Chapter 15
Free Flexural Vibrations of Axially Loaded Timoshenko Beams with Internal Viscous Damping Using Dynamic Stiffness Formulation and Differential Transformation

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Abstract The effects of axial compressive load and internal viscous damping on the free vibration characteristics of Timoshenko beams are carried out using the dynamic stiffness formulation and the differential transformation method. The governing equations of motion are derived using the Hamilton's principle. After the analytical solution of the equation of motion has been obtained, the dynamic stiffness method (DSM) is used and the dynamic stiffness matrix of the axially loaded Timoshenko beam with internal viscous damping is constructed to calculate natural frequencies. Moreover, an efficient mathematical technique called the differential transform method (DTM) is used to solve the governing differential equations of motion. The calculated natural frequencies of Timoshenko beams with various combinations of boundary conditions using the DSM and DTM are presented and compared with the analytical results where a very good agreement is observed.

Keywords Axial load · Timoshenko beam · Internal viscous damping
Differential transformation · Dynamic stiffness · Natural frequencies

15.1 Introduction

Axially loaded beams with distributed internal viscous damping are of great importance in a wide class of civil engineering and mechanical engineering structures. The effects of internal viscous damping and the axial load of a beam play an important role on its vibration characteristics and dynamic stability.

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Several studies on vibrations and stability of axially loaded beams with viscous damping have been reported. Gürgöze and Erol (2004) investigated the eigen-characteristics of multistep Bernoulli–Euler beams carrying a tip mass subjected to nonhomogeneous external viscous damping. Cai et al. (2006) studied on an analytical approach for vibration response analysis of a Bernoulli–Euler beam with a single active constraining layer damping patch. In another study, the modal analysis of nonhomogeneous Timoshenko beams with generalized damping distributions is investigated (Sorrentino et al. 2007). Dohnal et al. (2008) investigated a uniform cantilever beam under the effect of a time-periodic axial force by using a finite-element approach. In the other study, an enhanced beam model for constrained layer damping and a parameter study of damping contribution are studied by Xie and Shepard (2009). Bending-bending vibration equations of a twisted beam with internal damping of Kelvin–Voigt type are studied using Timoshenko beam theory (Chen et al. 2013). The dynamic response of a Timoshenko beam with distributed internal viscous damping is investigated (Capsoni et al. 2013). Lin (2014) studied the forced vibration of beam subjected to a harmonic external force and with the squeezing film and thermos elastic damping in non-Fourier model. In the other study, bending–bending vibrations of an axially loaded twisted Timoshenko beam with locally distributed Kelvin–Voigt damping are investigated by Chen (2014a). In the other study, Chen (2014b) studied the vibration behavior of a cantilevered twisted Timoshenko beam with partially distributed Kelvin–Voigt damping using a finite-element method.

DSM is an effective method for free and forced vibration analyses of structures such as beams, plates, and their assemblies. Banerjee (1997) noted that the Wittrick–Williams algorithm can be used as a nonlinear eigenvalue problem is experienced. Free vibration analysis of laminated composite beams under axial compressive force is performed by Jun et al. (2008) using DSM. The effectiveness of DSM for solving free vibration problem of laminated composite beams is observed. Dynamic stiffness approach is used for free vibration analysis of pipe conveying fluid according to Timoshenko beam theory (Bao-Hui et al. 2011). The first three natural frequencies of a multi-span pipe conveying fluid are calculated. Banerjee (2012) researched free vibrations of beams carrying spring–mass systems using the DSM. The natural frequencies of rotating tapered beams are obtained according to Rayleigh beam theory by using the DSM by Banerjee and Jackson (2013). The results are compared with the first natural frequencies obtained according to Euler–Bernoulli beam theory. Su and Banerjee (2015) calculated nondimensional natural frequencies of functionally graded Timoshenko beams for different boundary conditions. DSM is applied to in-plane free vibration problem and response analysis of isotropic rectangular plates. Different boundary conditions and length/width ratios are considered (Nefovska-Danilovic and Petronijevic 2015). Bozyigit and Yesilce (2016) applied dynamic stiffness approach for free vibration analysis of moving beams according to high order shear deformation theory. Different axial tensile force and axial speed values are used to reflect their effects on natural frequencies.

The concept of DTM was first introduced by Zhou (1986). Ozgumus and Kaya (2006) investigated the out-of-plane free vibration analysis of a double tapered Bernoulli–Euler beam using DTM. Çatal (2006, 2008) used DTM for the free vibration analysis of Timoshenko beams with fixed and simply supported ends. Çatal and Çatal (2006) calculated the critical buckling loads of partially embedded Timoshenko pile in elastic soil by DTM. Free vibration analysis of a rotating, double tapered Timoshenko beam featuring coupling between flapwise bending and torsional vibrations is performed using DTM by Ozgumus and Kaya (2007). In another study, Kaya and Ozgumus (2007) used DTM to analyze the free vibration response of an axially loaded, closed section composite Timoshenko beam. Yesilce (2010, 2013) investigated the free vibration analysis of moving Bernoulli–Euler and Timoshenko beams by using DTM. Yesilce (2015) described the determination of the natural frequencies and mode shapes of the axially loaded Timoshenko multiple-step beam carrying a number of intermediate lumped masses and rotary inertias by using the numerical assembly technique and the DTM. Previous studies have shown that the DTM is an efficient tool and it has been applied to solve boundary value problems in fluid mechanics, viscoelasticity, control theory, acoustics, etc. Besides the variety of the problems that DTM may be applied to, its accuracy and simplicity in calculating the natural frequencies and plotting the mode shapes make this method outstanding among many other methods.

The free vibration analysis of simply supported, one end fixed, the other end simply supported, fixed supported and axially loaded Timoshenko beams with distributed internal viscous damping is performed in this study. At the beginning of the study, the governing equations of motion are derived by applying Hamilton's principle. In the next step, the equations of motion, including the parameters for the damping factor and the nondimensionalized multiplication factor for the axial compressive force, are solved using an efficient mathematical technique, called DTM. Besides DTM, DSM is used for calculating the natural frequencies of the axially loaded Timoshenko beams with distributed internal viscous damping. The first four mode shapes are plotted and the effects of the parameters, mentioned above, are investigated. A suitable example that studies the effects of axial compressive load and internal viscous damping on the free vibration analysis of Timoshenko beam using DSM and DTM has not been investigated by any of the studies in open literature so far.

