



A Simple Minimum Cable-Tension Algorithm for a 2-DOF Planar Cable-Driven Robot Driven by 4 Cables

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Abstract. In this study, we propose an optimum non-iterative algorithm for the minimum cable tension solution of two degree-of-freedom cable-driven robots. The problem is specifically defined for a cable-driven robot with one end-effector connected to four motors by four cables. A two-cable algorithm and a three-cable algorithm are presented with examples, then the optimal two-cable and three-cable solutions are proven for the absolute value norm and Euclidean norm.

Keywords: Cable-driven robot · Cable tension · Projection method
Optimal solution

1 Introduction

A cable-driven robot consists of a moving point-mass end-effector/platform, fixed points with actuated motors/pulleys, and cables connecting the end-effector to the fixed points. Thanks to the advantages of lightweight cables, cable-driven robots have little inertia, high velocity/accelerations, a large workspace and high load capability. These advantages make them suitable to be used in force feedback haptic applications (e.g. [1–3]). However, the cable-driven robots have some disadvantages such as the fact that cables can only pull the end-effector but not push it. In addition, it is necessary to achieve positive tension in all the cables to avoid slacking, which might prevent proper winding around the pulleys.

According to the number of cables m and the number of degrees-of-freedom (DOF) n , cable-driven robots can be classified into two groups: (i) under-constrained if $m \leq n$ and (ii) fully- or over constrained if $m > n$. As pointed out in the literature [1, 2], in order to be able to produce an end-effector force in any direction while maintaining positive tension in all cables, the number of cables must be at least $n + 1$ or there must be an external force such as gravity or a spring if the cable-driven robots are under-constrained. When $m > n + 1$, cable-driven robots are redundantly actuated and there is an infinite number of feasible solutions with positive cable tensions.

Many methods have been proposed in the literature (see reviews in [1–4]) to deal with the minimization of positive cable tension and the determination of the feasible wrench workspace. These minimization methods can be classified as numerical and analytical. The numerical methods are based on iterative calculations and mostly give approximate solutions. They include a convex optimization method using Dykstra’s alternating projection algorithm to solve minimum L_2 -norm cable tensions of cable-driven robots [6]. They also include linear programming formulations for optimal tension distribution [7] as well as a quadratic programming formulation, which can yield good solutions even when the optimal solution is outside the feasible workspace [11]. In [9, 10], the authors used the Karush-Kuhn-Tucker algorithm to iteratively solve the redundancy resolution of cable-driven robots. Tang et al. [12] developed a geometrical-based convex analysis method with less iterative calculation and complexity to calculate the workspace of fully-constrained cable-driven robots and to optimize cable tension distribution. While numerical methods based on the iterative computation of the optimal solution are quite general and can be applied to a wide variety of cable-driven systems, they are also usually very computationally intensive and normally not suited for real-time control because of time constraints.

In contrast to numerical methods, analytical methods produce closed-form solutions. While they are usually less general, they are preferable for real-time implementations due to lesser complexity and time demands. In [5], Fang et al. presented an analytically-based method for optimum cable tension distribution of a 6-DOF cable-driven robot with seven cables. However, this method can only be applied to cable-driven robots having $n + 1$ actuators. Pott [8] focused on computational speed, real-time capability, maximum redundancy, and continuity of force distribution. He improved a closed-form method but the minimization of cable tension has weak solutions due to the medium feasible cable force. In [13], Mikelsons et al. focused on the continuous solution of the cable tensions and developed an algorithm without iterative steps for the real-time control of cable-driven robots. Since this algorithm needs to calculate the QR decomposition of the structure matrix for each vertex of the polytope, the approach is still computationally intensive and time inefficient. In order to avoid discrete cases and complex calculation of the cable tension distribution, Gosselin and Gernier [14] proposed a non-iterative algorithm to minimize the L_2 - and L_p -norm of the relative force vector. However, this method can only be applied to cable-driven robots having $m = n + 1$ cables because the n degree of formulation of cable tensions is symbolically written in terms of one extra cable and a unique solution is calculated for that cable tension, and then the remaining cable tensions are straightforwardly determined. In [15, 16], for computationally efficient tension distribution of the cable-driven robots having only two actuated redundancies, the authors extended the barycenter algorithm of [13] by using the centroid of a two-dimensional polytope as the desired cable tension distribution. However, this method is computationally costly to find the feasible continuous cable tensions. In [17], Williams et al. proposed a non-iterative method to maintain positive cable tension for a 4-cable planar device by utilizing the method of Shen et al. in [18].

