Additive Schwarz with Vertex Based Adaptive Coarse Space for Multiscale Problems in 3D

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1 Introduction

The choice of coarse spaces play an important role in the design of fast and robust Schwarz methods for problems of multiscale nature. Standard methods with standard coarse spaces have often difficulties to solve such problems, and even fail to converge due to computing in the finite precision arithmetic. The purpose of this paper is to propose a robust coarse space, adaptively enriched, for solving second order elliptic problems in three dimensions with highly varying coefficients, using the standard finite element for the discretization and the overlapping additive Schwarz method as the preconditioner. The coefficient may have discontinuities both inside and across subdomains. The convergence of the proposed method, as presented in the paper, is independent of the distribution of the coefficient, as well as the jumps in the coefficients, when the coarse space is chosen large enough. For similar works on domain decomposition methods addressing such problems, we refer to [7, 17] and the references therein.

Additive Schwarz methods for solving elliptic problems discretized by the finite element, which was proposed over 30 years ago, have been studied extensively over the past decades, see [16, 18] for an overview. It is known in general that if the coefficients are discontinuous across subdomains but are varying moderately with in each subdomain, then the standard coarse spaces are enough to generate additive

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P. E. Bjørstad et al. (eds.), *Domain Decomposition Methods in Science and Engineering XXIV*, Lecture Notes in Computational Science and Engineering 125, https://doi.org/10.1007/978-3-319-93873-8_45

Schwarz methods which are robust with respect to those jumps, cf. e.g. [16, 18]. This is however not true in the case when the coefficients may be highly varying and discontinuous almost everywhere, the fact which has in recent years drawn several researchers' attraction, cf. e.g. [2–15, 17].

In the present work, we extend some of the ideas presented in those papers, and propose to construct a coarse space based on the vertices of the subdomains and a twofold enrichment of the coarse space, which is done through solving two specially designed lower dimensional eigenvalue problems, one on each face common to two neighboring subdomains and one on each interior edge of the subdomains, and chosing the first few eigenfunctions corresponding to the bad eigenmodes. The analysis show that the condition number bound of the resulting system depends only on the threshold used to choose the bad eigenvalues.

The remainder of the paper is organized as follows: in Sect. 2 we introduce our differential problem, and its finite element discretization. In Sect. 3 a classical overlapping Additive Schwarz method is presented. Section 4 is devoted to the construction of our adaptive coarse space and Sect. 5 gives the theoretical bound for the condition number of the resulting system.

2 Discrete Problem

We consider the following elliptic boundary value problem: Find $u^* \in H_0^1(\Omega)$

$$\int_\Omega \alpha(x)\nabla u^* \nabla v \, dx = \int_\Omega f v \, dx, \qquad \forall v \in H_0^1(\Omega), \tag{1}$$

where $\alpha(x) \geq \alpha_0 > 0$ is the coefficient, Ω is a polyhedral domain in $\mathbb{R}^3$ and $f \in L^2(\Omega)$. Let $\mathcal{T}_h$ be the quasi-uniform triangulation of Ω consisting of closed tetrahedra such that $\bar{\Omega} = \bigcup_{K\in\mathcal{T}_h} K$. Let h_K denote the diameter of K, and $h = \max_{K\in\mathcal{T}_h} h_K$ the mesh parameter for the triangulation.

We will further assume that α is piecewise constant on T_h without any loss of generality. We assume that there exists a coarse nonoverlapping partitioning of Ω into open connected Lipschitz polytopes Ω_i, called structures, such that $\overline{\Omega} = \bigcup_{i=1}^N \overline{\Omega}_i$ and they are aligned with the fine triangulation, in other words a fine triangle of $\mathcal{T}_h$ can be contained in only one of the coarse substructures. For the simplicity of presentation, we further assume that these substructures form a coarse triangulation of the domain which is shape regular in the sense of [1].

