Chapter 5
Solution for a System of Hamilton–Jacobi Equations of Special Type and a Link with Nash Equilibrium

Ekaterina A. Kolpakova

Abstract The paper is concerned with systems of Hamilton–Jacobi PDEs of the special type. This type of systems of Hamilton–Jacobi PDEs is closely related with a bilevel optimal control problem. The paper aims to construct equilibria in this bilevel optimal control problem using the generalized solution for the system of the Hamilton–Jacobi PDEs. We introduce the definition of the solution for the system of the Hamilton–Jacobi PDEs in a class of multivalued functions. The notion of the generalized solution is based on the notions of minimax solution and M-solution to Hamilton–Jacobi equations proposed by Subbotin. We prove the existence theorem for the solution of the system of the Hamilton–Jacobi PDEs.

5.1 Introduction

The paper deals with a differential game, the dynamics of the game is entirely defined by the policy of the first player. The payoff functional of the first player is also determined by the control of the first player and the payoff functional of the second player depends on control of both players. Actually we investigate a bilevel optimal control problem. In considerable problem Nash equilibrium coincides with Stackelberg equilibrium [1, 5]. We restrict our attention to the case when the players use open-loop strategies and examine this problem applying the solution of the system of Hamilton–Jacobi equations.

The solution for a strongly coupled system of the Hamilton–Jacobi equations is open mathematical problem. For the general case there is no existence theorems. Furthermore the system of Hamilton–Jacobi equations is connected with the system of the quasilinear first order PDEs. The systems of quasilinear PDEs (the system

E. A. Kolpakova (✉)
Krasovskii Institute of Mathematics and Mechanics UrB RAS, Yekaterinburg, Russia

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of conservation laws) describe many physical processes. If we differentiate the system of Hamilton–Jacobi equations w.r.t. phase variable x, then we obtain a system of quasilinear equations. The existence theorems for a generalized solution are obtained only for initial values with a small total variation [3, 8]. Using this link Bressan and Shen [3] constructed Nash strategies in the feedback for some non-zero sum two players differential game on the line. The authors do not solve the system of Hamilton–Jacobi equations, but they solve the corresponding strictly hyperbolic system of quasilinear PDEs. This way can be applied only in the case of the scalar phase variable and the hyperbolic system of quasilinear equations. Analogous constructions for a differential game with simple motions were described in [4].

As we mentioned above the theory of the system of Hamilton–Jacobi equations is open mathematical problem, at the same time the theory of generalized solution for the single Hamilton–Jacobi equation is well-developed. Subbotin proposed the notion of minimax solution, he proved the existence and uniqueness theorems [13]. Crandall et al. introduced the viscosity approach [6]. Moreover Subbotin proved the equivalence of these approaches.

In the paper we consider the systems of Hamilton–Jacobi equations where the first equation of the system does not depend on the solution of the second equation, and the second equation depends on partial derivatives of the solution for the first equation. This implies that we can solve the equations of the system sequentially. This system is connected with a bilevel optimal control problem [16]. Using the minimax/viscosity approach we obtain the solution of the first equation of the system. Further we substitute the derivative of the minimax/viscosity solution in the second equation. The second equation is solved in the framework of M-solutions [9].

Our main result is the following. We show that the solution for the system of Hamilton–Jacobi equations of special type belongs to a class of multivalued map. We construct this multivalued solution and connects with a Nash equilibrium in a bilevel optimal control problem.

5.2 Bilevel Optimal Control Problem

A bilevel optimal control problem is a particle case of two-level differential games. Let us consider the bilevel optimal control problem with dynamics

$$\dot{x} = f(t, x, u),\ x(t_0) = x_0,\ u \in U \subset \mathbb{R}^n. \tag{5.1}$$

Here $t \in [0, T]$, $x \in \mathbb{R}^n$. The players maximize payoff functionals I_1, I_2:

$$I_1(u(\cdot)) = \sigma_1(x(T)) + \int_{t_0}^{T} g_1(t, x(t), u(t))dt,$$

$$I_2(u(\cdot), v(\cdot)) = \sigma_2(x(T)) + \int_{t_0}^{T} g_2(t, x(t), u(t), v(t))dt.$$