In this study, we present a simple algorithm to compute the minimum cable tension solutions for a 2-DOF planar robot driven by four cables ($n = 2$ and $m = 4$). We provide geometrical proofs of the validity of the algorithm. We consider both the absolute value (L_1) and the Euclidian (L_2) norms. Our algorithm is simpler than other non-iterative algorithms [13, 14, 16] and provides some geometrical insight about the solution.

2 Problem Definition

Let us consider a cable-driven robot with one end-effector connected to four motors by four cables. Each cable has a fixed entry point with the other extremity attached to a central end-effector. All entry points A_i are on the same plane. The end-effector position x corresponds to the point where the cables are connected together. We define the workspace as the area where it might be possible to move x by controlling the cable lengths. Typically, the four entry points are arranged to form a rectangle.

Let u_i be the cable directions (unit vectors) corresponding to some end-effector position x inside the workspace.

$$u_i = \frac{A_i - x}{\|A_i - x\|} \quad (1)$$

We want to find a set $\{t_i, i = 1, \dots, 4\}$ of cable tensions such that

$$f = t_1 u_1 + t_2 u_2 + t_3 u_3 + t_4 u_4 \quad (2)$$

with the constraint that cable tensions are positive or null $t_i \geq 0$ and the L_2 (or Euclidian) norm is minimum.

$$\|T\|_2 = \sqrt{t_1^2 + t_2^2 + t_3^2 + t_4^2} \rightarrow \min \quad (3)$$

Since the square function is monotonic, the minimum for this norm is the same as that of the squares of the cable tensions.

Another possible optimal solution might correspond to the minimum of the absolute value (L_1 or Manhattan) norm.

$$\|T\|_1 = |t_1| + |t_2| + |t_3| + |t_4| \rightarrow \min \quad (4)$$

In the following sections, we first present the two-cable algorithm and the three-cable algorithm, then consider the optimal cable tension solutions for the Euclidian norm and then for the absolute value norm. A priori, the optimal solution might involve any number of cables.

3 The Two-Cable Solution

Two cables are necessary and sufficient to produce an arbitrary force in the entire workspace. To show this, one can pick the two cables $T_{12} = [t_1, t_2]$ that bracket the force direction and project the force on them.

Let u_1 and u_2 be the direction of the two cables that bracket the force:

$$\varphi_1 = \tan^{-1} \frac{u_{1y}}{u_{1x}} < \varphi = \tan^{-1} \frac{f_y}{f_x} \leq \varphi_2 = \tan^{-1} \frac{u_{2y}}{u_{2x}} \quad (5)$$

By construction, the force is

$$f = t_1 u_1 + t_2 u_2 = \underbrace{\begin{bmatrix} u_1 & u_2 \end{bmatrix}}_U \begin{bmatrix} t_1 \\ t_2 \end{bmatrix} = UT_{12} \quad (6)$$

The solution for the cable tension is

$$T_{12} = \begin{bmatrix} t_1 \\ t_2 \end{bmatrix} = U^{-1}f \quad (7)$$

Geometrically, the matrix U^{-1} projects the force f on the cable directions u_1 and u_2 as seen in Fig. 1. The fact that the two cables bracket the force insures that t_1 and t_2 are always positive. For the same reason, U is always invertible (the two cable directions bracketing f are never collinear).

Remark 1. When the force is aligned with a cable direction (u_2), the two-cable algorithm yields a one-cable solution. This is easy to show geometrically since the oblique projections of f on u_1 and u_2 are 0 and $|f|$, respectively when f is aligned with u_2 .