Let $\mathcal{F}_{ij}$ denote the open face common to subdomains Ω_i and Ω_j, and let $\mathcal{E}$ denote an open edge of a substructure, not in $\partial\Omega$. We denote with Ω_h, $\partial\Omega_h$, Ω_{ih}, $\partial\Omega_{ih}$, $\mathcal{F}_{ij,h}$, and $\mathcal{E}_h$, the sets of vertices of the elements of $\mathcal{T}_h$, corresponding to Ω, $\partial\Omega$, Ω_i, $\partial\Omega_i$, $\mathcal{F}_{ij}$, and $\mathcal{E}$, respectively

Let S_h be the standard linear conforming finite element space defined on the triangulation $\mathcal{T}_h$,

$$S_h = S_h(\Omega) := \{u \in C(\overline{\Omega}) \cap H_0^1(\Omega) : \ v_{|K} \in P_1, \ K \in \mathcal{T}_h\}.$$

The finite element approximation u_h^* of (1) is then defined as the solution to the following problem: Find $u_h^* \in S_h$ such that

$$a(u_h^*, v) = (f, v) , \quad \forall v \in S_h. \tag{2}$$

Note that α can be scaled without influencing the solution, hence we can easily assume that $\alpha(x) \geq 1$. As ∇u_h^* is piecewise constant over the fine elements, we can further assume that α is piecewise constants over the elements of $\mathcal{T}_h$, since $\int_K \alpha \nabla u \nabla v \, dx = (\nabla u)_{|K} (\nabla v)_{|K} \int_K \alpha(x) \, dx$.

Since each subdomain inherits a local triangulation $\mathcal{T}_h(\Omega_k)$ from $\mathcal{T}_h(\Omega)$, two local subspaces can be defined as the following,

$$S_h(\Omega_i) := \{u_{|\overline{\Omega}_i} : u \in S_h\} \ \text{ and } \ S_{h,0}(\Omega_i) := S_h(\Omega_i) \cap H_0^1(\Omega_i),$$

along with a local projection operator $\mathcal{P}_i : S_h \rightarrow S_{h,0}(\Omega_i)$ as the following, find $\mathcal{P}_i u \in S_{h,0}(\Omega_i)$ such that

$$a_i(\mathcal{P}_i u, v) = a_i(u, v), \quad \forall v \in S_{h,0}(\Omega_i),$$

where $a_i(u, v) := a_{|\Omega_i}(u, v) = \int_{\Omega_i} \alpha(x) \nabla u \nabla v \, dx$.

The discrete harmonic part of $u \in S_h(\Omega_i)$ is defined as $\mathcal{H}_i u := u - \mathcal{P}_i u$, or equivalently as $\mathcal{H}_i u \in S_h(\Omega_i)$ which satisfies the following,

$$\begin{cases} a_i(\mathcal{H}_i u, v) = 0, & \forall v \in S_{h,0}(\Omega_i), \\ \mathcal{H}_i u(s) = u(s), & \forall s \in \partial \Omega_{ih}. \end{cases} \tag{3}$$

We say that a function $u \in S_h$ is discrete harmonic if it is discrete harmonic in each subdomain, i.e. $u_{|\Omega_i} = \mathcal{H}_i u_{|\Omega_i} \ \forall i$.

3 Additive Schwarz Method

In this section, we present the overlapping additive Schwarz method for the discrete problem (2). We refer to [16, 18] for a more general discussion of the method.

3.1 Decomposition of S_h

The space S_h is decomposed into the local subspaces $\{V_i\}_i$, and the global coarse space V_0, as follows.

