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Toward Equity and Social Justice in Mathematics Education

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Toward Equity and Social Justice in Mathematics Education

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To Jaedyn Bartell
toward a more just and equitable future
in which she may live

Preface

Contributions for this volume stem from the 2015 Annual Meeting of the North American Group of the Psychology of Mathematics Education (PME-NA) held at Michigan State University. The conference theme *Critical Responses to Enduring Challenges in Mathematics Education* invited mathematics education scholars to reflect upon and critically respond to enduring challenges in teaching and learning mathematics for all students. To ignite discussion within the field, the conference was organized around four focal enduring challenges: (1) addressing the needs of marginalized populations in school mathematics, (2) teaching as responsive to various conceptions of mathematics, (3) the role of assessment in teaching and learning, and (4) the impact of teacher evaluation in high-stakes assessment in teaching.

As suggested by these four challenges, PME-NA has moved beyond a primary focus on learning as dependent on a psychological representation of the mind alone to also consider, for example, the social and sociopolitical, attending to context, identity, culture, power, and the ways systems serve to privilege some and oppress others along the lines of class, race, gender, and so forth, within and beyond the boundaries of mathematics education. This book echoes these turns and is intended for all mathematics educators committed to ongoing work toward equity and justice in mathematics education. More specifically, authors whose papers explicitly attended to issues of equity and justice in mathematics education at the 2015 PME-NA conference were invited to contribute chapters. These chapters, then, reflect current efforts toward equity and justice in mathematics education. The authors' work spans across the ten strands of PME-NA (e.g., Student Learning and Related Factors, Theory and Research Methods, Teacher Learning and Knowledge), demonstrating a variety of perspectives.

The book is divided into four parts: (1) theoretical and political perspectives toward equity and justice in mathematics education, (2) identifying and connecting to family and community funds of knowledge, (3) student learning and engagement in preK-12 mathematics classrooms, and (4) supporting teachers in addressing the needs of marginalized learners. Each of these four parts addresses in some way the enduring challenge of meeting the needs of marginalized students in mathematics education and examines how race, class, culture, power, justice, and mathematics teaching and learning intersect in mathematics education to sustain or disrupt inequities. Each section includes contributions from scholars writing critically about mathematics education in diverse contexts. Further, scholars were invited to provide commentaries on each section so that the book might not only provide a glimpse of research at a moment in time but also continue to push our thinking and work forward toward equity and justice in mathematics education.

East Lansing, MI, USA Tonya Gau Bartell

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Notes on Contributors

Aditya P. Adiredja is an Assistant Professor of Mathematics at The University of Arizona. Adiredja's research is situated in the intersections between post-secondary mathematics, equity, and cognition. His work looks at ways that perspectives on epistemology and learning determine the kinds of knowledge and students that get privileged in the classroom. This focus is part of a larger goal of turning undergraduate mathematics classrooms into spaces where underrepresented students can personally, intellectually, and professionally succeed as they critically engage with and learn mathematics.

Annica Andersson is an Associate Professor at the Department for Mathematics and Science Teaching, Malmö University. Andersson earned her Ph.D. at Aalborg University, with a thesis focusing on teachers' and students' identities in the context of critical mathematics education. Her research is located at the intersection of mathematics education, language, cultural responsiveness, and social justice with a particular focus on equity, authority, discourses, and human relationships in mathematics education school contexts.

Tonya Gau Bartell is an Associate Professor of Mathematics Education at Michigan State University. Bartell's research focuses on issues of culture, race, and power in mathematics teaching and learning with particular attention to teachers' development of mathematics pedagogy for social justice and pedagogy integrating a focus on children's multiple mathematical knowledge bases. She served as a co-chair of the 2015 Annual Meeting of PME-NA from which this book arose, is on the editorial board of *Mathematics Teacher Education and Development*, and is a co-editor of the *Journal of Teacher Education*.

Dan Battey is an Associate Professor of Mathematics Education in the Graduate School of Education at Rutgers University. Battey's work examines engaging teachers in learning opportunities that generate and sustain change and which challenge deficit narratives that limit opportunities for African American and Latino students in mathematics. Battey is currently working on understanding mathematics education as a racialized space through researching relational interactions in classrooms.

Ruth Beatty is an Associate Professor at Lakehead University. For the past 5 years, Beatty has worked with research teams comprising First Nations community members and educators from Ontario school boards to research the connections between First Nations cultural activities and the Western mathematics found in the Ontario curriculum. The goal of this research is to collaboratively design culturally responsive mathematics instruction and to learn from and incorporate First Nations pedagogical perspectives in inclusive classroom settings.

Robert Q. Berry III Ph.D., is President of the National Council of Teachers of Mathematics and is a Professor of Mathematics Education in the Curry School of Education at the University of Virginia, with an appointment in Curriculum Instruction and Special Education. Berry teaches in the teacher education program, and his research focuses on equity issues in mathematics education, with a particular focus on Black children.

Kelley Buchheister focuses her research and teaching on enhancing students' thinking and reasoning in early mathematics by developing practicing and prospective teachers' understanding of the cultural contexts of learning and constructing appropriately challenging environments that provide the greatest opportunity for all students to achieve high-quality experiences in the STEM (science, technology, engineering, and mathematics) disciplines. Furthermore, through her research, Buchheister, along with her colleagues, closely examines the role of culturally relevant pedagogy and teacher beliefs in educational experiences and has prioritized equity and cultural contexts of learning as fundamental elements of early educational experiences.

Erika C. Bullock is an Assistant Professor of Mathematics Education in the Department of Curriculum and Instruction at the University of Wisconsin-Madison. Bullock's work exists at the intersections of curriculum studies, urban education, and mathematics education. She historicizes issues and ideologies within mathematics education to examine how power operates within mathematics education to create and maintain inequities. Bullock's work has been published in outlets including *Educational Studies*, *Educational Studies in Mathematics*, *The Journal of Education*, and *Teachers College Record.*

Marta Civil is a Professor of Mathematics Education and the Roy F. Graesser Chair in the Department of Mathematics at The University of Arizona. Civil's research examines cultural, social, and language aspects in the teaching and learning of mathematics; the connections between in-school and out-of-school mathematics; and parental engagement in mathematics. She has led funded projects working with children, parents, and teachers, with a focus on developing culturally responsive learning environments, particularly with Latinx communities.

Zandra de Araujo is an Assistant Professor in the Department of Learning, Teaching, and Curriculum at the University of Missouri. de Araujo's research examines teachers' use of curriculum, particularly with emergent bilingual students.

Corey Drake is a Professor and Director of Teacher Preparation at Michigan State University. Drake's research interests include teachers learning from and about curriculum materials, as well as the roles of policy curriculum, and teacher preparation in supporting teachers' capacity to teach diverse groups of students. Her work has been funded by the National Science Foundation and the Spencer Foundation and is published in venues including *Educational Researcher*, *Journal of Teacher Education*, and *Journal of Mathematics Teacher Education.*

Mary Q. Foote is a Professor Emerita of Mathematics Education in the Department of Elementary and Early Childhood Education at Queens College of the City University of New York. Foote's research attends to equity issues in mathematics education and broadly stated examines issues in mathematics teacher education. More specifically her interests are in cultural and community knowledge and practices, and how they might inform mathematics teaching practice. She is currently involved researching and developing/facilitating two professional development projects: one supports teachers to teach mathematical modeling using cultural and community contexts in Grades 3–5, and the other supports teachers to develop more equitable instructional practices through action research projects that incorporate examination of access, agency, and allyship in mathematics teaching and learning.

Lynette DeAun Guzmán Ph.D., is a mathematics education researcher and teacher educator at The University of Arizona. Motivated by experiences as a Latinx woman navigating academic spaces, Guzmán's scholarship centers on addressing inequities in education for historically marginalized students with attention to identity and power. She examines discourses and practices in mathematics education to interrogate narrow epistemological and ontological perspectives on teaching and learning that often exclude students of color. Guzmán enjoys working with prospective and practicing K-8 teachers to transform classrooms with equity-oriented and humanizing practices that value young people as knowers and creators.

Maren Hall-Wieckert is a former research assistant at Brooklyn College of the City University of New York. Hall-Wieckert's interests lie in data science, critical cartography, and how notions of space and place frame and reframe conversations in the social sciences, especially around notions of teaching and learning.

Frances K. Harper is an Assistant Professor of STEM Education at the University of Tennessee at Knoxville. Her research broadly focuses on issues of equity and social justice in mathematics education and teacher education, particularly within urban contexts. She aspires to collaborate with both teachers and students to understand the culturally and contextually specific nuances of equitable collaborative learning and critical mathematical explorations.

Beth Herbel-Eisenmann a former junior high mathematics teacher, is currently a Professor of Mathematics Education at Michigan State University. She uses ideas and theories from sociolinguistics and critical discourse literatures to research classroom discourse practices as well as the professional development of secondary mathematics teachers. She is especially interested in issues of equity that concern authority, positioning, and voice in mathematics classrooms and professional development. Much of her work has been done through long-term collaborations with mathematics teachers and involves considering how teachers make sense of, take up, and use ideas from these literatures in their practice through the process of action research.

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Christa Jackson Ph.D., is an Associate Professor of Mathematics Education in the School of Education at Iowa State University. Jackson's research focuses on teachers' knowledge and conceptions of equity in the teaching and learning of mathematics. Within this knowledge base, she seeks to understand the instructional strategies and teaching practices mathematics teachers use that afford opportunities for students from diverse cultures, ethnicities, and socioeconomic backgrounds to learn rigorous, challenging mathematics while simultaneously fostering productive mathematical identities. Jackson's work also centers on effective mathematics instruction at the elementary and middle levels, the preparation of prospective mathematics teachers, strategies to help students who struggle in mathematics, STEM education, and STEM learning.

Lisa Kasmer is an Associate Professor in the Department of Mathematics at Grand Valley State University. Currently, she teaches mathematics courses for undergraduate students pursuing degrees in secondary mathematics education and directs the Study Abroad: Tanzania program at GVSU. Through this program, prospective teachers teach in local mathematics classrooms while studying the cultural and historical aspects of education in Tanzania. Her research interests include developing intercultural competencies for prospective teachers and curricular reasoning of middle grades mathematics teachers.

Gregory V. Larnell is currently an Assistant Professor of Mathematics Education in the Department of Curriculum and Instruction at the University of Illinois at Chicago. Larnell studied mathematics and African American studies at Lake Forest College and later studied mathematics education and earned his Ph.D. from the Colleges of Natural Science and Education at Michigan State University. His work centers on sociological and sociopolitical inquiry in mathematics education, with particular attention to mathematics learning experiences and identity; postsecondary remediation and other math-curricular transition issues; urban mathematics education; and critical approaches to (in)equity and justice concerning mathematics learning and teaching. Larnell's work appears in several outlets, including the *Handbook of Urban Education*, the *Journal for Research in Mathematics Education*, and the *Journal of Education.*

Luis A. Leyva Ph.D., is an Assistant Professor of Mathematics Education at Vanderbilt University – Peabody College of Education and Human Development. Using intersectionality theory from Black feminist thought and counter-storytelling methodology from critical race theory, Leyva's research foregrounds the voices of marginalized undergraduate students to understand their strategies in navigating oppressive institutional and interpersonal contexts of STEM education toward the development of positive academic identities at intersections of their gender, race, sexuality, and other social identities. His work has been published in the *Journal for Research in Mathematics Education*, *The Journal of Mathematical Behavior*, and *Journal of Urban Mathematics Education.* Leyva was a past recipient of the National Academy of Education/Spencer Foundation Dissertation Fellowship and currently serves as the principal investigator on a research project entitled *Challenging, Operationalizing, and Understanding Racialized and Gendered Events (COURAGE) in Undergraduate Mathematics* funded by the National Science Foundation.

Vivian Lim is an Assistant Professor of Mathematics at Stella and Charles Guttman Community College, City University of New York. Lim's research interest is in teaching mathematics for social justice and the role of mathematics education in the development of youth as citizens.

Carlos A. LópezLeiva is an Associate Professor in Bilingual and Mathematics Education in the Department of Language, Literacy, and Sociocultural Studies at the University of New Mexico. LópezLeiva's work focuses on teaching and learning ecologies – in relation to social interactions through language uses and ideologies, task designs, relationship development, and what counts as mathematics – that mediate members' participation in and meaning making of mathematical practices. His research comprises three foci: (1) issues of equity in social interactions, (2) out-of-school interdisciplinary mathematics teaching and learning, and (3) in-school interdisciplinary mathematics teaching and learning.

Nirmala Naresh is a Senior Lecturer in the Department of Mathematics at the University of North Texas. She received her bachelor's, master's, and M.Phil. degrees in Mathematics from the University of Madras in Chennai, India. Naresh received her Ph.D. in Mathematics Education from Illinois State University in the USA. She currently teaches mathematics courses for undergraduate students pursuing a degree in mathematics and mathematics education. Naresh's research interests lie in the domains of ethnomathematics, teaching mathematics for social justice, and the teaching and learning of probability and statistics.

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Matthew Sakow completed his Masters of Education in Learning, Teaching and Curriculum: Mathematics Education at the University of Missouri. Sakow currently teaches mathematics and the AVID elective as a Knowles Fellow at Columbia River High School in Vancouver, Washington.

Erin Smith is a doctoral candidate in the Department of Learning, Teaching and Curriculum at the University of Missouri. Smith's interests include equity and identity in mathematics education, the academic success of emergent bi/multilingual learners, and the use of positioning theory to examine discursive practices.

David W. Stinson Ph.D., is a Professor of Mathematics Education at Georgia State University, Atlanta. Stinson's research interests include critical postmodern theory and identity. Specifically, he explores how mathematics teachers, educators, and researchers (might) incorporate the philosophical and theoretical underpinnings of critical postmodern theory into their education philosophies, pedagogical practices, and/or research methods. Additionally, he examines how students constructed outside the White, Christian, heterosexual male of bourgeois privilege successfully accommodate, reconfigure, or resist (i.e., negotiate) the hegemonic discourses of society generally and schooling specifically, including those found in the mathematics classroom. Stinson serves as the editor in chief of the *Journal of Urban Mathematics Education* and recently completed a 3-year term as a member of the editorial panel of the *Journal for Research in Mathematics Education* and a 2-year term as a member of the AERA Review of Research Award Committee.

Miwa Aoki Takeuchi is an Assistant Professor at the University of Calgary, Canada. Takeuchi's works are situated in an emerging and transdisciplinary field to address a complex educational issue: improving access to rich mathematics learning opportunities, with a focus on linguistic diversity, and cultural and historical contexts. She approaches this issue through longitudinal and horizontal analyses of mathematics learning trajectories (both in-school and out-of-school settings), students' identities and lived experiences in learning mathematics, students' collaboration for mathematics learning, and co-design of learning environments with learners, teachers, and parents.

Cynthia E. Taylor received her Ph.D. in Curriculum and Instruction with a specialty in Mathematics Education from the University of Missouri and is currently an Associate Professor of Mathematics at Millersville University of Pennsylvania. In addition to supervising student teachers, Taylor teaches methods courses, content courses for elementary teachers, general education mathematics courses, and graduate courses in the Master of Education in Mathematics program. Taylor's interests include studying mathematics teacher educators (what they say and do in their classroom) and the preparation of prospective mathematics teachers (instructional strategies and teaching practices). Prior to joining Millersville, she was a high school mathematics teacher for nearly 10 years.

Anita Wager is a Professor of the Practice at Vanderbilt University's Peabody College. Wager's teaching and research focus on culturally relevant and socially just (mathematics) teaching in early childhood and elementary school.

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Ayşe Yolcu is a Research Assistant at the Department of Mathematics and Science Education, Hacettepe University, Turkey. Yolcu studies the principles of knowledge as making up people related to issues of equity in mathematics education. Specifically, she questions how the practices of school mathematics fabricate particular human kinds and their differences as the politics of mathematics education.

Part I
Theoretical and Political Perspectives Toward Equity and Justice in Mathematics Education

Chapter 1
Disrupting Policies and Reforms in Mathematics Education to Address the Needs of Marginalized Learners

Robert Q. Berry III

Abstract This chapter uses a hybrid policy analysis-critical race theory (CRT) lens informed largely by the work of Derrick Bell to make the case that policies and reforms in mathematics education were not designed to address the needs of marginalized learners; rather, these policies and reforms are often designed and enacted to protect the economic, technological, and social interests of those in power. The chapter offers contrasting narratives between policy intentions and policy enactment, highlighting how the language of mathematics education policies, when enacted by educational professionals, positions marginalized learners as deficient within their cultures, families, and communities. This chapter is organized into four sections: (1) The Social Conditions of Marginalized Learners; (2) Theoretical Framework: CRT; (3) Historical Perspectives and Unpacking Policies and Reforms; and (4) Discussion and Conclusion. The Social Conditions of Marginalized Learners section describes central features of the social and historical context in which marginalized learners now function by contextualizing the school and mathematical experiences of marginalized learners. The Theoretical Framework section outlines CRT as a lens for critically examining policies and reforms. The Historical Perspectives and Unpacking Policies and Reforms section focuses on how marginalized students have been framed historically in policies and reforms. The Discussion and Conclusion considers features that are necessary in policies and reform documents when discussing the needs of marginalized learners.

Introduction

In its 1989 *Curriculum and Evaluation Standards for School Mathematics* (CSSM), the National Council of Teachers of Mathematics (NCTM) argued that schools were not meeting the economic needs of the time and called for new social

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goals for education (NCTM, 1989). The four social goals for education were (1) mathematically literate workers, (2) lifelong learning, (3) opportunity for all, and (4) an informed electorate. As ideals, these goals can be seen as unproblematic, with widespread appeal. They can appeal to mathematics educators who view mathematics as universal and desire mathematical literacy for all students, and they may also appeal to those who view mathematics as multicultural and desire contextualized interpretations of mathematics that all students can appreciate (Mutegi, 2011). These goals may also appeal to mathematics educators who are concerned with issues of equity in mathematics education as well as those who view mathematics education practices as independent of historical inequities (Mutegi, 2011). It is difficult to argue against the four social goals because they appear to be inclusive and contain generalizing language that suggests working to erase injustices and exclusion experienced by people who are identified as marginalized (defined here as Black[1], Latin@[2], Indigenous, and poor). The generalizing language in these goals includes all of the political, social, and cultural considerations for reformers to direct the practices and policies of mathematics education, almost without question. It is the use of such generalizing language that is problematic, however, because it varies in interpretation and offers few specific points for enactment.

The third goal in *CSSM*, opportunity for all, does not appear to be overtly problematic because it challenges mathematics educators by drawing attention to social injustices and full participation in society for all by stating:

> The social injustices of past schooling practices can no longer be tolerated. Current statistics indicate that those who study advanced mathematics are most often white males. Women and most minorities study less mathematics and are seriously underrepresented in careers using science and technology. Creating a just society in which women and various ethnic groups enjoy equal opportunities and equitable treatment is no longer an issue. Mathematics has become a critical filter for employment and full participation in our society. We cannot afford to have the majority of our population mathematically illiterate. Equity has become an economic necessity. (p. 4)

Combating injustices and creating a just society are noteworthy goals that many people would not question. This goal situates equity in mathematics education as protecting economic interests and creating workers, with little consideration for the moral grounds or the benefit of marginalized people and communities. In order to produce workers, a utilitarian perspective of mathematics literacy is needed, thus ensuring the economic and social interests of those with power. Given the high

[1] I use the term Black to acknowledge the Black Diaspora and to highlight that Black people living in North America have ancestry dispersed around the world. Black learners who attend schools and live in North America are racialized in similar ways regardless of country of origin.

[2] I borrow Latin@ from Rochelle Gutiérrez (2013a, 2013b) who stated that the use of the "@ sign to indicate both an 'a' and 'o' ending (Latina and Latino). The presence of both an 'a' and 'o' ending decenters the patriarchal nature of the Spanish language where is it customary for groups of males (Latinos) and females (Latinas) to be written in the form that denotes only males (Latinos). The term is written Latin@ with the 'a' and 'o' intertwined, as opposed to Latina/Latino, to show a sign of solidarity with individuals who identify as lesbian, gay, bisexual, transgender, questioning, and queer (LGBTQ)" (p. 7).

status that mathematical knowledge is given in business and society, we must consider the role that mathematics education plays in a democratic society. Mathematics education is not isolated from the larger society; in fact, it is connected to patterns of differential economic, political, and cultural power (Apple, 1992). Mathematical literacy is given high status in reforms because it has "socioeconomic utility for those who already possess economic capital" (Apple, 1992, p. 98). This raises questions concerning economic benefits for those who own, control, and profit from mathematical enterprises. It is plausible to raise questions about whether mathematics education contributes to social injustices and whether equity in mathematics education is an economic necessity or a moral obligation.

An examination of past research, policies, and reforms in mathematics education suggests that many are a response to economic, technological, and security threats. For example, the launching of *Sputnik* brought heightened concern about America's national security as well as concern that America was lagging behind the Russians in mathematics and science. Documents such as *A Nation at Risk* (Gardner, Larsen, & Baker, 1983), *Before It's Too Late* (National Commission on Mathematics and Science Teaching for the 21st Century, 2000), *Foundations for Success: The Final Report of the National Mathematics Advisory Panel* (National Mathematics Advisory Panel, 2008), and *Rising Above The Gathering Storm: Energizing and Employing America for a Brighter Economic Future* (Augustine, 2005) capture the driving forces that frame mathematics education policy (Basile & Lopez, 2015). These documents have demonstrated frequent concerns about maintaining America's economic privilege in a global market, technological interests in a technologically advancing society, and threats to national security.

Through a hybrid policy analysis-critical race theory (CRT) lens, this chapter makes the case that policies and reforms in mathematics education were not designed to address the needs of marginalized learners; rather, these policies and reforms are often designed and enacted to protect the economic, technological, and social interests of those in power. This chapter offers contrasting narratives between policy intentions and enactment. These contrasting narratives highlight how the language of mathematics education policies, when enacted by educational professionals, positions marginalized learners as deficient within their cultures, families, and communities. Consequently, policies and reforms frame marginalized learners as problems to be fixed and that if marginalized students adopt the values of the dominant culture, then the economic, technological, and security interests of those in power are maintained. Policies and reforms are about dominant culture interests rather than the needs and interests of marginalized students.

The Social Conditions of Marginalized Learners

When I use the term marginalized learners, I am not ascribing a sweeping set of attributes to the collectives of Black, Latin@, Indigenous, and poor peoples; I recognize that collapsing these groups into one does not acknowledge the intersectionality within these collectives. There are, however, shared histories and experiences among the collectives. Martin (2013) argued that there is a dominant discourse in research and policy documents about Black learners and mathematics focused on a fixed set of cultural and cognitive explanations for negative outcomes, including cultural differences or deficits, limited mathematical knowledge and problem-solving skills, family background and socioeconomic status, and oppositional orientations to schooling. Gutiérrez and Irving (2012) identified similar overlaps in the dominant discourse regarding research and policy documents that Latin@ and Black learners share. Barnhardt (2001) pointed out how Indigenous learners are positioned as "at risk" and how their culture and languages are positioned in opposition to schooling. This positioning, and the fact that the use of Indigenous languages and culture is discouraged in schooling, is associated with the legacy of federal policies aimed to "civilize" and "assimilate" Indigenous learners into an American culture (Barnhardt, 2001; Noel, 2002). Although there are differences among the collectives, they also share legacies of being positioned as deficient in research and policy documents and share values and beliefs that prioritize community and family, a respect for spirituality, and interconnectedness with the natural world (Barnhardt, 2001; Berry, 2008; Gutiérrez, 2013b).

When we consider a historical perspective of public education and policies, we see that there were intentional policies designed to keep marginalized people uneducated and/or undereducated. There were laws, particularly in Southern states, where it was illegal to teach Black people who were enslaved to read and write (Anderson, 1988). As public education developed and expanded, schools became institutions to civilize, Christianize, and control Black and Brown children to keep them passive to social change so that they would not contribute to social upheaval (Watkins, 2001). It was illegal to teach Indigenous peoples in their native tongues. Congress created genocidal policies to strip young Indigenous children from their homes, adopting "kill the Indian to save the man" (Churchill, 2004). There have been policies supporting linguistic nationalism, condemning multilingual and marginalized learners. Although during Reconstruction many Black children had access to education, the withdrawal of federal troops in 1877 led to generations of terror, legal segregation, and substandard educational opportunities. Given these collective legacies, reforms in education policies often overlook the generational impact that violence has had on marginalized people. Rather, a narrative of assimilation and control is ever present.

There is a body of research in mathematics education suggesting that marginalized learners experience devaluation, inequities, exclusion, and violence (Berry, 2008; Gutiérrez, 2002; Gutstein, 2003; Martin, 2015; McGee & Martin, 2011). Research, policy, and reform "has been violent to marginalized peoples, such as [I]ndigenous groups, who are represented by perspectives that are neither kind to their cultural worldview nor accurate regarding their priorities" (Leonardo, 2013, p. 5). Students from Black, Latin@, and Indigenous communities have disproportionately low representation in science, technology, engineering, and mathematics (STEM) fields at all levels of education compared to their representation in the general population of the USA (National Science Foundation [NSF], 2015). Due to projected population growth, people from marginalized groups are expected to be more than half of the US population by 2043 (Colby & Ortman, 2014). By the year 2020, more than half of the children in the USA are expected to be part of a minority race or ethnic group (Colby & Ortman, 2014). Yet, Black, Latin@, and Indigenous individuals collectively make up only 13% of the STEM workforce in the USA and only 16% of all STEM undergraduate degrees awarded (NSF, 2015). The number of Black, Latin@, and Indigenous people earning science and engineering bachelor's and master's degrees has been rising since 1993, but the number of doctorates earned in these fields has flattened at about 7% since 2002 (NSF, 2015). Since 2000, Black, Latin@, and Indigenous students earning degrees in engineering and the physical sciences have also been flat, and earned degrees in mathematics and statistics have dropped (NSF, 2015). The High School Transcript Study by the National Assessment of Educational Progress (Nord et al., 2011) found that high school graduates completing Algebra I before high school are twice as likely to successfully complete a Precalculus/Analysis course than students who take Algebra I in high school. Only 12% of Black students and 17% of Latin@ students, however, had taken Algebra I before high school. A report by the College Board (2013) found that even when Black, Latin@, and Indigenous students are equally prepared for Advanced Placement coursework, they are still less likely to experience these courses (p. 2).

The data presented above do not exist in isolation. Data are, however, often presented with little description of the conditions, contexts, and experiences of marginalized students. Research, policies, and reforms must consider the positionality of marginalized learners and the many conditions and contexts in which marginalized students exist. Schools are social institutions set up by those in power and are organized to support and value the types of cultural and social capital held by those in power (Bourdieu & Passeron, 1990). It is plausible to consider that since policies and reforms support the value of those in power, there may be differences in the ways marginalized learners are perceived. Policies and reforms in education often portray marginalized learners as in need of "fixing" and their cultures, families, and communities as deprived and deficit (Stein, 2004). For example, Basile and Lopez (2015) point out a deficit positioning and negative narrative in the document *Adding It Up* (National Research Council, 2001):

> The same survey found large differences between ethnic groups on the more difficult tests (but not on the Level 1 tasks) with 70% of Asian and 66% of non-Hispanic white children passing the Level 2 tasks, but only 42% of African American, 44% of Hispanic, 48% of Hawaiian Native or Pacific Islander, and 34% of American Indian or Alaska Native participants doing so. Other research has shown that children from lower socioeconomic backgrounds have particular difficulty understanding the relative magnitudes of single- digit whole numbers and solving addition and subtraction problems verbally rather than using objects … This immaturity of their mathematical development may account for the problems poor and minority children have understanding the basis for simple arithmetic and solving simple word problems. (p. 178)

The content of the quote from *Adding It Up* not only presented the data from a deficit position but also used the language surrounding the data to position marginalized learners in "a static racial hierarchy, via what may be inferred as a biological deficit, with White children" (Basile & Lopez, 2015, p. 534). Also, it implies that the conditions and contexts of marginalized students are the reasons for underperformance, rather than structural issues of schooling, ways these learners might be positioned, or variable access to jobs, healthcare, and other resources.

When examining how policies and reforms respond to the needs of marginalized learners, there is a constant pattern in which they are routinely given the least access to advanced mathematics content, the fewest opportunities to learn through methods other than memorizing facts and mimicking teacher-modeled procedures, and the least access to well-prepared mathematics teachers (Berry, Ellis, & Hughes, 2014). As a result, these learners experience the following conditions: (a) reduced access to advanced mathematics courses that prepare them for higher education and improved career options; (b) routine exposure to activities that focus primarily on rote, decontextualized learning through drill and practice with little to no engagement that promotes reasoning and using mathematics as a tool to analyze social and economic issues, critique power dynamics, and build advocacy; and (c) less access to qualified teachers of mathematics who both understand mathematics deeply and understand their students' cultural and community context deeply in order to give learners access to mathematical knowledge (Ellis, 2008; Flores, 2007; Martin, 2007). The effect of these conditions on marginalized learners' attainment in mathematics demonstrates well that such an approach constrains outcomes to a narrow range of proficiencies focused on basic skills.

While the disproportionality in conditions of marginalized learners is a cause for concern, it is important to understand that addressing the needs of these learners may not have been the primary goal of prior policies and reforms in mathematics education. Berry and colleagues (2014) argued that prior policies and reforms in mathematics education have failed due to having been developed to address the needs and interests of those in power. In fact, many past policies and reforms in mathematics teaching and learning have come at the expense of the needs and interests of marginalized learners by framing policies and reforms based on economic, technological, and security interests of those in power.

Theoretical Framework: Critical Race Theory

Derrick Bell, a former attorney with the National Association for the Advancement of Colored People (NAACP) during the civil rights era, employed his interest-convergence principle to explain how the US Supreme Court issued the landmark ruling in *Brown v Board of Education of Topeka, Kansas* (*Brown*) in 1954. The Supreme Court's ruling in the *Brown* case revoked the "separate but equal" doctrine, which legally sanctioned segregation in public education and daily life. Bell (2004) argued that the *Brown* decision was not the result of America coming to terms with its democratic ideals or moral sensibilities; rather, the Supreme Court was more interested in providing "immediate credibility to America's struggle with communist countries to win the hearts and minds of emerging third world people" than in doing what was morally right (p. 233). Under the interest-convergence principle, *Brown* is best understood as progress requiring the coincidence of a pressing issue, more than a commitment to justice (Donnor, 2005).

Brown provided the impetus for legislation, such as the *Elementary and Secondary Education Act* of 1965 and its reauthorizations *Improving America's School Act* of 1994, *No Child Left Behind Act* of 2001 and *Every Student Succeed Act* (ESSA) of 2015. Similar to its predecessors, the ESSA has a purpose statement, "To provide all children significant opportunity to receive a fair, equitable, and high-quality education, and to close educational achievement gaps" (2015, Sec. 1001). Stinson (2015) argues that decades of legislation, like ESSA, do not have the "will" to facilitate the violent reform necessary to change the conditions of marginalized learners. I argue that such reform is a grassroots proposition, rather than one facilitated by federal legislation; that is, activities similar to the Black Lives Matter (BLM) movement will demand that the perspectives and worldviews of marginalized peoples be given consideration such that the conditions, contexts, and experiences of those marginalized will be understood and appreciated. BLM is a grassroots effort that is unapologetic in its rhetoric and challenges structural racism, anti-blackness, and institutionalized violence in school reform, policy, and research. Grassroots efforts like BLM decentralize whiteness when discussing policies and reforms. An argument can be made that schools and schooling were created for maintaining the power and privilege of whiteness. Zion and Blanchett (2011) argued that the reason large-scale improvement in outcomes for marginalized learners has yet to be realized is that the problem has not been framed appropriately. It must be framed as part of the history of racism and, as an issue of civil rights and social justice, viewed through a critical lens.

Interest-convergence is an analytical viewpoint for examining how policies and reforms are dictated by those in power to advance their political, social, and economic interests (Donnor, 2005). Bell's (1980, 2004) interest-convergence principle theorizes that any empowered group will not help any disempowered group unless it is in their best interest to do so. For Bell, the historical advancement of Black people's needs and interests is a result of being fortuitous beneficiaries of measures directed at furthering aims other than racial equity and social justice (2004). Bell states, "Even when interest-convergence results in an effective racial remedy, that remedy will be abrogated at the point that policymakers fear the remedial policy is threatening the superior societal status of Whites, particularly those in the middle and upper classes" (2004, p. 69). Interest-convergence principle has its theoretical grounding in CRT, which draws from a broad literature in law, sociology, history, education, and women's studies (DeCuir & Dixson, 2004; Ladson-Billings & Tate, 1995; Matsuda, Lawrence, Delgado, & Crenshaw, 1993; Solórzano &Yosso, 2001). With respect to the use of CRT in education, as Solórzano and Yosso (2002) explained, "critical race theory in education is a framework or set of basic insights, perspectives, methods, and pedagogy that seeks to identify, analyze, and transform those structural and cultural aspects of education that maintain subordinate and dominant racial positions in and out of the classroom" (p. 25). In education, interest convergence provides a framework to discuss power dynamics as framed by systemic interests and a loss-gain binary (Milner, 2008). Interest-convergence principle has been used to examine policies and practices related to teacher education programs (Milner, 2008), practices for STEM education serving marginalized learners at universities (Barber, 2015), intercultural movements in multicultural education (Caraballo, 2009), inclusion in special education (Zion & Blanchett, 2011), intercollegiate athletics (Donner, 2005), the development of historically Black colleges/universities (Gasman & Hilton, 2012), and postsecondary access for Latino immigrant populations (Alemán & Alemán, 2010). This body of work provides a lens for using the interest-convergence principle to examine the motivating factors for policies and reforms in mathematics education in order to understand whose interests are served and the resulting fortuitous beneficiaries.

In mathematics education, Gutstein (2009, 2010) and Martin (2003, 2009, 2013, 2015) have examined several mathematics education policies using critical theory and the interest-convergence tenet of CRT. Gutstein argued that the policies and reforms are motivated primarily by the desire to maintain global economic superiority against the rising educational and intellectual infrastructure of other nations. For example, in President Obama's *Educate to Innovate* campaign:

> Whether it's improving our health or harnessing clean energy, protecting our security or succeeding in the global economy, our future depends on reaffirming America's role as the world's engine of scientific discovery and technological innovation. And that leadership tomorrow depends on how we educate our students today, especially in math, science, technology, and engineering … And that's why my administration has set a clear goal: to move from the middle to the top of the pack in science and math education over the next decade. (White House, 2009)

Gutstein (2010) pointed out that framing education as a US economic problem that affects us all, and therefore would benefit all, is based on the assumption that the benefits will "trickle down" from those whose interests are served. He suggested this is misleading, as US productivity has increased since the 1970s, yet income and wealth polarization has grown. Consequently, while the policies claim success will benefit all citizens, only those with the most wealth have received any benefit. Policies are more about the interests of those in power, and with wealth, and less about the interests of marginalized people. Martin (2003, 2008, 2009, 2013, 2015) has also examined the many ways in which mathematics education policy documents have worked to erase the lived experiences of marginalized learners. That is, within mathematics education research and policy, race has typically been invoked only as a categorical variable used to disaggregate data and to rank students in a racial hierarchy of mathematics ability; racism is rarely invoked. Martin found that mathematics education policy and reform documents promote a market enterprise working for the financial benefit of a select few. Martin (2003) stated that the

> status of African American, Latino, Native American, and poor students has not been a primary determinant driving mathematics education reform. When discussions do focus on increasing participation among these students, it is usually in reference to workforce and national economic concerns. (p. 11)

Often, education policy documents have aligned mathematics illiteracy with Black, Latin@, and Indigenous learners as a threat to the economic well-being, prosperity, and elite status of the USA (Martin, 2003, 2008, 2009, 2013).

Unpacking Historical Perspectives, Policies, and Reforms

A common theme among policy and reform documents is a call for increased participation of marginalized learners in STEM fields. These calls usually reference increased demands on the US economy, the drive to stay ahead technologically of international competitors, and a need to secure the USA from international security threats. Rarely are there references focused on the needs of marginalized communities. Positioning marginalized people's increased participation in mathematics to meet interests that may not include their own commodifies them by affixing a market value to their collective potential labor and intellectual property, or what Basile and Lopez (2015) describe as racial commodification. Racial commodification is the method by which racial hierarchies are replicated. From a CRT perspective, we must consider whose interests are protected and how policies and reforms maintain the protections of those with power. In *Foundations for Success: The Final Report of the National Mathematics Advisory Panel* (NMAP, 2008), we see examples of commodification:

> Moreover, there are large, persistent disparities in mathematics achievement related to race and income—disparities that are not only devastating for individuals and families but also project poorly for the nation's future, given the youthfulness and high growth rates of the largest minority population. (pp. 4–5)

In this example, and throughout the document, there is little discussion of the conditions, contexts, and experiences of marginalized students. It appears that this document is suggesting that the youthfulness of the largest minority population reflects a segment of the population that needs to be tapped to protect the interests of those with power. In fact, marginalized learners are commodified by suggesting that increased participation in mathematics among marginalized learners will provide benefit and potential resources for economic gain. It is not clear, however, how marginalized people will benefit from such gains. One might assume that benefits for marginalized people will "trickle down" from those in power or that benefits might be widespread, but historically this has not been the case.

Many policy and reform documents simultaneously commodify marginalized populations while blaming them for the disappointing performance of the USA on international mathematics assessments (Basile & Lopez, 2015; Martin, 2013). *Innovation America: Building a Science, Technology, Engineering and Math Agenda* (National Governors Association, 2007) described this simultaneous commodification and blaming by stating:

> Projected demographic shifts have the potential to magnify the U.S. problem if STEM achievement gaps are not rectified. As the U.S. domestic college population stabilizes at about 30 million students from 2010 to 2025, population groups currently underrepresented in STEM fields will attend college in growing numbers. If the achievement gap persists, increasing numbers of students will be unprepared to succeed in college and in STEM degree attainment. (p. 6)

Such language positions marginalized students as responsible for the entire performance of the USA on international measures; by discussing achievement gaps next to international test scores, this language, and other similar languages, suggests a causal link between the marginalized students and international performance (Basile & Lopez, 2015).

Policy documents and reforms in mathematics education build from a history and legacy of protecting economic, technological, and security interests of the USA rather than considering the needs and interests of marginalized communities and people. In their review of the history of school mathematics, Ellis and Berry (2005) noted a tension between reforms in mathematics education focused on efficiencies with an emphasis on procedural learning and a belief that mathematics beyond arithmetic should be reserved for those deemed capable of advancing to such heights. The focus on efficiencies is found in calls that focus on measuring knowledge attainment using efficient standardized assessments. Efforts to improve mathematics education

> situated many learners in an a priori deficit position relative to disembodied mathematical knowledge—meaning learning mathematics was taken to be harder for certain groups of students due to their backgrounds and/or innate abilities—and failed to acknowledge the importance of mathematics for all students. (Ellis & Berry, 2005, pp. 10–11)

Throughout this history, systems of standardized assessment were developed as a means to justify the separation of students within and between schools by race, class, and ethnicity. The use of assessments to stratify was built on the assumption that a distribution of mathematical ability exists that can be fairly measured and meaningfully interpreted as the basis for separating students and providing unequal access to opportunities to learn mathematics. The conflation of this with societal beliefs about race and intelligence cannot be overlooked; the interest of those with power was preserved. To that end, the remaining parts of this section focus on historical markers in mathematics education. The goal is to draw connections between the history of mathematics education and the positioning of marginalized learners in policy and reform documents.

Sputnik and the New Math Movement

The launch of the first artificial satellite *Sputnik* on October 4, 1957, by the Russians gave impetus to improve mathematics education in America. The response to *Sputnik* led to federal funds being allocated for mathematics education through the National Defense Education Act (NDEA) of 1958 intended to support US national security interests (Walmsley, 2003). NDEA provided funds to identify the "best and brightest" scientific minds and was designed to fulfill defense interests in mathematics, science, engineering, and foreign languages. The appeal of identifying the "best and brightest" was built on protecting national security and defense interests (Tate, 2004). As one of the most militarized countries in the world, we cannot overlook the fact that this military motivation for improved mathematics education is still part of the discourse of policy documents and reforms.

Approximately 3 years prior to the launching of *Sputnik*, the US Supreme Court issued the landmark ruling in *Brown* which revoked the "separate but equal" doctrine.

Black parents and community leaders sought desegregation based on the assumption that better school resources were available in schools where white children were taught and that better resources provided greater opportunities. The *Brown* decisions occurred in the midst of efforts to reform what mathematics should be taught and how it should be taught. This "new math" reform offered new mathematics content as well as new pedagogical approaches (Walmsley, 2003). One main idea of "new math" was to reduce focus on the drill and practice approach to teaching mathematics and increase focus on approaches where students could develop conceptual understanding of mathematics. These pedagogical approaches included the use of manipulatives, guided discovery learning, teaching practices, and the spiral curriculum (Walmsley, 2003; Willoughby, 2000).

When we consider that many schools remained segregated and the process of desegregation was slow and that schools serving Black children often received used textbooks handed down from schools serving white students (Snipes & Waters, 2005), the reforms of "new math" did very little to address the needs of marginalized children, specifically Black children (Tate, 2000). That is, Black children did not have access to the new content or the pedagogies associated with the "new math" reform. Within the interest-convergence framework, this era was characterized as one of "benign neglect" (Tate, 2000, p. 201) for marginalized students because the needs and interests of marginalized students were largely ignored. This does not imply that these learners did not have access to quality teaching in segregated schools; in fact, there is a body of research that suggests that many teachers in segregated schools "made do" with substandard materials and provided high-quality teaching (Foster, 1997; Siddle-Walker, 2000; Snipes & Waters, 2005; Standish, 2006). Rather, the "new math" reforms focused on identifying the "best and brightest" while ignoring the needs of marginalized learners.

The Back to Basics Movement

In the late 1960s and early 1970s, the "back to basics" reform movement in mathematics emerged in response to the perceived shortcomings of "new math" (Burrill, 2001). During this period, the NSF discontinued funding programs focused on "new math," and there was a call to go back to the "core curriculum" which was understood to be basic skills in mathematics. The "back to basics" movement called for teaching mathematics procedures and skills and was closely connected to the minimum competency testing movement used by states in the 1970s and 1980s (Resnick, 1980; Tate, 2000). Testing had a significant impact on the mathematics content that was taught and the methods used to teach. Typically, students were taught mathematics content deemed important for passing tests. Although the emphasis on skills did result in slightly improved standardized test scores for marginalized children, it did not adequately prepare these students for mathematics coursework requiring higher levels of cognition and understanding (Tate, 2000). This emphasis continued to limit marginalized learners' opportunities to achieve and access upper-level mathematics courses (Tate, 2000).

Considering the impact of emphasizing basic skills and testing, it is plausible that the pedagogies and the curriculum offerings during the “back to basics” reform were similar for marginalized students as during the “new math” reform. The pedagogies of “back to basic” were already a part of marginalized students’ mathematical experiences. The growing emphasis on testing during this period was used to legitimize the perception that many marginalized students were not capable of rigorous studies in mathematics (Perry, 2003). The “back to basics” movement provided more focus on using testing to pathologize marginalized students as being inferior, deficient, and deviant. During this period, we find the first of many research studies focusing on the achievement gap in mathematics that describe marginalized learners as deficient and in need of fixing (Perry, 2003). The analysis of achievement gap language provided no descriptions of conditions or the context of experiences of marginalized learners. In fact, these reports presented marginalized learners as static, with little within-group diversity. If one considers the context of the late 1960s and 1970s and the persistent limited educational opportunities available to marginalized learners, discussion of an achievement gap serves to reinforce an ideology about marginalized children’s intellectual inferiority.

A result of the research and policies of the back-to-basics period led to the National Commission on Excellence in Education report titled, *A Nation at Risk: The Imperative for Educational Reform* (1983). The report suggested that education reform is necessary because competitors throughout the world are overtaking America’s preeminence in commerce, industry, science, and technology. Furthermore, the report stated, “If an unfriendly foreign power had attempted to impose on America the mediocre educational performance that exists today, we might well have viewed it as an act of war” (p. 1). The inflammatory rhetoric of *A Nation at Risk* heightened concerns about national security and argued that America was lagging behind in mathematics and science when compared internationally. *A Nation at Risk* stated that through educational reform, American children’s promise of economic, social, and political security in the future would be earned by meritocratic ideals of effort, competence, and informed judgment.

The Standards Movement

In 1980, NCTM put forth its *Agenda for Action*, which diverged from “back to basics” and put forward recommendations that moved away from solely the notion of basic skills toward problem-solving, use of technology, measures other than conventional testing, and pedagogy and curriculum to accommodate the diverse needs of the student population. The *Agenda for Action*, not a standards document, was the foundation for the first standards document, *Curriculum and Evaluation Standards for School Mathematics* (*CSSM*) , developed by NCTM (1989). The standards supported conceptual understanding as a primary goal with algorithmic fluency following once conceptual understanding was developed. Critics of *CSSM* argued that the primary goal of conceptual understanding through an inquiry-based approach did not help children acquire basic skills efficiently nor learn standard algorithms and formulas (Klein, 2003). Tension between proponents and opponents of *CSSM* resulted in the "Math Wars," and there were proponents for improving mathematics instruction for marginalized children on both sides. The tensions of the "Math Wars" appear to have an underlying narrative focusing on the nation's technological interests, social efficiency, and perpetuation of privilege. There were intense debates focused on curriculum, teaching, and assessment, but little debate focused on understanding the realities of children's lives. For marginalized learners, issues of race, racism, and identity were not under consideration in the "Math Wars."

The NCTM revised its standards document in 2000 through the release of the *Principles and Standards for School Mathematics* (*PSSM*). *PSSM* was received as more balanced than *CSSM*, which led to some calming of, but not an ending to, the "Math Wars." *PSSM* highlighted equity as the first of its six principles for school mathematics. Martin (2003) critiqued the *PSSM's* equity principle, however, for not providing a sense of equity that considers the contexts of students' lives, identities of students, nor conditions under which mathematics is taught and learned:

> The Equity Principle of the *Standards* contains no explicit or particular references to African American, Latino, Native American, and poor students or the conditions they face in their lives outside of school, including the inequitable arrangements of mathematical opportunities in these out of school contexts. I would argue that blanket statements about *all* students signals an uneasiness or unwillingness to grapple with the complexities and particularities of race, minority/marginalized status, differential treatment, underachievement in deference to the assumption that teaching, curriculum, learning, and assessment are all that matter. (p. 10)

Following the release of NCTM's *Principles to Actions* (2014), Martin (2015) offered a critique similar to the one he had issued in 2003. He challenged policymakers and researchers to facilitate the kind of violent reform needed to change the conditions of African-American, Latin@, Indigenous, and poor students in mathematics education. Too often, race, racism, social justice, contexts, identities, conditions, and more are relegated as issues not appropriate for mathematics education when in fact these issues are central to the learning and teaching of mathematics.

In 2010, the Common Core State Standards for Mathematics (CCSSM, National Governors Association Center for Best Practices and the Council of Chief State School Officers) emphasized an inquiry-based approach to mathematics teaching and learning. The mission statement makes clear that these reforms were emerging from the same interests of college and career readiness by positioning American students to be able to compete in a global economy. Nowhere to be found is mention of the gross inequities within society that continue to be reflected in students' educational outcomes. Once again, framing the reform from the position of economic interest diminished the needs of learners to focus primarily on the acquisition of mathematics content and practices.

Discussion and Conclusion

This brief review of policies and reforms in mathematics education suggests that economic, technological, and security interests were, and continue to be, drivers of many policies and reforms. These policies and reforms situated mathematics education in a nationalistic position of being color-blind, in a context where race, racism, conditions, and contexts do not matter. This positions schools and communities as neutral sites rather than cultural and political sites. Despite the evidence that racism and marginalization exist in schools and communities, many still adhere to the belief that color-blind policies and pedagogical practices will best serve all students. It is difficult to argue against ensuring students' competitive place in the global marketplace. A careful look at policies and reforms focused on identifying the "best and brightest," identifying "high achievers," stratifying students based on characteristics, or identifying "failing" positioned some as competent while labeling others as being deficient. Policies and reforms have typically not attended to the social realities and needs of marginalized students in ways that lead to improvements in their life circumstances.

In order to have any meaningful policy gains, we must decentralize whiteness when discussing policies and reforms. There is significant evidence suggesting that whiteness is at the center of many educational policies and reforms. By decentralizing whiteness, we disrupt its power and privilege. Decentralizing whiteness opens the space and broadens opportunities to consider the roles that histories, contexts, and experience play in the development of reforms and policies.

Given the growing body of research focused on context, identities, experience, and conditions of marginalized learners (e.g., Gutiérrez, 2002; Jett, 2010; Martin, 2013; Noble, 2011; Stinson, 2015; Thompson & Lewis, 2005), mathematics education policies and reforms can draw from this body of work in establishing new policies and reforms. This body of research considers issues of race, racism, contexts, identities, and conditions as variables that impact the mathematical experiences of marginalized learners. This body of research challenges the dominant discourses and pushes the field of mathematics education to consider sociological, anthropological, and critical theories. It encourages researchers to consider outcomes other than achievement as the primary measure of success. One finding from this research is that educators must create opportunities for students to experience mathematics learning using the resources they bring to classrooms; teachers must know and understand learners' identities, histories, experiences, and cultural contexts and consider how to use these to connect students meaningfully with mathematics. There is a need for policies and reforms that focus on leveraging communities' and community-members' knowledge and experience in mathematics education. Mathematics teaching and learning not only occurs in classrooms but also occurs in other spaces. By leveraging these resources, we situate mathematics teaching and learning as a way to structure experiences that are contextual and provide opportunities for exchange of mathematical ideas. The use of context in mathematics education can help learners to recognize and build upon the cultural and social resources they bring to the mathematics classroom.

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Chapter 2
Making the Implicit Explicit: Building a Case for Implicit Racial Attitudes to Inform Mathematics Education Research

Dan Battey and Luis A. Leyva

Abstract Scholars continue to document that African American kindergartners bring the same competencies as their white peers (Ginsburg et al, Int J Psychol 16(1):13–34, 1981; O'Connor et al, Rev Res Educ 33(1):1–34, 2009). Research has found, however, that they experience low-quality mathematics instruction (Davis and Martin, J Urban Math Educ 1(1):10–34, 2008; Lubienski, J Negro Educ 71(4):269–287, 2002), which does not leverage the mathematical abilities of African American students. The mechanisms for how these disparities are produced are less clear (Battey, Educ Stud Math 82(1):125–144, 2013a; Lubienski, J Negro Educ 71(4):269–287, 2002). For instance, we do not understand the mechanism through which mathematics instructional quality or the cognitive demand of tasks is reduced for African American children. In this chapter, we argue that a potentially missing piece in understanding mechanisms that produce disparities in mathematics education is implicit racial attitudes. To make this theoretical case, we draw on work both inside and outside of mathematics education across four literatures: (1) the quality of mathematics instruction that African American students receive, (2) relationships developed with teachers, (3) racialized teacher perceptions of behavior and academic aptitude, and (4) racial microagressions in mathematics. The chapter ends with two examples of how implicit racial attitudes can be embedded in existing research in order to illustrate how the field could study ways to disrupt the perpetuation of deficit perspectives shaped by racial ideologies and systemic forms of oppression.

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Introduction

Disparities in the quality of instruction and achievement outcomes for African Americans are well documented (Davis & Martin, 2008; Ladson-Billings, 1997; Lattimore, 2005; Lubienski, 2002; Means & Knapp, 1991; Strutchens & Silver, 2000). Making these achievement and instructional quality differences even more problematic, numerous researchers have demonstrated that African American kindergartners bring the same mathematical competencies and achieve at similar levels as their white peers (Ascher, 1983; Cole & Scribner, 1974; Entwisle & Alexander, 1992; Ginsburg, 1978; Ginsburg, Posner, & Russsell, 1981; O'Connor, Hill, & Robinson, 2009; Phillips, Crouse, & Ralph, 1998). The mechanisms for how these disparities are produced, however, are less clear in mathematics education (Battey, 2013a; Lubienski, 2002). For instance, we do not understand the mechanism through which mathematics instructional quality or the cognitive demand of tasks is reduced for African American children. Here, we argue that a potentially missing piece in understanding mechanisms that produce disparities in mathematics education is implicit racial attitudes. To make this theoretical case, we draw on work both inside and outside of mathematics education and argue that research needs to begin exploring implicit racial attitudes within mathematics education to better understand the dynamics that produce lower-quality instruction and reduced learning opportunities for African American students.

Before outlining the chapter, it is important to situate work on implicit racial attitudes within a broader frame of racism. Racism is based on a racialized social system, "a network of racialized practices and relations that shapes the life chances of the various races at all levels" (Bonilla-Silva, Lewis, & Embrick, 2004, p. 558). Racism is then enacted through a dialectic that is both symbolic or ideological and material or resource-based (Lewis, 2004). When ideologies are internalized, they become implicit racial attitudes, which then can guide behavior and social interactions. These interactions can produce differential material consequences in the form of changing access to curriculum, lowering the cognitive demand of tasks, and in turn, producing differential achievement. We focus on implicit racial attitudes, how they guide behavior, and how they are situated within a broader system of racism to recognize the broader structures that produce these dynamics.

Four different literatures inform the argument of this chapter. We first define and examine the literature on implicit racial attitudes. The chapter then looks across scholarship on the four literatures: (1) the quality of mathematics instruction that African American students receive, (2) relationships developed with teachers, (3) racialized teacher perceptions of behavior and academic aptitude, and (4) racial microaggressions in mathematics. In looking across these literatures, we lay out how we see implicit racial attitudes interacting with existing findings. More specifically, we highlight how implicit racial attitudes may play a role in African American students having limited access to high-quality mathematics learning opportunities and supportive teacher-student relationships, being subjected to deficit expectations of academic ability and engagement, and dealing with microaggressions of mathematical ability. Such complex interactions between implicit racial attitudes and various educational constructs (e.g., instructional quality, achievement, students' identities as mathematics learners) that shape racialized mathematics learning environments and experiences for, in this case, African Americans, capture the need to adopt implicit racial attitudes as units of analysis in mathematics education research. By shedding light on the subtle operations of implicit racial attitudes in mathematics education, researchers can advance opportunities for teachers to be more critically aware of how they frame students and thus disrupt the perpetuation of deficit perspectives shaped by racial ideologies and systemic forms of oppression. The chapter ends with two examples of how implicit racial attitudes can be embedded in existing research.

Making a Case for Implicit Racial Attitudes

Explicit racial attitudes reside on a conscious level and are typically characterized by overtly racist talk and actions (Dovidio, Kawakami, & Gaertner, 2002). Researchers have found that although explicit racial attitudes have diminished, implicit racist attitudes have not (Dovidio & Gaertner, 2008). It is this decrease in explicit attitudes that many point to as a lessening of racism. However, as defined prior, racism is a social structure and therefore is not dependent on attitudes. The notion, then, that a decrease in explicit racial attitudes potentially lessens systemic racism is misleading at best. In contrast to explicit racial attitudes, *implicit racial attitudes* are characterized by unconscious feelings and beliefs, which can be orthogonal to publicly professed attitudes (Greenwald & Banaji, 1995). In this way, implicit racial attitudes can reside in individuals or teachers even though they profess overt beliefs in being equitable. Implicit attitudes are a result of exposure to stereotypes (ideologies) and characterized as evaluations and beliefs automatically activated by the presence of a particular stimulus, in this case, racial interactions (Dovidio, 2001).

Implicit attitudes are uncorrelated with explicit attitudes. Explicit attitudes about race are linked to deliberate behavior, while implicit attitudes about race guide spontaneous and unconscious behaviors (Dovidio et al., 2002). Because contemporary research has found that whites assert that racial prejudice, or antipathy toward a racial group, is wrong (Bobo, 2001), which is consistent with publically professed explicit racial attitudes, generally accepted values about fairness and equality inhibit the direct expression of implicit racial attitudes (Dovidio & Gaertner, 2008). Implicit attitudes, then, do not require an endorsement for prejudiced views; rather, they represent exposure to broad deficit ideologies. Therefore, contemporary prejudice, aligned with colorblind racism, is often complex since professed and internalized attitudes can be at odds. Individuals deny personal prejudice, but underlying this are unconscious negative beliefs about particular races (Wilson, Lindsey, & Schooler, 2000). Therefore, implicit racial attitudes could be impacting the quality of instruction and relationships in mathematics classrooms, since all teachers are subject to internalizing stereotypes.