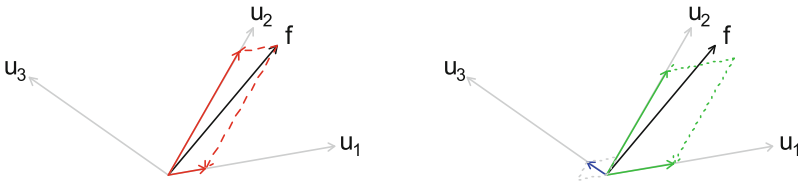


Fig. 1. Example of two- and three-cable solutions in a planar workspace. Angles between the force f and the cable directions u_1 , u_2 and u_3 correspond to -40 , 10 and 95° , respectively. *Left:* For the two-cable solution, the cable tensions are $t_1 = 0.227$, $t_2 = 0.839$. The L_1 and L_2 -norms are $\|T\|_1 = 1.066$ and $\|T\|_2 = 0.755$ respectively. *Right:* For the three-cable solution, the cable tensions are $t_1 = 0.403$, $t_2 = 0.714$, $t_3 = 0.135$, and the norms are $\|T\|_1 = 1.252$ and $\|T\|_2 = 0.691$, respectively. In this example, the three-cable algorithm is better than the two-cable algorithm. The decrease of the cable tension t_2 in the three-cable algorithm relative to the two-cable algorithm is enough to compensate for the increase in cable tension t_1 and t_3 .

Further on, we demonstrate that the two-cable algorithm is always optimal for the absolute value (L_1) norm. However, as illustrated in Fig. 1, the two-cable algorithm does not always provide a solution that minimizes the Euclidian (L_2) norm.

4 The Euclidian Norm and the Three-Cable Solution

In this section, we describe an algorithm to compute the optimal solution for the L_2 -norm. Depending on the position of the end-effector in the workspace and on the direction of the force, this solution might involve up to three cables. The problem is (i) to know how many cables are involved in the optimal L_2 -norm solution, (ii) which cables are involved and, (iii) to compute the tensions for all cables involved.

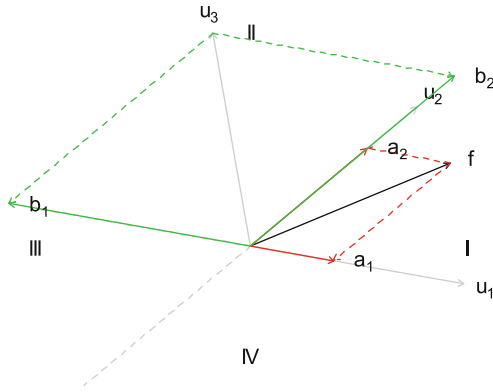


Fig. 2. Geometric interpretation of the components of vector A and vector B . Sectors I, II, III and IV define the cable directions u_i where u_1 and u_2 are the force directions that bracket the force and u_3 is the direction of the third that belongs to the optimal solution.

The complete algorithm is described below. For the moment, let us assume that we know that the optimal solution involves the three cables u_1, u_2 and u_3 . The question is how to compute the corresponding tension.

By assumption, the external force is a linear combination of the three cable forces:

$$f = [u_1 \quad u_2] \begin{bmatrix} t_1 \\ t_2 \end{bmatrix} + t_3 u_3 = UT_{12} + t_3 u_3 \tag{8}$$

Cable tensions t_1 and t_2 can be expressed as a function of cable tension t_3 :

$$T_{12} = U^{-1}(f - t_3 u_3) = U^{-1}f - t_3 U^{-1}u_3 = A - t_3 B \tag{9}$$

where U^{-1} is the inverse of the 2×2 matrix U . Written in components we have an expression for t_1 and t_2 as a function of t_3

$$\begin{bmatrix} t_1 \\ t_2 \end{bmatrix} = \begin{bmatrix} a_1 \\ a_2 \end{bmatrix} - t_3 \begin{bmatrix} b_1 \\ b_2 \end{bmatrix} \tag{10}$$

As shown in Fig. 2, geometrically, vector A is the projection of force f on cable direction u_1 and u_2 . Vector B is the projection of u_3 on u_1 and u_2 .

If u_1 and u_2 are the cable directions that bracket f , then a_1 and a_2 correspond to the two-cable algorithm and must be positive. b_1 and b_2 are the projection of u_3 on u_1 and u_2 . If u_3 is in sector II, b_1 is negative and b_2 is positive. The opposite is true if u_3 is in sector IV. b_1 and b_2 are negative if u_4 is in sector IV. Note that u_3 cannot be in sector I because we have assumed that u_1 and u_2 bracket the force.

