Chapter 13
Characteristic Functions in a Linear Oligopoly TU Game

Artem Sedakov

Abstract We consider a linear oligopoly TU game without transferable technologies in which the characteristic function is determined from different perspectives. In so-called γ-, δ-, and ζ-games, we study the properties of characteristic functions such as monotonicity, superadditivity, and supermodularity. We also show that these games have nonempty cores of a nested structure when the δ-characteristic function is supermodular.

13.1 Introduction

In the definition of a TU game, the characteristic function plays an important role as it measures the worth of any coalition of players, which, in turn, influences players' cooperative payoffs. When the game is initially formulated as a normal-form game, the characteristic function of the corresponding TU game has to be determined. The first study on this problem was done in [12] in which the concepts of so-called α- and β-characteristic functions were proposed. Later in [1], TU games based on these characteristic functions were called α- and β-games, respectively. When transiting from a normal-form game to the corresponding TU game, other studies devoted to the definition of the characteristic function include the concepts of γ-, δ-, and ζ-games proposed in [8, 9], and [5], respectively.[1] All these definitions of the corresponding TU games proceed from the assumption that any coalition of players maximizes the sum of the payoffs of its members.

[1]Characteristic functions considered in [9] and [8] were called later the γ- and δ-characteristic functions, in [2] and [10], respectively.

A. Sedakov (✉)
Saint Petersburg State University, Saint Petersburg, Russia
e-mail: a.sedakov@spbu.ru

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In this paper, we study the properties of the aforementioned characteristic functions applicable to linear oligopoly games where a finite number of firms producing a homogeneous product compete in a market. The literature on this topic covers two means of determining the cost of a group of firms (coalition): games with transferable technologies (weak synergy) [6, 13, 14], and games without transferable technologies [3, 4, 7]. Here, we follow the second approach as it is consistent with [12] in determining the profit of a coalition.

For the class of oligopoly TU games under consideration, the properties of α- and β-games have already been studied in [3, 7]. We continue studying the properties of γ-, δ-, and ζ-games such as monotonicity, superadditivity, and convexity. The remainder of the paper has the following structure. In Sect. 13.2, we consider a basic linear oligopoly game for which both noncooperative and cooperative solutions are presented. Next, Sect. 13.3 provides closed-form expressions for α-, β-, γ-, δ-, and ζ-characteristic functions, while their properties are examined in Sect. 13.4. The existence of the cores of linear oligopoly TU games based on the aforementioned characteristic functions is discussed in Sect. 13.5. Section 13.6 concludes.

13.2 The Model

We consider a market consisting of firms–competitors producing a homogeneous product. Denote the set of the firms by $N = \{1, \dots, n\}$ with $n \geqslant 2$. Each firm decides on its output, i.e., the quantity it must produce, $q_i \in Q_i = [0, a]$ with $a > 0$, thus the output is the firm's strategy. The market price for the product is determined by the profile of quantities $q = (q_1, \dots, q_n)$ according to the inverse demand function $P(q) = \left(a - \sum_{i \in N} q_i\right)_+ = \max\{0, a - \sum_{i \in N} q_i\}$. Under the assumption of linearity of the cost function $C_i(q_i) = c_i q_i$ with $c_i < a$ for any firm $i \in N$, we obtain the following expression of firm i's profit: $\pi_i(q) = (P(q) - c_i)q_i$. Thus we have a noncooperative normal-form game $(N, \{Q_i\}_{i \in N}, \{\pi_i\}_{i \in N})$. We note that π_i is not concave on $\prod_{j \in N} Q_j$ for any $i \in N$.

For any subset $S \subseteq N$, let $I_S = \{i \in N : i = \arg\min_{j \in S} c_j\}$, a firm belonging to I_S be denoted by i_S, and $c_S = \sum_{j \in S} c_j$.

13.2.1 *Nash Equilibrium*

A *Nash equilibrium* in the game $(N, \{Q_i\}_{i \in N}, \{\pi_i\}_{i \in N})$ is the profile $q^* = (q_1^*, \dots, q_n^*)$ such that $\pi_i(q^*) \geqslant \pi_i(q_i, q_{-i}^*)$ for any $i \in N$ and $q_i \in [0, a]$, where q_{-i}^* denotes the profile of outputs of all firms except firm i in q^*. For practical reasons, we suppose that the price $P(q)$ is positive under the equilibrium. It is well-known that the Nash equilibrium profile q^* has the form:

$$q_i^* = \frac{a + c_N}{n + 1} - c_i, \quad i \in N. \tag{13.1}$$

From the expression of equilibrium outputs, it follows that $q_i^* + c_i = q_j^* + c_j$ and therefore $q_i^* - q_j^* = c_j - c_i$ for any two firms i and j. To meet a positive equilibrium profile q^*, we additionally require that

$$(n+1)c_i < a + c_N \quad \text{for all} \quad i \in N. \tag{13.2}$$

Under the Nash equilibrium profile q^* we notice the following: $\sum_{i \in N} q_i^* < a$, the profit of firm $i \in N$ is positive and it equals $\pi_i(q^*) = (q_i^*)^2$; the equilibrium price for the product becomes $P(q^*) = \frac{a+c_N}{n+1}$ what exceeds the unit cost of any firm owning to inequality (13.2).

13.2.2 Cooperative Agreement

Now we shall consider the case when firms aim at maximizing the sum of their profits without being restricted in forming one alliance. This means that one must consider the following optimization problem:

$$\max_q \sum_{i \in N} \pi_i(q) \quad \text{subject to } q_i \in [0, a], \quad i \in N. \tag{13.3}$$

For practical reasons, we isolate the case when the price $P(q)$ is positive under the solution. Otherwise, when this price equals zero, the sum to be maximized will be nonpositive. The optimal solution of problem (13.3) will be denoted by $\bar{q} = (\bar{q}_1, \ldots, \bar{q}_n)$ and called the *cooperative agreement*. The optimization problem (13.3) may be written in an alternative form:

$$\max_{q_1,\ldots,q_n} \sum_{i \in N} (a - c_i) q_i - \left(\sum_{i \in N} q_i \right)^2 \tag{13.4}$$
$$\text{subject to} \quad q_i \in [0, a], \quad i \in N.$$

To analyze both the solution and the value of problem (13.4), we will use the following statement.