What often keeps these beliefs implicit are proxies that are used for race (Dovidio & Gaertner, 2004). Scholars within mathematics education have made the case that labels referencing culture, family values, or testing categories can serve as proxies for race since, socially, we have a difficult time discussing these complex issues (Diversity in Mathematics Education [DiME], 2007). These labels conceal patterns of behavior across racial lines, such as teachers working in one way with students stigmatized as “far below basic” (Battey & Stark, 2009; DiME, 2007). A teacher could justify a student’s lack of learning by placing blame on the cultural practices of a student’s family. This is even more problematic given that whites who would not discriminate explicitly tend to discriminate against African Americans when bias can be attributed to some factor other than race (Dovidio & Gaertner, 2004). These racial proxies cloud instances when teachers access racial stereotypes, which can unconsciously affect their behavior. Implicit racial attitudes could, then, reduce the quality of education for African Americans through teachers’ lowered expectations of the type of instruction appropriate for these students.

We do not want a focus on attitudes, however, to disconnect this work from broader forms of racism. Implicit racial attitudes are merely an internalization of symbolic racism in the form of racialized narratives. For example, given the prevalence of the ideology of a racialized hierarchy of mathematics ability (Martin, 2009), this could be internalized by mathematics teachers. Proxies, in turn, allow for the implicit attitudes to remain unconscious as teachers’ racial attitudes can be explained away in terms of “ability” or “competence.” This then allows automatic behaviors, such as reducing cognitively demanding tasks or focusing on rote mathematics, to go unchecked. These behaviors then contribute to reproducing disparities in mathematics (material racism), which subsequently reproduces the ideology of a racial hierarchy of ability. Implicit attitudes provide a way to explain how this ideology gets reproduced in mathematics classrooms despite teachers’ good intentions.

Quality of Mathematics Instruction

Research has shown that mathematics teachers of African American students are more likely to teach mathematics vocabulary out of context, disconnect procedures from students’ thinking, use unexplained procedures, assess students based on following steps rather than student thinking, and use fewer resources such as manipulatives even when available (Davis & Martin, 2008; Ladson-Billings, 1997; Lattimore, 2005; Lubienski, 2002; Means & Knapp, 1991; Strutchens & Silver, 2000). While the overall lack of quality of mathematics instruction in US schools has been noted in research from TIMSS (Hiebert et al., 2003), Lubienski (2002) found that even when controlling for socioeconomic status (SES), African American students were more likely to experience instruction that framed mathematics as primarily around the memorization of facts, having one correct strategy and being assessed using multiple-choice questions. To illustrate the lack of availability of high-quality instruction, a national sample of African American students in the USA reported that "there is one way to solve a math problem" and "learning math is mostly memorizing facts" (Strutchens & Silver, 2000). Given the history of unequal opportunities to learn, findings indicated that African American students often do not have access to quality mathematics education (DiME, 2007).

The field, however, still does not understand the mechanisms that deliver lower-quality mathematics instruction to African American students. For example, Hill and colleagues demonstrated that teachers' mathematics knowledge for teaching (MKT) is directly related to student SES and minoritized status, meaning that lower-income students of color have teachers with less mathematical knowledge (Hill, 2007; Hill & Lubienski, 2007; Hill, Rowan, & Ball, 2005). They have also separately shown that MKT is related to student achievement (Hill et al., 2005). Therefore, MKT certainly plays a role in students' mathematics achievement. More recently, however, research speaks to the indirect relationships between MKT, instruction, and student learning in mathematics (Shechtman, Roschelle, Haertel, & Knudsen, 2010). Across 181 classrooms over 3 years, Schectman and colleagues (2010) found that while MKT was correlated with student learning, it was not directly related to it. Additionally, MKT was not correlated with the quality of instruction. Shechtman et al. (2010) state that their "results suggest that mathematics knowledge for teaching may have a nonlinear relationship with student learning, that those effects may be heavily mediated by other instructional factors" (p. 317).

This research suggests that our notions of quality mathematics instruction may not be as directly related to learning as mathematics educators once thought and that these variables may be heavily mediated by factors that the field is not actively considering, especially for lower-income African American students. Several case studies demonstrate how teachers behaved in racialized ways with African American students through proceduralizing mathematics, hyper-focusing on misbehavior, and dismissing student contributions (e.g., Battey, 2013a; Jackson, 2009; Spencer, 2009). Researching a low-income, predominantly African American classroom, Jackson (2009) showed how the teacher framed students as deficient academically, behaviorally, and morally. This framing shaped instructional practices that focused on repeating steps to get a correct answer, having fun, and reserving understanding for later in their school careers because they needed basics now. The work offered a glimpse into a classroom where a teacher altered mathematical practices for reasons related to racially deficit views.

Spencer's (2009) study found similar deficit reasoning about African American students from teachers. She noted that African American students were disengaged from mathematics instruction and posed this observation to the teacher, Mr. Chung, in an interview. Mr. Chung dismissed the explanation of race to explain African American students' disengagement, instead explaining it through a lack of prior knowledge on the part of students. Spencer documented, however, that the level of middle school mathematics did not allow for much thoughtful engagement; the content was largely focused on basic skills. Mr. Chung went on to relate student disengagement to limited family involvement and poverty. Issues of culture, family involvement, and poverty can be used as proxies for race (Johnson & Martinez, 1999); we use these markers for race when we would otherwise be uncomfortable naming race as a reason (Dovidio & Gaertner, 2004).

A contrasting case is presented in Battey's (2013a) study about a classroom focused more on understanding mathematics. In this research, the teacher enacted many of the instructional practices aligned with reform efforts such as problem solving, student explanations, and mathematical discussions. However, interactions showed the teacher either ignoring or dismissing two African American boys' contributions. In instances of ignoring, it was as if the students were invisible, though she was looking directly at them. In instances of dismissing, the teacher rolled her eyes or challenged students on issues of language use (viz., the use of African American Vernacular English) rather than addressing their mathematical thinking. These interactions highlighted the way that racialized interactions can position African Americans as invisible in moments (Feagin & Sikes, 1994) and contradictorily hyper-visible in others (Higginbotham, 2001). The teacher explained these instances as related to students' prior knowledge, behavior, and laziness, each of which can be seen as proxies for race.

Across these case studies, teachers attributed their poor treatment of African American students to terms shown to serve as proxies for race (Wang, 2003): prior knowledge, behavior, laziness, poverty, and family involvement. Even though the studies themselves point to this link with race, without concluding or exploring implicit racial attitudes they all point to treatment by mathematics teachers consistent with holding implicitly racial attitudes toward African American students. Therefore, in addition to explaining the lower quality of instruction that African American students often receive, implicit racial attitudes could also explain the nonlinear relationship between MKT and student learning. For instance, a teacher with MKT may not enact high-quality instruction due to racially biased attitudes and therefore produce less student mathematics learning. Implicit racial attitudes could mediate the effects of MKT such that many students do not receive the high-quality mathematics instruction that they deserve.

Teacher-Student Relationships

In addition to the quality of instruction, teacher-student relationships have been found to be a critical component of both psychosocial and academic development. Jerome, Hamre, and Pianta (2009) determined that teachers in their study rated relationships with African American students as more conflictual than their relationships with white students. The study examined students longitudinally from kindergarten through sixth grade and, while they didn't parse the teachers out by race, the teachers across the grades ranged from 88% to 95% white. While teachers' and white students' ratings of their relationships were quite similar, teachers rated their relationships with African American students as more conflictual than African

American students did. Jerome and colleagues also noted that teachers rated their relationships with African American students as becoming more conflictual over time. While they note it would have been helpful to have a more racially diverse teacher sample, or to parse the findings by teacher race, the implications of their work are that white teachers perceive their relationships specifically with African American students as conflictual. Furthermore, Pianta and Stuhlman (2004) found that teacher-student conflict is a better predictor of standardized test performance in mathematics than teacher-student closeness. These findings raise the issue of how relationships relate to student learning, independent of the quality of instruction.

The conflict that teachers perceived in their relationships with African American students could be explained through implicit racial attitudes as well. In perceiving conflict in their relationships, stereotypes of African Americans as loud, aggressive, and violent could be accessed. This, in turn, could lead to more attention being paid by teachers on controlling students' behavior and so forth. As Gregory, Skiba, and Noguera (2010) noted, the "discipline gap" is closely related to racially biased teacher expectations resulting in differential discipline, suspension, and expulsion for African American students. Broader ideologies, internalized through implicit attitudes, could impact relationships between teachers and students that have critical implications beyond even mathematics learning.

We have found that the quality of relationships is not related to the quality of instruction in mathematics classrooms in urban schools with high populations of African American and Latinx[1] students (Battey & Neal, in press; Battey, Neal, Leyva, & Wiggins-Adams, 2016). In this work, the variability in the quality of teacher-student relationships across classrooms has been enormous. Some classrooms were consistent in handling student thinking positively, while others framed student ability and thinking in mixed ways or skewed negatively. While classrooms varied in handling mathematical thinking, over 90% of behavioral interactions were negative. The reasons, however, for such high frequencies of negative interactions are unknown in this work, though the negative focus on behavior could be a sign that teachers accessed stereotypes about African American misbehavior. Implicit attitudes hold potential for expanding our understanding of this relational phenomenon, especially in classrooms where high-quality relationships are not developed.

From the student side of relationships, research has shown how African American students perceive and cope with relationships of varying quality that differentiate access to high-quality mathematics instruction and support (Leyva & Strothers, 2014). More specifically, African American students often attribute stronger teacher-student relationships to placements in higher-tracked classes such as honors and Advanced Placement (AP), where students are perceived as being able to engage with challenging mathematics. African American students who are aware of their

[1] The term *Latinx* is used to decenter the patriarchal nature of the Spanish language that traditionally groups Latin American women and men into a single descriptor *Latino*, a masculine term denoting only men. The "x" in *Latinx* allows for gender inclusivity among Latin Americans (including those who identify as agender, gender-nonconforming or gender fluid, queer, and trans) compared to the either-or term *Latina/o* that implies a gender binary.

underrepresentation in these classes come to see themselves at a relational disadvantage with their mathematics teachers and can be dissuaded from seeking support. For example, one African American woman described her strategy of bringing an AP calculus peer as a "catalyst" to receive better support from her teacher during extra help sessions in high school (Leyva & Strothers, 2014). This illustrated the additional effort that this African American student had to exert in order to manage her teacher's possibly implicit racial attitudes. When teachers bring implicit racial attitudes unchecked into mathematics classrooms, the burden is placed on students to navigate these racialized spaces.

Across this research, the findings on teacher-student relationships more broadly, and within mathematics specifically, are consistent with implicit racial attitudes being a factor in producing more negative and conflictual relationships between teachers and African American students. This can be observed in the ways in which teachers chose to not provide instructional support, held low expectations, and focused on misbehavior among African American students.

Racialized Teacher Perceptions of Behavior and Academic Aptitude

The broader literature in education has found that African American students receive more negative consequences for their behavior. Ferguson (2000) found that black[2] and white boys were disciplined quite differently. White boys who broke school rules were treated as if their behavior was innocent. On the other hand, black boys who broke school rules were treated as if their behavior was intentional and fully conscious. Therefore, punishments of black boys were much more severe. The punishment of black students potentially removes them from mathematics instruction, but it can also impact teachers' perceptions of their cognitive abilities. To underscore this, Neal, McCray, Webb-Johnson, and Bridgest (2003) studied teachers' perceptions of African American movement styles. The middle school teachers rated students with African American movement styles as opposed to European American styles as lower in achievement, more aggressive, and more in need of special education services. While both studies presented above examined student behavior, the implications of the second study reveal that racial perceptions of behavior can impact teacher expectations of students' cognitive aptitude and thus lower the quality of instruction and learning opportunities made available to African Americans in the classroom.

These findings show how perceptions of behaviors impact teachers' intellectual expectations of students in ways consistent with implicit racial bias. In fact, Downey and Pribesh (2004) stated that it is more likely that racial bias impacts teachers' understanding of behavior than simple racial matching of teachers and students. A

[2] We use the term *black* here to remain consistent with the author's language in the research. We do this consistently throughout the paper.

recent study examined the impact of racial matching on teachers' expectations (Gershenson, Holt, & Papageorge, 2016). The authors examined intellectual expectations of the same black student by a black and a non-black teacher. They found that black teachers held significantly higher expectations for students than non-black teachers. For example, non-black teachers were 12–30% less likely to expect black students to go to college than black teachers. The effects were greater for black male students. Additionally, the major finding of this study was that "racial mismatch between students and teachers lowered teachers' expectations that students would earn a four-year college degree was largely driven by math teachers' expectations" (p. 221). Since the teachers in the study rated the same students, the authors argued that this is evidence that teachers are unconsciously racially biased. Beyond this work, a more detailed understanding of teachers' implicit racial attitudes would help the field better grasp how these attitudes shape teachers' expectations about, interpretations of, and reactions to African American students' mathematical behavior and thinking.

In a study more closely related to mathematics that specifically considered implicit racial attitudes, Dovidio (2001) assessed white college students' explicit and implicit prejudices. White students were tested into three groups: (1) prejudiced whites (high explicit prejudice/high implicit prejudice), (2) implicitly biased whites (low explicit prejudice/high implicit prejudice), and (3) non-prejudiced whites (low explicit prejudice/low implicit prejudice). The students were then asked to solve a logic problem together with black students and to report their perceptions. The study found that both implicitly biased and non-prejudiced whites perceived interactions as positive, while blacks only perceived interactions with non-prejudiced whites as positive. In addition, non-prejudiced whites grouped with blacks solved the problem quicker (4:35) than groups with prejudiced whites (5:42) or implicitly biased whites (6:10). Interestingly, the groups with implicitly biased whites solved problems the slowest, possibly because of the ambiguity of unconscious racial interactions. While Dovidio is not a mathematics educator, the implications are clear in these peer interactions during problem solving – a common practice across mathematics classrooms. Similarly, African American students' perceptions of teachers' implicitly biased attitudes potentially could impact their success in mathematics classrooms.

The studies discussed in this section show similar findings to those in prior sections, furthering the case that implicit racial attitudes are at play in classroom settings. For example, teachers perceived African American behavior as more aggressive and intentional, which is consistent with the findings from the section on relationships showing teachers focusing on misbehavior. Likewise, teachers perceived African American movement styles as exhibiting less intellectual capacity, which is consistent with teachers providing lower cognitively demanding instruction to African American students in urban schools. In each of these ways, behavior and academic capability could be seen as proxies for race, thus leaving teachers' attitudes implicit, resulting in the mistreatment of African American students and limiting their learning opportunities in mathematics classrooms.

Racial Microaggressions in Mathematics Classrooms

Another source of understanding how implicit racial attitudes affect classrooms is to ask African American students themselves about their mathematics experiences. Much of this work focuses on racial and mathematical identities (e.g., Berry, 2005; McGee & Martin, 2011; Martin, 2006), but microaggressions are also identified in this work. Microaggressions are insults or slights that are automatic and sometimes unconscious, directed toward oppressed groups (Solórzano, Ceja, & Yosso, 2000; Sue, Capodilupo, & Holder, 2008). Within the narratives of African American students' experiences in mathematics, interactions that would be considered microaggressions are identified. Many of these experiences stem from deficit discourses of African Americans' mathematics ability (Martin, 2009). The discourses, we argue, shape racial attitudes reflected in microaggressions that position students as illegitimate members of mathematics classrooms resulting in poor relationships with teachers, lower-quality instruction, and at times, dis-identification with mathematics (Spencer, 2009).

McGee and Martin (2011) discussed successful undergraduate black mathematics and engineering students as being constantly aware that their racial selves were undervalued. The stereotype of blacks as innately incapable in mathematics was identified as the primary force for students to adopt stereotype management strategies for their academic success. The students, for example, were driven to prove stereotypes wrong in persisting and succeeding academically, which was emotionally and psychologically exhausting. Participants reflected on teachers' and school administrators' lower academic expectations for them as black students in mathematics and engineering. One student shared, "These people [teachers and her peers] don't expect too much of me in this class, and so … if you tell me that I can't do something, then I want to prove to you that I can" (p. 1365). Here we see how the stereotype of blacks as not mathematically capable lowered academic expectations of ability and thus prompted this student to prove others wrong. The students in McGee and Martin's (2011) study also reflected on invoking evidence of their academic achievement, or "always being on point" (p. 1365), as a countering response to microaggressions framed by racial stereotypes such as blacks as academic underachievers. Another student recalled citing his high mathematics achievement scores and consistent homework completion when accused of stealing a classmate's phone – an accusation that the student perceived as being shaped by the stereotype of black males as thieves.

In addition, some students reflected on managing the stereotype of "acting white" (Fordham & Ogbu, 1986), by cultural code-switching and *frontin'* (McGee & Martin, 2011). McGee and Martin (2011) define *frontin'* as the "performance or demonstration of an act that is socially acceptable by the dominant culture but creates a series of self-sacrifices to one's own personal identity" (p. 1381). Students did this, in predominantly white spaces, including mathematics classrooms, to avoid being subjected to racial microaggressions of ability and belongingness. One student commented on nodding excessively in mathematics classes to manage feelings of hyper-visibility resulting from often being the only black student in mathematics courses:

> Sometimes it seems like they are watching me to make sure I get it or that I belong. It's like they are waiting for me to [#$%] up. So I just nod no matter what…Then at an inconspicuous hour I go find the TA. (p. 1370)

Other students adopted behaviors perceived by others as stereotypically black to resist racial judgments while also "manipulat[ing] and manag[ing] the stereotype on [their] own terms and for [their] own purposes" (p. 1371). McGee and Martin's (2011) study illustrated how the internalization of stereotypes about black students can give rise to implicit racial attitudes that shape microaggressions about belonging in mathematics classrooms, being smart, or completing work that African American student participants managed in a variety of ways.

Other research has also documented ways in which mathematics teachers have enacted microaggressions focused on beliefs about the lower intellectual capacity of African American students. For example, Martin (2006) discussed a teacher who told a student that they had gone as far as they could mathematically. Rather than offering support, advice, or understanding of mathematical struggles, the teacher assumed the student was simply not smart enough to do more mathematically. Another student reflected on how racialized views of mathematics ability brought teachers to overlook African American students as "being [among] the best and brightest" (p. 213) and thus limited their access to opportunities such as enrollment in gifted programs. Martin (2006) discussed how this student also saw how classrooms taught by white teachers can present mathematics as not belonging to African Americans through racial microaggressions of ability, or "teaching classroom interactions [that] often perpetuated the superiority of Whites in areas like mathematics and science" (p. 214). These examples capture the possibility of implicit racial attitudes of mathematical ability resulting in African American students' experiences of microaggressions about their academic competence through communications with teachers and mathematics instruction.

Berry (2005, 2008) also noted how teachers and administrators discriminated against African American males in three ways: (1) they resisted placing African Americans in gifted programs, (2) focused on behavior rather than cognitive abilities, and (3) pre-diagnosed students as having attention-deficit/hyperactivity disorder (ADHD). For example, African American males' behavior, rather than their academic performance, was used to assess their intellectual potential in mathematics (Berry, 2005). These student behaviors were often perceived, however, as being inappropriate and worthy of punishment due to racial or cultural differences between the students and the "school grounded in the ethos of the white middle-class culture that values and demands certain ways of talking, writing, dressing and interacting" (p. 56). Berry's work connects to the literature on teachers' racialized perceptions of African American behavior and academic aptitude. The difference here is that these are not merely perceptions but teachers' actions toward students in the form of microaggressions.

If mathematics education researchers are noting African American students experiencing racial microaggressions in classrooms, then someone must be enacting them. A sideways glance, questioning if they are in the right class, or surprise that they are good at mathematics are all noted in the literature. Each of these reactions is consistent with implicit racial attitudes in that they are automatic responses consistent with attitudes expecting less. Those enacting these microaggressions, if not holding explicitly racially biased attitudes, are certainly holding implicitly racially biased attitudes. Understanding more about this phenomenon can only increase the field's knowledge of ways of responding and transforming mathematics education for more socially affirming learning opportunities among African American students.

Embedding Implicit Racial Attitudes in Mathematics Education Research

Despite the promise that these four research literatures present, no study within mathematics education has yet explored implicit racial attitudes. A Dutch study in education more broadly, however, pointed to the significant potential of the construct in future mathematics education research. In the study, van den Bergh, Denessen, Hornstra, Voeten, and Holland (2010) examined the impact of teachers' implicit attitudes on their expectations for students and achievement differences between groups of students across 41 classrooms. They found that teachers with more implicit racial bias, as measured with the implicit attitudes test (IAT), held lower expectations of the intellectual capacity of their ethnic minority students. In addition, achievement differences between students of Dutch origin and ethnic minority students were larger in classrooms with teachers who were more implicitly biased. Finally, they found that students of Dutch origin actually showed an increase in achievement in classrooms where teachers were implicitly biased against ethnic minority students. This study highlighted the interconnected nature of implicit racial attitudes to other educational constructs, including teacher expectations and student achievement, which were explored in the reviewed literature on African American students in mathematics. But implicit racial attitudes could also be connected to research on, for example, cognitive depth, professional development (PD), and student identity. Here, we describe two studies to illustrate ways in which implicit racial attitudes could be embedded in future mathematics education research.

The goal here is not to critique prior work, as the notion of implicit racial attitudes was only in its infancy at the time that some of these studies were performed. Instead, we want to illustrate ways in which implicit racial attitudes could be embedded in different lines of research in mathematics education and in different study designs, both quantitative and qualitative. The studies we highlight focus on the space of teacher change and student identity, two key areas in thinking through the impact of implicit racial attitudes in mathematics education.

One area to better understand the impact of implicit racial attitudes would be around teacher change. In some of our own work examining the ways in which PD impacts the teaching of elementary mathematics, implicit racial attitudes could help us understand more about how to support teacher change, why some teachers resist taking on new practices, and how teacher change impacts student learning in mathematics. Specifically, we take the study of PD published by Jacobs, Franke, Carpenter, Levy, and Battey (2007) as a quantitative example of considering implicit racial attitudes in mathematics education research. In this work, the authors documented how PD, across a large district serving 99% African American and Latinx students, impacted teachers' mathematics knowledge and knowledge of student thinking as well as student learning of early algebraic concepts. Matching schools within the district, the study compared the effects of mathematics PD across treatment and control conditions. The main findings of the study were that, while treatment teachers' mathematics knowledge did not differ from the control group at the end of the PD, their knowledge of students' strategies around early algebra did. In addition, students in treatment classrooms better understood the equal sign and developed more strategies using relational thinking. Interestingly, the authors note in the manuscript that in the PD, "we also engaged teachers in telling stories about the mathematics their students could do rather than what they could not do" (p. 265) in order to challenge prevalent deficit narratives about the student population in mathematics. This raises the possibility that the PD had effects beyond teachers' mathematics knowledge and instructional practices. At the time, however, the authors did not measure the impact of the PD on teachers' notions of student capabilities or the impact of teachers' racial attitudes on mathematical learning.

If we were going to integrate work on implicit racial attitudes into this research, we could focus on two different goals. One goal would be to measure implicit racial attitudes pre-post or at intervals during the PD work, to see if the focus on challenging deficit narratives produced change in teachers' implicit racial attitudes. This would allow us to better know the extent to which mathematics PD influenced teachers' implicit racial attitudes. A second goal would be to better understand the impact of racial attitudes on teacher and student change. We could use the measure of implicit racial attitudes to examine which teachers gained more knowledge related to student thinking in mathematics as well as which teachers implemented the practices in their classrooms. Additionally, the regression model in the study could then use implicit racial attitudes to measure if they moderate change in teacher knowledge and practice or student learning of mathematics. Therefore, adding the construct of implicit racial attitudes would allow the field of mathematics education to measure the quality of PD, not just by the knowledge created or the practices implemented, but also by insights on when and why some teachers may or may not implement instructional practices based on the ways in which they frame African American students.

As noted in both van den Bergh and colleagues' (2010) study and the preceding section of this chapter on teachers' expectations, implicit racial attitudes can impact teachers' perceptions of student behavior and their academic aptitude. There is no reason to think that mathematics is immune to these forces. Teacher changes in expectations have real effects on classroom interactions, which can cause teachers to lower the cognitive demand of the mathematics, hyper-focus on negative behavior, or interact with students in a way that ignores or demeans their intellectual contributions. Students make meaning of these teacher behaviors in terms of the relationships they build with mathematics, subsequent student behavior, responses to microaggressions, and the energy exerted in classrooms. Therefore, the other area of research we consider for integrating work on implicit racial attitudes is student identity.

Implicit racial attitudes could also be explored with respect to student identity in a qualitative manner. This is akin to much of the work on student identity that situates the exploration of identities within instructional and relational dimensions of classroom contexts as well as a broader racialized society (e.g., Langer-Osuna, 2015; Leyva, 2016). One of our past studies, for example, used counter-storytelling methodology (Delgado & Stefancic, 2001) to examine co-constructions of mathematics and social identities among Latinx students pursuing engineering majors at a large, predominantly white university (Leyva, 2016). The analysis focused on how the Latinx students constructed their identities in making meaning of their mathematics experiences in and out of the classroom. Counter-stories were constructed using mathematics autobiographies, interviews, a focus group, and field notes from observations in their university mathematics classrooms. Stimulus narratives were presented during the focus groups to characterize the extent to which these dynamics, based on three research articles, shaped their identity constructions (taking up space (Hand, 2012), stereotypes of mathematical ability (Shah, 2017), and teacher-student relationships (Battey et al., 2016)).

Findings illuminated classrooms' constructions of a racial hierarchy of mathematics ability through teachers' instruction that limited the Latinx students' opportunities for participating compared to white and Asian American peers. In addition, the Latinx students reflected on how opportunities for building supportive teacher-student relationships at the university were limited because most instructors were not interested in connecting with students. The Latinx students noted some exceptional teachers who developed academically and emotionally supportive relationships that they viewed as motivating their success in challenging racial discourses in mathematics. Implicit racial attitudes were not explicitly adopted in this study's analysis; however, the findings capture how they could potentially explain racialized instruction and learning opportunities for classroom participation and supportive teacher-student relationships that influenced the Latinx students' mathematics identity development.

Future study designs may select teachers with more or less implicit racial bias to show the variation in their mathematics instruction and interactions with their African American students. In turn, the students themselves could be interviewed about their mathematics identity, racial identity, the quality of instruction, and their relationships with teachers. Links between implicit racial bias and effects on African American students and their agency could, therefore, be explored in more detail.

Similar to Dovidio's (2001) work around problem solving, mathematics education researchers could look more in-depth at the ways that non-biased teachers, implicitly biased teachers, and possibly even explicitly biased teachers impact African American students' identities in varying ways. For example, how do African American students respond similarly and differently in making sense of their mathematics classroom experiences in relation to their racial and mathematical selves when teachers are implicitly biased? What variation is observed among teachers who are non-, implicitly, and explicitly biased in their interactions with students including instructional decisions, building relationships, and responding to academic contributions and classroom behavior?

A teacher who has internalized the ideology of a racial hierarchy of mathematics ability would implicitly believe that African Americans are less capable mathematically. In this case, they could enact lower expectations of African American students in learning mathematics through posing tasks of low cognitive demand and classroom interactions that do not frame students as academically competent. African American students, in turn, could react to the quality of mathematics instruction that is available and expectations that they perceive their teacher to have. Students may respond by dis-identifying with mathematics, disengaging from "boring" mathematics, persevering due to the importance of mathematics, or proving the teacher's expectations wrong. Each of these responses impacts learning, but the teacher's lowering of cognitive demand and negative interactions with students limits their opportunities to understand the mathematics at a deeper level. A comparative case of a nonracially biased teacher would be interesting to show how students respond differently in this context. In looking at the interaction between implicit racial attitudes, student identity, and classroom interactions, we could detail the impact of varying levels of bias on a variety of mathematics classroom environments.

Discussion

Research in mathematics education is clear that African American students generally receive low-quality access to instruction and experience microaggressions in mathematical spaces. Outside of mathematics education, researchers have found that teachers perceive their relationships with African American students as more conflictual than the students themselves and that teachers have negative perceptions of student behavior and lower expectations when there is a racial mismatch. All of this work raises the possibility of a mechanism that is producing these racialized dynamics for African American students in mathematics classrooms. By not considering the possibilities of applying this lens to our work, we may be limiting our understandings of the subtle and not so subtle ways that race impacts mathematics learning and teaching. Such limited understandings about these processes also constrain the ways in which we can support teachers in changing their behaviors and attitudes for more socially affirming mathematical learning opportunities among African American students.

Examples of teachers expressing racially biased attitudes are occasionally evident in the literature within mathematics education (see, e.g., Battey, 2013a; Battey & Stark, 2009; Jackson, 2009; Spencer, 2009), though none of this work discussed teachers' views as manifestations of implicit racial attitudes. Another set of work specifically focused on PD efforts aimed at changing mathematics teachers' deficit views of students around race (Battey & Chan, 2010; Battey & Franke, 2015; Ho, 2010; Spencer, Park, & Santagata, 2010). Ho (2010) specifically discussed one white teacher who struggled to change her deficit-based views of students as she engaged in mathematics PD. While some change was evident, the tensions between changing her instructional practice and positioning students as unsuccessful continued through the end of the PD. Interestingly, as we considered research on mathematics PD that challenges deficit views, we noted that many studies highlighted changes in teachers' practices that result from a focus on social justice or multiple mathematical knowledge bases (e.g., Turner et al., 2012). While both of these threads of work are critically important, we still need new ways for engaging teachers' racially biased views of students in the field. By focusing mathematics PD work on implicit racial attitudes, new insights may be gained about how to support teachers in challenging biased views that limit African American students' access to and sense of belongingness in mathematics.

We are not arguing that teachers are aware of the ways in which racial attitudes are affecting their instructional practices, expectations, perceptions of behavior, relationships with students, and classroom interactions. Much the opposite, they may be largely unaware of the ways in which race impacts the quality of their teaching and interactions with African American students. And yet, high levels of exposure to low-quality mathematics teaching and negative interactions cause African American students to experience frustration and anxiety, leading to disengagement, greater dropout rates, and lower grades (Carroll, 1998; Feagin, Vera, & Imani, 2001; Solórzano, Allen, & Carroll, 2002; Tabibnia, Satpute, & Lieberman, 2008). Whether teachers are aware or not, their framings of mathematics and interactions with African American students have significant repercussions. Understanding the implicit influences on these classroom dynamics changes the way we might work with mathematics teachers to transform their instruction with African American students.

For instance, a recent psychological study looked at ways to reduce implicit bias toward blacks. It found that effective interventions included counter-stereotypical exemplars, evaluative conditioning methods, and specific strategies to override biases (Lai et al., 2014). Ineffective interventions had participants engage with others' perspectives, consider egalitarian values, or induce positive emotions. Many times in PD, we might engage teachers with considering students' perspectives as a way to better connect them with their African American students. While this can increase their knowledge of students and their everyday lives, it may not deconstruct their implicit biases. Therefore, this would need to be coupled with successful strategies from Lai et al.'s (2014) study such as countering stereotypes, associating positive attributes with African Americans (evaluative conditioning), or raising institutional racism as a constant check to internalized deficit views. These strategies have not yet been explored in addressing inequities in the context of mathematics PD (Battey & Franke, 2015).

Interestingly, the interventions in Lai et al.'s (2014) study were more successful when also raising the negative consequences and attributes of whiteness. Therefore, merely associating African Americans with positive attributes will not be successful if whiteness remains neutral and unchallenged (Battey, 2013b). Additionally, while educators consistently refer to changing student demographics in US schools as a reason to modify instruction, this will not produce the transformation we desire either. As one study found, exposing whites to the narrative about changing demographics in the USA produced more negative attitudes toward Latinxs, Blacks, and Asian Americans as well as *increased* pro-white bias (Craig & Richeson, 2014). Deconstructing whiteness, then, is critical in order to understand the ways in which race is a comparative hierarchy, as in the case of the discourse of a racial hierarchy of mathematics ability (Battey & Leyva, 2016; Martin, 2009). To date, no work in mathematics education has addressed ways to deconstruct whiteness in PD, either (Battey & Franke, 2015). Thus, both in order to deconstruct the mechanisms negatively impacting African American students in mathematics classrooms as well as identify ways of supporting equitable mathematics teaching, implicit racial attitudes may be key in future work in mathematics education.

Because of unconscious internalization of deficit ideas, we cannot explicitly ask teachers about these moments since much of the experience is implicit. This makes work on implicit racial attitudes so valuable. Implicit bias can play out in automatic responses and nonverbal behavior as well as be attributed to proxies for race. In looking for implicit bias, we might uncover the unconscious ways in which internalized racial narratives are shaping teachers' interactions in undesired ways. This potentially connects to classroom interactions of reducing cognitive demand, devaluing students' mathematical thinking, and publicly admonishing misbehavior – all interactions documented in work on relational interactions between teachers and students in mathematics classrooms (Battey, 2013a; Battey & Franke, 2015; Battey & Neal, in press; Battey & Stark, 2009). Connecting implicit racial bias with work on MKT, quality of instruction, cognitive demand, and relational interactions can move the field forward in understanding a complex dynamic that is underexplored. Additionally, researching the intersections of this work might shape new ways of thinking about transforming mathematics education through supporting teachers in actively resisting the consumption of stereotypes about African American students.

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Chapter 3
A Socio-spatial Framework for Urban Mathematics Education: Considering Equity, Social Justice, and the Spatial Turn

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Abstract In recent years, equity- and social justice-oriented discourses in mathematics education have been working to move the field toward an understanding of mathematics education as inherently and simultaneously social and political. Critical to this movement is the development of concepts that support scholarship, policy, and practice that are oriented toward equity and social justice. In this chapter, we propose a framework that engages scholarship in mathematics education, urban education, critical geography, and urban sociology. The resulting socio-spatial framework for urban mathematics education features a visual schematic that locates mathematics teaching and learning—vis-à-vis a mathematics-instructional triad—within a system of socio-spatial considerations relevant to US urban contexts. We situate the math-instructional triad amid a three-dimensional frame. Representing the first axis, we describe established categories of social significations or urban education: urban-as-sophistication, urban-as-pathology, and urban-as-authenticity. The concepts that represent the second (spatial) axis are drawn from considerations of how space is continually constructed: empirical space, interactive-connective space, image space, and place space. The third axis concerns the various moments and perspectives that have evolved and continue to unfold in mathematics education practice, scholarship, and policy. Particularly, we draw on recent depictions of the field's "moments": the process-product, interpretivist-constructivist, social turn, and sociopolitical turn. Finally, we use the urban system initiatives sponsored by the National Science Foundation in the 1990s to illustrate the elements of the framework and to demonstrate how the framework can help the field to clarify the relationship between the arrangements of spatial geography and distribution of social opportunity.

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Introduction

In this chapter, our goal is to spur the reconceptualization of urban mathematics education scholarship toward developing new and clearer pathways for exploring equity and social justice discourses within the broader field. Although our use of “urban” throughout this chapter is situated in the context of the United States, we intend for this reframing to have potential applications in, and implications for, other national urban contexts. Urban mathematics education has been a long-established, yet loosely organized, subdomain (and too often ill-defined or unnamed) for practice and scholarship—including research, curriculum, and policy development. Relatedly, the relationship between urban mathematics education and equity and social justice discourses has also been ambiguous over time. We argue that this ambiguity has resulted in disjointed periods of scholarship that—perhaps in the spirit of improving “urban”-oriented mathematics education research and practice—have primarily initiated and examined general mathematics education reforms within locales that typically feature high population densities, expanding demographic diversity, and/or large- and moderately sized municipal land areas.

Although the general notion that “urban” as a spatial consideration has been limited, historically, to conventional notions of the “big city,” our central claim in the chapter is that urban mathematics education has been defined tacitly as a *social* consideration in ways that veil a taken-as-given confluence with race and class discourses—often characterizing urban almost reflexively as “inner city.” These oversimplified renderings are more apparent within earlier scholarship, but we would suggest here that they have not yet given way fully to contemporary, more complex, and ever-evolving notions of urban. Our objective for this chapter is to reframe research within this domain by simultaneously clarifying and underscoring the complex relationships between mathematics teaching, learning, and curriculum; place and space; and the social, economic, and political conditions that accompany historical and contemporary understandings of urban life. As Martin and Larnell (2013) argued, the stakes for this work are high: “Because mathematics education is not a neutral enterprise, these conditions, significations, and potential necessarily implicate the values, ideologies, and power relations among its participants” (p. 373).

In the following sections, we review the extant trends in scholarship regarding urban mathematics education as a research- and practice-oriented domain. We then present our framework for urban mathematics education and explicate its multiple components. Finally, we provide an example of how the framework may be useful for analyzing the trajectory of urban mathematics education scholarship through a historical exemplar of urban mathematics education policy in the United States, the urban systemic initiatives of the 1980s and 1990s.

On the Trajectory of Urban Mathematics Education Scholarship

During the past few decades, urban mathematics education has emerged more prominently as a vibrant new area of scholarship in the United States—evinced most recently by the founding and proceedings of the *Journal of Urban Mathematics*

Education (*JUME*), founded in 2008. The roots of this subdomain of mathematics education extend back at least to efforts during the 1980s (Tate, 1996), concurrent with the development and publication of standards by the National Council of Teachers of Mathematics (NCTM) for mathematics curriculum and evaluation (1989), for the practice of mathematics teaching (1991), and for assessment (1994). These developments also coincided with commensurable shifts in research: mathematics education scholarship around the world was entering its much-discussed social turn (e.g., Meyer & Secada, 1989; also see Lerman, 2000; Martin & Larnell, 2013; Stinson & Bullock, 2012).

For researchers, teachers, policymakers, and education-interested foundations in the United States (e.g., the Ford Foundation, the National Science Foundation), a crucial new question emerged: How would the then-new version for school mathematics reform extend to and take shape in urban districts and classrooms (Tate, 2008)? We contend that this question has loomed large over the development of urban mathematics education as a domain. Both the question and the adjoining concern remain central given the most recent policy shift toward, development of, and state-level adoption of the Common Core State Standards for School Mathematics.

In his commentary in the inaugural issue of *JUME*, Tate (2008) issued the following charge:

> The challenge is to build theories and models that realistically reflect how geography and opportunity in mathematics education interact. If this challenge is addressed, the field will be one step closer to making scholarship in urban mathematics education visible. (p. 7)

Our aim in this chapter is to broaden the discourse in urban mathematics education in line with Tate's charge. Moreover, that aim is to develop theoretically and attend to the relationship between place, access and opportunity, and the progression of scholarship in mathematics education. Urban mathematics education scholarship has advanced to a point at which we may now begin to evaluate its production of knowledge—and, particularly, the building of "theories and models that realistically reflect how geography and opportunity in mathematics education interact" (p. 7). What has the study of urban mathematics education entailed? What can it become? The purpose of this chapter is to take "one step closer" toward addressing these questions and to consider the implications of a socio-spatial understanding of urban mathematics education on our understandings of the connections between mathematics and social justice.

Urban Mathematics Education, Equity, and Social Justice

This framework represents a departure from reductive notions of urban as "big city" or "inner city" toward the understanding that urban mathematics education is a complex, inherently interdisciplinary domain in its own right. It is *more than just* mathematics education performed with, or on, people who are labeled as "urban" based on race and/or class signifiers (Chazan, Brantlinger, Clark, & Edwards, 2013). Also, it is more than just a descriptor for situating traditional or reform-oriented mathematics teaching and learning in certain locales (i.e., the "inner city"). Thus, it is important that we address the need for a consideration of urban mathematics education that is separate from—yet connected to—prevailing equity discourses in mathematics education.

It is true that equity discourse and urban mathematics education have common interests, especially concerning commitments to underserved communities. Examining mathematics education in urban spaces through an equity-oriented lens appropriately centers conversations on children of color and their mathematical identities and experiences. However, engagement with the urban in such work is often limited either to contextual descriptors connected to racial demographic and free-and-reduced-lunch data or to situated applications of mathematics curricula or pedagogies in spaces inhabited by people who are largely Black and/or Brown and poor. These limits at the level of data representation do not adequately consider the ways in which the urban as its own domain may necessarily affect research design, data collection, and data analysis.

As a descriptor in research, therefore, "urban" functions as a sort of veil. This veiling allows the researcher to acknowledge race and class in superficial ways that obscure weightier systemic issues related to the deeply saturated influence of race- and class-based ideologies (Martin, 2009). This urban-as-veil perspective also frames our collective understanding of urban populations in ways that, perhaps ironically, obscure populations that do not align with static notions of urban educational contexts as Black, Brown, and/or poor. The challenge with this veiling is that it allows equity discourse to disengage from the enormously consequential and broadly societal issues concerning urban education, racism, and classism that inhabit mathematics classrooms and other aspects of the "network of mathematics education practices" (Valero, 2012, p. 374).

We argue that this framework for urban mathematics education encourages a more complex understanding of the urban that attends to the role of place and space in mathematics education and, additionally, unveils race and class as distinct categories that each warrant significant analysis in their own right. We propose that engaging the elements of this framework allows equity-oriented mathematics education researchers to remove the urban veil in a way that acknowledges the roles of place, race, and class as distinct and mutually constitutive. This positioning allows mathematics education researchers to explore the interactions between geography and opportunity within a multidimensional framework that acknowledges the political underpinnings of opportunity gaps that equity discourses reveal.

A Socio-Spatial Framework for Urban Mathematics Education Scholarship

In the spirit of addressing Tate's (2008) challenge (also see Rousseau Anderson, 2014), our objective is to posit a new theoretical framing for scholarship in urban mathematics education—the first of its kind (see Fig. 3.1). In this section, we detail the theoretical concepts undergirding the framework. We situate this framing squarely (but not entirely) in mathematics education scholarship—using as our

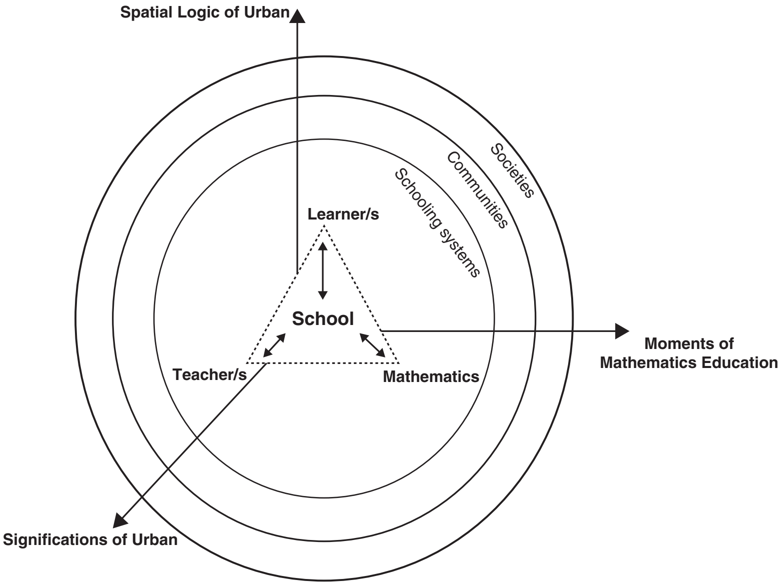

Fig. 3.1 Socio-spatial framework for urban mathematics education scholarship

central unit of analysis the well-regarded mathematics-instructional triad of teacher(s), learner(s), and mathematics (Cohen, Raudenbush, & Ball, 2003; NCTM, 1991; Stein, Smith, Henningsen, & Silver, 2009).

The NCTM Research Committee (Gutstein et al., 2005) argued that, in order for researchers to advance equity in mathematics education, we must "break with tradition, expand boundaries, and cross into fields outside of mathematics education *and* outside education" (p. 96; emphasis original). In this spirit, we extend beyond mathematics education, looking toward the interdisciplinary areas of urban sociology, critical geography, and urban education to consider the various forces that influence mathematics teaching and learning in urban spaces as well as the social significations that shape interactions in urban settings and to emphasize the ways in which urban spaces and their meanings are reciprocally constituted. We recognize, however, that the task of defining *urban* has been a challenge across disciplines, and our attempt here is to incorporate what is known, inasmuch as we can, given what is available to us at the moment (Milner & Lomotey, 2013).

In recent decades, there has been considerable momentum in the humanities and the social sciences to consider space as a social construction that is integral to social analysis (Arias, 2010). This *spatial turn* renders geographic considerations equal to, and mutually constructed with, temporal and social considerations in the analysis of social phenomena (Warf & Arias, 2009). This framework represents a spatial turn within mathematics education research in which temporal (i.e., the Moment of Mathematics Education axis), social (i.e., the Significations of Urban axis), and geographic (i.e., the Spatial Logic of Urban axis) elements are taken together as mutually constitutive of urban mathematics education (Fig. 3.1).

To inform the framework with respect to the social meanings that shape urban mathematics education, we draw on Leonardo and Hunter's (2007) typology of significations that circumscribe urban education (also see Martin & Larnell, 2013). We represent that typology as an axis of the framework that intersects with spatial considerations of urban, drawn from human and critical geography (e.g., Soja, 1980; Thrift, 2003) and urban sociology (e.g., Johnson, 2012). These two axes, when taken together, are intended to signal a "socio-spatial dialectic" regarding urban education (also see Soja, 2012). By socio-spatial dialectic, we mean that the social significations and spatial considerations necessarily interact to determine meaning for urban that, as Tate (2008) suggested, "realistically reflect[s] how [spatial] geography and [social] opportunity in mathematics education interact" (p. 7). We add a third axis to situate the socio-spatial elements in relation to the evolution of mathematics education. It incorporates the various theoretical orientations (e.g., cognitivism/behaviorism, constructivism, sociocultural perspectives) that have emerged amid "moments" of mathematics education during the past century (Stinson & Bullock, 2012). Next, we elaborate upon each of the three axes and the markers that characterize each as well as the mathematics-instructional triad that is central to the framework.

Spatial Axis of the Framework

To substantiate the spatial aspect of this framing, we draw on Thrift's (2003) four conceptions of space to establish four markers along the spatial axis: (a) empirical-constructing space, or the ways in which space is rendered measurable or objective; (b) interactive-connective space, or the pathways and networks that constitute space; (c) image space, or the visual artifacts that we readily associate with certain kinds of spaces; and (d) place space, or the everyday notions of spaces in which human beings reside—even if notions of "human" and "being" are actively being reconsidered (p. 102). Each of these types refers to ways in which space is conceptualized in relation to human geography, and not necessarily with respect to either a strictly geographical sense of urban spaces or the meanings that are derived from them. This allows us to avoid constraints of a spatial logic that is determined solely by, for instance, characterizations based on population density or physical geography (see Milner, 2012). Thrift's four conceptions of space allow for four distinctive conceptions of urban as space, which allows one to look across their various permutations in ways that provide a nuanced perspective on space.

Social-Signification Axis of the Framework

While urban is not easily defined, it is clear that it is not simply geospatial; it also carries social and political meanings. Therefore, considerations of the urban in mathematics education must engage these social and political dimensions directly because "'place matters' in the study of urban mathematics education" (Rousseau Anderson, 2014, p. 10). The markers along the social-signification axis are Leonardo and Hunter's (2007) three significations of urban: (1) urban-as-sophistication (or cosmopolitan space), (2) urban-as-pathological (or urban as "dirty, criminal, and dangerous;" p. 789), and (3) urban-as-authenticity (or the politics of authenticity). As we argued earlier regarding urban-as-veiling, urban does not always refer to the geographical urban space; rather, studies have used the label "urban" as a proxy descriptor for poor, Black, and/or Brown populations who inhabit these spaces and disproportionately fall victim to the segregation and concentrated poverty that often characterize these spaces (Darling-Hammond, 2013). Such employment of "urban" ignores the heterogeneity of urban space, its politics, its people, and their experiences (Fischer, 2013).

Theory-Moment Axis of the Framework

With a third axis in the framing, we attempt to construct (at least initially) what could be called a mathematical-socio-spatial dialectic. That is, we situate the mathematics-instructional triad within the dimensional space of not only the socio-spatial dialectic but also with respect to the ongoing "moments" of mathematics education theory and practice (Stinson & Bullock, 2012; also see Martin & Larnell, 2013). Put differently, the axes represent the intersectionality of geography (or spatiality), social opportunity, and the development of mathematics education, which is what Tate (2008) originally outlined. The moments of mathematics education—the "process-product," "interpretivist-constructivist," "social turn," and "sociopolitical turn" moments—are simultaneously occurring categorical periods of research, practice, and policy (also see Gutiérrez, 2013). These periods have often been indexed by a crisis metaphor within mathematics education scholarship (Washington, Torres, Gholson, & Martin, 2012); this notion of crisis also connects to particular significations of urban life and contexts.

Mathematics-Instructional Triad as the Center of the Framework

At the center of the framework is the principal set of relations involved in mathematics teaching and learning: the interactions and participation by and among learners, among teachers, learners and teachers together, and other interactions permuted according to learners, teachers, and mathematics curriculum (see Fig. 3.2). As Cohen et al. (2003) suggest, "teaching is what teachers do, say, and think with learners, concerning content, in particular organizations and other environments, in time" (p. 124). This depiction of a triadic relationship is traceable beyond mathematics education scholarship to the works of John Dewey, Jerome Bruner, Theodore Sizer, and others (Cohen & Ball, 2000).

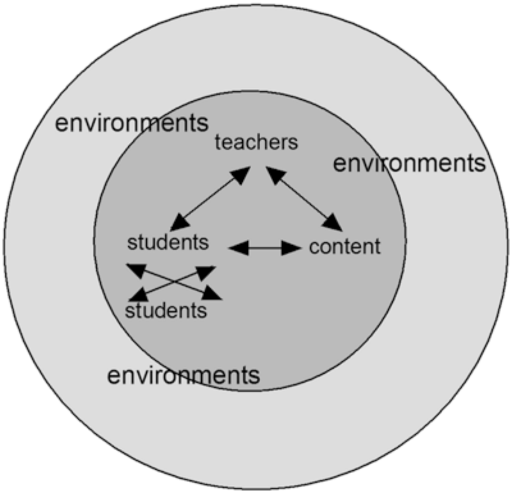

Fig. 3.2 Mathematics-instructional triad, with Cohen and Ball's (2000) focus on interaction

Not only does this positioning center the processes of formal and informal mathematics teaching and learning, but in terms of the diagrammatic representation of the framework, the triad represents a kind of coordinate point with respect to the social, spatial, and mathematics education "theory-moment" axes that we describe in the following sections. Furthermore, we embed this triad within these multiple levels and axes to acknowledge that the mathematics-instructional triad alone is a limited representation of the ways in which mathematics education unfolds amid sociohistorical and contemporary contexts (see Weissglass, 2002).

Ecological Rings of the Framework

In addition to the axes that situate the mathematics-instructional triad amid social, spatial, and sociohistorical considerations, we also locate the triad—and its associated network of practices—amid nested and reciprocally formative organizational fields in which mathematics teaching and learning occur (Arum, 2000; Martin, 2000; Weissglass, 2002). We consider the activity of mathematics teaching and learning within schooling systems (e.g., classrooms, schools, districts), communities, and at broader societal levels. Our attention to these levels also incorporates issues related to state regulation (e.g., Common Core State Standards, National Governors Association Center for Best Practices, and the Council of Chief State School Officers, 2010), professional associations (e.g., National Council of Teachers of Mathematics), market competition (e.g., choice and charter movements), and other institutional forces that shape and circumscribe school-level practices (Arum, 2000). We recognize, however, that this aspect of the framework should be further developed to address the nuances of particular contexts to which it may be applied—particularly, the various global contexts beyond the United States (from which this current articulation emerges).

Putting the Framework to Work: Analyzing Opportunity Initiatives

It has been (too) well documented that students in "veiled" urban schools (i.e., schools that are predominately Black/Brown and/or poor) experience inferior opportunities to learn mathematics (see, e.g., Berry, Ellis, & Hughes, 2014). Their race-classed counterparts in more affluent urban and suburban spaces more often enjoy both the human and material capital required for successful navigation of academic mathematics (Tate, 1996). Although race and class provide ready explanations for this disparity, what happens when we remove the veil to consider the particularizing role of the urban in this common narrative? What has it meant, historically, to consider urban mathematics education as a means toward addressing inequitable access to opportunity beyond the local mathematics classroom?

In a recent Iris M. Carl Equity Address at the NCTM annual meeting, William Tate (2016) revisited briefly a major urban mathematics education initiative during the 1990s, the urban systemic initiatives (USI) sponsored by the National Science Foundation (also see Borman et al., 2004; Knapp, 1997; Martin & Larnell, 2013; Stevenson, Dantley, & Holcomb, 1999). As Tate (2016) suggests, the USI represent the last major national effort that explicitly linked issues of mathematics teaching and learning with explicit intentions to address educational quality in urban school districts. The USI clearly exemplified the interpretivist-constructivist moment of the framework (also see Stinson & Bullock, 2012) in that it adopted the then-best practices concerning learner-centered theories of cognition and extended them into urban classroom environments yet did not fully acknowledge social factors related to the issues the USI sought to address. Although the connection to the "moments" axis is relatively clear, there remains a need to elaborate the connections to the spatial and social-signification elements. Our purpose here is not to exhaustively explore the utility of the framework but to consider how the USI may help to clarify the meanings of the various parts of the framework that are less familiar to the mathematics education audience.

Applying the Spatial Axis

Consistent with the central argument of this chapter, there was very little attention to spatial considerations within the original framing of the USI. Instead of drawing on complex themes such as connective pathways within urban boundaries or the various ways that urban spaces are rendered as empirical objects, locations were selected for inclusion in the USI strictly by way of child-poverty rates and population density (i.e., by selecting "22 of the 28 largest urban school districts in the United States" (Stevenson et al., 1999, p. 445)). This large-city formulation has persisted as a means to identify urban education spaces (see Milner, 2012).

Applying the framework's spatial axis—namely, the empirical-space element—we can problematize this limited empirical conceptualization of urban space and argue that child-poverty rates and population density are only two of many ways by which urban spaces are determinable. Other criteria combinations could be used to broaden or further delineate the types of spaces that are considered urban. Furthermore, we can also apply the place-space element of the framework and thereby question whether the organization of the USI incorporated any input or cooperation from the people who live in the communities being affected.

Lastly, applying the spatial dimension also yields questions and possibilities for new development related to image space and interactive-connective space, respectively: How did USI themselves contribute to the development of narrative representations for communities signified as urban within mathematics education, particularly deficit-oriented narratives? How did the USI address the disparate populations and resources across schools within districts by creating intradistrict resource and support networks?

Applying the Social-Signification Axis

The purpose of the USI was to create a district-level effort "to increase achievement in both mathematics and science for all students and close the achievement gap between mainstream and marginalized students throughout the district in question" (Borman et al., 2004, p. 250). Districts qualified to apply for USI funds based on their concentrations of child poverty. This formula systematically painted participating districts with a broad brush based on an *urban-pathological* view that portrays this space and its communities as impoverished, in need of external-interventional assistance, and perhaps more nefariously, portraying its adults as unfit to care for their children.

Paradoxically, however, there is an implicit reliance on an *urban-sophisticated* sensibility that these urban districts, representing the nation's largest populations as primary symbols of national progressive capacity, had the extant wherewithal to resolve achievement gaps in mathematics and science and that these spaces would then lead the country in equitable practice at a large scale. In other words, the USI relied on an understanding of urban spaces as pathologically poor and uneducated, yet the program also capitalized on a sophisticated sense of the urban district as "the meeting point of civilization" (Leonardo & Hunter, 2007, p. 782) from which success related to diversity in mathematics education would emanate. The USI's success was also attached to a sense of *urban authenticity* in that the project began with marginalized children in mind and was targeted toward increasing their opportunities for success without undermining their cultural practices. In this way, the USI differed from other efforts labeled as subtractive (Valenzuela, 1999) or ghetto (Anyon, 1997) schooling.

Reimagining Research Based on the Proposed Framework

Our intention here is not to evaluate USI directly or the research disseminated from the various USI projects. Rather, we use USI as a shared example from which we consider what such research could look like within our proposed framework.

It is essential to note, however, that while we have addressed how the USI projects connect to each axis of the framework it is not possible to retrofit these projects to align with the current framework. To do so would deny the framework's epistemological requirements and its power to not only evaluate extant research but also to create new questions and to propose new opportunities. Put differently, for an initiative like the USI to take hold today given the framing that we are proposing would require a *complete* rethinking of what those individual efforts would entail—ideologically and structurally.

