Singular Value Decomposition in Image Compression and Blurred Image Restoration

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Abstract. The singular value decomposition (SVD) is an important and very versatile tool for matrix computations with a variety of uses. The contribution briefly introduces the concept of the SVD and basic facts about it and then describes two classes of its applications in image processing - image compression and blurred image restoration. Calculations are implemented in MATLAB software. Our experiences and the results are presented in the text.

Keywords: Singular value decomposition · Matrix computations
Image processing · Image compression · Image deblurring

1 Introduction

The SVD is one of the most important and most versatile matrix computations tools. Its application can be found both in mathematical theory and in various practical areas. The SVD is related to many other concepts of linear algebra [3]. It is possible to use it for example to determine matrix rank, the Frobenius norm or spectral norm of a matrix, the condition number of a matrix, an orthonormal basis for the null space and the column space of a matrix, the approximation of a matrix by a matrix of lower rank. Further large application domain is in statistics in the context of principal component analysis and correspondence analysis. Another major application of the SVD is the area of signal processing including compression or data filtering [6], also it is used for data registration [1], recognition [7], steganography watermarking [5], latent semantic indexing and analysis [2] etc.

2 Basic Theoretical Facts About SVD

The following theorem states the existence of the SVD for any real matrix.

Theorem 1. *Let* $\boldsymbol{A} \in \mathbb{R}^{m\times n}$ *and* $p = \min\{m, n\}$*. Then there exist orthogonal matrices* $\boldsymbol{U} \in \mathbb{R}^{m\times m}$*,* $\boldsymbol{V} \in \mathbb{R}^{n\times n}$ *and a diagonal matrix* $\boldsymbol{\Sigma} \in \mathbb{R}^{m\times n}$ *with diagonal elements* $\sigma_1 \geq \sigma_2 \geq \cdots \geq \sigma_p \geq 0$ *so that it holds*

$$\boldsymbol{A} = \boldsymbol{U}\boldsymbol{\Sigma}\boldsymbol{V}^T. \tag{1}$$

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Diagonal elements $\sigma_{jj} = \sigma_j, j = 1, \ldots, p$, of the matrix $\boldsymbol{\Sigma}$ are called the singular values of the matrix $\boldsymbol{A}$. Let $\boldsymbol{u}_j$ resp. $\boldsymbol{v}_j$ denote the jth column of the matrix $\boldsymbol{U}$ respectively matrix $\boldsymbol{V}$. Vectors $\boldsymbol{u}_j, j = 1, \ldots, m$, are called the left singular vectors and vectors $\boldsymbol{v}_j, j = 1, \ldots, n$, are called the right singular vectors of the matrix $\boldsymbol{A}$.

The singular values are uniquely determined, and if we in addition to it suppose that they are written in sorted order ($\sigma_1 \geq \sigma_2 \geq \cdots \geq \sigma_p \geq 0$) then the matrix $\boldsymbol{\Sigma}$ is uniquely determined too. On the other hand, the left singular vectors and the right singular vectors and consequently the matrices $\boldsymbol{U}$ and $\boldsymbol{V}$ are not uniquely determined.

The rank r of a matrix $\boldsymbol{A}$ is equal to the number of non-zero (positive) singular values

$$\sigma_1 \geq \sigma_2 \geq \cdots \geq \sigma_r > 0,\ \sigma_{r+1} = \sigma_{r+2} = \ldots = \sigma_p = 0, \quad p = \min\{m, n\}.$$

However, the above-mentioned fact is correct only if we assume the SVD calculated in the exact arithmetic. When we apply numerical calculations on computer, it can be assumed that numbers $\sigma_{r+1}, \ldots, \sigma_p$ will not be exact zeros. In this case it is not clear which of the singular values are really zero and which are just almost zero. The concept of the numerical rank of a matrix is introduced from this reason. A matrix has the numerical rank of k if k singular values are greater than the chosen tolerance $\delta > 0$, the other singular values are considered to be zero.

3 Low-Rank Matrix Approximation and Data Compression

The task of finding for a matrix $\boldsymbol{A} \in \mathbb{R}^{m \times n}$ of rank $\operatorname{rank}(\boldsymbol{A}) = r$ another matrix $\boldsymbol{A}_k \in \mathbb{R}^{m \times n}$ having rank $\operatorname{rank}(\boldsymbol{A}_k) = k < r$ which is in some sense its best approximation is useful in many applications. Here, the best approximation will be considered in the sense of minimizing the spectral norm of error $\boldsymbol{A} - \boldsymbol{A}_k$, i.e.

$$\|\boldsymbol{A} - \boldsymbol{A}_k\|_2 = \min_{\substack{\boldsymbol{X} \in \mathbb{R}^{m \times n} \\ \operatorname{rank}(\boldsymbol{X}) = k}} \|\boldsymbol{A} - \boldsymbol{X}\|_2. \tag{2}$$

