depends on the calcium ions concentration [5]. For the numerical scheme to solve this coupled problem, we consider the partitioned scheme proposed in [8].

The numerical experiment was solved with the Finite Elements library LIFEV (www.lifev.org).

In Fig. 3, we report the electrical activity in the current deformed configuration of the myocardium at three instants along the contraction phase.

References

1. S. Abboud, O. Berenfeld, D. Sadeh, Simulation of high-resolution QRS complex using a ventricular model with a fractal conduction system. Effects of ischemia on high-frequency QRS potentials. Circ. Res. **68**(6), 1751–1760 (1991)
2. E. Behradfar, A. Nygren, E.J. Vigmond, The role of Purkinje-myocardial coupling during ventricular arrhythmia: a modeling study. PLoS ONE **9**(2), 1–9 (2014)
3. R.M. Bordas, K. Gillow, D. Gavaghan, B. Rodríguez, D. Kay, A Bidomain model of the ventricular specialized conduction system of the heart. SIAM J. Appl. Math. **72**(5), 1618–1643 (2012)
4. P. Colli Franzone, L.F. Pavarino, S. Scacchi, *Mathematical Cardiac Electrophysiology* (Springer, Berlin, 2014)
5. A. Gerbi, L. Dede', A. Quarteroni, A monolithic algorithm for the simulation of cardiac electromechanics in the human left ventricle. MOX-Report No. 51/2017 (2017)
6. G.A. Holzapfel, R.W. Ogden, Constitutive modelling of arteries. Proc. R. Soc. Lond., Ser. A, Math. Phys. Eng. Sci. **466**(2118), 1551–1596 (2010)
7. D.M. Jordan, A. Kiezun, S.M. Baxter, V. Agarwala, R.C. Green, M.F. Murray, T. Pugh, M.S. Lebo, H.L. Rehm, B.H. Funke, S.R. Sunyaev, Development and validation of a computational method for assessment of missense variants in hypertrophic cardiomyopathy. PLoS ONE **88**(2), 183–192 (2011)
8. M. Landajuela, C. Vergara, A. Gerbi, L. Dede', L. Formaggia, A. Quarteroni, Numerical approximation of the electromechanical coupling in the left ventricle with inclusion of the Purkinje network. MOX-Report No. 58/2017 (2017)
9. M. Lorenzo-Valdés, G.I. Sanchez-Ortiz, A.G. Elkington, R.H. Mohiaddin, D. Rueckert, Segmentation of 4d cardiac mr images using a probabilistic atlas and the em algorithm. Med. Image Anal. **8**(3), 255–265 (2004)
10. R. Mittal, J.H. Seo, V. Vedula, Y.J. Choi, H. Liu, H.H. Huang, S. Jain, L. Younes, T. Abraham, R.T. George, Computational modeling of cardiac hemodynamics: current status and future outlook. J. Comput. Phys. **305**, 1065–1082 (2016)
11. D. Mozaffarian, E.J. Benjamin, A.S. Go, D.K. Arnett, M.J. Blaha, M. Cushman, S.R. Das, S. de Ferranti, J.-P. Després, H.J. Fullerton, V.J. Howard, M.D. Huffman, C.R. Isasi, M.C. Jiménez, S.E. Judd, B.M. Kissela, J.H. Lichtman, L.D. Lisabeth, S. Liu, R.H. Mackey, D.J. Magid, D.K. McGuire, E.R. Mohler, C.S. Moy, P. Muntner, M.E. Mussolino, K. Nasir, R.W. Neumar, G. Nichol, L. Palaniappan, D.K. Pandey, M.J. Reeves, C.J. Rodriguez, W. Rosamond, P.D. Sorlie, J. Stein, A. Towfighi, T.N. Turan, S.S. Virani, D. Woo, R.W. Yeh, M.B. Turner, Heart disease and stroke statistics—2016 update. Circulation **132** (2015)
12. F. Nobile, A. Quarteroni, R. Ruiz-Baier, An active strain electromechanical model for cardiac tissue. Int. J. Numer. Methods Biomed. Eng. **28**(1), 52–71 (2012)
13. C. Petitjean, J.-N. Dacher, A review of segmentation methods in short axis cardiac mr images. Med. Image Anal. **15**(2), 169–184 (2011)
14. A. Quarteroni, T. Lassila, S. Rossi, R. Ruiz-Baier, Integrated heart—coupling multiscale and multiphysics models for the simulation of the cardiac function. Comput. Methods Appl. Mech. Eng. **314**, 345–407 (2017)

15. A. Quarteroni, A. Manzoni, C. Vergara, The cardiovascular system: mathematical modelling, numerical algorithms and clinical applications. Acta Numer. **26**, 365–590 (2017)
16. D.A. Rawling, R.W. Joyner, E.D. Overholt, Variations in the functional electrical coupling between the subendocardial Purkinje and ventricular layers of the canine left ventricle. Circ. Res. **57**(2), 252–261 (1985)
17. B. Rodriguez, L. Li, J.C. Eason, I.R. Efimov, N.A. Trayanova, Differences between left and right ventricular chamber geometry affect cardiac vulnerability to electric shocks. Circ. Res. **97**(2), 168–175 (2005)
18. N.P. Smith, D.P. Nickerson, E.J. Crampin, P.J. Hunter, Multiscale computational modelling of the heart. Acta Numer. **13**, 371–431 (2004)
19. C. Vergara, M. Lange, S. Palamara, T. Lassila, A.F. Frangi, A. Quarteroni, A coupled 3D-1D numerical monodomain solver for cardiac electrical activation in the myocardium with detailed Purkinje network. J. Comput. Phys. **308**, 218–238 (2016)
20. E.J. Vigmond, C. Clements, Construction of a computer model to investigate sawtooth effects in the Purkinje system. IEEE Trans. Biomed. Eng. **54**(3), 389–399 (2007)

Weird Gears

Gian Marco Todesco

Gears are a simple but extraordinary invention. They are not a recent attainment although it is difficult to date their discovery. The oldest finding containing gears (more than 30) is the *Antikythera mechanism*, made at the end of the second century BC. The Antikythera was an ancient Greek analog computer, capable to follow the sun's and the moon's astronomical positions and to predict eclipses [8]. Given the great complexity of the object it is reasonable to assume that the gears' theory and technique were already highly developed at the time. The first chapter of *Mechanical Problems* (school of Aristotle, ca 280 BC), which mentions gears, seems to confirm this supposition. It refers to a pair of connected wheels and notes that the direction of rotation is reversed when a wheel drives another wheel.

Another ancient device is the *Chinese South Pointing Chariot*, a sophisticated tool using a similar mechanism to modern cars differential. Some legends date it as far back as 2600 BC although the first well-documented evidences of its existence are much more recent (after 200 AD) [4, 17].

Surprisingly Nature too, in its ceaseless experimentation, has discovered gears. The nymphs of a small insect, the *Issus coleoptratus*, have small gear-like structures with inter-meshing teeth that synchronize the movement of their hind legs when they jump. These insects are among the fastest-accelerating creatures, reaching an acceleration of almost 400 g. Their movement is so fast that the nervous system alone can not balance the force impressed to the two legs. The bug can control the trajectory of its jumps only thanks to those natural cogwheels [2].

Nowadays humans use gears on a daily basis. They are present in virtually every device with moving parts: engines, bicycles, analog clocks, kitchen mixers, mills, bottle openers, various types of toys and countless other examples.

Gears are also used in kinetic art. It is worth mentioning *Machine with Concrete*, by Arthur Ganson: it is a kinetic sculpture driven by an electric engine turning at about 200 revolutions per minute. The sculpture has 12 pairs of gears that reduce

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the rotation speed to such an extent that the last gear would require well over two trillion years to do a whole rotation. The last gear is embedded in a concrete block, but this does not seem to affect the machine movement at all [1].

Another emblematic example is the "*Do Nothing*" machine built over a period of 15 years by Lawrence Wahlstrom, a retired clock maker. In 1948 he stumbled across a surplus WWII bombsight containing a complicated cluster of gears. He repaired the device and then started adding more and more gears. Today the machine has more than 740 gears; its complex operation is pleasantly hypnotic and totally useless [11].

The gear icon has a strong symbolic value. It may represent work in general (such as in the national emblems of many republics, including the Italian one), or more specifically the intellectual work with abstractly thinking or concentration. This last meaning is illustrated by the common pictogram of a head with gears inside or by small cogwheels spinning above the head of comics characters [3].

The connection between gears and thought is two-way. A couple of meshing cogwheels are an archetypical mechanism and therefore represents well the incredibly complex mechanism of the mind. On the other hand designing the optimal shape for gears is not an easy task; it requires much thinking and care and presents many subtleties.

The following pages will present some of them.

Tooth Profile

The simplest gear configuration is a pair of round gears arranged on the same plane so that their teeth engage. In this configuration the centers of the two gears are fixed and the gears can rotate around them. We assume that one gear transmits the motion to the other one; and we qualify the first one as the *driver* and the second one as the *driven*. Ignoring the teeth, the two gears can be represented by two externally tangent circles that roll without slipping. The circle representing a given gear is called the *pitch circle*. The rotational speeds of the two gears are inversely proportional to the radii of their pitch circles: the smallest gear spins faster.

The teeth serve to prevent slipping. Assuming that all teeth are equal in size (for both gears) then the number of teeth of each gear is proportional to the gear size. Therefore the ratio between the number of teeth of the two gears is equal to the inverse ratio between their rotation speed.

Teeth are essential in a real mechanism, but properly designing their shape is not an easy task. The squared teeth that often appear in pictograms and comics are easy to understand and to draw, but they are functionally unsuitable (see Fig. 1).

A first problem comes from the sharp corners. They happen to transmit the whole force: this generates friction and tends to wear down the tooth profile.

In addition this kind of tooth shape does not guarantee a constant speed ratio between the two gears: if the driver rotates at constant speed then the driven is subject to repeated abrupt accelerations and decelerations depending on the position of the contact point.

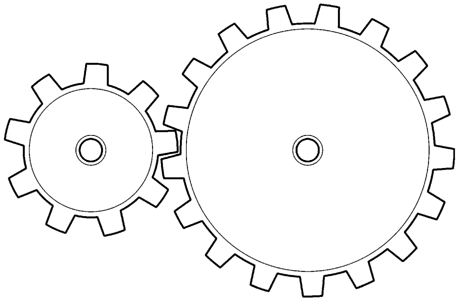


Fig. 1 Two circular gears with square teeth

Finally the two gears tend to wobble a bit. If the driver stays stationary then the driven can rotate freely through a small angle. It is possible to reduce this flaw by changing the teeth width, but beyond a certain limit the two gears get stuck together.

In a page of the *Madrid Codex I* (1490–1499) by Leonardo da Vinci, there are two pairs of gears: the first pair features squared teeth while the second pair has rounded teeth. The last design reduces the friction, but the other two flaws (non constant speed ratio and wobbling) are still present. Leonardo was aware that the movement of the contact point can affect the speed ratio, but this defect is negligible if the gears rotate slowly and under small load. It becomes much more serious a flaw in modern industrial applications [5].

Finding the most efficient shape for gear teeth became an important subject of mathematical interest: for more than two centuries many mathematicians worked on the problem, including Girard Desargues (ca. 1650), Girolamo Cardano (1557), Philip de La Hire (1694), Charles Étienne Louis Camus (1733) and Leonhard Euler (1754).

Eventually they found two different solutions that ensure a perfectly constant ratio between the rotation speeds. One solution requires a tooth profile based on *epicycloid* and *hypocycloid* curves, which are the curves generated by a circle rolling without slipping around the outside and inside of another circle respectively. The other solution is based on the *involute* curve that is the figure traced by the end an imaginary taut string as it is wound onto a circle.

These solutions are common today but were implemented extensively only since the nineteenth century. The involute gear profile is currently the most widespread: it is used in almost all industrial gears. The cycloidal gears are used predominantly in watchmaking [10, 16].

Let us analyse the involute gear profile a bit closer. Suppose that we have two circles next to each other but not in contact. A taut rope is rolled up clockwise around the right circle and anticlockwise around the left one. The straight segment of the rope is disposed along the common tangent to the two circumferences as in Fig. 2. Each circle rotates around its center. The two circles rotate in opposite directions with rotation speeds inversely proportional to their sizes. The rope unrolls from the first circle and rolls on the second, remaining taut.

Let us fix a point P on the straight rope segment. While the circles rotate, P moves along the common tangent. In the reference frame of each one of the two circles the

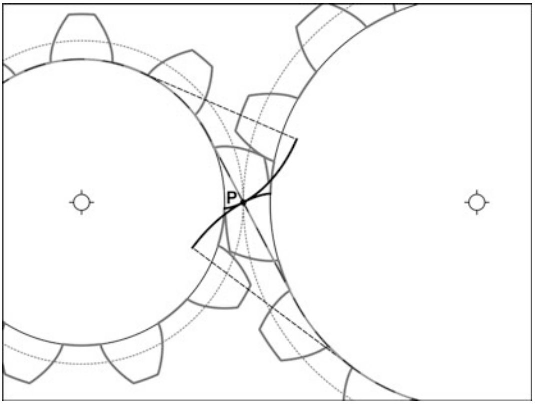


Fig. 2 Two involute gears: the teeth profiles are circles' involutes

point P describes an involute curve. If we add two matching teeth, with their profile shaped by these curves, we reckon that point P represents the contact point between the two teeth.

If we remove the rope, the relationship between the gears' speeds remains constant as we wanted.

In addition, the movement of the contact point is always perpendicular to the tooth profile which ensures the minimum friction and wear [6].

The Maltese Cross

We have seen that the teeth's shape affects the rotation speed ratio. Not surprisingly the shape of the whole gear affects this ratio even more. In the previous paragraphs we analyzed circular gears with involute teeth profile. This configuration ensures a constant ratio, that is normally required. Sometime we need a more complex relation between the movements of the two gears. For some applications we assume that the driver rotates at steady speed while the driven undergoes a sequence of accelerations and decelerations.

Let us consider, for example, the mechanism that controls the film movement in cinema projectors: each frame must remain still in front of the light beam for several milliseconds in order to impress the image on viewers' retinas. Then the film must move quickly to the next frame. The speed in this sequence should increase gradually to avoid breaking the film.

One special gear can meet these demands: it is a little masterpiece of mechanics developed for mechanical watches and called *Geneva drive* (from the city considered to be the birthplace of watchmaking). It is also known as *Maltese cross* because of its shape (Fig. 3), although there are variants with a number of arms other than four.

This mechanism has been used in projectors since 1896 and quickly became the most preferred system. It is still used today in the non-digital projectors.

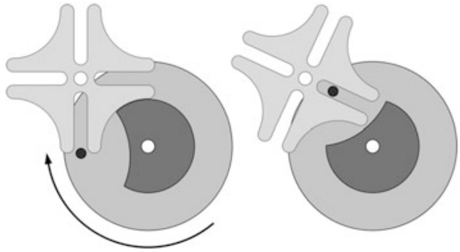


Fig. 3 Geneva drive (also known as Maltese Cross)

The driver features an eccentrically mounted pin. As the driver moves the pin slips into a groove in the driven gear and makes it turn to a fraction of a complete rotation (90° in the four-arm model). The driver also has a circular disc which lacks a section in correspondence of the pin. When the pin does not engage in the groove the disc locks the driven gear in position.

Elliptical Gears

Between the simple shape of a circle and the intricate profiles of the Geneva drive there are many other shapes that can be used to design pairs of gears with non-constant speeds ratios. A truly remarkable example are elliptical gears.

Oval-shaped gears are not a recent idea: in the *Madrid Codex I* there is a sketch of a mechanism with a small circular gear rolling over a larger oval-shaped gear. The pivot of the smaller gear can move along a line and the oval gears acts as a cam. Similar apparatuses were used in the past to control the movement of astronomical models or robots.

Surprisingly two identical elliptical gears can rotate around two fixed points and mesh perfectly during the whole movement. The pivot of the elliptical gear must be located in one of the two foci and the distance between the two pivots must be equal to the length of the ellipse's major axis. With these constraints the two gears' meshing is geometrically perfect.

Let us see how it works.

We can ignore the teeth and choose the reference frame of one of the two ellipses. The other ellipse rolls on the first one without slippage, perimeter-to-perimeter. The distance between the two pivots must remain the same during the whole movement.

In every moment the two ellipses share a common tangent line that passes through the contact point. The rolling ellipse is the mirror image of the steady one on the common tangent. The traces of the contact point along the two perimeters (highlighted in the Fig. 4) are equal for symmetry reasons: there is no slippage. We need only to check the distance between the pivots: it should remain constant.

One of the ellipse definitions tells us that the sum of the lengths of segments AP and PB must be constant and equal to the length of the major axis. For the reflective property of the ellipse and for symmetry reasons it is relatively easy to demonstrate that the segments AP and *PB'* are on the same line and that the sum of their length is also constant.

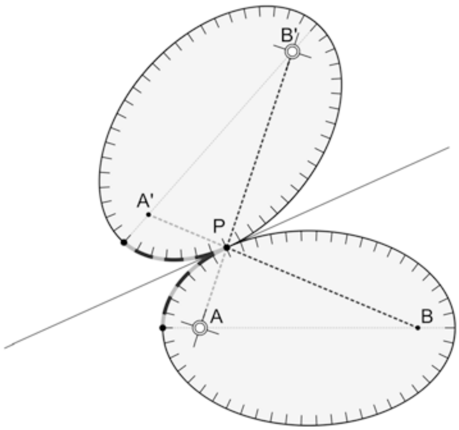


Fig. 4 Two identical ellipses. One rolls without slipping around the other

Therefore also the segment *AB'* length is constant, as we required.

The ratio between the rotation speeds varies continuously with a continuous smooth acceleration/deceleration. This makes elliptical gears well suited for a number of industrial applications. They are commonly used in bundling machines, conveyor systems, the textile industry, printing machines, etc.

The chain of five elliptical gears shown in Fig. 5 is mesmerizing when moving. The acceleration/deceleration of the movement is transmitted back and forth as a wave between the driver and the last driven gear. In some positions the mechanism seems doomed to get stuck, but surprisingly all the gears are able to disengage from each other and to continue their smooth movement.

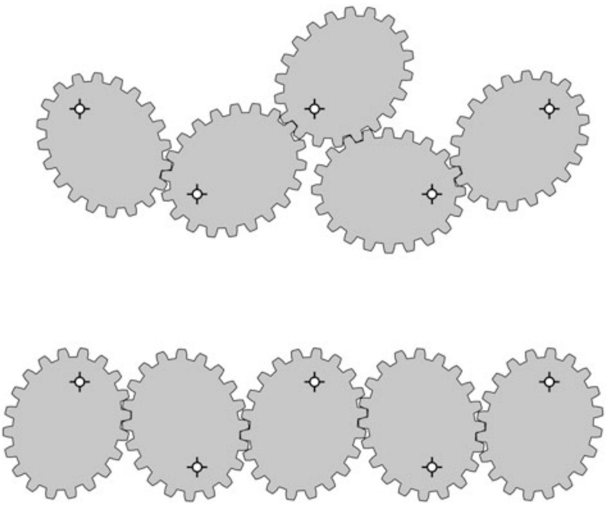


Fig. 5 Two frames of the animation of a train of five elliptical gears

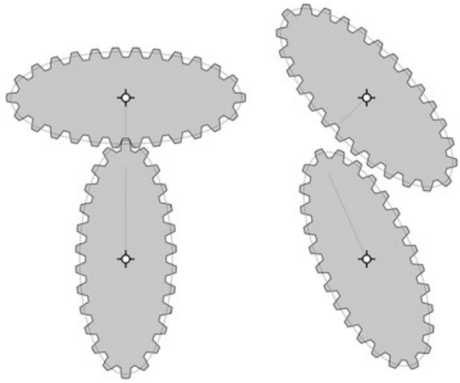


Fig. 6 Two elliptical gears with pivots at center. They lose contact during the rotation

An elliptic gear with the pin at the ellipse center does not work so well. Figure 6 shows a couple of gears that lose contact during the rotation. Nevertheless it is possible to find a curve that looks like an ellipse and works well. Figure 7 shows an example of two gears that roll smoothly without losing contact. This configuration is useful to make pumps or flowmeters. We will see in the following paragraphs how to build such a curve.

Nautilus Gears

In addition to circles and ellipses many other curves can create pairs of identical gears that mesh together with perfect fit. Let us examine another simple case like the spiral. There are many simple curves with a spiral-like shape. Do they meet our requirements?

The simplest spiral-like curve is perhaps the *Archimedean spiral*. In polar coordinates the radius grows proportionally to the angle: $r = a + b\theta$. Unfortunately this curve does not work: two identical Archimedean-spiral shaped gears loose contact during the rotation (see Fig. 8).

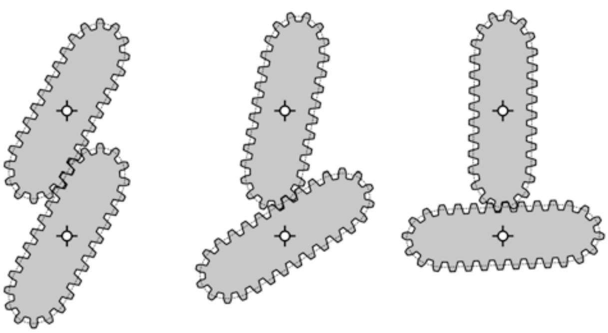


Fig. 7 Two perfectly meshing oval gears with pivots at center

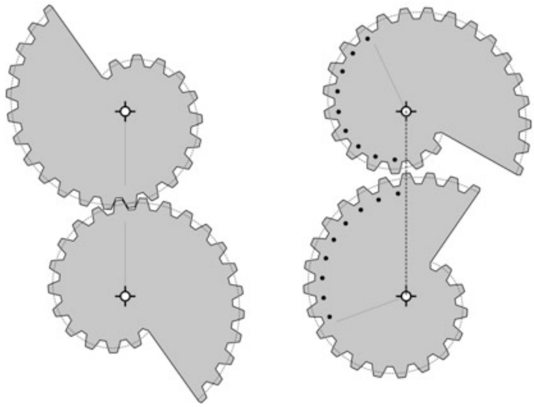


Fig. 8 Two identical Archimedean spiral shaped gears. They lose contact when they roll without slipping

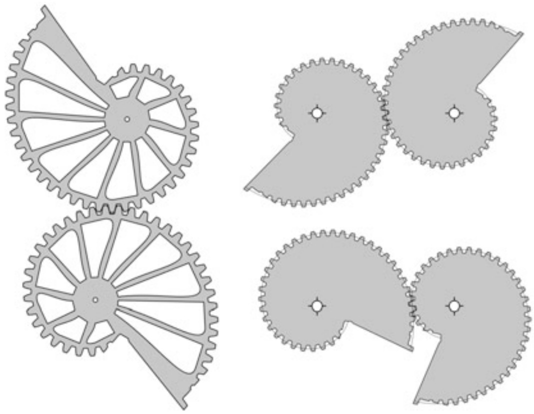


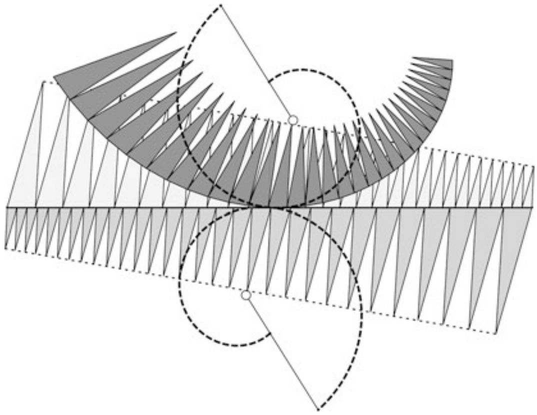
Fig. 9 "Nautilus" gears. Their shape is a logarithmic spiral and they mesh perfectly

Another spiral-like curve gives us an exact result: the *logarithmic spiral*. It is a beautiful curve which applies to many different natural phenomena among which the Nautilus shell, the pattern of sunflowers pistils, the configuration of a cyclone's clouds and even some galaxies' arms. The Swiss mathematician Jacob Bernoulli (not to be confused with other mathematicians and physicists of the exceptionally talented Bernoulli family) named this curve *Spira mirabilis* (Latin for *The marvelous spiral*). He wanted a similar spiral engraved on his tomb, along with the phrase *Eadem mutata resurgo* (*Changed and yet the same, I rise again*) which refers to the characteristic of the curve to remain identical subjected to different types of mathematical transformations. As we will see this feature also applies to the gear world: the shape of a gear that mates perfectly with a logarithmic-spiral-shaped gear is again a logarithmic spiral.

Unfortunately for Jacob, the math skills of the stonemasons was limited: the curve engraved on his tomb is definitely an Archimedean spiral.

Two identical gears shaped like two logarithmic spirals mesh perfectly for the whole movement (Fig. 9). Let us check that this is the case.

Fig. 10 Two "Nautilus" gears unfolded



We divide the spiral gear in a large number of slices, each with the same central angle. We can approximate each slice as a thin triangle. According to the logarithmic spiral properties these thin triangles have different sizes, but the same form. The ratio between the sizes of any two adjacent triangles is constant.

Now we unfold the spiral in such a way that the longer edges of the triangles are all parallel. The shortest sides, which composed the spiral profile create a straight line and the triangles vertices that were at the center of the spiral arrange themselves along another line.

Let us repeat this operation with the other gear and place the two figures in contact along the lines representing the two spiral curves. The final result is a rectangle cut into two equal parts along a diagonal line. All the points along the longest sides of the rectangle correspond to the centers of the two gears. The diagonal line corresponds to the profile of the two gears (Fig. 10).

In this configuration any point on the first curve coincides with a given point on the second curve. These couples of points are going to touch at a given moment during the gears meshing in the original configuration.

The distance between the two pivots is represented by the rectangle shorter side and therefore remains fixed regardless of contact point position.

The spiral gears (also known as *Nautilus gears*) have a certain aesthetic value, and can be easily found in toys and puzzles.

The pin of a single nautilus gear rolling on a rack (a linear gear) moves along a sloping line. This fact has an unexpected and interesting practical utilization in rock climbing where a piece of protection equipment called *Spring-loaded camming device* or *Cam* or *Friend* consists of two, three, or four cams shaped as logarithmic spiral.

The device is inserted in a crack in the rock; the cams engage with the rock and spread farther apart. If the climber falls, the pull force is converted into outwards pressure on the rock preventing the removal of the unit [7].

Free-Shape Gears

After these specific examples it is natural to scale up to the general case: given an arbitrarily shaped gear, which form must the other gear have in order to perfectly mesh with the first?

Let us define the problem more precisely and ignore the teeth (at first): the two gears are represented by two curves. The two curves should rotate, each one about its own center. At the contact point they should roll over each other without slippage.

For each curve we define a *starting point*: the two starting points touch each other at the beginning. It is convenient to describe the curves in polar coordinates: each curve point is identified by the *radius* (the distance between the point and the gear center) and the *polar angle* (the angle from a reference direction, e.g. the direction of the starting point). During the rotation the points of the first curve come in contact with the points of the second one. This define a relationship between points: a point on the first curve *corresponds* to a point (possibly more than one point) of the second curve if the two points happen to touch each other during the rotation. In the following paragraphs we suppose that the contact points lies always on the line which connects the two centers [12, 14, 15].

The problem is finding the second curve if the first one is given. The curves must meet two constraints: for each pair of corresponding points the sum of radii must be constant (the distance between the centers is fixed) and the distance from the reference points along the two curves must be equal (no slippage).

This problem lends itself to numerical integration. We consider a large number of points evenly spaced on the first curve. Then we want to compute the same number of points for the second curve. Each point must have the same distance from the previous one and as we know its radius, this allows us to compute its polar angle.

There are some other requirements that must be met, in particular: the curve must be closed, e.g. the last radius must be close to the first one and the last angle must be close to 360°.

Actually, the length of the second gear could be an integer multiple of the length of the first gear; so that the second gear would make one full turn while the first one makes N full turns. In that case our algorithm should generate only the first slice of the second curve, with a maximum angle close to 360°/N. Then this slice should be copied N times to generate the whole second gear.

We can now test this approach on the odd shape that tiles the floor in *Palazzo Franchetti* hall room, where the conference was held. Let us shape a gear with this profile. A digital animation builds the second gear in real time. Figure 11 shows the final result.

The curve shape depends on the distance between the two pivots. For some values of the distance the curve is closed. Different values generate a different number of lobes (and therefore a different number of complete rotations of the first gear corresponds to a complete rotation of the second one).

In general, we expect the shape of the mating gear to be different from the initial one, but it is possible to design gears that mesh with themselves as in the case of the circular, elliptical and spiral gears.

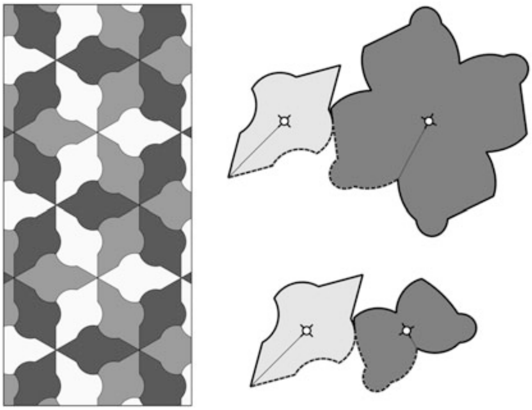


Fig. 11 A gear shaped as Palazzo Franchetti floor's tiles with two possible mating gears

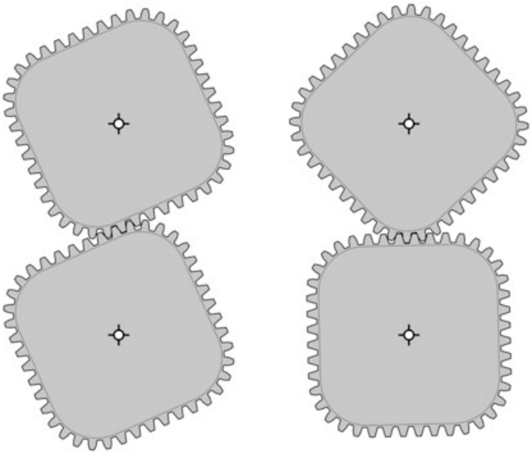


Fig. 12 Two perfectly meshing identical square-like gears

For example let us try to design an approximately square gear (Fig. 12). Of the infinite possibilities we chose a gear profile with four linear segments connected by four arcs and decide that the linear segments must have the same length as the curved segments.

The problem is to determine the length of the straight segments with respect to the gear size and the exact shape of the curved segments.

During the rotations we expect that for symmetry reasons the straight part of one gear mesh with the curved part of the other and vice versa.

If we knew the length of the linear part we could use the algorithm illustrated in the previous paragraph to determine the curved part length and shape. Then we could evaluate the angle spanned by the linear part and the curved part joined together. This angle should be 90° (four repetitions of the two parts should generate the whole shape), but if we start with a randomly chosen length for the straight part then the angle will be larger or smaller.

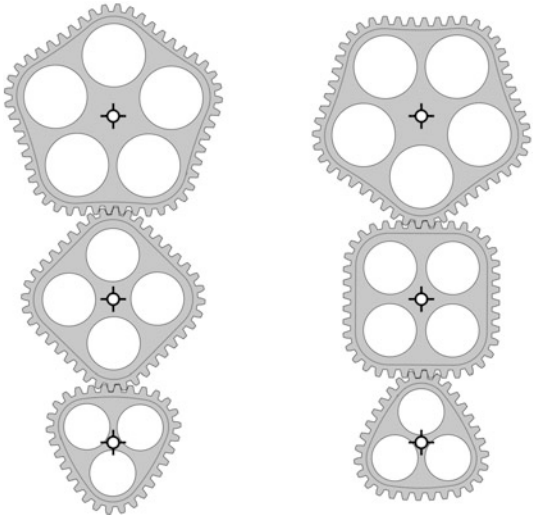


Fig. 13 Perfectly meshing polygonal gears

The spanning angle depends on the length of the straight part with larger straight segments yielding to larger spanning angles and shorter segments yielding to smaller angles.

The correct length can be determined iteratively, e.g. using a bisection algorithm. At each iteration we select a length, compute the curved part, measure the angle and then correct the length. The iteration terminates when the difference between the spanning angle and 90° is negligible.

With this kind of strategy it is possible to generate any kind of polygon-like gears (e.g. triangular gears, square gears, pentagonal gears, etc. as in Fig. 13) as well as the oval gears (with pivot at center) mentioned before: they correspond to a degenerate polygon with two sides only.

Free-Shape Gears Teeth

Cycloidal and involute tooth profile work well for circular gears only. Non-circular gears require different curves. Indeed each tooth in these gears can require its own different shape to ensure a smooth transmission. There is no general case as the correct profile is not defined by any specific curve: it must be computed and specified numerically.

A simple approach has been proven to be valid and suitable for most applications.

Let us start with an arbitrary closed curve representing the gear. We want to calculate the teeth profile that is another curve following the first one, crossing it in and out and generating teeth tips and roots.

We first need a reference gear: for example a circular gear with involute tooth profile, or even better a rack (a linear gear) with straight tooth flanks. We engage

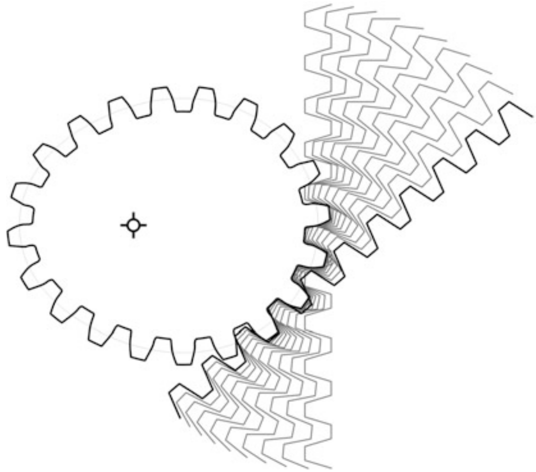


Fig. 14 Ten discrete rack positions. The complete movement of the rack around the pitch curve generates the teeth profile

this reference gear with the non circular gear. The latter has no teeth yet, therefore we just roll the pitch curves one onto the other, without slippage.

Figure 14 shows the reference gear teeth profile at different moments during the rotation. The envelope of these curves builds the desired teeth profile [19].

Non-circular Gears in Flesh and Bones

Today the manufacturing of non-circular gears is greatly facilitated, even for small productions. 3D printers or numerically controlled cutting machines can achieve reasonably accurate models suitable for non-critical applications. One does not need to own the appropriate tools as there are many services that accept a file with the model and send back the gears within few days at a very reasonable cost.

The gear shapes can be designed with any popular CAD program. Several plugins are available to do the needed computation. It can be very challenging and satisfactory programming exercise to try and write a program computing the gear shape. Complete applications that perform this task are already available and can easily be found on the web. An example is the excellent Gearify software [9].

In this context designing and building a simple mechanism with non-circular gears can be a very stimulating and interesting didactic or recreational activity.

The Naucleus

The last example illustrates a purely playful use of non-circular gears. Figure 15 depicts the *Naucleus*, a fantasy object that was the core of an urban adventure created and organized in 2014 by the author with some friends in partnership with the *Teatro Argentina of Rome* and the acting company *Il ratto d'Europa*. During the

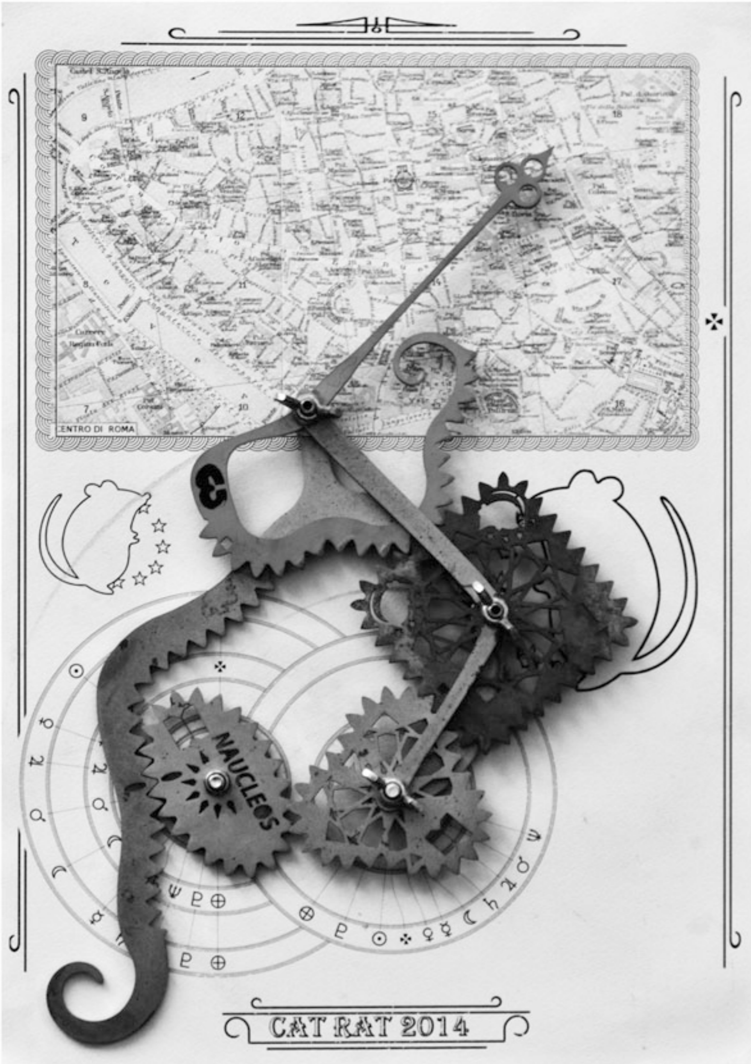


Fig. 15 The *Naucleus*, an alleged seventeenth century mechanism featuring non circular gears. Photo courtesy of Simone Cabasino

adventure a hundred players (selected among the theatre's regular audience) had to face unexpected and strange situations, interact with actors, solve puzzles and try to decipher the complex plot in which they were plunged. The event started as a regular representation, but the abrupt kidnapping of the main character by unknown black-dressed men dragged the participants into a quest to rescue her. During the adventure they had to find gears with bizarre shapes: one buried in a flower vase, another disguised as a pendant around a musician's neck, etc. Eventually they were able to reconstruct the whole Naucleus: an alleged seventeenth century mechanism, that, if properly used, would point at the refuge of the bad guys on an ancient map of Rome [13, 18].