We offer this framework for urban mathematics education as a means to avoid simply reforming our current practices by, instead, proposing a foundational shift. Stinson and Bullock (2015) remind us of the explicit, and nearly linear, connections among epistemology, theory, and methodology and, thereby, among data collection, analysis, and representation in research. This framework for urban mathematics education scholarship honors this connection and requires a holistic rethinking of the research process, not simply an intervention on one part of the process or another. To attend to urban mathematics education in ways that acknowledge mathematics, spatial-turn logic, and the social significations of urban, as well as the interplay among them, requires that scholars wrestle with these considerations at the center of their thinking throughout this process. For example, efforts to simply rethink representation, while important and possibly necessary, are not sufficient to meet this framework's requirements. Investigating urban mathematics education outside of the processes and practices of veiling that have become common in mathematics education research generates new questions and may require a rethinking of extant questions.

Given that the framework requires a full rethinking of the research process, our consideration of USI requires such a rethinking. The intention for the USI was to design mathematics education interventions targeted specifically toward urban spaces. However, as our framework has revealed, the USI did not fully engage the socio-spatial aspects of the urban. So, what could it mean to rethink it, and does the framework inform such a rethinking?

One possibility (again, re*imagining*) for approaching the USI through the framework could be to position urban-authentic elements of communities as central and the true source of how urban sophistication is conceptualized and conceived through policy efforts. In the same way that mathematics educators have argued against deficit views of communities in the broader field (e.g., Aguirre, Mayfield-Ingram, & Martin, 2013; Clark, Frank, & Davis, 2013), urban-pathologic views must also be relegated as unproductive for urban mathematics education (e.g., Gholson & Martin, 2014; Walker, 2007a, 2007b, 2012). Urban-pathologic views, however, should be recognized fully as outgrowths of systemic forms of oppression that rely on societal ideologies (e.g., capitalism) that also inform notions of urban sophistication. This is not to suggest that issues of privilege and oppression should not factor into urban mathematics education; rather, the opposite is true: instead of positioning these as issues that characterize communities, position them as systemic socialization forces with broad and generational effects that reinforce the fabric of urban life.

Conclusion

The terms "equity," "social justice," and "urban" may include conceptual points of intersection, but one little-explored commonality among them is the curious fact that in contemporary use the terms obscure as much as they signify. Many scholars have attempted to clarify the meaning and utility of equity within mathematics education discourse (e.g., Gutiérrez, 2002; Martin, 2003; Secada, 1989), and others have revisited the heritage of social justice discourse in mathematics education (e.g., Larnell, Bullock, & Jett, 2016), but few have engaged clarifying the conceptualization of urban mathematics education, despite its relatively quick growth as a domain of scholarship. In the absence of a clear and shared sense of what urban mathematics education is (i.e., what its contours and elements are, at least), the mathematics education research and practice community seems content with a taken-as-assumed sense of urban: that urban mathematics education represents efforts to import reformed teaching and learning practices and strategies to locales that have both densely populated spaces and a significant proportion of socioeconomically poor residents and/or residents of color.

In research particularly, consideration of the urban has been reduced to implied requirements related to data representation that include reporting demographic data that paints a picture of students and schools as stereotypically "urban" while neglecting a deeper interrogation of the space. This definitional shallowness may result from the nurturing of good intentions (or guilt), but the research community also may be close to hitting a wall in terms of the power of new contributions that may push development of urban mathematics education scholarship in ways that respond to ever-evolving concerns and contextual circumstances.

Our purpose in this chapter was—following Tate's (2008) call—to help the field to avoid this potential conceptual dead end by clarifying the relationship between the arrangements of spatial geography and distribution of social opportunity. Our framework draws on concepts from a variety of disciplines and places mathematics education concepts both at the center and as a central guiding dimension. The purpose is to allow researchers to pose and explore new kinds of questions related to elements of distinctively urban spaces—e.g., institutional structures, cultural practices, physical constructions, and capital production. As Tate suggested, the first step is theory building. The next step is challenge the "collective cognition" of those who would be responsible for carrying out the work of urban mathematics education (p. 7).

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Chapter 4
Building on "Misconceptions" and Students' Intuitions in Advanced Mathematics

Aditya P. Adiredja

Abstract The goal of this chapter is to challenge deficit perspectives about students and their knowledge. I argue that predominant reliance on formal procedural knowledge in most undergraduate mathematics curricula and the oftentimes focus on students' misconceptions contribute to the racialized and gendered inequities in mathematics education. I discuss my design of an instructional tool to learn the formal limit definition in calculus called the Pancake Story. The story builds on a misconception and student's everyday intuitions. A successful sensemaking episode by a Chicana student illustrates the utility of everyday intuitions leveraged in the story and the inaccuracy and harm of the notion of "misconceptions." Recognizing misconceptions as students' attempts to make sense of mathematics, solidifying such knowledge by finding an appropriate context for it, and leveraging other knowledge resources are explicit ways to challenge dominant power structures in our practice.

Introduction

Most instructional practices in undergraduate mathematics tend to primarily focus on developing students' formal mathematical knowledge. Learning is mainly seen as the process of building formal mathematical knowledge on top of formal mathematical knowledge (e.g., Cottrill et al., 1996). These perspectives reflect the general values of the advanced mathematical community for pure abstraction and logic, efficiency, and elegance (Harel & Sowder, 2005; Weber, Inglis, & Mejia-Ramos, 2014).

Scholars have, however, critiqued the implicit assumptions about the cultural objectivity and neutrality of these values (Bishop, 1990; Frankenstein, 1983). The dominant focus on formal knowledge plays a major role in demarcating and excluding students from learning opportunities. Those who do not have the requisite formal knowledge are positioned as "not ready" or not "math students." Students

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who are labeled as having the requisite formal knowledge are seen as "belonging" or "smart" and are given more opportunities to learn mathematics (Gutiérrez & Dixon-Roman, 2011; Herzig, 2004). This de-emphasizes the inequities that have made the requisite knowledge more available to some groups than others (Oakes, 1990). The use of formal mathematical knowledge as a gatekeeper (e.g., Adelman, 2006) serves as an example of how those deemed as having formal mathematical knowledge can exert power in society (Valero, 2004).

Deficit perspectives about students and their knowledge exacerbate the impact of demarcating students based on their possession of formal requisite knowledge. Students' attempts to make sense of this formal knowledge, at times without the support of the requisite formal knowledge, are often characterized as "misconceptions" or deficient. Despite some scholars' attempts to dispel the field's fixation with students' misconceptions, and their attempts to replace misconceptions with other terms (Smith, diSessa, & Roschelle, 1993), it is still used most prevalently in discussions about mathematics education.[1] This might reflect the utility of the phrase to describe our attempts to consider students' thinking in instruction.

In undergraduate mathematics education research, students' informal knowledge has been positioned as against or in competition with formal knowledge. Tall and Vinner's (1981) widely used framework, for example, of *concept image* and *concept definition*, focuses on the distinction between formal mathematical knowledge (definition) and other non-formal knowledge and intuitions (image). Przenioslo (2004) also provides an example of ways that the two types of knowledge have been discussed:

> Students would not notice a *contradiction* between [the formal] definition and his or her other, more "private" conceptions, and worse, would not try to *confront* the two parts of his or her knowledge. More importantly still, for the majority of students the definition was not the most significant element of the image … This could be a consequence of unsatisfactory understanding and inability to interpret the very formulation of the definition. (emphasis added, p. 129)

Davis and Vinner (1986) shared similar sentiments: "It was not possession of correct ideas that was in question, so much as the possible presence of incorrect naïve ideas, and the outcome of the competition between the two" (p. 294). These quotes reflect general deficit narratives about student thinking as generally naïve, unrefined, and full of misconceptions that are common narratives in undergraduate mathematics education. Some contemporary researchers have focused their work on building a formal mathematical understanding on students' informal knowledge (e.g., Wawro et al., 2012; Zandieh et al., 2008; Zandieh & Rasmussen, 2010). The work in this chapter builds on these existing works.

Our tendencies to prioritize requisite formal knowledge lead to deficit perspectives on student thinking which directly contribute to racialized, gendered, and classed inequities in mathematics education. Martin, Gholson, and Leonard (2010) similarly pointed out that research studies that focus on the mathematical content of teaching and learning often ignore the sociopolitical contexts of learning. This

[1] A search for "students' misconceptions in mathematics" on Google scholar resulted in 31,900 publications since 2000. Since 2016, there were about 13,100 results.

chapter discusses the way that mathematics has been used to demarcate students' intelligence. It recognizes that deficit perspectives about students intersect with broader deficit narratives about the intellectual and academic inferiority of students of color and/or women in mathematics (Solórzano & Yosso, 2002).

Studies of cognition share the power to determine what counts as productive knowledge, how learning is supposed to happen, and what kinds of students benefit in the process (Apple, 1992; Gutiérrez, 2013; Nasir, Hand, & Taylor, 2008). Because intellectual capacity in learning mathematics is the heart of deficit narratives about students of color and/or women, cognitive studies play a critical role in challenging this narrative. The goal of this chapter is to challenge dominant power structures in mathematics education. First, it challenges our tendency to rely on formal knowledge in teaching mathematics. Second, it challenges the fixation with misconceptions and deficiencies and its implication in the positioning of students. This chapter presents ways that knowledge from students' everyday experiences can be productive in learning formal mathematics beyond serving as a cover story. Specifically, this chapter uses an episode of successful sensemaking of the formal limit definition from calculus by a Chicana undergraduate student to evidence this position.

Theoretical Frameworks

Two theoretical frameworks guide the work of this chapter: the *sociopolitical perspective* and *Knowledge in Pieces*. The sociopolitical perspective (Gutiérrez, 2013, Valero, 2004) serves as the framework for equity and focuses the chapter on ways that issues of power are related to learning. Knowledge in Pieces (KiP; diSessa, 1993) serves as the framework for cognition. It guided the design of the instructional tool discussed in this chapter. More importantly, it is a cognitive framework that is explicit about its anti-deficit view on students' knowledge.

Sociopolitical Perspective

The sociopolitical perspective assumes that knowledge, power, and identity are interrelated and "arising from (and constituted within) social discourses" (Gutiérrez, 2013, p. 40). This perspective acknowledges mathematics as a human practice that is "inherently political, rife with issues of domination and power" (p. 40), and foregrounds the political aspects of research and practice in mathematics education, with particular attention to issues of power. Adopting such a perspective involves "uncovering the taken-for-granted rules and ways of operating that privilege some individuals and exclude others" (p. 40) and continuing to challenge threatening narratives (e.g., stereotypes) about students from particular backgrounds and their experiences (Larnell, 2013). In this chapter, the sociopolitical perspective supports my critique of mathematics instructors' practices primarily relying on students' formal requisite knowledge and their focus on misconceptions, which have implications for students' identity development. To challenge the focus on "deficits" of students from underrepresented groups in mathematics, I focus on understanding resources students use that support their sensemaking of advanced mathematical topics (Valencia, 2010). KiP guides this analysis.

Knowledge in Pieces (KiP)

KiP conceptualized knowledge as a system of diverse elements and complex connections (diSessa, 1993). The analysis in this chapter focused on the nature of the elements, their diversity, and connections. KiP viewed characterizing knowledge using ideas with larger grain size like "concept" or the commonly used idea of "misconceptions" as analytically uninformative and unproductive in teaching (Smith et al., 1993). KiP has taken an anti-deficit approach to knowledge and focused on the context sensitivity of knowledge to maintain the *productivity* of a particular piece of knowledge (Smith et al., 1993). This means that the productivity of a piece of knowledge is highly dependent on the context in which it is used. For example, the knowledge that "multiplication makes a number bigger" is productive in the context of multiplication with numbers larger than one. The knowledge is not productive in the context of multiplication with all real numbers.

In contrast to studies that focus on identifying and replacing students' misconceptions, KiP focuses on building new knowledge on students' prior knowledge (Smith et al., 1993). Studies using KiP have shown that students have intuitive knowledge that can be leveraged in instruction (Campbell, 2011; Pratt & Noss, 2002). KiP also paid particular attention to the continuity of knowledge, i.e., ways that knowledge gets used or built upon in new contexts. It is common for studies that have used this framework to have uncovered productive sensemaking behind students' use of nonnormative language to describe their reasoning (e.g., Campbell, 2011; diSessa, 2014). These theoretical assumptions are operationalized in the analysis framework section.

Methods

The data used for the illustration in this chapter came from a larger study investigating the role of intuitive knowledge in learning the formal definition of a limit in calculus (Adiredja, 2014). Twenty-four students at a large public research university participated in individual interviews that lasted 2–3 hours. The larger study also aimed to investigate the impact of an instructional tool, the Pancake Story, on students' understanding of the formal definition.

The analysis for this chapter focused on a sensemaking episode by one student, Adriana. At the time of the interview, Adriana was a mathematics and Chicanx studies major. Adriana self-identified as Chicana on a background survey administered at the end of the interview. Adriana was selected because she is representative of the majority of the students in the larger study whose understanding improved after engaging with the instructional intervention. She was also articulate about the way her understanding changed during the interview. I selected Adriana as the case in this chapter to centralize the sensemaking of a woman of color (Bell, Orbe, Drummon, & Camara, 2000). It is important to note that the goal was not to essentialize all Chicana students nor to uncover some commonality of "cultural ways of reasoning" (Gutiérrez & Rogoff, 2003) by Chicana students. Rather, the selection supports two broader goals. First, it maintains the connections between cognition and the sociopolitical contexts (e.g., experiences related to race and gender) associated with the student (Martin et al., 2010), even when such contexts might not be the focus of the analysis. Second, it allows for this study to draw implications about learning from the experience and behavior of a woman of color. The lack of reporting of students' backgrounds in much mathematics education research maintains the implicit assumption that findings are taken from studying White and male students and implications are drawn from the experiences and behaviors of students from this population (Martin, 2013; Nasir, 2013). The selection of Adriana's successful case thus challenged broader deficit narratives about women of color in mathematics (Adiredja, in press).

Mathematical Context and Brief Literature Review

The formal definition of a limit of a function at a point (hereafter "formal definition") has been an essential topic in mathematics majors' development. Students are often first introduced to the formal definition in calculus. The formal definition states that the limit of a function $f(x)$ as x approaches a is L (see Fig. 4.1), if and only if, for every positive number ε, there exists a positive number δ, such that all numbers x that are within δ of a (but not equal to a) yield $f(x)$ values that are within ε of the limit L.[2]

Informally, one might say, "If L is the limit, then for however close one wants $f(x)$ to be to L, one can constrain the x-values so that $f(x)$ would satisfy the given constraint." We return to this intuitive idea shortly.

Students' difficulty in understanding the formal definition has been widely documented in research and practice (e.g., Fernández, 2004; Tall & Vinner, 1981; Williams, 1991). Studies that have discussed students' difficulties with the formal definition have focused on broader misconceptions about limit that prevent students from understanding the formal definition (Davis & Vinner, 1986; Parameswaran, 2007;

Fig. 4.1 Limit of a function

$$\lim_{x\to a} f(x) = L$$

[2] This defining property is usually written as for every number $\varepsilon > 0$, there exists a number $\delta > 0$ such that if $0 < |x - a| < \delta$ then $|f(x) - L| < \varepsilon$.

Przenioslo, 2004; Tall & Vinner, 1981; Williams, 1991). A few studies that have taken an anti-deficit perspective of students' existing conceptions have uncovered important aspects of understanding the logical structure of the formal definition (Boester, 2008; Swinyard, 2011). These studies found that one of the more challenging yet important conceptual ideas was the relationship between epsilon and delta. The "temporal order" (Davis & Vinner, 1986, p. 295) of delta and epsilon (hereafter "temporal order") refers to the relationship between delta and epsilon, which reflects an important logical structure behind the formal definition. Within the definition, "for every number $\varepsilon > 0$, there exists a number $\delta > 0$" expresses this relationship.

The temporal order of δ and ε in the definition can be intuitively described using the idea of quality control in manufacturing an item. Given a permissible error in the measurement of the output (ε), one determines a way to control the input to achieve that result. One does so by determining the permissible error in the measurement of the input (δ) based on the given parameter for the output (ε).The error bound for the input is dependent on the given error bound for the output. Epsilon can be seen as the error bound of the output, whereas delta is the error bound for the input. Therefore, δ depends on ε.

Instructional Design: The Pancake Story

I designed the Pancake Story (see Fig. 4.2) to leverage students' intuitive knowledge about quality control in making sense of the formal limit definition. Boester's (2008) *Bolt Problem* and Oehrtman's (2008) approximation metaphor inspired The Pancake Story. The Bolt Problem leveraged intuitions about quality control of manufacturing

The Pancake Story

You work at a famous pancake house that's known to make pancakes with 5" diameter. To make the perfect 5" pancake you would use exactly 1 cup of batter. On your first day of work your boss told you that it is practically impossible for you to be able to use exactly one cup to make the perfect 5 inches given how many and how fast you will be making these pancakes. So for now, since you're new, as long as your pancakes are anywhere within ½" from the 5", he won't fire you. Your job is then to figure out the maximum you can be off from the 1 cup to still make pancakes that meet your boss' standard. Specifically, given that your boss gave you the ½ an inch error bound for the size, you need to figure out the error bound for the batter so that your pancakes won't be off more than the given error bound.

According to the work manual, there are two steps to do this. Based on the error bound for the size, you first need to guessan error bound for the amount of batter. *Then*, you have to check to see if using any amount of batter that is within the error bound from the 1 cup would make pancakes that are within the given error bound from the 5". For example, suppose based on the ½ inch error bound, you guessed ⅙ of a cup error bound for the amount of batter. Then you check to see if using any amount of batter that is within ⅙ of a cup from the 1 cup, so between ⅚ and 1⅙ of a cup would make pancakes with size somewhere between 4½" and 5½", that is within the ½" error bound from 5".

Over time, your boss expects you to be even more precise. So instead of ½" error bound from 5", he says he wants you to make pancakes that are within some ridiculously small error bound from 5", but you don't know what it's going to be. This means while he started by asking you to be within ½," later he might want ¼" or 1/1000" from 5". Your job then becomes for however close your boss wants the pancake to 5", you need to figure out the maximum you can be off from 1 cup of batter such that if you use any amount of batter that is within that error bound from the 1 cup then your actual pancakes will still be within whatever error bound your boss gives you from the 5".

Now, you don't want to spend time each morning to recalculate everything. So you will try to come up with a way to calculate an error bound for the batter based on whatever the given error bound for the size.

Fig. 4.2 The pancake story

bolts for different clients, like home improvement clients versus NASA. The Pancake Story illustrated the key elements in the formal definition in what I assume to be a more familiar context of pancake making than bolt making. The Pancake Story also incorporates language related to error and error bound as in the approximation metaphor.

I designed the story with two differing principles from the metaphor. First, the story focuses on the single topic of limit definition of functions. The approximation metaphor prioritizes the goal of identifying a generalizable abstract structure that can be used with other related concepts associated with limits (e.g., Taylor series). Guided by KiP's assumption that knowledge is context sensitive, the design included one particular topic, attending to the details of this definition (e.g., logical structure of the mathematical statements). Second, in addition to leveraging a "real-life" context, the story explicitly incorporated common ways students made sense of the topic. To assist with the design of the story, I interviewed students about different ways that they made sense of the formal definition (Adiredja & James, 2013). This allowed me to recognize dominant (intuitive) ideas in students' attempts to understand the limit definition.

These interviews revealed two dominant ideas. First, a number of students understood the logical structure of logical implication statements (If A, then B) but incorrectly applied it to the relationship between epsilon and delta. They read the statement "if $0 < | x - a | < \delta$ then $| f(x) - L | < \varepsilon$" to say if delta then epsilon, so epsilon had to depend on delta. Second, many students used the functional dependence between x and $f(x)$ to make sense of the relationship between epsilon and delta. They argued that since epsilon was associated with $f(x)$ and delta was with x, then knowing that $f(x)$ always depended on x, epsilon must also depend on delta. These dominant ideas could easily be dismissed as "misconceptions." I instead viewed them as misapplication of useful knowledge elements. So, instead of preventing their use, I determined what the useful idea(s) behind their assertion were and identified them as *knowledge resources*. I incorporated these resources into the story by finding a context where they could be used productively. The next section specifies the process of identifying knowledge resources.

The story specifically focuses on the logic behind the statement "for every number $\varepsilon > 0$, there exists a number $\delta > 0$" and distinguishing it from the statement "if $0 < | x - a | < \delta$ then $| f(x) - L | < \varepsilon$." The story offers the language of error and error bound to help students differentiate between ε and $| f(x) - L |$ and δ and $| x - a |$ and their respective relationships. Error in the output does depend on the error in the input, but the boss in the story gives or specifies the permissible error bound in the size to determine the error bound in the amount of batter. In this way, the story gives a productive context for the *functional dependence* resource while leveraging students' familiarity with the resource *quality control*. The story emphasizes that epsilon is given, highlighting the *givenness* resource.

The story treats the two statements "for every number $\varepsilon > 0$, there exists a number $\delta > 0$" and "if $0 < | x - a | < \delta$ then $| f(x) - L | < \varepsilon$," as two separate steps in the worker's manual. First, the employee is given the specification to find the error bound in the input. Second, the employee is to check whether using any amount of batter within the discovered bound would make pancakes within the boss' specification. The story also includes an example of the whole process with a particular error bound. The goal was to assist students to recognize the appropriate relationship between delta and epsilon as a structure underlying the formal definition.[3]

Analytical Framework: Knowledge Resources and Counter-Models

A knowledge resource is a unit of ideas consisting of a single or a small collection of interrelated knowledge elements with utilities in multiple contexts. Table 4.1 includes some of the relevant knowledge resources that I have identified (Adiredja, 2014). As part of a person's prior knowledge, a knowledge resource can be intuitive in nature and stem from their physical experiences in the everyday world. Other knowledge resources might be ideas with origin in a person's prior learning experiences. For example, the resource *proportional variation* might stem from people's

Table 4.1 Glossary of some knowledge resources

Knowledge resource	Description
Functional dependence	In an input-output relationship, the input directly determines the output. The relationship has a clear direction that the input determines the output, but not vice-versa The resource is reinforced by students' experiences learning about functions. The y or $f(x)$ is often treated as the dependent variable and x the independent variable
Function slots	These resources use the stipulation that a function is a relationship between x and $f(x)$ or y. This resource supposes that when two quantities share a functional relationship, one quantity is the x and the other is the $f(x)$ or the y
Givenness	A characteristic of being specified in advanced. A *given* quantity is one whose properties are previously stipulated, and therefore its determination is not relevant for further examination Any mention of a quantity being previously set is a version of this resource. Students often assume that the variable x or the domain of any function is a given
Proportional variation	A small change in the input leads to a small change in the output. This may be an application of the more physically intuitive *Ohm's p-prim* (diSessa, 1993) into input/output relationship. Ohm's p-prim says bigger effort begets bigger result, and consequently smaller effort begets smaller result
Quality control	The idea of controlling the input values of a function in order to meet a given specification (e.g., error bound) for the output values This resource involves the resource **givenness** of the constraint for the desired output, but it also emphasizes the modification of the input to satisfy the given constraint

[3] The story is not to be solved mathematically because it lacks sufficient information (e.g., constraint on the thickness of the pancakes). The numbers were selected to connect with the limit of a linear function, $f(x)= 3x+2$ at $x= 1$. I fully recognize that a function involved in the story is not linear and that one cup of batter makes an extremely thick 5 inch pancake!

physical experiences of more effort begets more result, whereas *functional dependence* is also often discussed in mathematics classrooms as functional relationship.

The utility of knowledge resources comes from their small grain size. It allowed my analysis to recognize the continuity of forms of Adriana's knowledge. Thus changes in claims were not treated as Adriana simply changing her mind. Instead, knowledge resources were repurposed and applied in a new context.

Theoretical assumptions from KiP about knowledge and some methodological orientations assisted in the identification of knowledge resources in the analysis:

Knowledge resources are context sensitive, and so they are neutral with respect to correctness. Incorrectness is a result of misapplications of a knowledge resource in a new context. The neutrality assumption also distinguishes knowledge resources from larger and more complex ideas. Assertions or claims are not a knowledge resource because they can be incorrect. For example, the assertion, "epsilon depends on delta because epsilon is y and delta is x" is not a knowledge resource because it is incorrect. However, the claim includes the use of the knowledge resource *functional dependence*, which is neutral with respect to correctness.

Unpacking ideas behind assertions and their development directed the analysis toward identifying knowledge resources. The analysis triangulated the meaning of a student's assertion with other instances during the interview where the student made similar assertions. It also used similar expressions of ideas by other students to triangulate the meaning of the student's assertion. The analysis did not assume that they reflect the same knowledge resource. In fact, analyzing the degree of consistency can reveal the use of a knowledge resource in the particular context.

The use and articulation of a knowledge resource might vary depending on the contexts in which they are cued. The use of a resource is influenced by their *cueing priority* and *reliability priority* (diSessa, 1993). High cueing priority means that an element is more likely to be activated when other elements that are consistent with it are already activated. For example, the idea of functional dependence might have a high cueing priority when students are talking about functions and relationships between variables x and y. When a knowledge resource is taken to be understood, its need to be articulated can diminish. However, a knowledge resource does not get "abandoned" from a student's conceptual ecology.

Building a "best fit" model using *competitive argumentation* (Schoenfeld, Smith, & Arcavi, 1993; VanLehn, Brown, & Greeno, 1984). The analysis constructed a model of the student's argument, which included the role for each knowledge resource in the particular context. Competitive argumentation is illustrated explicitly in this chapter with the use of *counter-models*. The counter-models serve as competing hypotheses for the way that the student put together different knowledge resources.

Knowledge resource as a construct is larger in grain size compared to phenomenological primitives (p-prims) (diSessa, 1993). P-prims are small, self-explanatory knowledge elements that describe and are established out of everyday experience. As described in diSessa (1993), p-prims are *phenomenological* in that the explanations are drawn from the behavior of things that people experience and observe. They are *primitives* in that they are self-explanatory, and so they are, in that sense, atomic level knowledge structures. In justifying the validity of a particular p-prim, people often respond with, "That's just the way it is." Some knowledge resources share the primitive nature of p-prims.

The analysis in the data illustration is technical and complex. This is because Adriana's way of making sense of the topic is complex. The theoretical tools and level of details helped us understand the reasoning behind Adriana's claim. In the discussion section, I compare the analysis with how labels of "misconceptions" would have misled our understanding of Adriana's sensemaking and could actually be harmful to our perception of Adriana as a student. The analysis illustrates the way that Adriana needed to align her prior knowledge with the new resources that the story leveraged. It was not until she was able to align her prior knowledge that she took up the resources that the story leveraged.

The Case of Adriana

Adriana's sensemaking about the temporal order after engaging with the Pancake Story is split into four episodes. The excerpt below is episode 4 and occurred toward the end of her discussion about the dependence between epsilon and delta. Adriana took some time to reconcile the ideas from the story and her prior knowledge in the first three episodes. This was her final explanation about the temporal order. Bolded terms are phrases or claims that represent ideas from which the analysis inferred knowledge resources. Italicized terms in the analysis are knowledge resources.

Interviewer: So, do they [epsilon and delta] depend on each other, is it just one way now?

Adriana: Um, see cus I was looking at it like … **the f of x [*f* (*x*)] depends on the *x*** and that's how I was like saying that epsilon depends on delta because **epsilon is related to the f of x** [$f(x)$]…. But that's just saying **the error of the *L* and the f of x [*f* (*x*)] depends on the *a* and *x*** but that's not to say that epsilon depends on delta.

Interviewer: Ok, so?

Adriana: So, I think that delta depends on epsilon now [*laughs*]. Just cus if it's given like this [reference unclear] and **you're trying to aim at getting … within a certain error bound**, then you're gonna **try to manipulate your entries … to be within a certain error bound** [*gestures a small horizontal interval with her palms*]

Interviewer: Ok. Alright, so and so you changed your mind it seems? Um, so how did that happen? Why did you change your mind?

Adriana: Because I was **given an epsilon** [*points at the inequality* $4.5 < f(x) - L < 5.5$[4]] and that's kinda like the main goal. **The main goal is to get the pancake, … and they gave me a constraint** … and … they **didn't give me an error bound for the batter** or for like the *a* or *x*, they didn't give me an error bound. But I know I **want to make it small so that it's within the error bound, the epsilon**. So then **I would kinda base my delta on what was epsilon**.

Counter Model

Adriana realized and corrected her mistake in using *functional dependence* to determine the temporal order for epsilon and delta. She realized that she had loosely associated epsilon with the function, $f(x)$, and delta with x (*function slots*), and had incorrectly concluded that epsilon depended on delta. With a new understanding of epsilon and delta as error bounds, she replaced her incorrect use of *functional dependence* to describe the temporal order with *quality control*. With resources from the story prioritized, Adriana concluded that delta depended on epsilon.

The counter model would have been a nice success story. Adriana shared a very common "misconception" among students in the study. She erroneously treated epsilon as $f(x)$ values and delta as x values and relied on *functional dependence* to describe their relationship. However, she realized her mistake and changed her mind. She ended with a correct understanding about the temporal order. The analysis using the framework for this chapter revealed a different model that more accurately represents her reasoning.

This is not a story about a student correcting a misconception but a process of a student aligning ideas she gained from a story with her prior knowledge. The analysis identified the actual conflict. By going to the knowledge resource level, the analysis revealed that the fact that epsilon was given in the story conflicted with the purpose of epsilon in providing a range of acceptable output values. A range of acceptable output values was given (because epsilon was given), but such range of values was also influenced by what input values one used. Adriana understood that outputs depended on inputs and that she could not use that idea to describe the relationship between epsilon and delta. She ultimately resolved the conflict only after she found a way to use the *functional dependence* relationship productively.

[4] This inequality is incorrect. The inequality should have been $4.5 < f(x) < 5.5$, but Adriana arrived at this inequality in using the $|f(x) - L| < \varepsilon$ from the statement of the definition.

Adriana had a particular conceptualization of epsilon as an error bound. In prior episodes, Adriana never once loosely related epsilon to $f(x)$ and delta to x as a result of using *function slots* resource as was suggested by the counter model. Adriana consistently inferred the error bound, epsilon, from the inequality $4.5 < f(x)–L < 5.5$ without explicitly stating that epsilon was 0.5. To Adriana, the inequality represented both a range of acceptable $f(x)$ values and the existence of an error bound. Adriana focused on what epsilon was for rather than what it was. To Adriana, epsilon's main purpose as an error bound was to constrain acceptable output values.

Excerpts from this episode suggested that this was the conception she continued to use until the end. Her use of "within a certain error bound" suggested the use of a range of values in conceptualizing epsilon. Adriana said, "You aim at getting within a certain error bound," instead of getting the errors to be less than the error bound. She also said, "manipulating your entries to be within a certain error bound," instead of manipulating your entries to be within a range of acceptable values. Epsilon was an error bound that was conceptualized through a range of acceptable $f(x)$ values it provided.

It is important to note that while there might have been other ways to accurately infer error bounds, Adriana's way of inferring error bounds from the range of values was also accurate. Aside from the incorrect simplification of $| f(x) - 5 | < 0.5$ to $4.5 < f(x) - L < 5.5$ instead of $4.5 < f(x) <5.5$, she was able to correctly and consistently infer the appropriate error bound from the range of acceptable values. This conceptualization influenced the way Adriana interacted with some of the ideas presented in the story.

The error $(f(x) - L)$, a range of acceptable $f(x)$ values, and the error bound ($\varepsilon=0.5$) were all encoded in the same inequality, $4.5 < f(x)\text{-}L < 5.5$. This fact prioritized epsilon's connection with $f(x)$ values and the dependence of $f(x)$ values on the x values. This conflicted with the fact that epsilon was given in the story. Adriana consistently noted that epsilon was given in the story in all three episodes leading up to episode 4, and this fact did not settle well with her. Back in episode 1, she said, "But, … if epsilon's already set then you'll manipulate your … delta so it's within an error bound and/…/then continue to manipu- wait [*long pause*] wait, so you're … hm." In episode 3, Adriana specifically mentioned her confusion with the fact that epsilon was given. She said, "I'm confused because they gave me an epsilon and it's already set." She pointed at the inequality, $4.5 < f(x) - L < 5.5$, when she stated that epsilon was set. The fact that epsilon was already given conflicted with the idea that $f(x)$ values depended on the x values. The range of acceptable $f(x)$ values was already given, yet it was also dependent on the x values used.

This conflict was noticeably absent in the fourth episode. Adriana did not mention any confusion. One possible explanation might be that she successfully distinguished error from error bound by episode 4. Adriana did spend some time earlier in the interview distinguishing between error and error bound. It was not deemed sufficient, however, to resolve the issue. In episode two, she made that distinction but was still unsettled about the givenness of epsilon. She said, "I thought that the epsilon and the delta were the errors but they're the error bounds. … I guess if your epsilon is already set then your delta would depend on epsilon [*silence*]." She ended that episode saying that she was confused.

Tracking the use of knowledge resources across the four episodes revealed that while the *givenness* of epsilon was present and caused conflict in all previous episodes leading up to episode 4, *functional dependence* coincidentally was only brought up in episode 1 and 4. In episode 1 the *functional dependence* resource actually drove her claim about the temporal order despite her mention of the *givenness* of epsilon. She said, "Mostly whatever you're putting in to your x is gonna determine what you get for $f(x)$ … Epsilon depends on delta." The only new information Adriana mentioned in episode 4 was "the error of the L and the f of x [$f(x)$] depends on the a and x." So Adriana's use of *functional dependence* to describe the relationship between the errors ($f(x)$-L and x-a) somehow allowed her to resolve the conflict with the *givenness* of epsilon. We turn to KiP to explain how the conflict was resolved.

KiP posits that *functional dependence* was a knowledge resource that had high reliability priority and high cueing priority in Adriana's discussion of the temporal order. It had high reliability because it was a piece of knowledge that Adriana likely had used in high frequency as a mathematics major. More importantly, it had been productive in different learning contexts. It also had high cueing priority because she encoded epsilon through the inequality that involved $f(x)$. That is, the "$f(x)$" in the inequality $4.5 < f(x)-L < 5.5$ likely cued knowledge about function, which included the *functional dependence* relationship. In non-theoretical terms, Adriana knew that this was productive knowledge and that it was relevant in this context. She did not have a context where she could use *functional dependence* productively in episodes 2 and 3.

Adriana, in considering the *givenness* of epsilon, recognized that the application of *functional dependence* to epsilon and delta was incorrect. This was because if epsilon was given, then delta would have to depend on epsilon. So Adriana needed a context where this generally productive and relevant resource could be used productively in this discussion. She found it with the errors. This effectively aligned her prior knowledge with productive resources from the story and leveraged *givenness* of epsilon and *quality control* to describe the temporal order.

It is worth noting that Adriana had already understood those resources from the story. She had been able to articulate them since episode 2: "If I said epsilon was an error bound and if they already give me an error bound, I want … my result to be within this error bound here [*circles the inequality* $4.5<f(x)-L<5.5$] then … I would try to manipulate my errors here [*points to a small range on the x axis on the graph*] to be within a smaller error bound [*points at delta in the delta inequality in the definition*]." She stated something similar in episode 3.

She also noticeably switched her language when explaining the reason for the change in her thinking. When I first asked the temporal order question, she did not mention pancakes or batter in her explanation. She did use the terms "error" and "error bound," which were terms introduced in the story. In explaining the change in her thinking, however, she used parts of the story more explicitly. For example, she referred to satisfying epsilon as the goal of making pancakes within a specified constraint.

Final Model

In a final model, Adriana repurposed *functional dependence* to describe the relationship between the errors. This move helped Adriana resolve a critical conflict that she had in previous episodes about the *givenness* of epsilon. Adriana understood the implication of both *givenness* of epsilon and *quality control* on the temporal order since previous episodes. Having addressed her conflict, she prioritized those resources and determined the relationship between ε and δ. Adriana also made a productive observation that delta was not given, showing her use of the *givenness* resource with delta. She used *proportional variation* in arguing that she wanted delta to be small to accommodate a small epsilon. Adriana recognized that epsilon and delta are error bounds, not errors, and she maintained her conceptualization of the error bounds based on a range of function and input values.

Discussion and Implications

The Pancake Story was designed explicitly to incorporate common ways of reasoning and to use a context and language accessible to students. It sought to build on and help reorganize students' thinking about the topic. The story was, however, just a tool for Adriana to use. The analysis illustrates the way Adriana successfully made sense of the temporal order using the story. In addition to getting access to knowledge resources in the story, like *quality control* and *givenness* of epsilon, Adriana benefitted from the context and the language of the Pancake Story. The story and Adriana's interaction with it show an example of the utility of knowledge (resources) from students' everyday lives in learning advanced mathematics.

The framework offered in this chapter proved productive in understanding Adriana's sensemaking of the topic with the story. Staying at the knowledge resource level allowed the analysis to recognize the versatility of knowledge resources and how they interacted with the context and with one another. We observed how Adriana's conceptualization of epsilon might have increased the priority of *functional dependence* for the temporal order. We also observed how *functional dependence* conflicted with *givenness* of epsilon from the story and the way Adriana repurposed that knowledge resource to describe the relationship between the errors. An important takeaway from Adriana's case is the critical importance of attending to students' ideas, especially those that have high reliability priority like the *functional dependence*.

The analysis highlights that knowledge elements with high reliability cannot be treated as "misconceptions" and simply replaced with the "correct" idea. Such knowledge elements might demand a context in which they can be used productively. Adriana knew how to use *quality control* and *givenness* from the story to make sense of the temporal order almost immediately after she engaged with the story. She also knew that *functional dependence* was relevant in this context because she was working with functions. It was not until she found a context in which she could use *functional dependence* productively that she prioritized the use of the productive resources from the story. This finding opens the discussion about the inaccuracy *and* harm of focusing on students' misconceptions.

There were a number of instances when Adriana's reasoning could be classified as involving misconceptions. Consider Adriana's use of *functional dependence* to determine the temporal order. Focusing on it as a misconception, we would look for the source of the mistake. We might guess that the issue was with her understanding of epsilon and delta. We would then learn that Adriana used the inequality $4.5 < f(x)\text{-}L < 5.5$ to infer epsilon. We first would notice that there was an error in algebraic manipulation. Then we might focus on how such a conceptualization might be the reason for her misconception. We might then try to fix those errors and suggest an alternate way to approach the problem by clarifying that epsilon and delta were error bounds. We might even reiterate the *quality control* idea from the story or the fact that epsilon was *given.*

The analysis in this chapter shows that this approach would not work to help Adriana make sense of the topic. What we identified as issues were not issues for Adriana. She understood the story and what epsilon was. Aside from an error in simplifying the expression, there was not really an issue in her understanding. Focusing on misconceptions fixates us on students' mistakes and positions us further away from identifying the real issue that students might be grappling with.

Identifying mistakes and errors seems to be a quick way to attend to our students' thinking. Attempting to do the analysis of this chapter in the classroom would also take a lot of time and energy. In addition to not identifying the real issue, such an approach also would not have recognized anything that Adriana was doing right. The more time spent on identifying misconceptions, the less time we have to recognize what students have understood.

Let us step back and consider the sociopolitical context and the interrelatedness of power, knowledge, and identity in the interaction with Adriana. Adriana is a Chicana student studying mathematics. I included Adriana's gender and ethnicity as a way to *centralize* the sensemaking of a woman of color in advanced mathematics (Bell et al., 2000) and partly to construct counternarratives about women of color in mathematics (Adiredja, in press). Including her background allowed for consideration of the impact of a fixation on misconceptions of students whose knowledge has historically been and continues to be marginalized. By focusing solely on their misconceptions, we are highlighting what they do not understand, thus reinforcing deficit narratives about students of color and/or women in mathematics. This is partly why challenging deficit narratives about mathematical learning in the broader field has been included as part of equitable teaching (Bartell et al., 2017) and in position statements about social justice in mathematics education (e.g., National Council of Supervisors of Mathematics (NCSM) & TODOS: Mathematics for All, 2016).

Considering the interrelatedness of knowledge, power, and identity, it is also worth noting that the Pancake Story privileges what I, as a teacher and researcher, considered to be helpful for students. I designed the story to prioritize certain ideas (e.g., quality control), but de-emphasized others (e.g., relationship between volume and diameter). Although I incorporated information from student interviews in designing the story, I still made assumptions about the degree of familiarity students might have and ways that students might make sense of the story. The story did help many students in the study to make sense of the temporal order (Adiredja, 2014), but it likely did not resonate equally with all the students (Tate, 1994). Allowing students to come up with similar stories could help support students' positive mathematical identity development by incorporating students' thinking in learning. In a different study, a colleague and I explored such ideas with a group of women of color STEM students. The women in the study constructed novel and useful stories to explain the concept of *basis* in linear algebra using contexts from their everyday experiences (Adiredja & Zandieh, 2017).

In closing, leveraging students' intuitions from their everyday lives can be helpful in learning advanced mathematics. At the same time, the analysis of Adriana's sense-making shows us that designing an instructional tool that incorporates those intuitions is complex, and the design is only part of the work. The main work lies in helping students engage with resources from instruction and attending carefully to their prior knowledge. Students want and need to connect what they are learning to what they have learned. Students likely use ideas that have been useful in the past, especially if there is something in the learning context that specifically cues such knowledge. The notion of misconception is convenient, but it does not help the students or the teachers to make sense of how students are making sense of the topic. Recognizing students as capable learners of mathematics, equipped with different kinds of knowledge, and broadening what counts as productive mathematical knowledge are ways that we as educators can engage in the politics of mathematics education.

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Chapter 5
Promoting Equitable Systems in Mathematics Education Starts with Us: Linking Literature on Allywork to the Work of Mathematics Teacher Educators

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Abstract Mathematics teacher educators (MTEs) within mathematics education systems have unearned assets that present strengths and challenges to the process of developing relationships with mathematics teachers (MTs), students, and their communities. Aware of such issues, we discuss in this chapter the concept and potential applications of allywork for the promotion of equitable systems in mathematics education. With this goal, we draw on literature from within and outside of mathematics education to (1) understand sociohistorical reasons for why MTEs should consider an ally stance in their work with MTs, (2) to consider who an ally is (and is not), and (3) to detail what allywork entails. An analysis of the coercive and hierarchical relations that the mathematics education field has inherited through the systemic feminization of education in the classroom and the masculinization of research and teacher education and the current diverse demographics of the US mathematics educational system frames the need and relevance of allywork. Allywork is understood as MTEs and MTs (as well as students and their community) working with each other in self, others, and systems (SOS) spaces. Description of an interior as well as exterior negotiation and disruption of issues of privilege and oppression highlight the personal and yet systemic dimensions required in MTEs' work and identities as allies. This chapter further contrasts allywork with other stances, links these ideas to mathematics education, and raises questions on how MTEs' work and identities critically address and intersect with the goals, needs, and actions of others in SOS spaces.

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Introduction

> We're just never, ever, ever, ever, ever treated with autonomy,
> Or to think that what we think would be best,
> Or to think about what's important and do it for a long time,
> Or to be supported in what you think is best over a long time…
> that structure was so foreign to me. (Teacher-Researcher Interview[1])

This teacher's comment depicts the sentiment of many teachers. Teachers are often not treated as professionals—as if they do not know what is best for their students—and are often degraded in the media and by policymakers (Adler, 1992; Cochran-Smith & Lytle, 1992; Felton & Koestler, 2015; Herbel-Eisenmann, 2015; Herbel-Eisenmann & Cirillo, 2009). As mathematics teacher educators (MTEs), we are aware of our being outsiders to K-12 classroom contexts, even if we have prior teaching experience. We have a strong call, however, to work with and make ourselves available to teachers and students, aiming to establish alliances with mathematics teachers (MTs) to promote more equitable mathematics systems. Equitable mathematics systems refer to levels of mathematics education that support fair distribution of opportunities to participate and learn. In this chapter, we draw on literature from within and outside of mathematics education to (a) understand sociohistorical reasons for why MTEs should consider an ally stance in their work with MTs, (b) to consider who an ally is (and is not), and (c) to detail what allywork entails.

We recognize that sociocultural and political "privilege doesn't derive from who we are or what we've done. It is … a [broader] social arrangement that depends on which category we happen to be sorted into by other people and how they treat us as a result" (Johnson, 2006, p. 15). Our positions as university faculty entitle us an "epistemic privilege" (p. 22) that might easily lead us to remain oblivious to these unearned assets and ignore the challenges that MTs and students face within the system of mathematics education. Overcoming this obliviousness requires effort and commitment. In the work toward equity and justice in mathematics education, this ignorance must be named and overtly worked on. For example, a group of teacher educators who identified themselves as allies and committed to anti-racist teaching confronted whiteness in their own teaching practices by conducting focus group interviews, self-study, and analysis of students' (teacher candidates) reflections and comments about teacher educators (Galman, Pica-Smith, & Rosenberger, 2010). They found that prospective teacher reflections affirmed the nonparticipation of white students in race-related topics and also silenced race talk.

Additionally, when we consider MTEs, the need for an ally approach becomes explicit. For example, if we focus on demographics, most (three-quarters) teacher educators are female and white; black and Hispanic MTEs make up only 13% (Goldring, Taie, & Riddles, 2014). Similarly, in the K-12 teaching population, the majority of teachers are female and white, and an even smaller percentage of mathematics teachers are black and Hispanic (about 7% and 8%, respectively).

[1] From Herbel-Eisenmann (2015, p. 7).

Secondary MTs are 57% female, but still over 75% are white with similar representation for MTs who are black and Hispanic. Female and white are predominant patterns in general education and mathematics education from K-university levels. Teachers' attrition rates over time follow a reverse pattern that reduces the percentage of underrepresented groups of teachers even more. From the K-12 teacher population, 15% of white teachers either left teaching definitively or moved to a new school within or outside of the school district. Respectively, and on the same variables, black and Hispanic teachers had higher attrition levels of 22% and 21% (Data USA, 2014; Goldring et al., 2014). Although statistics consistently describe higher percentages of female MTEs and MTs, historically the majority of people working in higher education have been white males, whereas the majority of people teaching in K-12 have been white women (Martin, 2015). These numbers partly depict the circumstances that MTs face, especially since one out of five black and Hispanic MTs leave their schools. Further, these numbers describe how students from minoritized groups are taught mathematics with limited access to teachers who look, act, and speak like them. Finally, these demographic patterns speak to a greater likelihood for white female MTEs to develop alliances with white female MTs who work with students from diverse backgrounds. Such a situation portrays the need to develop greater understandings, relationships, and nurturing alliances across interracial groups.

In the following sections, to address the different culturing and valuing of these various "spaces" (i.e., nature of institutions, work requirements of positions, what we focus on, how we think about things), we use a gender lens to illustrate the ways in which sexism sets up MTEs as more privileged participants in society by virtue of being in higher education and MTs as second-class citizens[2]. We make use of the gender divide not because we agree and support such a dichotomous divide but because this divide served historically as a way to exert control and reinforce systemic values and hierarchies. We then draw on literature about allies in the context of racism, sexism, and classism to understand how to disrupt systemic patterns of oppression in mathematics education, in particular, focusing on who an ally is (and is not) and what allywork involves. We use those ideas to propose ways to disrupt a system in mathematics education that continually diminishes the work of MTs. We argue that MTEs need to be allies for MTs, and at the same time, both groups need to extend alliances with more racially, culturally, and linguistically diverse MTs and communities to serve better together the diverse students that are in current mathematics systems. This approach seems especially relevant under the current influences of neoliberal ideologies and interest convergence practices.

[2] We recognize the complicated fact pointed out by Labaree (2003) that, as academics, faculty who prepare teachers are, within the systems of power at the university, considered second-class citizens. He describes many reasons for this, for example, a long history of producing many teachers at fairly low costs and with little attention to quality of programs and the link to preparing teachers, who are seen as low status by many in the United States.

Considering Higher Education and K-12 Teaching Through a Gendered Lens

We first draw on work about gender to argue that as academics in higher education (a space that was originally founded primarily by and for men), MTEs are in a space that is highly male-oriented; we see many remnants of patriarchy in this space that not only banned women for decades but also shaped many of the rankings and practices that currently exist[3]. In contrast to that space, we show that historically, teaching in preK-12 (and especially preK-5) has been constructed as "women's" work.

Higher Education as a "Male-Oriented" Hierarchical Space

Between 1636 and 1776, ten colleges in the United States were founded to offer higher education in the "new world," including Harvard, William and Mary, Yale, and others. These colleges, however, were "beyond the reach of most men, for lack of social status, and of all women, by virtue of their sex" (Solomon, 1985, p. 2). The 1890 signing of the second Morrill Land Grant Act affirmed the importance of public higher education by requiring "federal allocations to be 'fairly divided between Negroes and Whites'" (Solomon, 1985, p. 44). Women were not mentioned in this act, but it allowed women in greater numbers to establish their right to attend universities across the United States. This part of the history of higher education is included here mainly to make the point that colleges and universities were established, run, and organized by and for white, upper middle-class men as an elite way to develop and maintain their leadership in many domains of society. This history relates to and impacts the practices and hierarchies that are ingrained in institutions of higher education today.

Regardless of the progress being made toward including women in higher education, there were still fewer than 10% of doctorates granted to women across the 1960s (Harris, 1974). By 1970, reports on women in higher education, as reviewed by Peterson (1974), highlighted how women were still contending with differential judgment of their work (based on stereotypes rather than on the quality of the work they produced). They needed to argue for equivalent pay and experienced differential treatment, such as if they chose to have a family—a sign that patriarchy still reigned highly in the structures and systems of bias. The *College and University Faculty: A Statistical Description* (Bayer, 1970; as reported in Kreps, 1974) indicates that less than 10% of all colleges and universities had women serving in high-, senior-level positions. In fact, Fox (2001) mentions that even though the number of women obtaining doctorate degrees increased from 12% to 18% between 1960 and 1970, the

[3] We also recognize that people of color did not have access to higher education for many years. To keep our argument concise and to keep it aligned with the argument about the feminization of K-12 teaching, we limit our discussion to drawing on gender-related work.

numbers in science fields were disproportionately lower. “In 1973, women were 4 percent of professors across all science fields; in 1987, that proportion was 7 percent; in 1993, 10 percent; and in 1997, still just 11 percent” (p. 657). Solomon (1985) corroborates this trend arguing that by 1980, the percentage of women with doctorates was 30%, with only 26% of faculty being women in nonprofessional programs at that time (p. 133). As Harris (1974) pointed out: “Because more men than women are encouraged to attend graduate school, present papers at meetings, and publish them, the standard of competence has been established by male performance” (p. 11). She highlights the fact that the women’s liberation movement began outside of academia, stating that the university community was “unable to perceive the enormity of the problem … [in society and of] the subjugation of women in the university community” (p. 12) and, in fact, was probably one of the “most sexist institutions in this country” (p. 14). Women have been evaluated on sameness and difference from men. In science, for example, some valued attributes include objectivity, depersonalized thinking, analysis, and judgment, attributes that often were associated with men, masculinity, and “thinking like a man” (Fox, 2001, p. 662). Such perspectives othered women within scientific academic fields and reinforced fixed “feminine” traits of being caring and subjective.

More recently, scholars have used constructs like patriarchy (Bagihole & Goode, 2001), hegemonic masculinity (Bagihole, 2002), and gender schemas (Valian, 2005) to unearth the various ways in which this history of majority male participation in higher education may impact women in academia. This literature shows that not much has changed in the institution to support women, even when equal opportunity policies have been put in place (Bagihole, 2002; Bagihole & Goode, 2001). These authors suggest that academia perpetuates a myth of individualism and self-promotion, which can be problematic for women “in the pervasive culture where they may find their academic achievements very differently valued and evaluated from [their] male peers (Littin, 1983)” (Bagihole & Goode, 2001, p. 163). Although some research has focused on women as the source of the problem, more recent research has identified issues within the institution and academic practices instead:

> As the present study demonstrates, the reality is very different. There is nothing wrong with the women. They are in fact entering this competitive professional environment ‘from different starting points and with heavier burdens than their male peers’ (Aisenberg & Harrington, 1988, p. 50). They face questions, usually unvoiced but still tangible, such as, ‘Do women have what it takes to do this job?’ For men, on the other hand, entry is perceived as ‘natural.’ (Bagihole & Goode, 2001, p. 174)

These “male-dominated” practices exist, for example, in the ways academics network and self-promote, whom they choose to cite, and through barely visible, small disparities that accumulate over time to advantage men. As academia continues to perpetuate this male culture, it also benefits from it in terms of power systems in comparison to K-12 teaching. For example, there is an assumption that all academics should experience an elite-based system that allows, for instance, professional autonomy, fairly flexible allocation of time, and relatively low levels of administration. These kinds of assumptions are far from the reality for K-12 teachers where all of these resources are being more and more managed and restricted.

The Feminization of K-12 Teaching

In contrast, K-12 teaching has been regarded as a feminized profession as compared to other university-educated career choices. The continuous decline of males in the teaching workforce, however, is not a recent phenomenon. The feminization of public school teaching can be traced back to the end of the nineteenth century, which is a particular moment in the history of American public education. As Richardson and Hatcher (1983) argue, starting from the 1870s, public school teaching was transformed from predominantly male work to one identified as a female occupation due to two larger social forces: "the processes of state school system formation" and "the persistence of cultural constraints on women's work participation" (p. 82). The economic and demographic expansion in the United States during this time period facilitated the spread of compulsory common schools. The concern was to provide supply for the workforce while reducing the increased school costs. The industrial urbanization brought the shortage of teaching workforce as men were hired in factories, managerial positions, or various businesses. Female teachers, then, turned out to be a low-cost labor supply for the teaching and educating of children as the dominant culture necessitated. Nonetheless, according to Lather (1987), the history of a gendered workforce cannot be read by only the changes in the mode of production (i.e., from agriculture to industrialization). What is at stake is to consider the tenets of patriarchy and capitalism together that reproduce the power relations at the societal level. As Lather (1987) argues, "there is an interactive reciprocity and interweaving of the needs of patriarchy and capital which must be taken into account in understanding the work lives of teachers. How has gender shaped the nature of teacher work?" (p. 29).

The industrial revolution not only changed the nature of work but also transformed the US culture toward consumption of factory products from the domestic women-made ones. As a result, the social responsibility to educate citizens for this emerging society of consumption switched to women. Although women were getting out into the public sphere and participated in the workforce, the patriarchal authority did not shift because teaching still maintained women's moral obligations as mothers to raise a child as a productive member of the society (Grumet, 1981). This paradoxical situation resonates with Apple's (1983) positioning of teachers into a "contradictory class location" (p. 612). According to Apple, it is reasonable to think teachers are located in two classes simultaneously: Although they share the interests of the petty bourgeoisie (the particular culture that is imposed through the official curriculum), they are also part of the working class due to fiscal crisis where many teachers had to face poor working conditions, layoffs, unpaid hours, or a loss of control and voice for themselves. Nonetheless, Apple (1983) cautions us, similarly to Lather (1987), not to read teachers' working conditions only from a perspective of economic structures since teachers are gendered actors as well due to historically experienced sexist practices of recruitment and promotion in patriarchal societies where women's labor is colonized.

The relations of power and authority in K-12 schools have historically been formed as patriarchal, in which the school system is organized around primarily male educational leadership and a female teaching body (Apple, 1983). This means that teaching is considered both a feminine occupation and a kind of domestic service conducted by women that male leadership has to control. More specifically, policymakers and academics in the male-oriented hierarchical space of higher education intervened in school curricula and textbooks as part of the educational reforms (Apple, 1983). The manipulation and control of textbooks and curricular materials, however, yielded to the prescriptions for teacher practice, a feminine workforce, by the male-dominated academy. That is to say, the academicians were perceived as "experts" who were defining, controlling, and managing teaching, which was already a feminized profession. Academics were regarded as "experts" who corrected the "wrongs" of teaching in K-12 settings. Male-dominated expert control of the feminized teaching workforce in K-12 schools entered as another layer to the gendered power relations and contributed to the hierarchical structures.

In this section, we have used gender-related work to argue that higher education has been structured under the hegemony of men, while K-12 teaching was feminized as a result of economic and gendered expectations of women (who make up the majority of the teaching population, especially at the K-6 level). Thus, we see the historical aspects of these spaces as creating a hierarchical system whereby academics who work in higher education are seen as intellectual contributors who exist in a mythical "ivory tower" and "above" the primarily "women's work" of K-12 teachers who work with children and youth (Tate, 1994). Although we reiterate that we do not see this particular system hierarchy as being on the same level as systems that perpetuate racism, classism, heterosexism, etc., we recognize the impact of the history of these spaces as setting up a hierarchy that is important to the ways in which faculty (in our case MTEs) interact with teachers (in our case, MTs). In the following sections, we examine literature related to these other systems of privilege and oppression to explain what an ally is and what allywork involves. We use this literature to extrapolate some key characteristics of allywork when MTEs work with MTs. In an era of *Waiting for Superman* (Swalwell & Apple, 2011) and other societal and policy moves to further deprofessionalize teachers (e.g., Cochran-Smith, 2005; Emery & Ohanian, 2004; Sleeter, 2007), we see MTEs as important allies for MTs.