Proposition 13.1 *Let $z, z_1, \ldots, z_k$ be real numbers such that $z \geqslant z_1 \geqslant \ldots \geqslant z_k > 0$. The value of the constrained optimization problem*

$$\max_{x_1,\ldots,x_k} \sum_{i=1}^k z_i x_i - \left(\sum_{i=1}^k x_i \right)^2 \tag{13.5}$$
$$\textit{subject to} \quad x_i \in [0, z], \quad i = 1, \ldots, k,$$

equals $z_1^2/4$. This value is attained at $x = (z_1/2, 0, \ldots, 0)$.

Note that Proposition 13.1 does not list all optimal solutions. For example, when $z_1 = z_2$, the aforementioned optimization problem (13.5) admits any optimal solution $(\frac{wz_1}{2}, \frac{(1-w)z_1}{2}, 0, \ldots, 0)$ where $w \in [0, 1]$. When $z_1 = \ldots = z_{k_0}$ for some integer $k_0 \leqslant k$, the profile $x = (x_1, \ldots, x_k)$ where

$$x_i = \begin{cases} \dfrac{z_1}{2k_0}, & \text{if } i \leqslant k_0, \\ 0, & \text{if } k_0 < i \leqslant k, \end{cases}$$

also appears to be the optimal solution of the optimization problem (13.5) as it gives the same value of $z_1^2/4$. For the analysis that we carry out below, we are interested only in the value of the constrained optimization problem, therefore the optimal solution/solutions are not of much relevance.

The cooperative agreement $\bar{q} = (\bar{q}_1, \ldots, \bar{q}_n)$ can directly be found from Proposition 13.1:

$$\bar{q}_i = \begin{cases} \dfrac{a - c_{i_N}}{2|I_N|}, & \text{if } i \in I_N, \\ 0, & \text{otherwise.} \end{cases} \tag{13.6}$$

Under the cooperative agreement, only firms with the lowest unit cost produce positive output. Firm i's profit under this agreement equals

$$\pi_i(\bar{q}) = \begin{cases} \dfrac{(a - c_{i_N})^2}{4|I_N|}, & \text{if } i \in I_N, \\ 0, & \text{otherwise,} \end{cases} \tag{13.7}$$

and the sum of firms' profits is $\sum_{i \in N} \pi_i(\bar{q}) = (a - c_{i_N})^2/4$. The price on the product will be $P(\bar{q}) = (a + c_{i_N})/2$. Comparing the equilibrium and cooperative policies, we conclude that $P(\bar{q}) > P(q^*)$. Moreover since $\bar{q}$ maximizes the sum of firms' profits, it immediately follows that $\sum_{i \in N} \pi_i(\bar{q}) \geqslant \sum_{i \in N} \pi_i(q^*)$, yet there may exist a firm $j \in N$ that $\pi_j(\bar{q}) < \pi_j(q^*)$. At the same time, $\sum_{i \in N} \bar{q}_i < \sum_{i \in N} q_i^*$. Indeed,

$$\begin{aligned} \sum_{i \in N} q_i^* - \sum_{i \in N} \bar{q}_i &= \frac{na - c_N}{n+1} - \frac{a - c_{i_N}}{2} = \frac{(n-1)a - 2c_N + (n+1)c_{i_N}}{2(n+1)} \\ &> \frac{(n+1)c_{N \setminus i_N} - (n-1)c_N - 2c_N + (n+1)c_{i_N}}{2(n+1)} \\ &= \frac{(n+1)c_N - (n-1)c_N - 2c_N}{2(n+1)} = 0. \end{aligned}$$

The inequality is true because $(n-1)(a + c_N) > (n+1)c_{N \setminus i_N}$ owning to (13.2). Summarizing the above, under the cooperative agreement firms produce less product, its price is higher, and firms get more profit in total with respect to the Nash equilibrium agreement.

13.3 Characteristic Functions in a Linear Oligopoly TU Game

We have observed that under the cooperative agreement, a firm may receive zero profit, however its profit is always positive under the Nash equilibrium. To encourage firms to cooperate with each other, the joint profit of $\sum_{i\in N}\pi_i(\bar{q})$ should be allocated in an alternative way, differing from (13.7). For this reason, we first move from a noncooperative game $(N, \{Q_i\}_{i\in N}, \{\pi_i\}_{i\in N})$ to its cooperative version called a *cooperative game* or TU game, and then allocate that joint profit with the use of an appropriate cooperative solution. We denote the cooperative game by (N, v) where $v : 2^N \mapsto \mathbb{R}$ is the characteristic function assigning the worth $v(S)$ to any subset $S \subseteq N$ called coalition with $v(\varnothing) = 0$. In this section, we consider different approaches for determining the characteristic function v. To emphasize a particular approach, we will use a superscript for v, however for any of the approaches $v(N) = (a - c_{i_N})^2/4$ will denote the joint profit to be allocated.

13.3.1 α-Characteristic Function

The first measure determining the worth of any coalition $S \subset N$ and considered in the game-theoretic literature was the α-characteristic function v^α introduced in [12]. The value $v^\alpha(S)$ is interpreted as the maximum value that coalition S can get in the worst-case scenario, i.e., when the complement $N \setminus S$ acts against S:

$$v^\alpha(S) = \max_{q_i\in[0,a], i\in S}\ \min_{q_j\in[0,a], j\in N\setminus S} \sum_{i\in S}\pi_i(q). \tag{13.8}$$

From [3] it follows that $v^\alpha(S) = 0$ for any coalition $S \subset N$, and the profile of outputs that solves (13.8) is of the form:

$$q_i^{\alpha,S} = \begin{cases} 0, & \text{if } i \in S, \\ \dfrac{a}{|N\setminus S|}, & \text{if } i \in N \setminus S, \end{cases} \tag{13.9}$$

with $\sum_{i\in N} q_i^{\alpha,S} = a$.