Here u and v are controls of the players. Assume that $U, V \subset R^n$ are compact sets. Denote the set of all measurable controls of the first player by $\tilde{U}$:

$$\tilde{U} = \{u : [t_0, T] \to U, u \text{ are measurable functions}\},$$

and the set of all measurable controls of the second player by $\tilde{V}$:

$$\tilde{V} = \{v : [t_0, T] \to V, v \text{ are measurable functions}\}.$$

From [7, 12] it follows that the payoffs of the players satisfy the system of the Hamilton–Jacobi equations:

$$\frac{\partial c}{\partial t} + H_1(t, x, p) = 0, \;\; c(T, x) = \sigma_1(x); \tag{5.2}$$

$$\frac{\partial w}{\partial t} + H_2(t, x, p, q) = 0, \;\; w(T, x) = \sigma_2(x), \tag{5.3}$$

under condition

$$H_1(t, x, p) = \max_{u \in U} \langle f(t, x, u), p\rangle + g_1(t, x, u)$$

$$= \langle f(t, x, u^*(t, x, p)), p\rangle + g_1(t, x, u^*(t, x, p)),$$

$$H_2(t, x, q) = \langle f\Big(t, x, u^*\Big(t, x, \frac{\partial c(t, x)}{\partial x}\Big)\Big), q\rangle + \max_{v \in V} g_2\Big(t, x, u^*\Big(t, x, \frac{\partial c(t, x)}{\partial x}\Big), v\Big).$$

Here

$$u^*(t, x, p) \in \arg\max_{u \in U} \langle f(t, x, u), p\rangle + g_1(t, x, u), \tag{5.4}$$

$p = \frac{\partial c}{\partial x}, q = \frac{\partial w}{\partial x}$.

Further we shall assume that

A1. the function $H_1 : [0, T] \times \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$ is continuously differentiable, H_1 satisfies sublinear condition w.r.t. x, p, the function H_1 is strongly convex w.r.t. p for any $(t, x) \in [0, T] \times \mathbb{R}^n$.
A2. the function σ_1 is Lipschitz continuous.
A3. the function $H_2 : [0, T] \times \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$ is continuously differentiable, H_2 satisfies sublinear condition w.r.t. x, p, q, the function H_2 is strongly convex w.r.t. q for any $(t, x) \in [0, T] \times \mathbb{R}^n$.
A4. the function σ_2 is continuously differentiable.

From assumptions $A1$, $A3$ we get

$$g_1(t, x, p) = H_1^*\Big(t, x, \frac{\partial H_1(t, x, p)}{\partial p}\Big),$$

$$g_2(t, x, p, q) = H_2^*\Big(t, x, \frac{\partial H_1}{\partial p}, \frac{\partial H_2(t, x, p, q)}{\partial q}\Big).$$

Here H_1^*, H_2^* are conjugate functions to H_1, H_2, $\frac{\partial H_1}{\partial p} = \Big(\frac{\partial H_1}{\partial p_1}, \dots, \frac{\partial H_1}{\partial p_n}\Big)$. Hence g_1, g_2 are continuous functions w.r.t all variables. Since condition $A1$ holds a measurable function (5.4) $u^* : (t, x, p) \to U$ is well-defined.

Let us introduce the mapping

$$(t_0, x_0) \to \xi(t_0, x_0) = \{\xi \in \mathbb{R}^n : \tilde{x}(t_0, \xi) = x_0, \tilde{x}(T, \xi) = \xi,$$
$$\tilde{s}(T, \xi) = D_x\sigma_1(\xi), \ \tilde{z}(T, \xi) = \sigma_1(\xi), \ \tilde{z}(t_0, \xi) = c(t_0, x_0)\} \tag{5.5}$$