Linkages Between MTEs' Work and Definitions of Ally and Allywork

The work of MTEs as allies is complex because they need to support the equitable negotiation of the spaces between MTs (mostly white) and their students and communities (i.e., representing diverse communities, with significant percentages of minoritized groups). This work demands a process of finding a fair, balanced, equitable point between teaching and learning efforts and needs. A laden system of values can immediately oppress and limit any of the participants in a mathematics system. As allies, MTEs work not only *for* the MTs and students but also *with* them. Below, we present literature focused on the conceptualization of ally identity and allywork and make links to what we envision as equitable MTE allywork. We have organized this section into three subsections. We raise reflective questions related to MTEs' work as allies to nurture discussion. By acknowledging their historically privileged academic positions, MTEs have the possibility and responsibility of addressing the laden dynamics of their privilege when working with MTs. As allies, we envision the possibility of the MTEs disrupting issues of power, as related to race, class, and gender, during their interpersonal and professional work relations with MTs and the diverse student communities with whom they might work. With this purpose, and to develop deeper understandings of who an ally is, we consulted literature outside of mathematics education which we discuss in relation to *self*, *others*, and *systems* (SOS) spaces and present as follows: (a) Who is and is not an ally? (b) What are some processes involved in becoming an ally? and (c) What are some considerations for working as an ally?

Who Is and Who Is Not an Ally?

Initially introduced in the literature in relation to heterosexual people working in support of lesbian, gay, and bisexual issues (Broido, 2000; Washington & Evans, 1991) and white students addressing issues of racism (Bourassa, 1991), the term *ally* refers to "a member of the 'dominant' or 'majority' group [e.g., men, Whites, heterosexuals, able people, language dominant speakers], who works to end oppression in [their] personal and professional life through support of, and as an advocate for, the oppressed population" (Washington & Evans, 1991, p. 195). As such, allies aim to be a positive force to minoritized communities by providing external support and force against forms of oppression (Alimo, 2012; Bishop, 2015; Broido, 2000; Clark, 2010; Goldstein & Davis, 2010; Jenkins, 2009; Myers, Lindburg Jenkins, & Nied, 2014; Patton & Bondi, 2015; Washington & Evans, 1991). Being an ally, however, does not necessarily mean "help victims of racism, but rather … speak up [and act] against systems of oppression and … challenge other Whites to do the same" (Tatum, 1998, p. 109). In the different contexts of systemic oppressions, allies invite others to join, fight against, and end the "isms" that disempower minoritized communities (Schniedewind & Cathers, 2003).

From this perspective, we understand an ally as a person, or entity, who has gained awareness of issues in relation to SOS and who actively works across these spaces to promote social justice. Below, we describe how "being an ally is about attitude, awareness, and behavior" (Myers et al., 2014, p. 69) across SOS spaces.

Awareness of spaces of privileges for self (*S*OS) and oppression of others (S*O*S) is fundamental for an ally position. This knowledge is acquired through immersion in a community (within one's own community and outside communities) and the noticing of how differences affect the distribution of privilege and oppression (Clark, 2010; Jenkins, 2009; Myers et al., 2014). This awareness can turn into a source of discomfort through realization of self-privileges. An ally stance needs to reach beyond guilt and resist biased socialization behavior to embrace one's responsibility toward the community (Myers et al., 2014) by negotiating, creating, and promoting social change (Cochran-Smith, 1995; Patton & Bondi, 2015).

Alongside the relation between awareness and action, Bishop (2015) directs our attention to issues beyond an ally's episodic work and personal stances:

> It is ridiculous to claim you are not sexist if you are a man or not racist if you are white and so on. No matter how much work you have done on that area of yourself, there is more to be done; the oppressive messages that surround us, unconsciously absorbed, constantly undo some of our efforts … It is an ongoing task, like keeping the dishes clean … A white person never becomes non-racist but is always a 'recovering racist,' more often referred to as 'anti-racist.' … members of an oppressor group are always oppressors, no matter how much individual learning we have done: until we change the politics and economics of oppression, we are still 'living off the avails.' […] So, until we succeed in making a more humane world, yes, we are racist (or ageist, or classist, or heterosexist and so forth). Understanding this is part of learning to think structurally rather than individually. It is part of avoiding overpersonalization of the issues. (pp. 94–95)

Bishop highlights the relevance of a laden systemic space (SO*S*) as a source of hierarchical oppression and privilege. Systemic spaces of laden power nurture the privilege of nontarget (i.e., privileged; e.g., white, male, heterosexual, fit) members and the oppression of target (i.e., oppressed) members in our communities. Privilege results from systemic, acting paradigms that are set as norms and expectations that benefit those who have traits that align with such paradigms (e.g., a societal space with a dominant heterosexual paradigm or English-only paradigm); in turn, these dominant paradigms limit, oppress, or stigmatize others who engage in actions or have traits different from those of the dominant paradigms (e.g., being bisexual; speaking other languages than English). For that reason, episodic actions or personalization of these issues by nontarget (privileged) group members fall short toward effecting social change. Rather, a position for social change aims at disrupting systemic spaces that sustain and perpetuate coercive relations (Bell, 1997; Munin & Speight, 2010). Mindful of these dynamics of power, especially between researchers and teachers, some educational initiatives have addressed these issues by developing collaborative approaches, such as action research in the classroom, where teachers and researchers work as colleagues (Elliot, 1991). This approach to research in education has opened spaces of fruitful, collaborative, and equitable relations between MTs and MTEs (Adler, 1992; Felton & Koestler, 2015; Herbel-Eisenmann & Cirillo, 2009; Parks & Wager, 2015). Here is where we located SOS spaces for allywork in mathematics.

Therefore, we understand allywork between MTEs and MTs as working *with* each other. Our review of literature on allywork has led us to identify four different stances that one can take in SOS spaces; we portray them in Fig. 5.1.

Essentially, we understand allywork as working *with* others in SOS spaces. Allywork implies an *awareness* about others' (members from target groups') situation and oppression, as pointed out by them (others). This awareness is nurtured

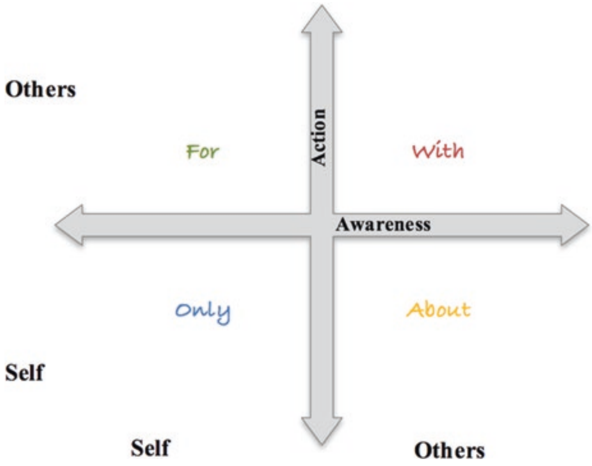

Fig. 5.1 Stances (*only, for, about*, and *with*) in systems of privilege and oppression

through interactions *with* them (others). At the same time, allywork implies engaging in *actions with* them (others) to disrupt systemic issues of oppression and privilege. Thus, the identification of the systemic issues as well as the development of plans to disrupt the systemic issues results from a collaborative work with others.

The Indigenous Action Media (IAM) (n.d.) goes even further naming allywork as insufficient to address systemic issues of oppression. IAM argues that having an anti-oppression stance requires counteracting current systems of oppression to the point that these liberatory actions might be condemned by the system as criminal. IAM, in fact, suggests an accomplice stance as a person who helps or works *with* another to commit a crime. We highlight the relevance of the work *with* as essential in allywork and that of an accomplice when the needed work reaches beyond official SOS spaces.

The remaining stances portrayed on the quadrant of Fig. 5.1 describe how, on the continua of awareness and actions, one can take a stance in the SOS spaces in relation to systems of privilege and oppression. The right lower quadrant on developing work *about* others depicts a stance of someone who is aware of issues of power in a community and who aligns with the needs of the oppressed community. Such a stance, however, is mainly rhetorical without explicit actions that lead to systemic change, for example, "the mere silencing of hate-talk" or "anti-work" (p. 712). Clark (2010) names this stance as an advocate, who partly interrupts oppression against minoritized communities by becoming a defender that pleads for a cause (Jenkins, 2009).

The left upper quadrant describes a stance in SOS spaces of work *for* others. This position involves engaging in actions that promote changes but not necessarily with enough awareness about the needs and goals of others or of the community. Clark (2010) names this stance "agents" who "work to create change within dimensions of society in which they may or may not have power" (p. 28). Finally, the lower left quadrant depicts a stance that we understand as centered *only* on awareness of self needs and issues and actions that *only* benefit the self. This stance remains impartial to or oblivious about issues of power affecting others in the community and implements no actions toward others. Through this passive stance, one becomes complicit with the perpetuation and systemic structure of oppression for others. An example related to this stance could be laughing at racist jokes without challenging these oppressive practices, as well as omitting the history of oppressed people or people of color in everyday life or the curriculum at school (Tatum, 1998). In our description of interactions in SOS spaces regarding awareness about and actions of self and others, we made use of prepositions (i.e., *only, for, about,* and *with*), because a preposition is a word that links other words, such as nouns or pronouns. SOS spaces are relational in nature, and an analysis on how we relate to and work with others and to the system is necessary especially when we aim at supporting equitable mathematics systems.

Considering allywork between MTEs and MTs, we—as MTEs—understand the need for a collaborative working process in which awareness about the circumstances and actions to be developed as part of this alliance needs to evolve through the interaction of MTEs *with* the MTs. MTEs are "[o]ppressors [in this system,] are always oppressors, no matter how much individual learning we have done: until we change the politics and economics of oppression, we are still 'living off the avails' [of MTs]" (Bishop, 2015, p. 94). Therefore, we are responsible for addressing consciously and responsively the power dynamics within our interactions with MTs. During allywork *with* MTs, we are to cultivate a *politicized trust* with them. This kind of trust refers to an "ongoing building and cultivation of mutual trust and racial solidarity. It is thus a trust that actively acknowledges the racialized tensions and power dynamics inherent in design partnerships" (Vakil, McKinney de Royston, Nasir, & Kirshner, 2016, p. 199). An ongoing building of trust should support a relational collaboration with MTs.

Allywork is not just about teaching, talking, and working *with* others but also on promoting systemic change through our actions. Allywork calls for a disruption on the use of deficit discourse and perspectives in work with MTs, schools, students, and communities; the goal should be to nurture collaborative and transformative relationships and to invite others in the mathematics education community to do the same. As MTEs, we wonder about the following questions, which we see as important to answer for the development of alliances with MTs: *Do MTs need, see, or want MTEs as allies? How would the field of mathematics education benefit from having MTEs become allies to MTs and school communities? What would that look like? How can MTEs develop deep awareness of the sociopolitical pressures that MTs and students face at school? What disruptive actions with MTs could better support minoritized students and the opportunities of their communities? How can white MTEs and MTs promote alliances with other MTEs, MTs, or community members from diverse racial, cultural, and linguistic groups that better resemble the student communities? How can white MTEs and MTs learn with the community to address issues that matter to them? Would the community be open to this relationship? How can MTEs disrupt systemic pressures that oppress or limit MTs' actions?*

While we are cognizant of MTEs' positions of power, we also acknowledge the plurality of their identities extending beyond mathematics education and intersecting with other socioeconomic identities. This diversity of identities provides potential intersections between MTEs and MTs through their shared identities. For example, most MTEs are former MTs, and this relation between self and others might ease the development of alliances. What is more, though "most allies tend to be those with privileged identities, and allies from the majority are necessary for change, the discussion of allies within a community is fundamental as well" (Myers et al., 2014, p. 70). Thus, allies may also belong to minoritized communities (Clark, 2010; Jenkins, 2009). This means that MTs themselves can also take up an ally role to other MTs or MTEs. Furthermore, some MTEs may also share racial identities with MTs from minoritized communities, links which can provide a set of shared lived experiences that may ease the development of stronger relationships across groups. Yet, it is important not to over-generalize since the sharing of or "the self-identification with an identity does not equate to automatic allyhood for the same community" (Myers et al., 2014, p. 74). Consequently, the fluidity of MTEs' and MTs' intersecting identities widens possibilities for building alliances. Ally relations across and within target and nontarget groups are diverse and not formed in expected patterns but on awareness and actions built in SOS spaces *with* others.

What Are Some Processes Involved in Becoming an Ally?

This section addresses more specifically the *self* in SOS spaces. Ally positions are not inherited. In fact, anyone can become an ally. Being an ally is not a title to be claimed. Instead, being an ally encompasses a continuous renegotiation process of "an identity that is achieved by acting on the moral imperatives of pursuing social justice and validating differences" (DeTurk, 2011, p. 575). Ally positions are constructed and developed both through awareness of self and others' situations *with* others in a system and through actions *with* others against systemic issues of power that privilege some and oppress others. This section describes some processes and strategies that nurture the development of ally awareness and action. Lastly, links to MTEs' process of becoming allies are discussed.

The process of developing an ally identity and allywork encompasses a cycle of awareness, healing, and action taking. For this, allies need to "travel the world" and cross borders in order to realize others' experiences. It starts through an inward questioning and internal transformation of personal dispositions, which often trigger discomfort for those of privilege when developing awareness (Reason, Scales, & Roosa Millar, 2005). Awareness often starts through direct and indirect exposure to issues of inequality and oppression that others face (Broido, 2000; Dillon et al., 2004; Goldstein & Davis, 2010; Reason et al., 2005; Roades & Mio, 2000; Stotzer, 2009; Zúñiga, Williams, & Berger, 2005). These exposures nurture awareness through "hearing others' perspectives, or being challenged by others with different views on social justice issues, [and] could serve as a catalyst for self-reflection" (Broido, 2000, p. 11). The perspective-taking process and self-reflection promote bidirectional, intergroup communication and bias reduction (Broido, 2000), as well as intercultural understanding (DeTurk, 2006), and genuine empathy. The levels of empathy lead to taking action and affecting change for and with others (Clark, 2010). Then, the renovated perspective expands outwardly toward others in joint action *with* others to support change. Allywork, then, addresses a twofold change, first by engaging *with* others in a "critical dialogue and discussion, interrogating perceived lines of difference and [then] inquiring into the possibilities for creating productive alliances across these lines" (Clark, 2010, p. 705).

Participants in a study on developing awareness about others who developed ally stances identified three areas that contributed to their deliberate commitment to disrupt oppressive systems: (1) increased information on social justice issues, (2) engagement in meaning-making processes, and (3) self-confidence (Broido, 2000, p. 7). The last area represents the most challenging task for allies from nontarget groups because understanding and accepting that self can be an oppressor are harder than accepting those situations in which one is the oppressed (Bishop, 2015). "Learning about race, racism, and systems of whiteness may provide the necessary information to be able to influence the behavioral choices in the future that contribute to racial justice in society" (Alimo, 2010, p. 38). It is through learning *with* others (especially target group members) that the variety of conditions in intergroup relationships contributes to bias reduction. Intergroup cooperation activities and dialogue, for example, create these conditions for interrace/ethnicity dialogues (Alimo, 2010). This is due to the fact that when oppression is not part of your experience, you can understand it only through others' experiences (Bishop, 2015). The experience of oppression is "hidden" because one is cut off from the ability to "live" and conceive the experiences of the oppressed. This lack of empathy makes oppression possible. Thus, a renewed perspective of self represents a struggle as it is linked to the acceptance of an "internalized domination" and of one's prejudices against others (Bell, 1997; Goldstein & Davis, 2010; Myers et al., 2014).

Although individuals respond differently toward issues of privilege and oppression, common to target and nontarget groups are the wounds that dehumanizing relations have inflicted on them (Memmi, 1991). Thus, experiences related to oppression require *healing*. If oppression is learned through unconscious pain, then the learning of liberation needs a *conscious healing* (Bishop, 2015). For a conscious, liberatory process to happen, sharing experiences is a vital process for individual and collective healing (Bishop, 2015; Philip, Martinez, Lopez, & Garcia, 2016). Allies need to struggle for their own liberation, and others from the same nontarget group could understand and support this process, so together they can unlearn oppression. Speaking out nurtures the breaking out of secrecy and shame and contact with others suffering of similar pain. The collective and individual healing processes for allies, whether from the oppressors or the oppressed groups, would require both an emotional expression shared with trusted people and action taking. "Without individual healing, a person might destroy the groups they join; without group healing, individual healing reinforces the private isolation that is the basis for 'divide and conquer'" (Bishop, 2015, p. 79). When individuals recognize emotions as a natural part of the developmental process, they more readily acknowledge, work through, and move beyond negative emotions (Reason et al., 2005).

Relatedly, MTEs' intersecting social identities represent a resourceful context that may nurture ally relationships. *What about MTEs with little or no intersectionality with the communities with which they work? How would familiarity with and awareness of the community's oppression develop? What would encompass a healing process of an MTE?* Being an ally is founded on the development of awareness of oppressive forces that feed action toward systemic change. College students and teachers developed awareness of oppressed sociopolitical identities by engaging in closer relationships *with* these communities (Alimo, 2010; Broido, 2000; Philip et al., 2016; Reason et al., 2005; Zúñiga et al., 2005). Perhaps the process of increasing proximity of MTEs and MTs, collaboration *with* others, and reflection might nurture insights and awareness not only on what issues need to be addressed to support others (e.g., MTs and students and student communities) but also on how those issues should be addressed and by whom. So these points speak to the relevance of allywork of becoming aware of issues and devising related action plans in coordination *with* others in SOS spaces, by collaborating *with* members of the target community. We hypothesize that MTEs' active and collaborative participation *with* MTs, students, and other members of the community might support MTEs' deeper understanding of the issues that these community members face on a regular basis. In turn, this process might simultaneously facilitate community members' understanding of MTEs' identities and work. We believe that this mutual learning process might nurture a healing process to the separation of MTs' and MTEs' work and relationships. We need to heal our mutual trust and nurture a *politicized trust* (Vakil et al., 2016) to be able to work together. Although this process requires changes in the systemic work and relationships of allies *with* target group members, it also requires changes in the emotions and *trust* that support such relationships. More specifically to MTEs' healing process alone, it might be that MTEs need to support each other through an in-group liberation process to unlearn oppression, always informed by the experiences of target group members and work *with* them. *In addition to a mutual healing process and the development of critical responsibility, what is it that allies are supposed to do?* The following section elaborates on these ideas.

What Are Some Considerations for Working as an Ally?

This section provides recommendations to address some challenges in allywork, more specifically, for the working with *others* within SOS spaces. Some of these challenges include managing emotional and ethical loads in the process of addressing privilege, distributing responsibilities, and negotiating action goals as allies. Links to MTEs' work are discussed at the end of the section. Emotionally, allies need to prepare for being vulnerable to self and others' confrontations regarding privilege and oppression and to the unknown journey as an ally, which is a dynamic, multilayered, and constantly changing process (Myers et al., 2014). Allies must face their emotions as they work *with* others toward social change. First, allies need to develop self-comfort in understanding systemic power and its relation to their privilege (Goodman, 2001; Katz, 2003). This point is critical because it relates to their ability to articulate their stances and act on social justice (Broido, 2000). Further, Bishop (2015) argues, "I don't believe it is possible to … become an ally without being involved in your own experience of liberation" (p. 92). So allies, as mentioned above, need to participate in a healing process allowing them to notice their privileges, keep a list, and help others see them to break the invisibility of privilege (Bishop, 2015). This identification would be useful for self and others to be mindful about the dynamics of power in the contexts they work in and how and when these power dynamics might need to be disrupted to work *with* others in SOS spaces. "Allies from dominant groups are essential to break the cycle of oppression, as are allies within community, given the varying sub-identities in communities" (Myers et al., 2014, p. 84). In support of oppressed groups and to work in action *with* these groups, allies (both from target and nontarget groups) need to make strategic use of their social and cultural capital to understand the context, influence others, and affect change. This means that social identities matter within their context of action (DeTurk, 2011) *with* others.

Second, allies also need to speak up when witnessing an act of oppression. Allies should not wait for the oppressed to point it out (Bishop, 2015). Here is where allies need to deal with the tension of risk (i.e., personal safety, relationships, or status) because, at times, allies can have their personal well-being threatened (DeTurk, 2011). As allies inhabit in-between spaces—which include others' spaces—embedded in a SOS space that is simultaneously an oppressive and a privileged system, tensions and not knowing always how to proceed next are common situations to be faced. Allies' successful efforts to confront racism and sexism depend "on how much power they had in a given situation" (DeTurk, 2011, p. 578) and how aware or familiar they are with those issues. The nature of power, however, is contextual and fluid since it is socially constructed, situated, multifaceted, and contingent (Myers et al., 2014). The arbitrariness of power dynamics becomes evident through the juxtaposition of privilege and oppression across systems. For example, an African American person "could act as [an] all[y] to Whites by interrupting anti-White prejudice" (DeTurk, 2011, p. 584); also an ally self-identified as a gay man could promote closer connection as an ally to other gay men. Consequently, membership to a social and cultural group does not determine an absolute power or lack thereof. Allies aware of these dynamics need to develop tactics and alliances *with* others that crisscross different groups to support one another and strategically work in SOS spaces to disrupt oppressive actions.

A greater ethical tension relates to speaking for others. Such action might disempower those to be empowered (DeTurk, 2011). Allies, as mentioned above, become aware of issues and take actions *with* others (members of target groups). As such, allies must build up some trust and equality through identity interactions and getting to know the community they work *with* in order to prevent falling into the trap of "*knowing what is good for them*" (Bishop, 2015, p. 96, italics added). It is fundamental that allies learn about the community *with* which they work. Bishop (2015) suggests that allies find members of the community to learn more about the community from an insider's perspective, as the ignorance of privilege about the conditions of the oppressed is a source of oppression. This speaks about developing a relationship of mutual learning. Only people from oppressed groups can figure out what is "good for them," and the development of their own leadership strengthens them (Bishop, 2015). In this context, an ally should only add thoughts or resources to the process in response to the target group members. Allies always need to listen and reflect, to be better allies: "sit down, shut up, and listen" (Bishop, 2015, p. 102) and seek to identify and acknowledge the community's skills and strengths (Russell, Hynes, Kane, & Bradeen, 2005). Allywork is then characterized by work that is guided by awareness gained *with* others (target group members) and actions *with* others (target group members). If in-group target members' perspectives are not at the center of allywork, we might be at the threshold of engaging in an interest convergence issue (Martin, 2015), in which "allies" work mostly toward their own goals and needs rather than to those of the targeted community, therefore, undermining the basis of an alliance. Nevertheless, the collective nature of an alliance nurtures the process of problematizing and solving together, *with* each other, issues that affect the target groups, so that both parties contribute to the development of transformative processes and equitable change. Here both parties become like one through the sharing of goals, problems, opinions, strategies, and solutions.

Thus, allies must understand the motives of their work specifically: that being an ally is *NOT* about "being one of the good ones." Rather, it is about acting for critical positive social change (Myers et al., 2014). But, *what is positive social change?* Martin (2015) describes mathematics education controlled by a dominant group that implements colonizing approaches. Such approaches are not liberating because the ideas of improving mathematics education "for all" are not necessarily addressing the needs of all, rather the interpretations and understandings of the needs of "all" were defined by a narrow group of people. Civil (2006), for example, describes how often educational innovations implement norms that vary from the norms of the school and of the teachers. MTs referred to in Civil's work thought that students resisted their new approaches, such as engaging in mathematical discussions, because students did not see discussions as mathematics work. Civil argues that even attempts at a liberatory curriculum can be oppressive. She states: "I often find myself wondering about innovations that try to contextualize the mathematics in situations that claim to be more relevant to the reality of the working-class Latino students with whom we work. *Who decides what is relevant?*" (p. 39; italics added). Furthermore, Civil also asserts that professional development (PD) content is often disconnected from what MTs know and do, so that the time invested in these PD meetings result pointless. Consequently, she questions: "efforts aimed at helping teachers teach using a 'reform-approach' leave me wondering whether teaching mathematics was becoming a smorgasbord of activities with no apparent road map" (p. 43). Related to Civil's points and Martin's argument, benevolent approaches to "education for all" are at times just a form of interest convergence. It is because MTs and students receive support only on those points that those at the top of the system decided as pivotal. Martin challenges us to think that no matter how good our intentions are, if we only acknowledge our own perspectives and goals, we are under a colonizing approach: "gains for minority groups coincide with White self-interests" (p. 20). Therefore, we believe that MTEs' work should involve more collaborative processes and perspectives inclusive of the input from MTs, students, and the communities *with* whom they work.

Having target and nontarget groups (e.g., researchers and teachers; researchers and community members) engaged in collaborative problem posing and inquiry-based approaches that are inclusive provide a promising and fruitful SOS space for allywork. For example, participatory action research (PAR) supports a group situation in which members communicate *with* one another, engaging in a "meta-dialogue" about their goals, actions, and distribution of work. These processes lead to insights for all participants and empower them as co-researchers, which they combine with collective reflection and action (Cammarota & Romero, 2009; Fals Borda, 2001; Vakil et al., 2016). Although we see the feasibility in such an approach that brings together MTs and MTEs (and perhaps students and community as well), we wonder how a PAR approach in mathematics education between MTEs and MTs would support allywork. It is especially relevant in the overarching demographic trend in the US society in which the majority of MTEs and MTs are white and female. When these groups come together in SOS spaces and if their work goal includes to work together to support students from minoritized groups, perhaps a way to move toward this goal could be through the development of intergroup relationships/alliances with MTEs, MTs, and community (in-group) members from diverse cultural, socio-economic, gender, racial, and linguistic backgrounds, who may support the process and nurture the development of close relationships with members of target/minoritized groups to collaborate together with greater personal and academic links. With this said, we also acknowledge that belonging to a group does not guarantee understanding of the issues from an emic perspective; not that belonging to a nontarget group makes one less sensitive or critical of oppressive situations but that new alliances and possibilities can emerge when an ally team itself includes diverse (target and nontarget groups) perspectives and together work to support the target groups. Some reflective questions, within this context of MTEs' allywork, include: *How can they (MTEs and MTs) prevent interest convergence in their allywork? Whose voices, interests, and feelings should matter? How can awareness and action on the related issues emerge in coordination* ***with*** *the target groups in the community and the schools? When issues of power and injustices have been normalized in a community, how can outsiders (MTEs) intervene while working as allies respecting the insiders' (MTs') work and perspectives? What should matter in such situations? How would MTs and MTEs need to respond to all: the community, the students, and the mathematics standards? How would this responsiveness be prioritized? Why? How would the educational system support sustainable relationships under this framework? Who is to lead? What would be the responsibilities for each member in this dynamic? Though some answers to these questions might have some evident theoretical answers, in praxis they might not be as clear*, *why? How might* ***systemic forces*** *in SOS spaces filter down into the collaboration between allies, MTEs, and MTs, and their work* ***with*** *students from minoritized groups?*

Discussion

Through these theoretical links across fields, we have explored how MTEs' identity and work as allies include their awareness, negotiation, and actions within SOS spaces. This means that awareness, negotiation, and actions extend across, within, and with self, others, and the system. We think that MTEs' work and identities as allies can only emerge within the power dynamics of this threefold space. Self-privileges and the oppression of others revealed in relationship to systemic oppressive structures can serve as a catalyst for MTEs' promotion of equitable mathematics systems. Awareness, empathy, and disruptive work coexist at the core of the development of an active and relational *with*-stance of MTEs' allywork in SOS spaces. Moreover, allywork and ally identities are fluid and never fulfilled until equitable systemic changes are achieved. Accordingly, we have discussed how MTEs—from an ally stance—may strategically coordinate actions *with* others depending on the contexts and the level of engagement of their work with MTs, students, schools, and systems in mathematics education.

Additionally, we have learned, raised questions, and provided suggestions on how MTEs' identities as allies may intersect with MTs' social identities. This intersection might nurture greater opportunities for alliances to evolve. Regardless of this intersection, MTEs need to learn about the community with which they work and develop relationships and familiarity *with* MTs, students, schools, school districts, and local communities. Through the development of awareness, familiarity, and a relation *with* target members of the community, consecutive actions of allywork *with* them will develop. From this point of view, we can also state that allies belong not only to dominant groups, but they also belong to target and oppressed groups. It is in this multiplicity of relations that allywork is strengthened. The strength of the relationship of MTEs with the community is not solely built on their awareness *with* the community about what matters to the community but also on how much MTEs' work, as allies, addresses and intersects *with* the goals and needs of MTs and those of the community. MTEs working under an interest convergence approach are not allies. Such approaches stifle relations with others, with MTs and especially the community of target groups, and favor oppressive systems. MTEs' relational work *with* MTs and the community in SOS spaces might be nurtured through participatory approaches as these approaches address the completeness of a *with*-stance on the awareness of the community goals and needs as well as in the collaborative disruptive action for equitable change.

Given MTEs' multiple actions and relations as allies, we view this work as an evolving third space (Whitchurch, 2010). As MTEs work within SOS spaces, they need to contest the laden power dynamics of the system, heal self and heal and reconcile relationships between self and with others, and then collaboratively reconstruct structures that would support new equitable relations within those SOS spaces. The relevance of third spaces in allywork resides in the promotion of collaborative and equitable relationships in SOS spaces. Because each SOS space requires a negotiation linked to the other two spaces, allywork must support connections of each SOS space with the other SOS spaces. Ideally, such an approach would weaken and dismantle the coercive and hierarchical relations that the mathematics education field inherited through the systemic feminization of education in the classroom and the masculinization of research and teacher education efforts. As MTEs, we recognize that the steady and collective work of the participants of SOS spaces in mathematics education would nurture relations that would reinforce the equitable facing of new challenges and tensions in the mathematics system. Thus, the *with us* in the title includes several stakeholders (allies) participating in a system (insiders and outsiders). And acknowledging our privilege as MTEs in mathematics education, we embrace our responsibility for renovating the SOS spaces for equitable mathematics education and conclude our discussion by using Bishop's (2015) statement related to the oppressors' role: "No matter how much work you have done on that area of yourself, there is more to be done" (p. 94).

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Chapter 6
A Commentary on Theoretical and Political Perspectives Toward Equity and Justice in Mathematics Education

David W. Stinson

Abstract This chapter presents a commentary on chapters in the section focused on theoretical and political perspectives toward equity and justice in mathematics education.

In July 2015, Ta-Nehisi Coates's book *Between the World and Me* was released. The book experienced a whirlwind of both positive and negative reviews, spanning from CNN to Fox News and *The New York Times* to *The Washington Post*. The book went on to top *The New York Times* best-seller list for nonfiction for several weeks in 2015 and then again for 1 week in 2016; it won the coveted 2015 National Book Award for Nonfiction and was a 2016 Pulitzer Prize finalist. The dustcover of the Spiegel & Grau imprint reads:

> In a profound work that pivots from the biggest questions about American history and ideals to the most intimate concerns of a father for his son, Ta-Nehisi Coates offers a powerful new framework for understanding our nation's history and current crisis. Americans have built an empire on the idea of "race," a falsehood that damages us all but falls most heavily on the bodies of black women and men—bodies exploited through slavery and segregation, and, today, threatened, locked up, and murdered out of all proportion. What is it like to inhabit a black body and find a way to live within it? And how can we all honestly reckon with this fraught history and free ourselves from its burden? … *Between the World and Me* is Ta-Nehisi Coates's attempt to answer these questions in a letter to his adolescent son.

I use two essays on Coates's (2015) book, written by two public intellectuals, David Brooks and Michelle Alexander, who hail from somewhat polarized theoretical and political perspectives—conservative and liberal, respectively—to frame this brief commentary of the five chapters contained in the first section of this edited volume. The ideas and arguments the authors put forth in each of the chapters, individually and collectively, similar to Coates's book, displace too often

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taken-for-granted ways of thinking and knowing. In that, the author(s) of each chapter provide different possibilities to think and rethink mathematics and mathematics teaching, learning, and policy.

For instance, Berry (Chap. 1) presents a critical race theoretical rethinking of mathematics education policy and reform, illustrating that too often policy and reform efforts do not address the needs of marginalized learners but rather reinforce the economic, technological, and social interests of the powerful. Battey and Leyva (Chap. 2) provide a rethinking of how the ideology of racialized hierarchy of mathematics ability might be disrupted, suggesting an explicit focus on deconstructing implicit racial attitudes as a means of moving toward equitable opportunities to learn. Larnell and Bullock (Chap. 3) offer a new socio-spatial turn to "urban" mathematics education research, rethinking urban within a framework that includes three axes: temporal, social, and geographic. Adiredja (Chap. 4) explores the politics of undergraduate mathematics education research, connecting the cognitive to the sociopolitical by rethinking intuitive knowledge in advanced mathematics. And LópezLeiva, Herbel-Eisenmann, and Yolcu (Chap. 5) provide a critical feminist reading on K–12 education and teacher education, rethinking the relationship—or better yet "allyship"—between the mathematics classroom teacher and the mathematics teacher educator.

Not only do the aforementioned chapter authors ask the reader to rethink taken-for-granted ways of thinking and knowing, but also, similar to Coates (2015), the intellectual engagement that the authors ask of readers can be both discomforting and comforting depending on where the reader positions herself or himself on the theoretical and political ideological spectrum: conservative or liberal or somewhere in between. It is important to note, however, that in naming the polar ends of an ideological spectrum that I am not suggesting some rigid binary, such a binary is a fiction, as most people are consistently shifting from end to end depending on both the issues and the contexts at hand. That is to say, theoretical and political ideologies are best understood as existing in the tensions of sliding signifiers in contexts. In short, theoretical and political ideologies are complex; the error in much of the current public discourse is that this inherent tension and complexity is often erased.

Nevertheless, for argument sake, I use Brooks's (2015) *The New York Times* Op-Ed essay "Listening to Ta-Nehisi Coates While White" and Alexander's (2015) *The New York Times* Sunday Book Review essay "Ta-Nehisi Coates's 'Between the World and Me'" as polarized ends to provide a model, if you will, which illustrates the benefits of opposing intellectual engagement. My aim is to seduce mathematics educators who position themselves on either end of the spectrum, or somewhere in between, to engage with the authors of the chapters contained herein, inviting readers from both ends and those somewhere in between into a thoughtful, intellectual discourse around crucial issues in mathematics education that are inherently filled with tensions and complexities.

Brooks (2015) begins his op-ed on Coates's (2015) book, writing: "The last year has been an education for white people. There has been a depth, power and richness to the African-American conversation about Ferguson, Baltimore, Charleston and the other killings that has been humbling and instructive" (para. 1). Recall, this is 2015.

Brooks's concern throughout the op-ed, however, appears to be what he considers to be Coates's rejection of the "American dream." The American dream, according to Brooks, provides for equal opportunity, social mobility, and an ever more perfect democracy that cherishes the future more than the past. Brooks is also taken aback (i.e., made to feel uncomfortable) by Coates's definition of White America: "a syndicate arrayed to protect its exclusive power to dominate and control [Black] bodies. Sometimes this power is direct (lynching), and sometimes it is insidious (redlining)" (Coates, as cited in Brooks, 2015, para. 9). In summation, Brooks writes:

> I read [*Between the World and Me*] like a slap and a revelation. I suppose the first obligation is to sit with it, to make sure the testimony is respected and sinks in. But I have to ask, am I displaying my privilege if I disagree? Is my job just to respect your experience and accept your conclusions? Does a white person have standing to respond? (para. 11) … This [American] dream is a secular faith that has unified people across every known divide. It has unleashed ennobling energies and mobilized heroic social reform movements. By dissolving the dream under the acid of an excessive realism, you trap generations in the past and destroy the guiding star that points to a better future. Maybe you will find my reactions irksome. Maybe the right white response is just silence for a change. In any case, you've filled my ears unforgettably. (para. 15–16)

Alexander (2015) begins her book review of *Between the World and Me* (Coates 2015), stating that as an African American, Coates's essays and blog posts for *The Atlantic* (where he is a national correspondent) over the years has made her proud: "There is no other way to put it. I do not always agree with him, but it hardly matters" (para. 1). Throughout her review, Alexander shows enthusiasm for Coates's prose, noting that she read the book twice before she was able to decide whether Coates actually did what she expected and hoped he would: "He did not. Maybe that's a good thing" (para. 4).

During her first reading, Alexander (2015) confesses that she was disappointed. She had hoped that Coates would carefully define the American dream and clearly show the difference between the universal dream that parents have for their children—good heath, security, quality education, and so on—and the "insidious Dream that is destroying the lives of [Black] children in Baltimore [Coates's childhood home] and threatening human existence on the planet itself" (para. 16). She also had hoped that Coates would reveal exactly what it might mean to choose "the Struggle over the Dream, and why so many black people…find themselves lost in the Dream" (para. 16). During her second reading, however, Alexander states, "I held no such expectation that the big questions would be answered" (para.16). But rather what she understood the second time around was that Coates was intentionally offering no answers but instead was critically challenging the reader to wrestle with probable questions and answers on her or his own: "Maybe this is the time for questioning, searching and struggling without really believing the struggle can be won" (para. 17). Nonetheless, in summation, Alexander writes:

> I tend to think we must not ask whether it is possible for a human being or society to become just or moral; we must believe it is possible. Believing in this possibility—no matter how slim—and dedicating oneself to playing a meaningful role in the struggle to make it a reality focuses one's energy and attention in an unusual way. (para. 18)

Hopefully, using excerpts from Brooks's (2015) and Alexander's (2015) essays as a model for productive opposing public intellectual engagement will provoke similar productive engagement from mathematics educators around the crucial issues discussed in these first five chapters of this edited volume. Neither Brooks nor Alexander agreed or disagreed "hook, line, and sinker," so to speak, with Coates (2015). The selected excerpts from the two essays illustrate how both Brooks and Alexander, during their respective readings of *Between the World and Me*, at times leaned in and at other times leaned back. Maybe that is what a "good" intellectual does; she or he gets readers to oscillate between *intellectually* leaning in and leaning back.

So no matter what the theoretical and political perspective(s) of the reader, the authors of each of the chapters in this section, as well as those throughout the edited volume, are asking the reader to intellectually engage with them while considering different possibilities for mathematics and mathematics teaching, learning, and policy. In the end, intellectual engagement, at times, might mean, "to sit with [the different possibility], to make sure the testimony is respected and sinks in," or "just silence for a change," while at other times, intellectual engagement might mean questioning, searching, and struggling while believing simultaneously in the possibilities and impossibilities of the struggle, "focus[ing] one's energy and attention in an unusual way." Intellectual engagement also means holding virtual conversations with the chapter authors, posing additional questions to the authors from whatever ideological perspective(s) the reader hails, shaking up habitual ways of working and thinking, dissipating conventional familiarities, and reevaluating rules and institutions (cf. Foucault 1984/1996). After all, that is the role of the *public* intellectual, isn't it?

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Part II
Identifying and Connecting to Family and Community Funds of Knowledge

Chapter 7
Connecting Algonquin Loomwork and Western Mathematics in a Grade 6 Math Class

Ruth Beatty

Abstract In this project we explored the connections between Algonquin ways of knowing and the Western mathematics that is represented in the current Ontario provincial mathematics curriculum. Using an ethnomathematics framework, we worked with community members from the Algonquins of Pikwakanagan First Nation to co-design and co-teach a Grade 6 lesson sequence based on Algonquin loom beading. As a research team made up of Algonquin and non-Native educators, we documented the mathematical thinking and cultural connections that emerged. Results indicate that the activity was both mathematically rigorous and culturally responsive. Creating and analyzing looming patterns supported students' algebraic, proportional, and spatial reasoning. Community members made connections between the environment created in the classroom, which was based on trust, humor, and proximity to available experts, and the safe learning contexts they had experienced as children. They also indicated that this type of experience supported students' pride in the Algonquin identity and strengthened their relationships with non-Native peers. This project illustrates the potential of co-designing and co-teaching mathematics instruction as a first step to creating meaningful community and classroom interactions.

Introduction

Ministries of Education across Canada have recognized the need to explicitly incorporate Indigenous content to support identity building and appreciation of Indigenous perspectives and values. There is a need in contemporary education to understand how to provide Indigenous students with meaningful connections to their learning. In recent years, work has been undertaken to incorporate First Nations, Metis, and Inuit perspectives in curriculum subjects such as social studies, but it is our belief that connections to Indigenous content should permeate all

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curriculum content, including mathematics. This project explored connections between the mathematical content knowledge required of the Ontario curriculum expectations and the mathematics inherent in Indigenous cultural practices.

The work outlined here was conducted in a Grade 6 classroom at a small public school near Pembroke, Ontario. The school population comprises approximately 20% Algonquin students from the nearby Algonquins of Pikwakanagan First Nation and 80% non-Native students. In this work, we brought together community and school in support of mathematics learning, with a long-term goal of transforming school-based education from a colonial past into a form that respects both Western and Indigenous traditions.

Theoretical Framework

We drew upon ideas from ethnomathematics research and culturally responsive education. In our education system, the dominant Eurocentric culture is manifest in the content of educational curricula, including mathematics curricula (Battiste & Henderson, 2000; Lipka, Mohatt, & the Ciulistet Group, 1998). Mathematics instruction, as reflected in the Ontario provincial curriculum, goes back to a Western European tradition of mathematical knowing (Bishop, 2002). Mathematics was, and still is to some degree, taught with the intention of scaffolding learners from elementary to more complex levels of mathematical thinking, through secondary school to university level. This comes from an historical tradition of educating the elite and using mathematics as a gatekeeper to higher education (D'Ambrosio, 1985). An elitist vision of mathematics education contributes to feelings of alienation that many students, particularly Indigenous students, feel toward mathematics (Barta, Jette, & Wiseman, 2003). This view does not allow students who may not wish to pursue higher-level mathematics to experience the beauty of mathematics for its own sake, nor does it allow them to develop a positive personal relationship with mathematics.

Ethnomathematics research includes a growing area of research about how Western mathematics curricula can and should connect to local culture (D'Ambrosio, 2006; Knijnik, 2002). Ethnomathematics has been thought of as reclaiming mathematics as part of Indigenous culture; however, previous researchers have had different interpretations about what mathematical reclamation entails (Barton, 1996). One interpretation of ethnomathematics has been that school mathematics is one of many diverse mathematical practices and is no more or less important than mathematical practices that have originated in other cultures and societies (Mukhopadhyay, Powell, & Frankenstein, 2009). Another interpretation was based on generating mathematical thinking from combining traditional Indigenous sources and conventional mathematical thinking, what Gerdes (1988) termed "unfreezing" the mathematics from cultural artifacts or activities. The idea of unfreezing the mathematics referred to those who analyze an activity or artifact and identified the "hidden" mathematics. Barton (1996) defined ethnomathematics as creating a bridge between mathematical thinking and the practices of other cultures, with the aim of both reconceptualizing cultural activities through a lens of mathematical interpretation and highlighting new ways of conceiving mathematical concepts. What these interpretations had in common was that the mathematical thinking identified in different cultural contexts, or in various cultural artifacts, was deemed to be mathematical because of its alignment with Western mathematical thinking.

Mathematical thinking is not, however, simply about participating in an activity. Also needed is a context within which students can reflect on the mathematical relationships embedded within the activity. Mathematizing is a way of articulating or highlighting the mathematical aspects of an activity by translating the material into mathematical terminology or relating it to existing mathematical concepts (Ascher, 1991). The mathematics inherent in the activity can be identified and extended to a creative, mathematical investigation. Looming is an activity that can be undertaken without consciously or explicitly focusing on mathematics. We were interested to explore, however, the consequences of having community members, teachers, and students explicitly mathematize an activity in order to explore the connections between Algonquin looming and mathematics and to create an opportunity for Algonquin students to see their culture reflected in mathematics instruction.

Culturally responsive mathematics education refers to efforts to make mathematics education more meaningful by aligning instruction with the cultural paradigms and lived experience of students (Castagno & Brayboy, 2008). Making connections between math instruction and Indigenous culture has had beneficial effects on students' abilities to learn mathematics (Cajete, 1994; Lipka, 1994; Lipka, Sharp, Adams, & Sharp, 2007). Long-term studies by Lipka (2002), Brenner (1998) and Doherty, Hilbert, Epaloose, and Thar (2002) found that culturally responsive education in mathematics had statistically significant results in terms of student achievement. Recent researchers have also explored the insights Indigenous epistemologies and practices provide for understanding ways of teaching mathematics (Barta & Barkley, 2001; Barta et al., 2003; Battiste, 2002, 2004; Hampton, 1995; Leavitt, 1995; Nielson, Nicol, & Owuor, 2008). While these studies suggested that Indigenous pedagogical approaches benefitted both Indigenous and non-Native students' mathematics learning, few studies have focused specifically on connecting Anishinaabe and Western mathematical perspectives. This is an important connection to make because Anishinaabe communities comprise one of the largest Indigenous groups in Canada.

Cultural knowledge can be defined as knowledge derived from settings outside of school, such as in the home or in the community. As outlined below, practices such as looming were (and are still) taught informally in students' homes, along with some of the historical significance and importance of these practices. In the Algonquins of Pikwakanagan First Nation, activities like looming were part of community teachings passed down from elders and were retained by some community members even during the time that other aspects of culture, such as language and ceremonies, were forbidden. The culture of the Algonquins of Pikwakanagan First Nation, as it pertains to this study, encompasses the community's emphasis on rediscovering and revitalizing cultural practices and language.

Most ethnomathematics studies have focused on the mathematical thinking of diverse cultures to make connections between Indigenous cultural activities and Western mathematics in contexts where local cultures are less influenced by

Westernized mathematics textbooks and pedagogy. We were interested in investigating the potential of cultural activities to support mathematical instruction for Indigenous students who live in an Algonquin community, who are educated in a publicly funded provincial school and are familiar with Western mathematics, but whose culture is not incorporated in their classroom experience.

Methods

One concern working as non-Native researchers within Indigenous cultures is the risk of appropriating culture. Cultural appropriation is the taking of Indigenous knowledge to use within a different cultural context, without truly understanding the cultural significance of the knowledge. Our core research team for this project, therefore, included cultural insiders and outsiders. The team was made up of two Algonquin teachers, including Jody Alexander, along with the operations manager of the Algonquins of Pikwakanagan Cultural Centre, Christina Ruddy, who is an expert loomer. The team also included three non-Native teachers from the school, including the Grade 6 teacher, Mike Fitzmaurice, in whose classroom the study took place. We followed a cyclical approach (consult, plan, teach, reflect, share) in the project to insure that we continually cycled back to members of the Algonquins of Pikwakanagan community for guidance and feedback. The cycle included a consultation phase during which we met with community leaders from Pikwakanagan, who were invited to share their insights and provide guidance.

A key theme raised by community members was the importance of revitalizing Algonquin culture. As one community member stated, “From the outside it’s perceived that we have all the culture. We don’t. Culture was taken from us. We are actually in the process of learning our culture again.” Like most Indigenous communities in Canada, the Algonquins of Pikwakanagan were forced to abandon many cultural practices, which they are now in the process of reviving. One focus of this cultural revitalization has been beadwork, including loomwork, which has always had significance for this community. This focus on cultural revitalization in the community was something the advisors believed should be reflected in students’ experiences in the classroom. They also spoke about the importance of engendering pride in identity by supporting students to navigate Algonquin and Western cultural perspectives.

Another key message was the importance of making mathematics instruction meaningful for the children, because mathematical understanding is important for effectively participating in Canadian society. Community members articulated a desire to afford their children the opportunity to find meaning in, and develop positive relationships with, mathematics by “seeing themselves in the math classroom,” because these students had not had a chance to explore mathematical thinking in a way that reflected priorities and experiences that are important in their community. Community members also identified the difficulty many students have learning mathematics, particularly those who struggle with pedagogical approaches that prioritize memorization.

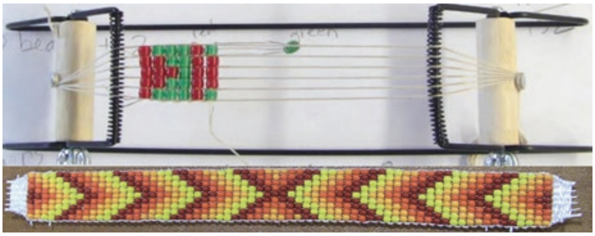


Fig. 7.1 Loom beading

Howard: Some kids still memorize times tables but there are other ways of attaching meaning to it and kids will remember because there is meaning to it. Rote memory is okay if your brain can do that but for some kids they can't. But if it is something that's meaningful for them they will be able to remember it, and it's maybe a longer process but they will be able to do it. So those are the kinds of things that we are looking at. How do we reach those kids?

Shirley: I think its hands on! They need to see and do!

This aligns well with current approaches to mathematics teaching, which emphasize the development of conceptual understanding in meaningful contexts rather than rote memorization and symbol manipulation (National Council of Teachers of Mathematics [NCTM], 2000).

For this study, we co-planned a unit of instruction based on Algonquin looming. Looming is a type of beading that is done on a loom and involves stringing beads onto vertical weft threads and weaving them through horizontal warp threads (see Fig. 7.1).

During the co-planning phase, Christina taught the rest of the research team how to loom and shared her insights about some of the inherent mathematics of looming:

> When you're deciding the width of the loomwork you count your beads, how long and how wide you want it. So, how long determines the number of columns and how wide determines the number of rows. There's a lot of counting. When you take your design and put it on the loom then you need one extra thread on the loom for every count of beads that you do. For example, a bracelet that is 9 beads wide will need 10 warp threads on the loom. Then you have to measure your wrist and decide how long your bracelet needs to be.

The team co-designed a sequence of lessons centered on looming. This study focused on approaches to mathematics instruction based on Indigenous activities, and the lesson designs prioritized exploration. Lessons were structured in terms of the sequence of looming patterns Christina introduced to the students. However, the processes to engage in mathematical thinking were generative; we had no predetermined paths of investigation for students to follow. Instead, teachers and students co-constructed avenues of inquiry, delving into mathematical ideas as they naturally arose during the activities.

The team also arranged for Albert Owl, a fluent speaker of The Language, to teach some Algonquin Language during the lessons. Ten to twenty minutes at the beginning of every lesson was devoted to teaching students Algonquin words and phrases related to the activity of beading. As Mr. Owl taught students words, he also explained some of the meaning behind the words. The word for bead, manidominens, comes from the word Manido, Creator or spirit, and minens, meaning small piece, so a bead is a small piece of the Creator, or a small spirit. Mr. Owl deconstructed The Language so students learned, for example, the suffix “tig” means that an object is made out of wood. The word for loom, mazinàbido-iganàtig, and pencil, ojibìhiganàtig, both end with this suffix. Mr. Owl also shared stories and some teachings about beading, including the teaching that if you drop a bead, it is important to pick it up because by doing this you show respect for the bead and the activity of beading. Introducing Algonquin Language through the activity of looming helped students develop a meaningful connection to The Language.

Data Collection

Mike and Christina co-taught the lessons over 2 weeks in a Grade 6 classroom. One-third of the 27 students in the class were from the community of Pikwakanagan and two-thirds were non-Native. All lessons were videotaped and field notes were completed each day by the author. These notes included an overview of the sequence of activities and a summary of the mathematical thinking evidenced either through students’ verbal answers or their written work. All student work, including patterns drawn on paper and beadwork, was photographed. We videotaped Christina’s and Jody’s reflections at the end of each of the 2 weeks to capture the cultural connections they perceived. Given that both are members of the Algonquins of Pikwakanagan First Nation, we wanted to ensure we honored their voices during the project.

Lessons were shared with community members at community meetings during which we showed compilation videotapes to give community members an overview of the activities in which we had engaged and examples of students’ mathematical thinking. Attendees spoke about the fact that, from watching the mathematical thinking of students, “everyone can learn math, really complicated math” and that “everyone has a mathematical voice.” One of the attendees, a band councilor whose grandchildren attend the school, stated that what she saw was “an awakening of a spirit of mathematics” and was proud that this awakening had come through the process of Algonquin looming.

Data Analysis

Lesson video was edited into segments that related to a specific learning episode generated from the looming activity, and those segments were transcribed. For example, when one student in the class posed the question, “What would the column before column 1 look like in a chevron pattern?” the ensuing discussion was edited into a segment that was then transcribed and analyzed for mathematical thinking. The video segments evidencing mathematical thinking, transcripts of those segments, field notes, and student artifacts were analyzed by the research team using a framework to identify mathematical thinking. The framework for identifying mathematical thinking consisted of the following areas:

Unitizing

- Working with a unit simultaneously as one unit and the number of elements that make up the unit
- Identifying and working with different sizes of units including the pattern core (unit of repeat), the pattern core made up of *n* columns, the pattern core made up of *n* beads, the template made up of *n* pattern cores, a bracelet made up of *n* templates

Algebraic Reasoning

- Identifying the unit of repeat in a two-dimensional pattern
- Identifying the covariation between the numbered columns and the beads within each column and the relationship of the numbered columns to the unit of repeat in a repeating pattern, to make predictions further down the sequence (e.g., the 64th column)
- Creating generalizations

Proportional Reasoning

- Using multiplicative relationships to compare quantities and to predict the value of one quantity based on the values of another. For example, comparing the number of columns in a pattern to predict the number of centimeters in beadwork

Spatial Reasoning

- Understanding relationships within and between spatial structures
- Understanding the relationship between visual and numeric representations of quantity
- Decomposing, for example, decomposing a pattern into identifiable units of repeat
- Mental rotation and transformation

Coding was carried out by the author. Each learning episode could receive more than one code, particularly since the concepts are intertwined. The author then developed brief case accounts describing each learning episode. The team then reviewed video segments and written case accounts and discussed each to reach consensus about the students' demonstrated mathematical thinking.

Results

Results focus on the mathematical thinking documented from week 1 of the lesson sequence, which focused on pattern design. In addition, some of the reflections from the Algonquin members of the research team are presented in which they identify aspects of the project that they found culturally relevant. We begin by providing an overview of the context in which this learning occurred.

Designing Patterns in the Classroom

During co-teaching, Christina taught the art of looming to the students and introduced them to the design process. Mike facilitated class discussions about the mathematical content of the work by asking students, for example, to identify the unit of repeat of different patterns. Inevitably, although the Indigenous members of the research team identified the need for authenticity in teaching the activity, the resulting experiences were both culturally authentic and inauthentic. We strove to create the kinds of learning experiences community members remembered from their own childhoods, when they learned beading techniques from community elders around kitchen tables. Our goal was also, however, to bring out mathematical thinking, and to this end we facilitated the exploration of mathematical ideas as they arose. For example, the template we created for students to use as they designed their patterns was based on the kinds of drawings Christina and others in the community created prior to looming. Some of the inquiries students engaged in, for example, calculating the numbers of beads needed for different designs, were questions that students asked during the course of the investigation but are not areas on which traditional loomers would necessarily focus. Christina told us that most beaders estimate the numbers of beads they require and that this focus on exact numbers seemed to her more of a Western idea.

Christina explained to the students that looming is a traditional Algonquin activity that pre-dates the arrival of Europeans to North America. Historically, sinew and porcupine quills and shells would have been used, but currently looming is done with nylon thread and plastic or glass beads. The pattern for each looming project is created on graph paper (see Fig. 7.2). The first step in creating a design is to define the space to be used. Based on Christina's work, we created a design template of 20 columns and 7 rows. The columns represent the weft threads, on which beads are threaded and woven onto the warp threads. The number of columns corresponds to the horizontal length of the beadwork. The rows represent the spaces between the warp threads, where the beads sit when they are woven into the work. The number of rows corresponds to the width of the beadwork.