In order to minimize the total cable tension for the Euclidian (L_2) norm, we can write the function for the three-cable algorithm as

$$\|T\|_2 = t_1^2 + t_2^2 + t_3^2 \rightarrow \min \tag{11}$$

This implies that

$$0 = \frac{\delta}{\delta t_3} (t_1^2 + t_2^2 + t_3^2) = 2t_1 \frac{\delta t_1}{\delta t_3} + 2t_2 \frac{\delta t_2}{\delta t_3} + 2t_3 \tag{12}$$

Substituting Eq. (10) with the partial derivatives into Eq. (12), can be solved for t_3 as follows:

$$t_3 = \frac{a_1 b_1 + a_2 b_2}{1 + b_1^2 + b_2^2} \tag{13}$$

t_1 and t_2 can now be computed using Eq. (10). Because we have assumed that the optimal solution involved these three cables, t_1 , t_2 and t_3 must be positive.

The complete algorithm addresses the remaining issues, i.e. how to select the cables involved in the optimal solution in addition to how to compute the tension for these cables.

Minimum Cable Tension Algorithm

1. Select the two cable directions u_1 and u_2 that bracket force f . Compute the inverse of $U = [u_1 \ u_2]$ and $A = U^{-1}f$. Note that A corresponds to the two-cable solution.
2. Compute $B = U^{-1}u_i$ and t_i ($i=3$ or 4) according to equation (13) separately for the two possible three-cable solutions $\{u_1, u_2$ and $u_3\}$ and $\{u_1, u_2$ and $u_4\}$
3. Check the sign of t_i for the two possible three-cable solutions.
 - a. If the two values of t_i are negative, the two-cable solution A is optimal.
 - b. If one of the two t_i has a positive sign, compute $T_{12} = A - t_i B$ for this value of t_i and return the corresponding three-cable solution $\{t_1, t_2,$ and $t_i\}$.

Remark 2. Geometrical intuition might suggest that a cable must make an angle of less than 90° with the external force as a necessary condition to be part of the optimal solution. The example of Fig. 1 shows, however, that this is not the case. A small contribution along cable direction u_3 can improve the L_2 -norm solution even though this cable makes an angle larger than 90° with respect to the external force. The above algorithm resolves the problem of determining when the three cable solution is valid by looking at the sign of the third cable in the two possible three-cable solutions.

Remark 3. As presented above, using the position of the end-effector and the direction of the force, minimum cable tension algorithm makes simple calculations with square-invertible matrix to solve how many and which cables with minimum tensions. In addition, our algorithm does not need to calculate all the intersection points of the feasible tension distribution polytopes like other non-iterative methods [13, 14, 16]. In terms of computational simplicity or efficiency, our algorithm is very fast and convenient for real-time implementations.

5 Proofs

In this section, we provide simple proofs, based on geometric reasoning, to establish the correctness of this algorithm. We also demonstrate that the two-cable solution is always optimal for the absolute value (L_1) norm.

We start by considering the optimal solution for a three-cable device.

Theorem 1. *The two-cable algorithm gives the optimal solution for a cable-driven system with only three cables. This is true for the L_1 and L_2 norms.*

Proof. Let u_1 and u_2 be the cable directions that bracket force f . We want to show that u_3 cannot contribute to the solution. In a system with three cables, the angle between u_3 and u_1 or u_2 is always less than π for any position inside the workspace.

We know that the optimal three-cable solution corresponds to $t_1 = a_1 - t_3 b_1$ and $t_2 = a_2 - t_3 b_2$. Let us select u_1 and u_2 as the cable directions that bracket force f . In this case, it is easy to see that a_1 and a_2 , the parallel projections of f on u_1 and u_2 , must be positive. Similarly, b_1 and b_2 , the parallel projections of u_3 on u_1 and u_2 , must be negative.

The L_1 and L_2 norms for the two-cable solution are $|a_1| + |a_2|$ and $a_1^2 + a_2^2$ respectively, since a_1 and a_2 correspond to the two-cable solution. Since b_1 and b_2 are negative and t_3 must be positive, it is easy to see that any positive value of t_3 will also increase t_1 and t_2 with respect to the two-cable solution for both norms. Therefore, the optimal solution corresponds to the two-cable solution.

Note that this demonstration does not hold for a four-cable device because there might be a case where b_1 and b_2 have a different sign with four cables (this is the case in Fig. 2 where the projection of u_3 on u_1 is negative and the projection of u_3 on u_2 is positive).