Here $(\tilde{x}(\cdot), \tilde{s}(\cdot), \tilde{z}(\cdot))$ is the unique and extendable solution of the characteristic system for Bellman equation (5.2):

$$\dot{\tilde{x}} = \frac{\partial H_1(t, \tilde{x}, \tilde{s})}{\partial \tilde{s}}, \ \dot{\tilde{s}} = -\frac{\partial H_1(t, \tilde{x}, \tilde{s})}{\partial \tilde{x}}, \ \dot{\tilde{z}} = \langle \frac{\partial H_1(t, \tilde{x}, \tilde{s})}{\partial \tilde{s}}, \tilde{s}\rangle - H_1(t, \tilde{x}, \tilde{s})$$

with a boundary condition

$$\tilde{x}(T, \xi) = \xi, \ \tilde{s}(T, \xi) = D_x\sigma_1(\xi), \ \tilde{z}(T, \xi) = \sigma_1(\xi), \ \xi \in \mathbb{R}^n.$$

It follows from [11, 15] that for any point $(t_0, x_0) \in [0, T] \times \mathbb{R}^n$ assumption $A1$ guarantees the existence of optimal open-loop control $u^0(\cdot; t_0, x_0)$ satisfying the relation

$$\max_{u(\cdot)\in\tilde{U}} I_1(u(\cdot)) = I_1(u^0(\cdot; t_0, x_0)) = c(t_0, x_0).$$

Pontryagin's Maximum principle implies that the optimal open-loop control $u^0(\cdot; t_0, x_0)$ of the first player for the initial point $(t_0, x_0) \in [0, T] \times \mathbb{R}^n$ is defined by the rule

$$u^0(t; t_0, x_0) \in \arg\max_{u \in U}[\langle \tilde{s}(t, \xi_0), f(t, \tilde{x}(t, \xi_0), u)\rangle + g_1(t, \tilde{x}(t, \xi_0), u)], \ \forall t \in [t_0, T] \tag{5.6}$$

Here $(\tilde{x}(\cdot), \tilde{s}(\cdot), \tilde{z}(\cdot))$ is the solution of the characteristic system for problem (5.2) for any $t \in [t_0, T]$, for any $\xi_0 \in \xi(t_0, x_0)$ defining by (5.5).

We determine the set of optimal open-loop controls of the first player

$$U^0(t_0, x_0) = \Big\{u(\cdot) : [t_0, T] \to U \text{are measurable functions, satisfying (5.6)}\Big\}.$$

Remark 5.1 Equivalently the first player's control can be considered in feedback strategies [14]. In this case the optimal feedback is given by

$$u(t, x) \in \arg\max_{u \in U}\Big[\frac{dc(t, x)}{d(1, f(t, x, u))} + g_1(t, x, u)\Big],$$

where c is the solution of Cauchy problem (5.2), $\frac{dc(t,x)}{d(1, f(t,x,u))}$ is the derivative of c at the point (t, x) in the direction $(1, f(t, x, u))$.

5.3 The Solution of the System of the Hamilton–Jacobi Equations

In this section we will focus on solution of system of Hamilton–Jacobi equations (5.2), (5.3). We begin with definition of a minimax/viscosity solution of Cauchy problem (5.2).

Definition 5.1 The continuous function $c : [0, T] \times \mathbb{R}^n \to \mathbb{R}$ is said to be the minimax/viscosity solution if $c(T, x) = \sigma_2(x)$, $x \in \mathbb{R}^n$ and the following inequalities hold for any $(t, x) \in (0, T) \times \mathbb{R}^n$

$$\alpha + H_1(t, x, \beta) \leq 0, \quad (\alpha, \beta) \in D^- c(t, x),$$

$$\alpha + H_1(t, x, \beta) \geq 0, \quad (\alpha, \beta) \in D^+ c(t, x).$$

Here $D^- c(t, x)$ and $D^+ c(t, x)$ are sub- and superdifferentials of function c at a point (t, x).

It is known from [13] that under conditions $A1$–$A3$ there exists the unique minimax solution $c(\cdot, \cdot)$ in problem (5.2).

We recall the properties of the minimax solution of problem (5.2) under conditions $A1$,A_2 from [13, 14]:

1. the minimax solution $c(\cdot, \cdot)$ is a locally Lipschitz function;
2. the superdifferential of the minimax solution $D^+c(t, x) \neq \emptyset$ for any point $(t, x) \in [0, T] \times \mathbb{R}^n$.