15.2 The Mathematical Model and Formulation

An axially loaded uniform Timoshenko beam with distributed internal viscous damping is presented in Fig. 15.1.

The total kinetic energy T and the total potential energy V of the axially loaded Timoshenko beam can be written as

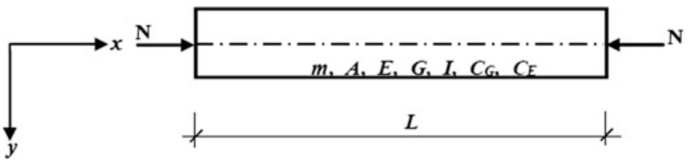

Fig. 15.1 Axially loaded Timoshenko beam with internal viscous damping

$$T = \frac{1}{2}\int_0^L \left\{ m\left(\frac{\partial y(x,t)}{\partial t}\right)^2 + \frac{mI}{A}\left(\frac{\partial \phi(x,t)}{\partial t}\right)^2 \right\} \mathrm{d}x \tag{15.1}$$

$$V = \frac{1}{2}\int_0^L \left\{ EI\left(\frac{\partial \phi(x,t)}{\partial x}\right)^2 + \frac{AG}{\bar{k}}\left(\frac{\partial y(x,t)}{\partial x} - \phi(x,t)\right)^2 - N\left(\frac{\partial y(x,t)}{\partial x}\right)^2 \right\} \mathrm{d}x \tag{15.2}$$

where $y(x,t)$ is the total transverse deflection, $\phi(x,t)$ is the angle of rotation due to bending, m is the mass per unit length of the beam, L is the length of the beam, N is the axial compressive force, A is the cross-section area, $\bar{k}$ is the shape factor due to cross-section geometry, I is the moment of inertia, E and G are the Young's modulus and shear modulus of the beam, respectively, x is the beam position, and t is time variable.

The concept of Rayleigh's dissipation function is utilized to express the dissipation function V_D of Timoshenko beam with distributed internal viscous damping as:

$$V_\mathrm{D} = \frac{1}{2}\int_0^L \left\{ C_E I\left(\frac{\partial^2 \phi(x,t)}{\partial x \partial t}\right)^2 + \frac{C_G A}{\bar{k}}\left(\frac{\partial^2 y(x,t)}{\partial x \partial t} - \frac{\partial \phi(x,t)}{\partial t}\right)^2 \right\} \mathrm{d}x \tag{15.3}$$

In Eq. (15.3), for an isotropic material, the relationship between the coefficients of the internal damping C_E and C_G is assumed to be similar to those between E and G, respectively.

The equations of motion for the axially loaded Timoshenko beam with distributed internal viscous damping are derived by applying Hamilton's principle, which is given by

$$\delta \int_{t_1}^{t_2} L_g \mathrm{d}t = 0 \tag{15.4a}$$

where

$$L_g = T - V - V_D \tag{15.4b}$$

is termed as the Lagrangian density function.

Taking the variation of the Lagrangian density function and integrating Eq. (15.4a) by parts, the equations of motion for the axially loaded Timoshenko beam with distributed internal viscous damping can be derived as $(0 \le x \le L)$:

$$\begin{aligned} &\frac{AG}{\bar{k}}\left(\frac{\partial^2 y(x,t)}{\partial x^2} - \frac{\partial \varphi(x,t)}{\partial x}\right) - N\frac{\partial^2 y(x,t)}{\partial x^2} - m\frac{\partial^2 y(x,t)}{\partial t^2} \\ &+ \frac{C_G A}{\bar{k}}\left(\frac{\partial^3 y(x,t)}{\partial x^2 \partial t} - \frac{\partial^2 \varphi(x,t)}{\partial x \partial t}\right) = 0 \end{aligned} \tag{15.5}$$

$$\begin{aligned} &EI\frac{\partial \phi^2(x,t)}{\partial x^2} - \frac{mI}{A}\frac{\partial \phi^2(x,t)}{\partial t^2} + \frac{AG}{\bar{k}}\left(\frac{\partial y(x,t)}{\partial x} - \phi(x,t)\right) \\ &+ C_E I\frac{\partial \phi^3(x,t)}{\partial x^2 \partial t} + \frac{C_G A}{\bar{k}}\left(\frac{\partial^2 y(x,t)}{\partial x \partial t} - \frac{\partial \phi(x,t)}{\partial t}\right) = 0 \end{aligned} \tag{15.6}$$

The bending moment function $M(x, t)$ and the shear force function $T(x, t)$ of the axial-loaded Timoshenko beam with distributed internal viscous damping are written as

$$M(x,t) = EI\frac{\partial \phi(x,t)}{\partial x} + C_E I\frac{\partial^2 \phi(x,t)}{\partial x \partial t} \tag{15.7}$$

$$\begin{aligned} T(x,t) = \frac{AG}{\bar{k}}\gamma(x,t) - \frac{C_G A}{\bar{k}}\frac{\partial \gamma(x,t)}{\partial t} = \frac{AG}{\bar{k}}\left(\frac{\partial y(x,t)}{\partial x} - \phi(x,t)\right) \\ - \frac{C_G A}{\bar{k}}\left(\frac{\partial^2 y(x,t)}{\partial x \cdot \partial t} - \frac{\partial \phi(x,t)}{\partial t}\right) \end{aligned} \tag{15.8}$$

where $\gamma(x,t)$ is the corresponding shear deformation.