13.3.2 β-Characteristic Function

Another measure determining the worth of coalition $S \subset N$ was also considered in [12]. The value $v^\beta(S)$ amounts to the smallest value that the complement $N \setminus S$ can force S to receive, without knowing its actions, and this value is defined as

$$v^\beta(S) = \min_{q_j \in [0,a], j \in N \setminus S} \max_{q_i \in [0,a], i \in S} \sum_{i \in S} \pi_i(q). \tag{13.10}$$

In [3] it was shown that $v^\beta(S) = 0$ for any coalition $S \subset N$ thus $v^\alpha(S) = v^\beta(S) = 0$, and the profile of outputs that solves (13.10) is the same: $q_i^{\beta,S} = q_i^{\alpha,S}$, $i \in N$ with $\sum_{i \in N} q_i^{\beta,S} = a$.

13.3.3 γ-Characteristic Function

Considered in [2, 9], the γ-characteristic function v^γ for any coalition $S \subset N$ assigns its equilibrium payoff in a noncooperative game played between S acting as one player and players from $N \setminus S$ acting as singletons. Hence we get the following result.

Proposition 13.2 *For any coalition $S \subset N$, it holds that*

$$v^\gamma(S) = \left(q_{i_S}^* + \frac{1}{n-s+2} \sum_{j \in S \setminus i_S} q_j^* \right)^2. \tag{13.11}$$

Proof According to the definition of the γ-characteristic function, coalition $S \subset N$ aims at maximizing the profit $\sum_{i \in S} \pi_i(q)$ over $q_i \in [0,a]$ for all $i \in S$, whereas each firm $j \in N \setminus S$ seeks to maximize its own profit $\pi_j(q)$ over $q_j \in [0,a]$. Maximizing $\sum_{i \in S} \pi_i(q)$ with respect to the profile of quantities of firms from S, we get the reaction of S (by Proposition 13.1):

$$q_i^{\gamma,S} = \begin{cases} \dfrac{a - \sum\limits_{j \in N \setminus S} q_j - c_{i_S}}{2|I_S|}, & \text{if } i \in S \cap I_S, \\ 0, & \text{if } i \in S \setminus I_S. \end{cases} \tag{13.12}$$

At the same time for any $j \in N \setminus S$, maximizing $\pi_j(q)$ with respect to the q_j, the first-order conditions imply $q_j = a - \sum_{i \in S} q_i - \sum_{i \in N \setminus S} q_i - c_j$. Summing these equalities over all $j \in N \setminus S$ and substituting expression (13.12) into this sum, we obtain that

$$\sum_{i\in N\setminus S} q_i^{\gamma,S} = \frac{(n-s)(a+c_{i_S})-2c_{N\setminus S}}{n-s+2},$$

where $s=|S|$. Thus

$$v^\gamma(S) = \frac{1}{4}\left(a-\sum_{i\in N\setminus S} q_i^\gamma(S)-c_{i_S}\right)^2 = \left(\frac{a-(n-s+1)c_{i_S}+c_{N\setminus S}}{n-s+2}\right)^2$$
$$= \left(q_{i_S}^* + \frac{1}{n-s+2}\sum_{j\in S\setminus i_S} q_j^*\right)^2.$$

The equilibrium profile of outputs which is used to find the value $v^\gamma(S)$ for $S\subset N$ is of the form:

$$q_i^{\gamma,S} = \begin{cases} \frac{1}{|I_S|}\left(q_{i_S}^* + \frac{1}{n-s+2}\sum_{j\in S\setminus i_S} q_j^*\right), & \text{if } i\in S\cap I_S,\\ 0, & \text{if } i\in S\setminus I_S,\\ q_i^* + \frac{1}{n-s+2}\sum_{j\in S\setminus i_S} q_j^*, & \text{if } i\in N\setminus S, \end{cases} \tag{13.13}$$

and $\sum_{i\in N} q_i^{\gamma,S} \leqslant \sum_{i\in N} q_i^*$. □

13.3.4 δ-Characteristic Function

Motivated by the computational complexity of α-, β-, and γ-characteristic functions known for that moment, Petrosjan and Zaccour [8] introduced the δ-characteristic function v^δ which for any coalition $S\subset N$ was determined as its best response against the Nash equilibrium output of singletons from $N\setminus S$, i.e.:

$$v^\delta(S) = \max_{q_i\in[0,a], i\in S} \sum_{i\in S} \pi_i(q_S, q_{N\setminus S}^*). \tag{13.14}$$

Here the equilibrium profile of outputs of firms from coalition $N\setminus S$ is given by (13.1).

Proposition 13.3 *For any coalition $S \subset N$, it holds that*

$$v^{\delta}(S) = \left(q_{i_S}^* + \frac{1}{2} \sum_{j \in S \setminus i_S} q_j^* \right)^2. \tag{13.15}$$

Proof By the definition of the δ-characteristic function, the expression of the Nash equilibrium output (13.1), and the result of Proposition 13.1, we obtain

$$\begin{aligned}
v^{\delta}(S) &= \max_{q_i \in [0,a], i \in S} \sum_{i \in S} \left(\left(a - \sum_{j \in S} q_j - \sum_{j \in N \setminus S} q_j^* \right)_+ - c_i \right) q_i \\
&= \max_{q_i \in [0,a], i \in S} \sum_{i \in S} \left(a - \sum_{j \in N \setminus S} q_j^* - c_i - \sum_{j \in S} q_j \right) q_i \\
&= \max_{q_i \in [0,a], i \in S} \sum_{i \in S} \left(a - \sum_{j \in N \setminus S} q_j^* - c_i \right) q_i - \left(\sum_{j \in S} q_j \right)^2 \\
&= \frac{1}{4} \left(a - \sum_{j \in N \setminus S} q_j^* - c_{i_S} \right)^2 \\
&= \frac{\left((s+1) q_{i_S}^* - c_S + s c_{i_S} \right)^2}{4} = \left(q_{i_S}^* + \frac{1}{2} \sum_{j \in S \setminus i_S} q_j^* \right)^2.
\end{aligned}$$