We solve the system of Hamilton–Jacobi equations sequentially. The minimax solution of the first equation (5.2) is a Lipschitz continuous function. Thus the partial derivative of the minimax solution can be discontinuous w.r.t. x. We substitute the superdifferential $D_x^+c(\cdot, \cdot)$ of function c for p in the second equation (5.3), therefore we obtain the multivalued Hamiltonian

$$\tilde{H}(t, x, q) = H_2(t, x, D_x c(t, x), q). \tag{5.7}$$

Hence, we have the Hamilton–Jacobi equation with the multivalued Hamiltonian:

$$\frac{\partial w}{\partial t} + \tilde{H}(t, x, q) = 0, \ \ w(T, x) = \sigma_2(x). \tag{5.8}$$

A.I. Subbotin proposed the notion of M-solution for Cauchy problem (5.8) with the multivalued Hamiltonian relative to x.

Consider the differential inclusion

$$(\dot{x}, \dot{z}) \in E(t, x, q), \ \ E(t, x, q) = \{(f, g) : f \in \frac{\partial H_2(t, x, p, q)}{\partial q}, \ \ p \in D^+c(t, x),$$
$$\langle f, q\rangle - g \in [H_{2*}(t, x, q), H_2^*(t, x, q)], q \in \mathbb{R}^n\}. \tag{5.9}$$

Here $\frac{\partial H_2(t,x,p,q)}{\partial q} = \left(\frac{\partial H_2(t,x,p,q)}{\partial q_1}, \ldots, \frac{\partial H_2(t,x,p,q)}{\partial q_n}\right)$,

$$H_{2*}(t, x, q) = \lim \inf_{(\tau,\xi)\to(t,x)} \tilde{H}(\tau, \xi, q), \ H_2^*(t, x, q) = \lim \sup_{(\tau,\xi)\to(t,x)} \tilde{H}(\tau, \xi, q). \tag{5.10}$$

It follows from [10] that differential inclusion (5.9) is an admissible characteristical inclusion. Recall some definitions and theorem from the work [9].

Definition 5.2 The closed set $W \subset [0, T] \times \mathbb{R}^n \rightrightarrows \mathbb{R}$ is viable w.r.t. differential inclusion (5.9), if for any point $(t_0, x_0, z_0) \in W$ there exist $\tau > 0$ and a trajectory $(x(\cdot), z(\cdot))$ of admissible differential inclusion (5.9) such that $(x(0), z(0)) = (x_0, z_0)$, $(t, x(t), z(t)) \in W$ for any $t \in [0, \tau]$.

Definition 5.3 The closed maximal set $W \subset [0, T] \times \mathbb{R}^n \rightrightarrows \mathbb{R}$ is called the M-solution of Cauchy problem for Hamilton–Jacobi equation (5.8), if W is viable w.r.t. admissible differential inclusion (5.9) and satisfies the condition

$$(T, x, z) \in W \Rightarrow z = \sigma_2(x) \ \forall \ x \in \mathbb{R}^n.$$

Definition 5.4 The closed set $W \subset [0, T] \times \mathbb{R}^n \times \mathbb{R}$ is said to be the epi-solution (hypo-solution) of problem (5.8) if W is viable w.r.t. admissible differential inclusion (5.9) and satisfies the condition

$$(T, x, z) \in W \Rightarrow z \geq \sigma_2(x)((T, x, z) \in W \Rightarrow z \leq \sigma_2(x))\ \forall\, x \in \mathbb{R}^n.$$

We introduce the definition for a generalized solution of the system of the Hamilton–Jacobi equations.

Definition 5.5 The multivalued map (c, w), where $c(\cdot,\cdot) : [0, T] \times \mathbb{R}^n \to \mathbb{R}$, $w : [0, T] \times \mathbb{R}^n \rightrightarrows \mathbb{R}$ is called a generalized solution of Cauchy problem for the system of Hamilton–Jacobi equations (5.2), (5.3), if the function $c(\cdot,\cdot)$ is the minimax solution of problem (5.2), the map $w(\cdot,\cdot)$ is the M-solution of problem (5.8).

