Modification of Interval Arithmetic for Modelling and Solving Uncertainly Defined Problems by Interval Parametric Integral Equations System

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Abstract. In this paper we present the concept of modeling and solving uncertainly defined boundary value problems described by 2D Laplace's equation. We define uncertainty of input data (shape of boundary and boundary conditions) using interval numbers. Uncertainty can be considered separately for selected or simultaneously for all input data. We propose interval parametric integral equations system (IPIES) to solve so-define problems. We obtain IPIES in result of PIES modification, which was previously proposed for precisely (exactly) defined problems. For this purpose we have to include uncertainly defined input data into mathematical formalism of PIES. We use pseudo-spectral method for numerical solving of IPIES and propose modification of directed interval arithmetic to obtain interval solutions. We present the strategy on examples of potential problems. To verify correctness of the method, we compare obtained interval solutions with analytical ones. For this purpose, we obtain interval analytical solutions using classical and directed interval arithmetic.

Keywords: Interval arithmetic · Interval modeling
Potential boundary value problems
Parametric integral equations system (PIES)

1 Introduction

Classical definition of boundary problems assumes that the shape of the boundary and boundary conditions are precisely (exactly) defined. This is an idealized way. In practice we can obtain measurement errors, for example. The Measurement Theory is widely presented in [1]. Additionally, even differential equations (used to define physical phenomena mathematically) do not model all of the phenomena properties. Therefore, the differential equations, as well as shape of boundary and boundary condition, do not model the phenomenon accurately.

Y. Shi et al. (Eds.): ICCS 2018, LNCS 10862, pp. 231–240, 2018.
https://doi.org/10.1007/978-3-319-93713-7_19

Nevertheless, the methods of solving so-define problems are still being developed. In the most popular techniques the domain or boundary is divided into small parts (elements). On these elements some interpolation functions are defined. Next, to obtain solutions for whole problem, the complex calculations (on so-define numerical model) were carried out.

Above mentioned strategy is used in finite element method (FEM) [2] and in boundary element method (BEM) [3]. The basic problem of such strategy is to ensure proper class of continuity of interpolation functions in points of segment join. However, these methods do not consider the uncertainty of problems. It is impossible to use these methods directly for solving uncertainly defined problems. There is no option to define the uncertainty using traditional mathematical apparatus.

Recently, there is growing number of well known techniques modifications for solving boundary value problems, which use different ways of uncertainty modeling. For example, interval numbers and its arithmetic [4] or fuzzy set theory [5] can be used. In element methods, their development in the form of interval FEM [6] and interval BEM [7] appeared. Unfortunately they inherit disadvantages of classical methods (used with precisely defined boundary value problems), which definitely affect the effectiveness of their application. Description of the boundary shape uncertainty is very troublesome because of discretization process. Moreover, in classical boundary integral equations (BIE), the Lyapunov condition for the shape of boundary should be met [8] and appropriate class of continuity in points of elements join should be ensured. It is significantly troublesome for uncertainly defined boundary value problems.

Contrary to traditional boundary integral equations (BIE), we separate approximation of the boundary shape from the boundary function in parametric integral equations system (PIES). Therefore, we decided to use this method in uncertainly modelled problems. Recently PIES was successfully used for solving boundary value problems defined in precise way [9,10]. PIES was obtained as a result of analytical modification of traditional BIE. The main advantage of PIES is that the shape of boundary is included directly into BIE. The shape could be modeled by curves (used in graphics). Therefore, we can obtain any continuous shape of boundary (using curves control points) directly in PIES. In other words, PIES is automatically adapted to the modified shape of boundary. It significantly improves the way of modeling and solving problems. This advantage is particularly visible in modeling uncertainly defined shape of boundary.

In this paper, for modeling uncertainly defined potential problems (defined using two-dimensional Laplace's equation) we proposed interval parametric integral equations system (IPIES). We obtain IPIES as a result of PIES modification. We include directed interval arithmetic into mathematical formalism of PIES and verify reliability of proposed mathematical apparatus based on examples. It turned out that obtained solutions are different for the same shape of boundary defined in different location in coordinate system. Therefore, to obtain reliable solutions using IPIES, we have to modify applied directed interval arithmetic.