Assuming that the motion is harmonic we substitute for $y(x, t)$ and $\phi(x,t)$ the following:

$$y(x,t) = y(x)e^{i\omega t} \tag{15.9a}$$

$$\phi(x,t) = \phi(x)e^{i\omega t} \tag{15.9b}$$

where $y(x)$ and $\phi(x)$ are the amplitudes of the total transverse deflection and the angle of rotation due to bending, respectively; ω is the natural circular frequency of the vibrating system and $i = \sqrt{-1}$. By using Eq. (15.9a, b) and introducing the dimensionless coordinate $z = x/L$, Eqs. (15.5) and (15.6) can be converted into the ordinary differential equations:

$$\left(\frac{AG + C_G Ai\omega}{\bar{k}L^2} - \frac{N}{L^2}\right)\frac{\mathrm{d}^2 y(z)}{\mathrm{d}z^2} - \left(\frac{AG + C_G Ai\omega}{L\bar{k}}\right)\frac{\mathrm{d}\phi(z)}{\mathrm{d}z} + \left(m\omega^2\right)y(z) = 0 \tag{15.10}$$

$$\left(\frac{EI + C_E Ii\omega}{L^2}\right)\frac{\mathrm{d}^2\phi(z)}{\mathrm{d}z^2} + \left(\frac{AG + C_G Ai\omega}{L\bar{k}}\right)\frac{\mathrm{d}y(z)}{\mathrm{d}z} + \left(\frac{m\omega^2 I}{A} - \frac{(AG + C_G Ai\omega)}{\bar{k}}\right)\phi(z) = 0 \tag{15.11}$$

$$y(z) = C \cdot \mathrm{e}^{isz} \tag{15.12}$$

$$\phi(z) = P \cdot \mathrm{e}^{isz} \tag{15.13}$$

Substituting Eqs. (15.12) and (15.13) into Eqs. (15.10) and (15.11) results in

$$\left[m\omega^2 - \left(\frac{AG + C_G Ai\omega}{\bar{k}L^2} - \frac{N}{L^2}\right)s^2\right]C - \left[\left(\frac{AG + C_G Ai\omega}{L \cdot \bar{k}}\right)is\right]P = 0 \tag{15.14}$$

$$\left[\left(\frac{AG + C_G Ai\omega}{L\bar{k}}\right)is\right]C + \left(\frac{m\omega^2 I}{A} - \frac{(AG + C_G Ai\omega)}{\bar{k}} - \left(\frac{EI + C_E Ii\omega}{L^2}\right)s^2\right)P = 0 \tag{15.15}$$

Equations (15.14)–(15.15) can be written in matrix form as

$$\begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}\begin{Bmatrix} C \\ P \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix} \tag{15.16}$$

where

$$A_{11} = m\omega^2 - \left(\frac{AG + C_G Ai\omega}{\bar{k}L^2} - \frac{N}{L^2}\right)s^2 \tag{15.17a}$$

$$A_{21} = -A_{12} = \left(\frac{AG + C_G Ai\omega}{L\bar{k}}\right)is \tag{15.17b}$$

$$A_{22} = \frac{m\omega^2 I}{A} - \frac{(AG + C_G Ai\omega)}{\bar{k}} - \left(\frac{EI + C_E Ii\omega}{L^2}\right)s^2 \tag{15.17c}$$

The nontrivial solution is obtained when the determinant of the coefficient matrix is set up to zero. Thus, we have a fourth-order equation with the unknowns, resulting in four values and the general solution functions can be written as:

$$y(z,t) = \left[C_1 e^{is_1 z} + C_2 e^{is_2 z} + C_3 e^{is_3 z} + C_4 e^{is_4 z}\right] e^{i\omega t} \tag{15.18}$$

$$\phi(z,t) = \left[P_1 e^{is_1 z} + P_2 e^{is_2 z} + P_3 e^{is_3 z} + P_4 e^{is_4 z}\right] e^{i\omega t} \tag{15.19}$$

The eight constants, C_1, …, C_4 and P_1, …, P_4 will be found from Eqs. (15.14), (15.15) and boundary conditions.

The bending moment and shear force functions of the axially loaded Timoshenko beam with distributed internal viscous damping can be obtained by using Eqs. (15.7) and (15.8) as:

$$M(z,t) = \left[\left(\frac{EI + C_E I i\omega}{L}\right)\frac{\mathrm{d}\phi(z)}{\mathrm{d}z}\right] e^{i\omega t} \tag{15.20}$$

$$T(z,t) = \left[\frac{(AG - C_G A i\omega)}{\bar{k}L}\frac{\mathrm{d}y(z)}{\mathrm{d}z} - \left(\frac{AG - C_G A i\omega}{\bar{k}}\right)\phi(z)\right] e^{i\omega t} \tag{15.21}$$

15.3 The Differential Transform Method (DTM)

DTM is a semi-analytic transformation technique based on Taylor series expansion and is a useful tool to obtain analytical solutions of the differential equations. Certain transformation rules are applied and the governing differential equations and the boundary conditions of the system are transformed into a set of algebraic equations in terms of the differential transforms of the original functions in DTM. The solution of these algebraic equations gives the desired solution of the problem. The DTM differs from Taylor series as Taylor series method requires symbolic computation of the necessary derivatives of the data functions and is expensive for large orders. DTM is an iterative procedure to obtain analytic Taylor series solutions of differential equations (Yesilce 2015).

A function $y(z)$, which is analytic in a domain D, can be represented by a power series with a center at $z = z_0$, any point in D. The differential transform of the function $y(z)$ is given by

$$Y(k) = \frac{1}{k!}\left(\frac{\mathrm{d}^k y(z)}{\mathrm{d}z^k}\right)_{z=z_0} \tag{15.22}$$

where $y(z)$ is the original function and $Y(k)$ is the transformed function. The inverse transformation is defined as

$$y(z) = \sum_{k=0}^{\infty} (z - z_0)^k Y(k) \tag{15.23}$$

From Eqs. (15.22) and (15.23), we get

$$y(z) = \sum_{k=0}^{\infty} \frac{(z - z_0)^k}{k!} \left(\frac{d^k y(z)}{dz^k} \right)_{z=z_0} \tag{15.24}$$

Equation (15.24) implies that the concept of the differential transformation is derived from Taylor's series expansion, but the method does not evaluate the derivatives symbolically. However, relative derivatives are calculated by iterative procedure that are described by the transformed equations of the original functions. In real applications, the function $y(z)$ in Eq. (15.23) is expressed by a finite series and can be written as:

$$y(z) = \sum_{k=0}^{\bar{N}} (z - z_0)^k Y(k) \tag{15.25}$$

Equation (15.25) implies that $\sum_{k=\bar{N}+1}^{\infty} (z - z_0)^k Y(k)$ is negligibly small. Where $\bar{N}$ is the series size and the value of $\bar{N}$ depends on the convergence of the eigenvalues.