Theorem 5.1 ([10]) *Let $w : [0, T] \times \mathbb{R}^n \to \mathbb{R}$ be a multivalued map and gr w is closed set. Suppose that $w(t, x)$ is not empty for $t \in [0, T]$, $x \in \mathbb{R}^n$ and put*

$$w_*(t, x) = \min_{z \in w(t,x)} z > -\infty, \quad w^*(t, x) = \max_{z \in w(t,x)} z < \infty.$$

The map w is the M-solution of problem (5.8) iff epi w_* and hypo w^* are the M-solutions of problem (5.8).

Given $t \in [t_0, T]$, $x \in \mathbb{R}^n$, $u \in U$ let

$$(t, x, u) \to Q(t, x, u) = \arg\max_{v \in V} g_2(t, x, u, v). \tag{5.11}$$

be the set of optimal controls of the second player. Consider the map $\Gamma(u(\cdot)) : \tilde{U} \to \mathbb{R}$ given by the following rule

$$u(\cdot) \to \sigma_2(x[T; t_0, x_0]) + \int_{t_0}^{T} g_2(t, x[t; t_0, x_0], u(t), Q(t, x[t; t_0, x_0], u(t)))dt, \tag{5.12}$$

$u(\cdot) \in U^0(t_0, x_0)$, the function $x[\cdot; t_0, x_0]$ is a solution of the problem

$$\dot{x} = f(t, x, u(t)), \ u(\cdot) \in U^0(t_0, x_0), x(t_0) = x_0. \tag{5.13}$$

Put

$$w(t_0, x_0) = \bigcup_{u(\cdot) \in U^0(t_0, x_0)} \Gamma(u) \tag{5.14}$$

Lemma 5.1 *Map (5.14) is compact-valued.*

Proof Let us choose $w_i = \Gamma(u_i(\cdot)) \in w(t_0, x_0)$. We show that if $w_i \to w_0$, $i \to \infty$, then $w_0 \in w(t_0, x_0)$.

Let us define the set generalized controls

$$\Lambda = \{\mu : [t_0, T] \times U \to [0, +\infty) \text{ is measurable },$$

$\forall\, [\tau_1, \tau_2] \subset [0, T] \quad \mu([\tau_1, \tau_2] \times U) = \tau_2 - \tau_1, \}$. Here λ is Lebesgue measure on $[0, T]$. Hence the trajectory $x(\cdot)$ under control μ has the form

$$x(t) = x_0 + \int\limits_{[t_0,t]\times U} f(\tau, x(\tau), u)\mu(d(\tau, u)).$$

In this case the first player's outcome is

$$I_1(\mu) = \sigma_1(x(T)) + \int\limits_{[t_0,t]\times U} g_1(\tau, x(\tau), u)\mu(d(\tau, u)).$$

We consider the set of generalized optimal controls

$$M_{t_0} = \{\mu \in \Lambda : \mu \text{ maximizes } I_1(\mu)\}.$$

It is known from [15] that the set M_{t_0} is a compact metric set. Now we show the link between $U^0(t_0, x_0)$ and M_{t_0}. If $u(\cdot) \in U^0(t_0, x_0)$ then there exists $\mu_{u(\cdot)} \in M_{t_0}$ such that

$$\forall\, \varphi \in C([0, T] \times U) \qquad \int\limits_{[0,T]\times U} \varphi(t, u)\mu_{u(\cdot)}(d(t, u)) = \int\limits_0^T \varphi(t, u(t))dt.$$

Hence from $u_i \in U^0(t_0, x_0)$ we obtain $\mu_i = \mu_{u_i(\cdot)} \in M_{t_0}$. Consider $\mu_i \to \mu^*$ as $i \to \infty$. Since M_{t_0} is a closed set we get $\mu^* \in M_{t_0}$. Let us construct $u^* \in U^0(t_0, x_0)$ such that $\mu^* = \mu_{u^*(\cdot)}$.

We have

$$\lim_{i\to\infty} w_i = \lim_{i\to\infty} \Gamma(u_i(\cdot)) = \Gamma(u^*(\cdot)) = w_0.$$

Hence $w_0 = \Gamma(u^*(\cdot)) \in w(t_0, x_0)$. Since $\Gamma(u)$ is bounded on the set $U^0(t_0, x_0)$ it follows that $w(t, x_0)$ is bounded.

We prove the following theorem.