$$V_i = \{u \in S_h : v(x) = 0 \ \ \forall x \in \Omega_h \setminus \overline{\Omega}_i\}, \quad i = 1, \ldots, N,$$

where $u \in V_i$ can take nonzero values at the nodes that are in Ω_i and on $\partial\Omega_i$ only, giving $\{V_i\}_i$ as subspaces with minimal overlap. The global coarse space V_0 is defined in Sect. 4. For $i = 0, \ldots, N$, the projection like operators $T_i : S_h \to V_i$ are defined as

$$a(T_i u, v) = a(u, v), \quad \forall v \in V_i. \tag{4}$$

Now, introducing the additive Schwarz operator as $T := T_0 + \sum_{i=1}^{N} T_i$, the original problem (2) can be replaced with the following equivalent problem: Find u_h^* such that

$$T u_h^* = g, \tag{5}$$

where $g = \sum_{i=0}^{N} g_i$ and $g_i = T_i u$. Note that g_i may be computed without knowing the solution u_h^* of (2): $a(g_i, v) = (f, v)$ for all $v \in V_i$.

4 Adaptive Vertex Coarse Space

We introduce our adaptive vertex based coarse space in this section. Each edge $\mathcal{E}$ inherits a 1D triangulation $\mathcal{T}_h(\mathcal{E})$ from $\mathcal{T}_h$. For each edge $\mathcal{E}_h$, let $S_h(\mathcal{E})$ be the space of traces of functions of S_h on the edge, that is the space of continuous piecewise linear functions on $\mathcal{T}_h(\mathcal{E})$, let $S_{h,0}(\mathcal{E}) = S_h(\mathcal{E}) \cap H_0^1(\mathcal{E})$ be its subspace with compact support, and let the edge bilinear form $a_{\mathcal{E}}(u, v) : S_{h,0}(\mathcal{E}) \times S_{h,0}(\mathcal{E}) \to \mathbb{R}$ be defined as

$$a_{\mathcal{E}}(u, v) = \sum_{e \in \mathcal{T}_h(\mathcal{E})} \int_e \overline{\alpha}_e u' v' \, ds, \tag{6}$$

where $\overline{\alpha}_e = \max_{e \subset \partial K} \alpha_K$ is the maximum value of the coefficient over the tetrahedra sharing the fine edge $e \in \mathcal{T}_h(\mathcal{E})$. Here u', v' are the weak derivatives of $u, v \in S_{h,0}(\mathcal{E})$. The definition of the form $a_{\mathcal{E}}(u, v)$, in particular the definition of $\overline{\alpha}$, is introduced in a way which enables us to estimate this form from above by the sum of energy norms over all subdomains which share this edge.

4.1 Vertex Based Interpolation Operator

We introduce the vertex interpolation operator $I_V : S_h(\Omega) \to S_h(\Omega)$ as follows. For $u \in S_h(\Omega)$

- $I_V u(x) = u(x)$ where x is a crosspoint (a subdomain vertex inside Ω),
- $I_V u$ on each edge $\mathcal{E}$ satisfies, cf. (6):

$$a_{\mathcal{E}}(I_V u, v) = 0, \quad \forall v \in S_{h,0}(\mathcal{E}). \tag{7}$$

- $I_V u(x) = 0$ at all $x \in \mathcal{F}_{ij,h}$ for each face $\mathcal{F}_{ij}$,
- $I_V u$ is discrete harmonic in the sense as described in Sect. 2.

Note that $I_V u$ is uniquely determined by the values of u at the crosspoints, as (7) uniquely determines $I_V u$ at the edge interior nodes, $I_V u$ is equal to zero at all face interior nodes, and then extended as discrete harmonic to the subdomain interior nodes, cf. (3). The auxiliary coarse space $\hat{V}_0$ is then defined as the image of this interpolation operator I_V, that is $\hat{V}_0 := Im(I_V) = I_V S_h$.The coarse space V_0 is the algebraic sum of $\hat{V}_0$ and a sequence of small subspaces built with functions that are extensions of certain eigenfunctions of the two particular classes of eigenvalue problems presented below.

4.2 Eigenvalue Problems

We start by introducing the two classes of local eigenvalue problems, one on the subdomain edges or the edge interfaces, and one on the subdomain faces or the face interfaces.