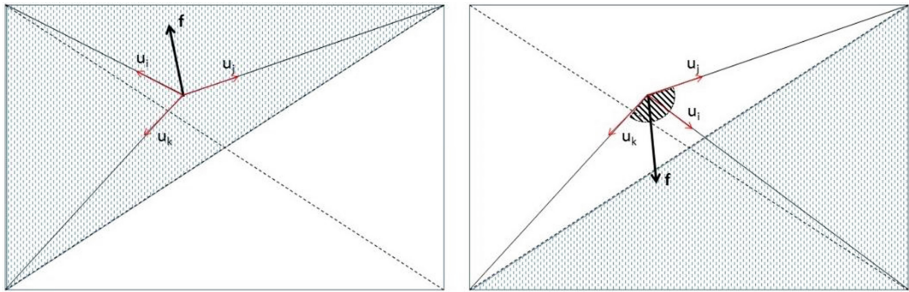


Fig. 3. Representation of the end-effector inside and outside of the triangular workspace (grey area) in the left and right panels respectively.

Lemma 1. *If $\{u_i, u_j, u_k\}$ is an optimal three-cable solution for a four-cable system, the end-effector must be outside the triangular workspace defined by the corresponding attachment points.*

Proof. The lemma is proven by showing that the end-effector cannot be inside the workspace. Let us assume that the end-effector is inside the triangular workspace defined by attachment points of the optimal three-cable solution (see Fig. 3, left panel). Since the fourth cable does not contribute to the optimal solution by assumption, the three-cable solution should be optimal for this reduced three-cable system. However, this is not possible because Theorem 1 states that all optimal solutions for a three-cable system are two-cable solutions. Therefore, the end-effector must be outside the workspace as shown in Fig. 3 (right panel).

Lemma 2. *If $\{u_i, u_j, u_k\}$ is an optimal three-cable solution for a four-cable system, then the direction of the force is bracketed by the cable directions of this reduced system.*

Proof. From Lemma 1, we know that the end-effector position must be outside the triangular workspace defined by the attachment points of the three-cable optimal solution. In this configuration, the external force is necessarily bracketed by two of the cables of this system (see gray filled area in Fig. 3, right panel).

Theorem 2. *For a four-cable system, the optimal solution always involves the two cables direction bracketing the force.*

Proof. To prove this theorem, we consider separately all possible cases. (1) If optimal solution involves only two-cables, then the force is necessarily bracketed by those two cables that are part of the solution because the cables can only pull. (2) The one-cable solution is a special case of the two-cable solution. (3) If the solution involves three cables, Lemma 1 tells us that the end-effector must be outside the corresponding workspace and Lemma 2 tells us that the force direction must be bracketed by two of the three cables involved in the corresponding reduced system. (4) If a four-cable solution existed (but see below), it would necessarily involve the two cables bracketing the force.

This theorem is key to justifying the initial choice of cable directions u_1 and u_2 in the above algorithm (see point 1).

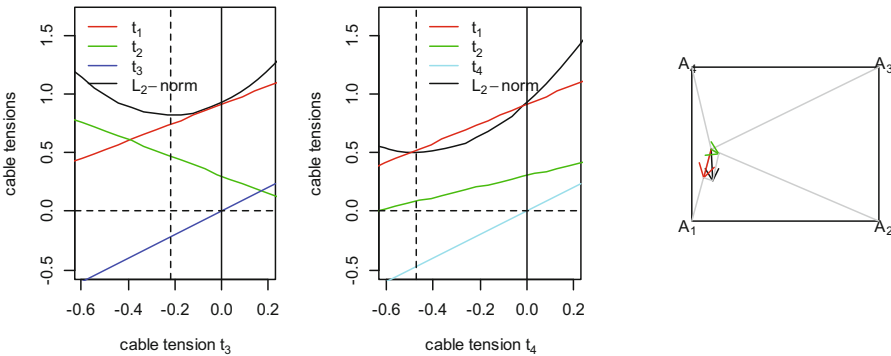


Fig. 4. Example of a configuration where the minimum L2-norm three-cable solution (vertical dotted line) is not valid because the tension for the third cable is negative. The first two cables correspond to the cables bracketing the force. The third cable (t_3 in the left panel and t_4 in the middle panel) corresponds to the cables attached to A_3 and A_4 respectively. The best valid solution is obtained by setting this cable tension to zero (vertical solid line), which corresponds to the two-cable solution plotted in the right panel.