Here we assumed that the total output does not exceed a, otherwise the maximum in (13.14) would be negative. Note that one of the profiles of quantities that solves maximization problem (13.14) is of the form:

$$q_i^{\delta,S} = \begin{cases} \frac{1}{|I_S|} \left(q_{i_S}^* + \frac{1}{2} \sum_{j \in S \setminus i_S} q_j^* \right), & \text{if } i \in S \cap I_S, \\ 0, & \text{if } i \in S \setminus I_S, \\ q_i^*, & \text{if } i \in N \setminus S, \end{cases} \tag{13.16}$$

and $\sum_{i \in N} q_i^{\delta,S} \leqslant \sum_{i \in N} q_i^{\gamma,S}$, but $\sum_{i \in S} q_i^{\delta,S} > \sum_{i \in S} q_i^{\gamma,S}$. Hence the proposition is proved. □

We notice a relationship between the γ- and δ-characteristic functions. For any S, it holds that

$$v^{\gamma}(S) = \left(\frac{2\sqrt{v^{\delta}(S)} + (n-s)\sqrt{v^{\delta}(i_S)}}{n-s+2} \right)^2,$$

i.e., $v^{\gamma}(S)$ is the square of the weighted average of $\sqrt{v^{\delta}(S)}$ and equilibrium output $q^*_{i_S}$ of firm i_S having the smallest unit cost in coalition S.

13.3.5 *ζ-Characteristic Function*

An approach for determining the worth of a coalition by means of the so-called ζ-characteristic function v^{ζ} was presented in [5]. For coalition $S \subset N$, the value $v^{\zeta}(S)$ measures the worst profit that S can achieve following the cooperative agreement $\bar{q}$ given by (13.6). In other words, $v^{\zeta}(S)$ is the value of the minimization problem

$$v^{\zeta}(S) = \min_{q_j \in [0,a],\, j \in N \setminus S} \sum_{i \in S} \pi_i(\bar{q}_S, q_{N \setminus S}). \tag{13.17}$$

Proposition 13.4 *For any coalition $S \subset N$, it holds that*

$$v^{\zeta}(S) = -|S \cap I_N| c_{i_N} \bar{q}_{i_N}. \tag{13.18}$$

Proof By the definition of the ζ-characteristic function and the expression of the cooperative output (13.6), we get

$$\begin{aligned} v^{\zeta}(S) &= \min_{q_j \in [0,a],\, j \in N \setminus S} \sum_{i \in S} \left(\left(a - \sum_{j \in S} \bar{q}_j - \sum_{j \in N \setminus S} q_j \right)_+ - c_i \right) \bar{q}_i \\ &= \sum_{i \in S \cap I_N} \left(\left(- \sum_{j \in S} \bar{q}_j \right)_+ - c_i \right) \bar{q}_i = -|S \cap I_N| c_{i_N} \bar{q}_{i_N}. \end{aligned}$$

A profile of firms' outputs that solves minimization problem (13.17) is given by:

$$q_i^{\zeta,S} = \begin{cases} \bar{q}_i, & \text{if } i \in S, \\ \dfrac{a}{|N \setminus S|}, & \text{if } i \in N \setminus S, \end{cases} \tag{13.19}$$

and $\sum_{i \in N} q_i^{\zeta,S} > a$. Thus the statement of the proposition is proved. □

13.4 Properties of the Characteristic Functions

In this section, we study properties of the characteristic functions that have been introduced and the relationships between them. Characteristic function v is *monotonic* if $v(R) \leqslant v(S)$ for any coalitions $R \subset S$. Characteristic function v is *superadditive* if $v(S \cup R) \geqslant v(S) + v(R)$ for any disjoint coalitions $S, R \subseteq N$. Characteristic function v is *supermodular* if $v(S \cup R) + v(S \cap R) \geqslant v(S) + v(R)$ for any coalitions $S, R \subseteq N$. When v is supermodular, the game (N, v) is *convex*. The properties of v^α, v^β such as monotonicity, superadditivity, supermodularity were examined in [3, 7]. Some results for v^γ were presented in [6, 9], for example, the existence of the γ-core for an oligopoly game either with transferable technologies or without transferable technologies but with $n \leqslant 4$. In the present section we study the properties of v^γ, v^δ, and v^ζ for the linear oligopoly game without transferable technologies.

Proposition 13.5 *Characteristic functions v^γ and v^δ are monotonic whereas v^ζ is not.*

Proof Let $R \subset S \subseteq N$ with $|R| = r$ and $|S| = s$. Therefore, $c_{i_R} \geqslant c_{i_S}$ and $q^*_{i_R} \leqslant q^*_{i_S}$. First, prove the monotonicity of v^δ. Since v^δ is nonnegative, it suffices to show that $\sqrt{v^\delta}$ is monotonic. Indeed,

$$\sqrt{v^\delta(S)} - \sqrt{v^\delta(R)} = q^*_{i_S} - q^*_{i_R} + \frac{1}{2} \sum_{j \in (S \setminus i_S) \setminus (R \setminus i_R)} q^*_j > 0.$$