References

1. C.C. Barratt, Time machines: steampunk in contemporary art. Neo-Vic. Stud. **3**(1), 167–188 (2010)
2. M. Burrows, G. Sutton, Interacting gears synchronize propulsive leg movements in a jumping insect. Science **341**(6151), 1254–1256 (2013)
3. N. Cohn, *The Visual Language of Comics: Introduction to the Structure and Cognition of Sequential Images* (Black, London, 2013)
4. W.P. Crosher, *A Gear Chronology, Significant Events and Dates Affecting Gear Development* (Xlibris Corporation, USA, 2014)
5. L. Da Vinci, *Codex Madrid* (Castrovilli Giuseppe, 1998). Available online: http://leonardo.bne.es/index.html. Accessed 6-Jan-2018
6. Fellow corp., Design of involute gear teeth, in *Issue of Gear Technology*, Oct./Nov. 1984. Available online: https://www.geartechnology.com/issues/1084x/back-to-basics.pdf. Accessed 7-Jan-2018
7. R. Ferréol, A. Esculier, *Roads and Wheel* (2017). http://www.mathcurve.com/courbes2d.gb/engrenage/engrenage2.shtml. Accessed 14-Jan-2018
8. T. Freeth, *Ancient Computer* (Scientific American, USA, 2009). Available online: http://www.cs.virginia.edu/~robins/Decoding_an_Ancient_Computer.pdf. Accessed 14-Jan-2018
9. Gearify, https://www.gearifysoftware.com/. Accessed 14-Jan-2018
10. R. Hessmer, *Cycloidal Gear Builder* (2012). http://www.hessmer.org/blog/2012/01/28/cycloidal-gear-builder/. Accessed 14-Jan-2018
11. Internet Craftsmanship Museum, The "Do Nothing" Machine—it produces nothing except smiles. http://www.craftsmanshipmuseum.com/Wahlstrom.htm. Accessed 14-Jan-2018
12. M. Masal, S. Ersoy, M.A. Güngör, Euler–Savary formula for the homothetic motion in the complex plane C. Ain Shams Eng. J. **5**(1), 305–308 (2014). Available online: http://www.sciencedirect.com/science/article/pii/S2090447913000920. Accessed 14-Jan-2018
13. Pure Emotions CAT-RAT, https://youtu.be/MMaNRZYQ2Hk. Accessed 14-Jan-2018
14. S.P. Radzevich, *Theory of Gearing: Kinematics, Geometry, and Synthesis* (CRC Press, Boca Raton, 2012)
15. Shuanghuan Transmission Research Institute, Gear University. http://www.precision-sh.com/en/yanfazhongxin/chilundaxue/38.html. Accessed 14-Jan-2018
16. H. Sparks, *Designing Cycloidal Gears* (2013). https://www.csparks.com/watchmaking/CycloidalGears/index.jxl. Accessed 14-Jan-2018
17. D. Taimina, Geometry and motion links mathematics and engineering in collections of 19 th century kinematic models and their digital representation, in *ICME*, 5-8 March 2008
18. Teatro di Roma, *CATRAT*, http://www.teatrodiroma.net/doc/2742/catrat. Accessed 14-Jan-2018
19. I. Zarębski, T. Sałaciński, Designing of non-circular gears. Arch. Mech. Eng. **55**(3), 275–292 (2008). Available online: http://prozamet.pl/art_2008_3_08.pdf. Accessed 6-Jan-2018

Part VI
Mathematics & Art

Historical Notes on Star Geometry in Mathematics, Art and Nature

Aldo Brigaglia, Nicla Palladino, and Maria Alessandra Vaccaro

- Gamma: "I can. Look at this Counterexample 3: a star-polyhedron I shall call it urchin. This consists of 12 star-pentagons. It has 12 vertices, 30 edges, and 12 pentagonal faces-you may check it if you like by counting. Thus the Descartes-Euler thesis is not true at all, since for this polyhedron $V - E + F = -6$".
- Delta: "Why do you think that your 'urchin' is a polyhedron?"
- Gamma: "Do you not see? This is a polyhedron, whose faces are the twelve star-pentagons".
- Delta: "But then you do not even know what a polygon is! A star-pentagon is certainly not a polygon!"

In the above dialogue from [25], Imre Lakatos used the example of star polyhedra to describe the complex path *definition–proof–refutation–new definition* that mathematical thought threaded before reaching a consensus on fundamental points of the main mathematical topics. In this paper we will try to examine how in the history of polyhedra (and in particular star polyhedra), a long period of "discovery" of individual types due to the observation of natural objects or due to artistic imagination preceded (and was connected with) the mathematical solution fixing the "right" definitions. Such long period of discovery—we will argue—influenced further investigations on nature and art. The paper will start from the thirteen century and will end with the publications of Pappo's work [30] and Kepler's *Harmonices Mundi* which provided solid mathematical foundations to the subject. We want to describe the geometric ideas and the theories of Adelard of Bath, Thomas Bradwardine, Luca Pacioli, Albrecht Dürer, Simon Stevin, Daniele Barbaro, Jan Brożek. We conclude with some short notes about the subsequent developments (Johannes Kepler, Louis Poinsot and Albert Badoureau).

A star polygon may be constructed by connecting with straight lines every h-th point out of n regularly spaced points lying on a circumference. We call order the

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number n and species the number h of the star polygon. It is obviously possible to have a star polygon only if n and h are co-prime integers.

In western culture, the mathematicians' interest for star figures originated as a result of the common use of these shapes by artists and artisans in their architectural decorations and mosaics, especially Islamic, in the medieval period. Symmetries and fascinating combinations of elementary figures, essentially inspired by the observation of shapes in nature and intuition, led artists to create increasingly articulated patterns, persuading mathematicians to progressively develop a study of their regularity and properties. The key to formalize these new concepts was the interest by artists-mathematicians as Pacioli, Piero della Francesca, Leonardo da Vinci and Dürer who influenced the mathematicians Stevin and Barbaro. From that time on, the systematic exploration and classification of the star figures through mathematical theories, allowed to suppose the existence of new shapes that neither nature nor art had previously shown. This is why it is important to study less known medieval treatises where star polygons and polyhedra appeared, even if their completed classifications occurred only in the 17-th century by Kepler and in the 19-th century by Augustin-Louis Cauchy and Poinsot.

In this paper, we do not analyse the well-known Greek period; similarly, we give only a brief mention of the very interesting contributions of Arab and Islamic art and mathematics. It seems that the influence of Arabic mathematicians on the western mathematicians of the Middle Ages was not crucial. We did not find manifest or declared Arab influence on the authors we present here. It is important however, to mention that in the texts of practical geometry by the mathematicians of Islamic culture, the needs of artists and artisans were significantly linked to the abilities of the mathematicians. The fact that star figures arose and were mathematically studied starting from artistic necessities is perfectly showed by a quote by Abū'l-Wafā' al-Būzjānī (c. 940–998)[1] and by a figure.[2]

> The geometer knows the correctness of what he wants by means of proofs, since he is the one who has derived the notions on which the artisan and the surveyor base their work. However, it is difficult for him to transform what he has proved into a [practical] construction, since he has no experience with the practical work of the artisan.

Boethius (c. 480–524) was probably the first to talk about star pentagon, in his *Geometria*, showing a pentagram (a well-known figure) inscribed in a circle. In his very popular translation of Euclid's *Element* (realized starting from a translation by Adelard of Bath [7]), Giovanni Campano (1220–1296) described the pentagram and the property that the sum of its interior angles is equal to two right angles. A first systematic study on star polygons is contained in Adelard of Bath's (1080–1152) Latin translation of Euclid's text (1120), from the Arabic text by al-Ḥajjāj ibn Yūsuf ibn Maṭar. Adelard had been in close contact with Arab culture, staying for many

[1] *Kitāb fīmā yahtāju ilayhi al-sāni min al-a'māl al-handasiya* (*On the Geometric Constructions Necessary for the Artisan*). We got it from [28].

[2] The picture is in the anonymous text (dating to the 14th century) *Fī tadākhul al-ashkāl al-mutashābiha aw al-mutawāfiqa* (*On Interlocks of similar or Corresponding Figures*) and in [12], p. 774. See also [13] and [32].

years in Spain, Cilicia and Syria.[3] In his text, he showed that in a convex polygon with n angles, the sum of the measures of the interior angles is: $(n-2)2R$ (where R means a right angle). He deduced the sum of the interior angles of a star polygon starting from the number of intersections of each side with the others [20].

The results of Adelard were probably developed independently in [4] (printed in 1496) by the English theologian and mathematician Thomas le Byer, known as de Bradwardine (c. 1290–1349). Bradwardine was one of the Oxford Calculators, a group of thinkers devoted to natural science, mainly physics, astronomy and mathematics. In 1331 Bradwardine was ordained sub-deacon; in 1337 he became Chancellor of St. Paul's Cathedral and finally, he was elected Archbishop of Canterbury. At the end of the first part of his work, Bradwardine included a chapter in which he studied figures of egredient angles (*figurae egredientium angulorum*), distinguishing them from the convex ones (*simplex*):

> I shall speak about figures of egredient angle [...]. Discussion about them is rare, and I have not seen a discussion of them, except only by Campanus, who only casually touches on pentagon a little. A figure is said to be of egredient angles when the sides of some polygonal figure from among the simple ones are produced until they meet outside.[4]

Star polygons produced in this way from simple polygons are those of the second species, which Bradwardine called "of the first order" (i.e. the first constructible, given the number of its sides). Bradwardine's first conclusion was that the star pentagon is the first figure of egredient angles; then he showed that the pentagon of egredient angles has five angles equal to two right angles. His third deduction was that of figures of egredient angles of given specie, each figure of successive order adds two right angles over the figure of precedent order. He also affirmed that, while this is immediately evident for all figures having an even order, for each of them is composed of two simple figures mutually entwined (the hexagon is worth four right angles for it is composed of two triangles; the octagon is composed of two quadrangles, and so on), the same is less evident for figures having an odd order:

> it is likely, however that the heptagon adds two right angles over the hexagon and the nonagon two right angles over the octagon and so for the others.

Bradwardine considered as star polygons figures such as the star hexagon that in the nineteenth century would be regarded as two triangles rather than as a simple polygon. He showed how to construct figure of higher species, producing the sides further until they would meet. His fourth statement was that the heptagon is the first figure of egredient angles of the second species. His fifth conclusion was that the first figure of the following order is always taken from the third member of the preceding species. Bradwardine was also interested in the sum of the angles, but he said that:

> To investigate here the value of the angles of such figures would be more laborious than fruitful, and so I do not set about it. But it sometimes seemed to me that all orders of figures

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[3] See http://turnbull.mcs.st-and.ac.uk.

[4] The quotes are from [27] for Bradwardine's geometry see also [26].

are agreed in this, that always the first has the value of two right angles, and each successor adds two right angles in value over its predecessor. But although this is near to it in reality, yet I do not assert it.

In 1542, the French mathematician and philosopher Charles de Bouvelles (1479–1567?) resumed Bradwardine's theory on star polygons in his "Géométrie pratique".[5] He did not show great originality, but it is noteworthy that he used proofs from the composition or decomposition of polygons into triangles. For example, in order to calculate the sum of the angles of a pentagram, he inscribed it in a regular pentagon and then he said that any angle of the pentagram is the third part of any angle of the pentagon. So the sum of the angles of a pentagram is equal to two right angles. He argued that the star hexagon is composed by two equilar triangles; the sum of its angles is four right angles and its area is twice the area of the original convex hexagon.[6]

In the 17th century, Jan Brożek (1585–1652), a Polish mathematician, astronomer, physician, poet, writer, and musician who was also rector of the Kraków Academy and biographer of Copernicus, further developed the theory. In [6], Brożek showed, by simple arguments on angles at centre and at circumference, the property that it is possible to create an infinity number of star polygons so that the sum of their interior angles is two right angles. The problem of the not univocal definition of star figures, however, is still present: for Brożek, the hexagon with six egredient angles was a not convex figure with twelve sides; the star heptagon was a not convex figure with 14 sides, and so on. For the first time, Brożek clarified that a star hexagon is built by two triangles and a star octagon (of II species) by two squares. Brożek conceived a special procedure for the construction of star polygons starting from isoperimetric convex polygons. From the regular pentagon *SDCBY*, he built, by extending its sides, the star polygon *SEDICOBVYA* (Fig. 1). He overturned the triangle *SAE* along the segment AE and made the same thing for all the triangles of the figure; finally, he proved that the new pentagon *HKLFG* is isoperimetric with the star pentagon. The transformations were even more for polygons with greater number of sides, and Brożek showed what was possible to do starting from a 14-sided figure.

It seems that in Europe the earliest representation of a star polyhedron (a small stellated dodecahedron) is in Venice, in a mosaic attributed to Paolo Uccello (1397–1475) located on the floor of the St Mark's Basilica (Fig. 2). The first systematic discussion of star polyhedra is [29] (completed as manuscript in 1498 and printed in 1509[7]) by Luca Pacioli (1445–1517).[8]

[5] We used the reprint of 1551 [17].

[6] See [17] p. 23. See also [16]. For other short studies of this period on star polygons, see [11] pp. 478–481.

[7] Two manuscripts of this work are in Milan (at Biblioteca Ambrosiana) and in Geneva (at Bibliothéque Publique et Universitaire); a third manuscript, from which the print copy was probably edited, is lost (see [39]).

[8] On Pacioli there is abundant literature; see [39].

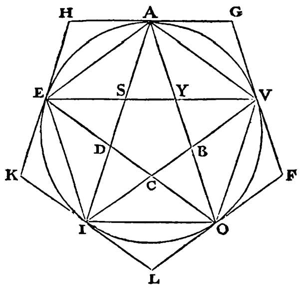

Fig. 1 Picture from [6], p. 47

Fig. 2 Picture of a mosaic in the St Mark's Basilica, Venice

Pacioli was, as is well known, under the influence of Piero della Francesca, who was not explicitly interested in star polygons and polyhedra [15]. His work represented a milestone and was responsible for reawakening the interest of mathematicians and artists on the problems associated with the geometrical shapes, in particular Archimedean polyhedra. Here, we do not discuss this topic, but we refer to the large bibliography in [19]. In his treatise, which contains the famous drawings by Leonardo, Pacioli showed not only the five regular polyhedra, but also six polyhedra created by cutting solid angles from the regular solids, later known as Archimedean (*polyhedron abscissum*). Pacioli also presented for the first time some new star polyhedra that he built by adding regular and equilateral pyramids on the faces of regular polyhedra (*polyhedron elevatum*), or by adding pyramids on the faces of Archimedean solids (*polyhedron abscissum elevatum*). Sometimes this procedure has been called *Kleetope* (in honour of the mathematician Victor Klee). Most likely, Pacioli's book was the main source of inspiration for the subsequent developments of the topic.

The first of Pacioli's new polyhedra is the *elevatum tetrahedron* with 12 faces, 18 edges and 8 vertices (now called *kistetrahedron*, a Catalan solid since it is the dual of the truncated tetrahedron); this is a tetrahedron with triangular pyramids added to each face. The second polyhedron is the *elevatum cube* with 16 faces, 36 edges and 14 vertices (now called *tetrakis hexahedron*, dual of the truncated octahedron). Then

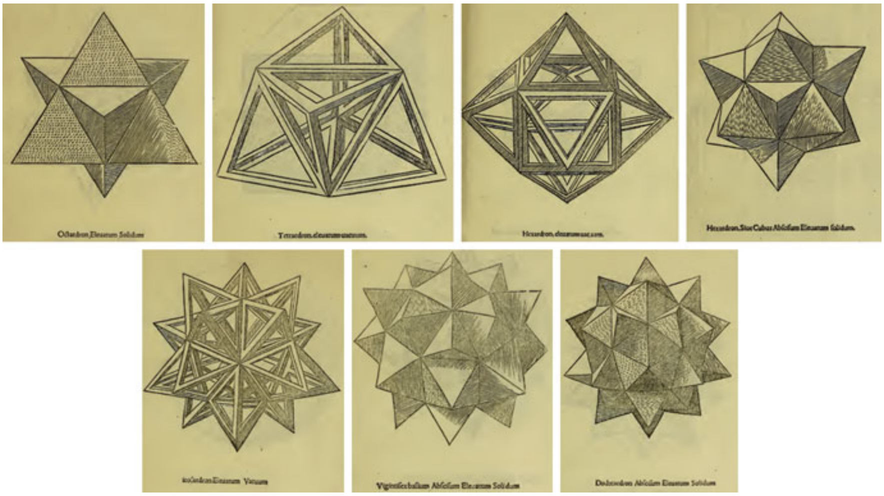

Fig. 3 Pictures from [29]

Pacioli argued that it is possible to create an *abscissum elevatum cube* (the *disdyakis dodecahedron*) with 48 faces, 72 edges and 26 vertices, dual of the cuboctahedron. From the octahedron, he continued, it is possible to create the *elevatum octahedron* (or *stella octangula*) with 24 faces, 36 edges and 8 vertices. He also showed the *elevatum icosahedron* (*triakis icosahedron*, dual of the truncated dodecahedron) and the *elevatum dodecahedron* (*pentakis dodecahedron*, dual of the truncated icosahedron), both with 60 faces, 90 edges and 32 vertices. Then Pacioli illustrated the *abscissum elevatum dodecahedron* (*elevatum icosidodecahedron*) with 120 faces, 180 edges and 62 vertices (Fig. 3). Probably Pacioli used material models of the solids. He said:

> E cascando in piano questo sempre si ferma in 6 ponte o coni piramidali [in 6 vertici]. De li quali coni uno sia de pyramide pentagona e li altri 5 sono dele pyramidi triangule. La qual cosa [. . .] pare a l'ochio absurda che simil ponte sieno ad un piano [. . .].[9]

Pacioli's text influenced Fra' Giovanni da Verona (c. 1457-1525) who created wonderful intarsia in the Roman Catholic church Santa Maria in Organo in Verona; pictures of star polyhedra inspired from Leonardo's ones are also present.[10]

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[9] "And falling flat this always stops in 6 pyramidal cones. One is the vertex of the pyramid with pentagonal base and the other 5 are of the triangular pyramids. It seems absurd that they are on the same plane". This propriety observed by Pacioli is not correct: the vertex of the pyramid with pentagonal base is not on the same plane as the five vertices of the triangular pyramids. The distance from the vertex to the plane, however, is very small, which suggests that Pacioli made some tests with material models of the solids. On this point, we would like to acknowledge the precious help of Maria Dedò.

[10] An interesting text on the relationships between Giovanni and Leonardo is Hayashi, S., I Poliedri nelle Tarsie di fra Giovanni da Verona: l'Influenza delle Illustrazioni nel De Divina Proportione (1509) di Luca Pacioli. *Studi Italici*, 59 (2009), pp. 97–117.

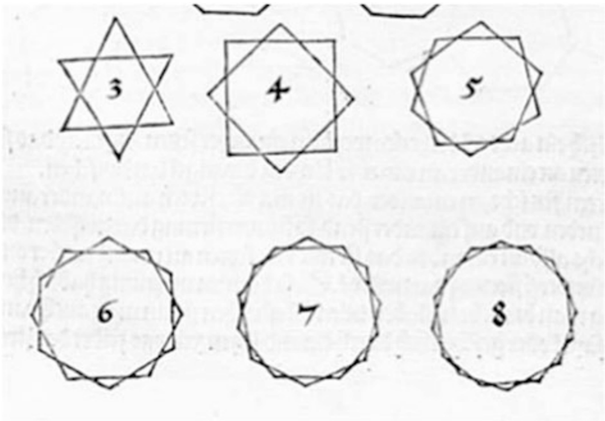

Fig. 4 Picture from [18], second part, Fig. 27

Without doubt, Pacioli influenced Albrecht Dürer (1471–1528), who lived in Venice between 1494 and 1495, and then again between 1505 and 1507.[11] His famous treatise [12, 18][12] includes some significant notes on star polyhedra. The book was intended to be a guide for young craftsmen and artists, giving them both practical and mathematical tools for their trade. Dürer, recalling the famous phrase of Leonardo, wrote:

> Many young people have been trained in the art of painting, and have been educated in them only with daily practice, without giving them the basis. They grew up in intelligence like the wild tree that was not carved.

In the second part of this book, Dürer provided compass and straight edge constructions (sometimes approximated) for the regular polygons from the triangle to the 16-gon. He also provided star figures made with arcs of circumference (not with line segments, so they are not star polygons), which are a variety of stars described within circles for ornamental purposes (Fig. 4). He talked about star polygons built by crossing over each other and rotating regular polygons:

> Sometimes it is convenient to superpose the figures [. . .] or even to allow them to penetrate each other, as I indicated in the six figures I built as result.

It is interesting to note that this construction of star polygons is quite different from those of Bradwardine, who extended the sides of regular polygons; so it is not by chance that among the six examples of star polygons, the most famous, the pentagram, is missing. Although without drawings and his famous developments of surfaces, the great artist provided some notes on star polyhedra; he built star polyhedra by interpenetration of regular polyhedra:

[11] On the relationships between Pacioli and Dürer, see [31] at preface.

[12] We are referring here to [31].

> You can look for the intersections two by two of these solids of the same size, making sure that the vertices of one pierce the faces of the other. This gives good results in construction.

Or, like Pacioli and Leonardo, adding pyramids on the faces of regular polyhedra:

> So you may want to apply on each side of these solids a pyramid, more or less high, with as many sides as the face on which it is placed.

Dürer defined a class of polyhedra larger than Pacioli's one, since his pyramids were not only equilateral but they had different altitudes.

It is also thanks to the work of Dürer, that the decorative use of star polyhedra in texts on the perspective drawing spread in the environment of Nuremberg during the sixteenth century. The work [21] of Wenzel Jamnitzer[13] (1507–1585) stands out amongst all. Jamnitzer was a goldsmith of Nuremberg, and his work is full of beautiful geometric shapes on copper plates, but without explanations and mathematical theories, which could be used to build his solids. His book, however, develops a study of 120 regular and semi-regular solids, obtained by modifying, starring or cutting the five platonic polyhedra (Fig. 5). Other significant texts are [37] by Lorenz Stoer (c. 1540–1620) and [3] by Daniele Barbaro (1513–1570). Coadjutor bishop of Aquileia (Italy), Barbaro, had held important roles in Venetian politics (ambassador in England, delegate of Venice at the Council of Trento etc.) and his treatise is one of the most significant of the century on perspective. Barbaro used Bradwardine's construction of star polygons extending the sides of regular polygons, as well as drawing their diagonals:

> Et se prolungherai i lati della superficie di 5, 6, 7 e più lati [. . .] farai simiglianti figure come appare nelle figure [. . .] se tirerai le linee dagli anguli agli anguli & dai lati ai lati & dai lati agli anguli.[14]

Influenced by Dürer and Pacioli, Barbaro for the first time showed the developments of star polyhedra; the first description was related to Pacioli's elevatum tetrahedron:

> Spiegatura d'Alcuni corpi fondati sopra la soperficie dei corpi sì regulari come irregulari, & prima di quello, il quale è fondato sopra la piramide. Molto dilettevole è la pratica seguente & ha di belle considerationi, imperocchè ella trova il modo con la quale sopra le superficie piane de i corpi regulari come irregulari si fanno le piramidi di molti lati come si vede della spiegatura di dodici trianguli di lati uguali rinchiusa & posta insieme forma un corpo di molte punte fondate sopra la piramide & si hanno a ponere insieme secondo i numeri notati nelle soperficie triangulari come appare nella figura.[15]

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[13] On its diffusion in Germany and beyond, see [40].

[14] And if you will extend the sides of the surface of 5, 6, 7 [. . .] and more sides you will make similar shapes as it appears in the figures [. . .] if you draw the lines from the angles to the angles & sides to sides & sides to angles.

[15] Developments of some shapes built on the surface of both regular and irregular & before that is built on the pyramid. The following practice is of great pleasure and it has beautiful remarks because it explains how to build pyramids of many faces on both regular and irregular solids, as we can see the development of twelve triangles of equal sides enclosed & placed together forms a solid of many vertices placed on the pyramid & we have to put together according to the numbers noted in the triangular surfaces as it appears in the figure.

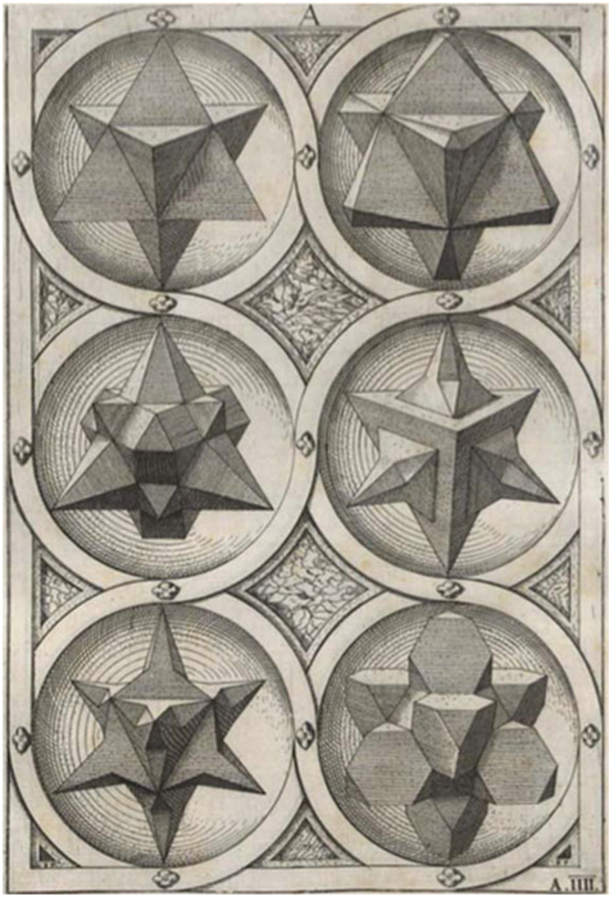

Fig. 5 Picture from [21], carta VIII r. Reproduced with permission from www.bncf.firenze.sbn.it

Barbaro also showed the elevatum cube, the *stella octangula*, the "Spiegatura d'un corpo sostenuto dallo icosaedro" (elevatum icosahedron) and the "spiegature" of other three star polyhedra among which the elevatum icosidodecahedron. Only for three polyhedra, Barbaro showed the perspective view (Fig. 6). He had, like Pacioli, the conviction that he could proceed indefinitely. This conviction was inevitable, in the absence of a clear definition and classification of the examined objects. For a first attempt in this direction, we will have to wait for Kepler's work. Barbaro's book, however, was the first real step forward on star polyhedra. While strictly following Pacioli's constructive idea, the Venetian mathematician laid the foundations for further developments by artists and artisans, thanks to his precise indications for the actual construction of such polyhedra. The mathematician Simon Stevin (1548–1620), too, was profoundly influenced by Dürer; in the third of the five books of [37], he described ways to build Archimedean polyhedra by cutting off or augmenting Dürer's solids:

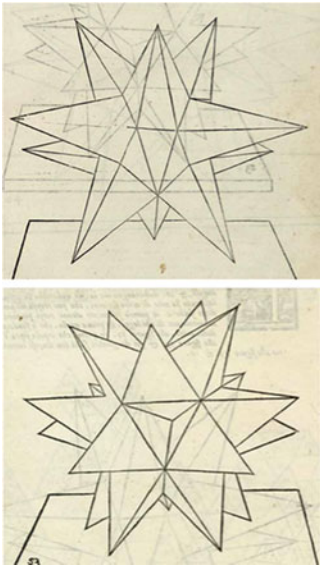

Fig. 6 Pictures from [3], pp. 111, 112

> Besides the five regular solids mentioned by mathematicians, we draw attention to some other solids which, though they do not have so great regularity as required in these five regular solids [...] nevertheless would be full of Geometrical speculations and of a remarkable arrangement of the correlative faces. Now six of these solids have been mentioned by Albert Dürer in his Geometry.

Stevin, starting from regular polyhedra, divided all the edges of the solid into two or three parts and cut each solid angle by a plane passing through the points of division of the edges adjacent to it. Stevin also described augmented regular solids obtained by placing on top of each face of a regular polyhedron a pyramid with equal edges. He mentioned a conversation with Frans Cophart, the leader of the Collegium Musicum Leiden. In our opinion, this conversation is a rare testimony not only of the development of ideas born from the intertwining between mathematical knowledge and artistic intuition, but also of the effort of mathematicians to clarify fundamental definitions, such as the regular polyhedron:

> the extraordinary lover of Geometry wanted to persuade me that he happened to have found a sixth regular solid whose construction was as follows: draw all the diagonals of all the squares of a cube, and then draw planes from all the solid angles of the cube through two diagonals up to the mid- points of said diagonals, and in this way cut off all the sides of the faces of the cube, with the adjacent solid part of the cube included between two intersecting planes. And thus the cube (since it has twelve edges) will have twelve incisions; there remains an elegant solid included... by twenty-four equal equilateral triangles.

The solid is what is now called the "*stella octangula*". Stevin, while admiring the discovery, had to deny this claim, because the vertices of Cophart's solid do not lie all on one sphere, but are distributed across two spheres. At the same time, he discovered another way of constructing the solid, by starting from an octahedron and then augmenting it by placing a pyramid on each face. By applying the same procedure to all regular bodies, he obtained four new polyhedra. Stevin then calculated the length of the edges of the polyhedra inscribed in the same sphere and showed their developments on the plane, adding a description of how to build the solids. For example, on the augmented tetrahedron he wrote:

> [...] dispose four [equilateral] triangles, as in the preceding first section, each of whose sides be equal to the line G. Subsequently four times three triangles such as the three triangles 1, 2, 3, each of whose sides be equal to the said G.

Stevin showed the same "augmented" polyhedra of Pacioli, but apparently, he did not know him and in any case, he did not mention him.

With Stevin, we consider concluded what can be defined as the prehistory of star polyhedra. The first chapter of the real mathematical history of polyhedra begins with Kepler. In 1619, Johannes Kepler (1571–1630) in [24] showed two star polyhedra, calling them regular ("most perfect regular"). See also [1, 5, 22, 23]. Kepler obtained these polyhedra by "stellation" from the dodecahedron and the icosahedron, so as to have star polygons as faces. Defining regular star polygons the ones created by extending the sides of regular polygons, Kepler recognised as regular the polyhedra having pentagrams as faces, i.e. the *small stellated dodecahedron* and the *great stellated dodecahedron* (both with 12 faces) (Fig. 7). These, with their duals discovered by Poinsot, are the only regular star polyhedra.

Following the discussion by Lakatos mentioned in the beginning, we could say that it is with Kepler that the dialectic *definition—proof—refutation—new definition* acquired a proper mathematical content. We should not forget, however, that these polyhedra already had illustrious predecessors in the field of art, with Uccello and Jamnitzer. It is only two centuries after Kepler, that we will have new studies (and rediscoveries) on star polyhedra. As rightly pointed out by Lakatos, the consistent application of the definitions led to explore this "new" mathematical topic. Louis Poinsot (1777–1859) was the first to study it in [33], see also [34]; he wrote:

> Cela posé, je dis que l'on peut construire de nouveaux solides parfaitement réguliers: ils ont tous leurs faces égales et régulières, également inclinées deux à deux, et assemblées en même nombre autour de chaque sommet. Ils peuvent être inscrits et circonscrits à la sphère; et quoiqu'ils présentent au dehors des cavités et des saillies, ils sont convexes suivant cette définition générale, que tous leurs angles dièdres sont au-dessous de deux angles droits. La différence essentielle de ces solides aux polyèdres ordinaires, est que, dans ceux-ci les faces étant projetées par des rayons sur la sphère inscrite ou circonscrite, les polygonés correspondant recouvrent une seule fois la sphère; au lieu que dans les autres, ces polygones la recouvrent exactement ou deux fois, ou trois fois, &c.; et cela d'une manière uniforme.[16]

[16] I say that we can construct new perfectly regular solids: they have equal and regular, also inclined two by two, faces, and assembled in the same number around each vertex. They can be inscribed and circumscribed to the sphere and although they have cavities and protrusions outward, they are convex according to this general definition that all their dihedral angles are smaller than two right angles. The essential difference between these solids and the ordinary polyhedra is that, in these the faces projected by the rays on the inscribed or circumscribed sphere, the corresponding polygons

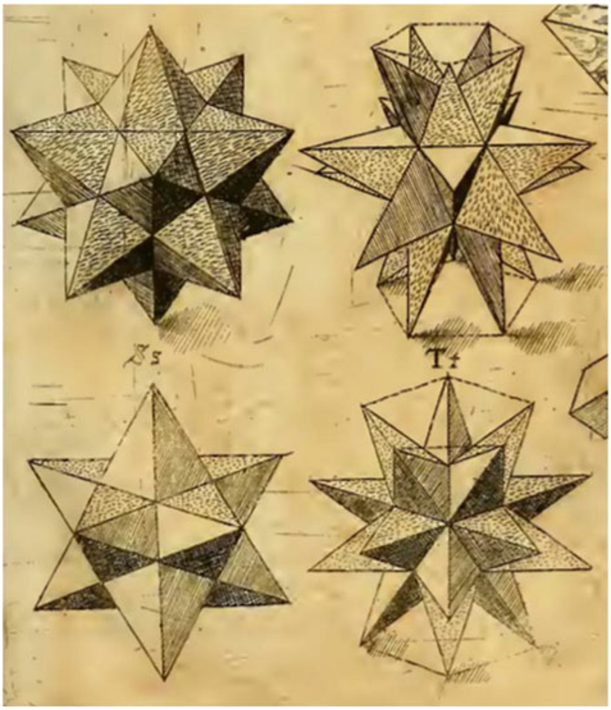

Fig. 7 Pictures from [24]

As we can see, there are several points in the quote above, where definitions are clarified. For example, the concept of "cover", where the projection of a polyhedron is assimilated to the tiling of a sphere, or the different ways of conceiving the term "convex". Poinsot did not know Kepler's text, thus he rediscovered two of Kepler's regular polyhedra and added their duals: the *great icosahedron* (with 12 vertices, 30 edges, 20 faces)—dual of the great stellated dodecahedron—and the *great dodecahedron* (with 12 vertices, 30 edges, 12 faces)—dual of the small stellated dodecahedron. Some years later Cauchy, following Poinsot, proved that these four are the only possible regular star polyhedra [9, 10].

As we can see, the discussion on regular polyhedra, born with Plato and developed by Stevin's musician friend, continued to develop until very recent times, with some unexpected results. Examining semi-regular polyhedra (here we use this term generically, because different authors use it in different way), the situation becomes more difficult. We go from thirteen Archimedean solids classified by Kepler and rediscovered several times, to their duals, discovered and classified in [8]. As to star polyhedra, Jean Paul Albert Badoureau (1853–1923) in [2] took a decisive step for-

cover the sphere only once; instead of the others, these polygons cover it exactly twice, or three times, etc.; always in a uniform way.

ward at the end of the century. He gave a new and more precise definition (as well as a new name) to semi-regular polyhedra:

> Je désigne sous le nom de polyèdres isocèles des polyèdres formés par des polygones réguliers, convexes ou étoilés, et tels qu'on puisse les faire coïncider avec eux-mêmes ou avec leurs symétriques, en plaçant un sommet sur n'importe quel autre.[17]

Symmetries gain a growing importance, in parallel with the development of group theory in mathematics. The inspiration for the most advanced mathematics came directly from the progress of crystallography:

> J'ai pu simplifier la théorie des polyèdres isocèles convexes, au moyen de considérations empruntées soit à la Géométrie élémentaire, soit à la Cristallographie, soit aux notions introduites dans la science par Bravais et développées par M. Jordan.[18]

Badoureau also highlighted the potential connection with Arab art culture:

> Les assemblages isocèles étoilés se déduisent des assemblages convexes [. . .] Les figures auxquelles il donnent lieu pourraient bien avoir été connues des géomètres arabes, si l'on en juge par leur analogie avec les dessins dont l'art oriental aime à orner ses créations.[19]

Afterwards, for many years, the history of the study of polyhedra went on in the absence of a clear mathematical strategy, but with the discovery of new individual figures; in 1881, Johann Pitsch added four new polyhedra to Badoureau's thirty-seven solids. Only in the 1950s, a new important progress occurred in connection with a mathematically coherent vision, thanks to the work of Miller, Longuet–Higgins and Coxeter [14]. The latter showed seventy-five uniform polyhedra, but could not yet prove that the list was complete, which was only done in the 1970s [35, 36].

This history, started with Platonic polyhedra, seems to be analogous to the history of the discovery and definition of "element" in physics and chemistry: from the discovery of single chemical elements, to the periodic table, up to the Bohr model of the atom, which allowed to frame new individual discoveries within an overall theory. In the end, we can assert that our Bohr's atom was the group theory.

In conclusion, we may ask whether the deepest beauty of this story lies in the aesthetics of forms or in the wonderful intellectual adventure that led to find a logic and rational order in the endless variety of these forms.

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[17] I designated with the name "isosceles polyhedra" the polyhedra composed by regular, convex or star polygons, and such that we can make them coincide with themselves or with their symmetries, placing a vertex on any other.

[18] I was able to simplify the theory of convex isosceles polyhedra by means of considerations from elementary geometry, crystallography, or notions introduced into science by Bravais and developed by M. Jordan.

[19] The assemblages of starred isosceles are derived from convex assemblies [. . .] The arising figures were well known by Arab geometers, if we judge by analogy with the drawings of which oriental art loves to adorn its creations.