2 Concept of Classical and Directed Interval Arithmetic

Classical interval number $\boldsymbol{x}$ is the set of all real numbers x satisfying the condition [4,12]:

$$\boldsymbol{x} = [\underline{x}, \overline{x}] = \{x \in \mathbb{R} | \underline{x} \leq x \leq \overline{x}\}, \tag{1}$$

where $\underline{x}$ - is infimum and $\overline{x}$ - is supremum of interval number $\boldsymbol{x}$. Such numbers are also called as proper interval numbers. In order to perform calculations on these numbers, the interval arithmetic was developed and generally it was defined as follows [4]:

$$\boldsymbol{x} \circ \boldsymbol{y} = [\underline{x}, \overline{x}] \circ [\underline{y}, \overline{y}] = [min(\underline{x} \circ \underline{y}, \underline{x} \circ \overline{y}, \overline{x} \circ \underline{y}, \overline{x} \circ \overline{y}), max(\underline{x} \circ \underline{y}, \underline{x} \circ \overline{y}, \overline{x} \circ \underline{y}, \overline{x} \circ \overline{y})], \tag{2}$$

where $\circ \in \{+, -, \cdot, /\}$ and in division $0 \notin \boldsymbol{y}$.

The development of different methods of classical interval number applications [13,14] resulted in the detection of some disadvantages of this kind of representation. For example, it is impossible to obtain opposite and inverse element of such numbers. Therefore in the literature, the extension of intervals by improper numbers was appeared. Such extension was called as directed (or extended) interval numbers [15].

Directed interval number $\boldsymbol{x}$ is the set of all ordered real numbers x satisfying the conditions:

$$\boldsymbol{x} = [\underline{x}, \overline{x}] = \{x \in \boldsymbol{x} | \underline{x}, \overline{x} \in \mathbb{R}\}. \tag{3}$$

Interval number $\boldsymbol{x}$ is proper if $\underline{x} < \overline{x}$, degenerate if $\underline{x} = \overline{x}$ and improper if $\underline{x} > \overline{x}$. The set of proper interval numbers is denoted by $\mathbb{IR}$, of degenerated numbers by $\mathbb{R}$ and improper numbers by $\overline{\mathbb{IR}}$. Additionally directed interval arithmetic was extended by new subtraction ($\ominus$) and division ($\oslash$) operators:

$$\boldsymbol{x} \ominus \boldsymbol{y} = [\underline{x} - \underline{y}, \overline{x} - \overline{y}], \tag{4}$$

$$\boldsymbol{x} \oslash \boldsymbol{y} = \begin{cases} [\underline{x}/\underline{y}, \; \overline{x}/\overline{y}] & \text{for } \boldsymbol{x} > 0, \; \boldsymbol{y} > 0 \\ [\overline{x}/\overline{y}, \; \underline{x}/\underline{y}] & \text{for } \boldsymbol{x} < 0, \; \boldsymbol{y} < 0 \\ [\overline{x}/\underline{y}, \; \underline{x}/\overline{y}] & \text{for } \boldsymbol{x} > 0, \; \boldsymbol{y} < 0 \\ [\underline{x}/\overline{y}, \; \overline{x}/\underline{y}] & \text{for } \boldsymbol{x} < 0, \; \boldsymbol{y} > 0 \\ [\overline{x}/\underline{y}, \; \underline{x}/\underline{y}] & \text{for } \boldsymbol{x} \ni 0, \; \boldsymbol{y} < 0 \\ [\underline{x}/\overline{y}, \; \overline{x}/\overline{y}] & \text{for } \boldsymbol{x} \ni 0, \; \boldsymbol{y} > 0 \end{cases}. \tag{5}$$