Second, prove the monotonicity of v^γ. Again, since v^γ is nonnegative, it suffices to show that $\sqrt{v^\gamma}$ is monotonic. We have:

$$\begin{aligned}\sqrt{v^\gamma(S)} - \sqrt{v^\gamma(R)} &= q^*_{i_S} + \frac{1}{n-s+2} \sum_{j \in S \setminus i_S} q^*_j - q^*_{i_R} - \frac{1}{n-r+2} \sum_{j \in R \setminus i_R} q^*_j \\ &\geqslant q^*_{i_S} - q^*_{i_R} + \left(\frac{1}{n-s+2} - \frac{1}{n-r+2} \right) \sum_{j \in S \setminus i_S} q^*_j > 0.\end{aligned}$$

Finally, show that v^ζ is not monotonic. Indeed, for $R \subset S \subset N$, it holds that $v^\zeta(S) - v^\zeta(R) = (|R \cap I_N| - |S \cap I_N|) c_{i_N} \bar{q}_{i_N} \leqslant 0$, but $v^\zeta(N) - v^\zeta(S) = (a - c_{i_N})^2/4 + |S \cap I_N| c_{i_N} \bar{q}_{i_N} > 0$, and this completes the proof. □

Proposition 13.6 *Characteristic functions v^δ and v^ζ are superadditive whereas v^γ is superadditive only in case of duopoly; i.e., when $n = 2$.*

Proof Let $S, R \subseteq N$ be two disjoint coalitions with $|S| = s$ and $|R| = r$. Without loss of generality, we suppose that $c_{i_S} \leqslant c_{i_R}$, therefore $q^*_{i_S} \geqslant q^*_{i_R}$. First, prove the superadditivity of v^δ. From (13.15), it can be easily verified that

$$\sqrt{v^\delta(S\cup R)} = \sqrt{v^\delta(S)} + \sqrt{v^\delta(R)} - \frac{1}{2}\sqrt{v^\delta(i_R)}. \tag{13.20}$$

Using (13.20), we obtain:

$$\begin{aligned}
&v^\delta(S\cup R) - v^\delta(S) - v^\delta(R)\\
&= \frac{1}{4}v^\delta(i_R) + 2\sqrt{v^\delta(S)v^\delta(R)} - \sqrt{v^\delta(i_R)}\left(\sqrt{v^\delta(S)} + \sqrt{v^\delta(R)}\right)\\
&= \frac{1}{4}v^\delta(i_R) + \sqrt{v^\delta(S)}\left(\sqrt{v^\delta(R)} - \sqrt{v^\delta(i_R)}\right) + \sqrt{v^\delta(R)}\left(\sqrt{v^\delta(S)} - \sqrt{v^\delta(i_R)}\right)\\
&\geqslant \frac{1}{4}v^\delta(i_R) + \sqrt{v^\delta(S)}\left(\sqrt{v^\delta(R)} - \sqrt{v^\delta(i_R)}\right) + \sqrt{v^\delta(R)}\left(\sqrt{v^\delta(S)} - \sqrt{v^\delta(i_S)}\right) > 0.
\end{aligned}$$

Second, to prove the superadditivity of v^ζ, we note that for $S\cup R \subset N$,

$$\begin{aligned}
v^\zeta(S\cup R) - v^\zeta(S) - v^\zeta(R) &= -c_{i_N}\bar{q}_{i_N}(|(S\cup R)\cap I_N| - |S\cap I_N| - |R\cap I_N|)\\
&= -c_{i_N}\bar{q}_{i_N}\,(|(S\cap I_N)\cup(R\cap I_N)| - |S\cap I_N| - |R\cap I_N|)\\
&= -c_{i_N}\bar{q}_{i_N}\,(|S\cap I_N| + |R\cap I_N|\\
&\quad -|(S\cap I_N)\cap(R\cap I_N)| - |S\cap I_N| - |R\cap I_N|)\\
&= c_{i_N}\bar{q}_{i_N}|(S\cap I_N)\cap(R\cap I_N)| = 0,
\end{aligned}$$

because $(S\cap I_N)\cap(R\cap I_N) = \varnothing$ when S and R are disjoint coalitions. At the same time, when $S\cup R = N$, we have $v^\zeta(S\cup R) - v^\zeta(S) - v^\zeta(R) = v^\zeta(N) + c_{i_N}\bar{q}_{i_N}(|S\cap I_N| + |R\cap I_N|) > 0$.

And finally, consider v^γ. The superadditivity of v^γ in case of duopoly is obvious. Let $S = i_S$ and $R = i_R$. Using (13.11), it follows that

$$\begin{aligned}
v^\gamma(i_S\cup i_R) - v^\gamma(i_S) - v^\gamma(i_R) &= \left(q^*_{i_S} + \frac{q^*_{i_R}}{n}\right)^2 - \left(q^*_{i_S}\right)^2 - \left(q^*_{i_R}\right)^2\\
&= q^*_{i_S}q^*_{i_R}\left(\frac{2}{n} - \frac{q^*_{i_R}}{q^*_{i_S}}\left(1 - \frac{1}{n^2}\right)\right),
\end{aligned}$$

which becomes negative when $n > q^*_{i_S}/q^*_{i_R} + \sqrt{1 + \left(q^*_{i_S}/q^*_{i_R}\right)^2} \geqslant 1 + \sqrt{2} > 2$. The statement is proved. □

Proposition 13.7 *Characteristic function v^ζ is supermodular; v^γ is supermodular only in case of duopoly, and v^δ is supermodular either when $n \leqslant 4$, or when firms are symmetrical.*

Proof We first prove the supermodularity of v^ζ. Using results of Proposition 13.6, we obtain: $v^\zeta(S \cup R) + v^\zeta(S \cap R) - v^\zeta(S) - v^\zeta(R) = c_{i_N}\bar{q}_{i_N}(|(S \cap I_N) \cap (R \cap I_N)| - |(S \cap R) \cap I_N|) = 0$ when $S \cup R \subset N$. If $S \cup R = N$, then $v^\zeta(S \cup R) + v^\zeta(S \cap R) - v^\zeta(S) - v^\zeta(R) = v^\zeta(N) + c_{i_N}\bar{q}_{i_N}(|S \cap I_N| + |R \cap I_N| - |(S \cap R) \cap I_N|) > 0$ because the expression in the brackets is nonnegative.