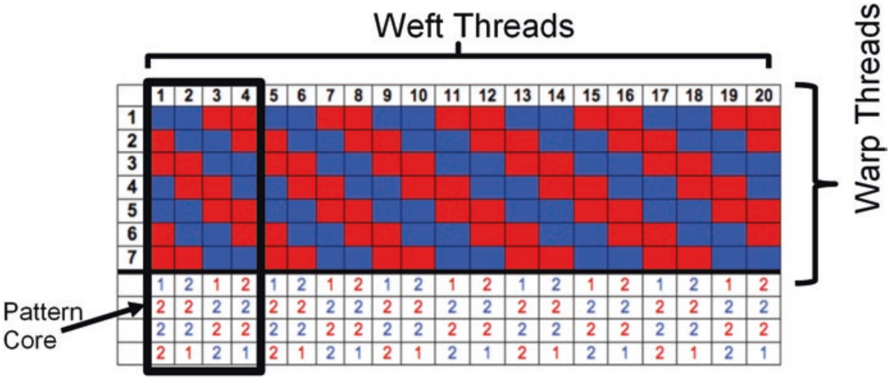


Fig. 7.2 Diagonal loom pattern

The rows and columns of the design space are numbered. Below each column, the number of each color of bead is entered (with the number representing the color of the beads, in this case blue numbers for blue beads and red numbers for red beads). This helps the beader know the order for stringing beads for each column, or line of beads on the weft thread. Each column should add to the total number of beads on each weft thread (so in the example in Fig. 7.2, each column should add up to 7).

Students were taught to copy and extend two patterns, the "diagonal" and the "chevron," which reflect the geometric nature of early Algonquin designs. Students explored these and other patterns through a series of investigations outlined below. Finally, students were invited to design their own pattern, which was then used to create their beadwork.

Diagonal Pattern Christina introduced the diagonal pattern using an interactive whiteboard as a way of orientating students to the columns and rows of the grid and to demonstrate the conventions of planning a pattern using a template. Christina filled in the template up to the 10th column, and students were asked to copy and extend the pattern to the 20th column. They were then asked to identify the section of the pattern that repeats (which was termed the "core" of the pattern). Identifying the unit of repeat in this pattern required students to analyze the visual and numeric structure of the pattern. The students identified the first four columns as the unit of repeat and had various ways of justifying their thinking. Some students looked at the numeric patterns and identified that number pattern for columns 1–4 repeated every four columns (see Fig. 7.2). Other students looked at the visual pattern of the first four columns and predicted that this four-column structure, when rotated 180 degrees, would be congruent. The first four columns represented the fewest number of columns for which this was true and so were identified as the core (e.g., the same is true if you rotate a core made up of columns 1–8, but this can be broken into two groups of four columns). Students tested their theory by copying the four-column structure on the interactive whiteboard and rotating the core. Still other students used "occluders" (strips of paper) to isolate different parts of the pattern to determine the unit of repeat and through visual trial and error identified the first four columns as the core of the pattern (see Fig. 7.3).

Christina then introduced students to a pattern of chevrons that were two beads wide and created using two alternating colors: a two-color two-bead chevron pattern

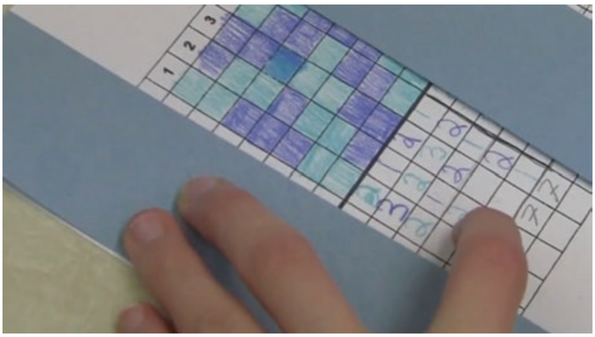


Fig. 7.3 Using "occluders" to find the unit of repeat exploring chevron patterns

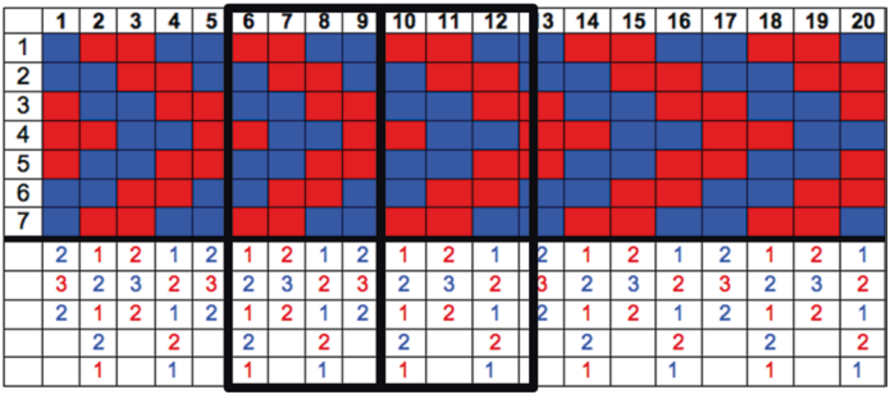


Fig. 7.4 Two-color two-bead chevron loom pattern

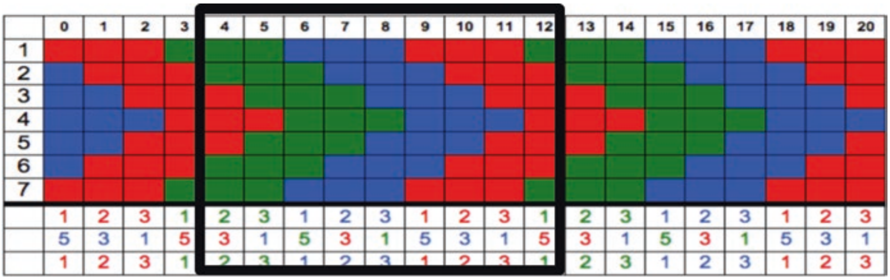


Fig. 7.5 Three-color three-bead chevron with column 0

(see Fig. 7.4). Students were asked to find the core of the pattern. Some students focused on the numeric pattern and noted that even though the numbers repeated after the second column, the colors were different, so the first four columns made up the core. Other students identified the visual pattern of the first four columns and described the central blue chevron surrounded by parts of red chevrons on either side.

Next, students were taught to create a three-color three-bead chevron (see Fig. 7.5). Students identified the core by finding the columns that were the same (either visually or numerically). Students found that column 10 was identical to column 1 and reasoned that the core comprised columns 1–9. Students described imagining superimposing the first nine columns onto the next nine columns to see if they "matched." They then used the interactive white board to copy the core and overlay it onto columns 10 to 18. Noticing that the pattern seemed to begin with a partial chevron, a few students wondered what would happen if they added a column "before" (to the left of) column 1 on the pattern (which they referred to as "column 0") and whether this would change the pattern core. The rest of the class then explored this line of inquiry. They discovered that if a column was added to the left of the core, and the new column was considered the beginning of the core, then the whole core "shifted" to the left by one column and comprised columns 0–8 (see Fig. 7.5). They continued to add columns to the left of column 1 and noticed the core shifted by as many columns as were added. They also

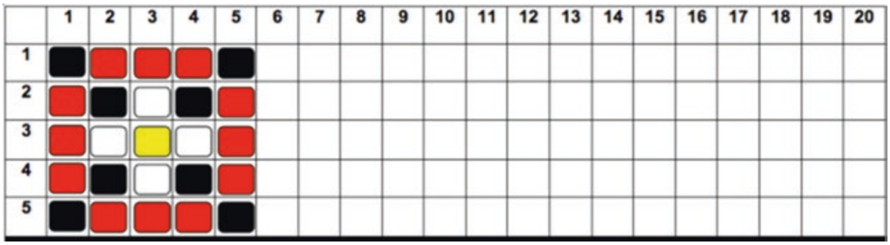


Fig. 7.6 Flower pattern

noticed, however, that the number of columns in the core did not vary. For a three-color three-bead chevron pattern, the core was always made up of nine columns.

The students identified that any nine consecutive columns in the pattern could be considered the core and that identifying the first nine columns as the core was arbitrary since the pattern could extend to the left as well as the right of the initial given element; the width of the core did not vary. This prompted a student-generated inquiry about whether it would be possible to predict the width of the core for any chevron pattern. Each student was asked to create a chevron pattern made up of any size chevron and any number of colors. Students then analyzed different chevron patterns to identify the core. They reviewed a two-color two-bead chevron with a core of four columns and a three-color three-bead pattern with a core of nine columns. Next they considered a two-bead four-color chevron and found the core was eight columns and a two-bead five-color chevron with a core of ten columns. Based on these experiences, the students generalized to predict the number of columns in the pattern core of any chevron pattern. They found that the width of the core was determined by a multiplicative relationship between the width of the chevron (the number of beads that made up the chevron) and the number of colors. To test this theory, they created a three-bead five-color chevron that they accurately predicted would have a core comprised of 15 columns. Students also discovered that their theory worked for the diagonal pattern as well, because when the top three rows of a seven-row diagonal pattern are reflected vertically, the diagonal pattern becomes a chevron pattern.

Other Patterns The students were then introduced to other patterns, like the flower pattern (see Fig. 7.6).

As they worked to determine the unit of repeat, students initially identified the core as column 1 to 5 and predicted that column 6 would be the same as column 1. They also argued, however, that the core might be considered as extending from column 1 to 4, and that column 6 would, therefore, look the same as column 2. Mike added some beads to the pattern so students could judge which of their conjectures were accurate (see Fig. 7.7). The students agreed the core was four columns.

They were then asked to make predictions about columns in the pattern further down the sequence, for example, what the 64th column would look like. Students had a few ways to solve this. Sylvie said, "I think it's column 4 because you just need to times 20 by 3 and that's 60 and go 4." Underlying this answer is Sylvie's understanding that the 20th column would be the same as the 4th column, and three full templates plus four more columns (or one pattern core) would be the 64th column.

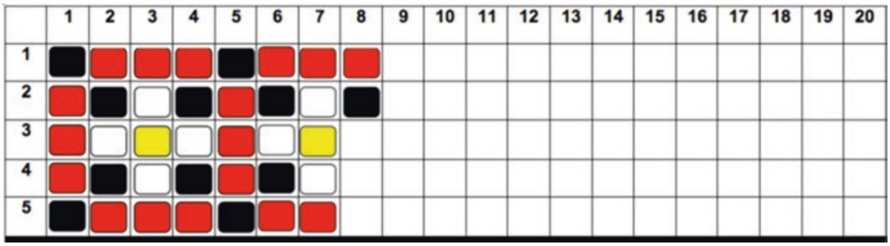


Fig. 7.7 Extended flower pattern

Adam stated, "It would be column 4 because 4 is a direct multiple of 64 and it would come around to be a full pattern core." This reasoning shows an integration of both the numeric relationships and the visual aspects of the pattern, since Adam justified his thinking both through stating a multiplication fact and an understanding that the 64th column would be the last column of a core. Mike then asked the students how many repetitions of the core would be included up to column 64. Again, explanations from students demonstrated an integration of numeric and visual reasoning. For example, Jonah's first explanation was grounded in multiplication facts. "It would repeat 16 times because 10 times 4 is 40 and 6 times 4 is 24 and add them together, it's 64." His response was challenged by Sam.

Sam: Sixty. You were thinking of 16 as 60. But there's one extra one, so 64, that would actually be 17.
Adam: One extra what?
Sam: One extra pattern core.
Jonah: No it's not, because 5 times 4 equals 20, which would make it 60, so that's 15. Then when you add one more it would be 16 to make it 64.
Sam: Oh, yea, ok.

Implicit in this argument is the understanding that the 20-column template will contain 5 repetitions of the 4-column core. Sam argued 16 repeats of the core would only reach column 60, and so one more core would be needed to get to column 64. Jonah, however, responded that "5 times 4 equals 20," meaning there will be 5 repetitions of the core in one 20-column template, "so that's 15" meaning for 3 full templates, there would be 15 cores, and adding one more pattern core would add 4 columns to 60 to end at column 64.

Individual Bracelet Designs At the end of the first week, the students designed their own pattern to use as the basis of their bead creations. By measuring some sample bracelets Christina had brought into the classroom, they discovered that five columns on the pattern template equaled 1 cm of beadwork. Students measured their wrists and then used the relationship of 5 columns = 1 cm to calculate how many columns long their bracelets would need to be in order to fit. They also calculated how many beads in total they would need and how many beads of each color they would require. Some students, like Ella, designed a pattern with a five-column core and reasoned that the number of repeats of the core would equal the size of her wrist in centimeters (see Fig. 7.8). Ella measured her wrist at 15 cm, so she would need 15 repeats of her core. Since each core comprised 35 beads, she calculated that

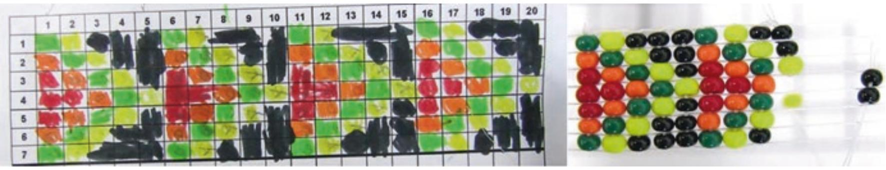


Fig. 7.8 Ella's five-column pattern core

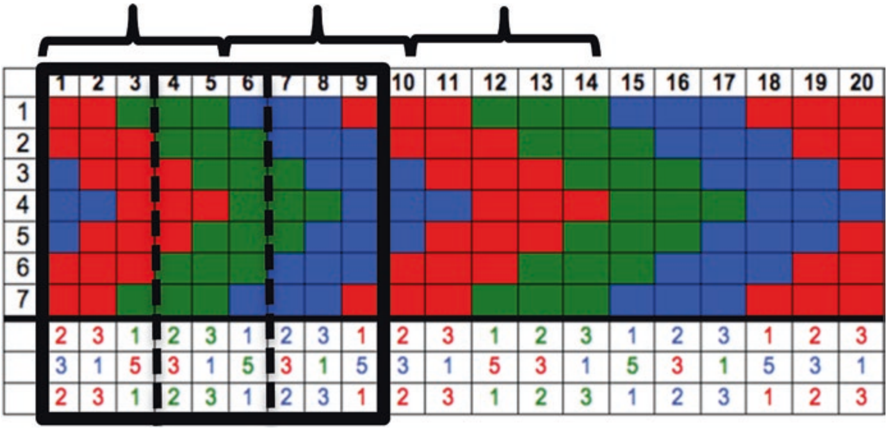


Fig. 7.9 Three "sub-cores" in a nine-column pattern core

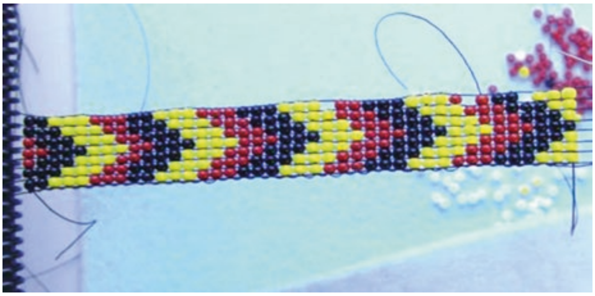
the total number of beads she would need would be 35 times 10 (350) plus 35 times 5 (175), for a total of 525 beads. She double-checked the total by calculating the number of columns (75) and multiplying that by the number of rows (7).

Many students, though, designed patterns that were not based on a five-column core. Julia, for example, measured her wrist at 15 cm and knew her final bracelet would need to be 75 columns in length. She found a numeric pattern in her three-color three-bead chevron that repeated every three columns; however, she saw that visually the core was nine columns wide, which could be broken down into three different "sub-cores" (see Fig. 7.9). She calculated the number of sub-cores that would fit into her 75 columns and determined that sub-core 1 would repeat nine times and sub-cores 2 and 3 would repeat eight times each. She then counted the number of different colored beads in each sub-core and multiplied that by the number of times the sub-core repeated. Sub-core 1 had 15 red, 4 blue, and 2 green, each of which was multiplied by 9. Sub-core 2 had 15 green, 4 red, and 2 blue, multiplied 8 times. Sub-core 3 had 15 blue, 4 green, and 2 red, multiplied 8 times. This resulted in a total of 525 beads.

Luke designed a patchwork design and found his pattern core was nine columns (see Fig. 7.10). His wrist measured 18 cm, so his design needed to be 90 columns with ten repeats of the pattern core. The design consisted of an overall 9 × 9 array, made up of smaller 3 × 3 arrays. Since he used only three colors, each color required 27 beads, so the total number for each color bead was 27 times 10 (units of repeat) for a sub-total of 270 beads. He then multiplied 270

Fig. 7.10 9 × 9 Luke's array pattern core

Fig. 7.11 Jack's chevron



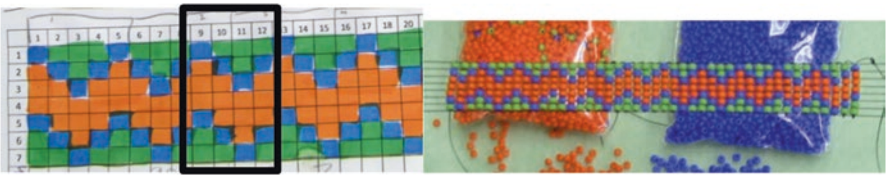


Fig. 7.12 Wyhatt's four-column pattern core

times 3 to figure out that he would need a total of 810 beads. He confirmed this by calculating the total number of beads in the 9 × 9 pattern core and multiplying the answer, 81, by the ten repeats of the pattern core.

Jack also needed 90 columns for his 3-color 3-bead chevron (see Fig. 7.11). He counted 14 beads in each chevron. He drew his pattern to the 90th column and found there were 15 yellow, 14 black, and 14 red chevrons. He then realized that each partial black and red chevron at the beginning and end of the pattern would, if put together, create a full chevron, so there were 15 chevrons of each color. He multiplied the number of chevrons by the number of beads per chevron to get 210 beads of each color, which multiplied by 3 meant 630 total beads.

Finally, Wyhatt's pattern core had four columns (see Fig. 7.12). He calculated that 5 pattern core units = 20 columns, or 4 cm; that 10 pattern core units = 40 columns, or 8 cm; that 20 pattern core units = 80 columns, or 16 cm; and that he would need one more core, plus a column, to reach 85 columns, or 17 cm.

The work of this project adheres to four components necessary for culturally responsive math education: (1) focus on important mathematics, (2) relevant content, (3) incorporate student identities, and (4) shared power (Averill et al., 2009). The mathematics the students explored was rigorous and complex. The content was relevant for students, both for those from the community and also for non-Native students who were equally interested and engaged in the activities and mathematical discussions. The Algonquin students saw their culture reflected in math instruction, and through our process of inquiry and discovery, all students contributed to the mathematical knowledge building – every student's ideas were important and acknowledged.

Mathematical Thinking

This sequence of lessons contributed to students' understanding of a number of key mathematical concepts inherent in beading, including identifying complex patterns, algebraic reasoning, proportional reasoning, and spatial thinking.

Patterning and Algebra Research has suggested that an important part of developing algebraic reasoning is the ability to identify the mathematical structure of a repeating pattern and to use that structure to make predictions about the pattern far down the sequence, which is an early form of constructing a mathematical generalization (e.g., Mulligan & Mitchelmore, 2009; Mulligan, Prescott, & Mitchelmore, 2004). Most of the visual patterns students typically encounter in elementary classrooms are created horizontally, with an emphasis placed on finding the pattern core and finding "what comes next" (i.e., to the right) of the given elements (McGarvey, 2013). In this project, students considered patterns that extended in two dimensions and were far more complex than a series of shapes or numbers. Copying, designing, and creating these patterns supported students to analyze this complex structure. We found that students were able to shift their focus to consider four different levels of the pattern: (1) the beads within each column, (2) the core of the pattern made up of a specific number of columns, (3) the number of beads in a core, and (4) the overall design of the pattern.

The idea of a "shifting" pattern core seemed to be a new idea for these students, as they would likely have never been previously prompted to consider that a pattern can extend to the left as well as the right of the given elements. This allowed them to recognize that the core of a pattern is not necessarily defined by the first set of elements and that what remains constant is the size of the core (in this case, the number of columns). This discovery prompted the students to capture the relationship between the size of the chevron and the number of colors to create a generalization in order to predict the size of the pattern core (bead × color = core width).

The students were also able to make predictions far down the sequence of different patterns, for example, predicting that the 64th column for a 4-column pattern would look like the 4th column because 64 is a multiple of 4, and the 64th column represents the final column of the 16th repetition of the core. These predictions were based on recognizing the covariation between the numbered columns and the elements in the unit of repeat (the beads in each column) and the number of columns comprising the core. What was interesting was how students justified their predictions using both their knowledge of multiplication (i.e., that 64 is a multiple of 4) and the structure of the pattern and its relation to the template, that is, describing the 64th column in terms of 3 completed 20-column templates (or 15 pattern cores) plus one more core.

Proportional Reasoning In this study, students used proportional reasoning to estimate the total length of their finished bracelets using a fixed ratio of 1 cm = 5 columns. Students whose pattern cores were also 5 columns were able to translate the size of their wrist in centimeters directly into the number of pattern cores required. Other students, however, used other units in their reasoning. For example, Wyhatt's reasoning used a composite unit: the 20-column template represented 5 pattern core units and 4 cm simultaneously. Students exhibited an ability to use single or composite units as the basis for multiplicative thinking and could make decisions about which unit to use for their calculations (e.g., using the unit of 5 columns for 1 cm, or using the unit of the pattern core of 4 columns related to an understanding that 20 columns = 5 cores *and* 4 cm). Many proportional reasoning activities found in elementary mathematics curricula are designed to encourage students to apply a memorized rule or algorithm; however, the application of memorized rules does not mean students are reasoning proportionally (Lamon, 1993). The problems posed during this unit were practical and engaging because the teachers needed to know how many beads to order and of what color. As one student put it, "we needed to figure out the number of beads pretty accurately, because if we didn't order enough beads we wouldn't be able to finish off our bracelets!" The students had not been taught specific strategies for reasoning proportionally but were able to do so because they understood the context of the problem.

Spatial Reasoning Numerous studies have suggested that spatial reasoning skills, including mental manipulation and spatial visualization, are linked to mathematical achievement (Gunderson, Ramirez, Beilock, & Levine, 2012; Mathewson, 1999). Here, we found that designing two-dimensional patterns on a grid, and identifying components of the pattern (like the pattern core), provided an opportunity for students to engage in visuospatial thinking. As they worked to discern the columns that made up the core of the pattern, the students were able to mentally visualize isolating the pattern core and rotating it, or superimposing one core onto the next to determine whether it "matched." These mental processes were then checked using the interactive whiteboard. In addition, the planning of patterns (and subsequent beading) required students to consider spatial relationships on many different levels: the relationship of the beads within a column, the relationship of a column to surrounding columns, the relationship of the columns representing the "core" to the bracelet as a whole, and the relationship of beads and columns with respect to the overall design of the piece. All of these relationships were considered within a two-dimensional grid (the paper designs) or in three dimensions on the loom.

Engagement and Mathematical Thinking

After reviewing transcripts of students' experiences in these lessons, we found that integrating a traditional cultural practice as the basis for mathematical instruction seemed to suggest enjoyment and engagement for all students, as well as high levels of mathematical thinking, which Christina believed were intertwined: "The thing that impressed me most about the videos of student learning is that they were *proud* of what they did. It wasn't hard for them to talk about the math because they enjoyed it so much." Although the lessons supported students to engage with complex concepts such as algebraic and proportional reasoning, the means of facilitating this learning was through the process of designing and constructing beaded bracelets. Students spent up to 3 hours at a time in the classroom not only working on designs but also solving mathematical problems that emerged from the activities, reflecting Howard and Shirley's advice to make the content meaningful and hands on.

The method of instruction also played a role in student success. For some looming lessons, Christina used direct instruction to teach students how to use a grid template to design their patterns and how to create certain designs. For a majority of the time, though, the classroom was more informal. Students designed their own bracelets and tackled the mathematical problems that arose naturally from the activity. They worked with Christina, or in small groups or individually, to find solutions. Jody likened this to her experiences as a child growing up in Pikwakanagan and learning how to bead from community members:

> Christina was sitting at the front table that was set up for the students. She was working with a group of girls, and it reminded me of when I was growing up. I must have been about 8, and my cousins and I went to an elder's house, and we did beading. I recall that there was a big wooden kitchen table, and we had beadwork all over the table. Christina working with the students just reminded me of that because we were just sitting, we were chatting, we were laughing. There was no worry about making mistakes; there was just, we were all learning together. And you were sitting in close proximity to someone who you trusted to teach you whatever it was you were doing. You know, a lot of humor and laughing among us, and it was a safe place, and I think that's the part that reminded me of when Christina created that environment in the classroom. Quite often we think of classrooms in schools as institutional, but I think that the community feeling was created in that classroom.

This feeling of safety, humor, and community provided an opportunity for students to begin to develop a positive, personal relationship with mathematics. Rather than focusing on the acquisition of math concepts as stepping-stones to even more complex mathematics, the students were able to explore the mathematical ideas inherent in the process of design as interesting lines of inquiry, rather than as sequences of memorized steps. Indigenous pedagogy is holistic, in that it emphasizes the need to address the intellectual, physical, emotional, and spiritual development of the student (Barnhardt & Kawagley, 2005; Cajete, 1994). We created an environment where students learned complex math by being active participants in activities that they seemed to care about and through which they made connections to Algonquin culture.

Importance of Indigenous Community Members in the Mathematics Classroom

Another aspect of the experience highlighted by the team was the importance of all students learning from community members who came into the classroom to teach and seeing that the knowledge brought by those members was honored and respected. Jody spoke about her excitement: “I’m excited to see more culture. The students really wanted to hear what Christina said and they wanted her help and I think that’s an important part too – the value of our own people bringing in their understandings.” Christina also spoke about the impact of this experience for Algonquin students and how this experience supported students’ pride in their identity and strengthened relationships with non-Native peers:

> I think it’s important for a First Nations person to teach First Nations skills. What are we doing this for if it’s not cultural? If it doesn’t have some kind of cultural significance then what are we doing this for, is it just to teach math? No. Not when you see the changes in the First Nations students. Seeing them more confident, and the pride in talking with their peers about their lives, their regalia and stuff like that. And it’s nice to be able to share with your best friend who might not be Native a little bit more about your life that they might not know about because they only ever see you in a school setting.

Bringing community members into the classroom expanded the students’ conceptions of “who does mathematics.” As Hatfield, Edwards, Bitter, and Morrow (2007) state, “pride and a sense of hope, rather than learned helplessness, are education’s goals for students who previously would never have considered mathematics as a viable option in their lives.” (p. 70).

Revisiting Ethnomathematics

As previously stated, one aspect of an ethnomathematics approach to Indigenous mathematics instruction has been the “extraction” of mathematics from Indigenous cultural activities. And while this stance has, for our study, yielded rich mathematical thinking that aligned with community goals to have Algonquin students become more proficient in “school math,” the fact is that this work is still framed by a Western perspective of what it means to think mathematically. The Algonquin activity of looming was, in effect, dominated by Western mathematical perspectives. Given that so much of the Indigenous mathematics teachings and language of the community had been taken, we found it difficult to determine what genuine “Algonquin mathematics” might look like. One of the goals of community members was for their students to become proficient in “school math,” and this goal was met within the context of a learning community that mirrored the childhood experiences of community members.

This study is aligned with an understanding of ethnomathematics as “interpretive mathematizing;” that is, identifying and developing the mathematics implied in the activity (Ascher, 1991; Barton, 1996). This contextualization of mathematical ideas meant students could connect with school math ideas through the process of engaging with a traditional Algonquin activity. Conversely, after instruction, Christina explained how she began to view her looming through more of a mathematical lens: "It's easier to design patterns once you understand there are numbers there, not just beads. I've started looking at it from a totally different perspective. Numbers, colors, and shapes in space."

Conclusion

Although this project represents an initial introduction of bringing together Algonquin community members and non-Native educators, we have been able to document the inclusion of mathematics content knowledge (informed by the Western perspectives and the mathematics inherent in the cultural activities explored), pedagogical knowledge (particularly the pedagogical practices brought to the experience by Christina), and contextual knowledge (connecting the school to the community of the Algonquins of Pikwakanagan First Nation). Co-designing and co-teaching units of math instruction are the first steps to creating classroom and community interactions in the hopes that mathematical and pedagogical knowledge will connect school and community contexts. Students learned techniques of looming and engaged in classroom discussions focused on selected ideas and concepts so that the mathematics was revealed.

One way of framing the results of this study is to consider it a "third space" (Gutiérrez, Rymes, & Larson, 1995; Haig-Brown, 2008). We attempted to create this space by documenting the Western mathematical thinking that is inherent in a traditional Algonquin activity and by explicitly privileging Algonquin content and pedagogical approaches alongside Western pedagogy and mathematical content. The resulting learning experiences were considered by the research team to represent a hybrid of Algonquin and Western cultural content and pedagogical approaches.

It was important that all students had an opportunity to learn from community members who came into the classroom to teach and to see that the knowledge brought by those members was honored and respected. Christina emphasized the cultural importance of the activity, which provided an opportunity for students from Pikwakanagan to connect to their own cultural heritage and to develop a sense of their mathematical identity through culture. This project was also important for the non-Native students who gained greater insights into the culture of their classmates and who extended their own mathematical thinking.

The focus of this project was specifically on integrating Algonquin culture and mathematics instruction, and it responds to the Canadian Truth and Reconciliation Commission's call to action to "develop culturally appropriate curricula" (*Canadian Truth and Reconciliation Commission: Calls to Action*, 2015, p. 2). We believe, however, that it also addresses the Commission's call to action outlined as "building student capacity for intercultural understanding, empathy, and mutual respect" (p. 7). Highlighting the teachings of the community members in the classroom gave students an opportunity to participate in knowledge systems that were valued on the same level as Western curricula. This experience can, we hope, form the foundation for lifelong intercultural learning and respect.

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Chapter 8
Conversions for Life: Transnational Families' Mathematical Funds of Knowledge

Miwa Aoki Takeuchi

Abstract In this chapter, I highlight mathematical funds of knowledge unique to transnational families by introducing a study conducted in an urban area of Japan, which is becoming increasingly linguistically and ethnically diverse. This chapter builds on sociocultural theory and the perspective of funds of knowledge (Moll et al., 1992), while paying attention to power dynamics, which is critical to interrogate the legitimacy of knowledge exchanged in school contexts. Based on the framework of tool-and-result methodology (Newman & Holzman, 1993), this study was designed to better understand the needs of Filipino transnational families and also to explore potential actions to collectively address these needs. In the interviews, Filipina transnational mothers commonly undervalued their knowledge and their involvement in school education for their children. Based on this finding, workshops with a group of these mothers were organized. The interactions during the workshops revealed one of the mathematical practices that they used daily: calculating international currency conversions. Interviews with their school-aged children suggested how these children were able to apply the knowledge of international currency conversion and ratios learned through discussions with their mothers. I conclude this chapter by discussing the possibilities of making explicit the lens of power to the study of funds of knowledge and also by providing pedagogical implications for mathematical teaching and learning in the context of globalization.

Introduction

How can we enrich mathematics teaching and learning at school by making a meaningful bridge between schools, families, and communities? This chapter approaches this question by highlighting mathematical funds of knowledge unique to some transnational families, with a focus on Filipina mothers, immigrated to Japan, and

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their children. Kitchen and Civil (2011) used the adjective *transnational* in depicting mathematics learning and teaching across multiple communities which transcend national borders. This descriptor, transnational, aptly captures the situation of Filipina immigrant mothers and their children. As described in detail later, Filipina women in this study were all born into lower socioeconomic status families and came to Japan to financially support their families in the Philippines. Some of the women were granted permanent residency; others remained in Japan on their working visas, and thus could not plan where they would be the following year. The languages these women used at home and in their communities were Tagalog, Bikol, Japanese, and English, and they purposefully switched back and forth among those different languages (Takeuchi, 2016).

Internationally, global population mobility has contributed to a growth in the number of transnational students and families. For example, in the United States, 21% of households reported using languages other than English (United States Census Bureau, 2013). In Canada, 20% of the population reported speaking a language other than the "official" Canadian languages at home (Statistics Canada, 2012). In Japan, where this study was conducted, some industrial areas are becoming ethnically and linguistically diverse, as represented in the percentages of registered immigrants in the following cities: Oizumicho, Gunma (14.5%); Minokamo, Gifu (7.7%); and Kikukawa, Shizuoka (5.4%) (Committee for Localities with a Concentrated Foreigner Population, 2012). In addition to these cities, larger cities such as Tokyo, Nagoya, and Osaka have relatively high percentages of registered immigrants (Ministry of Justice, 2016).

The increase of translational families and students in Japan and elsewhere has changed the picture of mathematics teaching and learning in schools. There is a growing international interest in mathematics teaching and learning in linguistically and ethnically diverse classrooms (e.g., Barwell, 2009; Bishop, Tan, & Barkatsas, 2015; Moschkovich, 2011). Supporting linguistically and culturally non-dominant students in the current globalized mathematics classroom is a multilayered process that addresses languages, ways of knowing, and cultural practices. One of the central goals of mathematics education is to challenge prevailing deficit views toward non-dominant students, including English language learners, in school contexts (Gutiérrez, 2007, 2013; Moschkovich, 2007).

In challenging deficit views toward non-dominant families, previous studies have recognized and affirmed immigrant parents' mathematics knowledge and resources that were embedded in their cultural practices (Civil, 2007; Willey, 2008). This line of research has also underlined some of the conflicts and struggles that immigrant parents experience (Abreu & Cline, 2005; Civil & Bernier, 2006; Crafter, 2012; Gorgorió & Abreu, 2009). Considering that certain parental involvement at home (such as engaging in conversations involving critical thinking) can positively influence students' academic engagement at school (Galindo & Sonnenschein, 2015; Lee & Bowen, 2006), it is important to foster a school context that can facilitate such parental involvement at home. Investigating the connection, as well as the disconnection, between children's out-of-school practices and their in-school learning is especially significant for transnational students, whose practices can be different from the norm assumed in school learning and whose competence can therefore be hidden in school contexts. This chapter aims to promote equity in mathematics education by addressing the power that shapes the (dis)connection between home and school for transnational families. Toward overcoming deficit views of non-dominant students and families, this chapter adds to the discussion about the informal mathematics knowledge that educators can learn from non-dominant family practices (Aguirre et al., 2012; Civil, 2007; Gutiérrez, 2013).

Sociocultural Theory of Learning and Funds of Knowledge

Sociocultural theory maintains that learning and development are fundamentally social and cannot be reduced to a phenomenon closed within an individual (Vygotsky, 1978). The human mind is perceived as being mediated by physical and psychological tools (Vygotsky, 1986). As represented by the concept of the zone of proximal development (ZPD), the distance between the level of learner's independent problem-solving and the level determined through problem-solving in collaboration with peers and under appropriate guidance, Vygotsky emphasized social relationships and interactions with others as essential resources for human development. Sociocultural theory, which stems from Vygotsky's theory, challenged the perspective of learning as being "a closed system" within individuals and as being measured merely by preset objectives. Rather, it sheds light on the "buds" of development, namely, the prospective development of a learner that can bloom through collaboration with others.

Because of this basic assumption, researchers drawing upon sociocultural theory analyze learning in its contexts: "what is around to be learned, in what circumstances, and to what end" (McDermott, 1993, p. 277). Learning should be understood in relation to its surrounding contexts, rather than separating the learner from the particular context (Gutiérrez & Rogoff, 2003). One of the central tenets of sociocultural theory is to highlight capabilities and competencies that people exhibit in the cultural practices they engage in their everyday lives (Lave, 1988; Nunes, Carraher, & Schliemann, 1985; Saxe, 1981, 2012; Scribner & Cole, 1981). For example, by examining an indigenous body numerational system among Oksapmin children and youth in Papua New Guinea, Saxe (1981) revealed the development of culturally specific forms and functions. By offering rich descriptions of cultural practices associated with mathematics, these studies together challenged the ethnocentric perspective on the development of mathematical ideas and offered emic accounts of intelligence linked to local historical practices.

One of the central concepts in recognizing the competencies and resources of non-dominant students and families is *funds of knowledge*, which is defined as "historically accumulated and culturally developed bodies of knowledge and skills essential for house-hold or individual functioning and well-being" (Moll, Amanti, Neff, & Gonzalez, 1992, p. 133). These competencies and resources are often neglected in school contexts, especially for non-dominant students and families (e.g., Abreu & Cline, 2005; Civil, 2014; Takeuchi, 2018). The funds of knowledge perspective challenged the deficit views toward non-dominant, working class families and demonstrated the possibility of transforming school practices and curriculum, where teachers act as ethnographers and identify the bodies of knowledge and skills that are embedded in family practices to support student learning.

The identification of mathematical funds of knowledge represents complexities because of the contrast between academic mathematics and mathematics embedded in everyday practices. Gonzalez, Andrade, Civil, and Moll (2001) described this tension as follows: “On the one hand, although the households we interviewed certainly developed mathematical concepts, the academic transformation of those concepts was elusive. On the other hand, academically validated school knowledge of mathematics seemed to obscure nonacademic forms of mathematical practices” (p. 120). Amidst this tension, Gonzalez and colleagues (2001) demonstrated the sophisticated geometric thinking observed among Latina women in the practice of sewing and the manufacture of clothing. Civil (2007) demonstrated ways in which Latino/a parents and children’s mathematical knowledge used in their everyday practices (i.e., gardening) could facilitate academic mathematic learning (i.e., exploring how area varies given a fixed perimeter or graphing the growth of amaryllis, which includes the concept of scale). The project described by Civil demonstrated a successful example of connecting with parents as “intellectual resources” (p. 117) for students’ learning at school.

This chapter draws on sociocultural theory and the perspective of funds of knowledge by focusing on the cultural practices that Filipina immigrant women engage in their daily lives and by revealing the mathematics knowledge exhibited through their practices. Careful attention was paid to the aspect of power dynamics, which is essential to interrogate the norm around legitimacy of knowledge exchanged in school contexts (Nasir & Hand, 2006). Examining power and sociopolitical dimensions of mathematics learning is also essential in rethinking relationships between school knowledge and knowledge gained through non-dominant students’ family practices (Gutiérrez, 2013). Power, as analyzed through Foucault (1980), is “something which circulates, or rather as something which only functions in the form of a chain” (p. 98). Power, from this perspective, is broader than a phenomenon of a certain group’s consolidated and homogenous domination over others. Power “reaches into the very grain of individuals, touches their bodies and inserts itself into their actions and attitudes, their discourses, learning processes and everyday lives” (p. 39). This lens allows us to examine how power produces self-monitoring and self-control, in the negotiation of forms of knowledge. The analytic lens of power provides a perspective on how non-dominant families value, or undervalue, their funds of knowledge.

Methodology

Tool-and-Result Methodology

The tool-and-result methodology introduced by Newman and Holzman (1993) is used to frame this project. In defining the tool-and-result methodology, Newman and Holzman built on Vygotsky’s (1978) thesis on methodology, which characterized methodology as an activity of searching both the tool and results of research.

Traditional research methodology sets out a linear and instrumental process to apply to the problem being investigated before initiating and implementing the research. Newman and Holzman characterized this as a *tool for result methodology*. Alternatively, based on Vygotsky's perspective, Newman and Holzman brought forth the *tool-and-result methodology*, where researchers continuously search the tool and results simultaneously. From this framework of tool-and-result methodology, a method is perceived as an activity that simultaneously explores and generates tools and results that are "elements of a dialectical unity/totality/whole" (Holzman, 2009, p. 9). The scope of the tool-and-result methodology fits well with the overarching agenda of this current research. In this research, interviews served as an exploratory tool to gain a better understanding of the needs of Filipino/a families living in urban areas of Japan and to search for the potential actions to collectively address these needs. Based on the emerging themes coming out of the individual interviews, workshops were organized to address some of the issues and needs identified by the participants. In this sense, rather than imposing the preset research agenda, appropriate tools and results are simultaneously generated through this research.

Context of Study and Participants

This project was conducted in an urban Japanese city that is becoming increasingly linguistically and ethnically diverse. Parent participants in this study were 12 Filipina women who came to work in Japan, the majority of whom had stayed in Japan after marrying Japanese men. Filipino/a constitute one of the largest and fastest growing ethnic groups in Japan (Ministry of Justice, 2016). Since the late 1970s, Filipina women have come to Japan to fill the bride shortage in farm village areas, to work as entertainers in urban cities, and, more recently, to work as nurses and caregivers (Suzuki, 2003). Some Filipina women I interviewed have been making the most of their English language proficiency and working as English language teachers and tutors. During the interviews, ten parent participants answered that Tagalog was their first language, and two answered that Bikol was their first language. All of the parent participants answered that they were comfortable in English as their second language because they had received education in subjects such as mathematics and science in English. Except for two participants, they all answered that they felt comfortable communicating orally in Japanese but were not comfortable in Japanese reading and writing; the majority of the participants chose to conduct the interviews in English. Child participants in this study were nine of the parent participants' children in the elementary grades. The majority of child participants had been born in Japan, but two children had been born in the Philippines. For six of the child participants who had one Japanese parent and one Filipina parent, the main home language was Japanese. For the other three child participants, the main home language was Tagalog.

Data Sources: Interviews and Workshops

In this study, the analysis was drawn from three types of data: (1) semi-structured individual interviews with Filipina women living and raising children in Japan, (2) semi-structured individual interviews with their school-aged children, and (3) post-interview workshops with the parents and children who had participated in the interviews. Participants for the interviews were recruited by circulating informational brochures at the child care center run by a local community church.

Each parent interview lasted approximately 90 minutes, and each child interview lasted approximately 45 minutes. During the interviews, participants' stories of coming to and living in Japan, their practices related to mathematics and language teaching at home, and their experiences with schools in the Philippines and in Japan were discussed. Many interview questions were based on ethnographic observations made in the participants' communities, for example, an after-school academic support service for immigrant students and a local church. Some of the interview questions were adapted from a study conducted by Guberman (2004), wherein semi-structured interviews were used to identify the everyday activities in which Grade 1 to Grade 3 children were involved. The interview questions used by Guberman were modified to reflect the situations and contexts that were relevant to the participants in the current study. For example, through ethnographic investigation, it was understood that talking about international currencies and time differences were part of everyday activities for Filipina/o immigrant families in Japan. As such, questions about these subjects were added to Guberman's interview questions. Participants were also asked who (e.g., teachers, parent participants, both, or none) played the most significant role in teaching the following: (a) how to use money, (b) how to tell time, (c) basic arithmetic (adding, subtracting, multiplying, dividing), (d) how to calculate international currency values, and (e) how to calculate time differences across countries. All of the interviews were audio-recorded.

Workshops were designed to address issues raised in the interviews, including Filipina mothers' undervaluing their role in their children's school education. The main participants in the workshops were those who participated in the interviews. Other Filipino/a parents who wanted to join the workshops were also welcomed and included. One of the main goals for the workshops was to foster a community of immigrant parents to share their common concerns, experiences, and knowledge. This chapter focuses on three workshops around one theme: parents' knowledge of ratio and currency conversion. Other themes that emerged have been discussed elsewhere (Takeuchi, 2015, 2018). Each workshop lasted between 90 and 120 minutes and was held during the evening in a community where the participants lived. The number of participants increased with each workshop, and six to nine participants joined every workshop. Interactions during the workshops were video-recorded; activity sheets and collective reflection sheets used during the workshops were photographed. After the workshops, I asked participants to complete anonymous surveys.

All of the parent participants were multilingual: they all spoke English fluently and the majority spoke Japanese. Because the interviewer's proficiency was limited in Tagalog and Bikol (the parent participants' native languages), interviews and workshops were conducted in either English or Japanese. All quotes were translated into English, and a bilingual person highly competent in Japanese and English confirmed the translations. Audio recordings of the interviews were fully transcribed, and video recordings of the workshops were content-logged.

Data Analysis

Data analysis focused on the following research questions: (1) How does Filipina immigrant mothers' positioning influence their involvement in their children's school learning? (2) What are the mathematical funds of knowledge exhibited through the practices of Filipina immigrant mothers? (3) How can the funds of knowledge of Filipina mothers contribute to their children's mathematics literacy? Semi-structured interviews were coded for positioning or positional identities defined as "the day-to-day and on-the-ground relations of power, deference and entitlement, social affiliation and distance with the social-interactional, social-relational structures of the lived world" (Holland, Skinner, Lachicotte, & Cain, 1998, p. 127). I specifically looked for linguistic markers such as "Filipino/a," "Japanese," and/or "foreigners." Participants' positioning was analyzed in relation to their involvement in their children's learning at school. Then, I analyzed parts of the interviews where I elicited the ways in which the parent participants communicated with their children about mathematics in everyday practices. The goal of this analysis was to identify the areas in which parent participants were involved in their children's educational activities related to mathematics learning.

Similarly, the analysis of the workshops focused on the interactions of how participants exhibited their informal mathematics knowledge through the conversation (e.g., knowledge of ratio in relation to converting the currency). To understand whether and how the children had appropriated informal mathematics knowledge from their parents, I analyzed whether and how the children solved a word problem that involved the topic of currency conversion. Analysis of commonalities and differences among participants and coding based on the above themes were conducted by using the qualitative analysis software, *MAXQDA* (Verbi GmbH, Berlin, Germany).

Findings

Four main themes that emerged from this study are highlighted. First, drawing from the interviews with parent participants, how Filipina immigrant mothers' positioning as a "foreigner" or "outsider" in Japanese society limited their active participation in their child's school learning is discussed. Next, I focus on the everyday practices involving mathematics unique to the transnational families who participated in this study and demonstrate Filipina mothers' sophisticated mathematics reasoning embedded in those practices. Subsequently, I consider how the informal mathematics funds of knowledge helped children's mathematics literacy.

Power and Parental Involvement

Interviews with parents revealed their decisions about their involvement in their children's school education. Here, I highlight two ways in which power and hierarchy manifested in parents' interviews: (1) parents' distancing themselves from the mainstream and (2) parents' discourse on what is considered to be legitimate in school mathematics learning. Initially, when participants were asked about their observations of Japanese school curriculum and pedagogy, the majority of parent participants indicated a perceived hierarchy between Japan and the Philippines and/or their marginalized identity as a "foreigner." For example, in her interview, Janice, one of the parent participants, repeatedly said Japanese mathematics teaching was more advanced as represented in the following quote: "Japan is at a high level … what to say, I think mathematics is also at a higher level. Filipino children learn division at around Grade 5 and Grade 6, but Japanese children learn it earlier." Nicole, another parent participant, also said, "I think Japanese education is more advanced," and she explained that the curriculum her son was learning in elementary school was taught at the high school level in the Philippines.

The interview also revealed that parents' positioning as a "foreigner" or "outsider" in Japanese society hindered them from playing an active role in their child's school learning, even when they desired to. Janice's story resonates with other parent participants' stories and highlights immigrant parents' struggles in Japan. Janice told about when her son could not go to school because of bullying: "I was really worried and really sad because I couldn't teach him. I couldn't read Japanese and it was impossible to teach school subjects to him. I am Filipina, so I couldn't even read important letters and felt helpless." Janice expressed a sense of powerlessness in supporting her child's school learning. Janice said she always taught the significance of a school education to her son by emphasizing her struggles in Japan. She said, "I always tell my kids, 'Look at mom, I can't read, it's really tough.' I'm Filipina, but you are Japanese, so study hard."

In an effort to maneuver their positional identities as an outsider, parent participants told their children to conform to the mainstream norms. For example, Michelle said, "I'm Filipina and I can't offer anything as a parent because I'm a foreigner. So, I always tell my child, 'You don't have to be number one, two, or three, but you have to follow what everyone else is doing.'" As these stories exemplify, parent participants adapted an assimilative discourse, where they tried to mold their children and themselves to fit the mainstream.

Everyday Practices Involving Mathematics

Even though the interviews showed that parents undervalued their involvement in their children's school learning, they also revealed that many of these parents engaged their children in conversations involving mathematics through everyday practices. Both parents and children reported that parents, especially mothers, played a major role in teaching children how to use money, tell time, calculate international currency values, and calculate time differences.

In particular, all of the parent participants reported that they played a major role in teaching their children how to use money. Parent participants described how their role was not limited to teaching how to calculate money but extended to the moral aspects of money (i.e., the importance of saving money and developing a reasonable habit of consumption). For example, Vanessa said, "My youngest always buys snacks—so, I always tell her you can't buy what you really need, if you keep doing that. And sometimes I ask her how much it'd be if you always, always buy this." Similarly, Irene explained her role as "basically teaching them how to organize money well because we don't have much money (with laughter)." Parent participants also reported that they taught the calculative aspects of using money, such as how to record children's allowances. In daily practices such as shopping together, these parents taught their children how to use money effectively. For instance, Fumi, a child participant in Grade 5, explained what she had learned from her mother: "When I make a payment and when the price is not even, my mom taught me how I can get minimum number of coins for change." This daily practice of shopping taught child participants basic arithmetic.

Conversations about international currencies and time differences were unique to the transnational families in this study and were embedded in the participants' practices of calling families and friends or traveling to visit families and relatives in other countries. Many child participants reported that they had learned about calculating international currency conversions and talking about time differences from the time when they were young (as young as Grade 2). International currency exchange was taught out of necessity. Irene taught about the currency exchange between the Filipino peso and Japanese yen to her children when traveling to the Philippines. She asked questions such as "If I have 100 pesos, then how much is it in Japanese yen?" Nicole said that her son asked related questions when they traveled in the Philippines or in the United States. She explained,

> Because he is very good at managing money, he is interested. When he bought a toy, then he looks at the price and then thinks how much it is in yen. He'd say something like, 'How much would that be in yen? 800 yen? Okay, that's cheap.'

As evidenced here, parents played a significant role in these everyday practices involving money and time, which were closely connected to the daily practices of these transnational families.

Calculating International Currency Conversion: Findings from the Workshops

As seen in the interview findings, power structures and immigrant parents' positioning were limiting their involvement in their children's school education. These interview findings led to workshop design addressing a need for parent participants' concerns regarding school education. As one of the discussion topics for the workshops, I asked participants about the mathematics they engage in their daily lives. As highlighted below, this discussion revealed the mathematical knowledge and reasoning that parents used in informal settings outside of school.

One commonly reported practice that involved mathematics was calculating international currency conversions. Five Filipina mothers attended the first workshop, and all stated that they constantly engaged in calculating international currency conversions. All of the Filipina mothers that I interviewed said that they were from a big family of lower socioeconomic status and that financially supporting their family was their main motivation for coming to Japan. Also, many of them were sending money back to family members in the Philippines. Because of this, engaging in international currency conversion was a daily practice for them.

During the interviews, Filipina mothers undervalued and underestimated their mathematical knowledge. During the process of the workshops, however, their mathematical knowledge was revealed. For example, when we were talking about a children's picture book, we talked about how much a Filipino picture book would cost in Japanese yen. As can be seen in the narratives from the workshops, participants had a strategy of doubling the Philippine pesos (to convert to yen) and halving yen (to convert to Filipino pesos) to get an approximate value. If a picture book is 300 Filipino pesos, they then figured how much it would be in Japanese yen by doubling the amount to make an approximate conversion. The following excerpts are narrative descriptions of selected video-recorded interactions from the workshops.

Narrative Description 1

When asked the average price of a picture book in the Philippines, Michelle said, "200 pesos." Irene answered "About 300 pesos. They are cheaper." They were asked, "200 to 300 pesos? How much would that be in Japanese yen?" Irene said "600 yen." Evelyn and Michelle in unison said "double it." Michelle said, "The Filipino price is the double of Japanese yen."

The subsequent conversation addressed the meaning of currency rate fluctuation. Filipina immigrant women used their common sense reasoning, which is strongly connected to their everyday experiences. Based on the conversation, the facilitator summarized and wrote, "yen: peso = 1: 0.5." One of the participants immediately responded by saying that the ratio had changed.

Narrative Description 2

The facilitator said, "So, what you just did is … (while pointing at the written note "yen: peso = 1: 0.5") 1 Japanese yen equals to 0.5 Filipino pesos?" Janice and Evelyn immediately responded, "That was before." Evelyn then said, "That was before. Now 0.45."

The conversation described in the following narrative depicts some of the ways in which Filipina mothers explained the implications of fluctuation. Filipina mothers not only understood the conversion between the two currencies but also demonstrated their understanding of the implications of fluctuation.

Narrative Description 3

The participants were asked, "What does this change in conversion rate mean to you?" Irene said, "For us, if the peso is 0.5, it's good for us but also it's good for people in the Philippines, too." Evelyn said, "In my case, I send 100,000 yen (approximately $1,000 USD) but now I need to add one (10,000 yen), and 110,000 yen, just to fill the loss." Other participants were nodding while listening to Evelyn. Evelyn continued, "But the price is the same. I told my mom, 'Mamma, the price is still there' but the currency change" (making a gesture of balancing the two with her hands).

Filipina mothers understood the meanings of fluctuations in the conversion rate (from yen: peso = 1: 0.5 to 1: 0.45) from their personal lives. They quickly calculated the change in the amount of money they would have to send to the Philippines. Participants presented the informal knowledge of doubling and halving to convert between the Philippine peso and the Japanese yen. They also demonstrated an understanding of the concept of fluctuating currencies and the consequences of this fluctuation.

The mathematical literacy presented here resembles what is tested in the Organization for Economic Co-operation and Development's (OECD) Programme for International Student Assessment (PISA) test for 15-year-olds. For example, the following problem was included in 2009 PISA test, and only 40.5% of students answered it correctly (OECD, 2010). The problem requires that the students understand currency value fluctuations and can communicate the meaning of this fluctuation.

> Mei-Ling from Singapore was preparing to go to South Africa for 3 months as an exchange student. She needed to change some Singapore dollars (SGD) into South African rand (ZAR). During these 3 months, the exchange rate had changed from 4.2 to 4.0 ZAR per SGD. Was it in Mei-Ling's favor that the exchange rate now was 4.0 ZAR instead of 4.2 ZAR, when she changed her South African rand back to Singapore dollars? Give an explanation to support your answer (OECD, 2010, p.113).

In the above interactions, which were closely related to their daily practices, Filipina mothers demonstrated understanding of the fluctuation of currency values and discussed the impact of the fluctuation on their lives and on their Philippine families' lives.

Children's Mathematical Literacy: International Currency Conversion

In this section, I discuss how the children of these Filipina mothers understood the concept of international currency conversion by examining how child participants engaged in related word problems. For example, I included the following word problem, which was a modified version of one of the problems from the PISA for 15-year-olds. This problem was presented in Japanese, the language of instruction at the school of the child participants, who were all elementary school students. The problem was modified from the original PISA problem to reflect the contexts that were more familiar to the participants. I also omitted the question about the implications of fluctuation, given that students as young as Grade 2 also engaged in this problem.

> Calvin from Japan was preparing to travel to the Philippines this summer. His grandmother gave him ¥35,000 yen. The exchange rate between the Philippine peso and the Japanese yen was 1 YEN = 0.52 PHP. If he exchanged the money he received from his grandmother, how many Philippine pesos would Calvin get?

When given this word problem that requires understanding of international currencies, two child participants accurately solved the word problem. Four child participants reported that they knew of the doubling and halving strategy to convert Japanese yen to the Philippine peso from their parents but could not complete the word problem. The child participants who solved the problem demonstrated an understanding beyond their grade level; they were in Grade 6 and the original PISA problem was targeted at Grade 9.

In the following section, it is demonstrated how May, one of the child participants, engaged in this problem. This example shows how having experience with and conversations about international currency conversion can facilitate children's problem solving. May said she learned about international currencies from her parents and often asked them how much a particular item in Japan would cost in the Philippines. May fluently decoded the word problem, except for the Japanese word for exchange rate. I thus explained to her what the exchange rate was, by using an example of exchanging currencies between Japan and the Philippines. Her informal knowledge of exchange rates, doubling, and halving to convert the two currencies helped her to better understand the word problem. May initially mumbled, "I know about the exchange rate, but I don't know how to solve this." The following narrative captures how she engaged in the problem from there.

Narrative Description of Interview 1

She wrote down "35,000 yen =? 1 yen = 0.52" in two rows, vertically (see Fig. 8.1). She said, "Should I use division? I don't know." Then she said, "Wait … got it." She then wrote "35,000 × 0.52=". She calculated the equation and asked, "It's 18,200, is it? I'm not sure."

Although her answer was correct, she was not confident in her identified answer, saying "It's 18,200, is it? I'm not sure." She then said, "Usually, I would divide by two, to get the estimation." She continued, "But this is now 0.52." When asked what would be the approximate value, she identified that 17,500 peso would be the answer.