4.2.1 Eigenvalue Problem on Edge Interface

Find the eigen pairs $(\lambda_j^{\mathcal{E}}, \psi_j^{\mathcal{E}}) \in \mathbb{R}_+ \times S_{h,0}(\mathcal{E})$

$$a_{\mathcal{E}}(\psi_j^{\mathcal{E}}, v) = \lambda_j^{\mathcal{E}} b_{\mathcal{E}}(\psi_j^{\mathcal{E}}, v), \qquad \forall v \in S_{h,0}(\mathcal{E}), \tag{8}$$

where $a_{\mathcal{E}}(u, v)$ is as defined in (6), and

$$b_{\mathcal{E}}(u, v) = h^{-4} \int_{G_{\mathcal{E}}} \alpha \hat{u}\, \hat{v}\, dx, \tag{9}$$

and $G_{\mathcal{E}}$ is a 3D layer around and along the edge $\mathcal{E}$, defined as the sum of all fine tetrahedra of $\mathcal{T}_h$ those touching $\mathcal{E}$ by a fine edge or a vertex, and $\hat{u}, \hat{v} \in S_h$ are the discrete zero extensions of $u, v \in S_{h,0}(\mathcal{E})$. The scaling in the form $b_{\mathcal{E}}(u, v)$, and in the form $b_{kl}(u, v)$ in (11) below, comes from an inverse inequality and the lines of the proof of Theorem 1, which will be provided in a full version of this paper published elsewhere. The functions $\psi_j^{\mathcal{E}}$ are extended inside as follows, taking zero values at the nodal points of all remaining edges and faces, and then extending further inside as discrete harmonic in the sense as described in Sect. 2. The extension is denoted by the same symbol. Writing the eigenvalues in the increasing order, i.e. $0 < \lambda_1^{\mathcal{E}} \leq \lambda_2^{\mathcal{E}} \leq \ldots \lambda_{M_{\mathcal{E}}}^{\mathcal{E}}$ for $M_{\mathcal{E}} = \dim(S_{h,0}(\mathcal{E}))$, we define the local edge spectral component of the coarse space as follows. Let $V_{\mathcal{E}} = \mathrm{Span}(\psi_j^{\mathcal{E}})_{j=1}^{n_{\mathcal{E}}}$, where $n_{\mathcal{E}} \leq M_{\mathcal{E}}$ is the number of eigenfunctions $\psi_j^{\mathcal{E}}$, whose eigenvalues $\lambda_j^{\mathcal{E}}$ are less then a given threshold prescribed for each subdomain by the user.

4.2.2 Eigenvalue Problem on Face Interface

Each face $\mathcal{F}_{kl}$ inherits a 2D triangulation consisting of triangles $\mathcal{T}_h(\mathcal{F}_{kl})$, and a local face finite element space $S_h(\mathcal{F}_{kl})$ being the space of traces of S_h onto $\mathcal{F}_{kl}$, and $S_{h,0}(\mathcal{F}_{kl}) = S_h(\mathcal{F}_{kl}) \cap H_0^1(\mathcal{F}_{kl})$. We introduce $\overline{\mathcal{F}}_{I,ij}$ as the sum of closed triangles of $\mathcal{T}_h(\mathcal{F}_{kl})$ such that all their nodes are not in $\partial\mathcal{F}_{kl}$.

The face eigenvalue problem is then to find the eigen pairs $(\lambda_j^{kl}, \psi_j^{kl}) \in \mathbb{R}_+ \times S_{h,0}(\mathcal{F}_{kl})$ such that

$$a_{kl}(\psi_j^{kl}, v) = \lambda_j^{\mathcal{F}_{kl}} b_{kl}(\psi_j^{kl}, v), \qquad \forall v \in S_{h,0}(\mathcal{F}_{kl}), \tag{10}$$

where

$$a_{kl}(u, v) = \sum_{\tau \subset \mathcal{F}_{I,kl}} \int_\tau \underline{\alpha}_\tau \nabla u(x) \nabla v(x), \quad b_{kl}(u, v) = h^{-3} \int_{G_{\mathcal{F}_{kl}}} \alpha \hat{u}\, \hat{v}\, dx, \tag{11}$$