Such operators allow us to obtain an opposite element ($0 = \boldsymbol{x} \ominus \boldsymbol{x}$) and inverse element ($1 = \boldsymbol{x} \oslash \boldsymbol{x}$). Therefore, it turned out, that direct application of both classical and directed interval arithmetic in modeling the shape of boundary failed. Hence, we try to interpolate the boundary by boundary points using directed interval arithmetic. We define the same shape of boundary in each quarter of coordinate system and, as a result of interpolation, we obtain different shapes of boundary depending on place of its location. Therefore, we propose modification of directed interval arithmetic, where all of arithmetic operations are mapped into positive semi-axis:

$$\boldsymbol{x}\cdot\boldsymbol{y}=\begin{cases}\boldsymbol{x}_s\cdot\boldsymbol{y}_s-\boldsymbol{x}_s\cdot y_m-x_m\cdot\boldsymbol{y}_s+x_m\cdot y_m & \text{for } \boldsymbol{x}\leq 0,\boldsymbol{y}\leq 0\\ \boldsymbol{x}_s\cdot\boldsymbol{y}-x_m\cdot\boldsymbol{y} & \text{for } \boldsymbol{x}>0,\boldsymbol{y}\leq 0\\ \boldsymbol{x}\cdot\boldsymbol{y}_s-\boldsymbol{x}\cdot y_m & \text{for } \boldsymbol{x}\leq 0,\boldsymbol{y}>0\\ \boldsymbol{x}\cdot\boldsymbol{y} & \text{for } \boldsymbol{x}>0,\boldsymbol{y}>0\end{cases}, \tag{6}$$

where for any interval number $\boldsymbol{x}=[\underline{x},\overline{x}]$ we define $x_m=\begin{cases}|\overline{x}| & \text{for } \overline{x}>\underline{x}\\ |\underline{x}| & \text{for } \overline{x}<\underline{x}\end{cases}$ and $\boldsymbol{x}_s=\boldsymbol{x}+x_m$, where $\boldsymbol{x}>0$ means $\underline{x}>0$ and $\overline{x}>0$, $\boldsymbol{x}\leq 0$ means $\underline{x}<0$ or $\overline{x}<0$ and $(\cdot)$ is an interval multiplication. Finally, application of so-defined modification allow us to obtain the same shapes of boundary independently from the location in coordinate system.

3 Interval Parametric Integral Equations System (IPIES)

PIES for precisely defined problems was applied in [9,10,16]. It is an analytical modification of boundary integral equations. Defining uncertainty of input data by interval numbers, we can present interval form of PIES:

$$0.5\boldsymbol{u}_l(s_1)=\sum_{j=1}^{n}\int_{\widehat{s}_{j-1}}^{\widehat{s}_j}\Big\{\boldsymbol{U}^*_{lj}(s_1,s)\boldsymbol{p}_j(s)-\boldsymbol{P}^*_{lj}(s_1,s)\boldsymbol{u}_j(s)\Big\}\boldsymbol{J}_j(s)ds, \tag{7}$$

where $\widehat{s}_{l-1}\leq s_1\leq\widehat{s}_l$, $\widehat{s}_{j-1}\leq s\leq\widehat{s}_j$, then $\widehat{s}_{l-1},\widehat{s}_{j-1}$ correspond to the beginning of l-th and j-th segment, while $\widehat{s}_l,\widehat{s}_j$ to their ends. Function:

$$\boldsymbol{J}_j(s)=\left[\left(\frac{\partial\boldsymbol{S}_j^{(1)}(s)}{\partial s}\right)^2+\left(\frac{\partial\boldsymbol{S}_j^{(2)}(s)}{\partial s}\right)^2\right]^{0.5} \tag{8}$$

is the Jacobian for segment of interval curve $\boldsymbol{S}_j=[\boldsymbol{S}_j^{(1)},\boldsymbol{S}_j^{(2)}]$ marked by index j, where $\boldsymbol{S}_j^{(1)}(s)=[\underline{S}_j^{(1)}(s),\overline{S}_j^{(1)}(s)]$, $\boldsymbol{S}_j^{(2)}(s)=[\underline{S}_j^{(2)}(s),\overline{S}_j^{(2)}(s)]$ are scalar components of vector curve $\boldsymbol{S}_j$ and depending on parameter s.