Theorem 5.2 *If conditions A1–A4 hold, then the multivalued map w, defining (5.14) is the M-solution of problem (5.8).*

Proof Put

$$w^*(t_0, x_0) = \max_{y \in w(t_0, x_0)} y,$$

where w is defined by (5.14). Let us show that hypograph w^* is viable w.r.t. differential inclusion (5.9).

We fix the position $(t_0, x_0) \in [0, T] \times \mathbb{R}^n$. Choose $(t_0, x_0, z_0) \in$ hypo w^*, $z_0 \leq w^*(t_0, x_0)$. If assumptions $A1$–$A3$ are true, then in the optimal control problem with payoff functional I_1 there exists an optimal open-loop control u^* in the class of measurable functions. And control u^* generates the trajectory ξ:

$$\dot{\xi} = f(t, \xi, u^*(t)), \quad \xi(t_0) = x_0.$$

The choice of point z_0 and Bellman's optimality principle yield the equality

$$z_0 \leq w^*(t_0, x_0) = w^*(t, \xi(t))$$

$$+ \int_{t_0}^{t} g_2(\tau, \xi[\tau; t_0, x_0], u^*(\tau), Q(\tau, \xi[\tau; t_0, x_0], u^*(\tau)) d\tau.$$

Further we have

$$z_0 - \int_{t_0}^{t} g_2(\tau, \xi[\tau; t_0, x_0], u^*(\tau), Q(\tau, \xi[\tau; t_0, x_0], u^*(\tau))) d\tau \leq w^*(t, \xi(t))$$

for any $t \in [t_0, T]$. Note that

$$z(t) = z_0 - \int_{t_0}^{t} g_2(\tau, \xi[\tau; t_0, x_0], u^*(\tau), Q(\tau, \xi[\tau; t_0, x_0], u^*(\tau))) d\tau,$$

hence the trajectory $(\xi(\cdot), z(\cdot))$ satisfies to differential inclusion (5.9). From definition of the Hamiltonian H_2 it follows that

$$g = \dot{z} = -g_2(t, \xi(t), u^*(t), Q(t, \xi(t), u^*(t))), \ \langle f(t, \xi(t), u^*(t), p\rangle - g$$

$$= \langle f(t, \xi(t), u^*(t), p\rangle + g_2(t, \xi(t), u^*(t), Q(t, \xi(t), u^*(t))) \in$$

$$[H_{2*}(t, \xi(t), p), H_2^*(t, \xi(t), p)].$$

Hence $(t, \xi(t), z(t)) \in$ hypo $w^*(t, \xi(t)), t \in [t_0, T]$. Therefore hypo w^* is a closed set, satisfying the definition of the hypo-solution.

Put

$$w_*(t_0, x_0) = \min_{y \in w(t_0, x_0)} y,$$

where w is defined by (5.14). We choose a point $(t_0, x_0, z_0) \in$ epi w_*, $z_0 \geq w_*(t_0, x_0)$. Let us consider the optimal trajectory $\xi(\cdot)$ of dynamical system (5.1), generated by control u_* and satisfying to initial condition $\xi(t_0) = x_0$. Since $\xi(\cdot)$ is the optimal trajectory we have

$$w_*(t, \xi(t)) + \int_{t_0}^{t} g_2(\tau, \xi[\tau; t_0, x_0], u_*(\tau), Q(\tau, \xi[\tau; t_0, x_0], u_*(\tau)))d\tau$$

$= w_*(t_0, x_0) \leq z_0$. Therefore

$$w_*(t, \xi(t)) \leq z_0$$

$$-\int_{t_0}^{t} g_2(\tau, \xi[\tau; t_0, x_0], u_*(\tau), Q(\tau, \xi[\tau; t_0, x_0], u_*(\tau)))d\tau = z(t),$$

that is the trajectory $(\xi(\cdot), z(\cdot))$ lies in the epigraph w_*. We show that $z(\cdot)$ is a solution of differential inclusion (5.9). Really

$$g = \dot{z} = -g_2(t, \xi(t), u_*(t), Q(t, \xi(t), u_*(t))), \ \langle f(t, \xi(t), u_*(t), p\rangle - g$$

$$= \langle f(t, \xi(t), u_*(t), p\rangle + g_2(t, \xi(t), u_*(t), Q(t, \xi(t), u_*(t))) \in$$

$$[H_{2*}(t, \xi(t), p), H_2^*(t, \xi(t), p)].$$

Consequently epi w_* is a closed set, satisfying the definition of the epi-solution.