Second, supermodularity implies superadditivity, and if v^γ is not superadditive, it cannot also be supermodular. By Proposition 13.6, in case of duopoly, v^γ is superadditive and therefore supermodular.

Finally, consider v^δ. Let $S, R \subseteq N$ be two coalitions with $|S| = s$ and $|R| = r$. Without loss of generality, we suppose that $c_{i_S} \leqslant c_{i_R} \leqslant c_{i_{S\cap R}}$, therefore $q^*_{i_S} \geqslant q^*_{i_R} \geqslant q^*_{i_{S\cap R}}$. It can be verified that

$$\sqrt{v^\delta(S \cup R)} = \sqrt{v^\delta(S)} + \sqrt{v^\delta(R)} - \sqrt{v^\delta(S \cap R)} - \frac{1}{2}\sqrt{v^\delta(i_R)} + \frac{1}{2}\sqrt{v^\delta(i_{S\cap R})}, \tag{13.21}$$

which is an extension of (13.20) when coalitions S and R are not necessarily disjoint. Using (13.21) and recalling that $\sqrt{v^\delta(i_R)} = q^*_{i_R}$ and $\sqrt{v^\delta(i_{S\cap R})} = q^*_{i_{S\cap R}}$, we have:

$$\begin{aligned} v^\delta(S \cup R) + v^\delta(S \cap R) - v^\delta(S) - v^\delta(R) = \frac{1}{4}\left(q^*_{i_R} - q^*_{i_{S\cap R}}\right)^2 \\ +2\left(\sqrt{v^\delta(S)} - \sqrt{v^\delta(S \cap R)}\right)\left(\sqrt{v^\delta(R)} - \sqrt{v^\delta(S \cap R)}\right) \\ -\left(q^*_{i_R} - q^*_{i_{S\cap R}}\right)\left(\sqrt{v^\delta(S)} + \sqrt{v^\delta(R)} - \sqrt{v^\delta(S \cap R)}\right). \end{aligned}$$

Due to the monotonicity of v^δ, the latter expression is positive when $q^*_{i_R} = q^*_{i_{S\cap R}}$, i.e., the supermodularity condition holds. This is also the case when firms are symmetrical, hence v^δ will be supermodular.

Now we show that v^δ is supermodular in a general case for $n \leqslant 4$. In case of duopoly supermodularity is obvious. Let $n = 3$, and without loss of generality, we suppose $c_1 \leqslant c_2 \leqslant c_3$, thus here there is only one case of our interest: $S = \{1, 3\}$, $R = \{2, 3\}$. The case when $S = \{1, 2\}$, $R = \{2, 3\}$ is not of much interest since $q^*_{i_R} = q^*_{i_{S\cap R}}$ and therefore $i_R = i_{S\cap R} = 2$. Similarly, when $S = \{1, 2\}$, $R = \{1, 3\}$, we have $q^*_{i_R} = q^*_{i_{S\cap R}}$ and therefore $i_R = i_{S\cap R} = 1$. Other cases lead either to the inequality for superadditivity or to the case when the supermodularity inequality becomes an equality. Consider the aforementioned case. Let $S = \{1, 3\}$, $R = \{2, 3\}$. We obtain: $v^\delta(S \cup R) + v^\delta(S \cap R) - v^\delta(S) - v^\delta(R) = \frac{1}{4}(-3(q^*_2)^2 + 3(q^*_3)^2 + 4q^*_1q^*_2 - 2q^*_2q^*_3) \geqslant \frac{1}{4}((q^*_2)^2 + 3(q^*_3)^2 - 2q^*_2q^*_3) = \frac{1}{4}((q^*_2 - q^*_3)^2 + 2(q^*_3)^2) > 0$.

Let now $n = 4$ and $c_1 \leqslant c_2 \leqslant c_3 \leqslant c_4$. There are ten cases when the inequality guaranteeing supermodularity should be verified (when coalitions S and R intersect, but $q^*_{i_R} \neq q^*_{i_{S\cap R}}$): $S = \{1, 3\}$, $R = \{2, 3\}$; $S = \{1, 4\}$, $R = \{2, 4\}$; $S = \{1, 4\}$, $R = \{3, 4\}$; $S = \{2, 4\}$, $R = \{3, 4\}$; $S = \{1, 3\}$, $R = \{2, 3, 4\}$; $S = \{1, 4\}$, $R = \{2, 3, 4\}$; $S = \{1, 3, 4\}$, $R = \{2, 3\}$; $S = \{1, 3, 4\}$, $R = \{2, 4\}$; $S = \{1, 2, 4\}$, $R = \{3, 4\}$; and

$S = \{1,3,4\}$, $R = \{2,3,4\}$. Prove for the case when $S = \{1,3,4\}$, $R = \{2,3,4\}$. We have: $v^\delta(S \cup R) + v^\delta(S \cap R) - v^\delta(S) - v^\delta(R) = \frac{1}{4}(-3(q_2^*)^2 + 3(q_3^*)^2 + 4q_1^*q_2^* - 2q_2^*q_3^* - 2q_2^*q_4^* + 2q_3^*q_4^*) \geqslant \frac{1}{4}((q_2^* - q_3^*)^2 + 2(q_3^*)^2 - 2q_4^*(q_2^* - q_3^*)) = \frac{1}{4}((q_2^* - q_3^* - q_4^*)^2 + 2(q_3^*)^2 - (q_4^*)^2) > 0$. All other cases can be examined in a similar way. Hence the proposition is now proved. □

Example 13.1 Consider an oligopoly with $N = \{1,2,3,4,5\}$ and the following values of parameters: $a = 10$, $c_1 = c_2 = 1$, $c_3 = c_4 = c_5 = 2$. From (13.1) we get: $q_1^* = q_2^* = 2$, $q_3^* = q_4^* = q_5^* = 1$. Let $S = \{1,3,4,5\}$ and $R = \{2,3,4,5\}$, therefore $i_S = 1$, $i_R = 2$, $i_{S\cup R} \in \{1,2\}$, and $i_{S\cap R} \in \{3,4,5\}$. Using (13.15), we obtain $v^\delta(S) = v^\delta(R) = 12.25$, $v^\delta(S \cup R) = 20.25$, and $v^\delta(S \cap R) = 4$ which means that $v(S \cup R) + v(S \cap R) < v(S) + v(R)$ and v^δ is not supermodular.