Integral functions $\boldsymbol{p}_j(s)=[\underline{p}_j(s),\overline{p}_j(s)],\boldsymbol{u}_j(s)=[\underline{u}_j(s),\overline{u}_j(s)]$ are interval parametric boundary functions on corresponding interval segments of boundary $\boldsymbol{S}_j$ (on which the boundary was theoretically divided). One of these functions will be defined by interval boundary conditions on segment $\boldsymbol{S}_j$, then the other one will be searched as a result of numerical solution of IPIES (7).

Interval integrands (kernels) $\boldsymbol{U}^*_{lj}$ and $\boldsymbol{P}^*_{lj}$ are presented in the following form:

$$\boldsymbol{U}^*_{lj}(s_1,s)=\frac{1}{2\pi}\ln\frac{1}{[\boldsymbol{\eta}_1^2+\boldsymbol{\eta}_2^2]^{0.5}},\quad \boldsymbol{P}^*_{lj}(s_1,s)=\frac{1}{2\pi}\frac{\boldsymbol{\eta}_1\boldsymbol{n}_1(s)+\boldsymbol{\eta}_2\boldsymbol{n}_2(s)}{\boldsymbol{\eta}_1^2+\boldsymbol{\eta}_2^2}, \tag{9}$$

where $\boldsymbol{\eta}_1 = \boldsymbol{\eta}_1(s_1, s)$ and $\boldsymbol{\eta}_2 = \boldsymbol{\eta}_2(s_1, s)$ are defined as:

$$\boldsymbol{\eta}_1(s_1, s) = \boldsymbol{S}_l^{(1)}(s_1) - \boldsymbol{S}_j^{(1)}(s), \quad \boldsymbol{\eta}_2(s_1, s) = \boldsymbol{S}_l^{(2)}(s_1) - \boldsymbol{S}_j^{(2)}(s), \tag{10}$$

where segments $\boldsymbol{S}_l, \boldsymbol{S}_j$ can be defined by interval curves such as: Bézier, B-spline or Hermite, $\boldsymbol{n}_1(s), \boldsymbol{n}_2(s)$ are the interval components of normal vector $\boldsymbol{n}_j$ to boundary segment j. Kernels (9) analytically include in its mathematical formalism the shape of boundary. It is defined by dependencies between interval segments $\boldsymbol{S}_l, \boldsymbol{S}_j$, where $l, j = 1, 2, 3, ..., n$. We require only a small number of control points to define or modify the shape of curves (created by segments). Additionally, the boundary of the problem is a closed curve and the continuity of C^2 class is easily ensured in points of segments join.

The advantages of precisely defined PIES are more visible in modeling uncertainty of boundary value problem. Inclusion of uncertainly defined shape of the boundary directly in kernels (9) by interval curves is the main advantage of IPIES. Numerical solution of PIES do not require the classical boundary discretization, contrary to the traditional BIE. This advantage in modeling uncertainty of the boundary shape significantly reduces amount of interval input data. Therefore the overestimation is also reduced. Additionally, the boundary in PIES is analytically defined by interval curves. That ensure the continuity in points of segments join.

3.1 Interval Integral Identity for Solutions in Domain

Solving interval PIES (7) we can obtain only solutions on boundary. We have to define integral identity using interval numbers, to obtain solutions in domain. Finally, we can present it as follows:

$$\boldsymbol{u}(\boldsymbol{x}) = \sum_{j=1}^{n} \int_{\widehat{s}_{j-1}}^{\widehat{s}_j} \left\{ \widehat{\boldsymbol{U}}_j^*(\boldsymbol{x}, s)\boldsymbol{p}_j(s) - \widehat{\boldsymbol{P}}_j^*(\boldsymbol{x}, s)\boldsymbol{u}_j(s) \right\} \boldsymbol{J}_j(s) ds, \tag{11}$$

it is right for $\boldsymbol{x} = [x_1, x_2] \in \Omega$.