The following Theorem 2 indicates the way of using the SVD to solve this problem. It is used the following form of the so-called economy-size SVD

$$\boldsymbol{A} = \boldsymbol{U}_r \boldsymbol{\Sigma}_r \boldsymbol{V}_r^T = \sum_{j=1}^{r} \sigma_j \boldsymbol{u}_j \boldsymbol{v}_j^T, \tag{3}$$

where $\boldsymbol{U}_r = \boldsymbol{U}_{\bullet,1:r}$, $\boldsymbol{\Sigma}_r = \boldsymbol{\Sigma}_{1:r,1:r}$ and $\boldsymbol{V}_r = \boldsymbol{V}_{\bullet,1:r}$.

Theorem 2 (Eckart, Young, Mirsky). *Let us have a matrix $\boldsymbol{A} \in \mathbb{R}^{m \times n}$ of rank r and let k be a natural number, $k < r$. Let us consider the SVD of the* *matrix* $\boldsymbol{A}$ *given by (3). Then the matrix*

$$\boldsymbol{A}_k = \sum_{j=1}^{k} \sigma_j \boldsymbol{u}_j \boldsymbol{v}_j^T$$

is the best rank k *approximation of the matrix* $\boldsymbol{A}$ *in terms of (2).*

A grayscale digital image can be represented by a matrix of type $m \times n$ whose element at position ij corresponds to the intensity of the ijth pixel of the image. Grayscale images will be taken for simplicity. The case of color image compression could be solved analogously by applying the further described procedure to each of the RGB color channels separately.

A photograph of the Large Square in the city of Hradec Kralove (Czech Republic) is used as a demonstration (see the leftmost image in Fig. 1). The image can be represented by the matrix $\boldsymbol{A}$ of size 706×670. In accordance with the Theorem 2 the matrices $\boldsymbol{A}_{200}$ and $\boldsymbol{A}_{20}$ are constructed. Only the elements of $\boldsymbol{u}_j$, $\boldsymbol{v}_j$ and the numbers σ_j, $j = 1, \dots, k$, are stored in the memory for each of these matrices when using the economy-size SVD, which represents the total amount of $(m+n+1)k$ values. The amount of $(m+n+1)r$ values must be stored for the initial matrix $\boldsymbol{A}$ of rank r. Thus, the compression ratio is given by fraction $\frac{r}{k}$.

Memory savings are 70.15 % for the approximation $\boldsymbol{A}_{200}$ and 97.01 % for $\boldsymbol{A}_{20}$ compared to the original image. However, the quality of approximation is getting worse with decreasing k. Figure 1 shows the obtained results.

Fig. 1. Initial image and compression of the image corresponding to the approximation matrices for $k = 200$, 20

4 Ill-Posed Problems and Regularization

The further described application of the SVD will concern blurred image restoration. A grayscale image represented by a matrix $\boldsymbol{X} \in \mathbb{R}^{m \times n}$ will be assumed. This matrix will represent the ideal sharp image and consequently at the same time the solution to be sought as the result of reconstruction (Fig. 2a). But in practice, we only have at our disposal an image given by a matrix $\boldsymbol{B} \in \mathbb{R}^{m\times n}$ that is degraded by blurring, for example due to the motion or poorly focused optical system of the camera (Fig. 2b). If the blurring is linear and the blurring of the columns in the image is independent of the blurring of the rows, the blurring process can be represented by the following matrix equation

$$\boldsymbol{A}_{\mathrm{C}}\boldsymbol{X}\boldsymbol{A}_{\mathrm{R}}^{T} = \boldsymbol{B}. \tag{4}$$

The matrix $\boldsymbol{A}_{\mathrm{C}} \in \mathbb{R}^{m\times m}$ represents the blurring of the columns and $\boldsymbol{A}_{\mathrm{R}} \in \mathbb{R}^{n\times n}$ the blurring of the rows.

Equation (4) can be rewritten to common form of a system of linear algebraic equations

$$\boldsymbol{A}\boldsymbol{x} = \boldsymbol{b}, \tag{5}$$

where $\boldsymbol{A} = \boldsymbol{A}_{\mathrm{R}} \otimes \boldsymbol{A}_{\mathrm{C}} \in \mathbb{R}^{mn\times mn}$ is a Kronecker product of the matrices $\boldsymbol{A}_{\mathrm{R}}$ and $\boldsymbol{A}_{\mathrm{C}}$. The vectors $\boldsymbol{x}$ and $\boldsymbol{b}$ represent vectorization of the matrices $\boldsymbol{X}$ and $\boldsymbol{B}$, i.e. $\boldsymbol{x} = \mathrm{vec}(\boldsymbol{X}) = (\boldsymbol{X}_{\bullet 1}^{T}, \ldots, \boldsymbol{X}_{\bullet n}^{T})^{T} \in \mathbb{R}^{mn}$ and $\boldsymbol{b} = \mathrm{vec}(\boldsymbol{B}) = (\boldsymbol{B}_{\bullet 1}^{T}, \ldots, \boldsymbol{B}_{\bullet n}^{T})^{T} \in \mathbb{R}^{mn}$.