Using Theorem 5.1 we obtain epi $w_* \bigcap$ hypo w^* is the M-solution of problem (5.8). We note that epi $w_*(T, x) \bigcap$ hypo $w^*(T, x) = \sigma_2(x)$, $x \in \mathbb{R}^n$.

Remark 5.2 We have proved that multivalued map (5.14) is the M-solution of problem (5.8). From definition 5.3 the M-solution is maximal-valued. Let us assume that there exist two M-solutions W and W' of problem (5.8). Then we have inclusions $W \subseteq W'$ and $W' \subseteq W$. Hence $W = W'$ and the M-solution is unique.

5.4 Design of Nash Equilibrium

Let us recall the definition of a Nash equilibrium in program strategies.

Definition 5.6 ([2]) A couple of strategies $(\bar{u}(\cdot), \bar{v}(\cdot))$ is a Nash equilibrium in two persons differential game if following inequalities hold for any $u(\cdot) \in \tilde{U}$, $v(\cdot) \in \tilde{V}$

$$\sigma_1(\bar{x}(T)) + \int_{t_0}^{T} g_1(t, \bar{x}(t), \bar{u}(t))dt \geq \sigma_1(x^{[1]}(T)) + \int_{t_0}^{T} g_1(t, x^{[1]}(t), u(t))dt,$$

$$\sigma_2(\bar{x}(T)) + \int_{t_0}^{T} g_2(t, \bar{x}(t), \bar{u}(t), \bar{v}(t))dt \geq \sigma_2(\bar{x}(T)) + \int_{t_0}^{T} g_2(t, \bar{x}(t), \bar{u}(t), v(t))dt,$$

$t \in [t_0, T]$, where

$$\dot{\bar{x}}(t) = f(t, \bar{x}(t), \bar{u}(t)),\ \dot{x}^{[1]}(t) = f(t, x^{[1]}(t), u(t)),\ \bar{x}(t_0) = x^{[1]}(t_0) = x_0.$$

Let us define the control $\bar{u}(\cdot)$ by formula (5.6). The control $\bar{u}(\cdot)$ maximizes the functional I_1 for optimal control problem (5.1), and therefore the first inequality holds in Definition 5.6.

Let $\bar{v}(\cdot)$ be given by

$$\bar{v}(t) \in \arg\max_{v \in V}\{g_2(t, \bar{x}(t), \bar{u}(t), v)\}, t \in [t_0, T], \tag{5.15}$$

where $\bar{x}(\cdot)$ is a solution of problem $\dot{\bar{x}}(t) = f(t, \bar{x}(t), \bar{u}(t)),\ \bar{x}(t_0) = x_0$. Since g_2 is a continuous function w.r.t. all variables, $\bar{x}(\cdot)$ is a differentiable function and $\bar{u}(\cdot)$ is measurable function we see that $g_2(\cdot, \bar{x}(\cdot), \bar{u}(\cdot), v)$ is a measurable function w.r.t. t and multivalued map $G(t) = \{g_2(t, \bar{x}(t), \bar{u}(t), v) : v \in V\}, t \in [t_0, T]$ is measurable w.r.t. t. The multivalued map $Gm(t) = \max_{v \in V} g_2(t, \bar{x}(t), \bar{u}(t), v)$ is upper semicontinuous therefore this map is measurable w.r.t t. Using this fact and Casteing's theorem [15], we get the map

$$\arg\max_{v \in V} g_2(\cdot, x(\cdot), \bar{u}(\cdot), v) :\ [t_0, T] \rightrightarrows V$$

is measurable. Hence from Neiman–Aumann–Casteing's theorem [15] the measurable multivalued map has a measurable selector $\bar{v}(\cdot) : [t_0, T] \to \mathbb{R}^n$.

By the definition $\bar{v}$ (5.15) the second inequality for integral parts holds in Definition 5.6.