Theorems that are frequently used in differential transformation of the differential equations and the boundary conditions are introduced in Tables 15.1 and 15.2, respectively.

15.3.1 Application of DTM for Solving Equations of Motion

Equations (15.10) and (15.11) can be rewritten as follows:

$$\begin{aligned} \frac{d^2 y(z)}{dz^2} = & \left[\frac{(AG + \eta AGi\omega) \cdot L^3}{(AG + \eta AGi\omega)L^2 - \bar{k}N_r EI} \right] \frac{d\phi(z)}{dz} \\ & - \left[\frac{m\omega^2 \bar{k} L^4}{(AG + \eta AGi\omega)L^2 - \bar{k}N_r EI} \right] y(z) \end{aligned} \tag{15.26}$$

Table 15.1 DTM theorems used for equations of motion

Original function	Transformed function
$y(z) = u(z) \pm v(z)$	$Y(k) = U(k) \pm V(k)$
$y(z) = a \cdot u(z)$	$Y(k) = a \cdot U(k)$
$y(z) = \frac{d^m u(z)}{dz^m}$	$Y(k) = \frac{(k+m)!}{k!} \cdot U(k+m)$
$y(z) = u(z) \cdot v(z)$	$Y(k) = \sum_{r=0}^{k} U(r) \cdot V(k-r)$

Table 15.2 DTM theorems used for boundary conditions

$z\ = 0$		$z\ = 1$	
Original boundary conditions	Transformed boundary conditions	Original boundary conditions	Transformed boundary conditions
$y(0) = 0$	$Y(0) = 0$	$y(1) = 0$	$\sum_{k=0}^{\infty} Y(k) = 0$
$\frac{dy}{dz}(0) = 0$	$Y(1) = 0$	$\frac{dy}{dz}(1) = 0$	$\sum_{k=0}^{\infty} kY(k) = 0$
$\frac{d^2y}{dz^2}(0) = 0$	$Y(2) = 0$	$\frac{d^2y}{dz^2}(1) = 0$	$\sum_{k=0}^{\infty} k(k-1)Y(k) = 0$
$\frac{d^3y}{dz^3}(0) = 0$	$Y(3) = 0$	$\frac{d^3y}{dz^3}(1) = 0$	$\sum_{k=0}^{\infty} k(k-1)(k-2)Y(k) = 0$

$$\begin{aligned}\frac{d^2\phi(z)}{dz^2} = &-\left[\frac{(AG+\eta AGi\omega)L}{(EI+\eta EIi\omega)\bar{k}}\right]\frac{dy(z)}{dz}\\ &+\left[\frac{(AG+\eta AGi\omega)L^2}{(EI+\eta EIi\omega)\bar{k}}-\frac{mI\omega^2L^2}{(EI+\eta EIi\omega)A}\right]\phi(z)\end{aligned} \tag{15.27}$$

where

$$N_r = \frac{NL^2}{EI}\text{(Nondimensionalized multiplication factor for compressive force)} \tag{15.28a}$$

$$C_E = \eta E \tag{15.28b}$$

$$C_E = \eta G \tag{15.28c}$$

$$\eta = \xi\sqrt{\frac{mI}{EA^2}}\text{(}\xi\text{ is nondimensionalized damping value called as damping factor)} \tag{15.28d}$$

The differential transformation is applied to Eqs. (15.26) and (15.27) by using the theorems introduced in Table 5.1 and the following expressions are obtained:

$$\begin{aligned}Y(k+2) = &\left[\frac{(AG+\eta AGi\omega)L^3}{(AG+\eta AGi\omega)L^2-\bar{k}N_rEI}\right]\frac{\Phi(k+1)}{(k+2)}\\ &-\left[\frac{m\omega^2\bar{k}L^4}{(AG+\eta AGi\omega)L^2-\bar{k}N_rEI}\right]\frac{Y(k)}{(k+1)(k+2)}\end{aligned} \tag{15.29}$$

$$\Phi(k+2) = -\left[\frac{(AG+\eta AGi\omega)L}{(EI+\eta EIi\omega)\bar{k}}\right]\frac{Y(k+1)}{(k+2)} + \left[\frac{(AG+\eta AGi\omega)L^2}{(EI+\eta EIi\omega)\bar{k}} - \frac{mI\omega^2L^2}{(EI+\eta EIi\omega)A}\right]\frac{\Phi(k)}{(k+1)(k+2)} \quad (15.30)$$

where $Y(k)$ is the transformed function of y(z) and $\Phi(k)$ is the transformed function of $\phi(z)$.

The boundary conditions of a simply supported Timoshenko beam with distributed internal viscous damping are given below:

$$y(z=0)=0 \quad (15.31a)$$

$$M(z=0)=0 \quad (15.31b)$$

$$y(z=1)=0 \quad (15.31c)$$

$$M(z=1)=0 \quad (15.31d)$$

Applying DTM to Eqs. (15.31a)–(15.31d) and using the theorems introduced in Table 15.2, the transformed boundary conditions of a simply supported beam are obtained as

$$\text{for } z=0; \quad Y(0)=\Phi(1)=0 \quad (15.32a)$$

$$\text{for } z=1; \quad \sum_{k=0}^{\bar{N}} Y(k) = \sum_{k=0}^{\bar{N}} \bar{M}(k) = 0 \quad (15.32b)$$

where $\bar{M}(k)$ is the transformed function of $M(z)$.