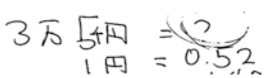

Fig. 8.1 Notes by May (a child participant)

In May's case, she used this knowledge for solving a new problem presented to her. In contrast, those children who reported not having learned about international currencies at home tended not to be able to interpret the meaning of the word problem, even when they were able to decode its texts. Interviews with Filipina mothers' children revealed how having a conversation about international currencies can help in solving a new mathematics word problem such as the one presented on the PISA.

Discussion

This chapter highlighted mathematical funds of knowledge unique to some Filipina immigrant mothers living transnationally. I drew from sociocultural theory and the perspective of funds of knowledge, while also carefully considering the repercussions of power structures experienced by Filipino/a immigrant families in Japan. Considering that all of the parent participants had sought employment opportunities in Japan in order to financially support their families in the Philippines, it is reasonable to think that the internalized hierarchical difference derived from the economic power between Japan and the Philippines was reflected in the parents' views toward mathematics teaching and learning in these two countries. Based on the interviews and interactions during the workshops, this chapter revealed how Filipina immigrant women engaged in calculating international currency conversion daily, in order to financially support their family members in the Philippines. The mathematical reasoning that the women exhibited is well-aligned with mathematical literacy that is defined as the "capacity to formulate, employ, and interpret mathematics in a variety of contexts." This includes "reasoning mathematically and using mathematical concepts, procedures, facts, and tools to describe, explain and predict phenomena" (OECD, 2010, p. 121). Interviews with child participants revealed how some were able to apply the knowledge of international currency conversion and ratio, gained from their conversations with parents, to mathematical problem-solving beyond their grade levels.

Despite its potential implications for facilitating children's mathematical reasoning, parents' funds of knowledge – in this case the knowledge about currency conversions – can be masked through school practices if only certain ways of knowing are treated as legitimate and valued. In fact, as Foucault's (1980) metaphor of power illustrates, parent participants were carefully observing and self-censoring what is considered to be legitimate and illegitimate in school mathematics learning. In this process, most of the Filipina mothers internalized and normalized their social positioning and undervalued their knowledge and experience, as represented in one participant's quote: "I'm Filipina, and I can't offer anything as a parent because I'm a foreigner." Filipina mothers' positioning in the society and the perceived economic and political hierarchy between their home country and host country led them to feel powerless as parents and made them doubt their mathematical knowledge.

The mathematical funds of knowledge presented here were revealed in conversations between a group of Filipina mothers and the workshop facilitator. Foucault (1980) named the process of "insurrection of subjugated knowledges" (p. 81) as a form of resistance. Subjugated knowledges are “a whole set of knowledges that have been disqualified as inadequate to their task or insufficiently elaborated: native knowledges, located low on the hierarchy, beneath the required level of cognition or scientificity” (p. 82). By participating in the space where those subjugated knowledges are manifested and affirmed, Filipino mothers in this study collectively resisted the repercussion of the power imbalance.

Findings from this study contribute to the growing body of research on equity in mathematics education (e.g., Aguirre et al., 2012; Bartell, 2013; Boaler, 2006; Civil, 2007; Esmonde & Caswell, 2010; Gutiérrez, 2013; Hand, 2012; Martin, 2013; Moses & Cobb, 2002; Nasir, 2002) and ethnomathematics (e.g., D’Ambrósio, 2006; Gilsdorf, 2015) by focusing explicit attention on power and identity in mathematics learning. Nasir and Hand (2006) corroborated sociocultural theories of learning and the theories and research on race and culture, and they maintained the significance of attending to how the hierarchy of power and societal structures are reproduced or challenged in the local activities of the classroom. Gutiérrez (2013) called for a sociopolitical turn in mathematics education by bringing the analysis of power and identity as a central agenda for mathematics learning and teaching. Students can internalize the hierarchy of power and social stratification through mathematics learning at home and at school. Analysis presented in this chapter responds to this call; this study extends the analysis of funds of knowledge by considering the issue of power and identity and by demonstrating possibilities for designing space, such as workshops, where students and their families’ funds of knowledge and their identities are affirmed.

Limitations

This study also has some limitations. For example, the workshops were held outside of the school and did not involve school teachers. There have been successful collaborations bringing students’ out-of-school funds of knowledge to the mathematics classroom (Aguirre et al., 2012; Civil, 2007; Foote, 2009). The challenge raised by these previous studies is how to leverage children’s mathematics thinking by embracing community funds of knowledge, without reducing its richness and complexity into simpler mathematics word problems. In this aspect, the current study demonstrates the possible connection between mathematics reasoning elicited in the OECD PISA tests and the mathematics reasoning that the particular transnational families engaged in their daily practices. Validating the community funds of knowledge in relation to school mathematics can help mitigate deficit views toward non-dominant families and children in the school. The potential challenges and possibilities of collaborating with schools and teachers, in order to incorporate the community funds of knowledge, should be addressed in future studies.

Pedagogical Implications

This work highlighted how Filipina mothers' mathematical understanding connected to their everyday lives. By drawing from children's interviews, the possibility of connecting non-dominant students' mathematical practices at home with their development of mathematical literacy in school was explored. This finding is particularly significant to the discussion on how to design educational practices which can meaningfully bridge out-of-school resources and in-school learning by considering non-dominant students' and their parents' positionality. As discussed briefly here, and elsewhere (Takeuchi, 2015, 2018), these Filipina mothers who are raising children in Japan perceived their mathematical resources to be less valuable, compared to the school knowledge. Countering the deficit views toward non-dominant students and identifying and noticing their intellectual resources through the workshops are ways to mitigate a power imbalance affecting Filipina mothers and students' identities in relation to mathematics learning. This study sheds light on the mathematical understandings of non-dominant parents and students, focusing on everyday mathematical practices that are familiar to them. As indicated through children's interviews, there is a potential that informal mathematical knowledge can facilitate broader mathematical literacy. Designing school lessons to affirm non-dominant transnational families' knowledge and experiences will contribute to embracing students' identities holistically. Such lessons can eventually enhance mathematics learning experiences for all the students.

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Chapter 9
A Commentary on Identifying and Connecting to Family and Community Funds of Knowledge

Marta Civil

Abstract This chapter presents a commentary on the pieces in the section of the book focused on identifying and connecting to family and community funds of knowledge. It underscores the importance of listening to and learning from those whose knowledge and experiences have often been unacknowledged (e.g., Indigenous communities; immigrant women). This commentary highlights potential tensions related to issues of valorization of knowledge and to the foregrounding of mathematics versus the foregrounding of the cultural knowledge. At the heart of these tensions is the concept of power captured in questions such as, whose mathematics are being represented? What approaches to doing mathematics are being valued and why? How is teaching taking place? Whose voices are being recognized?

While at some level one could think that the chapters by Ruth Beatty and by Miwa Aoki Takeuchi are quite different, in fact they both center on the importance of understanding and building on the mathematical knowledge of groups that have traditionally been ignored or marginalized. The two chapters provide insights into issues related to connecting in-school and out-of-school/community/home knowledge. Both chapters, however, offer different perspectives, theoretical frameworks, and methodologies. Beatty's chapter draws heavily on ethnomathematics, while Takeuchi draws on sociocultural theory, and in particular on the work around funds of knowledge.

The chapter by Ruth Beatty provides a detailed example of the mathematical learning opportunities in the practice of Algonquin loomwork. In so doing, this chapter is a good resource for readers who want to see the mathematics in everyday practices, such as in this case, looming, and in particular, the author provides a thorough description of possible connections between school mathematics concepts and the practice of looming. The chapter also touches on several potential tensions,

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including learning through apprenticeship by engaging in the practice instead of through direct transmission, bringing in a Western view to interpret the mathematics in Indigenous practices, and letting the ideas emerge organically versus imposing a predetermined path. In what follows, I turn my attention to these tensions.

Reading this chapter reminded me of the teaching of Nancy Sharp, a Yup'ik teacher, who engaged her young students in learning through apprenticeship. As Lipka, Sharp, Brenner, Yanez, and Sharp (2005) write, "expert-apprentice modeling and joint productive activity are one of the oldest forms of instruction: They have rarely been part of Western schooling in indigenous contexts, although they are part of the local culture" (p. 32). Similar to Beatty's chapter, the case of Nancy Sharp is also discussed as an example of "third space" (Gutiérrez, Baquedano-López, & Tejeda, 1999). Rogoff (1994, 2012) has been studying the differences between teaching and learning in middle-class communities of European origin in the USA and in Mayan communities in Guatemala. Once again, we see the idea of learning through some form of apprenticeship, through participation in a community of learners in the Indigenous settings. In writing about her interviews with mothers in a Mayan community, Rogoff (2012) would ask them, "How do you teach your daughters to weave?" and she notes that she "was puzzled when they replied simply, 'I don't teach them to weave, they learn'" (p. 234).

Beatty's chapter underscores the importance of a developing community feeling when learning. As she writes, "There was no worry about making mistakes, there was just, we were all learning together" (pp. 120–121). In Civil and Hunter (2015), we emphasize the importance of family feeling, humor, and building relationships as we describe the work in mathematics teaching and learning in two very different contexts, with Pāsifika students in New Zealand and with Mexican-American students in the USA.

The potential tension in our imposing a Western view of mathematics on Indigenous practices is not unrelated to the tension between everyday/out-of-school mathematics and academic/in-school mathematics. Whose knowledge is represented and valued? Several researchers have written about this topic (e.g., Civil, 2002, 2007, 2016; de Abreu, 1995; González, Andrade, Civil, & Moll, 2001; Nasir, Rosebery, Warren, & Lee, 2006). For example, in Civil (2016), I allude to the case of a teacher who was an experienced seamstress and dismissed the mathematics in sewing as being "just measuring"; or in Civil (2002), as we engaged a class of fifth graders in exploring the mathematics connected to games by drawing on their knowledge, some students questioned if this was really mathematics since they were used to filling out worksheets in mathematics classes.

The third tension in this chapter relates to the need to balance the authenticity of the activity from a mathematics point of view as well as from the cultural point of view. As Beatty writes:

> Our goal was also, however, to bring out mathematical thinking, and to this end we facilitated the exploration of mathematical ideas as they arose … Some of the inquiries students engaged in, for example calculating the numbers of beads needed for different designs, were questions that students asked during the course of the investigation, but are not areas on which traditional loomers would necessarily focus. (pp. 110–111)

This tension reminds me of what I have described as "preserving the purity of funds of knowledge, perhaps at the expense of mathematics" (Civil, 2007, p. 107). This observation was based on a year-long study in a second grade class where we developed a construction module. The teacher, Patricia Sandoval-Taylor, had had the students in first grade and had visited her students' homes during that year. Through her ethnographic home visits, she had uncovered a richness of funds of knowledge on the topic of construction. As we planned for the module, very animated conversations took place, as I, as a mathematics educator, was trying to push for mathematical activities, while the teacher (Patricia) wanted to make sure that the learning was grounded in authentic experiences that the children (or their families) encountered in construction. For example, I designed an activity that would involve proportional reasoning in having the students build a chair for a doll. Sandoval-Taylor (2005) writes:

> I was not sure that learning about proportion would emerge during the unit. From my experience in my students' community, I thought that the unit focus would more likely be on constructing buildings…. Students could be asked, for instance, how to build an additional room on their homes. (pp. 159–160)

This tension is nicely captured by Christina, an Algonquin teacher and expert loomer, in Beatty's chapter:

> If it doesn't have some kind of cultural significance then what are we doing this for, is it just to teach math? No. Not when you see the changes in the First Nations students. Seeing them more confident, and the pride in talking with their peers about their lives, their regalia and stuff like that. And it's nice to be able to share with your best friend who might not be Native a little bit more about your life that they might not know about because they only ever see you in a school setting. (p. 121)

These words from Christina capture the power of building on students' cultural practices, as doing this allows for bridges to be built between students and between in-school and out-of-school life. This idea is also central in Takeuchi's chapter, as she seeks to learn from the Filipina women about their uses of mathematics to build connections between their experiences and school mathematics that their children are learning.

Takeuchi's chapter adds to the body of research on immigrant parents' views of and experiences with mathematics. I have been conducting research on this topic for about 25 years. My work has focused on Mexican-American communities in the USA; Takeuchi's chapter focuses on Filipina women in Japan. It is important to note that while the contexts are different, there are some similarities in our findings (e.g., in terms of language of instruction as a potential barrier) but also some differences. In particular, the Filipina women in Takeuchi's study seemed to position themselves as outsiders to the Japanese system of education and viewed this system as superior to the one in the Philippines. This is different from my findings, and those of other researchers, where immigrant parents tended to view the system from their country of origin as more advanced and demanding (Civil, 2012). As I have discussed elsewhere (Civil & Planas, 2010), the point is not whether one system is more advanced than the other, as this is a complex question, but the point is that these parents' perceptions are likely to color their interactions with their children's schools and their views of the mathematics they are currently learning. In any case, both in my research with Mexican women in the USA and in Takeuchi's research with Filipina women in Japan, issues of marginalization due to not being familiar with the school system, not knowing the language of schooling, and difficult economic situations are very real.

The examples centered on the Filipina women (and their children's) understanding of currency exchange, based on their personal experiences going back and forth between Philippine Pesos and Yens, reminds me of my first experience with the Funds of Knowledge for Teaching project, where we devised a module around money, inspired in part by the fact that children were used to working with Mexican Pesos and US Dollars and that some of the parents had had a small grocery shop in Mexico (Civil, 1992). Similarly, in Moll, Amanti, Neff, and González (1992), Cathy Amanti describes how many of her students had experiences traveling across the border and engaging in currency conversion (Mexico/USA). Yet, those experiences are often not valued or recognized. This is addressed very explicitly in Takeuchi's chapter, as she looks at issues of power in her study of Filipina women in Japan. As Takeuchi writes, "Despite its potential implications for facilitating children's mathematical reasoning, parents' funds of knowledge, in this case the knowledge about currency conversions, can be masked through school practices if only certain ways of knowing are treated as legitimate and valued" (p. 138). And earlier in the chapter, Takeuchi writes, "During the interviews, Filipina mothers undervalued and underestimated their mathematical knowledge. During the process of the workshops, however, their mathematical knowledge was revealed" (p. 134). Of particular note is the fact that the workshops emerged from the themes in the interviews, thus responding to the needs and interests expressed by the Filipina women and promoting the concept of parents as intellectual resources (Civil & Andrade, 2003). The workshops in Takeuchi's study created spaces where the mothers could have their knowledge and experiences recognized, thus perhaps questioning the power issues they perceived based on their positioning in the host country.

The two chapters provide powerful images of what is possible if we focus on the knowledge and experiences that diverse groups bring with them, whether it is the cultural practices of Indigenous people or of immigrant women. In engaging with these practices with a mathematical eye, we can challenge several aspects related to the teaching and learning of school mathematics, such as: Whose mathematics is being represented? What approaches to doing mathematics are being valued and why? How is teaching taking place? Whose voices are being recognized?

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Part III
Student Learning and Engagement in PreK–12 Mathematics Classrooms

Chapter 10
"So We Only Have One We Share with More, and Then They Have Way More and They Share with Less": Mathematics and Spatial Justice

Laurie H. Rubel, Vivian Y. Lim, and Maren Hall-Wieckert

Abstract This chapter investigates student learning in the context of a module about a city's two-tiered financial system of banks and alternative financial institutions (i.e., pawnshops), held in ten sessions in a high school advisory class led by a mathematics teacher. The module exemplifies teaching mathematics for spatial justice, by extending teaching mathematics for social justice (Gutstein, 2006) with the idea that place matters (Gruenewald, 2003) and that justice has a geography (Soja, 2010). In this module, students use the concepts of percent to mathematize the costs of loans, and they analyze intensive variables to investigate spatial data about the density of these categories of institutions across their city. Spatial data is presented with GIS maps, layerable with demographic data and locations of financial institutions. The spatial distribution of the financial institutions reflects the inequalities of the spatial patterns in the city's social demographics. Contextualized in the disparity of interest rates across these financial institutions, the spatial pattern not only reflects but reinforces those social inequalities. This chapter presents findings from one round of piloting and pursues the question: How did the module's spatial justice orientation support the development of conceptual understanding of percent and ratio? Analysis focuses on growth with respect to mathematical understanding of percent and learning in class sessions organized around the use of ratios to understand the distribution. Findings include student adoption of strategies indicative of conceptual understandings of percent and development of critical opinions about their city's two-tiered personal lending system.

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Introduction

Social justice concerns are known to motivate the learning of mathematics by providing connections for learners between mathematics and relevant issues (Root, 2009). This potential is seen to be especially important for students from marginalized groups, for whom connections between mathematics and their everyday circumstances are typically not made by teachers or curriculum (Leonard, Brooks, Barnes-Johnson, & Berry, 2010; Remillard et al., 2014). This paper examines learning of mathematical concepts of ratio and percent in the context of a social justice investigation around the theme of a city's system of personal finance institutions, in which social justice is extended to spatial justice. In American cities, the mainstream bank industry is supplemented by an array of alternative financial institutions (AFIs), including pawnshops, wire transfer outlets, and check-cashing stores. Banks offer credit-building opportunities and charge lower rates for comparable services than their AFI counterparts. On the other hand, banks cater their services to people who already hold strong financial credentials and more flexible income (Servon, 2013). This chapter describes a ten-session curricular module ("Cash City") designed for high school students to pursue the complex theme of New York City's two-tiered personal finance system.

The foundation of "Cash City" was a mathematical analysis of interest rates to compare costs of personal loans at banks and AFIs. Differences in interest rates then contextualized a subsequent analysis of the spatial distribution of banks and AFIs using mathematical concepts of ratio. Interactive digital maps presented visualizations of relative densities of personal finance institutions, viewable along with visualizations of demographic data. Students conducted field research in the school neighborhood to explore the theme from street perspectives through a process of participatory mapping (Rubel, Lim, Hall-Wieckert, & Katz, 2016). The module culminated with articulations and justifications of opinions about the two-tiered system.

A hypothesis was that the spatial justice context of a city's two-tiered personal finance system would further students' conceptual understandings of percent and ratio. Aside from building financial awareness, perspectives about a city's spatial distribution of financial institutions speak to issues of access, equity, and spatial justice. This project aimed to provide students with an opportunity to use mathematics as a tool toward a deeper understanding of power relations that underlie this particular socio-spatial phenomenon. Together, the mathematical understandings of interest and of power relations are seen as pivotal in supporting students' sense of critical agency about systemic injustices (Gutstein, 2006). We pursue this argument by investigating how this spatial justice context supported (and complicated) students' development of conceptual understanding of percent and ratio.

Related Literature

Our review begins with a survey of research about student learning of percent and ratio. Next, we outline theoretical perspectives about teaching mathematics for spatial justice, attending to the potential role of percent and ratio. We conclude with questions from the research literature pertaining to the role of place in learning mathematics.

Student Thinking with and About Percent and Ratio

Percent and proportional reasoning are prominent across an array of common marketplace transactions (Parker & Leinhardt, 1995). The Common Core State Standards in Mathematics (National Governors Association Center for Best Practices, Council of Chief State School Officers, 2010) indicate that students should be able to solve multistep percent problems and be proficient with using percent to solve problems based on an array of marketplace scenarios by the end of middle school. Percent continues with an underlying but pervasive role in mathematics curriculum at the secondary level, essential for topics such as scaling, probability, statistics, and modeling. Proportional reasoning is so similarly significant to the development of mathematical thinking that it has been called by Lesh, Post, and Behr (1988), “a watershed concept, a cornerstone of higher mathematics, and the capstone of elementary concepts” (Lamon, 1993, p. 41). Mathematical competencies of proportional reasoning and reasoning with percent are essential for functional literacy in the marketplace, but also for critical literacy, for citizens to be able to evaluate information and analyze relationships as they relate to issues of power (Apple, 1992).

Despite the centrality of these mathematics topics in school curriculum, difficulties with problem solving related to proportional reasoning and percent are prevalent and persistent (Lamon, 1993; Lembke & Reys, 1994; Moss & Case, 1999; Parker & Leinhardt, 1995; Tourniaire & Pulos, 1985). Common errors with percent include ignoring the percent sign; difficulties translating between decimal, fraction, and percent; and confusion between multiplicative and additive relationships (Moss & Case, 1999; Parker & Leinhardt, 1995; Tourniaire & Pulos, 1985). Finding a given percent of a specific amount has been shown to be more straightforward for students than other problem types; however, even at age 17, most students have difficulty with, for instance, finding 4% of 75 (Parker & Leinhardt, 1995). Even among adult learners, while most master basic benchmark values like 100% and 50%, understandings of other values, even 25%, can be limited (Ginsburg, Gal, & Schuh, 1995). Since ratios are often expressed as fractions, the tendency to extend rules that pertain to whole numbers to fractions are implied – for example, to argue that 4/7 is bigger than 3/5 because 4 > 3 and 7 > 5 (Stavy & Tirosh, 2000).

An explanation for the persistence of these difficulties with percent and ratio tasks is that these concept areas are typically taught with a focus on procedural methods without accompanying understanding, without support of models, and without being contextualized in meaningful contexts. Even when teachers are successful at solving problems involving ratios and proportion or percent, they may still have difficulty explaining why particular operations make sense (Lo & Ko, 2013; Streefland, 1985). Representations that link procedures for solving ratio and percent problems to conceptual understanding of percent are area models (Haubner, 1992), halving or doubling models (Moss & Case, 1999), 100-board models (Wiebe, 1986), dual-scale number lines (Dole, 2000), or ratio tables (Middleton & Van den Heuvel-Panhuizen, 1995).

Mathematics and Spatial Justice

We draw on social definitions of space as encompassing social, historical, and geographical dimensions to define teaching mathematics for spatial justice (Rubel, Lim, Hall-Wieckert, & Sullivan, 2016; Rubel, Hall-Wieckert, & Lim, 2017). Abstract space comes to be bounded by human experience, imbued with historical meaning, and takes on character as place (Tuan, 1977). The associations that people have with a particular place, or "sense of place," can be personal or shared, and color their perceptions of the world (Lim & Calabrese Barton, 2006). Spatial justice is constituted by the extent to which the multidimensional relations that structure space and place are fair or unfair (Soja, 2010). Teaching mathematics for spatial justice entails reading and writing the world (Freire & Macedo, 1987) as in teaching mathematics for social justice (Gutstein, 2006), but with an emphasis on place and space: using mathematics for "reading" the political, social circumstances that structure the spatial world and for "writing" with agency to reimagine and transform such structures and spaces.

Spatial justice issues can be pursued with mathematical models and analysis. Since geographic areas vary in size and other properties, any comparison of allocation of resources across spaces demands normalization, which involves the mathematical construct of ratio (Slocum, 1998). Intensive variables compare two extensive quantities (e.g., population, area, number of banks) in a ratio (e.g., banks per square mile) (Lawvere, 1992). Some examples of intensive variables in the literature include comparing the distribution of continental gross domestic product by comparing it to continental population (Gutstein, 2006); comparing the number of liquor stores to the number of community centers as a way to contextualize the Rodney King riots (Brantlinger, 2005); comparing the amount of space to the size of the student body at two schools sharing a facility (Turner, 2012); or comparing the distribution of vacant lots and basketball courts in relation to area and population across neighborhoods to illuminate disinvestment in low-income parts of a city (Lim, 2016). Ratio and percent figure prominently across these examples. Although learners have been shown to demonstrate proportional reasoning more often with

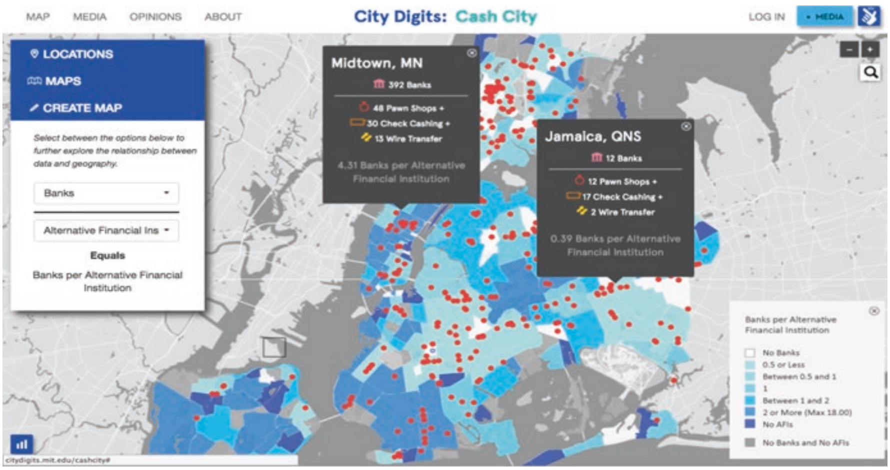

Fig. 10.1 Locations of pawnshops (red dots) layered atop a map showing banks per AFI (City Digits, 2013)

intensive variables (associated sets ratios) than other categories of ratios, these tasks remain difficult, especially when one of the variables being compared is continuous (Lamon, 1993; Tourniaire & Pulos, 1985).

With digital maps pinpointing the locations of financial institutions, extensive variables that quantify a neighborhood’s numbers of banks or AFIs demand normalization in relation to other properties like area (e.g., pawnshops per square mile) or number of households (e.g., households per bank), or by comparing them against each other (e.g., banks per AFI). These intensive variables can be represented on maps using choropleth coloring techniques, which correspond color shades to ranges in numerical data (Goodchild & Lam, 1980), as shown in Fig. 10.1.

Sense of Place to Learn Mathematics

Students are engaged by mathematics curriculum that makes clear connections to their experiences, and issues of fairness can motivate the learning of mathematics (Gutstein, 2003; Leonard et al., 2010; Remillard et al., 2014; Root, 2009; Rubel et al., 2016). We speculate that spatial justice contexts draw productively on students’ sense of place and can serve as a potential resource for engaging students, in this case with concepts of percent and ratio. There is evidence, on the other hand, that sense of place can complicate engagement with data pertaining to familiar places about which students hold prior knowledge. For example, Enyedy and Mukhopadhyay (2007) describe how some students’ privileging of their prior knowledge precluded negotiation with disconfirming data. Similarly, Wilkerson-Jerde and Laina (2015) showed students conforming data about their city to their prior knowledge instead of assimilating disconfirming data toward revision of those pre-existing views. This chapter contributes to the critical mathematics literature by pursuing the question of how a spatial justice context supports students' development of mathematical understanding – in this case, of percent and ratio.

Context and Methods

This module was piloted in a high school in a large city in the United States in 2014–2015. The school is located in one of the city's lowest income neighborhoods and provides free lunch to 100% of its students. Students at this school are identified as "Hispanic" (75%) or "Black or African-American" (25%), with about 20% of the students classified as English language learners. The school is characterized by the district as "persistently dangerous," which requires students to pass through metal detectors to enter school. Incoming students' test scores, on average, are "below proficient" and below city averages. The school suffers from low attendance; about half of its students are categorized as "chronically absent." Despite this array of statistics that portray the school and its students as struggling, the school consistently receives positive feedback on district surveys in which parents and students express enthusiasm and positivity about the school and its teachers.

We are part of a team of White and Asian educational researchers, urban planners, cartographers, and teachers, none local to the school's immediate neighborhood, which collaborated to design the module and its accompanying maps. The classroom teacher was in her 8th year of teaching, all at this school, identifies as White, and participated in a 4-day summer institute focused on this curricular module. The teacher piloted the curricular module with her advisory class of 16 tenth-grade students. Fifteen students consented to participate in the pre- and post-written assessments, 8 girls and 7 boys. Attendance throughout these sessions was reflective of the attendance in the school; about half (8) of the students were absent for 2 or more days over the course of the 10-day module. Four of these students (three boys) were absent for as many as 4 or more days, resulting in greater participation from the girls in the class. Only three students had reached the "college-ready" threshold on the state's entry-level algebra test the previous year, suggesting that nearly all of these students struggled with school mathematics.

We observed and audiotaped all 10 class sessions (each about 45 minutes long) and collected student written work. We wrote detailed field notes for each session and used recordings to clarify and enrich field notes. We coordinated field notes with corresponding audio to produce detailed, analytical memos. Table 10.1 contains a brief description of each session and how it pertains to student learning about percent and ratio. We selected sessions 2, 3, 5, and 7 for analysis because of the prominence of percent and ratio in those sessions. We used audio and the corresponding summary narratives to trace the development of students' understanding of percent and ratio in conjunction with opinions about the spatial distribution of financial institutions.

Table 10.1 Dates and description of class sessions

Session dates	Description
Session 1: 11/5/14	Elements of a transaction in a pawnshop through an improv skit. Concepts of interest and collateral are clarified
Sessions 2–3: 11/6/14 and 11/7/14	Calculations of interest using a ratio table tool (e.g., Middleton & Van den Heuvel-Panhuizen, 1995)
Session 4: 11/12/14	Comparison of various types of personal loans, including their annual percentage rate (APR)
Session 5: 11/13/14	Embodied activity atop an oversized map of their city (e.g., Edelson, 2011) to model their city's distribution of pawnshops and banks by county
Session 6: 11/14/14	Digital map exploration, focusing on layers showing the location of financial institutions and demographics by neighborhood
Session 7: 11/18/14	Ratio map exploration, which normalized the quantity of financial institutions by various variables for each neighborhood
Session 8: 11/19/14	Data collection (photographs and audio-recorded interviews) in the school's neighborhood using a participatory mapping tool
Sessions 9–10: 11/20/14 and 11/25/14	Creation of digital storyboards using images from the digital map and data from the field to express their opinions about pawnshops

Transcripts of whole class discussions totaling 29.5 minutes from sessions 5 and 7 were further analyzed using Dedoose software. We identified student utterances related to percent, ratio, or place, noting utterances in which students used a variable to describe place. In addition, focal students' work (three groups) with digital maps during session 7 was captured using Camtasia (Techsmith, 2010), which video-records students' screens, including the actions of their cursor, in sync with audio/video of students at work through the computer's camera. These recordings were used to produce narrative descriptions of students' actions with the digital maps with corresponding transcription of spoken utterances. We coded uses of variable in the whole-class discussions and during engagement with the digital maps in terms of whether the named variable was an extensive or intensive measure. Second, we identified statements in these whole-class discussions whereby students expressed opinions in terms of implications for spatial justice.

Student participants completed a pre- and post-written, individual assessment that included matched items on percent, ratio, and map reading. Fourteen of 15 participants completed the pre-assessment and 11 completed the post-assessment. We compared written work on these assessments in terms of strategy and correctness. Finally, a focus group session was conducted with five student volunteers (four of whom identify as female) after the module's implementation. The focus group session engaged the students in reflecting about their learning and on further probing their understandings of percent and ratio. Finally, we identified excerpts from the focus group video that focused on themes of percent and ratio.

Results

Conceptual Understandings of Percent

Students demonstrated procedural fluency with percent prior to the module. Nine of 14 students computed 4% of 150 correctly on the written assessment, but were limited to decimal-based strategies, such as multiplying 150 by 1.04, mostly aided by calculators. Students who computed 4% of 150 correctly on the pre-assessment using a decimal-based strategy demonstrated difficulty during the module's early sessions articulating a conceptual understanding of 4%. For example, Lina – a student who had used a decimal strategy and arrived at a correct answer on the pre-assessment – conjectured in class that 4% might mean one-fourth, or a quarter. On the post-assessment, 6 of the 11 students correctly computed 6% of 800, 4 of whom showed ratio table calculations. Nine students attempted to answer a second question, which asked them to model a pawnshop loan, and five had the correct answer. These results on the written assessments might seem underwhelming to readers but should be considered in light of the context of this group of underserved students who struggle with school mathematics. In what follows, we analyze the students' development of conceptual understanding of percent.

During the second session, the teacher introduced a ratio table tool (Middleton & Van den Heuvel-Panhuizen, 1995) with which to conceptualize percent by demonstrating finding the interest on a $150 loan with a 4% interest rate. To build an understanding of the process of a pawnshop loan, 4% simple interest was organized as a $4 amount for every $100 dollars borrowed in a ratio table (See Fig. 10.2). The table has multiplicative properties (i.e., 4% of 700 is 7 times 4) as well as additive

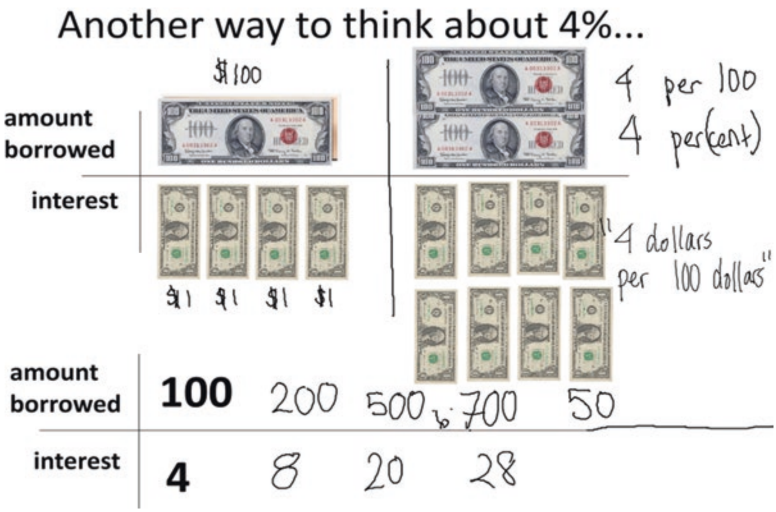

Fig. 10.2 Ratio table used in the classroom (11/6/14)

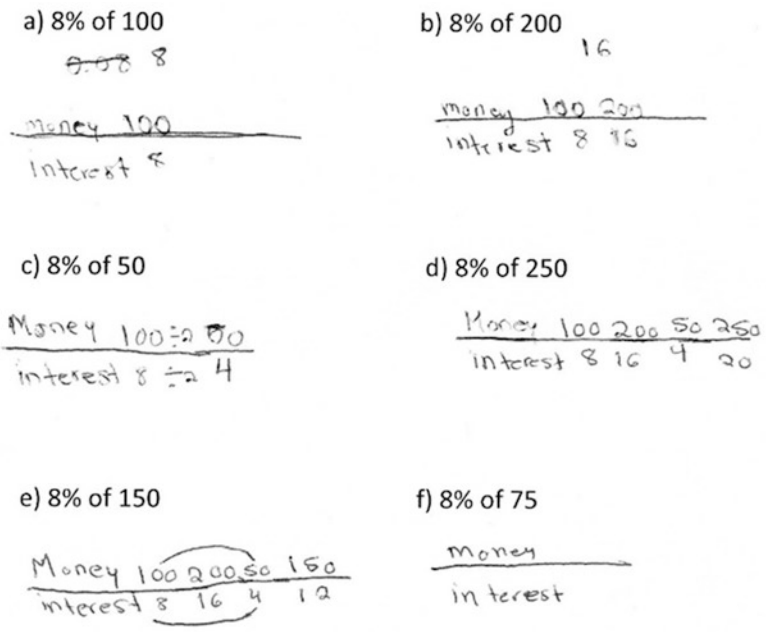

Fig. 10.3 Sample student work using ratio tables to find percent without a calculator (11/7/14)

properties (i.e., 4% of 150 is 4 + 2). A ratio table can be used to calculate any percent of any quantity but is especially efficient for benchmark values.

Students readily took up the ratio table tool in responding to the teacher’s subsequent questions. For example, when the teacher introduced $50 as the amount borrowed in the table (i.e., find 4% of 50), Sheeda offered two different strategies: (1) taking half of 4 since 50 is half of 100 or (2) multiplying 4 by 5 then moving the decimal point one place to the left. When the teacher asked students to use the table to find 4% of 150, Lina, the student who had previously said that 4% was a quarter, said that the answer must be 6 because it is in between 4 and 8 since 150 is in between 100 and 200. In session #3, 8 of 12 students in attendance showed that they could use ratio tables in their written work to compute a given percent of multiples of 100 (i.e., 8% of 100, 8% of 200). Many students could use ratio tables using doubling and halving to compute 8% of multiples of 50 (i.e., 8% of 50, 8% of 150), but only a few students computed 8% of 75, a slightly more challenging example (see Fig. 10.3).

By the end of the project, we observed more efficient and sophisticated use of ratio table strategies to calculate percent. More specifically, during the focus group, when asked to demonstrate how to compute 6% of 250, four of the five students quickly arrived at a correct answer, in some cases without writing anything down. When asked how to accommodate this strategy to a problem like 6% of 280, two students in the focus group demonstrated strategies that extended beyond the halving, doubling, and combining that had been worked on in class with the teacher.

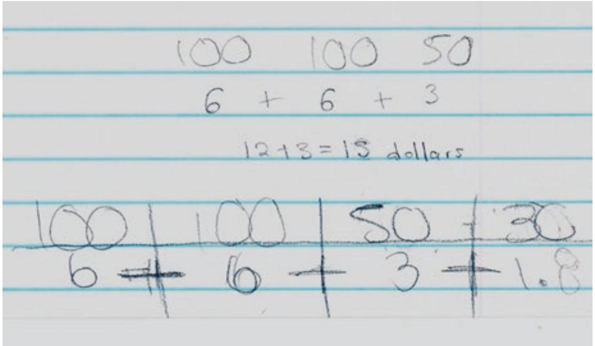

Fig. 10.4 Student work toward calculating 6% of 280 (11/25/14)

Lina scaled 100–280 by a factor of 2.8 and correspondingly scaled 6 by a factor of 2.8. Miguel first found 6% of 250 using doubling and halving and then used the entry for 50 to scale to 5 by dividing by 10 and reasoned that "as the money increases by five dollars, the interest also increases by 0.3." He then used this rate to scale up to 30 dollars toward finding the correct answer (see Fig. 10.4). In response to the question, "Did you learn any math?", all of the focal students quickly and emphatically responded affirmatively. Isabel described how her participation impacted her mental mathematics strategies and her confidence about those strategies; she elaborated: "I learned a lot. Because, like, before, when I was finding percent, I would only know how to find 50% like, oh -I'd be like- oh it's half. But then like, now, I know how to find, like for *any number*" (Focus group, 11/25/14).

The use of the ratio table to calculate percent was effective as a remediation tool. At many points, various students pointed out that it is important for a consumer to be knowledgeable about individual transactions, because "pawnshops get over on you." Students felt that understanding mathematics would help them to better navigate this process. In other words, mathematics would empower students as consumers. However, by virtue of the high rates of AFIs relative to other options, the quick access they provides to services, and their prevalence in low-income neighborhoods, AFIs can be considered as part of a predatory system. This is an argument with a spatial dimension, and pursuing it demands the use of ratios.

Conceptual Understandings of Ratio

As part of the pre-assessment, students were asked to respond to the following item:

Neighborhood A has 300 families and 3 grocery stores. Neighborhood B has 600 families and 5 grocery stores. In which neighborhood do you think people have better access to grocery stores? Explain your answer.

Five of 13 responses did not relate to ratios and related instead to extensive variables, such as neighborhood B having more stores (2 responses, example: "the more grocery stores the better"), or to neighborhood A having fewer families (4 responses, example: "Neighborhood A has a better access to the grocery stores since Neighborhood A has 300 families"). Among the eight students who demonstrated ratio reasoning, only Lina provided a complete justification: "NA 300/3 = 100 people per grocery stores; NB 600/5 = 120 people per grocery store. I think in neighborhood A people have better access to grocery stores because there's less people in each grocery store." The other seven responses were qualitative or less precise, such as "Neighborhood A. Because neighbor B has more stores but it has too many people for those stores" or "Neighborhood A because in Neighborhood B there are only 2 more stores and double the people so the first one is more accessible."

After the module, only 11 students completed the post-assessments, and of those, only 5 students answered the grocery store ratio item. The post-assessment took place before Thanksgiving holiday, so we speculate that students may have felt distracted by the impending holiday vacation. Nevertheless, there was evidence of progress in students' responses. All five students who responded to the ratio item demonstrated ratio reasoning and included calculations to illustrate their thinking and justify their responses, in contrast with only one student who had done so prior to the module. For example, prior to the module, Bo argued that neighborhood B is more accessible because it has "more stores" than neighborhood A, but after the module the student calculated the number of families per store for both neighborhoods and stated, "Neighborhood A because A have about 100 people to a store and B have about 120 people per store." Another student, who previously had written a vague and unjustified response, calculated the number of families per store for both neighborhoods and compared those ratios in his response. In what follows, we analyze students' development of conceptual understanding of ratios and, in particular, how this was supported by a spatial context.

Motivating Ratios The grocery store access task is limited to two extensive variables, the number of families and the number of grocery stores. If we consider the spatial distribution of a resource like grocery stores, other variables could be deemed salient, like the size of the neighborhood in area, in population, or relative to other, related businesses. The module intended to support students toward a conceptual understanding of data normalization and the coordination of such measures with associated demographics, supported by a walking-scale, laminated, 140-square-foot floor map of the city during the fifth session. Students were assigned to represent respective counties by standing in them and received scaled props to demonstrate relative numbers of pawnshops and banks in these spaces (Fig. 10.5). When a hypothetical equal distribution of pawnshops by county was introduced, Sheeda almost immediately refuted this as not making sense, drawing on an intensive measure of resource per area. She explained that "some places are bigger than others, and some places are poorer than others," suggesting that normalizations should be produced to compensate for these distinctions. The teacher highlighted the former part of Sheeda's observation to introduce the subsequent activity which modelled various,

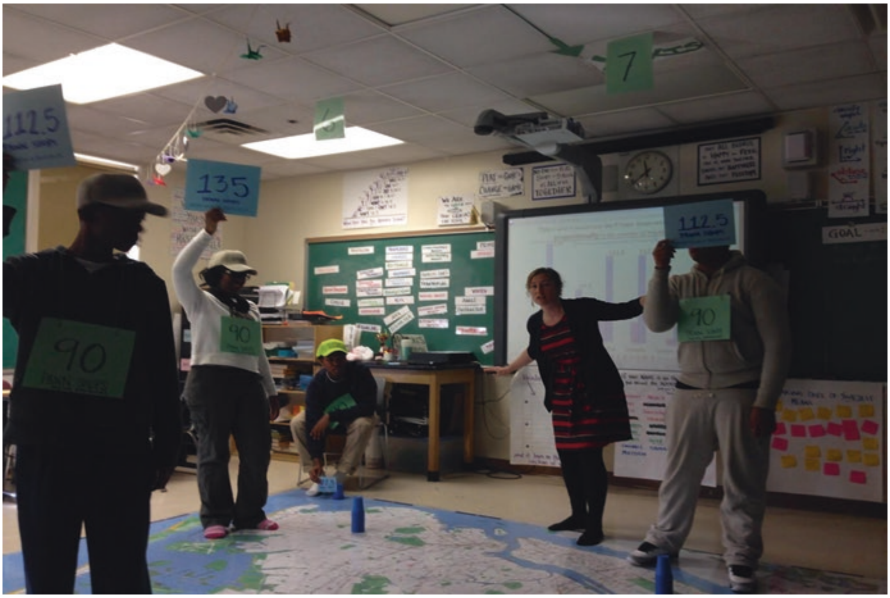

Fig. 10.5 Students participate in embodied distribution activity on large map

hypothetically fair distributions of the institutions, such as by area or by households, using scaled props. The actual distribution, which does not correspond to any of the hypothetically fair arrangements, was then modelled with the scaled props.

Students confirmed the actual, skewed distribution using their senses of place. For example, Sheeda explained the large number of banks in Manhattan by pointing out, “He (Manhattan) got a bank on every corner though.” Her confirmation suggests a normalization by area, using “corner” to represent the urban area measure of block. In addition, her statement adds nuance to the uniformity of banks within a specific county. Several students conjectured that places have more banks because of a corresponding higher density of stores, especially expensive ones. In other words, they sought to interpret the skewed distribution using their sense of place about an intensive variable, namely, retail density. Unfortunately, measures of retail density were not included on the predesigned digital maps, so this was not a relationship that students could further explore.

Another explanation of the distribution provided by students was in relation to their sense of place about income and spatial inequalities. For example, Rebecca ventured, “I think that the ones that have the most pawnshops is where people have less money.” Rebecca’s desire to explore a relationship between these two extensive variables, a neighborhood’s median household income and its amount of AFIs, foregrounded her later exploration with the digital maps. This particular hypothesis conformed to the maps’ design for students to seek patterns that relate locations and densities of banks and AFIs to patterns in provided socioeconomic measures. Later in that discussion, Rebecca responded to the unequal distribution of pawnshops and banks by pointedly asking the teacher, "Why they never fix that, Miss?" "Fixing" suggests more equitable access to financial services and would involve a transformative "rewriting" of the world. With her use of "they," it is unclear to whom Rebecca is attributing the power to make changes, and Rebecca's query was not further picked up by the teacher. Bo, another student, responded to her question and explained the skewed distribution with "'cuz they need the pawnshops," surfacing a counter-suggestion that pawnshops or AFIs can be seen as community resources. Rebecca revoiced her question, "Why they don't do it the way it's supposed to be?" indicating the idea that the distribution could be fairer, in a way that it is "supposed to be." Her question was again not picked up by the teacher, and instead a discussion among a small group of students ensued about pawnshops locating in parts of the city where "business is good," a reframing of the need to rewrite injustices in terms of capitalistic perspectives.

Using Ratios In the seventh session, students explored a set of predesigned maps showing intensive variables that normalize the distribution of these institutions, a set of maps referred to as "ratio maps." These maps relate the quantity of banks or AFIs to area, number of households, or relative to one another (i.e., banks per AFI, see Fig. 10.1). The teacher launched the session by orienting students to the ratio maps using an example of the distribution of McDonald's fast-food restaurants by sharing a city map colored according to McDonald's per square mile. The notion of a fair distribution of McDonald's is not equivalent in significance to a fair distribution of financial services, and some might consider the presence of McDonald's as a disservice, but the example was intended to serve as a familiar variable for students to situate their reasoning about distribution in relation to area and households.

Where on the floor map students' sense of place about locations in the city served to prompt their association of variables toward normalizing the distribution, here, two students recruited their senses of place as a way of interrogating the maps' accuracy. For example, upon the presentation of the map on the classroom SmartBoard, Lina immediately questioned the normalization by square miles and wanted to know the conversion factor to convert the areal unit to square blocks, reminiscent of Sheeda's comment from session 5 about banks "on every corner." Lina's and Sheeda's understanding of units of measurement in city distance did not conform to the traditional measure of square miles used in the maps. In this instance, Lina recruited her sense of place about the city's spatial arrangement to interrogate the decision to normalize density with square miles. Lina's skepticism of the map layers and interrogation of the presented data marks a shift from her participation on the 5th session when she had generated variables and conjectured relationships.

During the discussion about the distribution of McDonald's, students largely focused on the extensive variables included in the map rather than intensive variables. For example, when the teacher prompted students to consider how their neighborhood compared to another in the city, Sheeda and Jonny responded with (Sheeda): "there's more" (i.e., the other neighborhood has more) and (Jonny): "reeaaallly small." The teacher then had to push to get students to use the language of "per square mile" in their responses, to which Sheeda responded, "No, Harwood got less." This fixation on the language of "more" and "less" seems to suggest that students were noticing the absolute counts, or the extensive variables, rather than the intensive variables and data normalizations. An exception was when Rebecca expressed a conceptual understanding of the intensive measure of households per McDonald's, "So we only have one we share with more, and then they have way more and they share with less." Rebecca's observation prompted Lina to reduce the fraction 23,000 households/23 McDonald's to its unit ratio and follow up by asking, incredulously, "So a *thousand* people go to *one* McDonald's?" These combined observations fed into Rebecca's earlier expressions about unfair distributions, and, echoing that earlier question, she pointedly asked the teacher again, "So why doesn't anyone fix that?"

Rebecca's question suggests that she was again questioning spatial justice, using the ratios at work in the digital map. This time, the teacher pursued her idea, asking if she meant that they should open more McDonald's in that neighborhood, and Rebecca elaborated that "McDonald's is great" and "they have nothing over there" and that the neighborhood is under-resourced. As in the previous instance, when Rebecca had questioned how this injustice could be rectified, the teacher again did not address the need for "fixing" inequities. Instead, the teacher redirected Rebecca's statement about the distribution needing to be fixed by highlighting that it is producing quantification with these ratios is what enables comparisons. She said to the class, "Whether it's good or bad, it's something we can compare," and guided the students, working in pairs, to use the digital maps to choose one ratio map layer, interpret the map's data for Harwood, notice and examine patterns across the map, and compare the data for any two neighborhoods. The three focal groups engaged with the ratio maps in different ways, showing difficulties in interpreting and employing the intensive variables and a tendency to focus on the extensive variables instead.

Orange Pair Sheeda and Miriam compared Harwood with Montgomery, a familiar, adjacent neighborhood. They spent most of their time exploring the absolute numbers of financial institutions using various ratio maps. As a typical instance, upon examining the pawnshops per square miles map, Sheeda clicked on Montgomery and noted, "There's 3 pawnshops," by reading only the numerator from the fraction shown in the pop-up box. She ignored the square miles in the denominator and the resulting ratio of pawnshops per square mile. Using the ratio of AFIs per bank specifically for Harwood, Miriam again compared the quantities in the numerator to the denominator additively: "We got more pawnshops than banks." At no point did Sheeda or Miriam demonstrate thinking about the intensive quantities, and they persisted in engaging only with the extensive components. A researcher guided the students to note that Harwood was smaller in area but had more AFIs than Montgomery. When asked to explain the comparison of these ratios, Sheeda persisted with a limited focus on the extensive variables instead and responded, "That means Montgomery is bigger than Harwood."

Green Pair Miguel spent most of the session silently clicking through and across all of the ratio maps as his partner Rafaela watched, and connections to their senses of place were not apparent. Miguel's strategy was to click on each map's darkest shaded neighborhoods, that is, the neighborhoods with the highest ratios for each intensive variable. This does not necessarily mean he was looking at the most-serviced neighborhoods. For example, darker shading in the households per bank map layer signifies a lower rate of services. In his explanation for selecting the banks per square mile intensive variable, Miguel stated, "Because we wanted to see how many banks were in each square mile"; and he reported that in Harwood, "for each 1.81 square miles, there's only 3 banks." The use of the word "each" indicates an understanding of the multiplicative relationship between the extensive quantities, but Miguel's interpretation of the map shading referred only to the extensive variable of the counts of banks. He wrote, "The darker the color becomes, the more banks you will find in that location. And the more lighter the color becomes, the less banks you will find."

Purple Pair Rebecca worked alongside an assistant teacher and focused on the households per bank map. Rebecca wrote an analysis of Harwood's data that went beyond Miguel's by not only interpreting the ratio terms but also how it related to the unit ratio: "The data says that there are 35,521 households for each 3 banks. So 11,840 people share each bank. It says that Harwood shares each bank with a lot of people." After reading the number of households and institutions in her chosen neighborhoods, she stated, "So this one [Portmore] has less households and it has more banks. And this one [Easington] has more people and just one bank." She concluded, "They [Easington] should have put … had more [banks]." Rebecca did not generalize her interpretation beyond making sense of specific data points. When prompted to explain what a higher ratio meant for her variable, Rebecca said, "When it says the ratios are higher, I think it means where the banks are more at." She did not recognize that in the case of households per bank, a higher unit ratio would indicate a lower proportion of banks per household. In this case, Rebecca's sense of place confounded her analysis in that her hypothesis that the number of banks was related to a neighborhood's income level was a distraction from interpreting the given legend. Rebecca stated that she "just wanted to know where was the more banks for houses with less money" and she expected to be able to answer this question through a single map layer. This expectation led her to try to inject income into her analysis of the households per bank map and read the categories in the legend as referring to ranges in household income rather than ranges in number of households.

Discussion

Findings include student interest in acquiring a conceptual approach for percent calculations, widespread adoption of the ratio table model to calculate percent, and evaluation of spatial justice by normalizing data and comparing ratios. The spatial context, first at the scale of a transaction in an individual pawnshop, supported the use of a new representational tool to further develop students' conceptual understanding of percent. Although we cannot separate the use of the conceptually oriented ratio table from the spatial justice framing, our interpretation is that students were sufficiently interested in the context to be open to engaging with mathematics typically taught to younger students. Instead of positioning the ratio table and the work with percent as remediation of skills not yet mastered, the approach for a conceptual understanding of percent was encapsulated in a sophisticated and engaging context. The teacher was able to rely on the context of calculating the interest of a pawnshop loan to provide an accessible introduction of the ratio table tool to conceptualize and calculate percent, and the genuine context also provided motivation for students to perform the calculations. As a result, some students demonstrated not only ability to calculate percent using a ratio table but also flexibility in their approaches toward greater sophistication. Future research could focus on relationships between conceptual understanding of percent as a function, proportion, and statistic in mathematical investigations that draw on data that pertain to demographics and place.

In response to a researcher's follow-up question as to whether this mathematics was new to her, Sheeda contrasted her experience with this module with typical school mathematics learning:

Sheeda: Cuz basically we learned it, but basically we, didn't, like pursue more into it. Like, we just like, oh that's something that we learned in school, like, we-we like…

Lina*:* to learn it, just to learn it…

Sheeda: Yeah, it felt like, we felt like…

Lina*:* we did, applied learning.

Sheeda*:* Yeah, it felt like—exactly, that's like what I was saying—like, it felt like we really needed to, like, pursue it.

These students make the point here that the context was not only interesting to them, but they felt learning this mathematics was pertinent to their lives beyond school, supporting findings from the literature that suggest contextualizing mathematics in social justice issues contributes to student engagement (Gutstein, 2003, 2006; Leonard et al., 2010).

In this project, data was normalized in various ways to produce intensive variables to support spatial analysis of a resource distribution. Students' sense of place supported and complicated their engagement with these ratios. The activity atop the oversized floor map built on students' sense of place to generate thinking about the need for normalization as well as various intensive variables. Consideration of these variables provided an entry point for the teacher to expand upon and formalize the data normalization strategies used in the project's digital maps. Concrete representations of intensive variables, such as modeling and unpacking households per institution as the number of people sharing one institution, supported students in developing ratio-based arguments about fairness. However, with ratio data provided on digital maps, students tended to compare extensive variables and not ratios, echoing findings in the literature that identify challenges with proportional reasoning (e.g., Lamon, 1993).

One explanation of this tendency is that some hypotheses stated atop the oversize floor map were formalized in the digital maps but other variables suggested by students, like the retail density, were not available as data layers. The map tool did not allow students, therefore, to pursue their own conjectures about variables relating to their own senses of place to be able to engage in more authentic data explorations. Of course, a limitation produced by our analytic focus on whole class discussions is that these findings reflect the contributions not of the whole class but only its vocal participants. Nonetheless, these results suggest that further attention be paid toward how to better support students in understanding and using intensive variables to investigate spatial justice issues, especially abstract variables like AFIs per bank.

We have described three instances of a student (Rebecca) raising spatial justice critiques related to intensive variables in class discussions, critiques that were not addressed or followed up on by the teacher. We do not view this as a shortcoming of this particular teacher and instead use the trend to draw attention to the complexities involved in teaching mathematics for spatial justice (see Rubel, Hall-Wieckert, & Lim, 2016 for a broader discussion of these complexities). Better supports for the teacher to engage with this issue in terms of how the system could be transformed to be fairer could have resulted in more meaningful discussions of reimagining access to financial resources.

Conclusions

This module was organized for students to formulate a spatial justice analysis of their city’s two-tiered system of financial institutions. The module began at the spatial scale of a single transaction in a financial institution and proceeded outward to a broader perspective about the spatial distribution at the city scale. Our findings have demonstrated how this module engaged students’ sense of place. The successes demonstrated here suggest that contextualizing mathematics learning in investigations of themes of spatial justice is promising and we see potential across domains like access to healthcare, transportation, education, and more. Readers can find a heuristic for designing such investigations in the work of Rubel and colleagues (2016).

The open-endedness of the floor map enabled students to understand the conceptual basis of intensive variables and suggest their own data normalizations. This analog representation served as a building block toward making sense of the data represented more abstractly in the digital maps, which were more elusive for students since their closed nature did not as effectively engage students’ sense of place. The learning atop this analog floor map is a reminder that, despite the ubiquity of digital technologies, there are many ways to engage with place and that we should not limit our curricular designs only to conventional, single screen-sized map projections that represent place from an aerial view. Instructional designs that combine the open-endedness of the floor map activity with the abstraction and further mathematization of the digital maps seem needed. A future goal in this direction is the design and implementation of mapping tools that enable students to generate their own variables for visualization of intensive variables as ratio map layers. Not enough attention was paid in the design of the curricular activities described here to providing the students or teacher with supports for a critical reimagining of space. Since opportunities to "read the world" must be balanced with opportunities to "write the world," a future goal is to better connect youth to existing social movements that challenge the status quo related to the given spatial justice issue.