Theorem 3. *This theorem states the other assumptions on which the algorithm relies:*

- A1. A negative t_3 value obtained from Eq. (13) implies that the optimal solution involves two cables.
- A2. A positive t_3 obtained from Eq. (13) implies t_1 and t_2 must be positive.
- A3. The three-cable solution is unique.
- A4. For a four-cable system, the optimal solution cannot involve all cables.

Assumption 1 is used in point 3.a. Assumption 2 is used in point 3.b in the sense that the algorithm only checks the value of t_3 to identify a valid three-cable solution. Assumption 3 is used in the sense that the algorithm does not check for the validity of the second three-cable solution if the first one is valid.

Proof. By definition, we know that the value of t_3 obtained by Eq. (13) minimizes the L_2 norm. By the formula $t_i = a_i - t_3 b_i$, the changes in t_1 and t_2 are linearly related to the change in t_3 . Since the L_2 -norm is convex, we need to change t_3 (and t_1 and t_2) as little as possible.

To prove A1, it is enough to note that the smallest adjustment of a negative t_3 needed to make it compatible with a non-null constraint corresponds to $t_3 = 0$, which also corresponds to the two-cable solution since $T_{12} = A$ when $t_i = 0$ ($i = 3$ or 4 , see Fig. 4).

A2 is proven *by absurdum*. We start by assuming that t_3 is positive and t_1 negative and show that this leads to a contradiction. To keep the L_2 -norm as small as possible, t_1 and t_3 should change as little as possible. This means setting t_3 so that $t_1 = a_1 - t_3 b_1 = 0$. However, setting $t_1 = 0$ would mean that the optimal solution did not involve one of the two cables that bracket the force. Since this is not possible (Theorem 2), t_1 (or t_2) cannot be negative if t_3 is positive.

A3 is a consequence of the convexity of the L_2 -norm.

To prove A4, assume that an optimal four-cable solution exists, with u_1 and u_2 bracketing the force. We can consider the contribution of the two other cables u_3 and u_4 and replace this four cable system by a three cable system where the third cable u_{34} is placed so that its contribution is aligned with the sum of the two contributions $t_3 u_{34} = t_3 u_3 + t_4 u_4$. The L_2 and L_1 norms for this three-cable system are better than the corresponding norms for the optimal four-cable solution since $t_{34}^2 < t_3^2 + t_4^2$ or $|t_{34}| < |t_3| + |t_4|$. However, from Theorem 1 we know that the optimal solution for a three-cable system involves only the two cables bracketing u_1 and u_2 , therefore this three-cable solution cannot not be optimal. This implies that our assumption that a four-cable solution exists is not true.

Theorem 4. *The two-cable solution is optimal for the L_1 norm.*

Proof. From Eq. (10) and the L_1 norm, we have

$$\|T\|_1 = |t_1| + |t_2| + |t_3| = |a_1 - t_3 b_1| + |a_2 - t_3 b_2| + |t_3| \quad (14)$$

where $[a_1 \ a_2] = U^{-1}f$ and $[b_1 \ b_2] = U^{-1}u_3$ are the projection of f and u_3 on u_1 and u_2 respectively. The cable tensions t_1 and t_2 must be positive since u_1 and u_2 are the two cable directions that bracket the force and are part of any optimal solution (see Theorem 2). Moreover, t_3 must also be positive to be part of the optimal solution. Therefore, we can remove the absolute value and rewrite the objective function as

$$\|T\|_1 = a_1 - t_3 b_1 + a_2 - t_3 b_2 + t_3 = (a_1 + a_2) - t_3(b_1 + b_2 - 1) \quad (15)$$

with the constraint $t_3 \geq 0$.

$a_1 + a_2$ is the L_1 norm for the two-cable algorithm since a_1 and a_2 are the projections of f on u_1 and u_2 . The L_1 -norm for the three-cable solution will be smaller only if $b_1 + b_2 - 1$ is positive since t_3 cannot be negative. To prove that the two-cable algorithm gives an optimal solution, we therefore need to prove that the three-cable solution is impossible; that is, $b_1 + b_2 - 1$ is always negative or $b_1 + b_2 < 1$.