The boundary conditions of a fixed-fixed Timoshenko beam with distributed internal viscous damping are given below

$$y(z=0)=0 \quad (15.33a)$$

$$\varphi(z=0)=0 \quad (15.33b)$$

$$y(z=1)=0 \quad (15.33c)$$

$$\varphi(z=1)=0 \quad (15.33d)$$

Applying DTM to Eqs. (15.33a)–(15.33d), the transformed boundary conditions of a fixed-fixed beam are obtained as:

$$\text{for } z=0; \quad Y(0)=\Phi(0)=0 \quad (15.34a)$$

$$\text{for}\, z = 1; \quad \sum_{k=0}^{\bar{N}} Y(k) = \sum_{k=0}^{\bar{N}} \Phi(k) = 0 \tag{15.34b}$$

The boundary conditions of one end (z = 0) fixed and the other end (z = 1) simply supported Timoshenko beam are given below:

$$y(z = 0) = 0 \tag{15.35a}$$

$$\varphi(z = 0) = 0 \tag{15.35b}$$

$$y(z = 1) = 0 \tag{15.35c}$$

$$M(z = 1) = 0 \tag{15.35d}$$

Applying differential transformation to Eqs. (15.35a)–(15.35d), the transformed boundary conditions of one end fixed and the other end simply supported beam are obtained as

$$\text{for}\, z = 0; \quad Y(0) = \Phi(0) = 0 \tag{15.36a}$$

$$\text{for}\, z = 1; \quad \sum_{k=0}^{\bar{N}} Y(k) = \sum_{k=0}^{\bar{N}} \bar{M}(k) = 0 \tag{15.36b}$$

For simply supported beam, substituting the boundary conditions expressed in Eqs. (15.32a) and (15.32b) into Eqs. (15.29) and (15.30), and taking $Y(1) = \mathrm{c}_1$, $\Phi(0) = \mathrm{c}_2$; for fixed-fixed supported beam, substituting the boundary conditions expressed in Eqs. (15.34a) and (15.34b) into Eqs. (15.29) and (15.30), and taking $Y(1) = \mathrm{c}_1$, $\Phi(1) = \mathrm{c}_2$; for one end fixed and the other end simply supported beam, substituting the boundary conditions expressed in Eqs. (15.36a) and (15.36b) into Eqs. (15.29) and (15.30), and taking $Y(1) = \mathrm{c}_1$, $\Phi(1) = \mathrm{c}_2$; the following matrix expression is obtained:

$$\begin{bmatrix} \bar{A}_{11}^{(\bar{N})}(\omega) & \bar{A}_{12}^{(\bar{N})}(\omega) \\ \bar{A}_{21}^{(\bar{N})}(\omega) & \bar{A}_{22}^{(\bar{N})}(\omega) \end{bmatrix} \begin{Bmatrix} c_1 \\ c_2 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix} \tag{15.37}$$

where c_1 and c_2 are constants and $\bar{A}_{j1}^{(\bar{N})}(\omega)$, $\bar{A}_{j2}^{(\bar{N})}(\omega)$ (j = 1, 2) are polynomials of ω corresponding $\bar{N}$.

In the last step, for nontrivial solution, equating the coefficient matrix that is given in Eq. (15.37) to zero one determines the natural frequencies of the vibrating system as given in Eq. (15.38).

$$\begin{vmatrix} \bar{A}_{11}^{(\bar{N})}(\omega) & \bar{A}_{12}^{(\bar{N})}(\omega) \\ \bar{A}_{21}^{(\bar{N})}(\omega) & \bar{A}_{22}^{(\bar{N})}(\omega) \end{vmatrix} = 0 \tag{15.38}$$

The jth estimated eigenvalue, $\omega_j^{(\bar{N})}$ corresponds to $\bar{N}$ and the value of $\bar{N}$ is determined as

$$\left|\omega_j^{(\bar{N})} - \omega_j^{(\bar{N}-1)}\right| \leq \varepsilon \tag{15.39}$$

where $\omega_j^{(\bar{N}-1)}$ is the jth estimated eigenvalue corresponding to $(\bar{N}-1)$ and ε is the small tolerance parameter. If Eq. (15.39) is satisfied, the jth estimated eigenvalue, $\omega_j^{(\bar{N})}$ is obtained.

The procedure explained below can be used to plot the mode shapes of the axially loaded Timoshenko beam with distributed internal viscous damping. The following equalities can be written by using Eq. (15.37):

$$\bar{A}_{11}(\omega)c_1 + \bar{A}_{12}(\omega)c_2 = 0 \tag{15.40}$$

Using Eq. (15.40), the constant c_2 can be obtained in terms of c_1 as follows:

$$c_2 = -\frac{\bar{A}_{11}(\omega)}{\bar{A}_{12}(\omega)}c_1 \tag{15.41}$$

All transformed functions can be expressed in terms of ω, c_1 and c_2. Since c_2 has been written in terms of c_1 above, $Y(k)$, $\Phi(k)$ and $\bar{M}(k)$ can be expressed in terms c_1 as follows:

$$Y(k) = Y(\omega, c_1) \tag{15.42a}$$

$$\Phi(k) = \Phi(\omega, c_1) \tag{15.42b}$$

$$\bar{M}(k) = \bar{M}(\omega, c_1) \tag{15.42c}$$

The mode shapes can be plotted for several values of ω by using Eq. (15.42a).

15.4 Dynamic Stiffness Formulation

The dynamic stiffness matrix of a beam relates the amplitudes of end forces to the amplitudes of end displacement of a beam. The vector of end displacements of beam and the vector of coefficients are given in Eqs. (15.43) and (15.44), respectively.

$$\delta = [\, y_0 \quad y_1 \quad \phi_0 \quad \phi_1 \,]^{\mathrm{T}} \tag{15.43}$$

$$C = [\, C_1 \quad C_2 \quad C_3 \quad C_4 \,]^{\mathrm{T}} \tag{15.44}$$

where

$$y_0 = y(z=0), y_1 = y(z=1), \phi_0 = \phi(z=0), \phi_1 = \phi(z=1)$$

Equations (15.18) and (15.19) are used to obtain Eq. (15.45):

$$\begin{bmatrix} y_0 \\ y_1 \\ \phi_0 \\ \phi_1 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 & 1 \\ \mathrm{e}^{is_1} & \mathrm{e}^{is_2} & \mathrm{e}^{is_3} & \mathrm{e}^{is_4} \\ K_1 & K_2 & K_3 & K_4 \\ K_1\mathrm{e}^{is_1} & K_2\mathrm{e}^{is_2} & K_3\mathrm{e}^{is_3} & K_4\mathrm{e}^{is_4} \end{bmatrix} \begin{bmatrix} C_1 \\ C_2 \\ C_3 \\ C_4 \end{bmatrix} \tag{15.45}$$