and $\underline{\alpha}_\tau = \max_{\tau \subset \partial K} \alpha_K$ is the maximum value of the coefficient over the tetrahedra sharing the fine face $\tau \in \mathcal{T}_h(F_{I,kl})$, $G_{\mathcal{F}_{kl}}$ is a 3D layer of tetrahedra around and along the face $\mathcal{F}_{kl}$, defined as sum of all fine tetrahedra of T_h those touching $\mathcal{F}_{kl}$ by a fine face, a fine edge or a vertex, and $\hat{u}, \hat{v} \in S_h$ are the discrete zero extensions of $u, v \in S_{h,0}(\mathcal{F}_{kl})$. The functions ψ_j^{kl} are extended inside as follows, taking zero values at the nodal points of all remaining faces and edges, and then extending further inside as discrete harmonic in the same sense as in Sect. 2. The extension is denoted by the same symbol.

Again, by writing the eigenvalues in the increasing order as $0 \leq \lambda_1^{kl} \leq \lambda_2^{kl} \leq \ldots \lambda_{M_{kl}}^{kl}$ for $M_{kl} = \dim(S_{h,0}(\mathcal{F}_{kl}))$, we can define the local face spectral component of the coarse space as follows. Let $V_{kl} = \mathrm{Span}(\psi_j^{kl})_{j=1}^{n_{kl}}$, where $n_{kl} \leq M_{kl}$ is the number of eigenfunctions ψ_j^{kl} whose eigenvalues λ_j^{kl} are less than a given threshold provided by an user.

Finally, The coarse space V_0, after the enrichment takes the following form:

$$V_0 = \hat{V}_0 + \sum_{\mathcal{F}_{kl} \subset \Gamma} V_{kl} + \sum_{\mathcal{E} \subset \Gamma} V_{\mathcal{E}}. \tag{12}$$

Note that $\hat{V}_0 = I_V S_h$, as defined in Sect. 4.1.

Remark 1 The bilinear forms $b_{\mathcal{E}}(u, v)$, cf. (9), and $b_{kl}(u, v)$, cf. (11), can be defined in other ways. For instance, we can consider larger layers $G_{\mathcal{E}}$ or $G_{\mathcal{F}_{kl}}$, or even consider nonzero extensions of $u \in S_{h,0}(\mathcal{E})$ and $u \in S_{h,0}(\mathcal{F}_{kl})$, but with minimal energy. We can also take the bilinear forms to be equal to the restrictions of the scaled original energy form to their respective layers or to the whole substructures, that is following the ideas of [10–12]. In all cases, we will have similar estimates as in Theorem 1 in the next section.

5 Condition Number

Following the abstract Schwarz framework, cf. [16, 18], and the classical theory of eigenvalue problems, we can show the following theoretical bound on the condition number for the preconditioned system of our method.

Theorem 1 *For all $u \in S_h$, the following holds,*

$$c \left(1 + \max_{\mathcal{E}} \frac{1}{\lambda_{n_{\mathcal{E}}+1}} + \max_{\mathcal{F}_{kl}} \frac{1}{\lambda_{n_{kl}+1}} \right) a(u, u) \leq a(Tu, u) \leq C\, a(u, u),$$

where C, c are positive constants independent of the coefficient α, the mesh parameter h and the sudomain size H.

Acknowledgment The work of "L. Marcinkowski" was partially supported by Polish Scientific Grant 2016/21/B/ST1/00350.