Hence the couple of strategies $(\bar{u}, \bar{v})$ provides a Nash equilibrium. The first player solves the optimal control problem and the payoff does not depend on behavior of the second player. Choosing the control $\bar{u}(\cdot; t_0, x_0)$, the first player will obtain a payoff $c(t_0, x_0)$. We shall show how the choice of the control of the first player influences on the payoff of the second player.

Remark 5.3 Let us fix the point $(t_0, x_0) \in [0, T] \times \mathbb{R}^n$. Let (c, w) be the generalized solution of problem (5.2), (5.3), $\alpha \in w(t_0, x_0)$, then there exists a couple of Nash equilibrium strategies (u^*, v^*):

$$u^*(t) \in \arg \max_{u(\cdot)\in U^0(t_0,x_0)} \Gamma(u), \;\; v^*(t) = Q(t, x[t; t_0, x_0]), u^*(t)),$$

Γ is defined by (5.12), $x^*[\cdot; t_0, x_0]$ satisfies (5.1). From $A3$ we can use arg max instead of arg sup. The corresponding payoffs of players at the point $(t_0, x_0) \in [0, T] \times \mathbb{R}^n$ equal to $(c(t_0, x_0), \alpha)$.

5.5 Example

Let us consider the optimal control problem

$$\dot{x} = u, \;\; x(t_0) = x_0,$$

$x \in \mathbb{R}$, $t \in [0, T]$, $|u| \le 1$, $|v| \le 1$. Leader maximizes the payoff functional

$$I_1(u(\cdot)) = |x(T)| - \int_{t_0}^{T} \frac{u^2}{2} dt \to \max,$$

and the follower maximizes payoff functional

$$I_2(u(\cdot), v(\cdot)) = x(T) - \int_{t_0}^{T} v^2 + uv dt \to \max .$$

The system of Hamilton–Jacobi equations has the form

$$\frac{\partial c}{\partial t} + \max_{u\in U}[pu - \frac{u^2}{2}] = 0, \;\; c(T, x) = |x|,$$

$$\frac{\partial w}{\partial t} + qu^0(t, x, p) + \max_{v\in V}[-v^2 - u^0(t, x, p)v] = 0, w(T, x) = x,$$

$x \in \mathbb{R}$, $t \in [0, T]$, $p = \frac{\partial c}{\partial x}$, $q = \frac{\partial w}{\partial x}$. Using formula (5.4) we obtain

$$u^0(t, x, p) = \begin{cases} p, \;\text{if } |p| \le 1, \\ 1, \;\text{if } p > 1, \\ -1, \;\text{if } p < -1. \end{cases}$$

Lax–Hopf formula yields the solution of the first Hamilton–Jacobi equation

$$c(t, x) = |x| - 1/2(t - T).$$

Now by formula (5.6) the open-loop control of the leader

$$u^0(t; t_0, x_0) = \begin{cases} 1, & \text{if } x_0 > 0, \\ -1, & \text{if } x_0 < 0, \\ \{-1, 1\}, & \text{if } x_0 = 0. \end{cases}$$

Applying (5.11) we construct the map Q

$$Q(u) = -\frac{u}{2}.$$

Hence the open-loop control of the follower

$$v^0(t; t_0, x_0) = \begin{cases} -\frac{1}{2}, & \text{if } x_0 > 0, \\ \frac{1}{2}, & \text{if } x_0 < 0, \\ \left[-\frac{1}{2}, \frac{1}{2}\right], & \text{if } x_0 = 0. \end{cases}$$

Further we construct M-solution of the second equation

$$w(t, x) = \begin{cases} x + \frac{3}{4}(t - T), & \text{if } x < 0, \\ x - \frac{5}{4}(t - T), & \text{if } x > 0, \\ \left\{x + \frac{3}{4}(t - T), x - \frac{5}{4}(t - T)\right\}, & \text{if } x = 0. \end{cases}$$

We see that the solution of the second Hamilton–Jacobi equation is multivalued under $x = 0$.

The payoffs of the players at the point $(t_0, x_0) \in [0, T] \times \mathbb{R}^n$ equal to $(|x_0| - 1/2(t_0 - T), \alpha)$, where $\alpha \in w(t_0, x_0)$.

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