where

$$K_1 = \frac{-\left(\frac{AG + C_G A i\omega}{\bar{k}L^2} - \frac{N}{L^2}\right)s_1^2 + m\omega^2}{\left(\frac{AG + C_G A i\omega}{L\bar{k}}\right) i s_1}; \quad K_2 = \frac{-\left(\frac{AG + C_G A i\omega}{\bar{k}L^2} - \frac{N}{L^2}\right)s_2^2 + m\omega^2}{\left(\frac{AG + C_G A i\omega}{L\cdot\bar{k}}\right) i s_2};$$

$$K_3 = \frac{-\left(\frac{AG + C_G A i\omega}{\bar{k}L^2} - \frac{N}{L^2}\right)s_3^2 + m\omega^2}{\left(\frac{AG + C_G A\cdot i\omega}{L\bar{k}}\right) i s_3}; \quad K_4 = \frac{-\left(\frac{AG + C_G A i\omega}{\bar{k}L^2} - \frac{N}{L^2}\right)s_4^2 + m\omega^2}{\left(\frac{AG + C_G A i\omega}{L\bar{k}}\right) i s_4}$$

The closed form of Eq. (15.45) is presented in Eq. (15.46):

$$\delta = \Delta C \tag{15.46}$$

where

$$\Delta = \begin{bmatrix} 1 & 1 & 1 & 1 \\ \mathrm{e}^{is_1} & \mathrm{e}^{is_2} & \mathrm{e}^{is_3} & \mathrm{e}^{is_4} \\ K_1 & K_2 & K_3 & K_4 \\ K_1\mathrm{e}^{is_1} & K_2\mathrm{e}^{is_2} & K_3\mathrm{e}^{is_3} & K_4\mathrm{e}^{is_4} \end{bmatrix}$$

The vector of end forces of a member is given by

$$F = [\, T_0 \quad T_1 \quad M_0 \quad M_1 \,]^{T} \tag{15.47}$$

where

$$T_0 = T(z=0), T_1 = T\,(z=1), M_0 = M(z=0), M_1 = M\,(z=1)$$

It should be noted that the following sign convention is valid for DSM.

$$T_0 = -T_1, M_0 = -M_1 \tag{15.48}$$

Equations (15.20) and (15.21) are used to construct the matrix form below

$$\begin{bmatrix} T_0 \\ T_1 \\ M_0 \\ M_1 \end{bmatrix} = \kappa \begin{bmatrix} C_1 \\ C_2 \\ C_3 \\ C_4 \end{bmatrix} \tag{15.49}$$

The closed form of Eq. (15.49) is presented in Eq. (15.50):

$$F = \kappa C \tag{15.50}$$

where

$$\kappa = \begin{bmatrix} R_1 is_1 - R_2K_1 & R_1 is_2 - R_2K_2 & R_1 is_3 - R_2K_3 & R_1 is_4 - R_2K_4 \\ (R_1 is_1 - R_2K_1)\mathrm{e}^{is_1} & (R_1 is_2 - R_2K_2)\mathrm{e}^{is_2} & (R_1 is_3 - R_2K_3)\mathrm{e}^{is_3} & (R_1 is_4 - R_2K_4)\mathrm{e}^{is_4} \\ R_3 is_1 K_1 & R_3 is_2 K_2 & R_3 is_3 K_3 & R_3 is_4 K_4 \\ (R_3 is_1 K_1)\mathrm{e}^{is_1} & (R_3 is_2 K_2)\mathrm{e}^{is_2} & (R_3 is_3 K_3)\mathrm{e}^{is_3} & (R_3 is_4 K_4)\mathrm{e}^{is_4} \end{bmatrix}$$

$$R_1 = \frac{(AG - C_G Ai\omega)}{\bar{k}L}; \quad R_2 = \left(\frac{AG - C_G Ai\omega}{\bar{k}}\right); \quad R_3 = \left(\frac{EI + C_E Ii\omega}{L}\right)$$

The dynamic stiffness matrix of the beam can be obtained by using Eqs. (15.46) and (15.50):

$$F = \kappa(\Delta)^{-1}\delta \tag{15.51}$$

$$K^* = \kappa(\Delta)^{-1} \tag{15.52}$$

K^* denotes dynamic stiffness matrix of a Timoshenko beam with internal viscous damping and subjected to axial compression force.

15.5 Numerical Analysis and Discussions

Axially loaded Timoshenko beams with distributed internal viscous damping are considered in the numerical analysis. The first four natural frequencies, f_i (i = 1, …, 4 where $f_i = 2\pi\omega_i$) are calculated by using computer programs prepared in Matlab by the authors. The natural frequencies are calculated by equating the determinant of the coefficient matrix to zero for the analytical and differential transformation solutions. In the DSM, the natural frequencies are calculated by applying boundary conditions to K^* and using the equation below:

$$|K^{*}| = 0 \tag{15.53}$$

The numerical results of this paper are obtained based on uniform, rectangular Timoshenko beams with the following data:

m = 0.31250 kNs^2/m; $EI = 7.28086 \times 10^4$ kNm^2; $AG = 1.344159 \times 10^6$ kN; $\bar{k} = 6/5$; L = 3.0 m; N_r = 0.00, 0.50 and 1.00; ξ = 0.00, 0.10 and 0.20

Using the DTM and DSM, the frequency values of the simply supported Timoshenko beam for the first four modes are presented in Table 15.3. In Table 15.4, the first four frequency values of one end fixed, the other end simply supported Timoshenko beam can be seen. The fixed-fixed Timoshenko beam's first four frequency values are presented in Table 15.5. The first four mode shapes of the axially loaded Timoshenko beam with various boundary conditions for N_r = 1 and ξ = 0.2 are shown in Figs. 15.2, 15.3 and 15.4.

For all boundary conditions, as the axial compressive force acting to beams is increased with constant damping, the natural frequency values are decreased. This result indicates that the increasing axial compressive force leads to the reduction in natural frequencies for all types of boundary conditions. This result is very important for the effect of axial compressive force.

For the fixed supported axially loaded Timoshenko beams, an increase in natural frequency values is observed for the condition of N_r being constant and the values of the damping factor is increased. This result indicates that the increasing damping factor leads to an augmentation in natural frequency values for fixed-fixed boundary condition.