When working with real data, it often happens that the vector $\boldsymbol{b}$ contains noise (measurement inaccuracies, rounding errors, discretization errors, etc.). So it holds

$$\boldsymbol{b} = \boldsymbol{b}_{\mathrm{EXACT}} + \boldsymbol{b}_{\mathrm{NOISE}},$$

where $\boldsymbol{b}_{\mathrm{EXACT}}$ is the exact right-hand side of the system of equations and $\boldsymbol{b}_{\mathrm{NOISE}}$ is the noise vector. It will be assumed that $\|\boldsymbol{b}_{\mathrm{EXACT}}\|_2 \gg \|\boldsymbol{b}_{\mathrm{NOISE}}\|_2$.

If $\mathrm{rank}(\boldsymbol{A}_{\mathrm{C}}) = m$ and $\mathrm{rank}(\boldsymbol{A}_{\mathrm{R}}) = n$ then the matrix $\boldsymbol{A}$ is nonsingular and the solution of the system of equations (5), usually referred to as naive, can simply be written as $\boldsymbol{x}_{\mathrm{NAIVE}} = \boldsymbol{A}^{-1}\boldsymbol{b}$. This solution would correspond to the desired solution $\boldsymbol{x}_{\mathrm{EXACT}} = \boldsymbol{A}^{-1}\boldsymbol{b}_{\mathrm{EXACT}}$ if the vector $\boldsymbol{b}$ did not contain noise. In case the vector $\boldsymbol{b}$ contains noise, for $\boldsymbol{x}_{\mathrm{NAIVE}}$ holds

$$\boldsymbol{x}_{\mathrm{NAIVE}} = \boldsymbol{A}^{-1}\boldsymbol{b} = \boldsymbol{x}_{\mathrm{EXACT}} + \boldsymbol{A}^{-1}\boldsymbol{b}_{\mathrm{NOISE}}. \tag{6}$$

Reconstruction of a blurred image is a typical example of ill-posed problems. Figure 2c shows that the naive solution (6) is dominated by the inverse noise $\boldsymbol{A}^{-1}\boldsymbol{b}_{\mathrm{NOISE}}$, and this noise completely overlaps the solution $\boldsymbol{x}_{\mathrm{EXACT}}$. The solution $\boldsymbol{x}_{\mathrm{NAIVE}}$ can be written using the SVD of the matrix $\boldsymbol{A}$ as

$$\boldsymbol{x}_{\mathrm{NAIVE}} = \sum_{j=1}^{mn} \frac{\boldsymbol{u}_j^{T}\boldsymbol{b}_{\mathrm{EXACT}}}{\sigma_j}\boldsymbol{v}_j + \sum_{j=1}^{mn} \frac{\boldsymbol{u}_j^{T}\boldsymbol{b}_{\mathrm{NOISE}}}{\sigma_j}\boldsymbol{v}_j. \tag{7}$$

Furthermore, it is assumed that the vector $\boldsymbol{b}_{\mathrm{EXACT}}$ meets the discrete Picard condition. This condition can be formulated like this: Magnitude of the $\boldsymbol{u}_j^{T}\boldsymbol{b}_{\mathrm{EXACT}}$ components is with increasing j dropping to zero on average faster than the absolute value of the corresponding singular values σ_j. The noise vector $\boldsymbol{b}_{\mathrm{NOISE}}$ does not have to meet this condition. Values $\boldsymbol{u}_j^{T}\boldsymbol{b}_{\mathrm{NOISE}}$ do not usually decrease to zero, and their influence will increase as a result of division by small