References

1. E. Aiton, A. Duncan, J. Field, *Kepler. The Harmony of the World* (Am. Philos. Soc., Philadelphia, 1997)
2. A. Badoureau, Mémoire sur les figures isosceles. J. Éc. Polytech. **49**, 47–172 (1881)
3. D. Barbaro, *La Pratica della Perspettiva* (Camillo e Rutilio Borgominieri fratelli, Venezia, 1569). https://archive.org
4. T. Bradwardine, *Geometria Speculativa* (1496)
5. A. Brigaglia, Tassellazioni, solidi archimedei, poligoni stellati nell'Harmonices Mundi di Keplero, in *Conferenze e Seminari dell'Associazione Subalpina Mathesis 2015–2016*, ed. by F. Ferrara, L. Giacardi, M. Mosca (KWB, Torino, 2016), pp. 91–120
6. J. Brożek, Apologia pro Aristotele et Euclide Contra Petrum Ramum & Alios (Dantisci, 1652). https://books.google.it
7. B.L.L. Busard, *The First Translation of Euclid's "Elements" Commonly Ascribes to Adelard of Bath, Books I–VIII and Books X.36–XV.2* (Pontifical Institute of Medieval Studies, Toronto, 1983)
8. A. Catalan, Mémoire sur la théorie des polyèdres. J. Éc. Polytech., Paris **41**, 1–71 (1865)
9. A.L. Cauchy, Recherches sur les polyèdres: premier mémoire. J. Éc. Polytech. **IX**, 7–25 (1813), XVIe cahier
10. A.L. Cauchy, Sur les polygones et les polyèdres: second mémoire. J. Éc. Polytech. **IX**, 26–38 (1813), XVIe cahier
11. M. Chasles, *Aperçu Historique sur le Développement des Méthodes en Géométrie: Particulièrement de Celles Qui Se Rapportent à la Géométrie Moderne, Suivi d'un Mémoire de Géométrie sur Deux Principes Généraux de la Science, la Dualité et L'homographie* (M. Hayex, imprimeur de l'Académie royale, Bruxelles, 1837)
12. W.K. Chorbachi, In the Tower of Babel: beyond symmetry in Islamic design. Comput. Math. Appl. **17**(4–6), 751–789 (1989)
13. W.K. Chorbachi, A.L. Loeb, An Islamic pentagonal seal (from scientific manuscripts of the geometry of design), in *Fivefold Symmetry*, ed. by I. Hargittai (World Scientific, Singapore, 1992), pp. 283–305
14. H.S.M. Coxeter, M.S. Longuet-Higgins, J.C.P. Miller, Uniform polyhedra. Philos. Trans. R. Soc. Lond. Ser. A **246**, 401–449 (1953)
15. M.D. Davis, *Piero della Francesca's Mathematical Treatises* (Longo Editore, Ravenna, 1977)
16. A. de Bovelles, *Liber de Intellectu* (Stephanus & Parvus, Paris, 1510)
17. A. de Bovelles, *Geometrie Practique* (Regnault Chaudière, Paris, 1551)
18. A. Dürer, Underweysung der Messung, mit dem Zirckel und Richtscheyt, in *Linien, Ebenen unnd Gantzen Corporen*. Nuremberg (1525). https://books.google.it/
19. J. Field, Redescovering Archimedean polyhedra. Arch. Hist. Exact Sci. **50**, 241–289 (1997)
20. S. Gunther, Lo sviluppo storico della teoria dei poligoni stellati nell'antichità e nel medio evo, in *Bullettino di Bibliografia e di Storia delle Scienze Matematiche e Fisiche, Tomo VI*, ed. by A. Sparagna (Tipografia delle Scienze matematiche e fisiche, Roma, 1873)
21. W. Jamnitzer, *Perspectiva Corporum Regularium* (1568). https://bibdig.museogalileo.it/
22. J. Kepler, Mysterium Cosmographicum (Tübingen, 1596)
23. J. Kepler, Strena seu De Nive Sexangula (Francofurti ad Moenum. apud Godefridum Tampach., 1611)
24. J. Kepler, Harmonices Mundi (Frankfurt, 1619). https://books.google.it
25. I. Lakatos, *Proofs and Refutations* (Cambridge University Press, Cambridge, 1976)
26. A.G. Molland, *An Examination of Bradwardine's Geometry*. Archive for History of Exact Sciences, vol. 19 (Springer, Berlin, 1978), pp. 113–175, No. 2
27. A.G. Molland, T. Bradwardine, *Geometria Speculativa* (Franz Steiner Verlag Wiesbaden Gmbh, Stuttgart, 1989)
28. A. Özdural, Mathematics and arts: connections between theory and practice in the medieval Islamic world. Hist. Math. **27**(2), 171–201 (2000)
29. L. Pacioli, De Divina Proporzione (Venezia, 1509). https://archive.org/

30. Pappi Alexandrini, in *Mathematicae Collectiones*, ed. by F. Commandinus, Venezia (1588)
31. J. Peiffer, A. Dürer, *La Géométrie* (Seuil, 1995)
32. R. Penrose, The role of aesthetics in pure and applied mathematical research. Bull. Inst. Math. Appl. **10**, 266–271 (1974)
33. L. Poinsot, Mémoire sur les polygones et les polyedres. J. Éc. Polytech., Impr. Imp., Paris **IX**, 16–48 (1809)
34. L. Poinsot, Note sur la théorie des polyedres. C. R. Séances Acad. Sci., tome quarante-sixième (1858)
35. J. Skilling, The complete set of uniform polyhedra. Philos. Trans. R. Soc. Lond. Ser. A **278**, 111–135 (1975)
36. S.P. Sopov, A proof of the completeness on the list of elementary homogeneous polyhedra. Ukr. Geom. Sb. **8**, 139–156 (1970)
37. S. Stevin, *Problemata Geometrica Libri 5* (Anversa, 1583)
38. L. Stoer, *Geometria et Perspectiva* (Nuremberg, 1567). Translated in English in *The Principal Works of Simon Stevin*, ed. by D.J. Struik, Amsterdam C.V. Swets & Zeitlinger (1958), pp. 119–369
39. A. Ulivi, Luca Pacioli, una biografia scientifica, in *Luca Pacioli e la Matematica del Rinascimento*, ed. by E. Giusti, C. Maccagni, Giunti (1994), pp. 21–78
40. D. Wade, *Fantastic Geometry* (Squeeze Press, 2012)

Pentagonal Structures as Impulse for Art

Cornelie Leopold

Starting with the Pentagon

The present study of the art of Gerard Caris is based on a visit to his atelier in Maastricht, Netherlands, his homepage and some exhibition catalogues.

He showed and explained many of his works as well as his work processes. A discussion with him and studies on his art approach laid the basis for this research; see also [8]. Gerard Caris calls his art Pentagonism, because it is the concentration on the pentagon. A reason is for him the dominance of the rectangle and orthogonal structures in the built surrounding. An intuitively perceived irregular pentagon had been his starting point. The works *Birth of Forms*, 1968, and *Creation of the Pentagon*, 1969 (Fig. 1) as spontaneous compositions show the intuitive irregular pentagons. The pentagons are not obvious directly in Figs. 1 and 2, but they appear in mind, when the vertices get connected, lines extended or added, even outside of the image. The shown configurations in Fig. 1 are reminiscent of paper strips, which form pentagons, perhaps in analogy to the fact that the pentagon arises out of a superimposed knot of a paper strip, although the artist disclaims this interpretation and indicates them as spontaneous inspirations.

After explorations in tiling the plane with variations of irregular pentagons and hexagons (Fig. 2), he made the step from the irregular pentagon to the regular pentagon and traced the impossibility of tiling by regular pentagons. There are always gaps left. Caris' states that the pentagon had been for him the answer to the question: How to imagine something from nothing? [3].

Many patterns are developed by the various positions of the remaining rhombus in between the pentagons through the tiling attempts (Fig. 3, left). The *Eutactic Star Series* (Fig. 3, right) are further created out of the Pentagrid, which Caris finally generated as his fundamental pentagonal system, derived from the tiling experiments.

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Fig. 1 Gerard Caris, *Birth of Forms*, 1968. *Creation of the Pentagon*, 1969

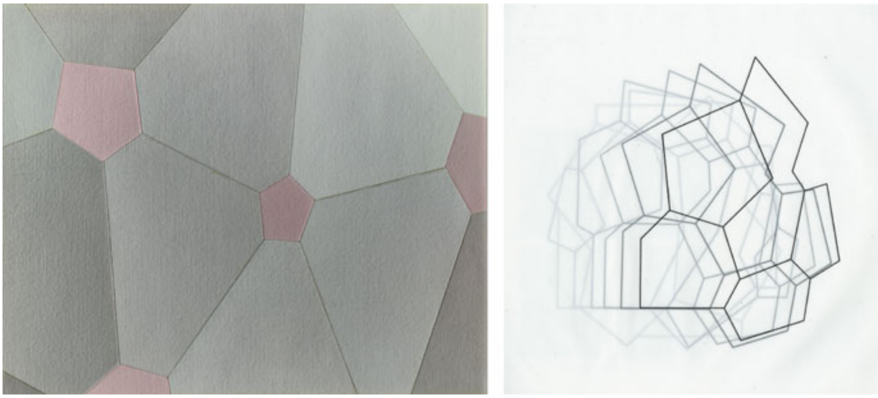

Fig. 2 Gerard Caris, *View of the Univers 1*, 1969. *Cosmic motion series*, 1971

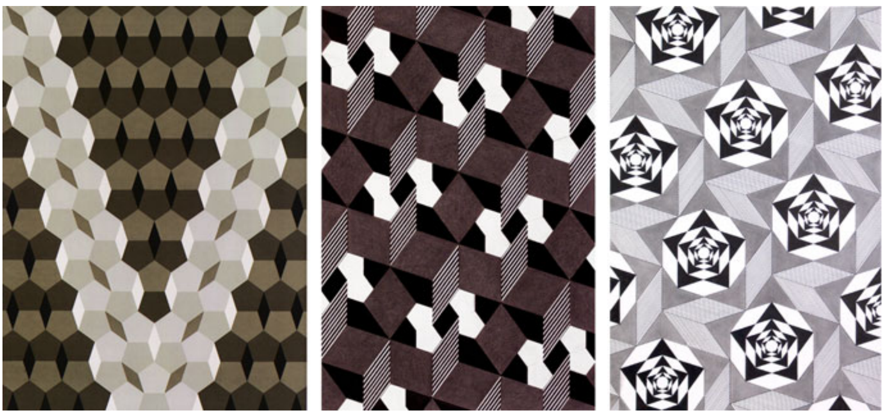

Fig. 3 Gerard Caris, *Structure 6C-2*, 1975. *Eutactic Star Series 31*, 1999. *Eutactic Star Series 113*, 2013

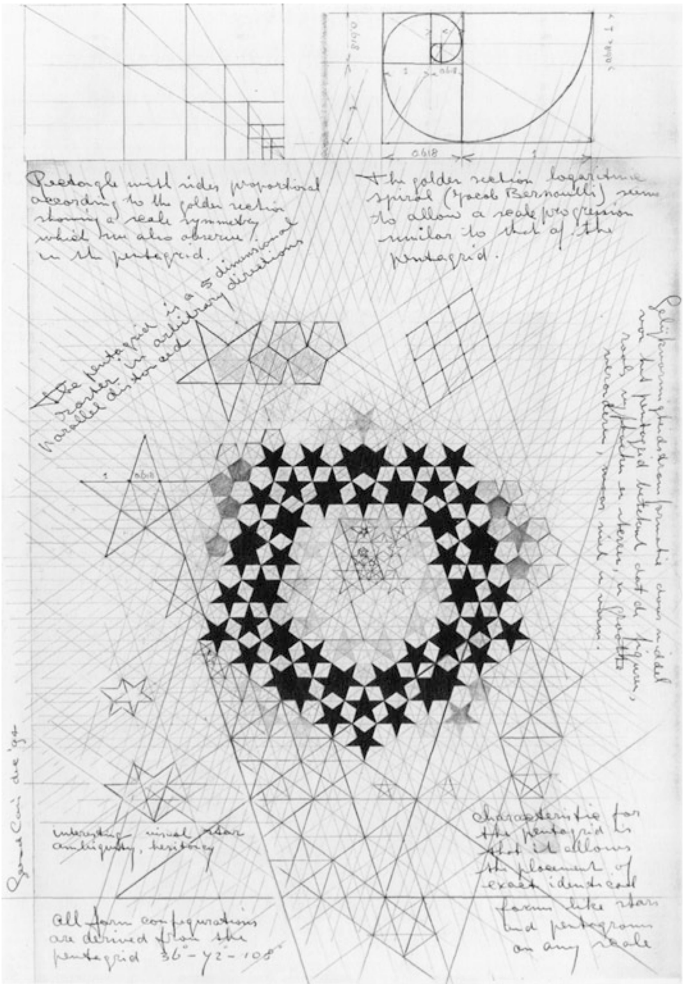

Fig. 4 Gerard Caris, *Pentagrid*, 1994

The received configurations suggest already spatial configurations, supported by the colours, which he created parallel to the two-dimensional works in reliefs and sculptures.

Development of the Pentagrid

Caris developed the Pentagrid out of various fillings of the regular pentagon configurations, which becomes the structural basis of his work. It includes the manifold relationships and configurations of the pentagonal structures, containing all possible derivable art works. The Pentagrid in Fig. 4 becomes the basic tool for his

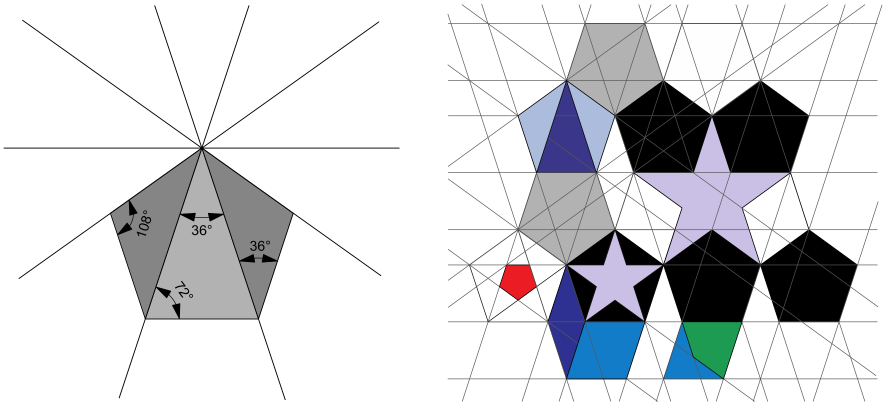

Fig. 5 Pentagon and five-dimensional grid. Figures in the Pentagrid

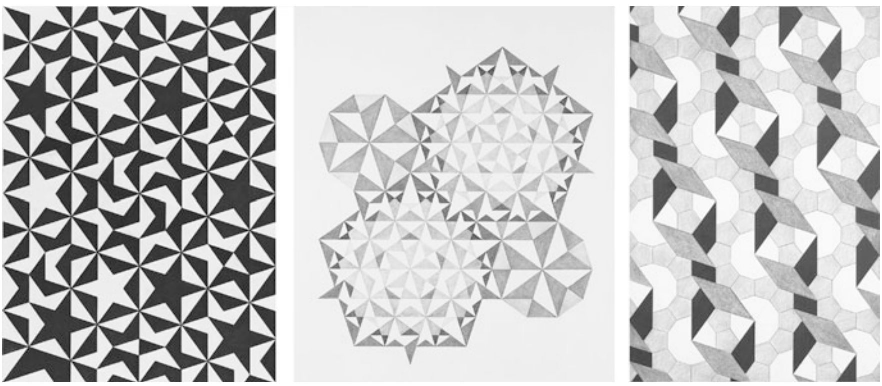

Fig. 6 Gerard Caris, *PC 25*, 1995. *PC 22*, 1995. *ETX 145*, 2014

creations and it reflects the outcome of his studies of the pentagonal structures. It is the repertoire for all possible creations within the pentagonal system. In this way, the intuitive approach led the artist to a systematic geometric development of his pentagonism.

He described the Pentagrid as a grid with five degrees of freedom. The five lines in one node form the fundamental structure of the Pentagrid. The analysis of the regular pentagon with the relation of side to diagonal according to the golden section leads to the grid with angles of 36°, 72° and their sum 108°, the interior angle of the pentagon (Fig. 5, left). The two possible golden triangles are enclosed in the pentagon. Fractal structures result from the infinite continuation of the subdivision according to the golden section. Gerard Caris explored also the fractal structures in the Pentagrid and applied them in his work (Fig. 3, right). The figures in the Pentagrid are the pentagon, the pentagram, the golden triangles and the rhombuses

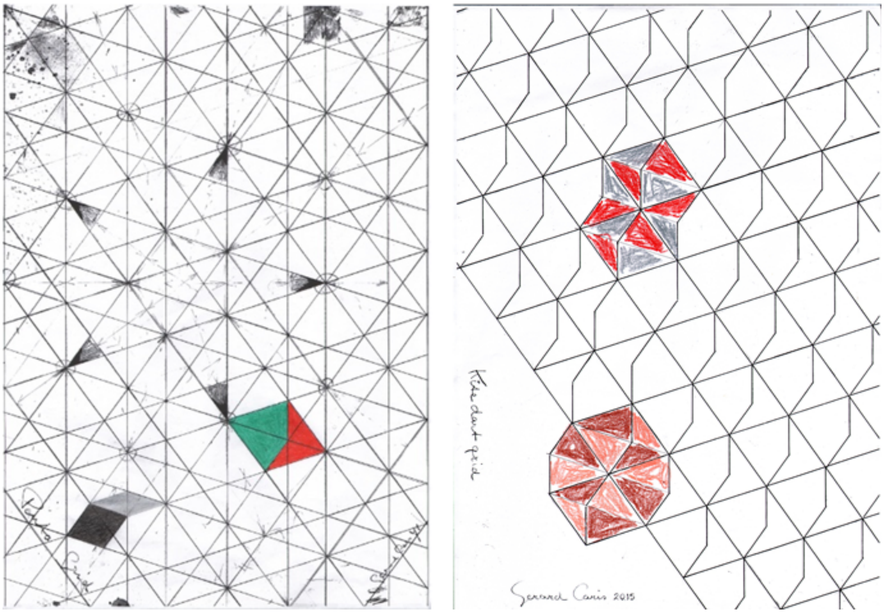

Fig. 7 Gerard Caris, Pentagrid with Kite and Dart, 1971 and derived Kite-dart-grid, 2015

(Fig. 5, right) derived from the golden triangles. The impulses for art by the Pentagrid, are expressed by Gerard Caris in series of paintings like *Structure*, *PC* (Pentagon Complex), *ETX* (Eutactic Star Series). Some examples are shown in Fig. 6 in their diversity, suggesting in some cases spatial interpretations or patterns in front and behind.

From the Pentagrid to the Kite-Dart-Grid

Almost automatically the kites and darts, developed by Penrose from rhombuses, are created. Caris deduced the Kite-Dart-Grid from the Pentagrid (Fig. 7). His newest works develop Kites and Darts series according to the possibilities to arrange kites and darts around a node, described by Roger Penrose and Robert Ammann. Conway [4] gave them the names Star, Ace, Sun, King, Jack, Queen and Deuce, as shown in Fig. 8.

The work of Caris on these pentagonal structures continues to be developed with new aspects. He drew in 2015 and 2016 a series of figurative Kites and Darts compositions (Fig. 9), concentrating on the included golden triangles, by coloring them in alternating colors. A new figurative aspect appears in addition to the creation of patterns.

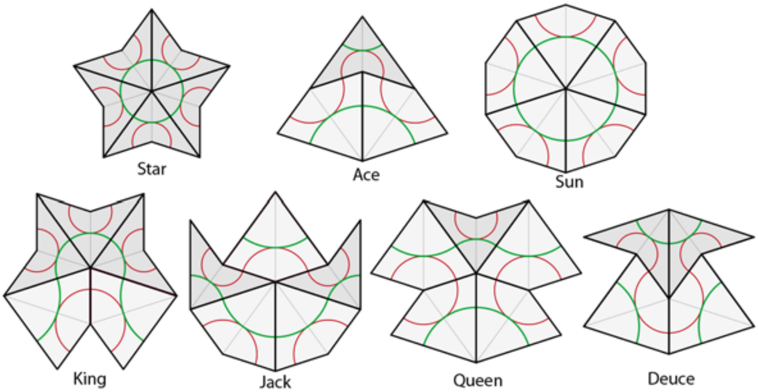

Fig. 8 The seven possibilities to arrange kites and darts around a node [13]

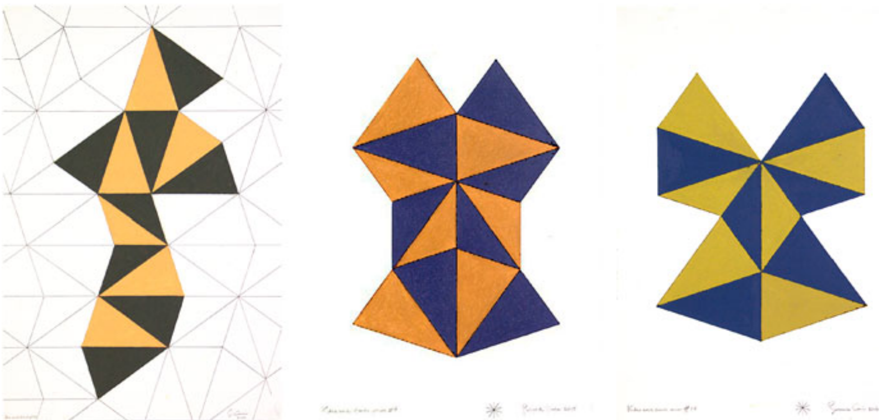

Fig. 9 Gerard Caris, *Kites and Darts series #55*, 2016. *#8*, 2015. *#54*, 2016

Spatial Structures Based on the Dodecahedron

Gerard Caris developed pentagonal structures in space in parallel to those on the plane. He started with the dodecahedron. Spatial configurations are arranged with dodecahedra, for example, by extending the edges of the dodecahedron. A spatial dodecahedra grid is the result. In this way, but also with other criteria for the distances of the dodecahedra, sculptures like Monumental Polyhedral Net Structure in Fig. 10, left, are created. The experiments with packings of dodecahedra in space led him to sculptures and reliefs, showing various gaps between the dodecahedra, dependent of the way of packing (Fig. 10, right). The studies correspond to the experiments tiling the plane with pentagons.

Euclid had explained this construction possibility in his *Elements*, Book XIII, Proposition 17.

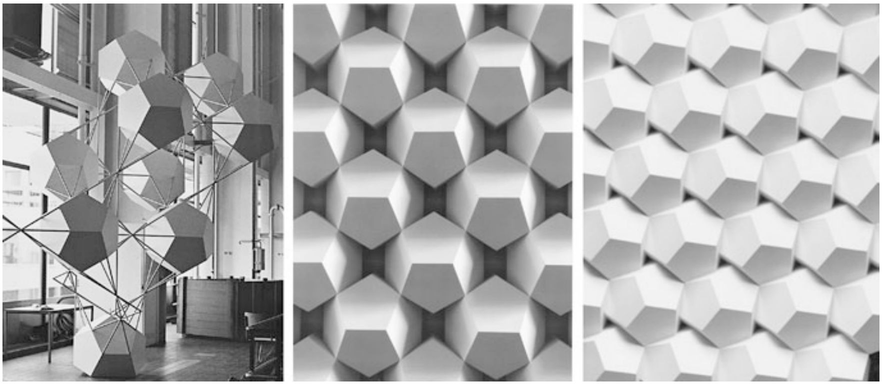

Fig. 10 Gerard Caris, *Monumental Polyhedral Net Structure*, 1977. *Reliefstructure 1K-1* Detail, 1988. *Reliefstructure 1E-2* Detail, 1985

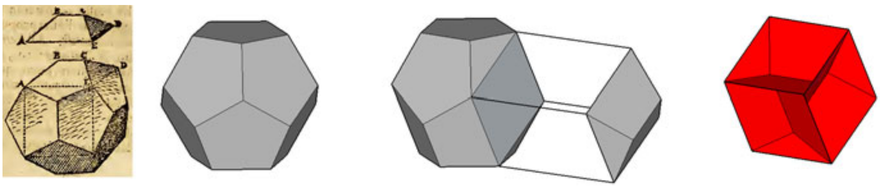

Fig. 11 Truncating the dodecahedron or putting hipped roofs on the cube faces, turning inside

> From this it is clear that when the side of the cube is cut in extreme and mean ratio, the greater segment is the side of the dodecahedron [5].

Johannes Kepler picked up the idea in his *Harmonices mundi* [7, p. 181] and illustrated it in a drawing (Fig. 11, left).

Other spatial structures were found by truncating the dodecahedron. The structures are based on the idea to inscribe a cube in the dodecahedron or starting with the cube and putting hipped roofs on the faces of the cube. If we take a cube and put the six hipped roofs inwards, we get a concave solid. For filling the space with dodecahedra, we can use this solid for the gaps. The space filling can be traced back to that one by cubes. Gerard Caris applied these studies for creating sculptures, reliefs and housing designs (Figs. 12 and 13).

The spatial counterpart of the Penrose patterns can be found in the so-called *golden diamonds* [11]. The structural elements are, corresponding to the rhombuses in the plane, the rhombohedra, where the diagonals cut each other in a golden ratio. Jan van de Craats [6, p. 44ff.] analyzed the two rhombohedra, developed out of the axes of the dodecahedron in the work of Gerard Caris. They play an important role for aperiodic space fillings in quasi-crystals. The six possible five-fold rotational axes in the dodecahedron through the faces and the center of the dodecahedron form its so-called eutactic star. Choosing three of the six axes leads to two different rhombohedra with edges along these axes. Each rhombohedron consists of

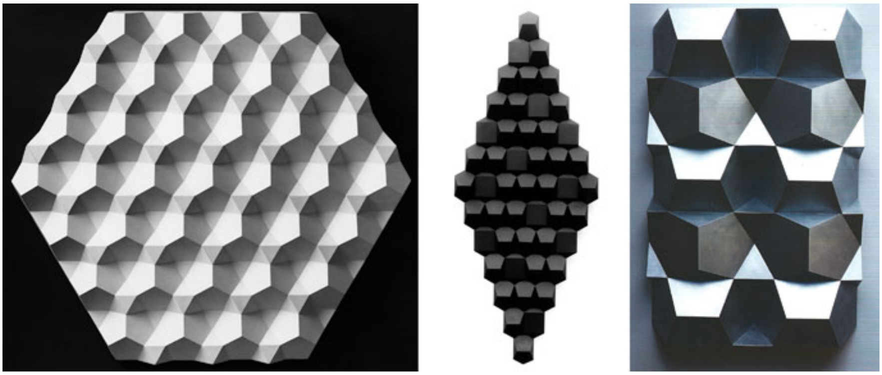

Fig. 12 Gerard Caris, *Reliefstructure 1D-3*, 1985. *Reliefstructure 2M-1*, 1989. *Reliefstructure 1O-2*, 2002

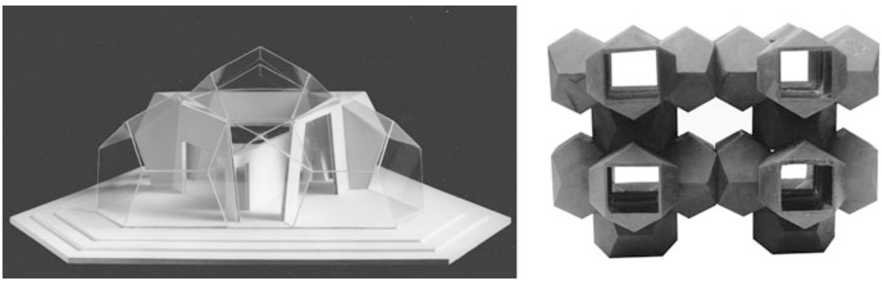

Fig. 13 Gerard Caris, *Model D House*, 1985. *Polyhedra sculpture 3* (truncated), 1979

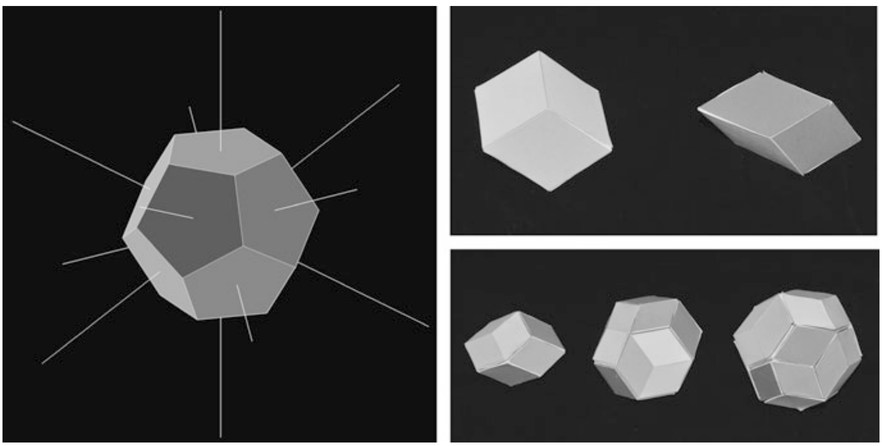

Fig. 14 Dodecahedron with its eutactic star, the two rhombohedra and compound rhombic polyhedral


Fig. 15 Gerard Caris, *Reliefstructure 14V-1*, 2003. *Sculpture 2X-1*, 1996

six congruent rhombi constituting the faces of them. Two of each different rhombohedra form together a rhombic dodecahedron and further on rhombic polyhedra with growing faces (Fig. 14). Gerard Caris transposes the design possibilities with the rhombohedra in sculptures and reliefs (Fig. 15).

Aesthetic Evaluation

Gerard Caris refers in his art to the geometry of pentagonal structures on the plane as well as in space. He uses the geometric structures for experiencing them in aesthetic compositional processes. In this way, he explores many geometric relationships in aesthetic expressions. His art cannot be reduced to these geometric order structures, but structural thinking turns out as an adequate method to consolidate design processes [10]. According to the information aesthetics, redundancy or respectively order and innovation have to be in an optimal relation to achieve an aesthetic state based on the theory of Georg David Birkhoff (1928)—the interplay of redundancy and innovation, order and chaos. More details on the aesthetic theory and the relation to geometry have been explained by the author in *Prolegomena zu einer geometrischen Ästhetik* [9]. Max Bense [1] explained in his *Kleine abstrakte Ästhetik*:

> A perfect innovation in which there were only new states as in chaos, would not be recognizable. Chaos is finally unidentifiable. The recognizability of an aesthetic state requires not only the recognizability of its singular innovation, but also its identifiability based on their redundant order characteristics [1, p. 356, translated from German by the author].

The idea to calculate the aesthetics of an art work may be obsolete, but the categories of redundancy and innovation are still useful tools. By applying these characteristics to the art of Gerard Caris, we are able to determine that the aesthetic states are identifiable by the geometric order. Patterns and ornaments with their basis in geometry have a fundamental role for art without identifying geometry with art. Paul Valéry [12] compared adequately the role of ornamental drawing for art with the role of mathematics for the sciences. Max Bill explained in *Die mathematische Denkweise in der Kunst unserer Zeit* [2], that the primary element of every art work is the geometry, the relationship of the layers on the surface, or in space. This description can be explored in Gerard Caris' art.

The viewer of Caris' art works can take part in his design processes by following the structures. But the innovative aspects too are not missed, because we experience manifold surprising and fascinating patterns, which catch the eye as images of spatial structures depending on the point of view. Gerard Caris' pentagonism is to be understood as a radical structural basis for art and design. With this claim, he works on all consequences for an alternative to orthogonal structures, showing a more complex and fascinating structural system.

Conclusion

By studying Caris' art work we become aware that his preference for the pentagon is not fixed on the form, but rather on the manifold geometric structures, derived from the pentagon. Using an aesthetic analysis, we can conclude that it is not the form of the pentagon that constitutes the aesthetic object in his art, but the pentagonal structures, which contain such a richness of art expressions. By applying the characteristics of redundancy and innovation to the art of Gerard Caris, we can determine that the aesthetic states are identifiable by the geometric order. The viewer of his works can take part in his structural design processes by following the steps. The manifold surprising and fascinating patterns and spatial configurations in his art manifest the innovative aspect besides the redundancy, so that a balance of both guarantees a perceptible aesthetics in his work.

Acknowledgements Many thanks to the artist Gerard Caris for the possibility to visit him in his atelier, showing and explaining me his work and to get an inside view of his creation processes. The images of his works of art in the figures here are used with his kind permission. I am grateful to Margriet Caris supporting me in all questions and demands.

References

1. M. Bense, *Aesthetica. Einführung in die neue Ästhetik* (Agis Verlag, Baden-Baden, 1965), 2nd expanded edition 1982
2. M. Bill, Die mathematische Denkweise in der Kunst unserer Zeit. Werk **36**, 3 (1949), Winterthur
3. G. Caris, Pentagonism. http://www.gerardcaris.com (as of Oct. 09, 2017)
4. J.H. Conway, J.C. Lagarias, Tiling with polyominoes and combinatorial group theory. J. Comb. Theory, Ser. A **53**, 183–208 (1990)

5. Euclid, *Elements*. Book XIII, Proposition 17. Corollary. https://mathcs.clarku.edu/~djoyce/elements/bookXIII/propXIII17.html (as of Jan. 16, 2018)
6. G. Jansen, P. Weibel (eds.), *Gerard Caris. Pentagonismus/Pentagonism* (Walther König, Köln, 2007)
7. J. Kepler, *Harmonices Mundi*. Libri V. 1619. https://ia600308.us.archive.org/33/items/ioanniskepplerih00kepl/ioanniskepplerih00kepl.pdf (as of Oct. 10, 2017)
8. C. Leopold, Geometry and aesthetics of pentagonal structures in the art of Gerard Caris, in *Proceedings Bridges Finland*, ed. by E. Torrence et al.(Tessellations Publishing, Phoenix, 2016), pp. 187–194
9. C. Leopold, Prolegomena zu einer geometrischen Ästhetik, in *Ästhetische Geometrie—Geometrische Ästhetik*, ed. by F. Kürpig (Shaker Verlag, Aachen, 2011), pp. 61–65
10. C. Leopold, Strukturelles Denken als Methode, in *Strukturelle Architektur. Zur Aktualität eines Denkens zwischen Technik und Ästhetik*, ed. by J.M. Warmburg, C. Leopold (Transcript Verlag, Bielefeld, 2012), pp. 9–29
11. K. Miyazaki, *An Adventure in Multidimensional Space: The Art and Geometry of Polygons, Polyhedra, and Polytopes* (Wiley, New York, 1986)
12. P. Valéry, *Introduction à la Méthode de Léonard de Vinci* (La Nouvelle Revue Française, Paris, 1895)
13. https://commons.wikimedia.org/wiki/File:Penrose_vertex_figures.svg, by Geometry guy, with thanks to Tovrstra (as of Jan. 16, 2018). Words added to the figures by the author according the description

Mathematics, a Closer Look

A Photographer's Perspective

Vincent Moncorgé

Introduction

This profession allows me to shed light on the world in which we live. Being a photographer is primarily about giving the subject all your attention and through the act of shooting the photograph, giving the subject meaning. Whether through a person, the environment we live in an event, I do as much as possible to offer the viewer a new look, a rediscovery of what might seem banal, too far from his interests or, on the contrary, too familiar.

I'm a specialized science photographer, more precisely a photographer documenting science.

My main goal is to demystify science and show it how it really is. I do confess that for a long time I imagined researchers as dressed in spotless white lab coats, all working in high tech laboratories, running out of their offices crying "Eureka!".

The beginning of the millennium, oddly enough, was marked by the crisis in French scientific research. Problems appeared before the public for the first time: inadequate funding, brain drain, pervasive unrest. Having been born in the late 60's, I belong to a generation that was long convinced that studying for many years was a guarantee of success and found myself surprised to see that it was not true anymore.

In 2005 I met Philippe Gillet, director of the *Ecole Normale Supérieure* de Lyon at that time and told him about my project of having a direct and true look to research. He gave me *carte blanche* and since then, I've kept the same photographic process that allows me to combine freedom of action and freedom of thought. It is important to come and go as you please and therefore discover things at your own pace. When I work on a new project, I always photograph without attaching a time or a date to the shots. I work without taking into consideration the hierarchy, without knowing who was more important than whom, scientifically speaking. I also take in

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consideration the ebb and flow of the human tide: visitors, lecturers, students, researchers, and so on. Some of those present in my books may have been to the place only once! I always keep in mind to be coherent and come as close as possible to an ethnological approach and to build a coherent and relevant corpus of photographs.

Little by little we discover that, far from the Manichean cliché vacillating between the images of the absent-minded professor and the mad scientist working to bring the world to an end, there is another reality. Even seen from the inside, the daily life of a researcher is a mystery. To try to understand, you must watch science as it happens, day after day, far from the high profile media events. It's a world made up of little things, of gestures repeated over and over, of imperceptible movements, of very little noise. The victories are all the more private, new pieces of an endless puzzle. They are the rare moments of beauty, often savoured alone, unnoticed by almost all.

To document science needs to keep a journalistic approach and disturb as little as possible the environment you *polute* by simply being here. The goal is to produce images that are capturing reality in contrast with the illustrating (and therefore fake) pictures that are published for example in magazines and pretend to highlight the laboratory life. These false iconographic materials share the same basic visual construction: blue or yellow colored lights in the background, a scientist with perfectly neat white lab coat is looking intensively to the test tube he holds or is looking through a microscope. Extra accessories are protective glasses and surgeon gloves.

Mathematicians

Naturally attracted by the abstract, I had wanted to get close to mathematics for many years. In 2010 we decided with Cedric Villani (Fields medal and director of the Institut *Henri Poincaré*) to work together on a book called *La Maison des mathématiciens* [1]. Founded in 1928, the institute is located in the heart of the 5th arrondissement of Paris. It is one of the oldest and most dynamic international structures dedicated to mathematics and theoretical physics.

The place has a strong emphasis on hospitality and sharing. Every year it welcomes hundreds of scientists from all over the world, and thousands of students and diverse math and physics sensitive publics.

The Institute serves multiple purposes: first of all, as a place for national and international scientific exchange, it hosts quarterly thematic programmes, high-level doctoral courses, conferences and seminars; it makes offices and logistic facilities available to its guests and visitors, including its renowned library. All mathematicians are welcome here to organise conferences and seminars, or just arrange to meet for a discussion or a work session.

It took me three years to study and document the life of this unconventional, free spirit tribe.

Photographically speaking, the visual mathematical environment (offices, corridors. . .) is so neutral that it drives you to work in a meditative state. The spaces are almost empty of decorations (except posters) or any elements on which the construction of the photograph could rest. In our constantly visually exciting societies, such a sobriety is exceptional. Up to them I had only experienced this in a monastery a few years ago (I was then commissioned by a news magazine for a story about silence).

Here the common language remains unknown and signs are indecipherable. The only key to enter this world is your ability to respect the inner rules. People are here to work and you have to keep in mind not to disturb them (courtesy, discretion, respect). Sometimes I felt like a wildlife photographer or a predator, trying to stay invisible and being very patient for the prey to get used to my presence.

I needed this operation mode to get close enough to be able to give a feeling of intimacy through my photographs. I understood that the pictures had to be simply visually constructed to enhance the essential aspects of my subject and to reveal this constant intellectual vibrancy.

Obviously mathematicians and field photographers share many common points. Key words are passion, patience, chance, persistence, encounters, loneliness, intuition, impermanence and rectangles (black board, paper sheet, computer screen for the scientists and viewfinder for photographers).

Mathematical Objects

Created in the late 19th century, they were made of plaster, wood or threads on a metallic structure mainly for an educational purpose. First made to answer a need to modelize scientific abstractions, (more precisely the geometry surfaces properties in space) by the end of the 20's enter on the shelves of the Institut *Henri Poincaré* most of them in the archival reserves.