For simply supported and one end fixed, the other end simply supported boundary conditions, when the damping factor is increased with constant axial compressive load, a decrease is observed in natural frequency values of the first mode and the fourth mode and an increase is observed in natural frequency values of the second mode.

For the constant N_r and increasing damping factor, it is observed that the natural frequency values of the third mode of simply supported beam is increased. However, the third mode frequency of fixed-simple supported beam is decreased when the damping factor is increased with constant axial compressive force.

In the application of the DTM, the natural frequency values of the axially loaded Timoshenko beams with internal viscous damping are calculated by increasing series size $\bar{N}$. In Tables 15.3, 15.4 and 15.5, convergences of the first four natural frequencies are introduced. It is seen that the series size varies between 14 and 30 for perfect convergence in the DTM application for the first four modes of axially loaded Timoshenko beams with internal viscous damping. Additionally, it is observed that higher modes appear when more terms are taken into account in the DTM applications. Thus, depending on the order of the required mode, one must try a few values for the term number at the beginning of the calculations in order to find the adequate number of terms.

It is observed that the DSM is a reliable method for free vibration analysis of axially loaded Timoshenko beams with internal viscous damping.

Table15.3 The first four natural frequencies of the simply supported Timoshenko beam for different values of N_r and ξ

		$\xi = 0$			$\xi = 0.1$			$\xi = 0.2$		
f_i (Hz)	Method	N_r			N_r			N_r		
		0	0.5	1	0	0.5	1	0	0.5	1
f_1	DTM ($\bar{N} = 14$)	80.5949	78.3774	76.0954	80.5949	78.3774	76.0954	80.5948	78.3773	76.0953
	ANM	80.5949	78.3774	76.0954	80.5949	78.3774	76.0954	80.5948	78.3773	76.0953
	DSM	80.5949	78.3774	76.0954	80.5949	78.3774	76.0954	80.5944	78.3773	76.0953
f_2	DTM ($\bar{N} = 20$)	289.2450	286.8833	284.5019	289.2472	286.8854	284.5040	289.2539	286.8919	284.5102
	ANM	289.2450	286.8833	284.5019	289.2472	286.8854	284.5040	289.2539	286.8919	284.5102
	DSM	289.2450	286.8833	284.5019	289.2429	286.8811	284.4997	289.2366	286.8747	284.0629
f_3	DTM ($\bar{N} = 24$)	569.4642	566.8363	564.1960	569.4742	566.8464	564.2062	569.5047	566.8772	564.2373
	ANM	569.4642	566.8363	564.1960	569.4742	566.8464	564.2062	569.5047	566.8772	564.2373
	DSM	569.4642	566.8363	564.1960	569.5492	566.9205	564.2795	569.8036	567.1727	564.5292
f_4	DTM ($\bar{N} = 28$)	883.6495	880.6716	877.6833	883.5194	880.5446	877.5593	883.1457	880.1796	877.2030
	ANM	883.6495	880.6716	877.6833	883.5194	880.5446	877.5593	883.1457	880.1796	877.2030
	DSM	883.6495	880.6716	877.6833	884.7772	881.7946	878.8015	883.4128	880.4127	877.4021

Table 15.4 The first four natural frequencies of one end fixed, the other end simply supported Timoshenko beam for different values of N_r and ξ

		ξ = 0			ξ = 0.1			ξ = 0.2		
f_i (Hz)	Method	N_r			N_r			N_r		
		0	0.5	1	0	0.5	1	0	0.5	1
f_1	DTM ($\bar{N} = 16$)	119.1143	117.4462	115.7516	119.1141	117.4460	115.7514	119.1136	117.4455	115.7509
	ANM	119.1143	117.4462	115.7516	119.1141	117.4460	115.7514	119.1136	117.4455	115.7509
	DSM	119.1143	117.4462	115.7516	119.1150	117.4469	115.7522	119.1171	117.4489	115.7543
f_2	DTM ($\bar{N} = 20$)	338.5120	336.4291	334.3328	338.5129	336.4298	334.3335	338.5156	336.4322	334.3355
	ANM	338.5120	336.4291	334.3328	338.5129	336.4298	334.3335	338.5156	336.4322	334.3355
	DSM	338.5120	336.4291	334.3328	338.3975	336.4734	334.3769	338.6911	336.6071	334.5099
f_3	DTM ($\bar{N} = 24$)	613.9415	611.4768	609.0019	613.9375	611.4729	608.9981	613.9263	611.4621	608.9877
	ANM	613.9415	611.4768	609.0019	613.9375	611.4729	608.9981	613.9263	611.4621	608.9877
	DSM	613.9415	611.4768	609.0019	614.5247	612.0577	609.5806	614.3902	611.9153	609.4303
f_4	DTM ($\bar{N} = 28$)	917.9218	915.0452	912.1592	917.7442	914.8706	911.9876	917.2354	914.3707	911.4965
	ANM	917.9218	915.0452	912.1592	917.7442	914.8706	911.9876	917.2354	914.3707	911.4965
	DSM	917.9218	915.0452	912.1592	918.4556	915.5612	912.6574	917.8426	915.0124	911.9886

Table 15.5 The first four natural frequencies of the fixed-fixed Timoshenko beam for different values of N_r and ξ

		$\xi = 0$			$\xi = 0.1$			$\xi = 0.2$		
f_i (Hz)	Method	N_r			N_r			N_r		
		0	0.5	1	0	0.5	1	0	0.5	1
f_1	DTM ($\bar{N} = 16$)	162.5348	161.2862	160.0261	162.5357	161.2871	160.0270	162.5385	161.2897	160.0295
	ANM	162.5348	161.2862	160.0261	162.5357	161.2871	160.0270	162.5385	161.2897	160.0295
	DSM	162.5348	161.2862	160.0261	162.6000	161.3459	160.0802	162.6746	161.4148	160.1435
f_2	DTM ($\bar{N} = 20$)	385.2480	383.3589	381.4597	385.2666	383.3772	381.4777	385.3221	383.4321	381.5319
	ANM	385.2480	383.3589	381.4597	385.2666	383.3772	381.4777	385.3221	383.4321	381.5319
	DSM	385.2480	383.3589	381.4597	386.2298	384.3244	382.4088	387.3330	385.4102	383.4772
f_3	DTM ($\bar{N} = 24$)	655.4122	653.0751	650.7295	655.4927	653.1554	650.8095	655.7342	653.3963	651.0497
	ANM	655.4122	653.0751	650.7295	655.4927	653.1554	650.8095	655.7342	653.3963	651.0497
	DSM	655.4122	653.0751	650.7295	656.2571	653.8838	651.5017	656.2989	653.8904	651.4732
f_4	DTM ($\bar{N} = 30$)	949.7997	947.0060	944.2036	949.8921	947.1003	944.3001	950.1785	947.3929	944.5986
	ANM	949.7997	947.0060	944.2036	949.8921	947.1003	944.3001	950.1785	947.3929	944.5986
	DSM	949.7997	947.0060	944.2036	950.4602	947.6031	944.7371	950.6756	947.7414	944.9369