Proposition 13.8 *For any coalition $S \subset N$, the condition $v^\zeta(S) \leqslant v^\alpha(S) = v^\beta(S) \leqslant v^\gamma(S) \leqslant v^\delta(S)$ is satisfied.*

Proof The fulfillment of two inequalities $v^\zeta(S) \leqslant v^\alpha(S)$ and $v^\beta(S) \leqslant v^\gamma(S)$ is obvious. Prove that $v^\gamma(S) \leqslant v^\delta(S)$. Since values $v^\gamma(S)$ and $v^\delta(S)$ are positive for all S, it suffices to show that $\sqrt{v^\gamma(S)} \leqslant \sqrt{v^\delta(S)}$. We have $\sqrt{v^\delta(S)} - \sqrt{v^\gamma(S)} = \frac{n-s}{2(n-s+2)} \sum_{j\in S\setminus i_S} q_j^*$ which is positive. The statement of the proposition is hence proved. □

13.5 Cooperative Solutions for a Linear Oligopoly TU Game

An imputation set of cooperative game (N, v) is the set $\mathscr{I}[v] = \{(\xi_1, \ldots, \xi_n) : \sum_{i\in N} \xi_i = v(N);\ \xi_i \geqslant v(\{i\})\}$. A cooperative solution is a rule that maps v into a subset of $\mathscr{I}[v]$. In particular, the core of the game (N, v) is defined as the set $\mathscr{C}[v] = \{(\xi_1, \ldots, \xi_n) \in \mathscr{I}[v] :\ \sum_{i\in S} \xi_i \geqslant v(S),\ \ S \subset N\}$. The Shapley value $\Phi[v] = (\Phi_1[v], \ldots, \Phi_n[v])$ is an imputation whose components are defined as $\Phi_i[v] = \sum_{S\subseteq N} \frac{(n-|S|)!(|S|-1)!}{n!}(v(S) - v(S \setminus \{i\}))$, $i \in N$. The core of the game (N, v^α) will be called the α-core and denoted by $\mathscr{C}[v^\alpha]$. Similarly, we determine β-, γ-, δ-, and ζ-cores and denote them by $\mathscr{C}[v^\beta]$, $\mathscr{C}[v^\gamma]$, $\mathscr{C}[v^\delta]$, and $\mathscr{C}[v^\zeta]$, respectively.

The existence of α-, β-cores was shown in [3, 7], thus in view of Proposition 13.8, the next result directly follows.

Corollary 13.1 *Let δ-core be nonempty. Then $\mathscr{C}[v^\delta] \subseteq \mathscr{C}[v^\gamma] \subseteq \mathscr{C}[v^\alpha] = \mathscr{C}[v^\beta] \subseteq \mathscr{C}[v^\zeta]$.*

The above result notes a nested structure of the cores when the δ-core is nonempty. The existence of γ- and δ-cores can be guaranteed when the number of firms does not exceed 4 and/or when firms are symmetrical since the δ-game becomes convex in these cases (see [11]). As to the ζ-core, it always exists, and its nonemptiness follows from the existence of $\mathscr{C}[v^\alpha]$.

Table 13.1 The values of α-, β-, γ-, δ-, and ζ-characteristic functions

S	$v^{\alpha}(S)$	$v^{\beta}(S)$	$v^{\gamma}(S)$	$v^{\delta}(S)$	$v^{\zeta}(S)$
{1}	0	0	0.0676	0.0676	−0.25
{2}	0	0	0.0676	0.0676	−0.25
{3}	0	0	0.0256	0.0256	0
{4}	0	0	0.0036	0.0036	0
{1, 2}	0	0	0.1056	0.1521	−0.5
{1, 3}	0	0	0.09	0.1156	−0.25
{1, 4}	0	0	0.0756	0.0841	−0.25
{2, 3}	0	0	0.09	0.1156	−0.25
{2, 4}	0	0	0.0756	0.0841	−0.25
{3, 4}	0	0	0.0306	0.0361	0
{1, 2, 3}	0	0	0.16	0.2209	−0.5
{1, 2, 4}	0	0	0.1344	0.1764	−0.5
{1, 3, 4}	0	0	0.1111	0.1369	−0.25
{2, 3, 4}	0	0	0.1111	0.1369	−0.25
{1, 2, 3, 4}	0.25	0.25	0.25	0.25	0.25

Example 13.2 We consider an oligopoly with $N = \{1, 2, 3, 4\}$ where $a = 2$, $c_1 = c_2 = 1$, $c_3 = 1.1$, and $c_4 = 1.2$. Here the set $I_N = \{1, 2\}$. Table 13.1 summarizes the values of α-, β-, γ-, δ-, and ζ-characteristic functions. Figure 13.1 demonstrates ζ-, α-, γ-, and δ-cores where the largest set represents the ζ-core and the smallest one is the δ-core (recall that the α-core coincides with the β-core). On Fig. 13.2, we demonstrate the same cooperative set solutions in a more detailed view.[2]