Interval integrands $\widehat{\boldsymbol{U}}_j^*(\boldsymbol{x}, s)$ and $\widehat{\boldsymbol{P}}_j^*(\boldsymbol{x}, s)$ are presented below:

$$\widehat{\boldsymbol{U}}_j^*(\boldsymbol{x}, s) = \frac{1}{2\pi} \ln \frac{1}{[\overleftrightarrow{\boldsymbol{r}_1}^2 + \overleftrightarrow{\boldsymbol{r}_2}^2]^{0.5}}, \quad \widehat{\boldsymbol{P}}_j^*(\boldsymbol{x}, s) = \frac{1}{2\pi} \frac{\overleftrightarrow{\boldsymbol{r}_1}\boldsymbol{n}_1(s) + \overleftrightarrow{\boldsymbol{r}_2}\boldsymbol{n}_2(s)}{\overleftrightarrow{\boldsymbol{r}_1}^2 + \overleftrightarrow{\boldsymbol{r}_2}^2}, \tag{12}$$

where $\overleftrightarrow{\boldsymbol{r}_1} = x_1 - \boldsymbol{S}_j^{(1)}(s)$ and $\overleftrightarrow{\boldsymbol{r}_2} = x_2 - \boldsymbol{S}_j^{(2)}(s)$.

Interval shape of boundary is included in (12) by expressions: $\overleftrightarrow{\boldsymbol{r}_1}$ and $\overleftrightarrow{\boldsymbol{r}_2}$ that are defined by boundary segments, uncertainly modeled by interval curves $\boldsymbol{S}_j(s) = [\boldsymbol{S}_j^{(1)}(s), \boldsymbol{S}_j^{(2)}(s)]$.

4 Interval Approximation of Boundary Functions

Interval boundary functions $\boldsymbol{u}_j(s)$, $\boldsymbol{p}_j(s)$ are approximated by interval approximation series presented as follows:

$$\boldsymbol{p}_j(s)=\sum_{k=0}^{M-1}\boldsymbol{p}_j^{(k)}f_j^{(k)}(s), \qquad \boldsymbol{u}_j(s)=\sum_{k=0}^{M-1}\boldsymbol{u}_j^{(k)}f_j^{(k)}(s) \qquad (j=1,...,n), \quad (13)$$

where $\boldsymbol{u}_j^{(k)},\boldsymbol{p}_j^{(k)}$ - are searched interval coefficients, M - is the number of terms in the series (13) defined on segment j, $f_j^k(s)$ - are base functions defined in the domain of segment. In numerical tests, we use the following polynomials as base functions in IPIES:

$$f_j^{(k)}(s)=\left\{P_j^{(k)}(s),H_j^{(k)}(s),L_j^{(k)}(s),T_j^{(k)}(s)\right\}, \quad (14)$$

where $P_j^{(k)}(s)$ - Legendre polynomials, $H_j^{(k)}(s)$ - Hermite polynomials, $L_j^{(k)}(s)$ - Laguere polynomials, $T_j^{(k)}(s)$ - Chebyshev polynomials of I kind.

In this paper we apply Lagrange interpolation polynomials. When the Dirichlet (or Neumann) interval boundary conditions are defined as analytical function, we can interpolate them by approximation series (14). In case of Dirichlet interval boundary conditions the coefficients $\boldsymbol{u}_j^{(k)}$ are defined, while for Neumann interval conditions $\boldsymbol{p}_j^{(k)}$ ones.

5 Verification of the Concept Reliability

We presented inclusion of the interval numbers and its arithmetic into PIES. Next, we need to verify the strategy on examples to confirm the reliability of IPIES. Firstly, we will test and analyze proposed strategy (application of modified directed interval arithmetic) on simple examples of boundary value problems modeled by Laplace's equation. Solutions obtained by IPIES will be compared with analytical solutions. However, analytical solutions are known only for precisely defined problems. Therefore, we apply interval arithmetic to obtain analytical interval solutions. These solutions will be compared with solutions obtained using IPIES.

5.1 Example 1 - Interval Linear Shape of Boundary and Interval Boundary Conditions

We consider Laplace's equation in triangular domain with interval shape of boundary and boundary conditions. In Fig. 1 we present the uncertainly modeled problem and the cross-section where solutions are searched. The shape is defined by three interval points ($\boldsymbol{P}_0, \boldsymbol{P}_1$ i $\boldsymbol{P}_2$) with the band of uncertainty $\varepsilon = 0.0667$.