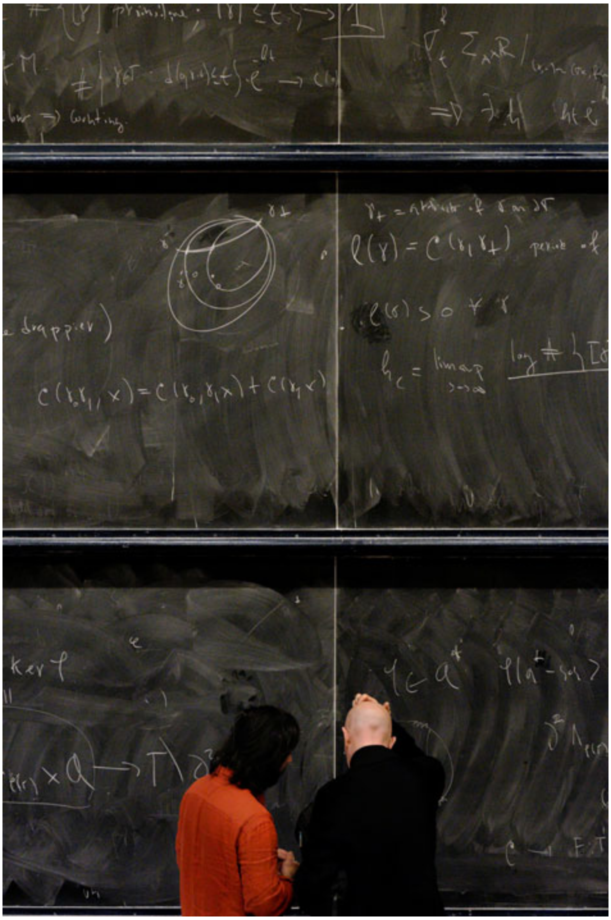
Objects made to render visible the invisible, to show complex shapes and phenomena not perceptible to the eye, they logically caught the attention of the surrealist artists. First of them is probably Max Ernst circa 1935. A year later Man Ray started to take pictures of these objects which were an encounter of art and science. Sculptures at the service of the surrealist mind revolution [2–5].

Man Ray with his lightning studio set is playing with lights and shadows to create a metamorphosis. On the pictures, coming out of the darkroom, the objects no longer belong to science but to the open fields of interpretation, influenced by human anthropomorphism and zoomorphism analysis reflexes.

I have to admit that my work is here more influenced by the work of George Orwell or Philip K Dick, by the drawings of Moebius (the French artist Jean Henri Gaston Giraud, who called himself Moebius, creator of cartoons) and Philippe Druillet (a French writer and comics artist, to form in 1975 with Moebius and two other artists the publishing house *Les Humanoides Associés* and the magazine *Métal Hurlant*) and by Fritz Lang's movie *Metropolis*. What appeared through my lens were primitive cult objects, architectural or viral modelisations. But each person is free to have his or her own interpretation.

Portraits

The third part of my work is made of straight portraits of mathematicians. While sitting in front of my camera, I asked my models to intensively think about their actual work. I had to get rid of the black boards, the symbols, or any elements that would make them directly recognizable. My goal here was to build an anthropologic ensemble of a community sharing the same faith.

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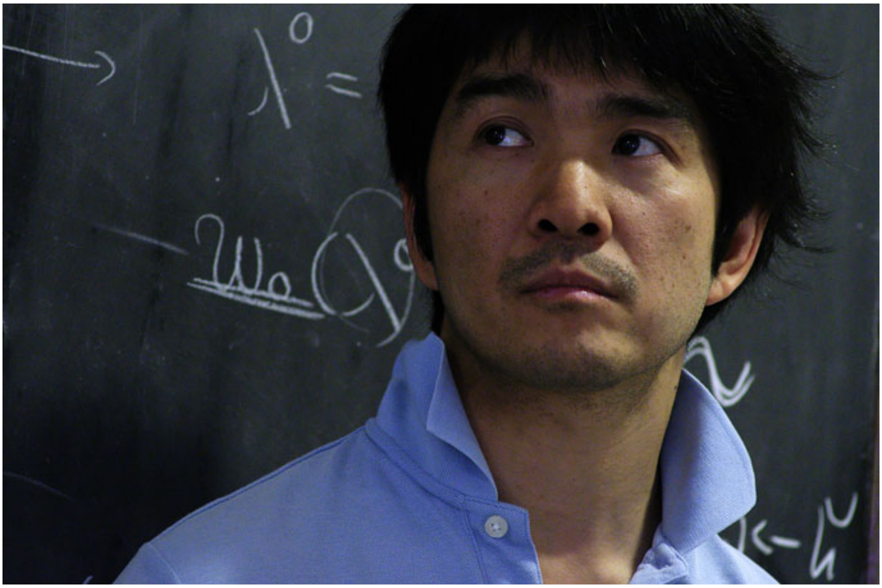

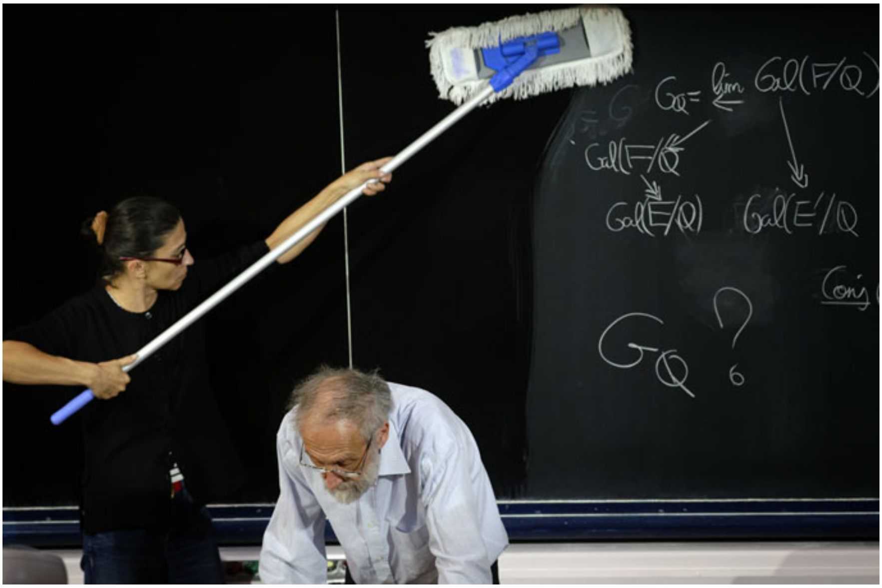

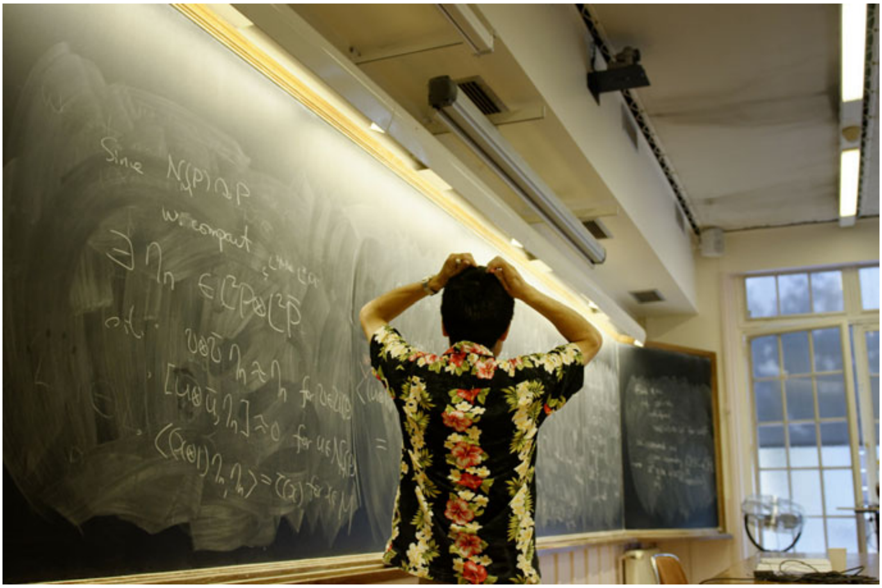


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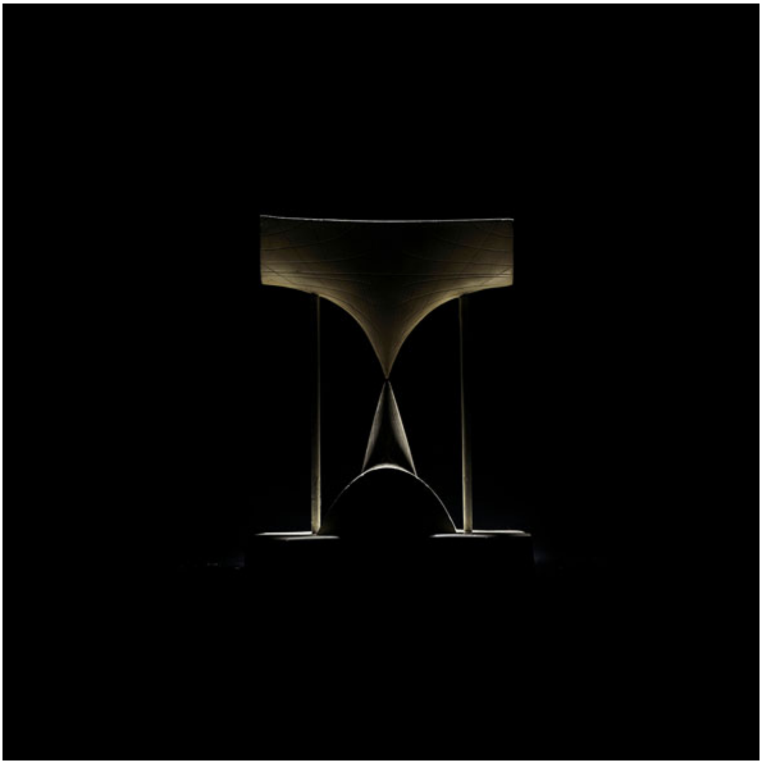

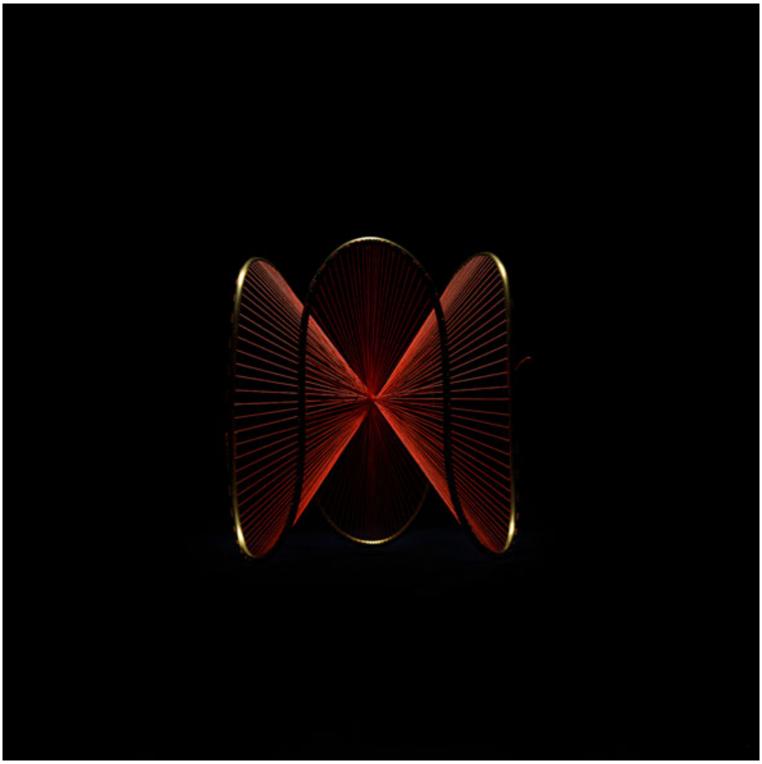

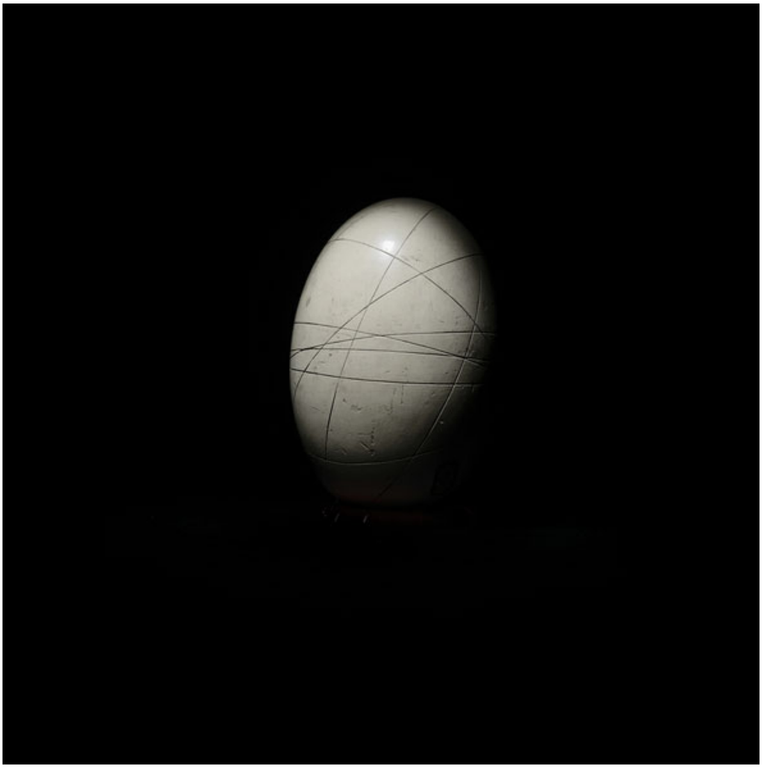

References

1. C. Villani, J.-P. Uzan, V. Moncorgé, *La Maison des Mathématiciens* (Le Cherche Midi, Paris, 2014). Eng. Ed., *The House of Mathematics* (Cherche Midi, 2017)
2. www.ihp.fr
3. I. Fortunè, *Man Ray et les objets mathématiques*, n. 6 mai 1999, Études photographiques, 6 mai (1999). http://etudesphotographiques.revues.org/190.
4. W. Grossman, E. Sebline, *Man Ray Human Equations*. Catalogue of the exhibition, The Phillips Collection, Washington D.C., Hatje Cantz verlag, Osefildern, Germany (2015)
5. M. Emmer, Visible harmonies: mathematica models, in *Imagine Math 3: Between Mathematics and Culture*, ed. by M. Emmer. (Springer, Berlin, 2014). Special section, articles by M. Emmer, E. Barisoni, J.-P.. Bourguignon, U. Bottazzini, L.D. Henderson, M. Rottmann

Other References

6. V. Moncorgé, *Portraits of Science*, Vuibert and ENS Lyon editors (2005)
7. V. Moncorgé, *The Goldsmiths of Light, a Visit to the Synchrotron Soleil*, Le Pommier editor (2010).
8. V. Moncorgé, *Physic Gardens*, Editions du CNRS editor (2017). www.vincentmoncorge.com

Part VII
Mathematics & Physics

Drowning by Numbers: Topology and Physics in Fluid Dynamics

Amaury Mouchet

Since its very beginnings, topology has forged strong links with physics and the last Nobel prize in physics, awarded in 2016 to Thouless, Haldane and Kosterlitz "for theoretical discoveries of topological phase transitions and topological phases of matter", confirmed that these connections have been maintained up to contemporary physics. To give some (very) selected illustrations of what is, and still will be, a cross fertilization between topology and physics,[1] hydrodynamics provides a natural domain through the common theme offered by the notion of vortex, relevant both in classical (second section) and in quantum fluids (third section). Before getting into the details, I will sketch in first section a general perspective from which this intertwining between topology and physics can be appreciated: the old dichotomy between discreteness and continuity, first dealing with antithetic thesis, eventually appears to be made of two complementary sides of a single coin.

The Arena of the Discrete/Continuous Dialectic

One century after Thales of Miletus had proposed that water was the natural principle of all things, the first atomists Leucippus and Democritus advocated for a discrete conception of matter. The existence of an ultimate lower limit of divisibility, materialised by the atoms, may have been a logical answer to the Zeno's paradoxes ([41, chap. VIII]; [4, chap. I]). In some westernmost banks of the Mediterranean sea, the Pythagorean school was concerned by a line of thought following quite an opposite direction: the discovery of the irrational numbers counterbalanced

[1] A more general review is proposed by [30] and a systematic presentation on the topological concepts used by physicists can be found in [29].

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the conception of a universe exclusively driven by the integer and rational—in the original acception of the word—numbers. For twenty-five centuries, the dialectic between continuity and discreteness has never stopped nurturing natural philosophy.

At our daily life scales, the ones for which the brains have been shaped by Darwinian evolution,[2] discreteness appears to be an inevitable way for intelligence to model the world.[3] Furthermore, operationally speaking, any measurement is reduced, in the last resort, to a reproducible counting [45, § 1.1]. Etymologically, "discrete", "critical", "criterion", and "discernment" share the same Greek root κρίνω (*krī́nō*, to judge).[4] However, the boundaries of macroscopic objects, considered both in space and time, remain inevitably blurred. For instance, consider one cherry; through absorption and desorption, a perpetual exchange of matter takes place at small scales on the skin of the cherry, and no one can really identify with a precision of one second the time when this cherry has appeared from a blossom or destroyed by natural deterioration.[5] This ambiguity was known from antiquity and supply the sorites paradox (what is the minimum number of grains in a heap of sand?)—and the paradox of the ship of Theseus (Plutarch asks if, after decades of restauration, once her last plank has been replaced, the ship remains the same Theseus's ship [33, The life of Theseus § XXIII.1]).

In the second part of the XIXth century, experiments allowed to move the debate beyond speculations into the microscopic world. In the same movement, mathematics saw the emergence of a new discipline, topology, where were identified some *discrete* classifications—first in geometry, then in analysis and algebra—up to *continuous* invertible transformations (homeomorphisms). The integer numbers upon which the classes of, say, graphs, knots, surfaces, fixed points of a flow, critical points of a real map, are discriminated provide, by essence, a robust quantization; they are topological invariant. To put it in a nutshell, there cannot be "half a hole". The dimension of a space,[6] its connectedness (π_0), its homotopy groups (π_1, π_2 and more generally π_n), the signature of the Hessian of a function at a critical point, are examples of such discrete quantities.

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[2]In modern times physics and chemistry were not, by far, the only scientific disciplines to be shaken by violent debates between discrete and continuous schools; in the XIXth century Lyell's uniformitarianism in geology, by contrast with catastrophism, had an important influence on the young Darwin. By the way, one can notice that the binary opposition between discreteness and continuity provides by itself a meta self-referring epistemological dichotomy, so to speak.

[3]However, neurology shows that numerical cognition is more analogical than numerical: beyond few units, the numbers are encoded and treated by the brain as fuzzy entities [11, specially part I and Chap. 9].

[4]The etymology lines of these words can be easily traced back with `www.wiktionary.org`.

[5]In a contribution to the previous volume of this series [28, § 5] I have tried to show how symmetries play a crucial role in the process of abstraction and conceptualisation of a macroscopic object like a cherry.

[6]In fractal geometry, the Hausdorff dimension of a set, which can be irrational, is not preserved by a homeomorphism.

In the beginning of the XXth century, quantum physics refuted so masterfully the Leibniz continuity principle (*Nature does not make jumps*) that it bears this claim in its very name. The general rule—known by Pythagoreans for music—according to which a stable wave in a bounded domain has its frequencies quantized (that is, function of integer numbers) now applied at a fundamental level to the Schrödinger waves, which described the states of elementary particles, when bounded. The discrete classification of chemical elements successfully proposed in 1869 by Mendeleev and the discrete spectral lines corresponding to the Balmer series, the Paschen series, the Lyman series etc. observed in radiation, could be explained within a unifying scheme offered by quantum theory. Even though it appears that each atomic energy level has actually a continuous bandwidth, due to the coupling to the electromagnetic field whose scattering states belong to a continuum (the photon has no mass), it is nevertheless quantum theory that conferred to "being an integer" a genuine physical property. So far, neither the quantification of the spin nor the quantification of the electric charge, say, can be seen as an approximation of a continuous model and the analogous of the Mendeleiev table in the Standard Model contains a finite number of species of elementary particles—about twenty, non counting as distinct a particle from its associated antiparticle—characterised by a handful of quantum numbers.[7] Many attempts have been made for finding a topological origin of these quantum numbers, one of the motivation being that topological invariance is much harder to break than symmetry invariance. In condensed matter, topology offers a protection against the effects of impurities or out-of-control perturbations and therefore participates to the reproducibility and the feasibility of measurements [45, § 1.3]. The seminal attempt in this direction is Dirac's model of magnetic monopole [13] whose existence would imply the quantization of the electric charge; however, so far, all the quantizations that have been explained find their root in *algebraic* properties of the symmetry groups used to build a basis of quantum states[8] (in the absence of evidence of elementary magnetic monopoles, the fact that the electric charges appear to be always an integer multiple of one unit remains mysterious).

Despite these (temporary?) failures of finding topological rather than algebraic roots for the discrete characteristics of what appears to be elementary particles, the quantum theory of fields offers the possibility of describing some collective effects of those particles whose stability is guaranteed by topological considerations. There exists some configurations of a macroscopic number of degrees of freedom that cannot be created or destroyed by a smooth transformation without passing through an intermediate state having a macroscopic, and therefore redhibitory, energy. Depending on the dimension of the space and of the field describing the model, several

[7] The discrete character of some observable properties is all the more strengthened that there exists some superselection rules that make irrelevant any continuous superposition of states differing by some discrete values of this observable.

[8] Topological properties of these Lie groups, obviously their dimensions but also their compactness, their connectedness and their simple connectivity, do play a role but the algebraic commutation relations of their generators remain the main characteristics, which are local ones, that allow to build the irreducible representations defining the one-particle states.

such *topological defects* can be considered (point, lines or surfaces) and have been observed in various condensed states [8, Chap. 9] including, of course, the quantum fluids where the defects are characterised by quantized numbers that can be interpreted as topological invariants. Vortices, which will be the object of the next two sections, provide typical examples of such topological defects along a line in a 3-dimensional space or localised at one point in a 2-dimensional space for a complex scalar field (or a real bidimensional vector field). Under certain circumstances, these collective effects share many properties with the so-called ordinary particles. Since, theoretically, the distinction between the quasi-particles and particles appears, after all, to be just a matter of convention on the choice of the vacuum and of the particles that are considered to be elementary, one may have the secret hope that at a more fundamental level, having the Standard Model as an effective theory, topology shall have the next, but presumably not the last, word.

Classical Vortices

... when I first opened my eyes upon the wonders of the whirlpool...
Edgar Allan Poe. *A Descent into the Maelström* (1841).

How Vortices Participate to the Dynamics of the World According to Leonardo and Descartes

By strong contrast with the still, rather mineral, backgrounds of his paintings, Leonardo da Vinci's interest for the dynamics of water is manifest in his drawings and writings all along his life. Vortices in water, in air, and even in blood [32, § 3.3], were a recurrent source of fascination for him.[9] Not only as esthetical motifs (Fig. 1), not only because of their crucial role for understanding hydraulics and flight, not only because they inspired him fear as a disordered manifestation of flooding or deluge, but also because they provided a central key for his global conception of the dynamics of the world: *l'acqua, vitale omore della terreste macchina, mediante il suo natural calore si move.* (water, vital humour of the terrestrial machine, moves by means of its natural heat)[10] [2, Chap. *Une science en mouvement*].

More than a century later, most probably without any influence from Leonardo, Descartes put the vortices in the very core of his cosmological model (Fig. 2). Rejecting the atomist concept of a vacuum separating matter [12, part II, 16th principle], he writes

[9] [18] saw in the exuberance of the terms used by Leonardo and in the profusion of his drawings an attempt to classify the vortices, a line of investigations he kept in mind throughout his life.

[10] Folio H95r, whose facsimile and transcription can be found on www.leonardodigitale.com.

Fig. 1 (Left) folio w12380, A Déluge, ∼1517-18. (Center) folio w12663r, Studies of flowing water, ∼1510-13. (Right) folio w12518, Head of Leda, ∼1504-1506. `Wikimedia Foundation`. On the folio w12579r Leonardo has drawn four studies of vortex alleys formed in water behind a parallelepipedic obstacle and writes *Nota il moto del livello dell'acqua, il quale fa a uso de' capelli, che hanno due moti, de' quali l'uno attende al peso del vello, l'altro al liniamento delle volte: così l'acqua ha le sue volte revertiginose, delle quali una parte attende al impeto del corso principale, l'altra attende al moto incidente e refresso.* (Observe the movement of the surface of water, like hair which has two movements, one due to its weight, the other following the lines of the curls: thus water has whirling eddies, in part following the impetus of the main stream, in part following the incidental and reversed motion, folio w12579r, trad. AM)

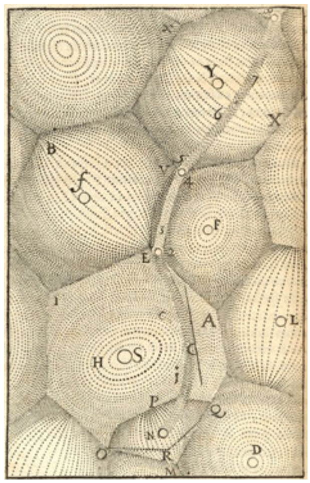

Fig. 2 Descartes' vortex-based cosmology. Each star denotes by F, D, etc. is at the center of a vortex. The Sun is denoted by S [12, § III.23, p. 78]

> *[...] putandum est, non tantum Solis & Fixarum, sed totius etiam coeli materiam fluidam esse.*
> ([...] we think that not only the matter of the Sun and of the Fixed Stars is fluid but also is the matter of all the sky, trad. AM)
>
> [12, § III.24, p. 79]

Being aware of the proper rotation of the Sun (it takes 26 days for the sunspots to complete one turn [12, § III.32, p. 83]) and of the different orbital period of the planets, he pursues further the hydrodynamical analogy

> *[...] putemus totam materiam coeli in qua Planetae versantur, in modum cuiusdam vorticis, in cuius centro est Sol, assidue gyrare, ac eius partes Soli viciniores celerius moveri quam remotiores [...]*
> ([...] we think that all the matter of the sky, in which the Planets turn, rotates like a vortex with the Sun at its center; that the parts near the Sun move faster than the remote ones [...], trad. AM)
>
> [12, § III.30, pp. 81–82]

Descartes' model was overruled by Newton's theory planetary motion but, somehow, in contemporary astrophysics, vortices are still present—in a complete different way, of course, from Descartes'— and triggered by gravitational field acting through the interstellar vacuum: one may think of protoplanetary accretion disks (turbulence plays a crucial role, in particular in the initial molecular cloud for explaining the scattered births of stars) and, at much larger scales, of galaxies, cosmic whirlpools spinning around a giant black hole.

Accompanying the Birth of Topology in the XIX*th Century*

His study of the physical properties of organ pipes led Helmholtz to scrutinize the motion of the air near sharp obstacles and the influence of viscosity. The memoir he published in German in 1858 on the subject had a decisive influence on the physicists of the Scottish school including Maxwell, Rankine, Tait and Thomson (who was ennobled in 1892 as Lord Kelvin), all the more that Tait translated it into English in 1867 under the title *On the integrals of the hydrodynamical equations, which express vortex-motion* [47]. Inspired by the parallel between mechanics of continuous media and electromagnetism [10, Chap. 4], Helmholtz showed that, given a field of velocities $\vec{v}$, its curl, the vorticity field,

$$\vec{\omega} = \overrightarrow{\text{curl}}\,\vec{v} \tag{1}$$

is a vector field proportional to the local rotation vector of the fluid. Helmholtz introduced the notion of vortex line (a curve tangent to $\vec{\omega}$ at each of its points) and vortex filament/tube (a bunch of vortex lines) and proved that during its evolution each vortex line follows the motion of the fluid. The dynamical equation of $\vec{\omega}$ allowed him to study precisely the dynamics of straight (Fig. 3) and circular vortex tubes (Fig. 4). A thin vortex ring whose radius R is much larger than the radius of the cross section of the tube that defines it moves perpendicularly to its plane with the velocity of its center increasing with R.[11] Based on the similar mathematical

[11] In particular, when two rings moving along the same direction get close, the flow created around the leading ring tends to shrink the following one which, conversely, generates a flow that tends to expend the ring ahead. Therefore the leading ring slows down while the second one is sped up until it overtakes the former by passing through it, and the role of the rings are exchanged. This tango, predicted and observed by Helmholtz, is described in the end of his 1858 memoir.

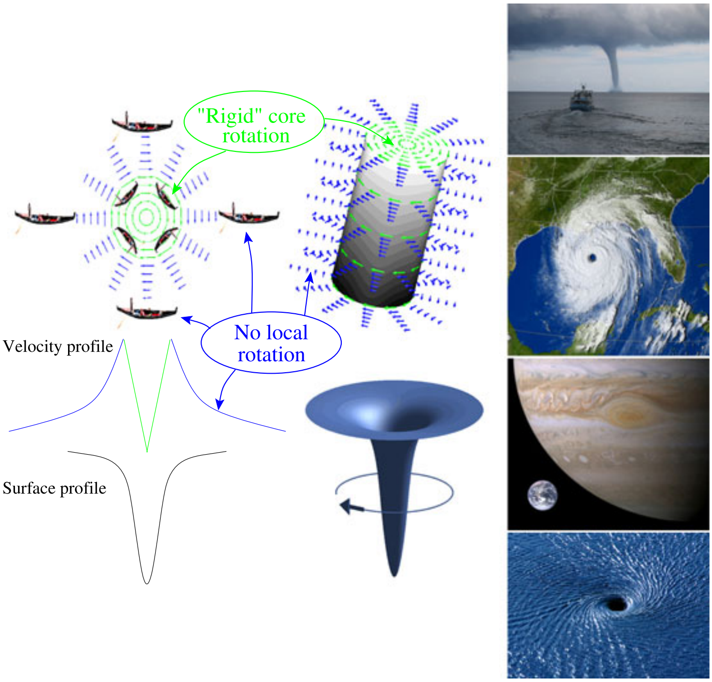

Fig. 3 The same year Helmholtz published his seminal memoir, the simplest model of vortex was explicitly proposed by Rankine in [37, pp. 629–633] who refers to some previous theoretical analysis made by the engineer and physicist James Thomson, inventor of the vortex wheel and brother of William. The vorticity (1) is constant and uniform inside a cylinder—in green, where the fluid rotates as a solid core and the particles rotate around themselves (the axis of the gondola rotates)—and zero outside—in blue, where the fluid particles do not rotate around themselves (the axis of the gondola keeps the same direction). When coming closer to the axis of the vortex, the velocity increases with the inverse of the distance outside the cylindrical core (and then producing a spiral-like shape) and then linearly gets to zero inside the core. In a more or less realistic way, Rankine's vortex models hurricanes, tornados or simply water going down a plughole (image credit: wikipedia, NOAA)

problem that arose in electrostatics and magnetostatics, Helmholtz understood that the topology of the irrotational part of the flow was essential to determine *globally* the velocity potential α: in the set of the points P where $\vec{\omega}(P) = 0$ one can always *locally* define a scalar field α such that

$$\vec{v} = \overrightarrow{\mathrm{grad}}\, \alpha \tag{2}$$

but

> If we consider [a vortex-filament] as always reentrant either within or without the fluid, the space for which [equation (2)] holds is complexly connected, since it remains single if we conceive surfaces of separation through it, each of which is completely bounded by a vortex-filament. In such complexly connected spaces a function [α] which satisfies the above equation can have more than one value ; and it must be so if it represents currents

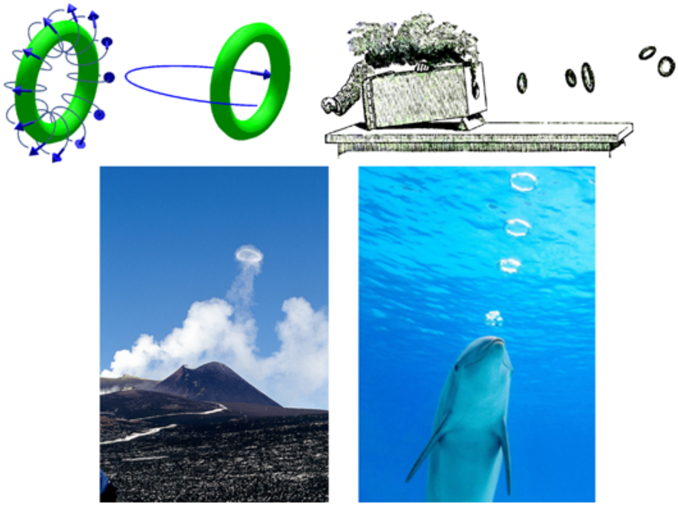

Fig. 4 When some vortex lines are bended into a circular tube (in green), each portion of the ring is dragged in the same direction by the fluid whose motion is induced by the other parts of the ring. As a result, a global translation perpendicular to the ring occurs. Helmholtz's study of the dynamics of the rings and the tango played by two interacting rings moving in the same direction, see footnote 11, p. 254, can be visualised with Tait's smoke box (upper right taken from [42, p. 292]). In exceptional circumstances vapour rings can be naturally produced by vulcanos (the lower left photograph is taken at Etna by the vulcanologist Boris Behncke, INGV-Osservatorio Etneo). Dolphins and whales are able to produce vortex rings in water (lower right from youtube)

> reentering, since the velocity of the fluid outside the vortex-filaments are proportional to the differential coefficients of [α], and therefore the motion of the fluid must correspond to ever increasing values of [α]. If the current returns to itself, we come again to a point where it formerly was, and find there a second greater value of [α]. Since this may occur indefinitely, there must be for every point of such a complexly-connected space an infinite number of distinct values of [α] differing by equal quantities like those of $\tan^{-1}\frac{x}{y}$, which is such a many-valued function [...].
>
> [47, § 3, translation by Tait].

The topological properties of vortices can also be understood from what is now known as Kelvin's circulation theorem [44, § 59d] which unified Helmholtz results: in an inviscid (no viscosity), barotropic (its density is a function of pressure only) fluid, the flux of the vorticity

$$\Gamma = \int_{\mathscr{S}} \vec{\omega} \cdot \mathrm{d}\vec{S} = \int_{\partial\mathscr{S}} \vec{v} \cdot \mathrm{d}\vec{l} \tag{3}$$

through a surface $\mathscr{S}$ following the motion of the fluid—or equivalently, according to Stokes' theorem, the circulation of the velocity through the boundary $\partial\mathscr{S}$ of $\mathscr{S}$—is constant. As a consequence, we recover Helmholtz statement that the non simple connectedness of the space filled by the irrotational part of the flow, i.e. the complementary of the vortex tubes, prevents the existence of a continuous globally-defined α and the circulation Γ depends on the homotopy class of the loop $\mathscr{C} = \partial\mathscr{S}$. In such an ideal fluid, the vortex lines were therefore topologically stable and Thomson's saw in this stability a key for the description of atomic properties without referring to the corpuscular image inherited from the atomists of antiquity, which was a too suspicious philosophy for Victorian times [25, §§ 2 and 9].[12] Since vortex tubes cannot cross transversally[13] otherwise it is easy to find a $\mathscr{C}$ that does not satisfy Kelvin's theorem, the knot formed by a closed vortex tube and the intertwining between several such closed loop remain topologically invariant.

> The absolute permanence of the rotation, and the unchangeable relation you have proved between it and the portion of the fluid once acquiring such motion in a perfect fluid, shows that if there is a perfect fluid all through space, constituting the substance of all matter, a vortex-ring would be as permanent as the solid hard atoms assumed by Lucretius and his followers (and predecessors) to account for the permanent properties of bodies (as gold, lead, etc.) and the differences of their characters. Thus, if two vortex-rings were once created in a perfect fluid, passing through one another like links of a chain, they never could come into collision, or break one another, they would form an indestructible atom; every variety of combinations might exist.
>
> Thomson to Helmholtz, January 22, 1867, quoted by [25, p. 38].

The theory of the vortex atoms offered to Thomson the possibility of making concrete his long-standing intuition of a continuous conception of the world, as he had confessed it to Stokes

> Now I think hydrodynamics is to be the root of all physical science, and is at present second to none in the beauty of mathematics.
>
> Thomson to Stokes, December, 20, 1857,
> quoted in [25, p. 35].

Despite the physical failure of Thomson's ambitious aim [15, 25, 40],[14] the identification of topological invariants on knots, upon which the classification of atoms and molecules would have been based, and the classification of the knots by Tait (see Fig. 5 for instance) remains a groundbreaking mathematical work, with direct repercussions in contemporary topology.

One of the Thomson's greatest hopes, while spectroscopy was gathering more and more precise data, was to explain the origin of the discrete spectral lines

---

[12] Some smoothness into the atom had already been introduced by Rankine in 1851 with his hypothesis of *molecular vortices* according to which "each atom of matter consists of a nucleus or central point enveloped by an elastic atmosphere, which is retained in its position by attractive forces, and that the elasticity due to heat arises from the centrifugal force of those atmospheres, revolving or oscillating about their nuclei or central points" [36, § 2]. It is worth noting that Rankine acknowledges the pertinence of William Thomson's comments on the first version of this 1851's proposal.

[13] But, it seems that neither Helmholtz nor Thomson have considered the possibility of a longitudinal merging of vortex tubes, forming a trousers-like shape [16, in particular Fig. 6].

[14] As far as classical hydrodynamics is concerned, some progress have been made in the XXth century with, for instance, the identification of new integrals of motion constructed from topological invariants like the Calugareanu helicity [27]; experimentally some not trivial knotted vortices could be produced only recently [22].