References

1. S.C. Brenner, L.Y. Sung, Balancing domain decomposition for nonconforming plate elements. Numer. Math. **83**(1), 25–52 (1999)
2. J.G. Calvo, O.B. Widlund, An adaptive choice of primal constraints for BDDC domain decomposition algorithms. Electron. Trans. Numer. Anal. **45**, 524–544 (2016)
3. T. Chartier, R.D. Falgout, V.E. Henson, J. Jones, T. Manteuffel, S. McCormick, J. Ruge, P.S. Vassilevski, Spectral AMGe (ρAMGe). SIAM J. Sci. Comput. **25**(1), 1–26 (2003). https://doi.org/10.1137/S106482750139892X

4. V. Dolean, F. Nataf, R. Scheichl, N. Spillane, Analysis of a two-level Schwarz method with coarse spaces based on local Dirichlet-to-Neumann maps. Comput. Methods Appl. Math. **12**, 391–414 (2012)
5. Y. Efendiev, J. Galvis, R. Lazarov, J. Willems, Robust domain decomposition preconditioners for abstract symmetric positive definite bilinear forms. ESAIM Math. Model. Numer. Anal. **46**, 1175–1199 (2012)
6. Y. Efendiev, J. Galvis, R. Lazarov, S. Margenov, J. Ren, Robust two-level domain decomposition preconditioners for high-contrast anisotropic flows in multiscale media. Comput. Methods Appl. Math. **12**(4), 415–436 (2012). https://doi.org/10.2478/cmam-2012-0031
7. J. Galvis, Y. Efendiev, Domain decomposition preconditioners for multiscale flows in high-contrast media. Multiscale Model. Simul. **8**(4), 1461–1483 (2010). https://doi.org/10.1137/090751190
8. H.H. Kim, E.T. Chung, A BDDC algorithm with enriched coarse spaces for two-dimensional elliptic problems with oscillatory and high contrast coefficients. Multiscale Model. Simul. **13**(2), 571–593 (2015)
9. H.H. Kim, E. Chung, J. Wang, BDDC and FETI-DP preconditioners with adaptive coarse spaces for three-dimensional elliptic problems with oscillatory and high contrast coefficients. J. Comput. Phys. **349**, 191–214 (2017). https://doi.org/10.1016/j.jcp.2017.08.003
10. A. Klawonn, P. Radtke, O. Rheinbach, FETI-DP methods with an adaptive coarse space. SIAM J. Numer. Anal. **53**(1), 297–320 (2015)
11. A. Klawonn, M. Kuhn, O. Rheinbach, Adaptive coarse spaces for FETI-DP in three dimensions. SIAM J. Sci. Comput. **38**(5), A2880–A2911 (2016)
12. A. Klawonn, P. Radtke, O. Rheinbach, A comparison of adaptive coarse spaces for iterative substructuring in two dimensions. Electron. Trans. Numer. Anal. 45:75–106 (2016)
13. J. Mandel, B. Sousedík, Adaptive selection of face coarse degrees of freedom in the BDDC and the FETI-DP iterative substructuring methods. Comput. Methods Appl. Mech. Eng. **196**(8), 1389–1399 (2007)
14. F. Nataf, H. Xiang, V. Dolean, A two level domain decomposition preconditioner based on local Dirichlet-to-Neumann maps. C.R. Math. **348**(21–22), 1163–1167 (2010)
15. F. Nataf, H. Xiang, V. Dolean, N. Spillane, A coarse space construction based on local Dirichlet-to-Neumann maps. SIAM J. Sci. Comput. **33**(4), 1623–1642 (2011)
16. B.F. Smith, P.E. Bjørstad, W.D. Gropp, *Domain Decomposition: Parallel Multilevel Methods for Elliptic Partial Differential Equations* (Cambridge University Press, Cambridge, 1996)
17. N. Spillane, V. Dolean, P. Hauret, F. Nataf, C. Pechstein, R. Scheichl, Abstract robust coarse spaces for systems of PDEs via generalized eigenproblems in the overlaps. Numer. Math. **126**, 741–770 (2014). https://doi.org/10.1007/s00211-013-0576-y
18. A. Toselli, O. Widlund, *Domain Decomposition Methods—Algorithms and Theory*. Springer Series in Computational Mathematics, vol. 34 (Springer, Berlin, 2005)