Interval Dirichlet boundary condition (precisely defined in [17]) are defined as follows:

$$\boldsymbol{\varphi}(\boldsymbol{x},\boldsymbol{y})=0.5(\boldsymbol{x}^2+\boldsymbol{y}^2). \quad (15)$$

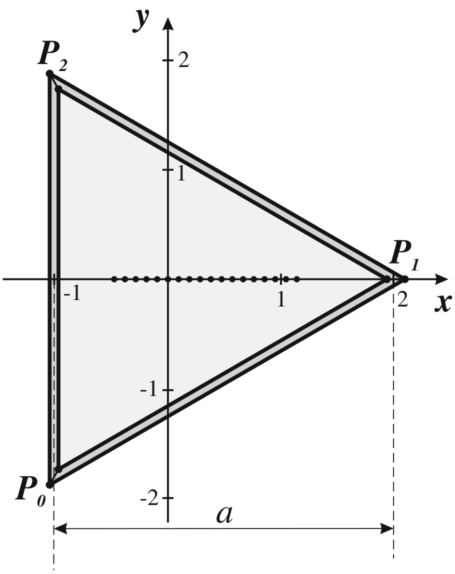

Fig. 1. Example 1 - uncertainty of linear shape of boundary.

So-defined interval boundary condition is defined by interval shape coefficients ($\boldsymbol{x} = [\underline{x}, \overline{x}], \boldsymbol{y} = [\underline{y}, \overline{y}]$) and its uncertainty is caused by uncertainly defined shape of boundary only. Solutions of so-defined problem with uncertainty are obtained in cross-section (Fig. 1) and are presented in Table 1. W decided to compare obtained interval solutions with analytical solution [17], which are presented, with uncertainly defined shape of boundary (interval parameter $\boldsymbol{a}$), as follow:

$$\varphi(\boldsymbol{x}) = \frac{x^3 - 3xy^2}{2\boldsymbol{a}} + \frac{2\boldsymbol{a}^2}{27}, \tag{16}$$

where $\boldsymbol{a} = [\underline{a}, \overline{a}]$ define interval shape of boundary and the middle value of $\boldsymbol{a}$ was marked as a on Fig. 1.

The analytical solution (defined by intervals) are obtained using interval numbers and their arithmetic. Analytical solutions obtained using classical as well as directed interval arithmetic are compared with IPIES solutions and presented in Table 1. We can notice, that obtained interval analytical solutions are very close to the interval solutions of IPIES. However, classical interval arithmetic solutions (in some points) are wider than in IPIES solutions.

Interval solutions obtained using IPIES are close to analytical solutions obtained using directed interval arithmetic. It confirm reliability of an algorithm. However, for more accurate analysis we decide to consider an example with uncertainly defined curvilinear shape of boundary.

Table 1. Interval solutions in domain.

Cross section		Interval analytical solutions		IPIES
x	y	Directed arithmetic	Classical arithmetic	Modified arithmetic
−0.4	0	[0.61264,0.700817]	[0.611928,0.701529]	[0.611936, 0.701502]
−0.1	0	[0.622802,0.711679]	[0.622791,0.711691]	[0.622783, 0.711675]
0.2	0	[0.624342,0.713142]	[0.624253,0.713231]	[0.624325, 0.713136]
0.5	0	[0.644515,0.732013]	[0.643124,0.733404]	[0.644492, 0.732014]
0.8	0	[0.711239,0.794432]	[0.705544,0.800128]	[0.711213, 0.794441]
1.1	0	[0.852446,0.926529]	[0.83764,0.941335]	[0.852411, 0.926579]

5.2 Example 2 - Interval Curvilinear Shape of Boundary and Interval Boundary Conditions

In the next example we consider Laplace's equation defined in elliptical domain. In Fig. 2 we present the way of modeling of the domain with interval shape of boundary. We defined the same width of the band of uncertainty $\varepsilon = 0.1$ on each segment.