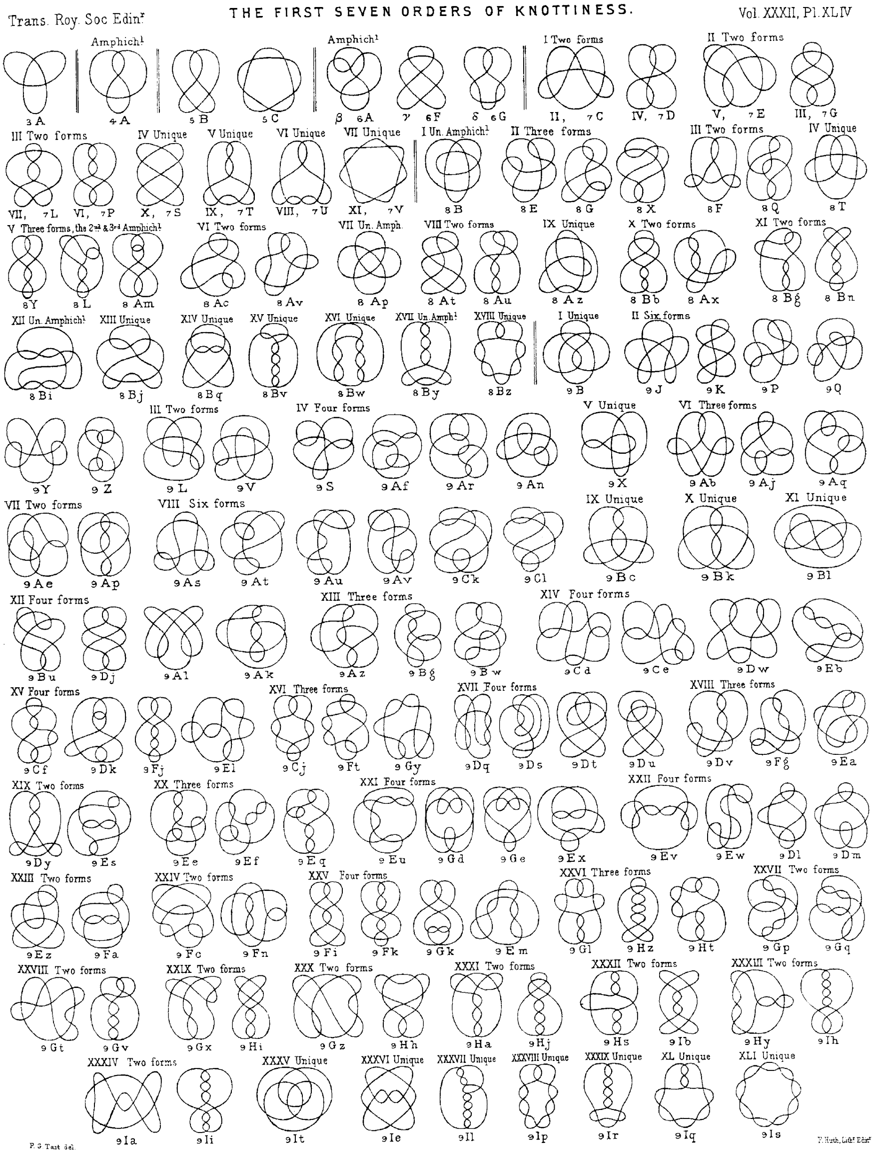

Fig. 5 List of knots up to the seventh order established by [42, Plate XLIV between pp. 338 & 339]

with “ [...] one or more fundamental periods of vibration, as has a stringed instrument of one or more strings [...]” [43, p. 96]. One cannot prevent to find an echo of this motivation in modern string theory where “each particle is identified as a particular vibrational mode of an elementary microscopic string” [49, § 1.2]—see also [7, in particular § 19]. Not without malice, [25] was perfectly right to qualify Thomson's dream as a "Victorian theory of everything".

Quantum Vortices

Topological Origin of Quantized Flux in Quantum Fluids

Unlike what occurs in classical fluids where viscosity eventually make the vortices smoothly vanish, quantum fluids provide a state of matter, much more similar to ideal fluids, where vortices are strongly protected from dissipative processes. Indeed, at low temperature, particles can condensate into a collective quantum state where transport can be dissipationless: this is one of the main characteristics of superconductivity (discovered in solid mercury below 4K by Onnes in 1911), superfluidity (discovered in liquid Helium-4 below 2K by Kapitsa and Allen & Misener in 1938), and Bose-Einstein condensate of atoms (discovered for rubidium below 170 nK by Cornell & Wieman and Ketterle in 1995).[15]

There is a second reason, of topological origin, that reinforces the stability of the vortices in quantum fluids: the scalar field α whose gradient is proportional to the current is not a simple mathematical intermediate as in the classical case (see (2)) but acquires the more physical status of being a phase (an angle) that may be measured in interference experiments like in the Aharonov-Bohm effect. As a consequence, on any closed loop $\mathscr{C}$, the circulation Γ given by (3) has to be an integer multiple of 2π:

$$w[\mathscr{C}] \overset{\text{def}}{=} \frac{1}{2\pi} \int_{\mathscr{C}} \overrightarrow{\text{grad}}\, \alpha \cdot \mathrm{d}\vec{l} \in \mathbb{Z}. \tag{4}$$

Since smooth transformations cannot provoque discrete jumps, w is therefore topologically protected. In other words, the flux of $\overrightarrow{\text{curl}}\,\vec{v}$—which keeps its physical interpretation of being a vorticity in superfluids as well as in Bose-Einstein condensates of atoms, whereas it represents a magnetic field in superconductors[16]—is quantized and naturally leads to elementary vortices carrying a unit flux quantum. As a matter of fact, the quantum fluid state is described by a complex field $\psi = |\psi|\mathrm{e}^{\mathrm{i}\alpha}$ (the order parameter) and $w[\mathscr{C}] \neq 0$ denotes a singularity of the order parameter on any surface $\mathscr{S}$ whose boundary is $\mathscr{C}$. Vortices constitute a particular case of what is generally called a *topological defect* whose dimension depends on the dimension of the order parameter and on the dimension of

[15]One can find many textbooks at different levels and more or less specialised to one type of quantum fluids. To get an introductory bird's-eye view on quantum fluids and other matters in relation to statistical physics, my personal taste go to [8, 20] and the particularly sound, concise, and pedagogical [39] (in French).

[16]Compare (1) with the relation $\vec{B} = \overrightarrow{\text{curl}}\,\vec{A}$ between the (gauge) vector potential $\vec{A}$ and the magnetic field $\vec{B}$.

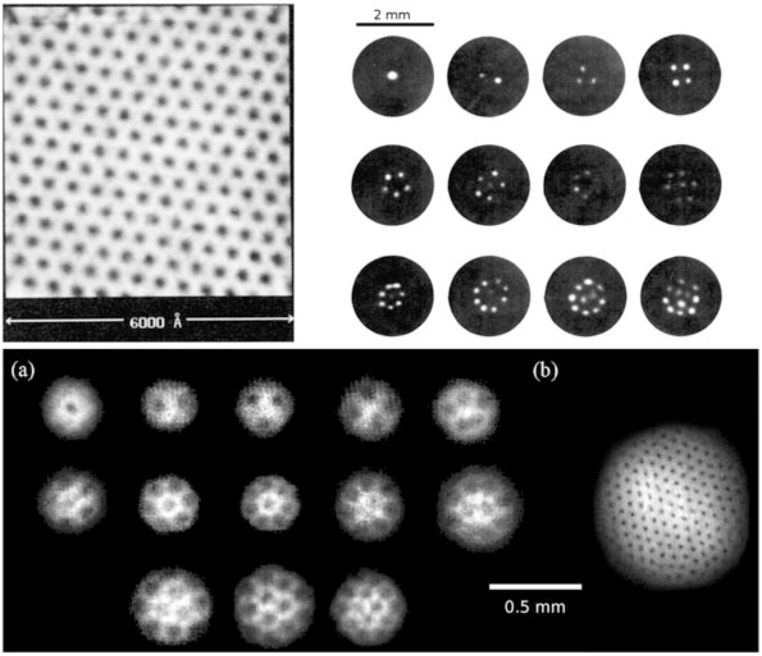

Fig. 6 (Up left) Abrikosov lattice of vortices in a superconductor [19, Fig. 2]. (Up right) Vortices in superfluid helium [48, Fig. 2]. (Below) Vortices in a rotating Bose-Einstein condensate obtained by (**a**) Dalibard's group ([26, Fig. 1] & [9, Fig. 4]); (**b**) Ketterle's group [35, fig. 4c]. ©European Physical Society and American Physical Society

the space. At microscopic scales, very much like in the Rankine model, the vortex is made of a core outside which $\overrightarrow{\text{curl}}\,\vec{v} = 0$; the vorticity/magnetic lines are trapped inside the core where the density of the superfluid $|\psi|^2$ tends to zero at its center. Not only, these vortices have been observed in all the three types of superfluids mentioned above but also the triangular lattice they form to minimize the (free) energy due to an effective repulsion between them first predicted by [1], see Fig. 6).

The XY*-Model*

When the fluctuations of $|\psi|$ in space and time are negligible, notably at sufficiently low temperatures, the quantum fluid is essentially described by the phase $e^{i\alpha}$ or equivalently by a bidimensional vector of unit norm oriented at angle α with respect to a given direction (Fig. 7). The latter picture is known as the XY-model, which is also relevant for some classical liquid crystals or for systems of classical spins [8, Chap. 6]. At macroscopic scales, some collective effects of such model are not very sensitive to the details of the interaction nor to the geometry of the elementary cell in

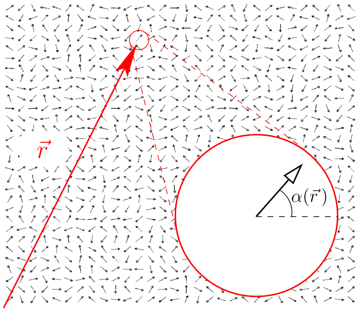

Fig. 7 The XY-model describes an interacting bidimensional vector field of constant and uniform norm. On a continuous space or on a lattice, the direction of the field at point $\vec{r}$ is given by one angle $\alpha(\vec{r})$

the case of a lattice but depend crucially on the dimension d of the space of positions (the number of components of $\vec{r}$). Typically, the energy of the system increases when some differences in the orientation α appears; more precisely the energy density contain a term proportional to $(\overrightarrow{\text{grad}}\,\alpha)^2$. It is not affected by a homogeneous rotation of all the spins,

$$\alpha(\vec{r}) \mapsto \alpha(\vec{r}) + \alpha_0, \tag{5}$$

where the angle α_0 does not depend on $\vec{r}$. The absolute minimum of the total energy is obtained when all the vectors are aligned, which is the configuration at temperature $T = 0$ K. When $T > 0$, the equilibrium corresponds to more disordered configurations but, for $d = 3$,[17] some non-zero average value of α can be maintained up to a critical temperature T_{critical} beyond which the average value of α is zero (Fig. 8). At $d = 2$, on the contrary, the correlations between fluctuations never decrease sufficiently rapidly at large distances and the average value of α is zero as soon as $T > 0$. However one can still identify, at some finite temperature $T_{\text{critical}} > 0$, a qualitative change of behaviour in the correlation lengths, from a power-law decay at large distances to an exponential decay and this phase transition has observable repercussions, notably in superfluids helium films [5]. The theoretical description of what appeared to be a new kind of phase transition, now known as topological phase transitions, was proposed by [23] who showed that vortices were a cornerstone of the scheme.

As soon as their first papers, Kosterlitz and Thouless, talked about "topological order" because they were perfectly aware that this type of phase transition, unlike all the phase transitions known at the time of their publication, relies on topology rather than on symmetry (breaking). As we have seen above on Eq. (4), each vortex (now a topological defect of one dimension) is characterised by an integer, called the topological index of the vortex which can be reinterpreted using the concepts introduced by Poincaré in a series of papers that can be considered as the foundations of topology as a fully autonomous research discipline [15, § 4]. Any direction far away

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[17] Surprisingly, as far as the computations are concerned, the integer nature of d becomes secondary and one can formally consider d as continuous. The condition for an order/disorder phase transition at $T_{\text{critical}} > 0$ to exist is $d > 2$.

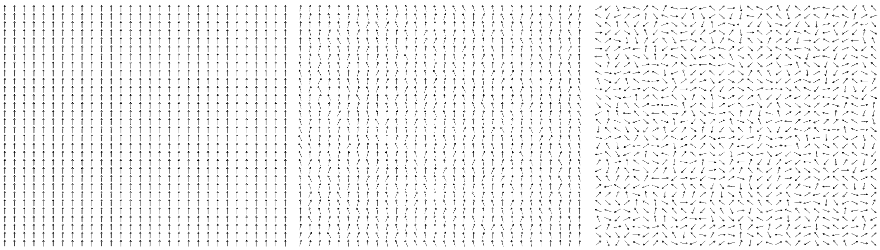

Fig. 8 In three dimensions the XY-model presents order/disorder phase transition, very similar to the familiar solid/liquid phase transition. Below a critical temperature $T_{\text{critical}} > 0$ some order is maintained throughout the system at macroscopic lengths (middle picture) with the perfect order obtained at $T = 0$ K (left picture). Above T_{critical}, the average orientation is zero and no more order at large scales can be identified (right picture)

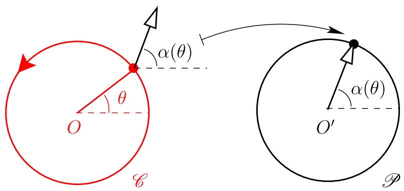

Fig. 9 For the XY-model, in $d = 2$, to each point on a loop $\mathscr{C}$ enclosing any given point 0 (in red, on the left) is associated the direction of the order parameter on the circle $\mathscr{P}$ (in black on the right)

a topological defect of dimension f in a space of dimension d is represented by an element of the rotation group in $n = d - f - 1$ dimensions, in other words such a defect can completely enclosed by a n-dimensional sphere S_n. In $d = 3$ dimensions a wall (a surface of dimension $f = 2$) cannot be enclosed ($n = 0$), a vortex-line ($f = 1$) can be enclosed by a circle ($n = 1$), a point ($f = 0$) can be enclosed by a $n = 2$-sphere. In $d = 2$ dimensions a wall (a line of dimension $f = 1$) cannot be enclosed ($n = 0$) and a point can be enclosed by a circle ($n = 1$). To each direction one can associate the value of the order parameter and therefore to each defect one gets a map from S_n to $\mathscr{P}$ where $\mathscr{P}$ denotes the space to which the order parameter belongs. In the examples above $\mathscr{P}$ is just the set S_1 of the angles α but much more different situations may be encountered. For $n = 1$, any loop $\mathscr{C}$ around a given point maps on a closed path $\mathscr{C}'$ in $\mathscr{P} = S_1$ and the topological index w of the point is just the winding number of $\mathscr{C}'$ (Figs. 9 and 10). More generally, the topological invariants are given by the group π_n of $\mathscr{P}$ (for $n = 0$ it provides the connectedness, for $n = 1$ it provides the first homotopy group that is the simple connectedness, etc.). A continuous transformation of the configuration cannot modify w at any point and physically it would require a macroscopic amount of energy to change w. On the other hand, one configuration having one defect can be deformed continuously at low cost of energy into any other configuration having a defect with the same w. In particular, the transformation (5) does not cost any energy at all.

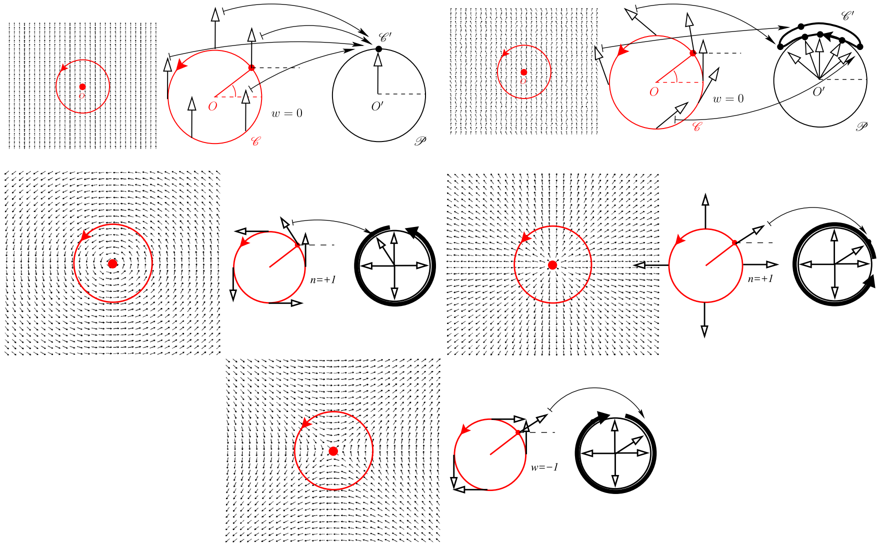

Fig. 10 In the XY-model, the topological index (4) of a point O is the winding number of the curve $\mathscr{C}'$ (thick black line) defined to be the image of a closed loop $\mathscr{C}$ (in red) in the circle $\mathscr{P}$ (thin black circle) that indicate the direction α of the order parameter. A smooth deformation deforms $\mathscr{C}'$ but do not change w (we stay in the same homotopy class). The upper row provides two examples having $w = 0$ (with, on the left, a uniform order parameter, $\mathscr{C}'$ is just a point). The central row provides two elementary vortices ($w = 1$) whose configurations differ from left to right by a rotation (5) with $\alpha_0 = -\pi/2$. The lower row provides an example of configuration having an elementary antivortex ($w = -1$)

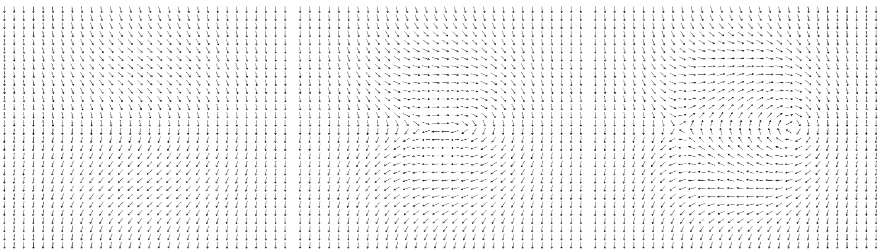

Fig. 11 A smooth transformation that does not require a macroscopic amount of energy can make a vortex/antivortex pair to spontaneously appear as a local fluctuation at non-zero temperature. The genericity and the structural stability of this scenario can be understood when considering the appearance of a fold (Fig. 12)

One cannot therefore expect *isolated* elementary vortex ($w = 1$) or *isolated* elementary antivortex ($w = -1$) to be spontaneously created from a perfect ordered state. Nevertheless, a pair of vortex-antivortex is affordable when $T > 0$ (Fig. 11). The continuous creation (or annihilation) of such a pair can be understood by considering the appearance of a fold on a drapery (back to Leonardo again?). One may intuitively see that this is a generic process, stable with respect to smooth transformations, that describes the creation or the annihilation of a pair of maximal-minimal

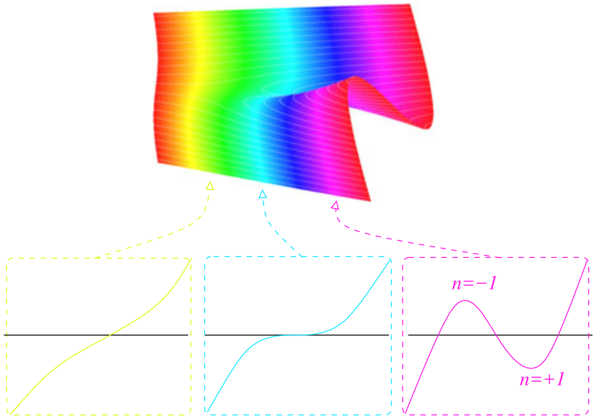

Fig. 12 The fold catastrophe is the simplest of the bifurcation scenario. It involves a one real parameter family of functions where two generic critical points (on the right), having opposite second derivatives, merge into a degenerate critical point (central graph) and disappear (on the left). It also represents how generically a non transversal crossing between two tangent curves (in the center) is unfolded from one (on the left) to three (on the right) transversal crossings

points on a smooth function (Fig. 12) or, equivalently, the creation or annihilation of intersection points when two curves that cross transversally are smoothly locally deformed.[18] The topological phase transition describes precisely how the creation of an increasing number of vortex-antivortex pairs as the temperature increases eventually lead from a topological order to a state where complete disorder reigns.

Concluding Remark

To come back to issues mentioned in the last paragraph of first section, in quantum theory, the fundamental elementary particles stem from algebraic symmetry considerations. However, we have some clues (topological defects, solitons, instantons, monopoles, etc.) that topology may offer a complementary ground. The parallel between creation/annihilation of particle-antiparticle pairs and creation/annihilation of vortex-antivortex pairs may be more than a simple analogy.

Thomson/Kelvin's intuition may take an unexpected but relevant form, after all.

Acknowledgements I am particularly grateful to Pascal Brioist (Centre d'Étude Supérieures de la Renaissance de l'Université de Tours) for his expert advices on Leonardo studies, to Boris

[18]Topology is fully at work here and the study of the stability of the critical points of smooth mappings is the object of catastrophe theory whose greatest achievement is to have classified the generic possible scenarios; the simplest one being precisely the fold catastrophe, depicted in Fig. 12 [3, 34, for a general survey].

Behncke (INGV-Osservatorio Etneo)) for letting me use his photo of the vapour ring created by the Etna (see Fig. 4) and to Michele Emmer who triggered the subject of this essay for the conference *Matematica e Cultura 2017*, Imagine Math, at Venice.

References

1. A.A. Abrikosov, The magnetic properties of superconducting alloys. J. Phys. Chem. Solids **2**(3), 199–208 (1957)
2. D. Arasse, *Léonard de Vinci* (Hazan, Malakoff, 1997). English translation: Konecky (1998)
3. V.I. Arnold, *Catastrophe Theory* (Springer, New York, 1984)
4. J.L. Bell, *The Continuous and the Infinitesimal in Mathematics and Philosophy* (Polimetrica, Milano, 2006). See also Bell's article with the same title in the Stanford Encyclopedia of Philosophy, plato.stanford.edu
5. D.J. Bishop, J.D. Reppy, Study of the superfluid transition in two-dimensional he-4 films. Phys. Rev. Lett. **40**(26), 1727–1730 (1978)
6. A.V. Borisov, V.V. Kozlov, I.S. Mamaev, M.A. Sokolovskiy (eds.), *IUTAM Symposium on Hamiltonian Dynamics, Vortex Structures, Turbulence*. IUTAM Bookseries, vol. 6 (Springer, Dordrecht, 2008)
7. A. Cappelli, E. Castellani, F. Colomo, P. Di Vecchia, *The Birth of String Theory* (Cambridge University Press, Cambridge, 2012)
8. P.M. Chaikin, T.C. Lubensky, *Principles of Condensed Matter Physics* (Cambridge University Press, Cambridge, 1995)
9. F. Chevy, J. Dalibard, Rotating Bose-Einstein condensates. Europhysicsnews **37**(1), 12–16 (2006). Free access on line at www.europhysicsnews.org
10. O. Darrigol, *Worlds of Flow. A History of Hydrodynamics from the Bernoullis to Prandtl* (Oxford University Press, Oxford, 2005)
11. S. Dehaene, *The Number Sense. How the Mind Creates Mathematics* (Oxford University Press, Oxford, 1997)
12. R. Descartes, *Principia Philosophiæ* (Ludovicum Elzevirium, Amsterdam, 1644). Latin original edition available on gallica.bnf.fr
13. P.A.M. Dirac, Quantised singularities in the electromagnetic field. Proc. R. Soc. Lond. Ser. A **133**, 60–72 (1931). Reproduced in § 2.1. of [45]
14. M. Emmer, M. Abate, M. Villarreal (eds.), *Imagine Maths 4. Between Culture and Mathematics* (Unione Matematica Italiana. Instituto Veneto di Scienze, Lettere ed arti, Venezia e Bologna, 2015)
15. M. Epple, Topology, matter, and space, I: topological notions in 19th-century natural philosophy. Arch. Hist. Exact Sci. **52**, 297–392 (1998)
16. V. Fuentes, On the topology of vortex lines and tubes. J. Fluid Mech. **584**, 147–156 (2007)
17. E.H. Gombrich, *The Heritage of Apelles*. Studies in the Art of the Renaissance, vol. 3 (Phaidon, Oxford, 1976)
18. E.H. Gombrich, The form of movement in water and air. In O'Malley [31], pp. 171–204. Reproduced in [17] pp. 39–56.
19. H.F. Hess, R.B. Robinson, R.C. Dynes, J.M. Valles Jr., J.V. Waszczak, Scanning-tunneling-microscope observation of the Abrikosov flux lattice and the density of states near and inside a fluxoid. Phys. Rev. Lett. **62**(2), 214–216 (1989)
20. K. Huang, *Statistical Mechanics*, 2nd edn. (Wiley, New York, 1987)
21. J. Ioan Mackenzie (ed.), *History of Topology* (North-Holland, Amsterdam, 1999)
22. D. Kleckner, W.T.M. Irvine, Creation and dynamics of knotted vortices. Nat. Phys. **9**, 253–258 (2013)
23. J.M. Kosterlitz, D.J. Thouless, Long range order and metastability in two dimensional solids and superfluids. J. Phys. C **5**, L124–L126 (1972). The detailed version of this work is [24]

24. J.M. Kosterlitz, D.J. Thouless, Ordering, metastability and phase transitions in two-dimensional systems. J. Phys. C **6**, 1181–1203 (1973)
25. H. Kragh, The vortex atom: a Victorian theory of everything. Centaurus **44**, 32–114 (2002)
26. K.W. Madison, F. Chevy, W. Wohlleben, J. Dalibard, Vortex formation in a stirred Bose-Einstein condensate. Phys. Rev. Lett. **84**(5), 806–809 (2000)
27. K. Moffatt, Vortex dynamics: the legacy of Helmholtz and Kelvin. In Borisov et al. [6], pp. 1–10.
28. A. Mouchet, Symmetry: a bridge between nature and culture, In Emmer et al. [14], pp. 235–244.
29. M. Nakahara, *Geometry, Topology and Physics*. Graduate Student Series in Physics (Hilger, Bristol, 1990)
30. C. Nash, Topology and physics—a historical essay. In James [21], pp. 359–415, Chap. 12, see also arXiv:hep-th/9709135
31. C.D. O'Malley (ed.), *Leonardo's Legacy. An International Symposium*, 2–8 May 1966. UCLA Center for Medieval and Renaissance Studies, vol. 2 (University of California Press, Berkeley, 1969)
32. A. Pasipoularides, *Heart's Vortex. Intracardiac Blood Flow Phenomena* (People's Medical Publishing House, Shelton, 2010)
33. Plutarch, *Lives*. Loeb Classical Library, vol. I (Harvard University Press, Cambridge, 1914). Bilingual edition commented and translated from the Greek by Bernadotte Perrin
34. T. Poston, I. Stewart, *Catastrophe Theory and Its Applications* (Pitman, London, 1978)
35. C. Raman, J.R. Abo-Shaeer, J.M. Vogels, K. Xu, W. Ketterle, Vortex nucleation in a stirred Bose-Einstein condensate. Phys. Rev. Lett. **87**(21), 210402 (2001)
36. W.J.M. Rankine, On the centrifugal theory of elasticity, as applied to gases and vapours. Philos. Mag., Supp. **2**(14), 509–542 (1851) (4th series). Reproduced in [38] § III, pp. 16–48
37. W.J.M. Rankine, *A Manual of Applied Mechanics*. Professor Rankine's Series of Practical Manuals on Civil Engineering and Mechanics (Richard Griffin and Company, London, 1858)
38. W.J.M. Rankine, *Miscellaneous Scientific Papers*. (Richard Griffin and Company, London, 1881)
39. N. Sator, N. Pavloff, *Physique Statistique* (Vuibert, Paris, 2016), in French
40. R.H. Silliman, William Thomson: smoke rings and nineteenth-century atomism. Isis **54**(4), 461–474 (1963)
41. M.C. Stokes, *One and Many in Pre-Socratic Philosophy*. Publications of the Center for Hellenic Studies (Harvard University Press, Cambridge, 1971)
42. P.G. Tait, On knots, Part II. Trans. R. Soc. Edinb. **32**, 327–342 (1884)
43. W. Thomson, On vortex atoms. Proc. R. Soc. Edinb. **6**, 94–106 (1867)
44. W. Thomson, On vortex motion. Trans. R. Soc. Edinb. **25**, 217–260 (1869)
45. D.J. Thouless, *Topological Quantum Numbers in Nonrelativistic Physics* (World Scientific, Singapore, 1998)
46. H. von Helmholtz, Über Integrale der hydrodynamischen Gleichungen, welche der Wirbelbewegungen entsprechen. J. Reine Angew. Math. **55**, 22–55 (1858). English translation by P.G. Tait in [47]
47. H. von Helmholtz, On the integrals of the hydrodynamical equations, which express vortex-motion. Philos. Mag., Supp. **33**(226), 485–512 (1867) (4th series). English translation by P.G. Tait of [46]
48. E.J. Yarmchuk, J.V. Gordon, R.E. Packard, Observation of stationary vortex arrays in rotating superfluid helium. Phys. Rev. Lett. **43**(3), 214–217 (1979)
49. B. Zwiebach, *A First Course in String Theory* (Cambridge University Press, Cambridge, 2004), 2009 for the 2nd edition

Part VIII
Mathematics & Literature

Éloge des Mathématiques

Michele Emmer

> O stern mathematics, I have not forgotten you since your learned teachings, sweeter than honey, filtered through my heart like a refreshing wave. From the cradle I instinctively aspired to drink from your spring more ancient than the sun, and, most faithful of your initiates, still I continue to tread the sacred court of your grave temple. . . .
> Arithmetic! Algebra! Geometry! Grand trinity! Luminous triangle! He who has not known you is a dolt! He deserves the test of the greatest tortures, for in his ignorant thoughtlessness there is blind contempt. . . .
> O concise mathematics, by the rigorous series of your tenacious propositions and the constancy of your iron laws, dazzle the eyes, shining forth a powerful reflection of that supreme truth whose imprint is discernible in the order of the universe... for the Almighty revealed himself and his attributes completely in this memorable task which consisted in bringing forth from the bowels of chaos your treasures of theorems and your magnificent splendors. . . .
> the mind wonders how mathematics happen to contain so much commanding importance and so much incontestable truth, while comparison between mathematics and man only uncovers the latter's false pride and mendacity. . . .
> Yet you remain the same forever. No change, no pestilential blast grazes the steep rocks and vast valleys of your identity.
> Thank you for the countless services you have rendered me. Thank you for the unfamiliar qualities with which you have enriched my intellect. . . .
> You gave me logic, the very soul of your wise instruction, and through its syllogisms whose involved maze makes them still more comprehensible, my intellect felt its bold strength redouble.
> O holy mathematics, would you might, by your perpetual commerce, console my remaining days for the wickedness of man and the injustice of the Most High!

This surprising *Éloge des Mathématiques* was contained in *Les Chants de Maldoror*, written during 1868–1869 by the Comte de Lautréamont, the pseudonym of the Uruguayan-born French writer Isidore-Lucien Ducasse [1].

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It became years later a major source of inspiration for many of the surrealist artists like Salvador Dalí, André Breton, Marcel Duchamp, Man Ray and Max Ernst.

L'Éloge des Mathématiques: Évariste Galois is the title of the film by Alexandre Astruc, based on the book by the same author, realized in France in 1965. A film dedicated to the genius of Galois, the last day of his life before his death in a sort of duel-suicide in 1832 [2, 3].

Just two ancient examples of an *Éloge des mathématiques*.

More recently, in France again, a book by the title *Éloge des mathématiques* was published by the French philosopher Alain Badiou. On October 20th, 2014, he was invited to a debate in a festival of literature to talk on mathematics. The presentation was a dialogue between the philosopher and a journalist. The book was published in 2015 always as a conversation with Gilles Haéri, director general of the *Éditions Flammarion*. The English version, translated by Susan Spitzer, was published in 2016 by the title *In Praise of Mathematics* [4, 5].

Philosophy and Mathematics

Why did a philosopher decide to make an *Éloge des mathématiques* in the twenty-first century? Are there still interesting connections between philosophy and mathematics? Do they both have an important impact on culture, at least in France?

The book is divided in chapters. In the first one, *Mathematics must be saved*, Badiou replies to the first question, addressing where the very strong relationship with mathematics comes from. The answer is very simple:

> It is something that goes back to before I was even born! Simply because my father was a math teacher. So there was the mark of the name of the father, as Lacan would say.

It is interesting to remember the great interest for mathematics of Jacques Lacan, the French psychiatrist and psychoanalyst, in particular for *Topology*. The Möbius band was the logo of his journal *Scilicet* from the sixties [6–8].

> I was fascinated by mathematics as soon as we started doing a few really complex proofs.... I was stuck early on by the quasi-esthetic feeling about mathematics.... Next I studied contemporary mathematics in depth by taking courses during the first two years of university at the ENS (*École Normale Supérieure*) in Paris.... Probably also because of the atmosphere of structuralism in the 1960s where there was a lot of buzz about formal disciplines, that I became really convinced that mathematics was in a very closed dialectical relationship with philosophy [9].

(Badiou is talking about the influence of the Bourbaki group on mathematics).

The new discovery is of course the possibility to give a complete proof of a theorem:

> There's the idea of a real discovery, of an unexpected solution, even if it means you have to make your way along a path that's sometimes a little hard to follow but where you're ultimately rewarded. Later, I often compared mathematics to a walk in the mountains: the approach is long and hard, with lots of twists and turns and steep climbs... the reward is truly beyond compare: that amazement, that ultimate beauty of mathematics, that hard-won,

utterly unique beauty.... That's why I continue to promote mathematics from this esthetic perspective, too [10].

A similar feeling is true for most mathematicians, and in almost all their writings on mathematics and culture, the esthetic of mathematics is one of the main topics. For Badiou his main interest in mathematics, what convinces him that mathematics is still in a very close dialectical relationship with philosophy is that "structures are first and foremost the business of mathematicians."

In an example cited by Badiou, the famous anthropologist Lévi-Strauss referred to the mathematician André Weil to show that the exchange of women in an Australian aboriginal tribe could be understood by using the algebraic theory of groups in his book *The Elementary Structures of Kinship* [11]. Weil recalled this episode in his autobiography *Souvenirs d'apprentissage* [12] published in 1991. Lévi-Strauss was for many years at the *Faculty of Philosophy* in *São Paulo University* in Brazil and he was able in 1944 to obtain for Weil a chair in mathematics. Weil wrote an appendix to the Lévi-Strauss' book explaining how the really complex rules regarding admitted and forbidden marriages in Australian tribes could be explained using very simple mathematical principles using the structure of group [13]. On March 17th, 2010 in Paris the mathematician Michel Broué gave a talk on this topic entitled *Des lois du marriage à Bourbaki*. In the presentation of the talk Broué wrote that structuralism had a great importance in the Humanities, in particular the notion of mathematical structures.

"Structure! It's just a question of structure! The structure of this element, and of the group that is generated by it". This is the young Galois' exclamation in the seventh scene of the theatrical piêce *Galois* by the mathematician Luca Viganò, staged a few years ago in Italy [14].

André Weil wrote that there were simple structures of groups, of topological space and of the more complex structures, from rings to real numbers and vector spaces, but Weil pointed out that before Bourbaki all this had never been said and it was also necessary to find a name for these mathematical objects.

Weil does not remember how the term "structures" was coined in this context even if he thinks the term was already used by linguists. According to Weil, the group of mathematicians named Bourbaki originates in a dialogue between himself and Elie Joseph Cartan, another famous mathematician, in December 1934 in Strasbourg.

The concept of group is the first example of an algebraic structure.

Badiou remembers that when he was at the ENS "my philosophical approach required mastering enormous conceptual constructions.... When you work on a mathematical problem, the discovery of the solution, and therefore the creative freedom of the mind, is not some sort of blind wandering but rather the determination of a path that's always lined by the obligations of overall consistency and demonstrative rules. You fulfil your desire to find the solution not in spite of the law of reason but thanks to both its prohibitions and its assistance.... This is what I had begun to think, first in conjunction with Lacan: desire and the law are not opposites but dialectically identical. And mathematics combines intuition and proof in a unique way, which the philosophical text must also do, as far as possible" [15].

Badiou's conclusion is that, as the back-and forth movement between philosophy and mathematics produced in him a sort of split, "all my work may be nothing but the attempt to overcome this split."

The second chapter of the book is entitled *Mathematics and Philosophy, or the Story of an Old Couple*.

> Philosophy was born in Greece, beginning in the fifth century BC there were some totally new ideas about mathematics (deductive geometry and arithmetic), artistic activity (humanized sculpture, painting, dance, music, tragedy and comedy), politics (the invention of democracy) and the status of the emotions (Transfer love, lyric poetry, and so on).
> So I suggested that philosophy really only develops when advances emerge in a set of truths of four different reasons: science, art, politics and love [16].

The reason for Badiou's interest lies in the fact that philosophy should be able to preserve the dimension of the *subject*, and yet to integrate mathematics in all its rational force and splendor, particularly as regards the doctrine of *being*.

From this point of view the relationship between mathematics and culture is very important. This is one of the factors that motivated me twenty years ago to organize the conferences of the *Mathematics and Culture* series, and to publish the *Proceedings* that will serve as a basis for discussion on the topic, I hope, for the next years.

Mathematics and Culture

Speaking of math and culture there are differences between the various cultures that must be taken into account. From my experience, the relations between mathematics and culture, considering mathematics an integral part of culture, are much more developed in France than in Italy. Badiou does not think so.

One of the topics that often return regarding the problem of the relationship between mathematics and culture is that the mathematical community with very few exceptions is not very interested in making itself known, believing the vast majority of humanity is not able to understand what mathematicians are dealing with today. And they are largely right.

Andrè Weil was convinced of the uselessness of the attempts and compared speaking of mathematics to non-mathematicians to making music understandable to the deaf. Badiou writes: [17].

> The vast majority of mathematicians have an extremely elitist relationship to their discipline. They're fine with thinking that they're the only ones who understand it, and that that's just the way it is.... A very exclusive world, which occasionally attempts to reach out to a somewhat wider public, as does Cédric Villani, and as did renowned mathematician Henri Poincaré well before him, but that's still the exception.
> In France mathematics isn't part of ordinary culture. And that, as far as I'm concerned, is scandalous. Like fine arts, like cinema, it should be an integral part of our general culture....
> To put an end to the mathematicians' elitism, a middle way has to be found between the understanding of formalism and the conceptual aim. And for that to happen, I think there is a need for philosophy, which should therefore be taught a lot sooner.

Philosophy is a way to capture the interest and cultural value of mathematics. Cédric Villani in the last years after winning the Fields medal in 2010 has been very busy spreading everywhere mathematical culture. He wrote a book about his experience as a mathematician [18], he has organized exhibitions, participated in meetings, written in newspapers and acted in films [19], until he was elected in the list of President Macron *La France en Marche* in the last political elections (see [8]).

But Badiou's goal does not only concern elitist mathematicians but also philosophers. Regarding elitism, philosophers have the opposite problem: [20].

> The problem in philosophy is the exact opposite, since just anyone can be considered a philosopher now.... Ever since philosophers have become *new* (he refers to the so called *nouveaux philosophes*) people are undemanding where they're concerned, even at a basic level. So the divergence between mathematics and philosophy also stems from the fact that philosophy has undergone an incredible trivialization of its status.
> The mathematician was somebody who, for the first time, introduced a universality completely free of any mythological or religious assumption and that no longer took the form of a *narrative* but of a *proof*.
> Truth based on a narrative is *traditional* truth, of a mythological or revealed type.... The proof depended only on a *rational* demonstration, shown to everyone and refutable in its very principle, so that someone who had put forward a hypothesis that was ultimately proved to be false had to accept that he was wrong....
> A proof must be a proof, and that was all.

Mathematics and Democracy

> In that sense mathematics is part of democratic thought, which moreover appeared in Greece at the same time. So it's true that there were close links between mathematics, democracy (in the sense of political modernity) and philosophy [21].