Example 13.3 (Symmetric Firms) As one of the special cases of oligopoly often considered in the literature, we examine a symmetric game where unit costs of the firms are equal, i.e., $c_1 = \ldots = c_n = c$ with $c < a$. Under these assumptions, the condition (13.2) holds for any n. We also note that $I_S = S$ for any $S \subseteq N$. The equilibrium output of firm $i \in N$ determined by (13.1), takes the form $q_i^* = \frac{a-c}{n+1}$, whereas under the cooperative agreement, the output determined by (13.6) becomes $\bar{q}_i = \frac{a-c}{2n}$. The equilibrium profit of firm $i \in N$ equals $\pi_i(q^*) = \left(\frac{a-c}{n+1}\right)^2$ and the profit of i under the cooperative agreement becomes $\pi_i(\bar{q}) = \frac{(a-c)^2}{4n}$ exceeding $\pi_i(q^*)$. This fact means that all firms take advantage from cooperation even without reallocating the total profit of $\sum_{i\in N} \pi_i(\bar{q})$ according to a cooperative solution. We note that this result does not hold in a general case. However if firms come to a cooperative solution, the characteristic function should be determined. Then $v^{\alpha}(N) = v^{\beta}(N) = v^{\gamma}(N) = v^{\delta}(N) = v^{\zeta}(N) = (a-c)^2/4$. Further, for any $S \subset N$, we have $v^{\alpha}(S) = v^{\beta}(S) = 0$ while using (13.11), (13.15), and (13.18), it

[2]The figures were obtained with the use of TUGlab toolbox for Matlab http://mmiras.webs.uvigo.es/TUGlab/.

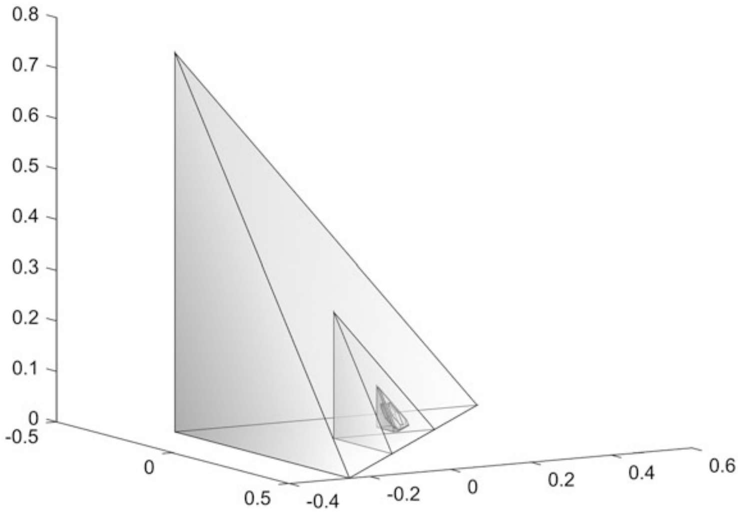

Fig. 13.1 ζ-, α-, γ-, and δ-cores (from largest to smallest)

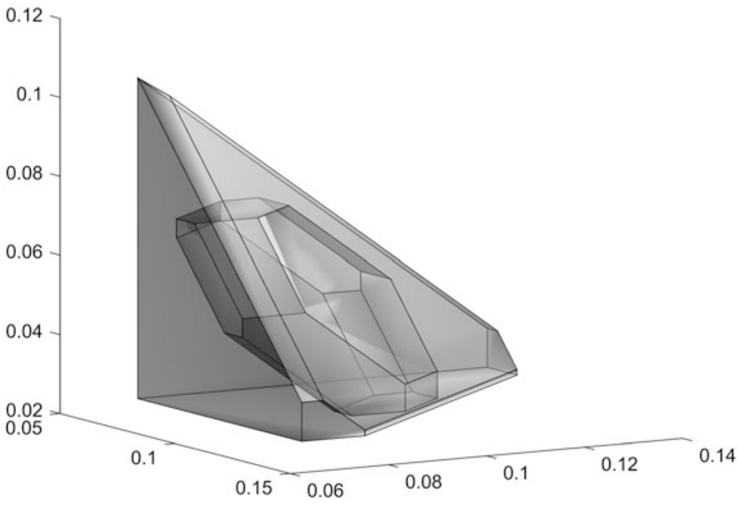

Fig. 13.2 γ-core (superset) and δ-core (subset)

follows that $v^{\gamma}(S) = \left(\frac{a-c}{n-s+2}\right)^2$, $v^{\delta}(S) = \left(\frac{(s+1)(a-c)}{2(n+1)}\right)^2$, and $v^{\zeta}(S) = -\frac{sc(a-c)}{n}$ where $s = |S|$.

By Proposition 13.7, where characteristic function v^{δ} is supermodular, the corresponding TU game (N, v^{δ}) is convex and therefore has a nonempty core $\mathscr{C}[v^{\delta}]$. From [11], the Shapley value $\Phi[v^{\delta}]$, whose components equal the cooperative profits $\pi_i(\bar{q})$, $i \in N$, belongs to $\mathscr{C}[v^{\delta}]$ being the center of gravity of its extreme points. Since firms are symmetrical, $\Phi[v^\alpha] = \Phi[v^\beta] = \Phi[v^\gamma] = \Phi[v^\delta] = \Phi[v^\zeta]$. From Corollary 13.1, it follows that the Shapley value $\Phi[v^\delta]$ belongs to any of the cores.

13.6 Conclusion

We have examined the properties of γ-, δ-, and ζ-characteristic functions in linear oligopoly TU games. We found that the γ-characteristic function is monotonic, however it is superadditive and supermodular only in case of duopoly. The δ-characteristic function is monotonic, and superadditive, but it is supermodular either when $n \leqslant 4$, or when firms are symmetrical. As to the ζ-characteristic function, it is superadditive and supermodular but not monotonic. When δ-characteristic function is supermodular, we also found that the γ-, δ-, and ζ-games have nonempty cores with a nested structure that is also expressed in their relationship to the α- and β-cores.

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