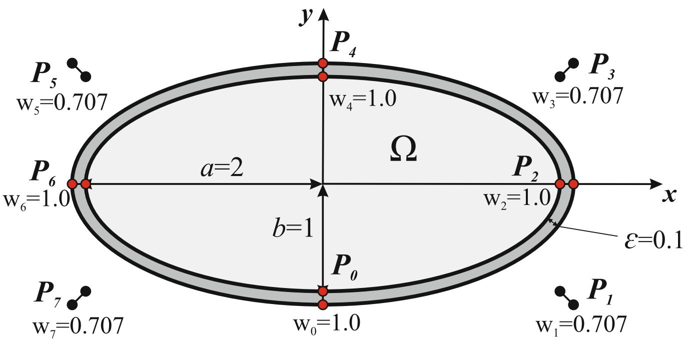

Fig. 2. Uncertainly defined shape of boundary.

The shape of boundary is modeled by interval NURBS curves of second degree. Uncertainty was defined by interval points ($\boldsymbol{P}_0 - \boldsymbol{P}_7$). Interval Dirichlet boundary condition is defined in the same way as in previous example:

$$\boldsymbol{u}(\boldsymbol{x}) = 0.5(\boldsymbol{x}^2 + \boldsymbol{y}^2) \tag{17}$$

Analytical solution for precisely defined problem is presented in [17], therefore it could be define as interval assuming, that the parameters $\boldsymbol{a}, \boldsymbol{b}$ are intervals, i.e.: $\boldsymbol{a} = [\underline{a}, \overline{a}]$ and $\boldsymbol{b} = [\underline{b}, \overline{b}]$:

$$\boldsymbol{u}_a = \frac{x^2 + y^2}{2} - \frac{\boldsymbol{a}^2\boldsymbol{b}^2(\frac{x^2}{\boldsymbol{a}^2} + \frac{y^2}{\boldsymbol{b}^2} - 1)}{\boldsymbol{a}^2 + \boldsymbol{b}^2}. \tag{18}$$

Interval solutions in domain of so-define problem with the interval shape of boundary and interval boundary conditions are shown in the Fig. 3. Similarly like in previous example, we obtain interval analytical solutions using classical and directed interval arithmetic. As we can see in Fig. 3, the widest interval is obtained using classical interval arithmetic. Directed interval arithmetic is slightly shifted from IPIES solution, but with similar width.

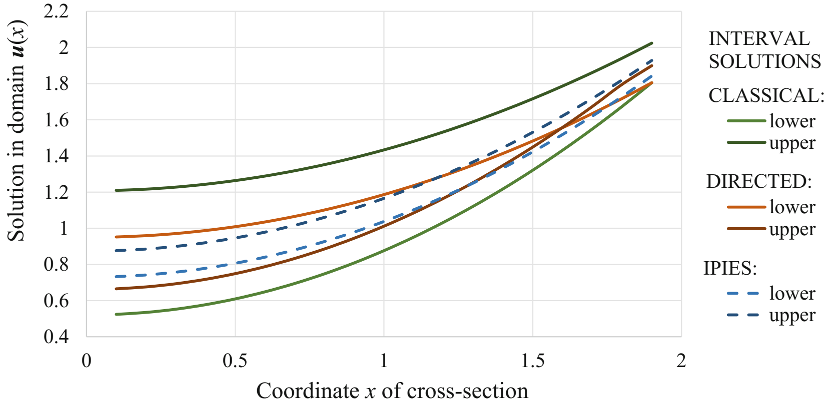

Fig. 3. Comparison between interval analytical solutions and IPIES.

6 Conclusion

In this paper we proposed the strategy of modeling and solving uncertainly defined potential boundary value problems. We modeled uncertainty using interval numbers and its arithmetic. The shape of boundary and boundary conditions were defined as interval numbers. Solutions of so-defined problems are obtained using modified parametric integral equations system. Such generalization includes the directed interval arithmetic into mathematical formalism of PIES. Additionally, we had to modify mentioned arithmetic mapping all of operations into positive semi-axis. Reliability of modeling and solving process of the boundary problems using IPIES was verified on examples with analytical solutions. Applying interval arithmetic to analytical solutions, well known for precisely defined (non-interval) problems, we could easily obtain interval analytical solutions. We used such solutions to verify reliability of proposed algorithm (based on obtained IPIES).

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