And Badiou gives examples:

> Descartes, the founder of modern philosophy, a very great mathematician. What he took from mathematics in terms of his specifically philosophical project so clear: it was the ideal of proof.
> Let's take Spinoza. He began his *Ethics* saying that if mathematics hadn't existed, man would have remained in ignorance... and he organized his book in the exact way of a mathematical treatise (*Ethica more geometrico demonstrata*) on the model of Euclid's *Elements:* definitions, postulates, propositions.
> Ideally this procedure should play the logical order of the truth, so that the first truths are those on which rests the entire set of demonstrations and every proposition is based and refers to the previous one.
> In any case mathematicians have the right to do mathematics for just their personal satisfaction or for showing their results to other mathematicians. Badiou observes, and it is really true, that mathematicians are interested in difficult problems and are not asking themselves if mathematics is an ontology or a language game. So Badiou forgives mathematicians' negligence of philosophy, because "they are rendering an invaluable service to humanity as a whole" [22].

Curiously, after these observations, the philosopher provides examples of mathematicians who are weirdos, or have tortured or strange personalities. Like Grigory Perlman, who solved the Poincaré conjecture and refused the Fields medal and million dollar of the *Millennium Prize Problems* of the *Clay Math Research Institute* and lives with his mother in a Moscow suburb with his pension as his only income.

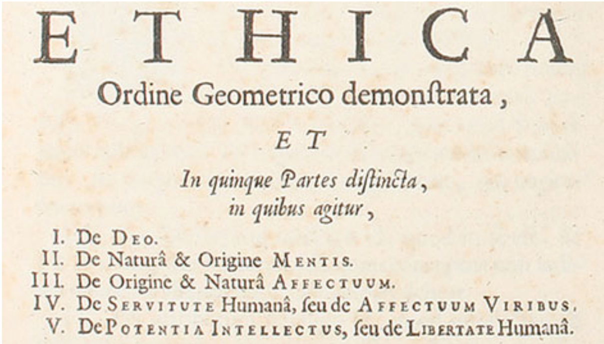
ETHICA
Ordine Geometrico demonſtrata,
ET
In quinque Partes diſtincta,
in quibus agitur,
I. De DEO.
II. De Naturâ & Origine MENTIS.
III. De Origine & Naturâ AFFECTUUM.
IV. De SERVITUTE Humanâ, ſeu de AFFECTUUM VIRIBUS.
V. De POTENTIA INTELLECTUS, ſeu de LIBERTATE Humanâ.

Fig. 1 B. Spinoza, *Ethica Ordine Geometrico demonstrata*, 1677

Other examples, all well-known, include Cantor and Gödel. Or that "typically Romantic character who, arrested for rebellion, wrote down his amazing thoughts in prison and died at the age of 20 in a stupid duel over a girl who wasn't really worth it (?)". He talks of Évariste Galois, of course. In any case Badiou recognizes that there are many mathematical geniuses, like Gauss and Poincaré, who were "serious academics, thoughtful people who were well established in their social world" [23].

Of course a serious genius is not so interesting, much better to note that mathematicians, like poets, can also be anarchists and romantics, or counter impulsive and withdrawn people, because what ultimately matters in mathematics is inventiveness, which often comes to them after long nights of slow and uncertain work, in the form of a lucky intuition.

Badiou surely read *A Mathematician's Apology* by G. H. Hardy, written in 1940 and knows well the pages of Poincaré and of Hadamard on intuition [24–26].

From what Badiou says, one understands very well that he considers philosophy very out fashioned in recent times, "since philosophers decided that there was no need for an idea of *specialized philosophy*, even more so as this idea made no sense. That philosophy might be the philosophy of this or that, what Lacan called the *discourse of the university* in the worst sense of the term. Philosophy is philosophy. So there has been a serious capitulation on the part of philosophers" [27].

And he gives two examples regarding when the capitulation of the philosophers materialized. They did not understand that mathematics in use from the late nineteenth century on was in fact mathematics that drastically changed many things in the most essential philosophical concepts.

Examples are the concept of infinity. If you are not familiar with the essential importance of research carried out in mathematics in the last fifty years, "when you say the word *infinity* you actually have no idea what you are talking about. . . . If you don't know anything about certain theorems from the 1970s and 1980s on the new figures of mathematical infinity, there is no point in using the word infinity" [28].

The second example is logic. "Logic, or rather logics, have become part of mathematics today. It is clear that philosophers cannot be unaware of logic and therefore of mathematized logic today".

This is what Badiou wants to do, to remedy to this situation, promoting the pleasure of mathematics.

It is clear that the philosopher cannot avoid the question of what the connections between reality and mathematics are.

Mathematics and Reality

The real is what is imposed on us. The demonstration is imposed on us more than the sensation, but it involves a part of convention. And it is necessary to grasp the unconventional mathematics.
Simone Weil

Badiou is a Platonist, as most of the mathematicians are, even if they are not really interested to a profound discussions of all this kind of topics in any case.

> There is a real *content* to mathematical thought.... Mathematicians have the impression that the path to the solution of a problem is a path that makes you touch a real and a sort of intrinsic complexity [29].

Obviously, the examples that mathematicians and even the few non-mathematicians who are interested in mathematics show must be simple, not complicated, understandable, since they are not addressed to mathematicians, who know very well others that are more complex and not narratable.

> How can we not be convinced that the infinity of natural numbers *exists*, in a sense that would need to be clarified?
> How is it possible to think that there's no real here other than our own playful invention?

The conclusion of the philosopher is that, in reality, "mathematics is simply the science of being *qua* being, i.e. what philosophers traditionally call *ontology*.

Mathematics is the science of everything that is, grasped at its absolutely formal level, and that's why paradoxically inventions of mathematics may be used in physical investigation."

Another classic example in this type of argumentation are imaginary numbers. Badiou believes they were invented just as a game, but in reality they were necessary in order to address all solutions of second degree algebraic equations. But in any case they seemed to be a curiosity, certainly necessary but very instrumental to that sole purpose to which they serve.

This is what Badioud wrote on this topic:

> Later, in the nineteenth century, they became a basic tool for example used in electromagnetism. If you want to know what it means to think only its *being*, the only way to do so is obviously to think purely formal structures, structures indeterminate as to their physical characteristics. And the science of these indeterminates as to their physical characteristics is mathematics [30].

It is well known that, in addition to the obvious questions generated by the idea of infinity, imaginary numbers have stimulated writers and ultimately also authors of theater and film directors. Even the name, as often happens for the names chosen by mathematicians for some objects they introduce and define, has had a great metaphorical impact in European culture.

Suffice it to mention the case of Robert Musil's short novel *The Confusions of the Young Törless* [31], the theatrical pièce, inspired by the novel, by Juan Mayorga, *El chico de la ultima fila* [32] and the film by François Ozon *Dans la Maison*, the cover of Musil's book appears in a scene [33].

The conclusion of Badiou is clear: [30].

> This why I reject the theory that mathematics derive from sensory experience. It's the other way around: the real of sensory experience is thinkable only because mathematical formalism thinks, ahead of times, the possible forms of everything that is.
> My thesis is: mathematics is ontology, i.e. the independent study of the possible forms of multiple as such, of any multiple, and therefore of everything that is—because everything that is, is in any case a multiplicity. This ontology can be developed for its own sake.

An interesting theory on which mathematicians will not agree. As many others I believe that at the beginning mathematics and geometry in particular were born from very practical needs and then gradually the structure which today we call mathematics was built. But the interesting point here is that it is a philosopher who makes these observations in 2015.

Pure and Applied Mathematics

And among the examples that Badiou uses, obviously there are those of the infinite and the notion of limit in mathematics. A problem which took thousands of years to be formally resolved (in mathematics), closing the long-running question of infinity that remained open since Achilles had to reach the turtle.

The definition of limit allowed to formally define derivatives, differential equations, integrals and open the way to contemporary science. But with the advent of computers, things have changed. Computers are able to quickly handle billions of data, provided to them through integers and rational numbers. Computers do not know limits and derivatives.

All mathematical simulations and models that allow us to live today use the numbers already known by Greek mathematicians. Not the real numbers that are those of continuity and differential equations. And here the methods of approximate calculation have become of enormous and essential importance.

In the vast majority of cases we are not able to find explicit solutions of the equation models that simulate a complex problem (typically atmospheric weather). In some ways calculation has become central again in mathematics.

Until a few years ago, mathematics students were told that true mathematics are totally abstract and devoid of practical applications. An emblematic example is given by G. H. Hardy in his autobiography written in the forties of the last century: [34].

> Is mathematics *unprofitable*? In some sense, plainly, it is not; for example, it gives great pleasure to quite a large number of people. I was thinking of *profit*, however, in a narrower

> sense. Is mathematics *useful*, *directly* useful, as other sciences such as chemistry and physiology are?... I shall ultimately say No....
> There are then two mathematics. There is the real mathematics of the real mathematicians, and there is what I will call the *trivial* mathematics, for want of a better word. The trivial mathematics may be justified by arguments that would appeal to several thinkers, but there is no such defense for the real mathematics, which must be justified as art if it can be justified at all.

For many years now the difference between pure and applied mathematics has completely disappeared. Paolo Zellini states that most probably we can not deny the *divine origin* of mathematics, (not in the literal sense of the term, of course) let us say of pure mathematics. It is actual men and women who have developed the art of calculating, or the algorithms that solve a given problem.

> It is important to realize that it is not the algorithms that decide the mathematical truth itself. The validity of an algorithm must always be established by external means. Computability is a mathematical idea, independent of any particular concept of a machine for calculation, but also because it illustrates the effectiveness of abstract ideas in mathematics.

La matematica degli dèi e gli algoritmi degli uomini (The mathematics of gods and the algorithms of men) [35] is the title of the book by Zellini published in 2016. A book that talks about mathematics and calculation procedures, algorithms in fact. But why should a non-mathematician be interested in thinking about what mathematics is, how it can be originated, what does a calculation process and its reliability mean?

> Which reality does mathematics tell us about? It is widely believed that mathematicians deal with abstract formalisms and that only for unexplainable reasons these formalisms apply to all areas of science. We conceive immaterial entities that seem to be destined to define models of phenomena that actually happen in the world.
> Reality is something that depends on making, by actually performing itself with action. The solution of a mathematical problem depends on the ability to calculate it efficiently in space and time through a physical automatic execution, which is the only possible strategy due to the size of the problems. There appears to be nothing more certain than a process that, in a finite number of steps, performs the necessary calculations on the basis of the given data. But does the reality of mathematical entities really sum up, in a comprehensive manner, in this conclusion?
> The method that from the mathematical model reaches up to the digital calculation is very unconventional: the physical nature of the event is necessarily transmitted in the structure of the equations and mathematical entities to resolve them, till the latest lists of numbers; wrote Zellini [36].

Mathematics and Politics

> *Mathematics: abstract universe in which I depend solely on me. Kingdom of justice, since every good will finds his reward.*
> Simone Weil

A question which Badiou answers concerns exactly whether mathematics can be a useful tool, besides counting the votes and making it clear whether an electoral system is fair or not [37].

> In my view, it's the following question that matters: Do you think it's possible, in politics, to reach decisions that really result from rational deliberations? Can there be such a thing?... A rational method of political discussions remains an ideal, even if everyone who has ever been an activist knows that there can be thrilling meetings, particularly in working-class circles, precisely because the conclusion, the operative, unifying slogan, was the result of a long and very efficient process.... Unfortunately, rhetoric is today's political language.

After having stated that mathematics is a key to happiness, (a chapter of the book is dedicated to this topic) he clearly states that mathematics, due to its intrinsic rules and the way the mathematicians proceed in their researches, can and in some way must be a model for public and political behavior, a sort of ethic and moral model:

> There is no obvious connection between mathematics and politics. The zero degree of connection is the computing of the votes on election night. A rational method of political discussion remains an ideal (...). If I must praise mathematics, I would say this: a sustained and ongoing exercise of true discursive rationality would counteract or mitigate our exposure to seductive rhetoric devoid of real substance.
> Actually, mathematics is the best of human inventions for practicing something that's the key to all collective progress and individual happiness: rising above our limits in order to touch, luminously, the universality of the true.

There is no doubt that it is of great interest nowadays that a philosopher arrives at these conclusions looking for a way out of the many-sided dramatic cultural, moral and political situation in which a large part of today's world lives. One more reason to continue on the road of the links between mathematics and culture.

References

1. I.L. Ducasse, *Chants de Maldoror* (A. Lacroix, Parigi, 1869), comte de Lautrémont. https://thanatologist.wordpress.com/ode-to-mathematics/
2. A. Astruc, *Evariste Galois* (Flammarion, Parigi, 1964)
3. A. Astruc, film maker, *L'Éloge des Mathématiques. Evariste Galois*, 30 m. (France, 1965)
4. A. Badiou, G. Haéri, *Éloge des Matématiques* (Flammarion, Paris, 2015)
5. A. Badiou, G. Haéri, *In Praise of Mathematics* (Polity Press, Cambridge, 2016)
6. J. Lacan (ed.), *Scilicet, n. 1* (Editions du Seuil, Paris, 1968)
7. E. Ragland, D. Milovanovic (eds.), *Lacan: Topologically Speaking* (Other Press, New York, 2004)
8. M. Emmer, *Racconto matematico memorie impersonali* (2019, to appear)
9. In [5], pp. 2–3
10. In [5], p. 4
11. C. Lévi-Strauss, *Structures Élémentaires de la Parenté* (Presses Universitaires de France, Paris, 1949). Eng. ed. *The Elementary Structures of Kinship* (Beacon Press, Boston, 1969)
12. A. Weil, *Souvenirs d'Apprentissage* (Birkhäuser, Basel, 1991)
13. A. Weil, *Appendice*, in [11]
14. L. Viganò, *Galois* (Il Nuovo Melangolo, Genova, 2005)
15. In [5], pp. 5–6
16. In [5], p. viii
17. In [5], pp. 8–9
18. C. Villani, *Théorème Vivant*, (Grasset, Paris, 2013). Eng. ed. *The Birth of a Theorem: A. Mathematical Adventure* (The Bodley Head, London, 2015)
19. O. Peyon, film maker, *Comment j'ai détesté les Maths*, 104 m. (Canope Editon, France, 2013).

20. In [5] p. 15
21. In [5], p. 22
22. In [5], pp. 26–27
23. In [5], p. 40
24. G.H. Hardy, *A Mathematician's Apology* (Cambridge University Press, Cambridge, 1941)
25. J.H. Poincaré, La Science et l'Hypothèse (Parigi, 1902)
26. J. Hadamard, *The Psychology of Invention in the Mathematical Field* (Princeton University Press, Princeton, 1945)
27. In [5], pp. 41–42
28. In [5], p. 43
29. In [5], pp. 62–64
30. In [5], p. 65
31. R. Musil, *Die Verwirrungen des Zöglings Törleß*, (Wiener verlag, Wien, 1906). Eng. ed. *The Confusions of Young Törless* (Penguin, London, 2001)
32. J.A. Mayorga Ruano, *El Chico de la Ultima Fila* (Madrid, 2006)
33. F. Ozon, film maker, *Dans la Maison*, script F. Ozon, with Fabrice Lucchini, Kristin Scott Thomas, Emmanuelle Seigner, Ernst Umhauer, (France, 2012).
34. In [24], p. 139
35. P. Zellini, in *La Matematica degli dei e Gli Algoritmi degli Uomini* (Adelphi, Milano, 2016)
36. In [35], p. 213
37. In [5], pp. 101–104

Luca Pacioli: Letters from Venice

Paola Magnaghi-Delfino and Tullia Norando

Euclid in Venice

Luca Pacioli had always maintained close contact with the Venetian cultural environment since he was pupil of Domenico Bragadin in the Rialto Gymnasium and tutor of Antonio Rompiasi sons.

He returned to Venice in 1494 for the printing of the *Summa*, dedicated to Marco Sanudo and in 1508 for printing *De Divina Proportione* [22].

In 1508, we find him at Rialto school [21], for the first time as a teacher. Sebastiano Foscarini, director of the Gymnasium Rivoaltinum, probably invited him to hold a course on the fifth book of Euclid's *Elements*. The inaugural lesson had wide appeal since its subject fitted the expectations of the Rialto philosophers and scholars [10, 16].

On 11 August 1508 in the San Bartolomeo Church in Venice, Luca Pacioli made the inaugural lesson of the Accademia di Rialto to a large gathering composed of theologians, philosophers, physicians, scholars, artists, architects and illustrious Venetians. Pacioli spoke about the great value and many applications of the proportion and proportionality. Pacioli published the text of this lecture as introduction to the Euclid's *Elements*, edited a few months later in Venice by the press of Paganinus de Paganinis.

The subject of the lecture was fashionable for the period for many good reasons. First, the Renaissance scholars were interested in scientific and mathematical works, because the ancient Greek book had become available, coming from Byzantium or from the collections of some famous Humanists, as the Cardinal Bessarione, who gave his library to Venice.

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The lower cost of the printed books greatly extended the diffusion of the mathematical learning to a wider community of merchants and artists.

The Johannes Campanus' Latin translation of Euclid from an Arabic text in 13th century is generally considered the main source for the 14th century scholars and the authoritative medieval text of the *Elements* until the 16th century, when it was replaced by translations made directly from the Greek.

According to Boethius, Campanus adopted the Latin translation *proportion* and *proportionalitas* for *logos* and *analogia*, respectively, that influenced the choice of terminology, contributing to the misinterpretation of ratio and proportion.

The first printed *Elements*' edition, based on Campanus' manuscript, appeared in Venice in 1482. This edition was severely criticized by Bartolomeo Zamberti, who published in 1505 a Latin translation from the Greek. Zamberti restored the previous terminology for ratio and proportion, however, his book was more philologically than mathematically correct.

Pacioli's edition is based on Campanus' book, but contains his own emendations and annotations. It was published in order to vindicate Campanus, apparently at the expense of Erhard Ratdolt, the publisher of Campanus' translation.

A summary of *Elements* and, in particular, excerpts from Book V appeared already in his most famous book: *Summa de arithmetica, geometria, proportioni et proportionalita*, written in vernacular language and edited in 1494 by Paganinus. The *Summa* is a review of known mathematics covering arithmetic, trigonometry, algebra, and tables of moneys, weights and measures, a large number of merchants' problems. Pacioli transformed the material in a way, which should be suited for a public with practical but only modest theoretical interests; there is no clear separation between definitions and enunciations and proofs are mostly replaced by explanations with reference to diagrams. Probably the *Summa* is a summary of the lessons that Pacioli gave when he taught at the University of Perugia, Naples, and Rome.

Euclid Book V

In the late Middle Ages, the character of the theory of ratios underwent important changes. Together with other 14th century sources, Campanus had an arithmetical terminology that was not derived from the geometrical ratio theory of Book V of Euclid, but instead from different sources, like *Arithmetic* of Jordanus de Nemore. In most part of references to the Book V, whose Greek version contains the theory of ratios in respect to magnitudes, the version of the definition for the proportionality of ratios is not the Eudoxian and geometrical definition of Book V. Instead, this definition is equivalent to treating ratio, as it was a fraction, which has important implication concerning ratios between incommensurable magnitudes. One possible solution would be to consider rational approximations. The arithmetical theory of Campanus' version of *Elements* provided the foundation for understanding of ratios in mathematical contexts and, on the other hand, had an important repercussion on the theoretical Renaissance music.

In *Elements*, Euclid furnishes two definitions of proportionality: first a general definition, valid for any magnitudes whatever, which is based on the notion of equimultiples (Book V), that we could call *equimultiple proportionality*. After (Book VII), a more restricted definition, valid for numbers, which is based on the notion of "being the same multiple, or the same part, or the same parts" that we could call *rational proportionality*.

In Book V, Euclid discusses the equality of numbers and magnitudes and never refers to ratio as being equal, but says that they are "in the same ratio" or that one ratio "is as" another one in a proportion. The idea of equality of ratios is probably not as natural as that of numbers or magnitudes. Probably, Euclid conceived of the theory of ratio as a generalization of music, which was a relevant part of mathematics from Pythagoreans until Euclid. In this period, the procedure of evaluation of ratios allowed through successive subtractions between their terms (Book VII, definition 20) that is not applicable to incommensurable quantities.

The definition 5 in Book V for proportion, including incommensurables, can be read as "*two ratios are proportional if they provide or underlie the same standard—in this case geometrical—established by the occurrence of two periodical phenomena*" [7]. The interpretation of Fowler is related to anthyphairetic theory: "*ratio in the Greek sense is an idea associated with the underlying abstract standards that, by means of proportions or analogies, make possible a link between contexts that are otherwise quite different and distinct*". In the light of this interpretation, "*ratios are proportional if and only if the same pattern underlies them*".

Analogia (proportion) was a concept strongly linked to logos (ratio) and acquired a more general theoretical status in Book V. Plato and Aristotle knew the doctrine of analogia, because it represented a medium by which one could conceive of an identical form in distinct contexts [5].

The attraction, or the repulsion, for the equimultiple definition and the search for a general meaning of the perpetual accord of the equimultiples was to fascinate mathematicians up until the nineteenth century. In many scholars' opinion, the proportion and proportionality definitions appear at once excessively obscure, verbose and counter-intuitive, whereas the arithmetical notion of ratio seems so natural. In his massive commentary on Euclid's *Elements*, first published in 1574 Christoph Clavius, one of the most respected mathematicians of his century was concerned with the obscurity of equimultiple proportionality [23]. Even Galileo criticized the Euclidean approach to proportions, considered too abstract and not very constructive. He provided another definition, which refers in some way to the continued fraction representation of a real number, so the result of a measurement is expressed as a "rational number", possibly with a remainder that can be as small as you want to be [9]. In a passage of his early *De Motu*, Galileo proved that the gravitates of different bodies of equal [specific] weight are proportional to their volumes (*Gravitates inaequalium molium corporum aeque gravium eam inter se habent proportionem, quam ipsae moles*) by means of the equimultiples. For this reason, Galileo came face to face with the problem of transferring the notion of equimultiple proportionality from the domain of pure mathematics to that of natural philosophy.

Pacioli and Euclid's Elements

In 1497, Pacioli was invited to the court of Ludovico Sforza, duke of Milan, to teach mathematics. Here he met Leonardo da Vinci, who was already in Sforza's employment. Leonardo consulted Pacioli on matters relating to mathematics: that is evident from entries in Leonardo's notebooks. Pacioli composed the first part of *De Divina Proportione* during 1496–1497, and Leonardo drew the figures of the solid bodies for this book, as the same Luca wrote. They leaved Milan in 1499 with the entry of the French army and journeying through Mantua and Venice, they arrived in Florence, where they shared quarters. In 1500, Pacioli was appointed to teach Euclid's *Elements* at the University of Pisa, the appointment was renewed annually until 1506. In 1504, he made a set of geometrical figures for the Signoria of Florence. During his stay in Florence, Pacioli also held an appointment at the University of Bologna as *lector ad mathematicam* (1501–1502).

Since his arrival in Florence, Pacioli had been preparing a Latin edition of Euclid's *Elements* and an Italian translation. Pacioli's Italian translation of Euclid's *Elements* was not published and there is no trace of the manuscript. The Pacioli's two works had two different aims.

His first purpose was the dissemination of the mathematical culture and concepts to the public of merchants and artisans who did not the Latin language. In the geometric part of the *Summa* that he wrote in a curious blend of Italian regional dialect and Latin, Pacioli introduced his readers to proportions with many examples from the real life.

However, the second aim is the fundamental purpose: the task Pacioli had set himself was to convince the scholars and the learned people that mathematics is the foundation of all the knowledge and arts and that all artists shared the same relevance as intellectuals.

In the figure of the Friar of Sansepolcro converge two aspects that characterize the history of the *Elements* in this period: on the one hand the transmission of the text returned to its integrity, amending errors and correcting or adding figures; on the other hand the diffusion of the Euclidean geometry in non-academic environments, by means vernacular translations [11].

Luca Pacioli embodies both aspects, being curator of a Latin Edition "learned" and tireless populariser of Euclidean geometry.

The humanist thought carried on meta-mathematical character concerned a new role that mathematics acquired within the philosophy of the Platonic and Pythagorean instances. On that the role played by Pacioli, which was in the same time teacher of abacus and *magister theologiae*, was crucial. This job allowed him to mediate the culture of technicians and learned men.

Pacioli's Inaugural Lesson

On 11 August 1508, Pacioli was in Venice, where he read to a large gathering in the Church of San Bartolomeo in Rialto an introduction to Book V of Euclid's *El-*

ements. Pacioli began his talk with the phrase "*Arduarum difficiliumque rerum omnium, Reverendi domini, venerandi patres, excellentissimi Doctores, Magnifici viri, Acutissimi cuiuscumque facultatis studentes, vosque caeteri praestantissimi cives, difficilissima est proportio*" (Of all things difficult and hard, the hardest is the proportion, venerable Sirs, Fathers, eminent Teachers, illustrious Men, each faculty's sharpest students and you others most excellent citizens).

Pacioli allowed that Book V is one of the most difficult in all of the Elements; his purpose is to convince the scholars and the learned people, participant at his lesson, that mathematics is the foundation of all the knowledge and proportion is the foundation of the mathematics.

Pacioli presented himself as skillful theologian in addition to eminent mathematician, and claimed that it is necessary to understand these statements in order to develop the knowledge of all human activities. Indeed, the second sentence in the prolusion states "*Haec est illa quia sola intima altissimae individuaeque trinitatis penetrat et a sacris theologis solertissime investigatur. Haec enim est quae saepius in eorum voluminibus relatio dicitur, aliquando respectus, nonnunquam habitudo*." (This is because it only penetrates the most intimate parts of the highest persons of the Trinity and saint theologians it diligently investigated. We often find it in their books: sometimes they named *relatio*, other times *respectus* or *habitudo*).

The Pacioli's philosophical and educational program consisted in understanding the universe by means of mathematics in general and by means the theory of proportions in particular. Proportion is the structure of the reality: "*Non enim aliud in rebus universis, superioribus scilicet et inferioribus, quam debita earum adinvicem proportio seu habitudo quaeritur*" (Nothing in all things, whatever kind, we look for, except proportion) [4].

In the Lucas' thought, we find the Plato's influence. Logos is the indestructible form of wisdom comprehensible only by the intellect [3, 6].

The reasoning capacity of a human mind is a portion of the all-pervading Divine Logos. Mind is a special gift to humans from God and it has divine essence. Uninitiated minds are unable to apprehend the Existent by itself; they only perceive it through its actions.

Then, Pacioli addressed to audience to ask they reflect on the frequent use of the proportions in all jobs or liberal arts in which they engaged. He reminded all scholars, they knew, that wrote on proportions applied to their branch of knowledge, from Egyptian, Greek, Islamic antiquity. It was not difficult for Luca to expect the audience's interest in his arguments, because he well knew most of them. In the Latin edition of *Elements*, Luca published the text of his prolusion and a short list of the people that attended at St. Bartholomew Church (*Omnes hi sunt qui interfuere in divi Bartholomei aede cum ego Lucas Paciolus Burgensis Sancti Sepulchri ex minoritana Francisci familia Quintum Euclidis profiteri solemniter caepi praefatione hac prius habita M.D.VIII. Augusti die XI*). He estimated that the listeners were about 500 illustrious or learned men (*Aliique plurimi quorum nomina sigillatim referre ad quingentos et amplius operosum nimis foret; florem tantum hominum decerpsi*) [2].

The first on the list is *Clarissimus vir Joannes Lascaris ad senatum Venetum Christanissimi Francorum Regis Orator* (the French ambassador), then we have *vir* *clarissimus Philippus Ferrerius Barchinonensis Catholici Hispaniarum Regis ad eundem Senatum Orator* (the Spanish ambassador). Up to the twentieth place, we find priests, and then we have eleven philosophers and nine doctors. We find also Franciscus and Jacobus fratres Cornelii, sons of the richest man in Venice. We have also Aldus Manutius Romanus, printer and humanist, and the Rompiasi brothers [1].

De Divina Proportione

Few months after the inaugural lesson, Pacioli published in Venice *De Divina Proportione*, in which he recalls the arguments of the *Summa* and the prolusion and he adds that the Universe is written in mathematical language. This sentence must be understood in a very really sense, since the five elements, that explain the nature and complexity of all matter, are made of the regular solids, named for the ancient Greek philosopher Plato (*Thimeus*). Therefore, the reality is understandably and the world is ordered according on clear parameters. The order induces harmony and beauty; the beauty is the right and God generates the right. The Universe is organically organised; the law that organises the Universe is the proportionality. The proportion is not only geometric figure nor quantitative ratio nor whichever ratio but it is the Divine Proportion, because God creates the world. Man can understand the order of the Universe by way of the mathematics and this path is not only scientific but especially ethical way.

De Divina Proportione comprises three independent works. At the beginning, Pacioli places the *Compendium de Divina Proportione*, the book about the Golden Sectio, which he dedicated to Duke Ludovico Sforza of Milan. Pacioli added a small *Tractato de l'architectura*. He states that he has written this work at the request of some "*respectable masons, most diligent friends of sculpture and architecture*", named personally as his disciples. He promises them "*norms and methods of arriving at the desired effect in architecture*" [19].

The role of the *Tractato* is very important and alone justifies the composition of the book. Pacioli's connection with architecture dates from his permanence in Rome, as a guest of Leon Battista Alberti. Later in Urbino, he meets Francesco di Giorgio Martini and Bramante and in Milan (1494–1499), he collaborates with Leonardo da Vinci. The text of *De Divina Proportione* clearly depends on the close collaboration of these Renaissance scholars. The interest of Leonardo in mathematical aspects and his artistic point of view has an important influence on the book. Leonardo himself draws the geometrical illustrations for the manuscript.

The *Tractato* is based on Vitruvius' book [25] and is probably a translation into Italian of Piero della Francesca's *Libellus de quinque corporibus regularibus*. The work begins with a discussion on the proportions of the human body, in which Pacioli inserts the side profile of the head. The human body serves as example for perfect proportions, but also as a concrete model. Pacioli understands the figure of the man in circle and square; a third geometrical form, the equilateral triangle, drawn in a profile of a head, with some gridlines lacking, introduces the illustrations at the end of the book [8].

Letters from Venice

In the *Tractato*, Luca also fits the tables with the construction of the capital letters of the alphabet with the compass and the straightedge. His construction is based on the same square and circle construction that had guided his predecessors, but he inserted some fundamental differences under which he marked the beginning of many researches from calligraphers and printers.

Pacioli is a little less dogmatic of his predecessors, he used the thickness of the strokes to somewhat ease the distortions involved in fitting letters like the N, R and H into a perfectly square scaffold [29]. His letters are calligraphic rather than epigraphic; Pacioli drops the claim of Feliciano and others that the constructions are faithful copies of the Roman Capital letters. We notice for example the unusual choice to make the middle bar of the E shorter than that of others, in contrast to the *scriptura monumentalis* do. Pacioli does not offer complete hints as to how the geometric construction was applied to artistic lettering. Probably Pacioli thinks that geometric construction concept is so clear to readers that he does not need further explanations and he thinks that the type-cutters can follow their eyes and their judgement according to their own taste. The last supposition is supported by the caption with he ends discussion on the perfection of the two O and concludes *you can take which you like, and form from it, as you will find set out in its place* [14].

The stylistic choice of Luca of the proportion of the Corinthian column indicates the choice of the thickness of the letter I that is similar to a small column. The ratio 1:9 was definitively established for the construction of all letters, instead of the classical 1:10. Pacioli does not consider necessary to justify it. Maybe he adopts this proportional ratio pursuant to the tradition of the Byzantine artistic model, establishing the size of the human body height 9 faces, although he suggests to architects the Vitruvian proportions. In the design scheme of the head's profile, the presence of the equilateral triangle is fundamental to the construction of the figure and justifies the choice of the Ternary subdivision [24, 26].

Simple fractions of the square's side determine the linear dimensions of each letter. For example all thick stems of letters (such as the verticals I and H) are to be one-ninth, and all-thin stems (such as the horizontal cross stroke in H) are to be one-third as broad as the thick stroke. In particular, in the letter B, the ternary division is evident for the horizontal side of the square; the vertical side is divided into two parts, whose ratio is 5/4.

The choice of the lowered position of the middle limb of the letter A and the drawing the letter M, with external strokes tilt from the vertical, suggests a relationship with Leonardo's study on human body's Centre of Gravity.

The real importance of the scale of Pacioli is that it gives the considerable advantage of a looser construction, not compelled by necessity of subdivision into 100 component squares as, for example, in the diagrams of some following scholars, like Geoffroy Tory [30].

Pacioli presents two different drawings of the letter O, in both he engages with the construction of ovals. In the Middle Ages masons use build oval form fitting any measure by trial-and-error adjustment, in Renaissance we find the first published oval layouts. Serlio solves the problem of laying out a surbased arch and explains it by an affine transformation of the circumscribed and inscribed circumferences [15, 17, 27].

In *Codex Atlanticus* (Atlantic Codex), folio 318 recto, we see project on weight, illustrated by various diagrams of scales; on the left, the procedure to obtain an oval figure from a circle and, on the right, study on the perspective of a sphere. Leonardo drew ovals by stretching a circle [28].

The caption of the first O is "*that O is perfect*" and the caption of the second O gives the tips for drawing: symmetry, thickness of the protuberances, but in Mardesteig's opinion the captions are swapped [18].

In the first O, Pacioli probably draws two ovals with orthogonal axes, using the method now called "by polycentric curves"; in the second, he draws one oval with ten circles, using the method of the circumscribed and inscribed circumferences.

The letter S is the "manifesto" of the alphabet of Luca Pacioli. The caption claims that the letter is made from eight circles, whose centres are on parallel lines and those below are one—third of one-ninth of the square greater than those above. The protuberance at the middle must be one-ninth of the height; the thin portions a third of the thickness, ending the heads with their ornament.

Since 1550, when Giorgio Vasari wrote a biography of Piero della Francesca, many scholars accused Pacioli of plagiarism and many others defended. In particular, many scholars accused Pacioli of plagiarism against Leonardo about Alphabet. The typographer and editor Frederic Goudy [13] reproduced two tables (C.L. Ricketts collection) from an unpublished manuscript alphabet of Roman capitals, attributed to Leonardo da Vinci and drawn probably about 1480.

Goudy did not believe that "*capitals constructed on any geometrical system of lines, squares, circles and angles possess the spontaneity and variety of the freely drawn capitals that preceded them, nor that these particular letters were other than studies in form and proportions.... They had not be drawn with that wonderful freedom of line for which Leonardo was justly famous*". Documents do not support the ascription to Leonardo, but, even if that were the case, according to some experts of Leonardo, the similarity to those of Pacioli does not exist [31].

Leonardo's notebooks contain no drawings of letters; only one example is in the scrolled motto on the reverse of portrait of Ginevra de' Benci (1474). It is hard to believe that, twenty-two years old, Leonardo could have developed a lettering's theory and, in any case, the similarity is not convincing. The only examples of ovals in *Codex Atlanticus* are drawn by stretching a circle and there are numerous studies for compasses or dividers, in particular compasses to draw ellipses.

A relationship could be seen in the double portrait of Pacioli (1495), but only if you accept attribution and meaning of inscription, both questionable [12].

Conclusion

The Pacioli's alphabetic tables reveal cultural values in which he believed and his philosophy or ideology of the language. Despite the lack of originality, his contributions to mathematics are important, particularly because of the influence that his books had over a long period. Many scholars accused Pacioli of plagiarism. This is an unfair accusation: Pacioli relied heavily on the work of others, but he never claimed the work as his own but acknowledged his sources. The accusations survive to this day also about the Alphabet. Some specialists claimed to have the proof of the authorship of Leonardo about the letters. The typographer and editor Goudy published two tables (C.L. Ricketts collection) attributed to Leonardo by anonymous art's critic, even though in his opinion these letters have not the Leonardo's elegance. In any case, documents do not support the ascription, but, even if that were the case, according to some experts of Leonardo, the similarity to Pacioli's letters does not exist.

Leonardo's notebooks contain no drawings of letters; only one example is in the scrolled motto on the reverse of portrait of Ginevra de' Benci (1474). It is hard to believe that, twenty-two years old, Leonardo could have developed a lettering's theory and, in any case, the similarity is not convincing. Other scholars claim to see a relationship with Leonardo's style in the letters drawn in the double portrait of Pacioli (1495), but only if you are ready to accept his attribution and meaning of inscription, both questionable.

We affirm, as jointly agreed with Goudy, that is very unlikely that Leonardo drew the letters of *De Divina Proportione*, in particular if we consider that its appear only in printed Venetian edition, not in the manuscript. Therefore, we could need to think that Leonardo drew these tables in the period of the collaboration with Luca, when his cultural interests were changed.

We think that the Pacioli's alphabetic tables reveal his cultural skills, so the hypothesis of plagiarism must be rejected for artistic and cultural reasons.