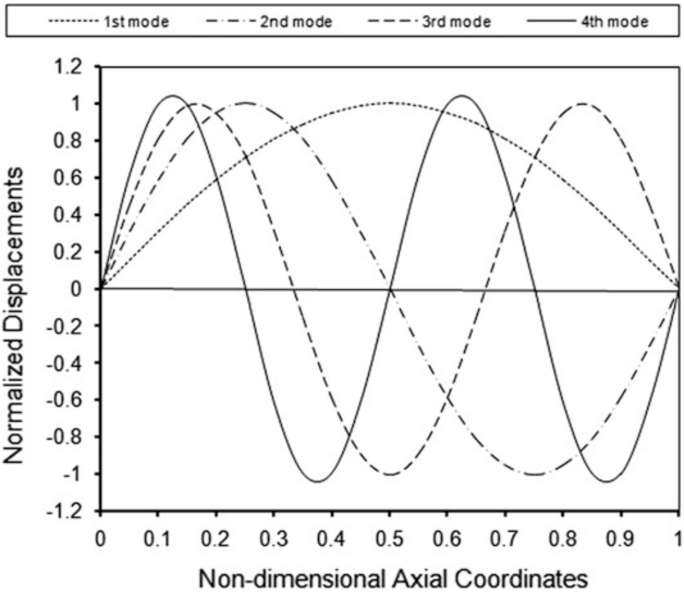

Fig. 15.2 The first four mode shapes of the simply supported and axial-loaded Timoshenko beam with internal viscous damping, $N_r = 1.00$ and $\xi = 0.20$

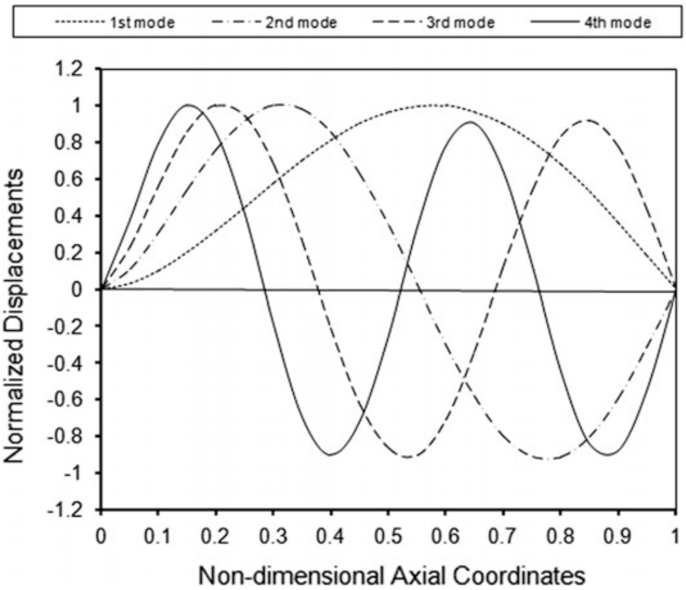

Fig. 15.3 The first four mode shapes of one end fixed, the other end simply supported and axial-loaded Timoshenko beam with internal viscous damping, $N_r = 1.00$ and $\xi = 0.20$

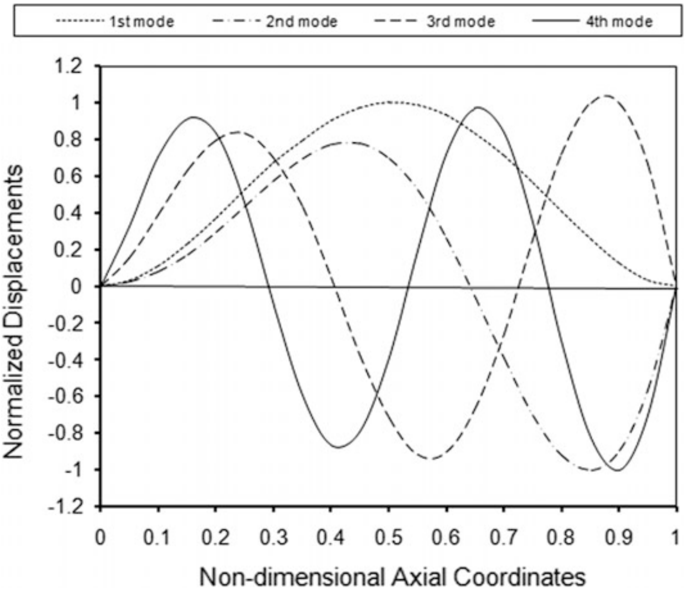

Fig. 15.4 The first four mode shapes of the fixed-fixed and axial-loaded Timoshenko with internal viscous damping, $N_r = 1.00$ and $\xi = 0.20$

15.6 Conclusions

The effects of viscous damping with axial compressive load on natural frequencies of Timoshenko beams are observed for different support conditions. This study reveals that DSM and DTM can be used effectively for free vibration analysis of axially loaded Timoshenko beams including internal viscous damping. The procedure of DSM is simple when compared to DTM. The application of the DTM to both the equations of motion and the boundary conditions seem to be involved computationally. However, all the algebraic calculations are finished quickly using symbolic computational software. Besides all these, the results show that DTM solutions converge fast. When the results of the DTM and DSM are compared with the results of analytical method, very good agreement is observed.

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