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Chapter 11
Supporting the Development of Bilingual Learners' Mathematical Discourse in a Multilingual, Technological Context

Oi-Lam Ng

Abstract In this chapter, I explore the use of dynamic, touchscreen technology for supporting the development of bilingual learners' mathematical discourse during pair-work exploratory activities in calculus. This study emerged from increasing linguistic diversity in mathematics education and a new mathematics curriculum movement which emphasizes fluency of mathematical communication and the consistent use of technology for learning mathematics. In response to the changing needs and curriculum, this study investigates bilingual high school students' communication when they interact with touchscreen dynamic geometry environments (DGEs) during calculus discussion and exploration. Specifically, I address bilingual learners' linguistic and nonlinguistic means of communication in the activity with touchscreen DGEs and the mathematical competence demonstrated by the students in the activity. Using the theoretical framing of thinking-as-communicating, I provide a vignette with qualitative analyses of one pair of bilingual learners' communication, focusing on the students' language, gestures, and touchscreen-dragging actions with touchscreen DGEs during the mathematical activity. Results suggest that a multimodal lens for understanding bilingual learners' mathematical thinking, along with the touchscreen interface and dynamic features of the DGEs, facilitated productive discussion and exploration about calculus for the students. This study raises implications for achieving equity in today's increasingly multilingual and technological learning environments.

Introduction

My research interest in linguistic diversity in mathematics education emerged through my life journey of becoming an English language learner (ELL) in my early teens, and later a mathematics educator in one of the world's most multilingual

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cities in Vancouver, British Columbia, Canada. From 10 years of teaching mathematics in a multilingual context and growing up as a multilingual learner, I learned that besides facing everyday teaching tasks, the role of a mathematics teacher demands grappling with multiple challenges in facilitating communication for learners from different mathematical and linguistic backgrounds. By different linguistic backgrounds, I mean the diversity of home languages spoken by students in the mathematics classroom, which in my local context usually exceeds three and ranges up to ten in a typical class of up to thirty students. The "2012 BC Education Facts," published by the British Columbia Teachers' Federation (BCTF, 2012), sketches the context of teaching and learning mathematics in the province currently: "In 2011–12, one in four (23.8%) of public school students spoke a primary language at home other than English. Almost double this number of ELL students (135,651) live in families where the primary language spoken at home is other than English, an increase of 16,874 students since 2001–02 and 8,676 students since 2007–08." In a global context, linguistic diversity in mathematics education has steadily increased due to globalization and an ever-expanding intra- and international movement (Morgan, 2007). From an equity standpoint, this gives rise to an emerging tension between addressing learners' lack of proficiency in the language of instruction and emphasizing communication as an essential process for mathematics learning: "They communicate to learn mathematics and they learn to communicate mathematically" (National Council of Teachers of Mathematics [NCTM], 2000, p. 60). The value of communication is further emphasized by NCTM (2000) as a *process standard* for mathematics learning, pointing to the value of rich conversations about worthwhile mathematical tasks for students' learning. It is suggested that "support for students is vital" (p. 60) as they learn to participate in these conversations and that teachers need to build a community in which the exchange of ideas can freely occur—a potentially challenging task in multilingual classrooms. This raises questions about equitable teaching practices in classrooms where students are encouraged to grapple with concepts by communicating in their own words, yet the abilities to do so vary among students with different experiences and language proficiencies. Currently, research focused on linguistic diversity in mathematics education has provided tremendous insights into the complexities of teaching and learning mathematics in multilingual contexts: the language tensions in multilingual classrooms (Barwell, 2014), the dilemmas of teaching in multilingual contexts (Adler, 2001), the role of code switching in learning mathematics (Clarkson, 2007), as well as associating mathematics learning with socioeconomic and epistemological access (Setati, 2005). Yet, there is increasing need to examine and understand how to support mathematics learning in a non-native language both locally and globally.

While the complexities of teaching and learning in multilingual mathematics classrooms have been more widely explored, studies which focus on examining and supporting bilingual learners' communication are limited. In particular, Moschkovich (2010) called for studies to consider broader ways of using language as the basis for understanding and facilitating bilingual learners'[1] communication, by calling attention to mathematics learning as a discursive, embodied (Gutiérrez, Sengupta-Irving, & Dieckmann, 2010), and multi-semiotic (O'Halloran, 2000) activity that includes the use of diagrams and gestures. Moschkovich's premise of addressing the mathematical competence of students from non-dominant communities without evaluating their language per se relates to *access* and *identity*, proposed by Gutiérrez (2009, 2012) as two dimensions reflecting research addressing equity. In Gutiérrez's view, *access* pertains to the tangible resources that students have available to them to participate in mathematics, and *identity* attends to whose perspectives and practices are valued, such as whether students have opportunities to draw upon their cultural and linguistic resources in mathematics learning.

In this chapter, I examine how recent advances in touchscreen technology and gestures as a form of communication may serve as *access* for bilingual learners to engage meaningfully in mathematical activities and communication. I examine the learning of high school calculus to highlight the potentials of my work to provide *access* for students from non-dominant communities to "important mathematics" in the sense of opportunities to participate science, technology, engineering, and mathematics (STEM) disciplines (American Educational Research Association, 2006). My research questions are informed both by studies that have examined the effect of technology-mediated learning of calculus concepts (Hong & Thomas, 2013; Yerushalmy & Swidan, 2012; Yoon, Thomas, & Dreyfus, 2011) and research on bilingual learners' nonlinguistic forms of communication, such as gestures and diagrams (Gutiérrez et al., 2010; Moschkovich, 2007, 2009). As neither lines of work have addressed the particular affordances of the touchscreen mode of interaction on calculus learning or bilingual learners' discourse, my study proposes to fill this gap. Specifically, I hypothesize that a touchscreen environment may offer additional affordances for learning by providing tactile and kinesthetic modes of interaction—hence, further facilitating bilingual learners' communication in calculus. My research questions are threefold: (1) How do bilingual learners utilize linguistic and nonlinguistic means to communicate calculus ideas during pair-work on mathematical activities within a touchscreen, technological learning environment? (2) What are the significance of gestures and the touchscreen in this communication? (3) How does this analysis identify bilingual learner's mathematical competence in the activity and participation as members of the classroom community?

In the pages that follow, I discuss the role of communication in mathematics, particularly the roles of gestures and dragging in mathematical thinking and learning. Next, I describe my methods in conducting the study and provide details about the setting and participants. Then, I share a vignette about a mathematical discussion

[1] Moschkovich (2016) used the term "bilingual" or "multilingual" learners instead of "English language learners" because it focused on what students know and can do (speak two or more languages) instead of what they do not know (English). In this research, I use the term "bilingual learners" for the same reason, and I reserve the term "multilingual" to describe the classroom contexts in which learners come from diverse language backgrounds and often do not share the same home language.

between two bilingual learners using touchscreen and dynamic technology, highlighting their mathematical competence through the use of language, gestures, and the touchscreen, and I conclude with a discussion of the implications of the results.

Theoretical Framing

In this section, I discuss the theoretical framing of *thinking-as-communicating* for studying bilingual learners' mathematical discourse within a technological environment. Anticipating a multimodal discourse, I have chosen Sfard's (2008, 2009) commognitive framework to highlight the role of talking and gesturing in mathematical thinking. I establish the notion of touchscreen dragging as a multimodal feature in one's mathematical discourse by drawing on what Sfard calls *routine* in communication.

Mathematical Thinking as a Discourse

Sfard's commognitive theory (2008) was based upon the social dimensions of learning and highlighted the communicative aspects of thinking and learning. In her theory, Sfard redefines thinking as an "individualised version of (interpersonal) communicating" (p. 81). The term *commognition* stressed the fact that cognition (intrapersonal communication) and interpersonal communication are manifestations of the same phenomenon. This perspective suggested that mathematics learning could be evidenced through a change in one's mathematical communication (discourse). For example, in terms of the development from arithmetic thinking to algebraic thinking, it is a case of engaging the discourse about arithmetic at the object level so that it can be used in the discourse about algebra. Sfard (2008) proposed four features (word use, visual mediator, routines, and narratives) of the mathematical discourse which could be used to analyze mathematical thinking and changes in thinking. For this paper, the first three features are used to examine one's language, gestures, and dragging—a multimodal mathematical discourse. *Word use* is a main feature in mathematical discourse; it is "an-all important matter because … it is what the user is able to say about (and thus to see in) the world" (p. 133). As a student engages in a mathematical problem, her mathematical discourse is not limited to the vocabulary she uses. For example, her hand-drawn diagram and gestures can be taken as forms of *visual mediator*, a visual realization of the object of a discourse. *Routines* are meta-rules defining a discursive pattern that repeats itself in certain types of situations. In learning situations, teachers may use certain words or gestures repeatedly to model a discursive pattern, such as looking for similarities and what it means to be "the same."

Multimodality: Gestures and Touchscreen Dragging

Besides theorizing mathematical thinking as a discourse and characterizing its four features, Sfard (2009) examined the relationship between talking and gesturing in mathematical thinking. She explained that utterances and gestures are different modalities that serve different functions in the commognitive process. Utterances enable humans to engage in meta-level discourse, or talk about talking. On the other hand, gestures ensure all interlocutors to "speak about the same mathematical object" (p. 197) and are essential for effective mathematical communication: "Using gestures to make interlocutors' realizing procedures public is an effective way to help all the participants to interpret mathematical signifiers in the same way and thus to play with the same objects" (p. 198). In relation to my study, it is anticipated that students would make use of gestures and touchscreen dragging while interacting with dynamic geometry environments (DGEs). For example, a student may repeatedly use her hand to signify certain slopes (horizontal, positive, negative) routinely or to compare slopes of different line segments. This is an example of the student using gestures as a *routine* to look for patterns and relationships. The same can be said if a student uses touchscreen dragging to move the tangent along a curve on the DGEs for comparing slopes of tangent. Hence, both gestures and touchscreen dragging can be taken as a *routine* for defining a discursive pattern of the students' mathematical communication. Besides identifying the general characteristics of gestures, I also distinguish gestures that are of *dynamic* nature from those of *static* nature, in order to understand when gestures are used to convey temporal relationships (Núñez, 2006). For example, when a person makes a gesture to realize the signifier, a linear function, it could be performed with the arm or hand enacting the function (static), or with the hand or finger tracing the motion of the function's path (dynamic). More than conveying the shape of the linear function in a static sense, the gesture in the latter case communicates temporal relationships and a sense of change, which may be relevant for calculus communication. It is also important to note that temporality and continuous change are both embedded in the act of touchscreen dragging; elsewhere, I have drawn some interesting parallels between gestures and touchscreen dragging in calculus communication (Ng, 2014).

Methods

During 2013–2014, I undertook a research study involving twelve participants who were bilingual learners enrolled in two sections of a calculus course in a culturally diverse high school in Western Canada. The participants, aged 16–18, were selected for their bilingual background. They were born outside of Canada and had recently moved to Canada from a non-English-speaking home country at the age of 10–16. The class size for each section was 23 and 26, respectively, with roughly one-half of the students enrolled in each section bilingual learners. The participants were paired up with each other on the basis that they were comfortable working with each other during pair-work activities in the regular classroom. For this chapter, I focus on the part of the study designed to address bilingual learners' patterns of communication while exploring a calculus idea that they had not yet learned in a touchscreen, dynamic calculus environment. The target calculus concept, area-accumulating function, was chosen for three reasons. First, the function $A(x)=\int_x^a f(t)\,dt$ could be represented geometrically, and it was possible for one to explore the change in $A(x)=\int_x^a f(t)\,dt$ without knowing the corresponding symbols. This can be achieved by thinking of the change of $A(x)=\int_x^a f(t)\,dt$ as area accumulation. Second, the timing for introducing the concept was appropriate because the students would have had some experience with learning calculus in a dynamic, technological environment in their regular classroom before the time of study. In particular, they would have used a similar dynamic sketch for exploring derivative functions by interpreting derivative as the slope of tangent to a curve geometrically. Third, the area-accumulating function was chosen as research suggests that the simultaneous change of the variables *x, f(x)*, and *A(x)* was difficult to grasp among calculus learners (Thomas, 1995; Thompson, 1994; Weber, Tallman, Byerley, & Thompson, 2012). The task used in the study invited participants to explore and discuss any emerging mathematical relationship while using an iPad-based DGE, *SketchExplorer* (Jackiw, 2011). For the purpose of examining students' routines during an exploratory activity with the touchscreen DGEs, they were given a sketch containing four pages and a "Try" page, all related to the concept of area-accumulating functions. Each page contains the same functionalities of enabling the user to drag two values, *a* and *x,* on the touchscreen and observe the corresponding change in $A(x)=\int_x^a f(t)\,dt$ numerically, geometrically as a shaded region, and graphically in the form of *(x, A(x))* which appeared in green. For example, Fig. 11.1a shows that the area under the function *"f(t) = 1"* is *"A = 4.70"* when the bounds are set to *"a = 0"* and *"x = 4.70,"* and the traces in green represent the corresponding area-accumulating function. Each page differs by the particular function, *f(t)*, shown; Page 1 shows the function *f(t) = 1* (see Fig. 11.1a), Page 2 shows *f(t) = t*, Page 3 shows $f(t) = t^2$, and Page 4 shows *f(t) = sin(t)* (see Fig. 11.1b).

At the end of the task, the participants were asked to move onto the "Try" page of the sketch where a problem related to sketching an area-accumulating function was posed. They were asked to solve the problem on a dry-erase whiteboard with their partners. A digital camera with video recording function was placed at an angle in front of the students' desks where they were discussing and interacting with the iPads and which captured the tablet screen at an angle. Before the start of the task, it was stressed to the participants that the activity was not about finding the right answer but more about exploring the concept, looking for patterns, and communicating what they saw with each other. Adopting a discursive research methodology tradition (Sfard, 2012), I attended to the use of linguistic (oral and written language) and nonlinguistic (gestures and touchscreen dragging) features in the students' communication. In order to investigate how they were utilized (research question 1) as well as the significance of gestures and touchscreen dragging

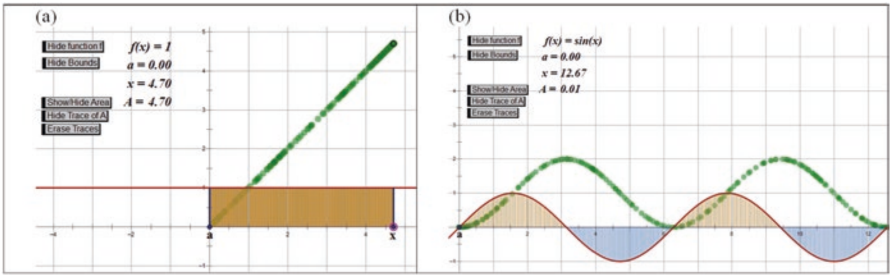

Fig. 11.1 (**a**) Page 1 of the sketch conveying the area under a constant function with $a = 0$ and x dragged from zero to its current positions. (**b**) Page 4 of the sketch conveying the area under a sine function

(research question 2), I needed to observe the way that multimodal communication was used simultaneously and in succession. Therefore, I devised a special transcript convention in order to keep a record of who spoke, gestured, and dragged, as well as any overlapping speech. Specifically, I introduced three columns, "S-er" (speaker), "G-er" (gesturer), and "D-er" (dragger), to track beyond "who spoke what" in a conventional transcript, and I implemented underlining of the transcript to record which words were spoken while a gesturing and dragging action was performed simultaneously by one of the students. Upon transcribing, the data was analyzed in terms of (1) specific words, phrases, visual mediators, and routines used; (2) turn-taking, instances of simultaneous speaking, gesturing, and dragging (either by the same person or by different persons), and instances of dragging or gesturing without accompanying speech; and (3) the mathematical ideas communicated and mathematical practices demonstrated (exploring, conjecturing, verifying, etc.) in the communication. It was intended that the analysis of (1) and (2) would address the first two research questions, while the analysis of (3) would address the final research question. According to Arzarello (2006), a *synchronic analysis* enables the study of relationship among different semiotic sets activated simultaneously, while a d*iachronic analysis* studies the same phenomenon in successive moments. Following Arzarello (2006), I used a *synchronic* lens to examine the interrelationships between linguistic and nonlinguistic modes of communication and a *diachronic* lens to investigate how this communication changed over time (see also Chen & Herbst, 2012). By performing these analyses, my goal is to highlight the use of touchscreen, dynamic technology for providing bilingual learners with access to calculus, opportunities to engage in mathematical communication, and possibilities to participate as members of the classroom community.

Vignette

In this vignette, I provide in-depth analyses of three different moments in the communication between two participants, Jay and Katie, during the task. I have chosen to share this vignette because Jay and Katie had the least experience with

Table 11.1 Transcript of Jay and Katie's discussion

Turn	Timespan	What was said	S-er	D-er	G-er
16	1:20.8–1:25.8	What is this dot?[a]	K	K	
17	1:24.0–1:25.8	[Area. Area.][b]	J		
18	1:25.8–1:28.5	Are you sure it's the area?	K		
19	1:28.5–1:35.7	Look… [Three, times three, equals nine right?]	J	J	
20	1:35.7–1:41.4	It's a triangle, so halve it. What is the area? It's four point five	J		J
21	1:41.4–1:42.4	Hm.	K		
22	1:42.4–1:52.0	Look. Two times two, area two. One, one, should be zero point five	J	J	
23	1:52.0–1:55.2	One, one, is zero point five. Oh ya it is	K		

[a]The symbol (?) denotes a rising intonation at the end of a sentence
[b]Since the participants spoke in English occasionally, it was necessary to differentiate the language spoken after the translation. The actual English words spoken by the participants were written in squared brackets "[ ]"

mathematics instruction in English out of all the participant pairs, having studied in an English-speaking environment for only 2 years. Therefore, in addition to learning calculus, which was new to them, they also had to grapple with the English language more so than their peers. In the regular classroom setting, they were described by the classroom teacher as students who often used their home language (Korean) for discussing calculus ideas and who seldom participated in whole class discussions. Given their backgrounds and experience, it was not surprising to observe that Jay and Katie communicated in their home language during the task. Indeed, they spoke in Korean for the majority of the task and used English words occasionally in their utterances, except when I approached to interact with them in English. The video data underwent two rounds of translation/transcription to ensure validity of the process. In each round, I asked a Korean-Canadian (one of whom was a mathematics educator in the local community) to translate the Korean words spoken in the video into English while I transcribed the data. The following transcript (Table 11.1) was taken 1 minute and 21 seconds into Jay and Katie's discussion, which also marked the first time that the word "area" appeared in their discussion. At this time, they had already turned to the second page of the sketch which showed the function, $f(t) = t$. While they were on the first page, the word "area" did not appear, but they talked about "multiplication" and "width," which suggests that they were noticing something about the rectangular region under $f(t) = 1$ from a to x.

In the above transcript, Jay and Katie used numerical values as shown on the sketch to explore the relationship between x, $f(x)$, and $A(x)$. Their discussion could be considered highly valued in the mathematics community because it resembled

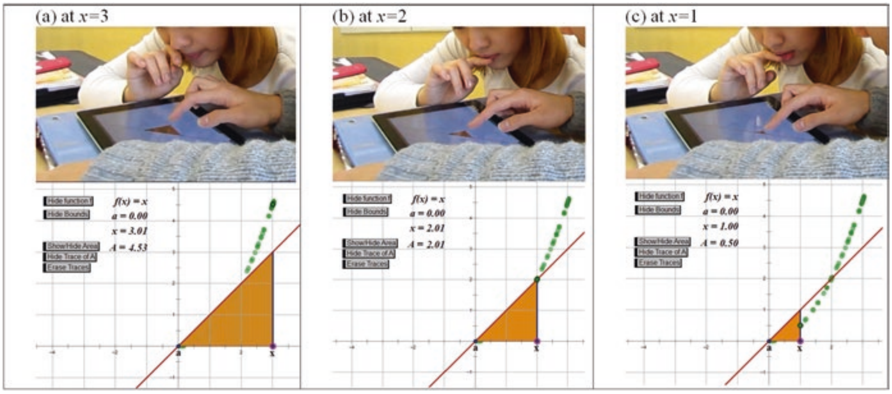

Fig. 11.2 (**a–c**) Jay combined dragging and speech when explaining to Katie the calculation of area of three different triangles

two interlocutors actively engaging in developing mathematical thinking. It began with Katie's dragging of *x* and her questioning of "what is this dot" (Turn 16). The question was likely facilitated by her interaction with the DGE after noticing that her dragging had made the green point move in a parabolic path. Jay responded to Katie's question by the word "area," spoken in English (Turn 17). Katie prompted Jay to explain his reasoning at Turn 18, and so Jay took over the role of the dragger and began explaining at Turns 19, 20, and 22. His dragging was purposeful, as he dragged x to $x = 3$ to describe the product of x and $f(x)$ in his English utterance, "three times three equals nine right." Then, he changed his mode from dragging to gesturing while his discourse reflected a change from a numerical to geometrical approach to area. Specifically, he used his left index finger to point toward the iPad screen near $x = 3$ while he spoke of, "It's a triangle, so halve it. What is the area? It's four point five" (Fig. 11.2a). In this utterance, Jay was formulating and computing the area of the triangle that was located under his finger. Katie acknowledged the calculation, and Jay continued to reason that the green dot was related to "area" by providing two more numerical verifications at Turn 22. His verification made use of the touchscreen DGE, as he was dragging while uttering the area of the triangle at that moment in time: at $x = 2$ (Fig. 11.2b), and then at $x = 1$ (Fig. 11.2c). Katie responded by using a similar sentence structure to Jay's at Turn 23 and later at Turn 28: "When it's five … when it's five, yes, twelve point five." This shows that the two students had gained an understanding of the green point as "area" under the function from a to x upon interacting with the DGE. They did so with a routine of combining dragging and exploring the numerical values displayed on the DGE. It can be said that the DGE supported their development of mathematical discourse by offering multiple representations and dynamic interactions for the students to explore change.

Jay and Katie struggled for a while with Page 3 of the sketch, at the beginning, because they could not find a way to calculate area under a quadratic function by geometrical means, as well as Page 4 of the sketch when they encountered “negative area.” They were able to resolve their conflicts, however, by observing consistency among all pages of the sketch through dragging. After roughly 20 minutes of interaction, they completed the whiteboard task of drawing the area-accumulating function for the given function *f(t) = cos(t)* with *a = 0*. They completed the task by using a strategy that no other pair of students had used. When I asked them to explain their drawing, Jay uttered in English accompanied by gestures:

> What we did was since the cosine graph is, like shifted to, left or right, half pi… we get the same as cosine graph, we move *‘a’* … to half pi, pi over two. And we use the graph provided to get the area. (20:36.0–21:02.9)

What Jay was referring to above was that they had used Page 4 of the sketch as a reference for sketching their area-accumulating function. Specifically, they used *f(t) = sin(t)*, shifted *a* from *a = 0* to *a = π/2*, and then used the green traces obtained from dragging *x* as a guide for sketching the area-accumulating function on the “Try” page. This was mathematically correct since the accumulation of area under *f(t) = sin(t)* with *a = π/2* is identical to the accumulation of area under *f(t) = cos(t)* with *a = 0*. As I was interested in finding out more about the students’ realization about area-accumulating functions, I prompted the students to “explain why,” upon which Katie provided a 40-second explanation incorporating speech and eleven acts of gestures, followed by another 20-second explanation incorporating speech and six acts of gestures. The transcript below (Table 11.2) illustrates her communication with me (named “R” for researcher) during this span.

At Turns 125 and 127 alone, Katie gestured 14 times while also speaking in English. Of the fourteen gestures, nine were used as visual mediators or routines that accompanied the word “this.” These gestures communicated significant mathematical ideas, such as positive and negative areas (Fig. 11.3a), the change of area

Table 11.2 Transcript of Katie’s explanation with researcher

Turn	Timespan	What was said	S-er	D-er	G-er
24	21:14.9–21:18.3	Can you explain why does <u>it goes up and down</u>?	R		<u>R</u>
125	21:18.3–21:57.2	When the cosine graph is at pi, the area of <u>these, area, equals this area right</u>? And they, somewhat <u>cancel</u>? Each other? So the area becomes <u>zero</u>. The area of <u>this</u> graph… kind of go <u>like this right</u>? And <u>so it looks like this</u>. <u>And this part</u>, at the same rate, it <u>goes down</u>. So it looks <u>like this</u>. <u>Symmetrical</u>. And it’s <u>like this</u>	K		<u>K</u>
126	21:57.2–22:00.3	You <u>got this part, can you explain this part</u>?	R		<u>R</u>
127	22:00.3–22:15.8	And, <u>this. For this part</u>, the area <u>decreases</u> right? <u>At this rate.</u> So… And it keeps… <u>increasing</u> after	K		<u>K</u>
128	22:15.8–22:17.0	After what? After…	R		
129	22:17.0–22:20.9	<u>After this point</u>. So…	K		<u>K</u>

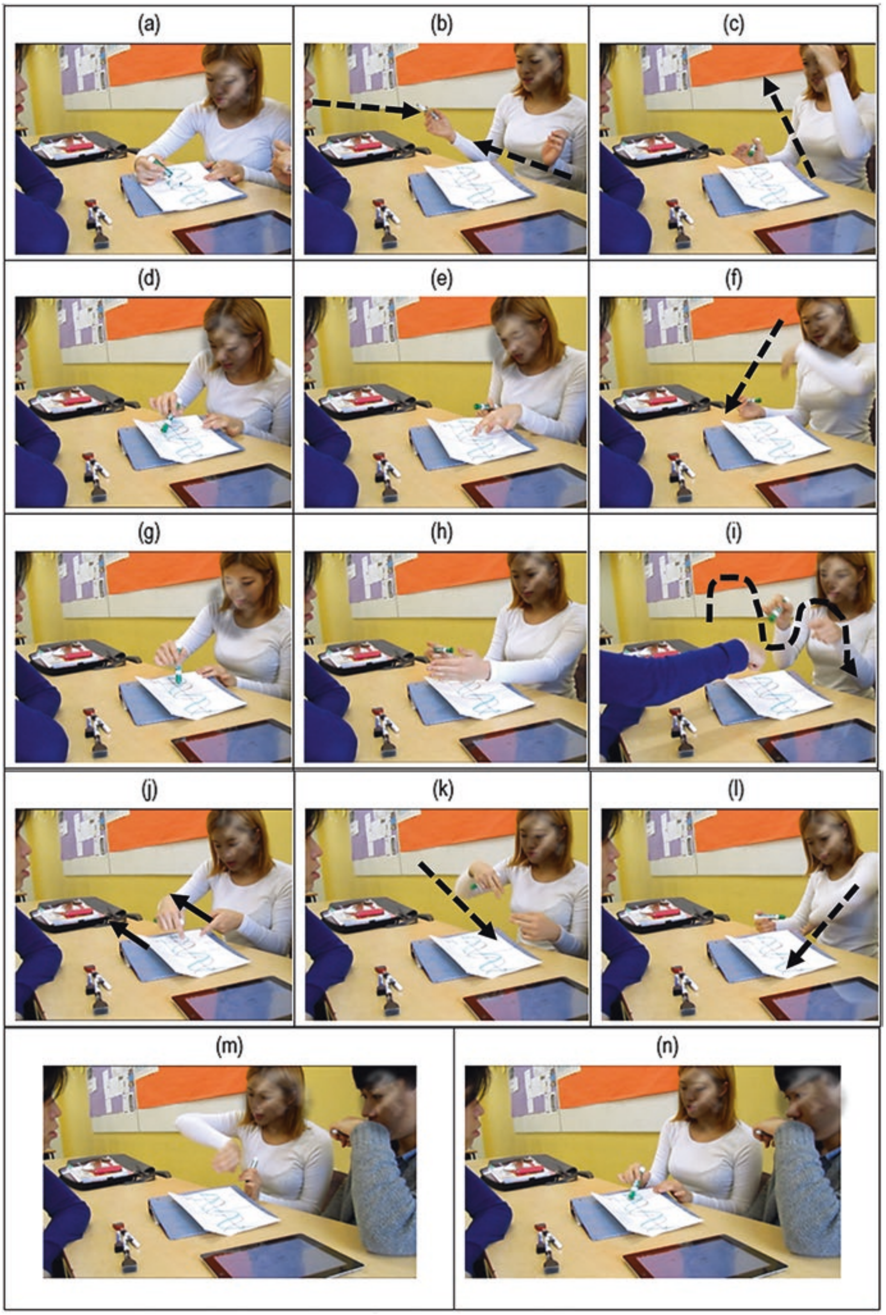

Fig. 11.3 (**a**–**n**) Snapshots of Katie's gestures during Turn 124–129

(Fig. 11.3c, f) at different parts of $f(t) = cos(t)$ (Fig. 11.3e), and the periodic behavior of the area-accumulating function (Fig. 11.3i). Other gestures communicated the "cancelling" of positive and negative area (Fig. 11.3b), the symmetrical nature of sinusoidal functions (Fig. 11.3h), and the location of which the area changed from increasing to decreasing (Fig. 11.3j). These gestures contained so much information that it would be impossible to interpret what Katie was communicating without looking at her gestures. By examining her speech and gestures synchronously, it was possible to understand Katie's reasoning of the shape of the area-accumulating function in her discourse. First, she explained that the area "becomes" zero at $x = \pi$ because "these area … somewhat cancelled" (Fig. 11.3a, b). Then, she gestured that the area in the interval $(0, \pi/2)$ "go like this," (Fig. 11.3c) and pointed to her drawing with her whiteboard pen and said, "And so it looks like this" (Fig. 11.3d). Having provided an explanation for the interval $(0, \pi)$, she proceeded to the interval $(\pi/2, \pi)$ with a similar combination of speech and gesture, adding that "at the same rate, it goes down" (Fig. 11.3e, f); Katie's word use "becomes," "go like this," and "go down" is accompanied by her hand gestures enacting the change of area, suggesting a dynamic and temporal realization of the area-accumulating function. Upon my request to continue explaining her drawing in the interval $(\pi, 2\pi)$, at Turn 126, Katie reasoned that the area would decrease, and then it would start increasing "after this point" while pointing to the point $(3\pi/2, 0)$ on the cosine graph (Fig. 11.3j). In summary, Katie was able to communicate the change of area-accumulation as x changes from $x = 0$ to $x = 2\pi$. She also described the shape of the area-accumulating function in terms of the "rate" of which area was increasing and decreasing. This suggests that she was thinking about *how* area was changing, for example, that area was not changing at a constant rate but at a varying rate, as shown in the movement of her gestures (Fig. 11.3m).

It looked like the two students had finished the task, when Jay became excited about a new discovery. Katie was already standing and talking on the phone after I had thanked the two for their participation, but Jay kept his hands and eyes on the iPad. He silently turned the pages from one page to the next upon dragging x back and forth rapidly on each page. After 25 seconds of doing so, he snapped his fingers three times which drew Katie's attention.

The episode as shown in Table 11.3 captures Jay and Katie's new discovery about the area-accumulating function and their excitement over it. At Turn 151, Jay snapped his fingers as if he had noticed something significant. His rapid dragging of x during this time suggests that he was no longer observing the discrete location of the green traces, but he was likely attending to the final shape of the green traces. Perhaps, dragging rapidly allowed him to see the shape sooner, and this dragging routine suggests that Jay was thinking of the set of green traces as one—the graph of the area-accumulating function. In other words, Jay's "rapid dragging" routine seems to suggest his *encapsulation* of the set of $(x, A(x))$ into a singular object, the area-accumulating function. Using this "rapid dragging" routine, he identified the relationship between $f(x)$ and the area-accumulating function $A(x)$, in that the former was the derivative of the latter. At Turns 152, 154, and 156, he verified this

Table 11.3 Transcript of Jay and Katie's new discovery

Turn	Timespan	What was said	S-er	D-er	G-er
151	23:29.0–23:55.0	<Jay drags *'x'* back and forth rapidly on different pages>		J	
152	23:55.0–24:07.5	<Jay snaps his fingers three times> <u>Look</u>. If this is <u>degree one</u>, then the [area] is degree two. <Jay turns to Page 3>	J	<u>J</u>	<u>J</u>
153	24:06.1–24:07.0	That's true	K		
154	24:07.5–24:14.5	<u>If you find the [derivative] of degree three, then you get degree two.</u> <Jay turns to Page 4>	J	<u>J</u>	
155	24:14.5–24:19.2	[d]?	K		
156	24:15.5–24:23.4	If you take the [derivative] of something then you get [sine]	J		
157	24:18.4–24:31.6	[Derivative] of something. Then if this graph is [negative cosine] then this is right	K		
158	24:19.2–24:30.0	Derivative of something, <u>it is [negative cosine]</u>	J	<u>J</u>	
159	24:31.6–24:38.3	<u>It is [negative cosine], right! Right!</u>	K	<u>J</u>	
160	24:38.3–24:49.5	<Katie grabs the whiteboard> If you take the [derivative] of something, you get [cosine x], that is [negative sine x], no no [sine x] it is. <Jay turns to the "Try" page>	K		
161	24:49.5–24:51.5	<u>Yes…</u> <Katie drags *'x'* rapidly>	K	<u>K</u>	
162	24:51.5–24:57.0	Antiderivative. <Jay and Katie clapped each other's hands>	K		J/K

newly discovered relationship with Katie, using gestures to specify $f(x)$ and "rapid dragging" to trace the shape of $A(x)$. It was observed that Katie was quick to react to the relationship proposed by Jay. At Turn 157, she used the proposed relationship between $f(x)$ and $A(x)$ to predict that $A(x)$ needed to be "negative cosine x" for the relationship to hold true. She said this before Jay dragged x rapidly to reveal the final shape of the area-accumulating function, and she was quite excited to see that her prediction was correct upon Jay's dragging, exclaiming, "it is negative cosine, right! Right!" Without saying the word antiderivative, she successfully communicated the idea that the antiderivative of cosine was sine: "If you take the derivative of something, you get cosine x" and "sine x it is" (Turn 160). In terms of mathematical processes demonstrated, Katie was actively predicting and verifying the relationship about the two graphs proposed by Jay. This shows that Katie had also developed her discourse by encapsulating the set of $(x, A(x))$ into an object—the area-accumulating function. The students gave each other "high fives" at the end of the episode which, again, marked their excitement.

Discussion and Reflection

With regard to the first and second research questions, my study concurs with Sfard (2009) that linguistic and nonlinguistic communications serve complementary functions in the commognitive process. As shown in the vignette, gestures took on a prevalent role in the students' communication both with each other and with me. For example, gestures were used extensively as a dynamic visual mediator to communicate the temporal movement of the green traces, while deictic gestures accompanied pronouns ("this" or "it") to ensure the interlocutors spoke about the same mathematical object (Sfard, 2009). Significantly for bilingual learners, these gestures could reduce the number of words or even replace the words to be spoken in a sentence, simultaneously reducing the language demands on bilingual learners. Equally important, touchscreen dragging emerged within the touchscreen interface of the DGEs and fulfilled the dual function of dragging (moving objects onscreen) and communicating (as a routine). It was repeatedly utilized by Jay and Katie for developing routines of questioning, exploring, conjecturing, and verifying calculus relationships. Initially, the students seemed unsure as to what to make of the sketch, particularly about the green "dot;" dragging on the touchscreen DGE enabled them to formulate questions and verify what it meant. Then, they began to explore and conjecture the relationship of the two functions in both geometrical and algebraic terms by exploiting the functionality of the DGE, particularly the dynamism and multiple representations offered by the technology. As bilingual learners do not have the luxury of a comprehensive English vocabulary, the touchscreen afforded them a nonverbal form of communication, by enabling them to gesture what was on the screen and drag objects dynamically. As shown in Katie's explanation to me in English, the DGE provided a visual means for Katie to communicate mathematically with limited English vocabulary. The DGE, along with her gestures, supported Katie's calculus communication about change and temporality (Núñez, 2006).

With regard to the third research question, Jay and Katie demonstrated important mathematical processes while speaking in their home languages and with "broken" English. For example, they were able to express the encapsulation of the set of all $(x, A(x))$, predicted the shape of $A(x)$, and conjectured that $A(x)$ was the antiderivative of $f(x)$ by using a combination of gestures, rapid dragging, and speech. These findings support Grosjean's (1985) analogy, in the sense that bilingual learners communicate by blending speaking, dragging, and gesturing, like hurdlers who blend jumping and sprinting competence. Given my own experience and background, I find this analogy extremely applicable and helpful to understand the importance of nonlinguistic communication for bilingual learners.

Returning to the issue of equity in mathematics education, the vignette illuminates what it means to engage in mathematical communication in a multilingual and technological context. In this context, there is a need to widen the view of *language*, defined by Sfard (2009) as tools for communication, to include nonlinguistic tools. Moreover, my study points to the use of touchscreen technology and pair-work activities for facilitating meaningful discussion of mathematical ideas and development in one's mathematical discourse in today's increasingly multilingual classrooms. Thus, drawing on Gutiérrez (2009, 2012), it highlights several elements of providing *access* to mathematics: touchscreen mode of interaction, DGE, and pair-work exploratory activity.

In terms of *identity,* the students were invited to draw upon their cultural and linguistic tools in the form of gestures and their home language for exploring calculus. Methodologically, the organization of the transcript was helpful to illuminate their participation in a multilingual and technological environment, by identifying the "speaker," "gesturer," and "dragger," and the utterances that were spoken in the midst of a dragging or gesturing act. Through this analysis, it was shown that bilingual learners may be participating actively as members of the classroom communities and not as "quiet" as they seem.

In terms of future directions, my study responds to Moschkovich (2011), who recommends future research to avoid deficit models of learners and their communities, by identifying bilingual learners' competence in mathematical activities. Moschkovich reminds us to exercise caution when comparing monolingual with bilingual learners; one must not assume that monolingual learners have an advantage over bilinguals or that monolinguals are the norm because of their proficiency in the language of instruction. This line of work is much needed in the field of linguistic diversity in mathematics education to achieve equity and to challenge a view that focuses on what bilingual learners cannot do and do not know. Moreover, I suggest that this discussion can be extended for cross-disciplinary work toward equity for other minority groups of learners, where much research has already been done to empower mathematics learners with special needs (Peltenburg, van den Heuvel-Panhuizen, & Robitzsch, 2012; van den Heuvel-Panhuizen, 2015). The potential for a "proficiency-based approach" in future research is promising, and it is also valuable for critiquing a normative paradigm in mathematics education.

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Chapter 12
The Micro-Politics of Counting

Annica Andersson and David Wagner

Abstract When we count, we have to decide what counts and what does not count. Thus, counting is a political act. Certain language repertoires are necessary to convey the ideas and perhaps even to perform counting actions. At the same time, the language used to describe these ideas and enact the processes shapes the way we conceptualize them. Our interest in the experience of counting includes the way counting and its communication position people. In this chapter we identify how micro-political moves are manifested in language and counting situations, including reciting numbers, counting things present and not, and subordinating counting to another goal. In our analysis, we look for language strategies that enable the process of deciding what to (not) count as the process of establishing boundaries or categories, and we consider how these processes work as political acts.

Introduction

When you count, you have to decide what counts and what does not count. In this chapter, we identify how such micro-political moves are manifested in language and how language shapes the moves. People use and invent language to make distinctions that they want to make in their interactions. This general phenomenon is also at work in mathematical problem-solving contexts, including out-of-school problem contexts for which mathematics is used. Thus, in our research, we look for language strategies that enable and/or direct the process of deciding what to (not) count. We consider how such communication strategies form political acts. This

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kind of analysis could, and probably should, be done with any mathematics, but we choose counting here because it is the most fundamental mathematical act.

We begin this chapter unconventionally; we begin with reflections on a counting exercise, which we would encourage others to try some time with their friends or others. Our exercise involved trying to decide among ourselves who has been in the most countries. We had to count the countries we visited, of course. Counting countries seems quite straightforward at first, but it doesn't take long to find controversy. For example, both of us have visited Yugoslavia before it was divided into smaller countries. Shall we count one for Yugoslavia because it was one country when each of us was there? Or can we count three or four (different for the two authors) for the current countries represented by parts we visited? We may decide it counts as one, because it was only one country when each of us was there.

A clearer question would be to ask how many national political entities we have visited. Germany complicates such reasoning. We have both spent time in East Germany, West Germany, and modern, unified Germany. Shall we count all three of these entities that we have visited even though it was never three countries (in our lifetimes)? Furthermore, one of us traveled through some countries by train. Does it count to travel through a country on a train if one doesn't get off the train? What about flying over a country? Or landing in a country to refuel but staying on the plane? What about a one-hour stop that includes a passport stamp? Or crossing the border and being escorted out by police? And then there are disputed territories, like the West Bank (of the Jordan River), or First Nations (Aboriginal lands never conceded to colonialist governments). These were just some of our political controversies when counting countries. Others who try this experiment are likely to have some of these controversies and others.

When we count, we have to decide what counts and what does not count. For example, what counts as being "in a country," and what counts as "a country?" This is political because different people will have unique reasons for wanting a country to count or not. The status of a country is a question in geopolitical dialogue—for example, Palestine has recently been recognized as a country by the United Nations—but the definition of a country is also a political question in a group of friends who may want to one-up each other. It is possible that someone has not traveled outside their country, which highlights yet another political aspect. Comparing numbers of countries visited may privilege people for whom travel has been possible and thus exclude others (if traveling is considered to be a good thing in the light of global environmental sustainability). Nevertheless, someone has to decide what counts and that decision sets up certain people's experiences as normative. For example, if a group of friends tried this exercise on our behest, we would be the ones deciding what is normative, but the group may make the question their own as they start dialoguing about it. Maybe it would be better to count our meaningful interactions with diverse people, or it may be even more appropriate to reflect on the qualities of those interactions instead of quantifying them. Indeed, it is possible to travel to many countries and remain insular.

Research on Counting

Counting has fascinated researchers for some time. One of the oldest domains in experimental psychology investigated children's counting abilities—called *numerosity* (Luwel, Lemaire, & Verschaffel, 2005, p. 449). Many studies have investigated people's ability to identify the number of objects in a group without counting (Dehaene, 1997)—called *subitization*. Subitization and numerosity are complemented with operations on number to form number sense (though this term is used in diverse ways). Others have compared human and animal number senses (e.g., Barrow, 1992). Linguists have been interested in counting as well, with a focus on the words used for numbers—for example, Gordon (2004) reported on the limited vocabulary for numbers among the Pirahã Amazonian Indians of Brazil, and Owens, Lean, Paraide, and Muke (2017) reported on the rich vocabulary for numbers in Papua New Guinea.

Mathematics education research covers these interests as well, generally with a pedagogical orientation. A large body of research has distinguished between cardinal and ordinal numbers especially in preschool education—for example, Bruce and Threlfall (2004) and the special issue of *Mathematical Thinking and Learning* (2015), "The Acquisition of Preschool Mathematical Abilities: Theoretical, Methodological and Educational Considerations." Tangentially, our literature review directed us to a rich research body on chimpanzees' learning of cardinal and ordinal numbers (Biro & Matsuzawa, 2001; Matsuzawa, 2003) and on parrots counting (Pepperberg, 2006). We note here that the choices of whose counting to research is political; why did the researchers choose chimpanzees and parrots to research?

Mathematics education research goes beyond the cardinal-ordinal distinction. Wagner and Davis (2010) distinguished between number sense and quantity sense, and they provided approaches to teaching to support the development of this quantity sense. Sinclair (2015), who focused on embodied learning and technology, proposed "an alternate way of approaching number that places much more emphasis on ordinality, rather than cardinality, and that highlights number's temporality" (p. 347). Also Radford's (2014) research on gestures supported this claim that counting is an embodied, but also cultural, activity.

There are politics involved in any of this research. Some of the researchers recognized the politics in their work. For example, Wagner and Davis (2010) suggested that people may numb themselves from disparities by working with numbers without awareness of their quantities. And as noted above, there is politics in choices about what to research. This is in line with Frankenstein's (2008) presentation of ways to help people understand large quantities in political contexts. Similarly, we note that there are politics involved in our choices for this chapter; we acknowledge that we chose to do this in Canadian English homogeneous first-language contexts, which may suggest or support the idea that English is a critically important language, and that children's experiences in Canada are important for others to know about. Most typically, researchers do not identify the politics in their work. For example, when languages are compared by the extent of their numbers, it is a way of identifying simplicity and development in the compared cultural contexts, whether or not that was the intention of the researchers.

The Politics of (Mathematical) Language

Here, we extend the research that recognized the politics of counting with a focus on language because language mediates power relations. For this, it is important for us to say what we mean with the word *politics*. We understand mathematical education as a network of created and re-created practices within particular social and cultural contexts. These practices are further networked with other practices outside the mathematics classrooms (Valero, 2007). Thus they are political, indicating that power is distributed among the different networking practices. In line with Valero (2004), we understand power as situational, relational, and in constant transformation. Power works among these practices in the network as macro-level processes. Power also works at the micro-level, however, in the immediate situational contexts among participants and (un)available materials. These micro-level actions are the focus in this chapter.

The relations between the macro-level practices and participants' micro-level actions are dialectical. Macro-level practices give meaning to micro-level actions, offering participants subject positions. The participants' speech and other actions, however, also give meaning to the mathematics practices, and they position the participants in ways that are relational to the discipline and to other individuals in their learning contexts (Wagner & Herbel-Eisenmann, 2009). Thus, participants are implicated in the construction and circulation of power within mathematical practices (Gutiérrez, 2013). We emphasize the importance of connections between the power relations at the macro- and micro-levels, which operate through positionings and discourses (Gutiérrez, 2013; Wagner & Herbel-Eisenmann, 2009). In our analysis, the negotiation and distribution of power on the micro-level are foregrounded, while we acknowledge and connect to the related macro-level power distributions, discourses, and negotiations (Morgan, 2006).

In our analysis of micro-politics in mathematical contexts, we choose to focus on counting. In order to count, language is necessary, both for communicating our counting choices and actions to others and perhaps for performing certain kinds of counting. At the same time, the language used to describe our choices and enact the processes shapes the way we conceptualize them. We are interested in the language of counting because it is the medium through which people position each other in counting situations.

Positioning theory (e.g., Harré, Moghaddam, Cairnie, Rothbart, & Sabat, 2009) points us to the distribution of rights and duties, or, in other words, the politics of counting. Careful language analysis helps us understand how these politics work. As noted by linguists who theorize language as functional grammar, language arises when people feel the need to make distinctions: "By 'text' … we understand a continuous process of semantic choice. Text is meaning and meaning is choice" (Halliday, 1978, p. 137). Thus the language and grammar of counting helps us understand the distinctions people have desired and continue to desire. At the same time, language and grammar shapes the distinctions people see. As noted by Gee (2011), language is inherently political:

> [Politics] is about how to distribute social goods in a society: who gets what in terms of money, status, power and acceptance on a variety of different terms, all social goods. Since, when we use language, social goods and their distribution are always at stake, language is always "political" in a deep sense. (Gee, 2011, p. 7)

We claim that attention to the distinctions and positioning in the language of counting can help us to understand ourselves and hopefully to consider alternative ways of positioning. We use this sensibility from systemic functional linguistics (SFL) to help us identify the qualities of interpersonal interaction as they appear in the distinctions made through grammar and lexicon. SFL is built on the recognition that language involves the interconnectedness among construction of experience (*ideational* metafunction), relationships with others (*interpersonal* metafunction), and connection with other circulating text (*textual* metafunction) (Halliday, 1973). Thus we look for grammatical and lexical choices and consider how they mediate experience and relationships.

Number and Power

At the macro-level, number is often associated with power. School curriculum is positioned as equipping children to be powerful in and outside of school, and number skills are generally taken as central to such power. For example, New Brunswick Curriculum explicitly expects children to "become mathematically literate adults, using mathematics to contribute to society" (New Brunswick Department of Education, 2008, p. 5). We refer to this kind of positioning as *equipping for the future*. Bishop (1990) has gone so far as to show how advanced counting systems—exponential-based number systems in particular—made colonialism possible. With rudimentary counting it is hard to organize and hold control over vast resources. Bishop's observation emphasized for us how such equipping for the future is not neutral. If students are to gain power and control, they are going to be exercising that power and control in particular contexts, presumably with the aim of having an advantage over others in those contexts, whether they be global or relatively local contexts. Such advantage may aid in one's domination of others, but it may also aid critical citizenry and related work on behalf of others.

Historically, technologies of counting (quantification) can be associated with certain political structures. Porter (1995) traced the history of quantification in the developing democracies of France and the United States of America. He noted the necessity of quantification and mathematical reasoning in democracies because they allow administrators to make "decisions without seeming to decide. Objectivity lends authority to officials who have very little of their own" (p. 8). In democracies, as opposed to other forms of political structure, arguments based on quantification and taken-as-shared number operations replace personal status as justification for decisions. We refer to this kind of positioning as *justifying an action*. Such justifications of action may well be the kind of future events imagined in the *equipping for the future* positioning.

We have not found research that is focused on the micro-level politics of number. Based on our experiences in classrooms as students, teachers, supervisors, and parents, we recognize the fairly common explicit discussion about classroom practices equipping students for the future, but we have not seen attention to the way politics plays out in counting situations. We ask, when children count, who "gets what in terms of … status, power and acceptance on a variety of different terms" (Gee, 2011, p. 7)?

Methodology

In order to illustrate the way micro-politics works in counting situations, we use examples from our research data to identify a range of language strategies associated with counting. Our key questions are these: what language strategies are used for counting, and what are political implications of this kind of languaging? To be clear, our primary goal is not to describe or document the situation in those contexts. Rather, our use of these data is motivated by our wish to use a range of different counting situations to identify various language strategies that are used for counting. We chose episodes from our data that struck us as being different kinds of communication related to counting. Some explanation of those contexts is necessary to make sense of the language strategies, but we do not go into depth describing the contexts though we acknowledge that the contexts were rich learning contexts in their own right.

To focus on our primary questions, we draw on excerpts of interactions from a larger research project called *students' language repertoires for investigating mathematics*, which oriented around the quest to identify linguistic resources mathematics students use to express conjecture. For that research, we collected data in a range of contexts, working with 4- to 18-year-olds, in both English first language and French immersion mathematics classroom contexts and in Swedish multilingual mathematical contexts. We began with contexts involving risk and prediction because we had noticed similarity in language repertoires between prediction and conjecture (Wagner, Dicks, & Kristmanson, 2015). We also identified political implications of the ambiguity in meaning of words that have different meanings depending on the context identified by people in an interaction—conjecture, prediction, risk assessment, and establishing authority. We extended the work to consider other contexts of straightforward conjecture, which involved activities designed to get student participants predicting numbers (e.g., Andersson & Wagner, 2016). These contexts, of course, required students to count things.

We have been especially interested in presenting students with situations that we expect to push them to the limits of their language resources. We engaged them in counting in increasingly challenging ways. We draw on the rice-counting tasks suggested by Wagner and Davis (2010), in their article distinguishing between quantity and number sense. We were interested to see how participants construct and negotiate roles and responsibilities as they decide what counts (e.g., how big does a tree have to be to be a tree? What kind of plant counts as a tree?), how to talk about a quantity when the numbers exceed individuals' quantity sense, and what benchmarks are used for communicating sense of quantity. Selecting benchmarks requires identifying something that has a taken-as-shared meaning by one's interlocutors. We identify how these micro-political moves are manifested in language.

In the interactions with participants, we avoided using specialized mathematical language ourselves and refrained from suggesting to participant students how they might perform the tasks and/or communicate their ideas. We aimed to identify their strategies for communicating their counting and consider what these language strategies say about the process of counting and the related politics. Thus, we wanted to avoid a deficit perspective that would rate the students on the basis of which skills and language they know. We have already noticed the problems with such deficit approaches—for example, we have evidence that a participant not using a language skill does not indicate inability (Wagner et al., 2015).

We also draw on examples of students counting in the following task because it presented a different kind of counting situation and thus would reveal different language strategies associated with counting. This task was also given to students as part of the project described above. The excerpt used below came from a group of grade 10 students (approximately 15 years old). They were given a page with the following task and some images of cubes and cut-up cubes (e.g., Fig. 12.1):

A cube was painted red and then cut into smaller cubes, 3 × 3 × 3.
How many of the small cubes have no red faces?
How many have one red face?
Two red faces?
Three red faces?
Four red faces? Five? Six?
How about a cube cut into 4 × 4 × 4?
Or 5 × 5 × 5?
Or 10 × 10 × 10?
Or $n \times n \times n$?

Each group of students was also given a set of 27 solid cubes (all of the same color) snapped together in a 3 × 3 × 3 configuration. It was possible for students to

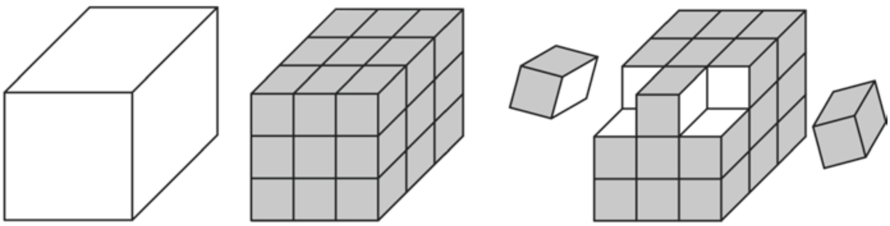

Fig. 12.1 Cube task images

pull the cubes apart and reorganize them, but they still had to visualize what sides would be painted and what sides would not be painted.

With the interactions that came from the above settings, we look closely at the language students used and to identify micro-politics in action. As noted, orientation to the data was to identify distinctions in the word choice (the orientation of SFL). Language is used to make distinctions that are relevant to the people in an interaction. For example, the prevalence of gender distinctions in personal pronouns in many languages signifies that people in those cultures have considered it important to make such distinctions. An individual may find a way to avoid making such a distinction and find this a challenge because the English language does not have some gender-inclusive personal pronouns—e.g., using “they” instead of “he” or “she.” Such practices may become acceptable to others and enter into a culture’s language repertoire, as has happened with the new Swedish gender neutral word *hen* to be used as an alternative to *hon* (she) or *han* (he) (The Swedish Academy, 2015). Just as we invent ways to avoid a distinction, we can invent ways to make distinction when no language strategy is established for that distinction. This is a phenomenon at work in mathematical problem-solving contexts (Wagner, 2009) and also in scholarship (e.g., when someone invents a new word or set of categories). Thus, in our research, we look for language strategies that enable the process of deciding what to count and what not to count (the process of establishing boundaries or categories), and we consider how these processes are political acts.

The next four sections present four different kinds of counting contexts, chosen to reveal different kinds of language used in counting situations and to prompt reflection on the politics of the language strategies. The contexts are reciting numbers, counting objects that are not present, counting objects that are present, and subordinating counting to a different goal. In each section, we give a brief overview of the context before presenting an excerpt of transcripts from the context. We follow this with an account of the language strategies used in the excerpt and close each section with an application of these strategies in different contexts, which serves to illustrate their political significance. We note that in school and home contexts, children are prepared to navigate these other contexts in which the politics may be more obvious. The language patterns may either prepare children to be aware of politics or to ignore the politics. Each of the contexts presented here reveals some language strategies, but we are aware that there are other strategies in other contexts, and that these strategies are likely to work out differently with different languages (though many languages have similar grammar and vocabulary structures to English). Our goal is to expand our understanding of the way counting works. In our reflection we call for more work that would further expand understanding.

Micro-Politics of Reciting Numbers

We started a conversation with a class of 4-year-olds to focus on counting. It began with Dave interacting with the whole class in their reading space. The children were familiar with him, as he had interacted with this class before—for example, to talk about a stick he likes when they were constructing a stick museum. We wondered if the children thought of counting as reciting numbers or an action done on objects. If we asked them to show us how they count, would they simply recite numbers or find something to count and then count those things? This question underpinned the conversation. (Participant names are pseudonyms.)

A1	Dave:	I would like to talk to you today about counting. Is that okay?
A2	Jenna:	I know how to count.
A3	Dave:	Do you!? Can you show me?
A4	Jenna:	Well, I can actually count past twenty.
A5	Dave:	Can you!?
A6	Jenna:	Yeah.
A7	Dave:	Can you show me?
A8	Jenna:	One, two, three, four, five, six, seven, eight, nine, ten, eleven, twelve, thirteen, fourteen, fifteen, sixteen, seventeen, eighteen, nineteen, twenty, twenty-one, twenty-two, twenty-three, twenty-four, twenty-five, twenty-six, twenty-seven, twenty-eight, twenty-nine, thirty.
A9	Dave:	Wow!
A10	Adam:	I can count.
A11	Dave:	Can you? Can you show me?
A12	Adam:	Uh huh. One, two, three, four, five, six, seven, eight, nine, ten, eleven, twelve, thirteen, fourteen, sixteen, seventeen, eighteen, nineteen, twenty.
A13	Reece:	I can show you.
A14	Dave:	Can you? Okay [*raises eyebrows with a nod*].
A15	Reece:	One, two, three, four, five, six, seven, eight, nine, ten, eleven, twelve.
A16	Jenna:	There are twelve stars on my clock.