References

1. G. Benzoni, Venezia, 11 agosto 1508: mille orecchie per Luca Pacioli, in *Studi Veneziani (LXIX)* (Fabrizio Serra, Pisa, 2014), pp. 59–325
2. E.M. Black, La prolusione di Luca Pacioli del 1508 nella chiesa di San Bartolomeo e il contesto culturale veneziano, in *La Chiesa di San Bartolomeo e la Comunità Tedesca a Venezia* (Marcianum Press, Venezia, 2011), pp. 87–104
3. J. Cazeaux, De Migratione Abrahami, in *Works of Philo of Alexandria*, vol. 14 (1965), p. 103. Cerf
4. A. Ciocci, Luca Pacioli e l'albero delle proporzioni, in *Pacioli 500 Anni Dopo: Atti Convegno di Studi* (Aboca, SanSapolcro, 2009), pp. 57–95
5. M. Crubellier, T. Hidber, Analogie (1. Philosophisch), in *Der Neue Pauly: Enzyklopädie der Antike*. Alterum, vol. 1 (Cancik, H., Schneider, H., Stuttgart, 1996), pp. 650–651
6. I. Feuer, Quod deterius potiori insidiari soleat, in *Oeuvres de Philon d'Alexandrie* (1965), pp. 75–76. Cerf
7. D.H. Fowler, Anthyphairetic ratio and eudoxus proportion. Arch. Hist. Exact Sci. **24**, 69–72 (1981)
8. M. Frings, The golden section in architectural theory. Nexus Netw. J. **4**(1), 9–17 (2002)
9. G. Galilei, Discorsi e dimostrazioni matematiche intorno a due nuove scienze: Quinta Giornata, in *A Cura di Giusti, E.* (Einaudi, Torino, 1990)

10. V. Gavagna, Euclide a Venezia, in *Pacioli 500 Anni Dopo: Atti Convegno di Studi* (Aboca, SanSapolcro, 2009), pp. 97–123
11. V. Gavagna, La tradizione Euclidea nel Rinascimento (2010). https://www.researchgate.net/publication/296195394_La_tradizione_euclidea_nel_Rinascimento. Accessed January 2010
12. C. Glori, Il misterioso doppio ritratto di Luca Pacioli e dell'allievo e la duplice decrifrazione del cartiglio di Capodimonte, in *Fogli e Parole d'Arte* (2013). http://www.foglidarte.it/il-rinascimento-oggi/299-luca-pacioli.html. Accessed 21 April 2013
13. F.W. Goudy, The Roman Alphabet, its origin and estethic development, in *Ars Typographica II* (1926), pp. 185–208
14. F. Graziadio, De l'origine dele lettere de ogni natione, in *La Costruzione Geometrica di Luca Pacioli*. Fontes (VI), vols. 11–12 (2003), pp. 95–126.
15. S. Huerta, Oval domes: history, geometry and mechanics. Nexus Netw. J. **9**(2), 211–248 (2007)
16. F. Lepori, La Scuola di Rialto dalla fondazione alla metà del Cinquecento, in *Storia della Cultura Veneta dal Primo Quattrocento al Concilio di Trento 3/II* (Neri Pozza, Italy 1980), pp. 539–606
17. A. Lopez Mozo, Ovals for any given proportion in architecture: a layout possibly known in the sixteenth century. Nexus Netw. J. **13**(3), 569–597 (2011)
18. G. Mardesteig, Leon Battista Alberti e la rinascita del carattere lapidario romano nel Quattrocento, in *Italia Medioevale e Umanistica* (Editrice Antenore, Roma, 1959)
19. G. Masotti Biggioggero, Luca Pacioli e la sua divina proportione. Rend. Ist. Lomb. Sci. Lett. (A) **94**, 3–30 (1960)
20. S. Morison, *Fra Luca de Pacioli of Borgo S. Sepolcro* (The Grolier Club, New York, 1933)
21. B. Nardi, La scuola di Rialto e l'Umanesimo Veneziano, in *Umanesimo Europeo e Umanesimo Veneziano* (Sansoni, Firenze, 1960), pp. 93–139
22. L. Pacioli, *De Divina Proportione* (Scriptorium Sassoni, Milano, 1998) (Ristampa Anastatica)
23. P. Palmieri, The obscurity of the equimultiples. Arch. Hist. Exact Sci. **55**, 555–597 (2001)
24. E. Panofsky, *Meaning in the Visual Arts* (Doubleday Anchor Books, New York, 1955)
25. A.M. Piemontese, Vitruvio. Fra gli alfabeti proporzionali arabo e latino. Litt. Caelestes **2**(1) (2007)
26. A. Rossi, *Alphabeto Dignissimo Antico di Luca Pacioli* (Silvana, Milano, 1960)
27. S. Serlio, Sebastiano Serlio on architecture, in *Tutte l'Opere d'Architettura et Prospetiva (1)*, ed. by V. Hart, P. Hicks (Yale University Press, New Haven, 1996)
28. M. Simona, Ovals in Borromini's geometry, in *Mathematics and Culture II. Visual Perfection: Mathematics and Creativity.*, ed. by M. Emmer (Springer, Berlin, 2005), pp. 45–52
29. G. Smedresman, *Alphabetic Architects. Geoffroy Tory and the Renaissance Reconstruction of the Roman Capital Alphabet* (2006). http://www.smedresmania.com/archives/Paper.pdf. Accessed 28 April 2006
30. G. Tory, *Champfleury* (Dover, New York, 1967). Trans. Ives
31. A. Uccelli, Sopra due presunte carte vinciane esistenti nella raccolta C.L. Ricketts di Chicago. Raccolta Vinciana XV **36**, 185–190 (1934)

Dante's Underworld

The Geometry of the Three Realms of the *Divine Comedy*

Riccardo Schiavi

Preface

The *Divine Comedy* is one of the greatest masterpieces in world's poetical literature, and it counts as a collection of the whole culture of an era. It contains notions of philosophy, science, geometry, astronomy, theology, psychology and so on. Therefore, we should not be surprised if we can talk about the *Divine Comedy* in mathematical terms too.

The places described in the Comedy are not only a backdrop for events; they are a fundamental part of the narration. Dante wanted to imagine a perfectly ordered underworld, erected according to God's purposes. Therefore, the geometry, for Dante, is the direct result of God's will.

We shouldn't forget that the real world in which Dante lived is Italy in XIII century, afflicted by civil wars, corruption, and an insecure political situation. Chaos and injustice were everywhere and Dante himself was forced to leave his city, Florence. It is reasonable to expect that, thinking about the afterlife, he imagined a world perfectly symmetric in its shape and directed by God's justice, corresponding to a strict sin-punishment or worth-recompense connection. This is the reason why such a great attention is given to the configuration of the underworld.

On the other hand, we shouldn't give in to temptation of interpreting Dante's considerations in the light of current scientific knowledge; neither to force our deductions to describe the three Realms with an exaggerated precision. Our purpose is not to reconstruct a detailed map of Hell, Purgatory or Heaven, because it would be impossible; what we want is to demonstrate that Dante's imagination deserves a more precise depiction, definitely better than some schematic illustrations shown in school textbooks.

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The Medieval Earth

Before looking at the shapes and dimensions of Hell, Purgatory and Paradise, whose structure are completely products of Poet's imagination, we need to understand the cosmological ideas and beliefs in the XIII century.

According to Aristotle's philosophy and Ptolemy's system, the Earth was thought as the centre of the universe, fixed and spherical, surrounded by the planetary spheres. The spherical shape of the Earth was known since ancient Greece, when Eratosthenes measured its circumference with an incredible accuracy. Nevertheless, in the early Middle Ages, when a primitive form of Christianity imposed a literal reading of the Bible, this idea began to be questioned. Moreover, the ancient texts of Aristotle, Ptolemy and Euclid, containing all the "scientific" results of the Greek world, gradually became rare and difficult to find in Europe. At the end of VIII century, thanks to some open-minded men of culture, the conception of a spherical Earth encircled by celestial spheres was finally restored.

However, those aforementioned ancient texts weren't completely dissolved from the face of the Earth: they were still available in Eastern kingdoms, where the Arabs inherited the legacy of the "scientific" works of the Greeks. In XII century, some Arabic translations of Aristoteles's books and Arabic treatises containing Ptolemy's discoveries were brought to Europe. Therefore, the Western world finally rediscovered classical knowledge after more than eight centuries.

In Dante's time, between the end of XIII century and the beginning of XIV century, almost all the knowledge of cosmology came from Arabic manuscripts. According to what was written in these texts, especially in the works of the Arabian astronomer Alfraganus, Dante believed that the Earth's radius was about 3,247 miles, or 4,870 km.[1] The reason of this underestimation is a conversion mistake between Arabic and Latin units while translating these manuscripts: therefore Europeans had a wrong conception of the size of Earth.

In addition, the distribution of the lands on the surface of our planet was poorly known. Inhabited lands, according to the most widespread idea in that time, occupy only half of the Northern hemisphere, and all the rest of the globe is covered by water. At the centre of the so called *Ecumene* there is Jerusalem, the most important city in Jesus's life, and in the middle of the great ocean, on a little island at the antipodes of Jerusalem, rises up the mount of Purgatory.

Starting from the geographic and cosmological conception at Dante's time, we will try to deduce the structure and the dimensions of the three worlds imagined by the Poet.

Dante's Hell

It is impossible to deduce something about the shape of Hell from the descriptions given in the Bible: the Holy Book only talks about a place full of fire, pain and dark-

[1]This value is reported in Dante's *Convivio* (book III, v.11).

ness. Conversely, Hades in Greek Mythology had a certain structure and a specific collocation in the real world: Hades' doors were located under the Lake Avernus, near Naples, and the whole realm extended underground.

The Christian conception recovered a part of the Classical idea and mixed it with the Jewish tradition. In the early Middle Ages, Hell was represented, in manuscripts and on the mural paintings, as a dark place in which the damned humans are indifferently exposed to the same horrible tortures. At the end of the XIII century, Hell started to be thought as divided in different sections, each dedicated to a particular sin and associated to a specific punishment. Dante's Hell belongs to this contest, but its shape is not simply suggested.

Several authors [6, 9], since the XV century, tried to reconstruct Dante's Hell, with particular attentions to the physical measures of its parts. The most famous among them is Galileo Galilei [5], who commented the results of his predecessors in two lectures for the Florence Academy.

The general shape of Dante's Hell is a conic cavity, its axis passing across Jerusalem and the centre of the Earth. Therefore, Hell's depth is equal to the radius of the globe. How much is the angular aperture of this cone? We could suppose that the dark forest at the beginning of the poem is around the Lake Avernus, where there were Hades' doors according to the description in *Aeneid*. This hypothesis is supported by the fact that Virgil's works were a strong source of inspiration for Dante and it is reasonable to expect that the Poet has wanted to reproduce his guide's original idea. Given this condition, we can establish that the aperture of the cone is about 60 degrees.

To obtain more physical details about the dimensions of this great cavity, let's start from the bottom: Dante talks about the chained giants in the large pit connecting the eighth circle, called *Malebolge*, to the ninth and last one. It is possible to calculate the size of a giant in three different ways, using Poet's detailed descriptions in If. XXXI, 58–66 and If. XXXI, 112–114. Taking the average of these measures we can establish that the height of a giant is about 27 m. Using the size of a giant we can obtain Lucifer's stature through the verses:

Lo 'mperador del doloroso regno
da mezzo 'l petto uscìa fuor de la ghiaccia;
e più con un gigante io mi convegno,

che i giganti non fan con le sue braccia:
vedi oggimai quant'esser dee quel tutto
ch'a così fatta parte si confaccia.[2]

The Emperor of the kingdom dolorous
From his mid-breast forth issued from the ice;
And better with a giant I compare

Than do the giants with those arms of his;
Consider now how great must be that whole,
Which unto such a part conforms itself.[3]

[2] If. XXXIV, 28–33.

[3] Translation (here and below) by Henry Wadsworth Longfellow [1].

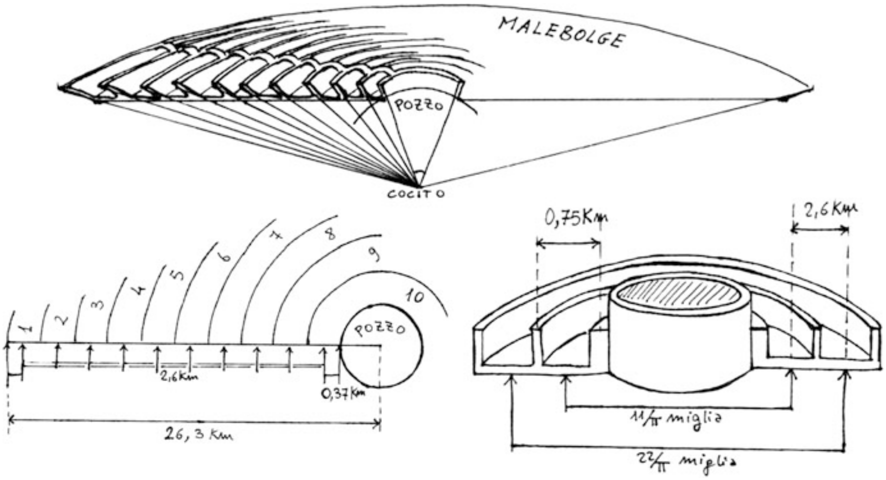

Fig. 1 Shape and dimensions of *Malebolge*, the eighth circle, on the basis of the information contained in *Divine Comedy*

Dante is inviting us to estimate the height of the Beast, according to his comparisons. He said that his own stature (D) can be compared with a giant (G) better than a giant can do with an arm of Lucifer (B_L). In formulas:

$$\frac{D}{G} > \frac{G}{B_L}. \tag{1}$$

If we assume for Dante an average stature ($D = 1.70$ m), and considering that, according to the classical standard of the human body, the length of an arm is one third of the total height ($B_L = L/3$), we can write

$$\frac{D}{G} > \frac{3G}{L}, \tag{2}$$

and hence

$$L > \frac{3G^2}{D} \simeq 1{,}286 \text{ m}. \tag{3}$$

To avoid any risk of exaggeration, we can fix Lucifer's stature at 1,300 m and, using the same proportions, we can infer the diameter of his chest: $d = L/6 \simeq 216$ m.

Lucifer is imprisoned in the frozen lake of Cocytus and the circumference of his chest coincides with the inner edge of the ninth circle, his navel being the centre of the Earth. This is the starting point to define the measures of *Malebolge*, according to the information given by Dante (If. XXX, 86–87; If. XXIX, 8–9) and using some goniometric relations. This analysis would require a longer discussion, but for sake of brevity, I just show the final results in the Fig. 1.

These are all the quantitative data which Dante provides: to go further in our deductions we should use some unsubstantiated assumptions. In any case, to obtain a general idea about the dimensions of other circles we could divide in equal parts the remaining portion of the abyss. Approximately, the upper circles have diameters of thousands of kilometres and a width of hundreds.

Even if a complete and accurate reconstruction of the whole infernal cavity is impossible, this study can show a more accurate map of Hell, closer to Alighieri's imagination.

Dante's Purgatory

Purgatory is a medieval invention. Nothing about this Realm, between Hell and Heaven, is explicitly written in the Bible or in other ancient books. In XIII Century, theologians and philosophers spoke about a place in which the souls can purify themselves from their tendencies to sin, suffering in flames. The configuration of a tall mountain, placed at the antipodes of Jerusalem, is exclusively a Dante's creation.

Nothing about physical dimensions of the seven terraces can be found in the poem, but, using some consideration, we can estimate the height of the whole mountain.

We should start to exclude some very common but wrong ideas about Purgatory. First, it isn't tall as Hell is deep, as if Purgatory were a sort of negative copy of the infernal cavity: in this case Earth wouldn't be actually spherical, but it would have a giant bulge on one side. Second, it is not high up to the orbit of the Moon, because, as Dante surely knew, it would be 33 Earth's radii tall, and such a structure couldn't be called a "mountain" [10].

I believe that Purgatory must be much shorter than an Earth's radius. In fact, if it was too high, its top would be out of Earth's shadow cone: therefore, at the summit it would be never night [2]. Conversely, Dante describes the day-night transition also when he and his guide are at Purgatory's top. With this argument, we can obtain an upper limit on the mountain's height using some goniometric relations and the latitude of Jerusalem: it can't be taller than 4,300 km.

It is possible to obtain a more accurate estimation using the details about Purgatory's formation during the fall of Lucifer on our planet. Dante says (If. XXXIV, 121–126) that Lucifer plummeted on the Southern hemisphere, where once there were all the land masses of the Earth; to avoid the contact with the Beast, these lands sunk under the ocean and emerged on the Northern hemisphere, where they are still now. Lucifer stopped his fall in the centre of the Earth and hence the soil in that region moved South and surfaced to form the Mount of Purgatory.

According to this description, in my opinion, it is evident that the soil forming now the mountain is the same soil which once was occupying the cave around Lucifer's legs (the so called *natural burella*) and the long tunnel connecting the centre of the Earth to Purgatory's island (*cammino ascoso*). Therefore, the volume of the mountain should be equal to the volume of the almost cylindrical

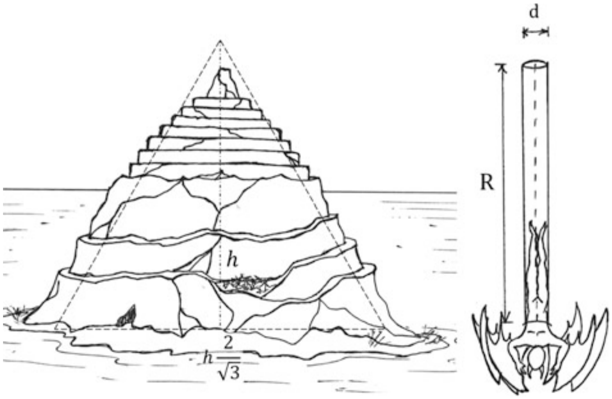

Fig. 2 The two equivalent volumes: the equilateral cone fitting the profile of Purgatory and the so called *cammino ascoso*, an almost cylindrical cavity extending from the centre of the Earth to the Southern hemisphere

cavity extending for an Earth's radius and having the same diameter of Lucifer's chest (Fig. 2):

$$V_{cyl} = R\pi \left(\frac{d}{2}\right)^2. \tag{4}$$

In this study I have assumed a conic shape with equilateral section for the mountain, but this is just an upper limit, because real mountains' profiles are very much wider than tall.

The equivalence of these two volumes allow us to obtain the height of Purgatory:

$$h = \sqrt[3]{\frac{9V_{cone}}{\pi}} \simeq 8 \text{ km}. \tag{5}$$

In this way I have found that the mountain is not taller than about 8 km, comparable with the Mount Everest. This is a reasonable result: Purgatory is very tall, but it is, anyway, just a mountain on the surface of the Earth.

Dante's Paradise

Describing the geometry of Heaven seems to be a superb human blasphemy. However, we cannot forget that the structure of Paradise, in Dante's conception, follows the Aristotelian cosmology. In Ptolemy's studies the radii of planetary spheres are calculated, therefore, even if Dante doesn't give any quantitative information, we can easily imagine the shape of his Paradise. The Earth is surrounded by nine concentric spheres: the first seven are assigned to the motion of the planets, the eighth is the sphere of the fixed stars and the last one, devoid of any celestial body, is the so called *Primum Mobile*, with the only function to transfer the motion from the angelical intelligences to the planetary spheres.

However, a great problem rises up when Dante and his beloved Beatrice cross the *Primum Mobile*, which encloses all the other spheres. What they see, looking outside, is a bright spot encompassed by nine angelical choirs. God appears, then, exactly in front of them, in form of a Point in the endless space of the Empyrean. Classic illustrations of Dante's Paradise show a sort of "flying disk", sometimes surrounded by a flower, outside the *Primum Mobile*. We can easily realize that such a structure cannot have a spherical symmetry, because a point outside a sphere identifies a particular direction in the space. In this framework God doesn't occupy a central position in the universe, He is located indeed in a random point in the Empyrean. Is this coherent with all the effort of Alighieri's imagination to create a perfectly symmetric underworld? I don't think so.

Moreover, to see God in front of them, Dante and Beatrice should have crossed the *Primum Mobile* in a specific point, whereas they wouldn't have been able to see it if they had crossed the last sphere, we say, in the diametrically opposed point.

However, in Dante's verses there are some clues that he actually thought things quite differently:

Le parti sue vivissime ed eccelse
sì uniforme son, ch'i' non so dire
qual Beatrice per loco mi scelse.[4]

Its parts exceeding full of life and lofty
Are all so uniform, I cannot say
Which Beatrice selected for my place.

This means that Dante and Beatrice cross the *Primum Mobile* not in a particular point. But how this could be coherent with the depiction of Heaven we have imagined? Later the Poet says something else which seems not to match the common idea:

E com'io mi rivolsi e furon tocchi
li miei da ciò che pare in quel volume,
quandunque nel suo giro ben s'adocchi,

un punto vidi che raggiava lume
acuto sì, che 'l viso ch'elli affoca
chiuder conviensi per lo forte acume;[5]

And as I turned me round, and mine were touched
By that which is apparent in that volume,
Whenever on its gyre we gaze intent,

A point beheld I, that was raying out
Light so acute, the sight which it enkindles
Must close perforce before such great acuteness.

---

[4]Pa. XXVII, 100–102.

[5]Pa. XXVIII, 13–18.

A shade of the original meaning is maybe lost in the translation; in the original text, anyway, the verse *quandunque nel suo giro ben s'adocchi* can be interpreted as "wherever you look", or, in other words, that God can be seen from any point on the surface of the *Primum Mobile*. This is clearly impossible for ordinary geometry. If you want to see something from any point on the surface of a sphere, this "something" should be a spherical shell encompassing the sphere.

Therefore, the condition expressed in Dante's verses would be satisfied only if God was not a point, but a spherical shell. In this vision all the angelical choirs should be centred on the Earth, and not on God! Is this so absurd?

It seems that Dante wants to instil in us exactly this idea:

Luce e amor d'un cerchio lui comprende,
sì come questo li altri; e quel precinto
colui che 'l cinge solamente intende.[6]

Within a circle light and love embrace it,
Even as this doth the others, and that precinct
He who encircles it alone controls.

Non altrimenti il triunfo che lude
sempre dintorno al punto che mi vinse,
parendo inchiuso da quel ch'elli 'nchiude,
a poco a poco al mio veder si stinse:[7]

Not otherwise the Triumph, which for ever
Plays round about the point that vanquished me,
Seeming enclosed by what itself encloses,
Little by little from my vision faded;

God seems to be enclosed by angelical choirs, but instead He encloses them and the *Primum Mobile* too, as if He were an immense sphere (Fig. 3). How can these two layouts coexist together? In traditional geometry we can't confuse an inner sphere, the enclosed one, with an outer sphere, the enclosing one.

It is clear that the structure of Heaven is more complex than we thought.

The first person who realized that Dante's Paradise couldn't be understood with ordinary geometry was Pavel Florenskij, theologian, philosopher and mathematician, in his work *Imaginary numbers in Geometry* [4]. He, such as other later authors [7, 8], suggested a four-dimensional geometry to describe the shape of the Heaven. Of course Dante couldn't know other types of geometry besides the Euclidean, but it is clear that his intention was to construct a Paradise with a symmetric structure but with a shape unintelligible for the human mind.

Accidentally, Dante was trying to describe a hypersphere, that is a four-dimensional sphere. The whole structure of Paradise begins to make sense if we think in four dimensions, as well as a deformed map of the Earth drawn on a paper is coherent only if we remember its real spherical shape. According to this analogy we can imagine a hypersphere as an ordinary three-dimensional globe: the Earth and

[6] Pa. XXVII, 112–114.

[7] Pa. XXX, 10–14.

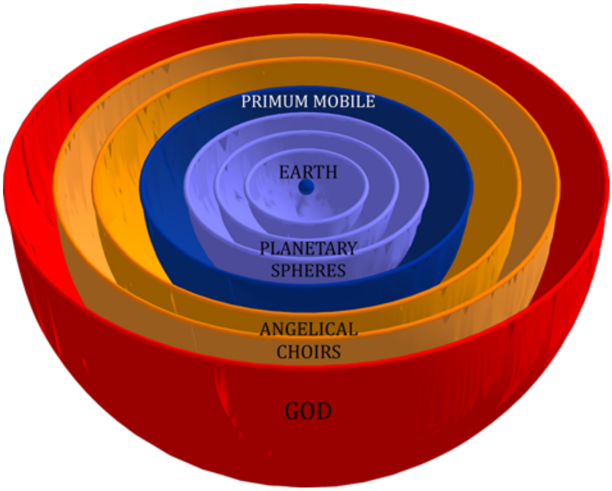

Fig. 3 God would be visible from any point on the surface of *Primum Mobile* only if He were a spherical shell encompassing the Earth and all the other spheres

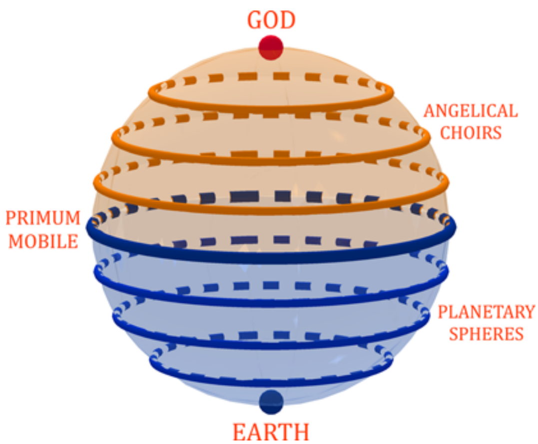

Fig. 4 A representation of the hypersphere in analogy with an ordinary sphere: circles in the illustration represent spheres, all concentric each other. The orange hemisphere is the Empyrean; the blue one is the physical world

God correspond to North and South Poles, the *Primum Mobile* is the analogue of the equator and parallels are the image of planetary spheres on one hand and of the angelical choirs on the other hand. In this beautiful vision, Dante's Paradise recovers a perfect symmetry which otherwise would be lost: God and Earth, with Lucifer inside it, are two distinct centres, wonderfully balancing Good and Evil (Fig. 4).

Evidently, the vision of Heaven changes while Dante crosses it, because of the change of perspective, exactly as a bi-dimensional reproduction of a solid warps if we modify the point of view. We could imagine the three-dimensional shape of Paradise as a stereographic projection of the hypersphere; with the observer positioned in the point of tangency between the hypersphere and the hyperplane.

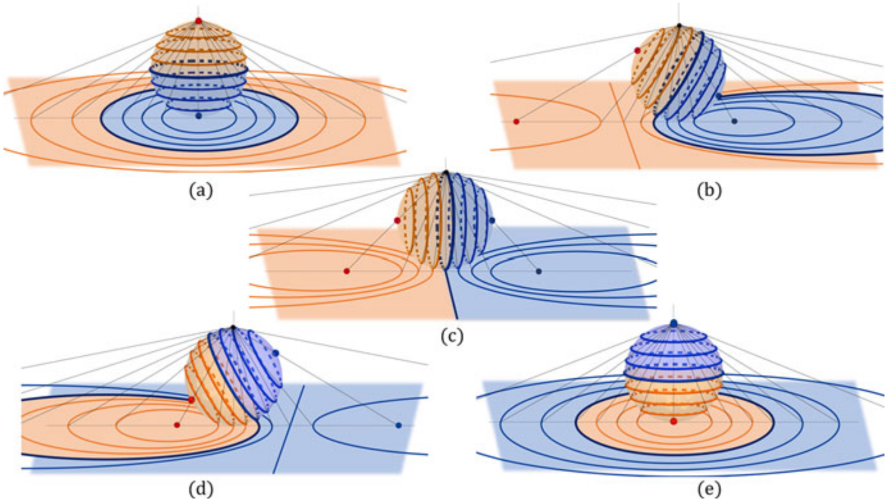

Fig. 5 Stereographic projections of the hypersphere. Curves on the plane represent surfaces in the three-dimensional space. The observer is in the point of tangency between the sphere and the plane. The colours correspond to those in Fig. 4. The explanation of the five different projections is in the text

In this scenario, in Dante's perspective at the beginning of his travel, all the planetary spheres encompass the Earth, and the same applies to angelical choirs and God himself, Who is now a great sphere with an infinite radius, enclosing all the rest. Poetically, we could say that, looking from the Earth, God seems to be infinitely far away (Fig. 5a). While Dante is crossing the planetary spheres, in his perspective some angelical choirs enclose God, and others encompass the Earth; the borderline case is the sphere diametrically opposite to the observer, whose projection is just an infinite plane (Fig. 5b). When Dante is placed on the *Primum Mobile*, all the planetary spheres are centred in Earth, and all the angelical choirs are centred in God. Only from this point of view the whole Paradise is balanced in the three-dimensional space, and this is coherent with the Poet's description (Pa. XXII, 133–153; Pa. XXVIII, 25–36) (Fig. 5c). Crossing the angelical choirs, Dante sees the Empyrean as a mirror image of the physical world: all the choirs are centred in God, and planetary spheres gradually leave the Earth to enclose Him (Fig. 5d). Finally, when Dante reaches God, all the universe is centred on Him, including our planet, which is now an infinite spherical shell (Fig. 5e).

All these assumptions may sound bizarre: obviously Dante did not have suitable mathematical instruments to define such a geometry. We are speaking about "hyperspheres" and "stereographic projections" because it is our current way to express these ideas, but we should never forget that a pure intuition is beyond words. Dante has wanted to depict a Heaven with a complex structure, impossible to be completely understood by the human mind. However, at the same time, he wished to construct an absolutely symmetric universe, governed by the laws of God: mysterious but perfect.

Conclusions

Did Dante, who lived more than seven hundred years ago, really imagine all this?

I think it is absolutely possible. However, the purpose of this study is not to demonstrate that Dante wanted to design his underworld on the basis of strict measures. I don't want to suggest either that Dante had scientific or mathematical knowledges more advanced than those of his time. My first aim is to provide more detailed depictions of Hell, Purgatory and Heaven, closer to the Poet's original idea. These three realms, with their structures and symmetries, are more than simple scenic settings for the poem, they rather are essential parts of Dante's creation. He has built in his mind a believable underworld, perhaps without all the physical dimensions, but coherent and wonderfully well-organized. And when he thought about the kingdom of God, he didn't feel to completely give up the human intellect, so he tried to elevate it to a higher level.

I am aware that this study is located in a hybrid area, between philology and mathematics, often neglected and not taken seriously, but I think it could be very useful for readers and academics to visualize what the Poet wanted to tell us, and it should be combined with the usual literary comments to the *Comedy*.

References

1. D. Alighieri, *The Divine Comedy* (Race Point Publishing, New York, 2015)
2. F. Angelitti, *Discussioni Scientifico-Dantesche su le Stelle Che Cadono e le Stelle Che Salgono, su le Regioni dell'aria, su l'altezza del Purgatorio* (Tipografia Domenico Vena, Palermo, 1901)
3. C. Antonova, Spazio iconico, geometria non euclidea e cultura della visione del mondo, in *Omaggio a Florenskij*, ed. by M. Emmer. Matematica e cultura 2010 (Springer, Milano, 2010), pp. 3–14
4. P.A. Florenskij, *Il Simbolo e la Forma, Scritti di Filosofia della Scienza* (Bollati Boringhieri, Torino, 2007)
5. G. Galilei, *Due Lezioni all'Accademia Fiorentina Circa la Figura, Sito e Grandezza dell'Inferno di Dante* (Sillabe, Livorno, 2011)
6. F. Gregoretti, *Sul Sito Forma e Grandezza dell'Inferno e Sul Sito e Forma del Purgatorio e del Paradiso nella Divina Commedia* (Tipografia del Commercio, Venezia, 1865)
7. R. Osserman, *Poesia dell'Universo* (Longanesi, Milano, 1996)
8. M.A. Peterson, Dante and the 3-Sphere. Am. J. Phys. **47**, 1031–1035 (1979)
9. V. Russo, *Nell'Inferno di Dante, Nuove Osservazioni e Ricerche Con Due Tavole in Litografia per Ricostruire la Valle d'Abisso* (Niccolò Giannotta Editore, Catania, 1893)
10. V. Russo, *Per un Nuovo Disegno del Purgatorio Dantesco* (Tipografia Sicula di Monaco e Mollica, Catania, 1895)

The Simpsons and Their Mathematical Secrets

Simon Singh

Just a Few Words

Michele Emmer

Simone Singh directed his film *Fermat's Last Theorem* for the *Horizon* series of the BBC in 1996. In 1997 he wrote the book by the same title, *Fermat's Last theorem*. In 1997 I was planning the second Venice conference on *Mathematics and Culture*, and as I had the occasion to see the film, I had the idea to invite Simon Singh to the meeting. The conference took place at the end of march 1998 at the *Auditorium Santa Margherita* of the *University of Ca' Foscari*, where I was professor for 7 years. He gave a talk on *L'ultimo teorema di Fermat. Il racconto di scienza del decennio*. The second volume of the Proceedings was published in Italian only by Springer Italia in 1999. The English version of the proceedings started only with the conference of the following year, in 1999. With Simon we became friends almost immediately.

The film *Fermat's Last Theorem* was presented to the *Math Film Festival* in Bologna in 2000 together with a special conference for the *World Year of Mathematics* and the Proceeding were published by Springer in Italian and in English, *Mathematics, Art, Technology and Cinema*, Springer verlag, 2003. John Lynch, producer of the series *Horizon*, came to present the film.

Singh came again in 2001 and he gave a talk on *L'emergere della narrativa scientifica*, in a special session on *Literature* together with Apostolos Doxiadis and Denis Guedj. The Proceedings were published in Italian in 2002. Singh's paper was published in English in the special issue *Mathematics and Culture II. Visual Perfection: Mathematics and Creativity*, Springer verlag, in 2005. This volume was in reality the second part of the volume *The Visual Mind 2: Art and Mathematics* I edited for MIT Press, published in 2004. When I asked authors from all over the world to send me papers for this volume I received so many interesting papers that at the end the number of pages was more than 1200. So I was obliged to divide the original volume in two books, one published by MIT Press the other by Springer.

Simon Singh came again in 2007 and gave the talk *Mettere in scena la matematica* published in Italian in the Proceedings by Springer Italia in *Matematica e cultura 2008*. We made also a video during the conference, 35 minutes, and Simon gave a short interview in it.

And again in 2017, Simon accepted to come once more to speak of his book on mathematics and the Simpsons. He was very kind to give the permission to reproduce the first chapter[1] of the book in the Proceedings.

I want to thank again Simon for his kindness and friendship.

[1]Reproduced with permission from © Simon Singh, 2013, 'The Simpsons and Their Mathematical Secrets', Bloomsbury Publishing Plc.

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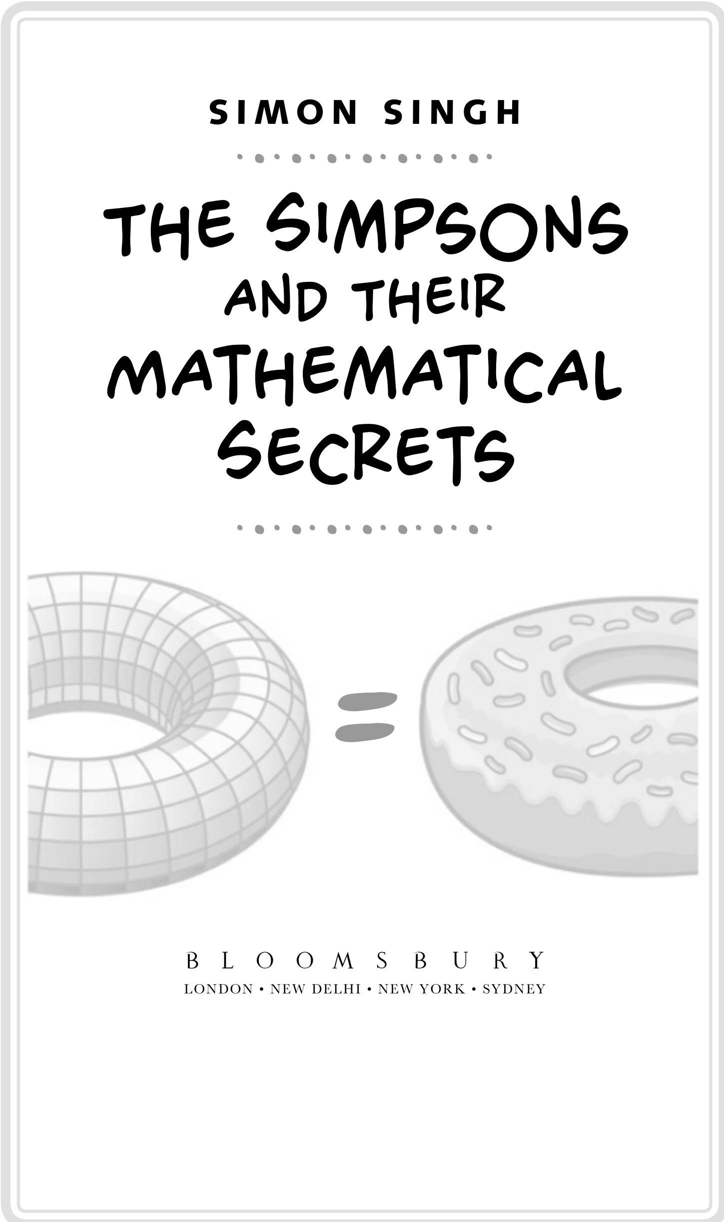

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S. Singh

CHAPTER 1

BART THE GENIUS

In 1985, the cult cartoonist Matt Groening was invited to a meeting with James L. Brooks, a legendary director, producer, and screenwriter who had been responsible for classic television shows such as *The Mary Tyler Moore Show*, *Lou Grant*, and *Taxi*. Just a couple of years earlier, Brooks had also won three Academy Awards as producer, director, and writer of *Terms of Endearment*.

Brooks wanted to talk to Groening about contributing to *The Tracey Ullman Show*, which would go on to become one of the first big hits on the newly formed Fox network. The show consisted of a series of comedy sketches starring the British entertainer Tracey Ullman, and the producers wanted some animated shorts to act as bridges between these sketches. Their first choice for these so-called bumpers was an animated version of Groening's *Life in Hell*, a comic strip featuring a depressed rabbit named Binky.

While sitting in reception, waiting to meet Brooks, Groening considered the offer he was about to receive. It would be his big break, but Groening's gut instinct was to decline the offer, because *Life in Hell* had launched his career and carried him through some tough times. Selling Binky to Fox seemed like a betrayal of the cartoon rabbit. On the other hand, how could he turn down such a huge opportunity? At that moment, outside Brooks's office, Groening realized that the only way to resolve his dilemma would be to create some characters to offer in place of Binky. According to the mythology, he invented the entire concept of *The Simpsons* in a matter of minutes.

Brooks liked the idea, so Groening created dozens of animated shorts starring the members of the Simpson family. These were sprinkled through three seasons of *The Tracey Ullman Show*, with each animation lasting just one or two minutes. Those brief appearances might have marked the beginning and the end of *The Simpsons*, except that the production team began to notice something strange.

Ullman often relied on extraordinary makeup and prosthetics to create her characters. This was problematic, because her performances were filmed in front of a live audience. To keep the audience entertained while Ullman prepared, someone suggested patching together and playing out some of the animations featuring the Simpsons. These animations had already been broadcast, so it was merely an opportunistic recycling of old material. To everyone's surprise, the audiences seemed to enjoy the extended animation sequences as much as the live sketches.

Groening and Brooks began to wonder if the antics of Homer, Marge, and their offspring could possibly sustain a full-length animation, and soon they teamed up with writer Sam Simon to work on a Christmas special. Their hunch was right. "Simpsons Roasting on an Open Fire" was broadcast on December 17, 1989, and was a massive success, both in terms of audience figures and with the critics.

This special was followed one month later by "Bart the Genius." This was the first genuine episode of *The Simpsons*, inasmuch as it premiered the famous trademark opening sequence and included the debut of Bart's notorious catchphrase "Eat my shorts." Most noteworthy of all, "Bart the Genius" contains a serious dose of mathematics. In many ways, this episode set the tone for what was to follow over the next two decades, namely a relentless series of numerical references and nods to geometry that would earn *The Simpsons* a special place in the hearts of mathematicians.

• • •

In hindsight, the mathematical undercurrent in *The Simpsons* was obvious from the start. In the first scene of "Bart the Genius," viewers catch a glimpse of the most famous mathematical equation in the history of science.

The episode begins with a scene in which Maggie is building a tower out of her alphabet blocks. After placing a sixth block on top, she stares at the stack of letters. The doomed-to-be-eternally-one-year-old scratches her head, sucks her pacifier, and admires her creation: EMCSQU. Unable to represent an equals sign and lacking any numbered blocks, this was the closest that Maggie could get to representing Einstein's famous scientific equation $E = mc^2$.

Some would argue that mathematics harnessed for the glory of science is somehow second-class mathematics, but for these purists there are other treats in store as the plot of "Bart the Genius" unfolds.

While Maggie is building $E = mc^2$ with her toy blocks, we also see Homer, Marge, and Lisa playing Scrabble with Bart. He triumphantly places the letters *KWYJIBO* on the board. This word, *kwyjibo*, is not found in any dictionary, so Homer challenges Bart, who gets revenge by defining *kwyjibo* as "a big, dumb, balding North American ape, with no chin . . ."