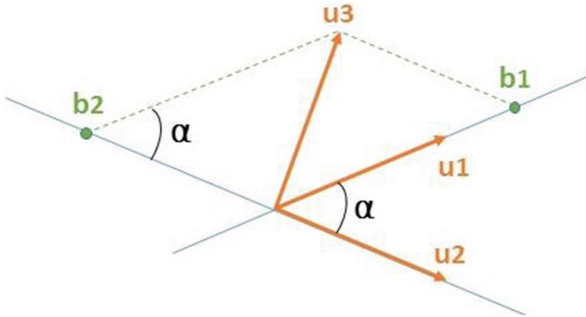


Fig. 5. Projection of u_3 on u_1 and u_2 .

First, b_1 and b_2 cannot both be positive since u_3 must be outside the sector defined by u_1 and u_2 , which brackets the force. If both b_1 and b_2 are negative, the demonstration is finished since $b_1 + b_2 < 1$ because $b_1 < 0$ and $b_2 < 0$. Thus, we need to consider only the case where b_1 and b_2 have a different sign.

To that end, we consider the parallelogram formed by the projection of u_3 on u_1 and u_2 as shown in Fig. 5. The length of the diagonal corresponding to u_3 is 1

$$|u_3| = \sqrt{b_1^2 + b_2^2 - 2b_1b_2 \cos \alpha} = 1 \tag{16}$$

Thus, we have

$$1 = b_1^2 + b_2^2 - 2b_1b_2 \cos \alpha \tag{17}$$

We want to demonstrate

$$b_1 + b_2 < 1 \tag{18}$$

or

$$(b_1 + b_2)^2 = b_1^2 + b_2^2 + 2b_1b_2 < 1 \tag{19}$$

Substituting (17) into (19), we need to verify that

$$b_1^2 + b_2^2 + 2b_1b_2 < b_1^2 + b_2^2 - 2b_1b_2 \cos \alpha \tag{20}$$

Since $b_1 b_2$ is always negative, the division will change the sign of the inequality (20) and we can rewrite it as

$$1 > -\cos \alpha \quad (21)$$

which is always true for $\pi \geq \alpha > 0$.

Extensive simulations showed that this algorithm works. We think that we have also proven that it is correct. It would be nice to have a geometric criterion that indicated when there is a two- or three-cable solution for the L_2 -norm rather than relying on the sign of t_3 but the algorithm is also quite computationally efficient. It only requires inverting a 2×2 matrix and a few additional operations and tests.

6 Conclusion

The main contribution of this study is the description of an efficient algorithm to minimize the cable tensions of a four-cable system with two DOFs. We have also shown that (i) the optimal solution always involves the two cables bracketing the force, (ii) the two-cable solution is always optimal for a three-cable system, (iii) the two-cable solution is always optimal for the L_1 norm. For a four-cable system, we described the algorithm to compute cable tensions that minimize the L_2 norm, and also identified conditions for the three-cable solution.

The reason why the L_1 norm solutions involve only two cables is that the increase in tension of the third is greater than the decrease in tension of the cable(s) bracketing the external force. Compared to the solutions for the L_1 norm, the solutions for the L_2 norm tend to avoid having a relatively large cable tension. In other words, the L_2 norm will distribute the force in a way that equalizes the tension across cables more so than the L_1 norm.

The difference between the L_2 -norm for the two- and three-cable solutions is generally relatively small. One might wonder from a practical point of view whether it is worth it to consider the optimal three-cable solution for the L_2 norm. One reason might be that three maximal tension-limited cables are able to produce markedly larger end-effector forces in some configurations than two cables.

We readily acknowledge the limits of our algorithm. In particular, we did not consider the fact that cable tensions must generally be somewhat above zero to avoid slacking when the end-effector moves. This problem could be handled by assuming a minimum tension t_{\min} on all cables and calculating the net force f_{\min} resulting from the minimum cable tensions. Then our algorithm could be applied to the difference $f - f_{\min}$ with the desired force f to find the optimal cable tensions t_i that correspond to this difference. These cable tensions could then be added to t_{\min} to produce the desired force.

When we compared our algorithm against other non-iterative algorithms [13, 14, 16], from geometric point of view it provides the minimum and faster solution with a few simple mathematical calculations. In the future, we intend to implement the proposed algorithm in a planar cable-driven force-feedback device with four motors under development for haptic applications in an educational context.

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