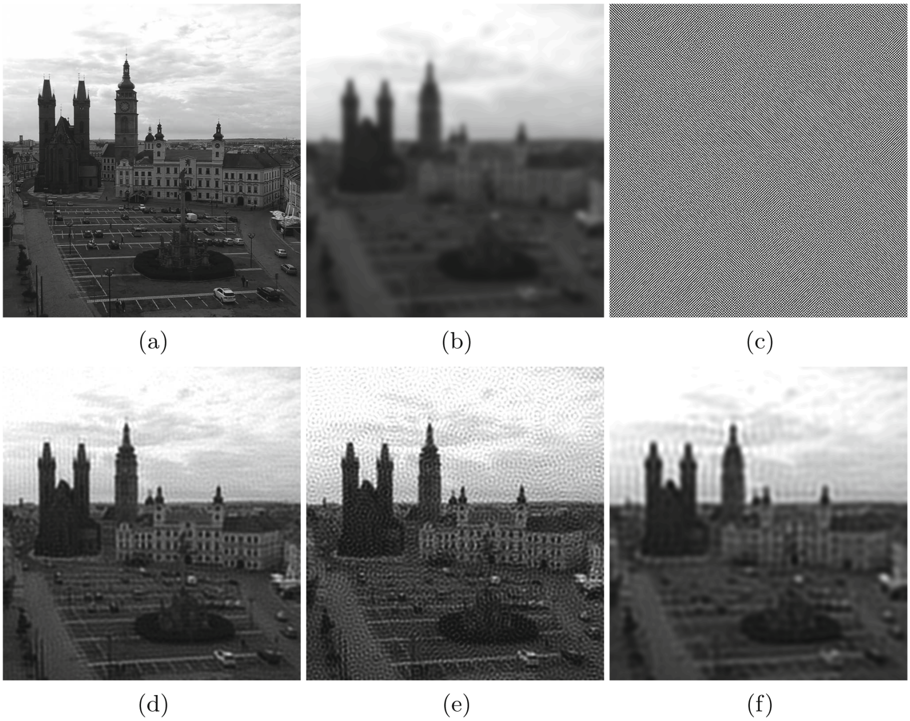

Fig. 2. Blurred image restoration: (a) initial sharp image (exact solution), (b) blurred image with additive noise, (c) naive solution, (d) solution for the appropriate choice of the truncation parameter ($k = 12693$), (e) undersmoothed solution with influence from high-frequency components of noise ($k = 15351$), (f) oversmoothed solution missing high-frequency information (noise, but also image details) ($k = 3602$)

singular values. The second sum dominates in the naive solution (7), which explains the result in Fig. 2c.

The way in which ill-posed problems can be solved is regularization. The goal is to find an approximation of the exact solution $\boldsymbol{x}_{\text{EXACT}}$ in order to suppress the impact of noise. One of the classic regularization methods is Truncated SVD (TSVD). TSVD regularization is based on the replacement of the matrix $\boldsymbol{A}$ by its best approximation of lower rank according to the Theorem 2

$$\boldsymbol{A}_k = \sum_{j=1}^{k} \sigma_j \boldsymbol{u}_j \boldsymbol{v}_j^T, \qquad k < mn.$$

The corresponding solution of the problem (5) (in the least squares sense) has the form

$$\boldsymbol{x}_{\text{REG}(k)} = \boldsymbol{A}_k^{\dagger} \boldsymbol{b} = \sum_{j=1}^{k} \frac{\boldsymbol{u}_j^T \boldsymbol{b}}{\sigma_j} \boldsymbol{v}_j.$$

The components mostly dominated by noise, corresponding to division by small singular values $\sigma_j, j = k + 1, \ldots, mn$, are simply removed.

Figure 2d – f shows solutions corresponding to various values of the truncation parameter k, for more details see [4].

5 Conclusion

The SVD has a variety of uses, some have been known for many years and have been developed by various authors over the course of time, while other applications are related to relatively new domains associated with the development of computing. The article briefly presents our experience with the SVD in the field of image compression and blurred image restoration. Additional outputs could not be included due to the limited scope of the text.

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References

1. Arun, K.S., Huang, T.S., Blostein, S.D.: Least-squares fitting of two 3-D point sets. IEEE Trans. Pattern Anal. Mach. Intell. **9**(5), 698–700 (1987). https://doi.org/10.1109/TPAMI.1987.4767965
2. Deerwester, S., Dumais, S.T., Furnas, G.W., Landauer, T.K., Harshman, R.: Indexing by latent semantic analysis. J. Am. Soc. Inf. Sci. **41**(6), 391–407 (1990). https://doi.org/10.1002/(SICI)1097-4571(199009)41:6⟨391::AID-ASI1⟩3.0.CO;2-9
3. Golub, G.H., Van Loan, C.F.: Matrix Computations, 4th edn. Johns Hopkins University Press, Baltimore (2013)
4. Hansen, P.C., Nagy, J., O'Leary, D.P.: Deblurring images: Matrices, Spectra, and Filtering, 1st edn. Society for Industrial and Applied Mathematics, Philadelphia (2006)
5. Liu, R., Tan, T.: An SVD-based watermarking scheme for protecting rightful ownership. IEEE Trans. Multimed. **4**(1), 121–128 (2002). https://doi.org/10.1109/6046.985560
6. Sadek, R.A.: SVD based image processing applications: state of the art, contributions and research challenges. Int. J. Adv. Comput. Sci. Appl. **3**(7), 26–34 (2012). arXiv:1211.7102
7. Turk, M.A., Pentland, A.P.: Eigenfaces for recognition. J. Cognit. Neurosci. **3**(1), 71–86 (1991). https://doi.org/10.1162/jocn.1991.3.1.71