During this somewhat bad-tempered Scrabble game, Lisa reminds Bart that tomorrow he has an aptitude test at school. So, after the *kwyjibo* fiasco, the story shifts to Springfield Elementary School and Bart's test. The first question that faces him is a classic (and, frankly, rather tedious) mathematics problem. It concerns two trains leaving Sante Fe and Phoenix, each one traveling at different speeds and with different numbers of passengers, who seem to get on and off in odd and confusing groups. Bart is baffled and decides to cheat by stealing the answer sheet belonging to Martin Prince, the class dweeb.

Bart's plan not only works, it works so well that he is whisked into Principal Skinner's office for a meeting with Dr. Pryor, the school psychologist. Thanks to his skulduggery, Bart has a score that indicates an IQ of 216, and Dr. Pryor wonders if he has found a child prodigy. His suspicions are confirmed when he asks Bart if he finds lessons boring and frustrating. Bart gives the expected answer, but for all the wrong reasons.

Dr. Pryor persuades Homer and Marge to enroll Bart at the Enriched Learning Center for Gifted Children, which inevitably turns into a nightmarish experience. During the first lunch break, Bart's classmates show off their intellects by offering him all manner of deals couched in mathematical and scientific terms. One student makes the following offer: "Tell you what, Bart, I'll trade you the weight of a bowling ball on the eighth moon of Jupiter from my lunch for the weight of a feather on the second moon of Neptune from your lunch."

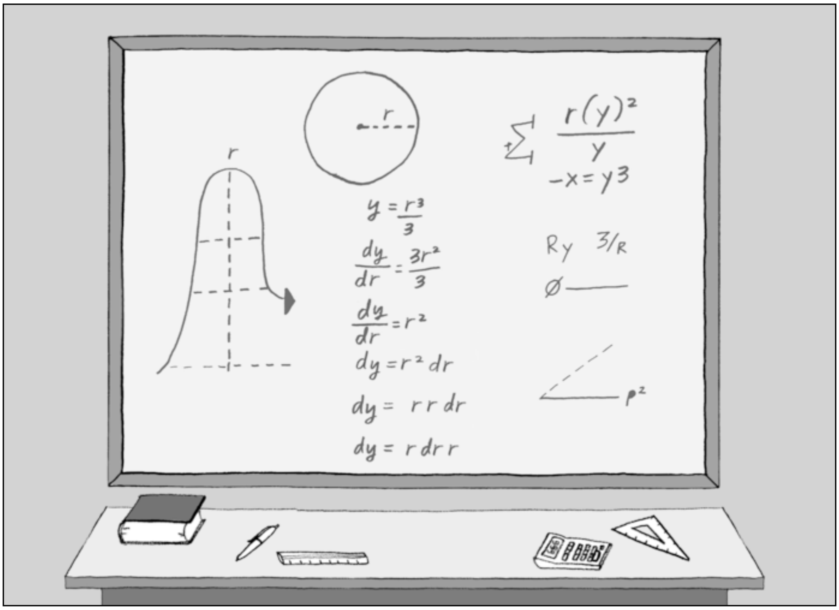
Before Bart can decipher the implications of Neptunian moons and Jovian bowling balls, another student makes a fresh and equally confusing offer: "I'll trade you one thousand picoliters of my milk for four gills of yours." It is yet another pointless puzzle, merely designed to belittle the newbie.

The next day, Bart's mood deteriorates even further when he realizes that the first lesson is mathematics. The teacher gives her students a problem, and it is at this point that we encounter the first example of an overt mathematical joke in *The Simpsons.* While at the board, the teacher writes up an equation and says: "So *y* equals *r* cubed over three, and if you determine the rate of change in this curve correctly, I think you will be pleasantly surprised."

There is a short pause before all the students—except one—work out the answer and begin to laugh. The teacher tries to help Bart amid the guffaws of his classmates by writing a couple of hints on the board. Eventually, she writes down the solution to the problem. Bart is still perplexed, so the teacher turns to him and says: "Don't you get it, Bart? Derivative *dy* equals three *r* squared *dr* over three, or *r* squared *dr*, or *r dr r.*"

The teacher's explanation is displayed in the sketch opposite. However, even with this visual aid, I suspect that you may be as bewildered as Bart, in which case it might help to focus on the final line on the board. This line (*r dr r*) is not only the answer to the problem, but also the supposed punch line. This prompts two questions; why is *r dr r* funny and why is it the answer to the mathematics problem?

The class laughs because *r dr r* sounds like *har-de-har-har*, an expression that has been used to indicate sarcastic laughter in reaction to a bad joke. The *har-de-har-har* phrase was popularized by Jackie Gleason, who played Ralph Kramden in the classic 1950s TV sitcom *The Honeymooners.* Then, in the 1960s, the phrase became even more popular when the Hanna-Barbera animation studio created a cartoon character named Hardy Har Har. This pessimistic hyena with a pork-



When the teacher poses a calculus problem in "Bart the Genius," she unhelpfully uses an unconventional layout and inconsistent notation, and she also makes an error. Nevertheless, she still obtains the correct answer. This sketch reproduces the content of the teacher's board, except that here the calculus problem is laid out more clearly. The important equations are the six lines below the circle.

pie hat starred alongside Lippy the Lion in dozens of animations.

So, the punch line involves a pun based on *r dr r*, but why is this the answer to the mathematics question? The teacher has posed a problem that relates to a notoriously nasty area of mathematics known as calculus. This is a topic that strikes terror into the hearts of many teenagers and triggers nightmarish flashbacks in some older people. As the teacher explains when she sets out the problem, the goal of calculus is to "determine the rate of change" of one quantity, in this case *y*, with respect to another quantity, *r*.

If you have some recollection of the rules of calculus,* then you will be able to follow the logic of the joke fairly easily and arrive at the

* Readers with a rusty knowledge of calculus may need to be reminded of the following general rule: The derivative of $y = r^n$ is $dy/dr = n \times r^{n-1}$. Readers with no knowledge of calculus can be reassured that their blind spot will not hinder their understanding of the rest of the chapter.

correct answer of *r dr r*. If you are one of those who is terrified of calculus or who suffers flashbacks, don't worry, for now is not the time to embark on a long-winded lecture on the nitty-gritty of calculus. Instead, the more pressing issue is why were the writers of *The Simpsons* putting complicated mathematics in their sitcom?

The core team behind the first season of *The Simpsons* consisted of eight of Los Angeles's smartest comedy writers. They were keen to create scripts that included references to sophisticated concepts from all areas of human knowledge, and calculus was particularly high on the agenda because two of the writers were devotees of mathematics. These two nerds were responsible for the *r dr r* joke in particular and deserve credit more generally for making *The Simpsons* a vehicle for mathematical tomfoolery.

The first nerd was Mike Reiss, whom I met when I spent a few days with the writers of *The Simpsons*. Just like Maggie, he displayed his mathematical talents while playing with building blocks as a toddler. He distinctly recalls a moment when he observed that the blocks obeyed a binary law, inasmuch as two of the smallest blocks were the same size as one medium block, while two of the medium blocks were the same size as one large block, and two of the large blocks equaled one very large block.

As soon as he could read, Reiss's mathematical interest matured into a love of puzzles. In particular, he was captivated by the books of Martin Gardner, the twentieth century's greatest recreational mathematician. Gardner's playful approach to puzzles appealed to both young and old, or as one of his friends once put it: "Martin Gardner has turned thousands of children into mathematicians, and thousands of mathematicians into children."

Reiss began with *The Unexpected Hanging and Other Mathematical Diversions* and then spent all his pocket money on other puzzle books by Gardner. At the age of eight, he wrote to Gardner explaining that he was a fan and pointing out a neat observation concerning *palindromic square numbers*, namely that they tend to have an odd number of digits. Palindromic square numbers are simply square numbers that are the same when written back to front, such as 121 (11^2) or 5,221,225

($2{,}285^2$). The eight-year-old was absolutely correct, because there are thirty-five such numbers less than 100 billion, and only one of them—698,896 (836^2)—has an even number of digits.

Reiss reluctantly admitted to me that his letter to Gardner also contained a question. He asked if there was a finite or infinite supply of *prime numbers*. He now looks back on the question with some embarrassment: "I can visualize the letter so perfectly, and that's a real stupid, naïve question."

Most people would consider that Reiss is being rather harsh on his eight-year-old self, because the answer is not at all obvious. The question is based on the fact that each whole number has *divisors*, which are those numbers that will divide into it without any remainder. A prime number is notable because it has no divisors other than 1 and the number itself (so-called trivial divisors). Thus, 13 is a prime number because it has no non-trivial divisors, but 14 is not, because it can be divided by 2 and 7. All numbers are either prime (e.g., 101) or can be broken down into prime divisors (e.g., $102 = 2 \times 3 \times 17$). Between 0 and 100 there are 25 prime numbers, but between 100 and 200 there are only 21 primes, and between 200 and 300 there are only 16 primes, so they certainly seem to become rarer. However, do we eventually run out of primes, or is the list of primes endless?

Gardner was happy to point Reiss toward a proof by the ancient Greek scholar Euclid.* Working in Alexandria around 300 B.C., Euclid was the first mathematician to prove that there existed an infinity of primes. Perversely, he achieved this result by assuming the exact opposite and employing a technique known as *proof by contradiction* or *reductio ad absurdum*. One way to interpret Euclid's approach to the problem is to begin with the following bold assertion:

Assume that the number of primes is finite
and all these primes have been compiled into a list:

$$p_1, p_2, p_3, \ldots, p_n.$$

* Incidentally, and coincidentally, Gardner was living on Euclid Avenue when he replied that Euclid had the answer to Reiss's question.

We can explore the consequences of this statement by multiplying all the primes on the list and then adding 1, which creates a new number: $N = p_1 \times p_2 \times p_3 \times \cdots \times p_n + 1$. This new number N is either a prime number or not a prime number, but either way it contradicts Euclid's initial assertion:

(a) If N is a prime number, then it is missing from the original list. Therefore, the claim to have a complete list is clearly false.

(b) If N is not a prime number, then it must have prime divisors. These divisors must be new primes, because the primes on the original list will leave a remainder of 1 when divided into N. Therefore, again, the claim to have a complete list is clearly false.

In short, Euclid's original assertion is false—his finite list does not contain all the prime numbers. Moreover, any attempt to repair the claim by adding some new prime numbers to the list is doomed to failure, because the entire argument can be repeated to show that the enhanced list of primes is still incomplete. This argument proves that any list of prime numbers is incomplete, which implies that there must be an infinity of primes.

As the years passed, Reiss developed into a very accomplished young mathematician and earned a place on the state of Connecticut mathematics team. At the same time, he was developing a flair for comedy writing and was even receiving some recognition for his talent. For example, when his dentist boasted to him about how he always submitted witty, but unsuccessful, entries for *New York* magazine's weekly humor competition, young Michael trumped him by announcing that he had also entered and been rewarded for his efforts. "I would win it a lot as a kid," said Reiss. "I didn't realize I was competing against professional comedy writers. I found out later all the *Tonight Show* writers would be entering the contest and here I was, aged ten, and I would win it, too."

Mike Reiss (second in the back row) on the 1975 Bristol Eastern High School Mathematics Team. As well as Mr. Kozikowski, who coached the team and appears in the photograph, Reiss had many other mathematical mentors. For example, Reiss's geometry teacher was Mr. Bergstromm. In an episode titled "Lisa's Substitute" (1991), Reiss showed his gratitude by naming Lisa's inspirational substitute teacher Mr. Bergstromm.

When Reiss was offered a place at Harvard University, he had to decide between majoring in mathematics or English. In the end, his desire to be a writer eclipsed his passion for numbers. Nevertheless, his mathematical mind always remained active and he never forgot his first love.

The other gifted mathematician who helped give birth to *The Simpsons* went through a similar set of childhood experiences. Al Jean was born in Detroit in 1961, a year after Mike Reiss. He shared Reiss's love of Martin Gardner's puzzles and was also a mathlete. In 1977, in a Michigan mathematics competition, he tied for third place out of twenty thousand students from across the state. He even attended hothousing summer camps at Lawrence Technological University and the University of Chicago. These camps had been established during the cold war in an effort to create mathematical minds that could rival those emerging from the Soviet network of elite mathematics training programs. As a result of this intense training, Jean was accepted to study mathematics at Harvard when he was only sixteen years old.

Once at Harvard, Jean was torn between his mathematical studies and a newly discovered interest in comedy writing. He was eventually accepted as a member of the *Harvard Lampoon*, the world's longest-running humor magazine, which meant he spent less time thinking about mathematical proofs and more time thinking up jokes.

Reiss was also a writer for the *Harvard Lampoon*, which had become famous across America after it published *Bored of the Rings* in 1969, a parody of Tolkien's classic. This was followed in the 1970s by a live theater show called *Lemmings*, and then a radio show titled *The National Lampoon Radio Hour*. Reiss and Jean forged a friendship and writing partnership at the *Harvard Lampoon*, and it was this college experience that gave them the confidence to start applying for jobs as TV comedy writers when they eventually graduated.

Their big break came when they were hired as writers on *The Tonight Show*, where their innate nerdiness was much appreciated. As well as being an amateur astronomer, host Johnny Carson was a part-time debunker of pseudoscience, who from time to time donated $100,000 to the James Randi Educational Foundation, an organization dedicated to rational thinking. Similarly, when Reiss and Jean left *The Tonight Show* and joined the writing team for *It's Garry Shandling's Show*, they discovered that Shandling himself had majored in electrical engineering at the University of Arizona before dropping out to pursue a career in comedy.

Then, when Reiss and Jean joined the writing team for the first season of *The Simpsons*, they felt that this was the ideal opportunity to express their love of mathematics. *The Simpsons* was not just an entirely new show, but also an entirely new format, namely a prime-time animated sitcom aimed at all ages. The usual rules did not apply,

A photograph of the mathematics team from the 1977 Harrison High School yearbook. The caption identifies Al Jean as the third student in the back row and notes that he won gold and third place in the Michigan state competition. Jean's most influential teacher was the late Professor Arnold Ross, who ran the University of Chicago Summer program.

which perhaps explains why Reiss and Jean were allowed—and indeed encouraged—to nerdify episodes whenever possible.

In the first and second seasons of *The Simpsons*, Reiss and Jean were key members of the writing team, which enabled them to include several significant mathematical references. However, the mathematical heart of *The Simpsons* beat even stronger in the third season and beyond, because the two *Harvard Lampoon* graduates were promoted to the roles of executive producers.

This was a crucial turning point in the mathematical history of *The Simpsons*. From this point onward, not only could Jean and Reiss continue to parachute their own mathematical jokes into the episodes, but they could also begin to recruit other comedy writers with strong mathematical credentials. Over the coming years, *The Simpsons* script-editing sessions would occasionally take on an atmosphere reminiscent of a geometry tutorial or a seminar on number theory, and the resulting shows would contain more mathematical references than any other series in the history of television.

From Algebra to the Secrets of the Universe: The Fascinating Life of Mary Somerville

Elisabetta Strickland

There are many good reasons for investigating the life and work of Mary Somerville (see Fig. 1), the Scottish mathematician who, more than two centuries ago, after a very insignificant childhood, slowly became one of the most prominent scientists in Britain, so well known that when she died in 1872, the obituaries defined her "The Queen of Sciences".

Indeed, the title was appropriate, as although she conducted few scientific investigations herself, Somerville spent a great amount of time of her life exchanging ideas with researchers in the most advanced areas of mathematics, physical science and geography of her time and published four well-reputed books concerning these disciplines. She was commissioned to write them by the Society for the Diffusion of Useful Knowledge, an interesting organization which, during the Victorian age, promoted the production of readable texts on scientific topics for an increasingly literate and educated population, a reflection of the growing focus on science which occurred in the nineteenth century. Yet her success as an author, as a mathematician and an astronomer, was only possible because of the stubborn way in which she pursued her studies, ignoring her lack of academic background and giving a successful example of self-taught education.

Her father, Admiral William Fairfax, began working as a midshipman at the age of ten and was never formally educated, while her mother, a daughter of the Scottish Solicitor of Customs, used to read only the Bible or newspapers. Since her father was constantly employed at sea, her education was in the hands of her mother, who was quite indulgent and only taught her the catechism of the Kirk of Scotland and how to read the Bible, plus some domestic cores such as how to preserve fruit, shell peas and beans, feed poultry, care for the family's cow.

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Fig. 1 Portrait of Mary Somerville, oil on canvas, by James Swinton (1816–1888), Somerville College, Oxford, UK. (Permission granted by the Librarian and Archivist of the Somerville College)

She was raised in Burntisland, a small seaport of the coast of Fifeshire in Scotland and, by her own admission, at nine years old she was "a wild creature", who spent her time roaming the countryside and seashore near her home, observing sea creatures and birds, collecting things and learning the name of the plants around her house. Throughout her life, the associations acquired during her childhood were the strongest, solitude and her intense interaction with nature were the hallmarks not only of her early days, but a deep investment for her future years as a scientist. At night, the stars she could see from her window held equal fascination.

But when she was ten years old, this carefree existence came to an end, as her father returned from a long voyage and discovered that her reading skills were minimal and she couldn't write.

So Mary was sent to a school run by a Miss Primrose, were the teaching techniques were closed to torture. After one year at the school, she returned home and continued her wandering existence, but at least she had increased reading skills that allowed her to enjoy a small number of books in their home.

Mary's interest in mathematics had an outburst during a party: she was paging through a fashion magazine and came across a puzzle. When she looked at the answer, it had x's and y's in the solution. Curious, she asked a friend who told her that it was something called algebra, but she couldn't tell her what it was. She decided to study the subject and in order to get the required books she conspired with her brother's tutor who gave her what she needed, Bonnycastle's *Algebra*, which at that time was in use to teach mathematics in schools. Mary was from that point on her way, as each night after the rest of the household retired, she would study mathematics by candlelight. After some time, the candle supply started to diminish and her mother discovered her peculiar habits, so she couldn't use anymore the night hours to satisfy her interest in science. In general people during that time believed that women's minds couldn't handle intellectual pursuits, as they could drive them crazy. Mary recalls in her autobiography [7] her father saying to her mother: "Peg, we must put a stop to this, or we shall have Mary in a straight-jacket one of these days!".

But the family couldn't ignore for ever her attitude for studying, so she was allowed to move to the house of her grandfather in Edinburgh and there Mary turned into a charming and elegant young lady, without neglecting her studies, as she concentrated on understanding Euclid in addition to practicing the piano, painting and following cooking lessons.

Her family's dislike for her serious studies forced her to marry her cousin Samuel Greig at the age of twenty-four, and her pursuit of mathematics became even more difficult, for she had her own house and soon two sons to look after and her husband shared her family's low opinion of her passion for science. Luckily her marriage was not a long one, as Greig died in 1807, so she gained economic independence and she could go back to her studies. In order to do this properly, she consulted with mathematician William Wallace of the University of Edinburgh and assembled a library of all important works in mathematics and astronomy at that time: the treatises of Newton, Poisson, Euler, Lagrange, Monge, plus the volumes of Laplace's *Mécanique Céleste* and many more. She was not discouraged and, as all these books were written in French, she had to teach herself to read in that language in order to understand their contents.

Her family was deeply worried by her interest in differential calculus, which seemed to them at least eccentric. So in 1812 Somerville was convinced to marry another cousin, William Somerville: he was totally different from Greig, being a handsome and learned army surgeon who was supportive of her studies and defended her against criticism from her family. Together, William and Mary studied geology, collected minerals, travelled and began a family. After a period in Edinburgh, they moved to London in fashionable Hanover Square in Mayfair, where Mary became acquainted with a new set of learned friends, such as the astronomer Sir William Herschel and his son John, whose twenty-foot telescopes she had the chance to see in their house in Slough, twenty miles from London. He had used that telescope for nine years to carry out sky surveys and investigate double stars. And some years later it was from their telescope that she could observe the "glorious appearance of Jupiter" for the first time.

These personal contacts were invaluable in building her career and reputation, today we would say that she was an excellent public relations woman for herself.

During her first trip to Paris, in 1817, she had the privilege of being introduced to Pierre Simon Laplace, who acknowledged and assimilated her into the scientific elite long before she had published anything of lasting significance.

In 1826 Somerville published a study on magnetism in the Philosophical Transactions of the Royal Society of London entitled *On the Magnetizing Power of the More Refrangible Solar Rays*, [1].

Her experimental setup included, of all things, a needle, which she used for totally different purposes than those intended by her Aunt Janet who wanted to transform her in a lady in charge of the linen of her household. Her paper earned her admission into the Royal Astronomical Society in 1835, first woman together with the astronomer Caroline Herschel.

Mary had also a fruitful friendship with another woman in science, Ada Lovelace, the only daughter of Lord Byron and the only person at that time who understood the importance of Charles Babbage early mechanical computer, the so called "analytical engine": for this reason she is regarded as the first computer programmer.

After Somerville published her study on magnetism, her husband received a letter from Lord Brougham, founder of the Society of Useful Knowledge, and at that time member of the House of Commons as Whig Lord High Chancellor. He commissioned her an account of Laplace's *Mécanique Céleste*.

Somerville spent much of the next four to five years (1827–1831) writing the commissioned book. She had studied the *Mécanique Céleste* in detail, but remained self-conscious about her understanding, which she believed could be considered limited, as it didn't take place in an academic setting.

So she extracted a promise from Lord Brougham and her husband that if her work wasn't sufficient, it would be burned. But inside herself she felt she should try, as she believed that the middle class sought to change society and secure their positions through the promotion of science. At the same time she wanted to develop her own abilities and demonstrate that women could succeed in any field of culture.

She organized her work and she began to write for hours, taking care at the same time of the education of her children, visiting with friends and looking after the household. When it was complete, her manuscript, *The Mechanism of the Heavens*, [2], was sent to her friend Sir John Herschel, son of William, who praised it greatly. His only suggestion was that she explain the calculus at the beginning more thoroughly to accommodate readers not as proficient in mathematics as she was.

So she wrote a *Preliminary Dissertation*, [3], which became extremely popular, especially among students at Cambridge. Maria Edgeworth, the Irish novelist and writer of moral tales for children, wrote in a letter to Mary:

"I was long in the same state of a boa constrictor after a full meal... my mind was so distended by the magnitude, the immensity of what you put into it!".

Her work earned her the Victoria Medal and an encouraging review [11] from William Whewell, President of Trinity College in Cambridge.

In 1835 a letter arrived from Sir Robert Peel, a British statesman, who had advised the Crown to grant a civil pension of 200 pounds a year for Mary Somerville in recognition of her eminence in science and literature and the Prime Minister Lord John Russell added 100 pounds a year to the pension, thereby coming to the rescue of the brilliant scientist's battered fortune, almost completely lost through mismanagement.

She continued writing after this initial success, claiming she couldn't stand to be idle, and published *On the Connexion of the Physical Sciences* [4] in 1846, which became famous not only for its contents, but also because it appears that one of her observations, noting that perturbations in the orbit of Uranus might be indicating the existence of an as yet unknown planet, led the astronomer John Couch Adams to his discovery of the planet Neptune; *Physical Geography* in 1848 [5] and *On Molecular and Microscopic Science* [6] in 1869.

Somerville's extensive writing earned her the respect of the scientific community in England and also in France, where she travelled twice.

During her first Grand Tour with her husband in 1817, she had visited also Italy, and fell in love with Rome. Later, at the beginning of the forties, when her husband was forced to retire for health reasons and compelled to move to a warmer climate, they both settled in Rome with their two daughters.

There she concentrated on her book *Physical Geography*, not only because she was deeply interested in the subject, but also because she was under increasing pressure to write, as her husband, now retired, had half of his former pay. Her Italian life became nomadic: after Rome she spent a period in Florence, were she obtained the permission from the Grand Duke of Tuscany, Leopoldo II, to borrow the books from his private library in the Pitti Palace. The Duke was delighted by these visits from Mary Somerville, as he found someone who could get interested in his work on the drainage of the Maremma, the Tuscan marches.

After Florence it was the turn of Siena, were they were provided with a nice apartment decorated with frescos. But Rome was in her heart, so she moved back to the eternal city were they settled in Palazzo Lepri in Via dei Condotti. Her arrival was greeted by a terrible flooding of the streets caused by the Tiber which burst its banks, but when the situation went back to normal, the Somervilles went on with their usual lives, inviting illustrious visitors and exchanging ideas with all the scientists in town, such as Padre Francesco de Vico, a Jesuit and astronomer, director of the Observatory of the Collegio Romano, who had discovered six comets.

Rome as usual was very hot in the summer, so the family escaped on the hills near Rome, in a villa in Albano, were Mary could write in peace and paint her beautiful landscapes, which are now kept in the Somerville College in Oxford.

Among her favorite friends in Rome, Mary appreciated Don Michelangelo Caetani, Duke of Sermoneta, who loved to invite at his beautiful Palazzo artists and writers. Mary's daughters had the privilege of learning to draw in the studio of John Gibson, a Welsh neoclassical sculptor, who was living in Rome in order to study Canova.

The summers of those years were spent in Perugia and after in Venice, where the Somervilles enjoyed the hospitality of Countess Mocenigo, whose family gave six doges to Venice, writing and painting as usual, while their daughters went to the Academy of Belle Arti for painting lessons.

Mary sent regular reports about her visits to the Italian cities she visited to her son from her previous marriage, Woronzow Greig. The other son had died in infancy and she also lost a daughter, Margaret, in her second marriage, both for infectious diseases.

The letter concerning Venice is among the most charming ones [7]; as a matter of fact, both Mary and William were in very good health, the terrible headaches which tormented her in London had disappeared, and William had recovered from the infectious disease which compelled him to leave his country.

When they went back to Rome, Mary started to carry out experiments to investigate the effect of the solar spectrum on the juices of plants and Sir John Herschel was so enthusiastic about her discoveries, that convinced her to publish her observations in another scientific paper in the Philosophical Transactions of the Royal Society.

While Mary was concentrating on her studies, the situation in the Papal State was in turmoil. Giuseppe Mazzini was working hard towards the unification of Italy, but the Pope, at that time Gregory XVI, didn't agree with the project.

As Mary at that point had finished her book on *Physical Geography*, the family decided to make a trip to London; William was now in good health and this interruption of their Italian lives seemed possible.

But her arrival in England was not a lucky one, as Mary was ill for a while and moreover she discovered that a book similar to her own had just been published under the name of *Kosmos*, by Alexander von Humboldt, a Prussian geographer, naturalist and explorer. Mary was deeply upset by this circumstance and almost threw the manuscript on the fire, but again her friend John Herschel convinced her to print it. When the book came out, Humboldt wrote to her stating that he admired her work.

After this successful outcome, Mary, William and their two daughters went back to Italy and found themselves in the middle of the War of Independence in 1848. They travelled to Turin and there Mary could talk with Baron Giovanni Plana, director of the Observatory, and remembered as a mathematician for the Abel-Plana formula in analysis. Baron Plana found a lovely apartment for the illustrious family in Casa Cavour, which belonged to the brothers Camillo, Minister of the Interior, and Gustavo.

Mary was very fond of the Italian statesman, who was a pleasant companion and they became good friends. She suffered later when he died, leaving people in deep mourning.

After Turin the family moved to Florence where they settled in a nice house in Via del Mandorlo.

Mary could entertain her guests in a beautiful garden, and so she could meet Frances Power Cobbe, a writer who was fighting against vivisection, and the British poetess Elizabeth Browning, who, like William Somerville, was suffering of bad health and needed a warm climate.

Living in Florence during those years meant that Mary witnessed the effects of the war of independence against the Austrians and she actually saw with her own eyes the Imperial family leaving Florence for good, escorted by the Austrian Minister and a part of the diplomatic corps [7].

There was luckily no blood or disorder and the people could walk around as usual, while Bettino Ricasoli, the so called "Iron baron", earned sufficient support to form a government and decide to annex Tuscany to the Kingdom of Italy. Of course the atmosphere in Florence was tense and explosions could be heard all the time, but Mary Somerville was not worried, she was playing the role of a reporter, rather than a scientific writer, as she noted some amusing details about French troops camped at Le Cascine, where she and her daughters saw a soldier skin a rat and put it in the soup-kettle [10].

The Somervilles were invited to watch King Victor Emmanuel's entry into Florence from the balcony of the Corsini Palace, a colourful event which she never forgot.

But fate was again preparing a severe blow on Mary, as on June 26, 1860, William Somerville passed away for another infectious disease. In those times no medicine had yet been found to fight dangerous bacteria, so nothing could be done to save his life.

The loss of her husband was a real tragedy; their marriage had been a happy one, as he was a congenial companion, who had supported and admired her during all the years they has spent together, allowing her to prove her extraordinary capacities, quite unusual for a woman of the Victorian era.

After this sad event, Mary's health was not so good and she left Florence for La Spezia, where she desperately tried to cope with her grief in all possible ways; she even came to the point of hiring a boat at night and rowing away from the shore in order to observe a beautiful comet which had appeared in the sky exactly on the day when she arrived in La Spezia.

At that point she could have decided to go back to London for good, since the main reason for her years in Italy, William's health, was now irrelevant. But she and her two daughters, Martha and Marie Charlotte, loved the country so much that she decided to stay and start a new book, *On molecular and Microscopic Sciences*: her age at that time was almost eighty!

A positive effect on her in that period was produced by her friendship with Lord William Acton, who had been the Minister of Marine and Admiral in the service of the Italian Navy for two years and in La Spezia was director general of the arsenal. He had a deep interest in natural history and therefore he represented a perfect interlocutor for the inexhaustible Scottish lady.

During her staying in La Spezia, in the south of Italy things were moving fast towards unification. Giuseppe Garibaldi had become a legend with his strategic expedition of one thousand volunteers, who landed in Sicily and conquered the Kingdom of the Two Sicilies, ruled by the Bourbons. In 1862, after the battle of Aspromonte which stopped Garibaldi from marching on Rome and the Papal States, the hero was imprisoned by the regular troops and sent to the fort Varignano in La Spezia. His leg was severely injured and he was in a state of great suffering, until a French surgeon saved him from amputation, by extracting a bullet from his ankle. People were asking to visit him in order to express their gratitude for his enterprises, but Mary decided to skip this, she admired him enormously but didn't want to disturb him [10]. So she never actually met Garibaldi, notwithstanding her deep understanding of his actions and his presence in her same city.

Her new book was now finished and she sent it to her editor John Murray who had some doubts due to her advanced age, so he turned to John Herschel for advice, who convinced him to publish it and it was another great success: she took ten years to write it, between the ages of seventy-nine and eighty nine, not an easy task. When the book came out in print, she was living in Naples, the final destination of her long life. Despite her old age and her shaking hands, she started to write her biography, entitled *Personal Recollections* [7], which was published by her daughter Martha after her death. Her handwritten notes are kept in two of the main collections in the Bodleian Library of Somerville College at Oxford [9].

Her friend Lord Acton introduced her in Naples to some of the leading Italian scientists in town, among whom she appreciated the astronomer and mathematician Annibale de Gasparis, who was looking after the Observatory of Capodimonte and had discovered nine asteroids.

After a while Mary became acquainted with all the Neapolitan intelligentsia and she was elected honorary member of the Accademia Pontaniana. She also had the chance to engage in scientific conversations with John Phillips, professor of geology at Oxford University, devoting their time to the geological aspects of the activity of the Vesuvius, which had been intriguing her since the first visit to Naples in 1817. She gave beautiful reports of the eruptions of the volcano in her *Recollections*, as during her last years in Naples some really dangerous ones took place, threatening the villages when the lava pushed its way downs its slopes.

Some interesting news arrived from Britain while she was in Naples regarding issues that were close to her heart.

One was certainly represented by the battle that John Stuart Mill, English philosopher and political economist, was conducting to show the iniquity of British laws regarding women, who were not allowed to vote. He presented a petition to the British parliament for the extension of suffrage to women, and Mary was very glad to be the first one to sign it, even if she did not agitate for woman rights: she just took them!

She received a letter in July 1869 from John Stuart Mill praising her "inestimable service to the cause of women by affording in her own person so high an example of their intellectual capabilities" [7].

Mary was obviously in favor of a proper education for women, something she had never had herself and from which she suffered most extensively. Her enormous effort to become a self-taught scientist convinced in that same year to decide to leave to the newly founded Ladies's College at Girton her scientific library, which obviously represented a treasure for all the women who would enter in that institution.

Moreover she was deeply interested in the theory of evolution to explain biological change developed by the English biologist and naturalist Charles Darwin. She had read his book *On the origin of species by means of natural selection* and she enjoyed it greatly, because of her interest in nature, even if she didn't agree with his theory. One of the main reasons why she liked his work was that Darwin devoted much study to birds, her favorite living creatures. And in any case she believed Darwin to be a man of genius who had contributed a great deal to knowledge of the natural world and its history. She was deeply in favor of the protection of animals and gave her support to the antivivisection movement.

One sad thing that happened when she was in Naples was the loss of her dear and affectionate friend John Herschel, though twelve years younger than her. Mary after his death realized that few of her friends remained and she was nearly left alone.

Strangely enough, one of the last pieces of news to reach her was that foreign troops had broken into Laplace's house in Arcueil, near Paris, in the Val-de-Marne, and that his original handwritten version of *Mècanique Cèleste* had been thrown into the river. Luckily though, someone had rescued the book that had started her career in science.

She kept studying mathematics to the very last day of her life, as she passed away in her sleep on the morning of November 1872, after devoting her last evening to the study of the "quaternions", a number system which extends the complex numbers. She indeed said to her daughter Martha that she had the same readiness and facility in comprehending the formulae which she possessed when she was young.

She was buried in the English Cemetery in Naples, after her friend Frances Power Cobbe made an effort to have her buried in Westminster Abbey in London: but the Astronomer Royal didn't want to sign the authorization to proceed, claiming that he had never read Somerville's books!

After her death, the only members of her family who were left were her daughters Martha and Charlotte, who had no family of their own. Maybe this was the effect of thirty years of nomadic life, but she couldn't change her living style and in any case the education of the two ladies was complete, they were cultured, happy and independent. Mary Somerville maybe could have tried harder to let them follow a path of their own.

In any case, after the death of her mother, Martha decide to accomplish her mother's wish to tell the truth about her life by publishing her *Recollections*. However, Martha didn't use everything her mother had written, but preferred to use only those writings which gave the best possible image of her, avoiding whatever represented a possible damage to a perfect image.

The book came out in 1873, one year after Mary's death. Much later, in 2001, Dorothy McMillan of the University of Glasgow re-edited Mary's Personal Recollections, [8], using the already mentioned manuscripts held in the Somerville Collection in the Bodleian Library in Oxford [9].

Surprisingly this edition doesn't mention the Italian period, which is rather an unforgivable fact, since the last hundred pages of her autobiography are all devoted to Italy and point out in a charming way how much this country meant to her.

In any case her memoir is a very good source of information on her life, so it's really important to read it in order to understand her polyhedric personality and capture the magic of her thoughts.

Those who knew her more superficially often found it difficult to match the height of her intellectual achievements with her rather simple style of conversing and conducting herself, but this features of her character were the reason of her skill in recreating the experience of the scientific sublime for her readers, providing them with a great deal of pleasure.

In a certain way she was a woman of paradoxes, a female intellectual but not a suffragette, a beauty and a charmer hiding controlled pragmatism and mental clarity, plus an extreme precision of thought.

The number of honors she received in her life was outstanding; she became member of all the important academies of her times even in Italy, such as the Accademia di Scienze Naturali in Florence, the Accademia dei Risorgenti in Rome, the Accademia di Scienze, Letteratura e Arti in Arezzo and the Accademia Pontaniana in Naples; this facts were quite unusual at that time for a woman and prove the esteem in which her contemporaries held her and the eminence she achieved in her own way.

Moreover the Somerville College in Oxford was named after her in 1879; initially it was intended for female students (Indira Gandhi, Margaret Thatcher and Dorothy Hodgkin, the British chemist who won the Nobel Prize in 1964, studied there), but in 1992, after extended debate, it ceased to be a women's college and began to admit men [10].

We would be not far from truth if we say that tracing the life and the work of Mary Somerville, who was barely taught to read as a child, but did succeed in learning and mastering science by herself, becoming one of the most outstanding British women scientists and furthermore a popular writer, gives a wonderful example of how some women succeeded in developing their own abilities, demonstrating that they are capable of assuming a higher place in the intellectual world than that usually assigned to them.

References

1. M. Somerville, On the magnetizing power of the more refrangible solar rays. Philos. Trans. R. Soc. Lond. **116**, 132–139 (1826)
2. M. Somerville, *Mechanism of the Heavens* (Murray, London, 1831)
3. M. Somerville, *A Preliminary Dissertation on the Mechanism of the Heavens* (Murray, London, 1832)
4. M. Somerville, *On the Connexion of the Physical Sciences* (Murray, London, 1834)
5. M. Somerville, *Physical Geography* (Murray, London, 1848)
6. M. Somerville, *On Molecular and Microscopic Science* (Murray, London, 1869)
7. M. Somerville, *Personal Recollections from Early Life to Old Age of Mary Somerville, with Selections from Her Correspondence, by Her Daughter Martha Somerville* (Roberts, Boston, 1873)
8. M. Somerville, *Queen of Science. Personal Recollections of Mary Somerville, Edited and Introduced by Dorothy McMillan* (Canongate Classics, Edinburgh, 2001)
9. Somerville Collection. Bodleian Library. Somerville College, Oxford
10. E. Strickland, *The Ascent of Mary Somerville in 19th Century Society*. Springer Biographies (Springer, Berlin, 2016).
11. W. Whewell, Review of "On the connexion of the physical sciences". Q. Rev. **51**, 54–68 (1834)

Imagine Math[1]

1. M. Emmer (ed.). *Imagine Math, Between Culture and Mathematics* (Springer, Milan, 2012)
2. M. Emmer (ed.). *Imagine Math 2, Between Culture and Mathematics* (Springer, Milan, 2013)
3. M. Emmer (ed.), *Imagine Math 3, Between Culture and Mathematics* (Springer, Switzerland, 2015)
4. M. Emmer, M. Abate, M. Villarreal (eds.), *Imagine Maths 4, Between Culture and Mathematics* (UMI & IVSLA, Bologna & Venice, 2015)
5. M. Emmer, M. Abate, M. Falcone, M. Villarreal (eds.), *Imagine Maths 5, Between Culture and Mathematics* (UMI & IVSLA, Bologna & Venice, 2016)
6. M. Emmer, M. Abate (eds.), *Imagine Math 6, Between Culture and Mathematics* (Springer, Switzerland, 2018)

[1]List of published books (all volumes are available on Amazon) as of 2012.

M. Emmer, M. Abate (eds.), *Imagine Math 6*,
https://doi.org/10.1007/978-3-319-93949-0