The three children who wanted to demonstrate their counting each chose to demonstrate by reciting numbers without reference to objects. Though it could be that Adam and Reece's way of showing their counting mimicked Jenna's, it seems that for these children, counting was more about saying correct numbers than it was a skill to apply to objects. Counting is abstract in this way. It does not make sense to identify the grammar of the number words in this case, as they do not appear in sentences. By contrast, in "two plus three is five," the number words are nouns, and in "there are five cats," the number word is an adjective. The numbers in an abstract recitation are agrammatical here because they are not connected to objects.

Reece's decision to conclude on twelve, however, prompted a connection to objects for Jenna, a recollection that her clock has twelve stars (turn A16). Still, she was referring to objects outside the classroom space. We wonder whether she was visualizing the quantity of twelve or recollecting a conversation in which the number twelve was central (perhaps a parent pointing out the twelve stars).

With an abstracted counting like this, we cannot see the counting as a *justification* for an action or decision. We could, however, say that the rehearsal of reciting counting numbers is part of a culture of *equipping* children for future action. With that, it seems evident that this is connected with a school culture of performance.

Even though this was technically a "preschool" class, and though we know the teachers in the class endeavored to construct a play-based environment, there are likely pervasive elements of a performance culture in these children's lives from popular media and their parents and even creeping into a classroom that tries to resist it. Dave did not even have to ask the children to demonstrate their counting; they volunteered (turns A2, A10, and A13).

Now we turn to the application of these language strategies in different contexts to illustrate the political significance of these strategies. There is political significance in abstraction, which we describe as an action that seems to be performed without context. In a conflict situation, a move to abstraction is a way of ignoring particulars of the context. People might make such a move when the conversation is discussing particulars they do not want to be considered. One can also move to abstraction as a tool with the intention of applying the abstracted work to the context, as a way of developing an alternative perspective or insight. Abstract work in school is often different though. It is not a move to ignore, nor is it a move to gain perspective. It is a rehearsal of skills that might be used for such intentions. In that way, it has a second level of abstraction—doing work outside of context with the expectation that such work might be used in any context.

We turn our attention again to the linguistic markers of this abstraction in the above context. We notice that the reciting of numbers is performed outside of sentence structure. There is no subject and there is no action recognized in the recitation. As noted by Stubbs (1996), a linguist, masking the presence of a subject is often identified as a political move because the identification of who has agency (or omission to identify this) has significant implications for the positioning of people in any interaction. MacLure (2003) also explained how the masking of action can be political. She showed how a speech from her prime minister, Tony Blair, featured lists of noun phrases without subject and without verbs. With this language choice, his speech associated him with these good things, without saying who would have to act to make them happen, nor saying what kind of action would be required. We have noticed this technique among current politicians and upper administration in universities as well.

Micro-Politics of Counting Something Not Present

After inviting the 4-year-old children to show us how they counted, we wanted to get them counting things not present. So Dave asked the children how many animals they had in their homes: "Think of how many animals there are in the place that you live. Don't tell me yet. I want everyone to think a little bit." This question was crafted also to address the politics of distinctions, as illustrated in the country-counting activity at the beginning of this chapter. What counts as an animal in the home? Many of the children responded saying one or two and then telling Dave the names of these pets. After engaging with the children about their pets, Dave said, "Yesterday, I was walking in my house and I saw a spider in the corner." He described what it looked like sitting between the wall and ceiling. Then he asked one of the children, who hadn't yet responded how many animals were in his house? The boy responded, "One." When asked what kind of animal it was, he said, "The spider." The conversation then diverted to discussion about whether spiders are good to have in the house or not.

For the children, pets are counted as animals in their homes but not spiders or other bugs. When Dave drew attention to a spider in his home, a boy recognized that he, too, had a spider in his house, but only one. Then they discussed how good spiders are; they talked about spiders eating other bugs, which could have drawn attention to the fact that there are many bugs in people's houses. But the children did not seek to revise their answers. Our impression was that the children simply did not think about bugs when asked about animals. When Dave noted one spider in his house, a boy also noted a spider in his own house but did not think about other bugs beyond that.

Again, we turn to the application of such language in different contexts to illustrate the political significance. We believe that the politics of exclusion often works like this. It is not that the children explicitly decided to exclude bugs from their counts. Rather, the bugs never came to mind. Similarly, when reporting on war casualties, reporters tend to count their nationals and not casualties from the opposing country. It may be the case that such exclusions form deliberate war journalism rhetoric in war writing discourses as exemplified by MacLure (2003). It may also be that the reporters do not think to include the casualties from other countries. Either way, whether calculated or by habit, the exclusion shapes and sustains ideas about what and whom are valued. Crowley (1989) called this an *economy of exclusion*.

Back to the classroom discussion, Dave tried another approach to counting things not present. He asked, "How many rooms are there in your house?" The first child to respond held up four fingers and said "four." He touched one finger at a time as he identified whose room each of the four was. Other children in turn listed the different rooms in their homes but without giving numbers. There was much excitement when they started sharing how many toilets they had. One boy identified that he had two houses (probably a result of his parents' separation). When Jenna (the girl who first volunteered to show off her counting) listed the rooms in her house, Dave asked her how many that was. She said she didn't know. Dave asked her if she could count them, and she said she could not because there were too many. This is interesting to us because the list she gave included fewer than thirty and she already showed us that she could count to thirty. This suggests that other qualities of the rooms in their houses were more important to the children than the number of rooms (however, for toilets, the number of them did seem important). We had hoped that there might be some discussion about what counted as a room, but the children did not raise this discussion, though some counted only bedrooms and others counted living rooms, dining rooms, and open spaces in their basements, etc. Perhaps if the number became important—for example, if there is some kind of benefit to having a certain number—then the issues around what rooms count and what rooms do not would become important. In such a case, counting would become *justification for action*, but without such motivation, it can only be *equipping for future action*. And so we wonder how one becomes equipped to use number to justify choices without experiences in which number is used to justify choices.

We find it a challenge to think about the language features that mediate the politics in these two conversations about counting things not present. This challenge is due to the general lack of language. SFL tools draw attention to grammar and lexicon, but the answers to questions about how many do not follow conventional grammar. How many animals? “Two.” As with the reciting of numbers, the one-word answer has no subject/agent, and it has no verb. Here we are reminded that this analysis is situated in the English language, because in some languages numbers are verbs (Lunney Borden, 2010). Even so, though numbers are verbs in Mi’kmaq, it is our understanding that there is often no subject for the verb, or that the language gives agency to the objects being counted, not the person doing the counting.

The masking of agency in answers to “How many?” has implications for the politics of categorization. The language structure does not recognize that categorization is occurring, nor the fact that people (with their own agendas and/or values) are making those choices about what to categorize. In the conversation about animals and rooms, there were, however, some full sentences. For example, the first boy to answer how many rooms said “Four. There’s my room, my brother’s, my Momma’s room, my Mom and Dad’s room, and my sister’s room.” Here we have at least a verb—*is* in “there is”—but the sentence is still passive voice and thus said in a way that ignores the categorization and the people doing the categorization.

To illustrate the political significance, we turn again to the application of such language strategy. A recent high-profile sexual assault trial in Canada prompted much discussion in the news and social media. In this context, people (mostly men) pointed to “due process” to defend the traumatic questioning of victims who came forward as witnesses: “there is due process.” This “there is” masks the truth that these processes (laws) are crafted by people to favor the rights of certain people at the expense of the rights of others.

Micro-Politics of Counting Something Present

After the discussion with the full class of 4-year-olds, children were invited to join a table hosted by Annica or by Dave, among other choices (including playing at the sand table, etc.). The excerpt below is from the table hosted by Annica, and we note that she is an English second language speaker. Annica had a container full of dry, uncooked kidney beans, pinto beans, and navy beans. We (Annica and David) had chosen a variety of beans to prompt the possibility of children distinguishing among the different kinds of beans. We also had broken many of the beans into pieces to provoke the children to decide what counts as a full bean. The variety of beans and presence of partial beans were intended to highlight aspects of the politics of counting; the decision of what to count is more apparent when there is variety of and “abnormal” items being counted.

When children came to the table, Annica dumped some beans into a pile and asked the children if they could count them or if they wanted to count them (Fig. 12.2).

Fig. 12.2 Counting beans

B3	Annica:	Can you tell me how many beans [*Mia starts her counting while Annica finishes her request.*] there are on this paper? I am so curious.
B4	Mia:	[*Pointing at a bean with each number,* see Fig. 12.2] One, two, three, four, five, six, seven, eight, nine, ten, eleven, twelve, thirteen, fourteen, fifteen, seventeen, eighteen, nineteen, twenty, twenty-one, twenty-two, twenty-three, twenty-four, twenty-five, twenty-six, twenty-seven, twenty-eight, twenty-nine, twenty-ten, twenty-eleven, twenty-twelve, twenty-thirteen, [*her voice now becomes quieter and quieter until it is no longer possible to discern what she says though her fingers keep pointing out beans*], twenty-fourteen, twenty-fifteen…
B5	Annica:	How many are there?
B6	Mia:	I don't know. [*She hides her face in her hands*.]
B7	Annica:	Why don't you know?
B8	Mia:	There are so many, there are too many!
B9	Annica:	There are too many…
B10	Tim:	I'll count [*with a smug expression*]
B11	Annica:	Can you count? Let's see how you will figure it out. I am so curious how many there are, I have been thinking of this….
B12	Tim:	[*Points at beans for each number, a very quiet voice*] One, two, three, four, five, six, seven, eight, nine, fifteen, nineteen, twenty, twenty-one, twenty-two, twenty-three, twenty-four, twenty-five, twenty-six, …
B13	Mia:	There are little pieces too. [*pointing to small piece of a bean*]
B14	Annica:	Should we count all these small pieces too, do you think?
B15	Mia:	No. [*she starts organizing the beans*]
B16	Annica:	Why don't you want to count those, Mia?
B17	Tim:	There are ninety beans.
B18	Annica:	Ninety beans? That is very many. Are you sure? How can we figure it out, if there really is ninety?
B19	Amanda:	Do you think we can count these ones? How many are here? [*pointing*]
B20	Mia:	One, two, three. Three!
B21	Annica:	Why did you choose to pick up those?

B22	Mia:	They are brown…
B23	Annica:	Now I see you are picking out all the brown ones. Hmm… now I'm curious, why do you do that [*to Tim*]. And now you choose white ones [*to Mia*].
B24	Mia:	No, it's a brown one.
B25	Annica:	Is it?
B26	Mia:	Yes. Yes, the skin comes off.

As with Adam reciting numbers (turn A12), both Mia and Tim did not count conventionally. Mia missed sixteen, and she generalized unconventionally (but brilliantly) with "twenty-nine, twenty-ten, twenty-eleven …" (turn B4). And Tim missed a few numbers between nine and nineteen (turn B12). We do not see such idiosyncrasies as political. (We rather wondered whether the children made more mistakes when counting actual objects or when reciting numbers, but our data does not allow us to come to a conclusion on this question, nor is the question significantly relevant to our primary research questions). We note, however, that the experience of counting things is very different than the experience of reciting the numbers in the absence of referents. Without referents, there seems to be no reason for someone to make choices about how to count. In this episode with Mia and Tim, such choices become apparent. Here, Mia became interested in the distinction-making in the counting.

First, we identify our choice for the mix of beans to pour out for the children as a communication act, and thus with political implications. We *wanted* them to think about what to count and what not to count, or at least we wanted the opportunity. Second, we notice the language in the conversation between Annica and Mia about her choices (Tim appears unaware of their conversation). Mia's language choices positioned her decisions about what to count as statements of fact: "They are brown" (turn B22), "No, it's a brown one." (turn B24), "Yes, the skin comes off" (turn B26). The verb family *to be* is significant in statements of fact—particularly the forms *is* and *are*. Also significant are unmodified action verbs—in this case, "the skin *comes*." There is no human agency in Mia's statements. Annica used an unmodified verb as well "now you choose white ones" (turn B23), but it is different because Mia is the agent in this sentence. Annica also modeled an alternative structure by explicitly recognizing her own agency: "I see you are picking out all the brown ones" (turn B22). Annica did not ask Mia to explain her choices of what to count but raised the issue. We think it is significant that Mia did not seem motivated to *justify her actions*. This may suggest that Mia had not had such justification modeled in her experience; counts for her had not been justified.

Paralinguistic aspects of communication also feature strongly in this episode. First, the pointing gesture that accompanies the counting seems to expose agency. When Tim was counting, Mia watched and listened, and she noticed that he was choosing only the whole beans (turn B13). Without the gesture, Mia could not have known that Tim was making decisions about what to count. We think it likely that she would not have even considered the question about what he was choosing.

Apparently Mia did not notice her own decision-making when she herself was counting, though she too was gesturing.

Another set of paralinguistic features includes the gestures and the tone. Both of these seem to serve to identify modality. Mia covering her face showed something like embarrassment. Both Mia and Tim quieted their voices when they became unsure of themselves. These indications of modality demonstrated awareness of positioning in the relationship. They probably saw themselves as being judged by Annica and each other, as part of a school culture of performance (discussed above) and thus indicated their feelings about their counting. The presence of judgment indicates a political presence that would have to be associated with larger discourses, such as performance discourses in school or competition discourses among peers.

Our turn to the application of these language strategies in different contexts again illustrates the political significance of the strategies. We consider, for example, the counting of resources, such as trees or cod fish. When numbers are reported in the press or scientific studies, they are often reported as statements of fact. We contrast this with contexts in which one sees someone doing the counting of these resources, in which case it becomes obvious that there are decisions being made about what to count and what not to count, and the recognition of humans making these decisions invites questions about the dependability of these numbers. Sometimes news reports reveal the identity of people counting a resource that is otherwise reported without the agency revealed. This choice to reveal the agency may undermine or strengthen a reader's confidence in the counts, depending on their trust of these people doing the counting.

Micro-Politics of Subordinated Counting

Finally we consider a case of the counting being subordinated to a more challenging task. We could not find an example of the 4-year-olds counting for another purpose, so we looked to our data from other contexts. We consider this subordination in Hewitt's (1996) terms, as he contrasted that a mathematical process is different when used for something else (the process is "subordinated" to a higher level process), as opposed to the process being the objective. For this chapter, the above contexts had counting as the explicit objective—coming up with the correct number. But in the task given to the grade 10 students (described above in the Methodology section), the counting was subordinated to pattern recognition and prediction. We pick up this group's interaction when they had come to their conclusions for the $3 \times 3 \times 3$ cube and begun imagining the $4 \times 4 \times 4$ cube, for which they had no diagram and not enough cubes to model it. They were counting cubes that were not physically present and doing so on the basis of extending a set of cubes they had at hand.

C157	Andrea:	There's only four that won't be covered, right?
C158	Carly:	These four. There's these four and the ones below it.
C159	Andrea:	The ones below it are the bottom, aren't they?
C160	Carly:	Because look, let's say it's these, and then there's these, too. Not those.
C161	Andrea:	Oh, so it would be eight.
…		
C175	Lynn:	So there's four here.
C176	Andrea:	These four. One, two, three, four.
C177	Lynn:	So it's six per side, and there's six sides?
C178	Andrea:	There's four per side.
C179	Lynn:	No, because there's a block here?
C180	Andrea:	That would have more than one side covered. Four times four. Sixteen?
C181	Lynn:	Four times six is twenty-four.

Considering the language of the girls in the interaction, we noticed the significant presence of the language structures identified in the above three interactions. When they counted, they usually counted outside of sentence structure, as in turn C176 above. And they often used a passive voice statement of fact that masks agency and action, as in "there's only four" (turn C157), "There's these four" (turn C158), "There's four here" (turn C175), "it's six per side" (turn C177), "there's six sides" (turn C177), and "there's four per side" (turn C178). Strikingly different, however, was a pervasive presence of hedges, which identify the possibility of different views by modulating the sense of certainty. The modal verb "would" is very common in this entire group interaction, as in Andrea's "it would be eight" (turn C161) and "That would have more" (turn C180). All three girls used this word extensively to indicate that they are extending beyond what is present among them. Andrea tagged her statement of fact with "right" (turn C157) and "aren't they" (turn C159) to invite the others to check. Also significantly, Carly identified her agency in the counting with "Let's say" (turn C160). These hedges and identification of agency demonstrate that is it possible for people to position themselves differently in counting situations.

We also found the girls' articulation of numeric operations fascinating. These are instances of technologies of counting. For example, Lynn noted that "Four times six is twenty-four" (turn C181). Throughout the interaction, when the girls did operations, the articulations appeared with this grammar. The verb is *is*. Hence there is no human agent. And thus the language does not recognize any alternative. This is significant politically in the same way as the passive voice statements of counts and outside-of-sentence structure counting identified above. We recall the technique used by politicians to mask agency and action.

To illustrate the politics of the hedging language described in this context, we consider its use in other contexts. We consider again the counting of resources and note that it is usual practice to avoid direct counts of fish, trees, and other numerous things (especially when they are living, growing, and dying). Those counts are usually built on mathematical models that involve sampling and prediction. Predictions are usually multilayered models as they depend on counts built on models and then apply other models for mapping the trends. We reiterate that reports on the count of such resources tend to mask the modeling and prediction. Such reports make it difficult for citizens to identify the choices that are behind these numbers. By contrast, political polls are more often reported with identification of degrees of confidence, and people who deny climate change also point to differences among scientists' numbers and to their hedging as if these are indicators of flaws in counting. When many scientifically based numbers are reported as statements of fact, exceptions to such statements of fact—for example, revelations of uncertainty or decision-making—appear more questionable.

Reflection

We came to this analysis already recognizing that counting is inherently political, as demonstrated with our country-counting activity and similarly in our planning for the interaction with the 4-year-olds. When people count, we have to decide what to count and what not to count. The analysis helped us see how this politics expresses itself in human communication specifically in the contexts of reciting numbers, counting present and not present objects, and lastly subordinated counting.

The language for counting is typically outside sentence structure and thus hides the presence of human action and the point of view of the human agent. Similarly, the statement of a counting result is typically given with a passive voice, again hiding the action and agent. Again, statements of operations mask actor and agent with the "… is…" structure. The analysis nevertheless showed us that people can use language and gesture (e.g., pointing at counted objects) to identify their points of view and their action. Not coincidentally, this appeared most often in a situation where students were solving a relatively complex problem.

We reiterate our point of view (and the point of view in SFL) that language specificities are designed to identify distinctions people in a culture want to identify and to ignore what they want to ignore. Generally speaking, many language strategies associated with counting mask the human activity and suggest only one point of view (which is to deny the existence of point of view). There are language strategies, however, that enable the recognition of point of view in counting. Either way, a denial of point of view or an identification of point of view can be seen as motivated by human relationships that are inherently political. In our examples, we found the identification of point of view prompted by contexts that were designed to prompt differences in views (three kinds of beans, including incomplete beans) and that required modeling. By contrast, point of view was masked in the most basic mathematical operations and straight counting. Our illustrative applications of the counting language to other contexts helped us realize that typical media reporting reflects the basic, single point of view language for reporting counting.

We ask ourselves why society would want to present counting as if interlocutors share assumptions. We notice that if people are encultured to take counts at face value, and not think about the choices behind the counting, they become ripe for manipulation by others who are savvy with their counting. Nevertheless, to understand better why our culture has developed in a way that masks agent and action in counting, we claim that further analysis and reflection is needed from contexts in which people are counting for their purposes. All four of our episodes above focus on tasks we initiated. We asked the 4-year-olds to count and indicated what they should count. We gave the cube task to students, who would not likely wonder how many cubes fit the given criteria without our task.

Thus we encourage further work that attends to counting. We want to pay attention to our own counting and others' counting and to the language used in those situations. We encourage others to focus attention in this way as well—in both research contexts and everyday activity. We invite reflection on the following questions in such counting contexts. (1) What motivates the counting? (2) How are these motives recognized (or not) in the communication? (3) What is counted? And what might alternatively be counted? (4) How are the decisions about what to count recognized (or not) in the communication? (5) How do others respond to the person making the decision about what to count? (6) How does all this relate to power relations in the moment and in larger discourses?

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Chapter 13
A Commentary on Student Learning and Engagement in Pre-K–12 Mathematics Classrooms

Anita A. Wager

Abstract This chapter is a commentary on the chapters in the book focused on student learning and engagement in pre-K–12 mathematics classrooms.

The chapters in this section on student learning and engagement in pre-K–12 mathematics classrooms shed light and invite discussion on the power of mathematics as a tool to reify existing structures or, when used in new ways, challenge existing structures. The authors raise critical issues for the field to consider with respect to (a) the political nature of mathematics in the classroom, community, and world, (b) possibilities for supporting equity with technology, and (c) how we can use mathematics to recognize and challenge what is normalized.

The Political Nature of Mathematics

Every decision we make with respect to teaching mathematics is political (Gutstein, 2006), including the curriculum we choose, the activities we assign, and the participation structures we organize. In Chap. 10, Rubel, Lim, and Hall-Wieckert direct us to the spatial political nature of where we live by extending the notion of teaching mathematics for social justice to "spatial justice." The researchers purposefully design a module to engage students to learn about ratio and percent by exploring New York City's personal finance system. Throughout the unit it is apparent that careful attention was given to curriculum that provides the students with access to rich mathematics and the realities of who has access to various financial resources. The activities in the module incorporate multiple ways of participating so that all students have opportunity to demonstrate the ways they are "smart" in mathematics

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(Featherstone et al., 2011). In Chap. 11, Ng's study provides an argument for broadening our notion of participation to include bilingual students' linguistic and nonlinguistic means of communicating calculus ideas. The choices we make about "what counts" as participation are political as these decisions change who has access to mathematics learning. By recognizing gestures and other nonlinguistic moves as evidence of students' understanding, we not only gain a more holistic picture of students' strategies and knowledge but can provide greater learning opportunities for bilingual students. In Chap. 12, Andersson and Wagner suggest that the choices we make about "what to count" is politically motivated – whether we are aware of it or not. They ask the reader to think about the political implications of everyday counting activities adults may do as well as the tasks asked of children in classrooms. Interestingly, the researchers' decision about what should be counted in one of the tasks they engage children is also political. Counting dry beans (or any food object) might be uncomfortable for children who experience hunger and are asked to count objects that could serve as a meal. Although the classroom is featured in each of these chapters, as Andersson and Wagner point out, the notion of mathematics as political resonates beyond the classrooms to the politics in our world. This is demonstrated in the module studied by Rubel, Lim, and Hall-Wieckert as students began by examining a single transaction and then eventually zooming out to explore the financial system of the city.

Possibilities for Supporting Equity with Technology

Two of the chapters (Chaps. 10 and 11) provide examples of how technology can be used to provide greater access to mathematics learning. Rubel, Lim, and Hall-Wieckert used interactive digital maps as a core feature of their instructional module. These maps provided students with a new way to examine the phenomena being studied and made the mathematics more dynamic. Beyond the participatory mapping, students were provided several other structures to move between the mathematics, the issue under study, and how mathematics could be used to study it. Ng (Chap. 11) examined the competence demonstrated by bilingual students as they used dynamic geometric environments. By attending to how students interacted with the touch screen, Ng was able to provide evidence of understanding that might not be apparent in verbal communication. Both of these studies point to the power of technology as a way of providing access to learning – if we attend to learning in a different way.

Recognizing and Challenging Normalization

All three chapters remind us to recognize how mathematics is used to normalize particular practices and ways of knowing. And, each chapter offers ways to challenge that power. Students can learn not only what is happening in the world but how mathematics can be used to interrupt what is “normal” by learning mathematics through an examination of local phenomena that have direct impact on their lives (Rubel, Lim, & Hall-Wieckert, Chap. 10). Recognizing that students’ nonlinguistic means of communication provide a greater understanding of their understanding interrupts what is considered “normal” with respect to mathematics learning (Ng, Chap. 11). Finally, thinking about and responding to “what counts” when we are counting or reading about what has been counted can start to challenge the normalized “truths” in school and society (Andersson & Wagner, Chap. 12). The three takeaways from these chapters should push us to constantly challenge what is taken as true and how we think about teaching and learning mathematics.

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Part IV
Supporting Teachers on Addressing the Needs of Marginalized Students

Chapter 14
Preservice Teachers' Strategies for Teaching Mathematics with English Learners

Zandra de Araujo, Erin Smith, Ji-Yeong I, and Matthew Sakow

Abstract Although English learners (ELs) are one of the fastest growing groups of students in the United States, many teacher preparation programs have yet to require preservice teachers (PSTs) to receive training in effective practices for teaching ELs. We examined four elementary PSTs' instructional practices when implementing cognitively demanding mathematics tasks with ELs during a 4-week field experience. Through interviews, observations, and written reflections, we found that the PSTs tried to support the ELs, with varying degrees of success, by allowing for multiple modes of communication, including visual supports, pressing for explanations, and checking for understanding. The PSTs' use of these strategies during the field experience was largely in response to the ELs' use of language rather than mathematics. Furthermore, although the PSTs' attention to linguistic supports was well intentioned, it often resulted in the PSTs taking on much of the mathematical thinking or failing to consider different student mathematical conceptions. We conclude that explicit instruction in and reflection on effective instructional strategies with ELs, set in authentic experiences, could help PSTs to more effectively develop the knowledge and skills necessary to meet the needs of ELs in the mathematics classroom.

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Introduction

Recent curricular reforms have emphasized the importance of student engagement in mathematical tasks and practices that facilitate a deep understanding of mathematics (e.g., National Council of Teachers of Mathematics, 2000; National Governors Association Center for Best Practices and the Council of Chief State School Officers, 2010). This shift has come with increased linguistic demands in mathematics classrooms as students are expected to justify their solutions and critique the reasoning of others. Such engagement requires extensive communication skills in both everyday English and the academic language of mathematics.

In conjunction with curricular reforms, the demographics of US public schools are undergoing significant changes. English learners (ELs) are one of the most rapidly increasing groups of students (Wolf, Herman, & Dietel, 2010). In states such as Texas, California, and Florida where ELs have made up a sizeable portion of the school population for a number of years, teacher education programs are required to prepare preservice teachers (PSTs) to teach ELs. Even states with historically low numbers of ELs are now experiencing dramatic increases in their EL population (Office of English Language Acquisition, 2015). This has driven the need for these states to consider ways to prepare teachers to provide instruction for an increasingly diverse student population.

Researchers in mathematics education have increasingly attended to the preparation and development of teachers to work with diverse student populations (e.g., Aguirre et al., 2012; Foote et al., 2013; Wager, 2012). There are relatively few studies, however, that focus specifically on the preparation of teachers to work with ELs in mathematics. Of these studies, most have focused on preparing primarily monolingual, English-speaking PSTs to work with ELs (e.g., Bunch, Aguirre, & Téllez, 2015; Fernandes, 2012; Kasmer, 2013). There have also been studies that have focused on preparing multilingual PSTs to work with ELs (e.g., Chitera, 2011; Vomvoridi-Ivanovic, 2012). In both cases, examining experiences that facilitate PSTs' development of their own and/or ELs' academic language in mathematics was a primary focus.

One such experience researchers have examined is field work that provides PSTs opportunities to work specifically with ELs in mathematics (Fernandes, 2012; Kasmer, 2013). Field experiences can support PSTs' development of skills and dispositions necessary to teach mathematics with ELs. Moreover, they can enable PSTs to become more aware of the inextricable relationship between mathematics and language (Fernandes, 2012). In instances where PSTs taught ELs in whole-class settings, researchers found PSTs enacted adaptations to accommodate ELs' mathematical learning, despite having no formal training (Kasmer, 2013). Because field experiences have the potential to either counter or reinforce unproductive beliefs about ELs (Fernandes, 2012; Kasmer, 2013), however, it is important to carefully design and facilitate field experiences to develop PSTs' productive beliefs about and effective pedagogy for ELs.

Although studies have examined experiences that support PSTs in learning to teach mathematics with ELs, we have found none which considered PSTs' instructional strategies as a starting point. As such, in this study we examined the instructional strategies PSTs employed as they implemented cognitively demanding mathematical tasks with ELs during a field experience. More specifically, we were interested in the strategies employed by PSTs who had received no formal instruction on effective mathematics pedagogy specifically for ELs. We were interested in this population because, as of November 2014, over 30 states did not require general classroom teachers to have training in teaching strategies for ELs (Education Commission of the States, 2014). Thus, a large proportion of teachers may leave their teacher preparation programs without learning about effective pedagogy for linguistically diverse students generally and in mathematics specifically. The following research question guided the present study: *What instructional strategies do PSTs enact when implementing cognitively demanding mathematics tasks with ELs*? We believe that by understanding the strategies PSTs employ, educators can better understand the needs of PSTs regarding teaching strategies specific to the mathematics education of ELs and researchers can study how these strategies may be used to more effectively enhance ELs' mathematics and language learning.

Theoretical Perspectives

We frame our study in a situated, sociocultural perspective (Moschkovich, 2002) of learning. We view field experiences as situated activity for PSTs (Putnam & Borko, 2000) because they provide a space to engage in the type of thinking and practice needed for their work as teachers. Putnam and Borko made the case for the benefit of similar experiences stating, "Thoughtfully combining university- and field-based experiences can lead to learning that can be difficult to accomplish in either setting alone" (2000, p. 7). In our context, the low proportion of ELs in the local community led us to develop a field experience that would provide PSTs with situated opportunities to engage in mathematics with ELs.

A situated, sociocultural perspective emphasizes not only the situation in which experiences occur but also the social interactions and semiotic activities that occur as part of those experiences—in this case, between the PSTs and ELs. Therefore, to anticipate the strategies PSTs might employ, we drew on the work of Chval and Chávez (2012) who described seven research-based strategies as keys to enhancing ELs' mathematical learning. The strategies included (1) connecting mathematics with students' prior knowledge, (2) fostering a classroom environment that is rife with language and mathematics, (3) allowing for the use of multiple modes of communication, (4) including visual supports, (5) connecting mathematical representations to language, (6) recording key ideas and representations, and (7) discussing students' writing (Chval & Chávez, 2012). These strategies capture social interactions, including the written and spoken resources and activities, that occur between teachers and students and the realities of ELs' simultaneous development of language and mathematics.

As a whole, these strategies facilitate ELs' use and development of academic language and mathematical understanding and served as a framework to guide our examination of the PSTs' work with the ELs. Though the PSTs were not instructed on these strategies as a means of facilitating ELs' learning, several of these strategies were discussed broadly in the PSTs' mathematics methods course. For example, the course emphasized eliciting and responding to student thinking in order to connect mathematics with prior knowledge and the use of multiple representations in the teaching and learning of mathematics.

Method

We examined the strategies elementary PSTs employed when enacting mathematics tasks with ELs. Four PSTs—Kimberly, Hannah, Morgan, and Fiona (all names are pseudonyms)—voluntarily participated in the study. The PSTs were juniors in an undergraduate elementary education program at a large research university. Although the PSTs varied in their confidence regarding teaching mathematics, each student had earned high marks in her prior mathematics method course, and their instructor described them as typical of elementary PSTs at her institution. Similar to the vast majority of US elementary teachers, each PST was a monolingual, English-speaking, white woman (Goldring, Gray, & Bitterman, 2013). The PSTs had limited prior experiences working with and learning about ELs; the mathematics method courses they had taken did not directly address teaching mathematics with ELs. Furthermore, the PSTs in this study, like all of the PSTs in the program at that time, had no coursework in general pedagogical strategies for ELs. In order to fill this gap, the field experience was purposely designed to provide the PSTs with opportunities to implement cognitively demanding mathematical tasks with ELs. Previously, the PSTs had learned about cognitive demand and various strategies for planning and enacting cognitively demanding mathematics tasks. Moreover, the PSTs had experience implementing cognitively demanding mathematics tasks with elementary students. None of the PSTs, however, had specifically discussed pedagogical strategies for enacting such tasks with ELs, nor had they enacted such tasks with ELs.

The four ELs were native Korean speakers. Kyeong-Tae, Jin, and Ho-Min had each been in the United States for about 6 months. Hwa-Young had been in the United States for a little over a year and was also a fluent Japanese speaker. At the time of the study, Jin and Ho-Min were in 4th grade, and Kyeong-Tae and Hwa-Young were in 5th grade. We purposefully selected (Patton, 2002) students enrolled in classes specifically for ELs and who were at or above grade level in mathematics. In selecting students in this way, we hoped the PSTs would gain experience with students who were not yet fully fluent in English and whose mathematical ability was not a confounding factor.

The 4-week field experience centered on each PST's weekly one-on-one meeting with her assigned EL. Prior to each of the first three meetings, the PSTs were

Week 1: Vehicles Task

Three classes at an elementary school are going on a fieldtrip to the zoo. Mrs. Ruiz's class has 23 people, Mr. Yang's class has 25, and Mrs. Evans' class has 24 people (all numbers include the teacher). They can choose to use buses, vans, and/or cars. Buses have 20 seats, vans have 16 seats, and cars have 5 seats.

You are in charge of deciding how to transport all of the classes to the zoo. Explain how you would choose how many of each type of vehicle to take and why. Write a response and explain your thinking.

Week 2: Space Creatures Task

The two-eyed space creatures, three-eyed space creatures, and four-eyed space creatures are having a contest to create a group with 24 total eyes.

If you have to include only two-eyed space creatures and three-eyed space creatures, how many of each kind are needed to make a group with 24 total eyes? If it is possible, list all possible combinations and explain your strategy. If it is impossible, explain why.

Week 3: Rounding Task

Baseball stadiums have different numbers of seats. Giants' stadium in San Francisco has 41,915 seats and Nationals' stadium in Washington has 41,888 seats. Padres' stadium in San Diego has 42,445 seats. Compare these statements from two students.

Jeff said, "I get the same number when I round all three numbers of seats in these stadiums."

Sara said, "When I round them, I get the same number for two of the stadiums but a different number for the other stadium."

Can Jeff and Sara both be correct? Explain how you know.

Fig. 14.1 The tasks given to PSTs for weeks 1–3

given a cognitively demanding mathematics task (Fig. 14.1) they were to enact with their student during the weekly 30 min meetings. The first and third tasks were adapted from the released items from the Partnership for Assessment of Readiness for College and Careers (2014), and the second task was adapted from the Smarter Balanced Consortium (2014). For the final meeting, the PSTs were asked to select their own tasks. For the entire experience, the PSTs were instructed they could modify the tasks as needed and make use of any additional resources. We also asked them to develop a lesson plan based on the given task to guide their weekly meeting with the EL.

We used qualitative methods in order to gain rich descriptions of the PSTs' interactions with the ELs (Patton, 2002). Prior to the start of the field experience, we administered a survey to each PST. For the present study, we used the survey data only to understand the PSTs' backgrounds and demographic information. The primary data for this study were collected each week in a cycle. Prior to each meeting with the ELs, the PSTs completed a lesson plan that detailed learning objectives, procedures, planned modifications to the task, EL strategies, and assessments. For each field session, the PSTs would then arrive before their student to participate in a pre-meeting interview. During these interviews, we drew on the lesson plan data as we asked each PST to discuss her planned instructional strategies in depth. We then video recorded each meeting the PSTs had with the ELs and took field notes to note moments on which to follow up on during the post-interview. The video-recorded post interviews were conducted with the PSTs immediately after each session with the ELs. These interviews investigated the PSTs' immediate reactions to the meeting, things she would do differently if she could redo the session, and thoughts for subsequent meetings. All video data were fully transcribed. Lastly, the PSTs wrote reflections following each meeting that described what stood out to them from their session.

The analysis began with each member of the research team coding one of the PST's interview and meeting data. For this initial round of analysis, each researcher coded instances in which the PSTs used or discussed planning to use the seven aforementioned strategies from Chval and Chávez (2012). For example, when Fiona told Jin during the first week's meeting, "So I also have these cubes you can use them if it can help you," we coded this as *including visual supports*, which is one of the seven strategies from Chval and Chávez (2012). We also used analytic memos to note any additional strategies the PSTs employed.

Following this initial coding, we met as a team to finalize a list of codes to use in subsequent iterations of analysis of the interview and meeting data. In this meeting, we collapsed and refined the initial strategies identified by Chval and Chávez (2012) in order to capture the instructional strategies the PSTs implemented. After considering only those strategies used during implementation and common across all PSTs, four strategies emerged. Two of these strategies aligned with those described by Chval and Chávez (2012)—*includes visual supports* and *multiple modes of communication.* Two other strategies, *presses for explanation and meaning* and *checks for understanding*, were not explicitly referenced in Chval and Chávez's (2012) work. We perceived these two strategies as more specific means through which the PSTs sought to access and understand the ELs' thinking than those described in the literature. For example, checking for understanding is related to connecting language with mathematical representations, discussing examples of students' mathematical writing, and creating a mathematically and linguistically rich environment. However, the PSTs often checked for understanding simply to know if the EL understood a term or phrase. Thus, we thought *checking for understanding* was a clearer term for the strategy they employed.

Following this meeting, we recoded the data. For example, Kimberly planned to use cutout pictures of vans, buses, and cars for a task about a class field trip to a zoo as manipulatives with her student, Kyeong-Tae, to help him understand the context and as a way to communicate his thinking. We coded this as *includes visual supports* because Kimberly provided the manipulatives as a form of visual support so that Kyeong-Tae could understand the task context. We also coded this as *allowing for multiple modes of communication* because she provided the manipulatives so that Kyeong-Tae could use them to show his thinking in addition to his oral explanation. Two coders worked to code each data source independently and then met to resolve any disagreements. In the following section, we discuss the four strategies that most commonly occurred in all four of the PSTs' data sets to better understand which strategies they used and how they facilitated ELs' engagement in the tasks.

Findings

Each of the PSTs employed a number of strategies, intentional and unintentional, to facilitate engagement in the weekly tasks. In this section we first provide a summary of the strategies enacted by the PSTs across all 4 weeks. Then, we provide two vignettes that exemplify the use of these strategies.

Common Strategies

Although each PST's use of strategies varied, four strategies—using multiple modes of communication, including visual supports, pressing for explanations and meanings, and checking for understanding—were frequently employed across all four weeks by the PSTs. This section describes these strategies and details the ways in which they were employed.

Multiple Modes of Communication The PSTs thought ELs would find it challenging to explain their mathematical thinking in English. As a result, the PSTs encouraged students to use other modes to communicate. We defined the use of multiple modes of communication as teacher actions that encouraged/allowed students to communicate meaning and thinking in modes other than spoken English. More specifically, we included writing, drawing, gesturing, manipulating, and/or using a first language (Chval & Chávez, 2012). Although the PSTs discussed the utility of such modes, they typically did not plan for this strategy. Rather, the PSTs typically encouraged the ELs to use an alternative form of communication in the moment when students appeared to face challenges communicating their thinking in English.

Aside from speaking, writing and drawing were the most common modes encouraged by the PSTs. The PSTs perceived writing as a way to not only communicate explanations, strategies, and thinking but to also scaffold language development. Although the PSTs shared this view, they encouraged ELs' use of multiple modes of communication in different ways. For example, in week 3 Hannah assured Hwa-Young, who was hesitant to respond, to "go ahead write on the paper." This encouraged Hwa-Young to respond, seemingly in a way she felt more comfortable. This prompt also avoided leading Hwa-Young toward a particular strategy and maintained the task's requirement for strategic reasoning and to explain one's thinking. Hannah's use of the strategy contrasted with Fiona's. During week 2, Fiona asked Jin, "How can you write a sentence to explain how each stadium is compared to each other? So are they all the same number? Is one stadium bigger than the other stadium?" In spite of Fiona's similar request to write, she also funneled Jin to a solution that ultimately lowered the task's cognitive demand. Furthermore, Fiona's question asking Jin to "write a sentence" implies that he is to provide words on a paper to explain his thinking. This seemed to lead him toward a particular mode of communication. In contrast, Hannah's request to "go ahead, write on the paper" did not explicitly connect to words in the same way. Thus, in this way, the PSTs not only funneled toward a solution strategy but also toward a particular mode of communication through which the students were to explain their use of the strategy.

The PSTs also described drawing as an effective means for their students to explain their thinking. In week 3, Hannah asked Hwa-Young how she solved a homework problem about "belly-button monsters." Hwa-Young began writing before hesitating and asking, "It's creepy, but can I draw a picture?" Hannah assured her, "Yes, of course." In this case, the student suggested an alternate mode of communication, which the PST supported. With this, Hannah gave power to her student to communicate in a way she felt most comfortable. In the following week, Hannah further encouraged Hwa-Young to use alternative modes of communication. She provided markers and gave Hwa-Young options, stating, "scissors here, or you can use these [markers] to draw. It's up to you. Let's split this [two brownies] between me and you. How would you do that?" In response, Hwa-Young combined speech and drawings to explain her thinking stating, "My whole is (draws a line)."

The PSTs also encouraged ELs' use of gestures and manipulatives to communicate mathematical thinking. For example, in week 3 Fiona's partner, Jin, frequently pointed to his written work to clarify his verbal explanation. The students commonly gestured in this way as they explained their thinking. Although unplanned, these gestures appeared to aid the students and PSTs in developing a shared understanding of the students' thinking. In less frequent instances, the PSTs encouraged students to explain their thinking with manipulatives, even though some PSTs hoped students would do this without prompting.

The PSTs' use of multiple modes of communication enhanced ELs' ability to communicate their thinking. In some instances, the PSTs actively encouraged this communication, while in other instances they accepted students' decisions to use writing, drawing, gesturing, and manipulatives to communicate. Interestingly, in contrast to research recommendations (e.g., Moschkovich, 2002), none of the PSTs drew on ELs' first language. Furthermore, the PSTs' use of these strategies was largely unplanned, which contrasted with their planned use of visual supports, as described in the next section.

Includes Visual Supports Throughout the field experience, each of the PSTs included visual supports in her lesson plans. We defined this strategy as the planned use of concrete objects, videos, illustrations, or added emphasis (e.g., bolding, color-coding). Whereas the use of multiple modes of communication was often student-driven and enacted in real time, the use of visual supports was typically teacher-driven and planned. Although the PSTs provided a variety of visual supports for the ELs, they most commonly used images or manipulatives.

Each of the PSTs added images to tasks. Frequently, the PSTs included pictures to increase students' understanding of the task contexts. Hannah, for example, included images of American and Japanese desserts on the 4th week's task to increase Hwa-Young's interest, engagement, and to provide a cultural connection. There were also instances in which the PSTs included images as scaffolds for mathematical learning. For instance, Morgan provided Ho-Min with a simplified shape in response to difficulties he faced calculating the perimeter of the original shape. Morgan ultimately took on the mathematical thinking of the task when she used her drawing to show Ho-Min his mistake and how to correctly solve the problem. When the PSTs used images in ways similar to Morgan, they lowered the cognitive demand of the task.

Each of the PSTs planned for the use of manipulatives at least once. These manipulatives included items such as alien or vehicle cutouts and connecting cubes. The PSTs intended for students to use these manipulatives during their exploration of the problem. Although the PSTs had two main reasons for including visual supports—to support students' mathematical thinking and understanding of the task context—in some cases, the PSTs (not the students) used manipulatives to demonstrate a certain solution strategy, as in the following excerpt from Fiona's second meeting.

Fiona: It's impossible? Let's work through this, Okay, because it's possible. It's kind of confusing, I know. So, if I have (*moves one of each 2, 3, and 4 eyed cutouts of creatures into center of table*), how many eyes do I have?

Jin: Nine eyes.

Throughout the lesson Fiona used the manipulatives in this way, which led to an imposed solution strategy. Such occurrences were not unusual when the PSTs used manipulatives.

Overall, the PSTs included visual supports to enhance their students' mathematical thinking or understanding of the task. In many situations, however, we found evidence that the PSTs used images or manipulatives to clarify or communicate their *own* thinking and not to support the student's mathematical work. Thus, the PSTs seemed to be more proficient at using visuals as linguistic supports rather than in support of mathematical understanding.

Presses for Explanation and Meaning In an interview Fiona said, "I want to be able to ask questions in a way that will help guide their thinking and not tell them what their next step should be." To accomplish this, she and the other PSTs regularly pressed the ELs to explain their solution strategies and mathematical thinking. As such, we defined this strategy as instances when the PST pressed the student for explanations and/or meaning, which most commonly occurred via questions or imperatives (e.g., "Explain what you mean by that" or "Tell me about how you figured that one out"). Throughout the field experience, we saw the PSTs use this strategy to elicit student thinking with various degrees of success.

Most commonly, the PSTs asked students to justify their solution and/or provide more details about their solution strategies. Hence, the PSTs' implementation of this strategy usually came after students finished or stopped their solving process. There were several instances when the PSTs pressed their students for explanations regarding the task context, though these instances were far less frequent. When soliciting meaning from their students, the PSTs used various types of questions ranging from general inquiries (e.g., "Why do you think so?" or "How did you find out?") to more specific ones (e.g., "Why did you decide to add these and not something else?" or "Where does the 12 come from?"). Other questions that were used asked the meaning of the task statement, students' current thinking, students' plans to solve the task, and students to think further.

When the PSTs pressed for explanation and meaning it generally provided students with greater opportunities to communicate or think more deeply about their solutions. This, however, was not always the case, as seen in the following excerpt.

Kimberly: Okay. (*Kyeong-Tae writes*) Okay, how did you find that?
Kyeong-Tae: Mm, (*long pause*) I don't know. I just found.
Kimberly: Yeah, well I saw you do it, and I was thinking about how I would have done that in my head, so I saw what you did, first was you found the two-eyed creatures right? And you found five of those and that gives you how many eyes?

In the first line, Kimberly asked Kyeong-Tae to explain his strategy; however, he was unable to formulate a response after a long pause. Unfortunately, Kimberly did not use guiding questions or scaffolding to facilitate the challenges Kyeong-Tae faced. Instead, she provided her interpretation of his work, which resulted in her taking on much of the thinking.

The PSTs' use of this strategy evidenced their attention to the core teaching practice of eliciting students' mathematical thinking, a strategy that was addressed in their mathematics methods course. When the PSTs pressed for explanations after the students arrived at a solution, the PSTs were typically able to hold them accountable for their mathematical thinking. There were several instances, however, as in Kimberly's excerpt above, where this strategy did not support students' mathematical thinking or hold them accountable to the task. This led to the PSTs suggesting specific solutions or moving to another task without further discussion.

Checks for Understanding While pressing for explanation and meaning occurred after the ELs had begun working on a task, the PSTs often checked for understanding immediately after presenting a task. We defined this strategy as instances when a PST checked whether a student understood the problem, context, word, or phrase (e.g., asking if there are words they don't know). Prior to meeting the ELs, the PSTs expressed interest in knowing more about their students' mathematical abilities and culture so they could anticipate potential challenges or better align the tasks with the students' culture. Fiona said, "I want to know as much about my EL student's culture and home life as possible before I start working with them. I believe this will allow for fewer barriers to be created." The PSTs thought that knowing more about their students' previous mathematical and linguistic knowledge, experiences, and abilities would help them as they tried to enact tasks that may include new ideas or contexts.

Each of the PSTs frequently checked the ELs' understanding during task implementation, usually after the PSTs presented the student with the task. Kimberly, for example, gave Kyeong-Tae a task about space creatures in week 2 and said, "Okay, well this is the first one that I put together for you. And I want you to read it before anything else. ((*very short pause))* Okay, are there any words on there that you don't know?" Asking whether there were unfamiliar words in a task was a common way in which the PSTs checked for understanding.

When a student did express difficulties with unfamiliar language or mathematical concepts, the PSTs often attempted to make connections to the student's prior knowledge or past experiences. For example, when Ho-Min did not readily understand a problem concerning empty seats on a bus in week 1, Morgan asked, "Do you ride a bus to school? Is there ever any seat, no one sitting on the bus? Or is it already full?" Similarly, during week 4 when Kimberly asked Kyeong-Tae, "Okay, so the first thing I wanted to talk to you about was what do you know about circles?" In this latter example, Kimberly not only checked for knowledge of vocabulary related to the mathematical topic, such as *radius* and *circumference*, but she also probed for Kyeong-Tae's understanding of pi's relationship to these features. Such uses intended to increase access to the task and determine aspects for which pre-instruction or clarification was needed.

Strategies to facilitate language development were commonly employed among the PSTs after checking for understanding. We found few situations where the PSTs provided mathematical support after checking for understanding. As such, this strategy was primarily used to check for language and contextual understanding, not mathematical understanding. This resulted in the PSTs missing opportunities to support students' mathematical thinking.

Illustrative Vignettes

In this section we provide two vignettes that illustrate the PSTs' use of the aforementioned strategies. Each vignette draws from the PSTs' second meeting with their student where they enacted the Space Creatures Task (Fig. 14.2). We focused on this meeting because it best evidenced the variety of instructional strategies the PSTs employed to support ELs' mathematical learning.

Morgan Morgan heavily modified the written task by including visuals. The most noticeable change was the addition of pictures of space creatures, which she gestured to as she read the task aloud. Morgan explained her reasons for the pictures,

> I created like a little chart with little space creatures. And so I say space creatures in each one and each one says, like, what's in here [the text] is put right here [the picture] so that he can connect it [word] with it [image]. And since the eyes are really important and it talks about eyes, I made sure to like box them and write like "eyes" so you can talk about that and make sure he understands.

Morgan color-coded relevant text to match the pictures of the space creatures (Fig. 14.2). Her modifications also included color-coded answer blanks as further visual support for Ho-Min. She made these modifications to provide Ho-Min with visual resources to support his understanding of the task's language and context. She drew on these visual resources heavily by gesturing and referencing them throughout the enactment. Morgan seemed to have found value in this strategy as evidenced by her continued use of it in subsequent meetings. It should also be noted that her

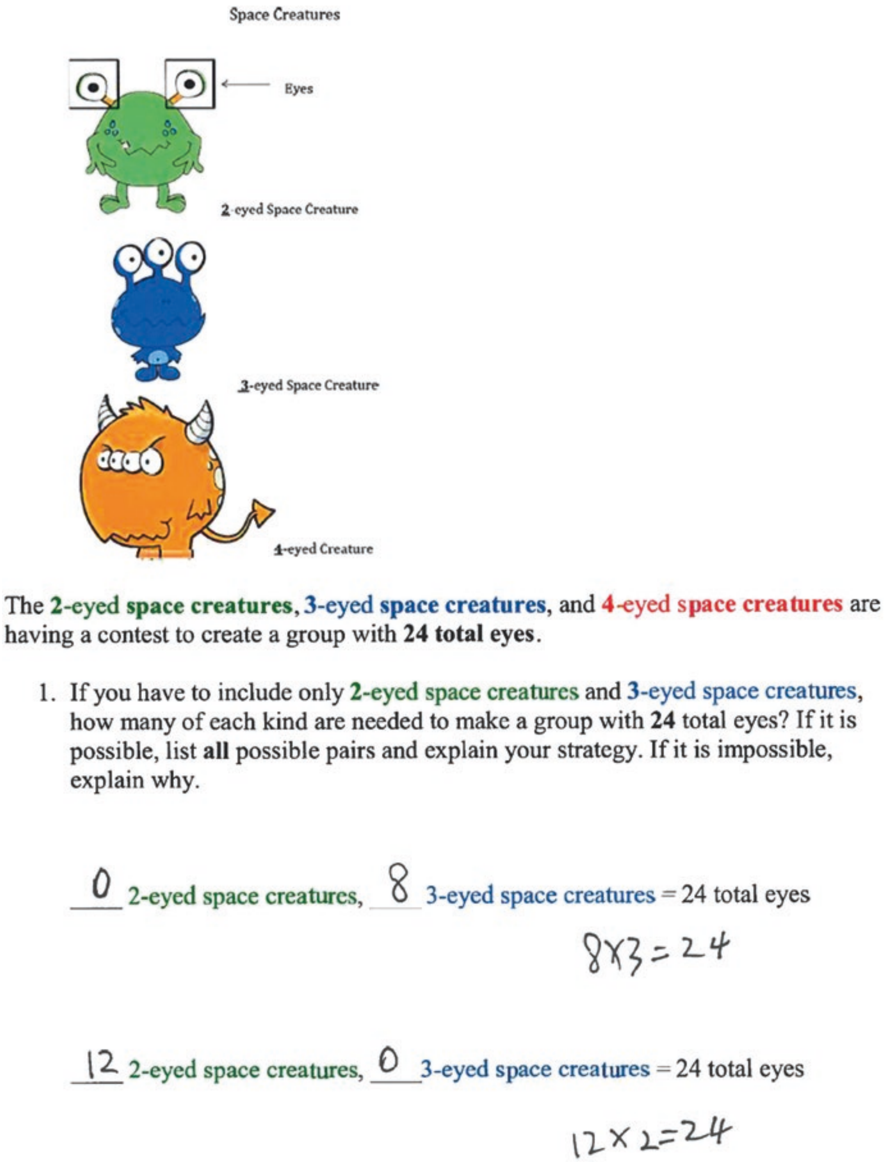


Fig. 14.2 Morgan's modified Space Creatures Task

inclusion of the color-coded answer spaces suggested a particular solution path (and an incorrect use of the equal sign). The blanks led Ho-Min to identify two numbers that, when paired with the factors 2 and 3, resulted in products that summed to 24.

Before Ho-Min began the task, Morgan rephrased the problem and gave him an example of how to combine groups of space creatures. Such practices were emblematic of Morgan's approach to set up a task and to check for Ho-Min's understanding of the task.

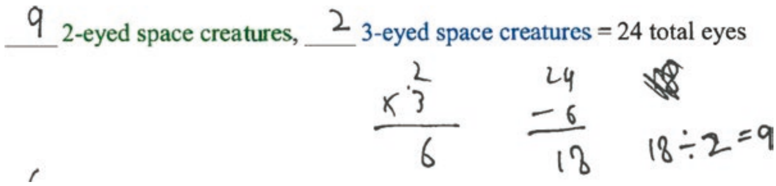


Fig. 14.3 An excerpt of Ho-Min's work on the Space Creatures Task

> They want to come altogether to make twenty-four eyes. So, if you have to include only two-eyed and three-eyed creatures, how many different ways, how many different ways could you put them together to have twenty-four total eyes? So you could do, you could have, a couple, you could have two of these [two-eyed] creatures have four eyes together, right? Because two plus two is four? Does that make sense?

When Ho-Min did not respond, Morgan continued to provide more examples and check his understanding, "What if you had two of these [three-eyed] creatures, how many eyes would you have? If you had two [more] of these [three-eyed creatures]?" Ho-Min responded with "three times four," and Morgan excitedly praised him before providing additional examples. It was only after some iterations of this questioning pattern that she revisited the task question and asked, "How many eyes do you have to put together to make twenty-four [eyes]?" Given Ho-Min's reservations to speak and Morgan's insufficient wait time, she often asked numerous follow-up questions to check for understanding before Ho-Min began writing.

Ho-Min's quietness encouraged Morgan to offer alternative modes of communication, which often entailed a written explanation in lieu of a spoken one. For instance, when Morgan asked how Ho-Min reached a solution of eight three-eyed space creatures, he wrote the number sentence $8 \times 3 = 24$ (Fig. 14.3) to explain his thinking. Often, Morgan would ask him what the various numbers in such equations referred to, and Ho-Min would point to the given pictures. Communicating through gestures helped Morgan to better understand Ho-Min's thinking.

Morgan frequently pressed Ho-Min to explain his thinking. She asked many questions in an effort to better understand his solution strategies during their meetings. Many of these interactions were similar to the transcript following Fig. 14.3.

Morgan: How'd you do that?
Ho-Min: (*writes multiplication problems and number sentences*)
Morgan: Okay, yeah. So you did two times three? And so where does, where did those numbers come from? Where?
Ho-Min: Two.
Morgan: Two? Mhmm. Which one, what's that two stand for?
Ho-Min: Two monster. (*Indicates picture see* Fig. 14.2)
Morgan: Two monsters? With how many eyes?
Ho-Min: Three eyes.
Morgan: With three eyes, okay. So you have, so what's the six down here mean?
Ho-Min: Six?
Morgan: Mhmm, what's the six stand for?
Ho-Min: Uh. Two monster eyes.

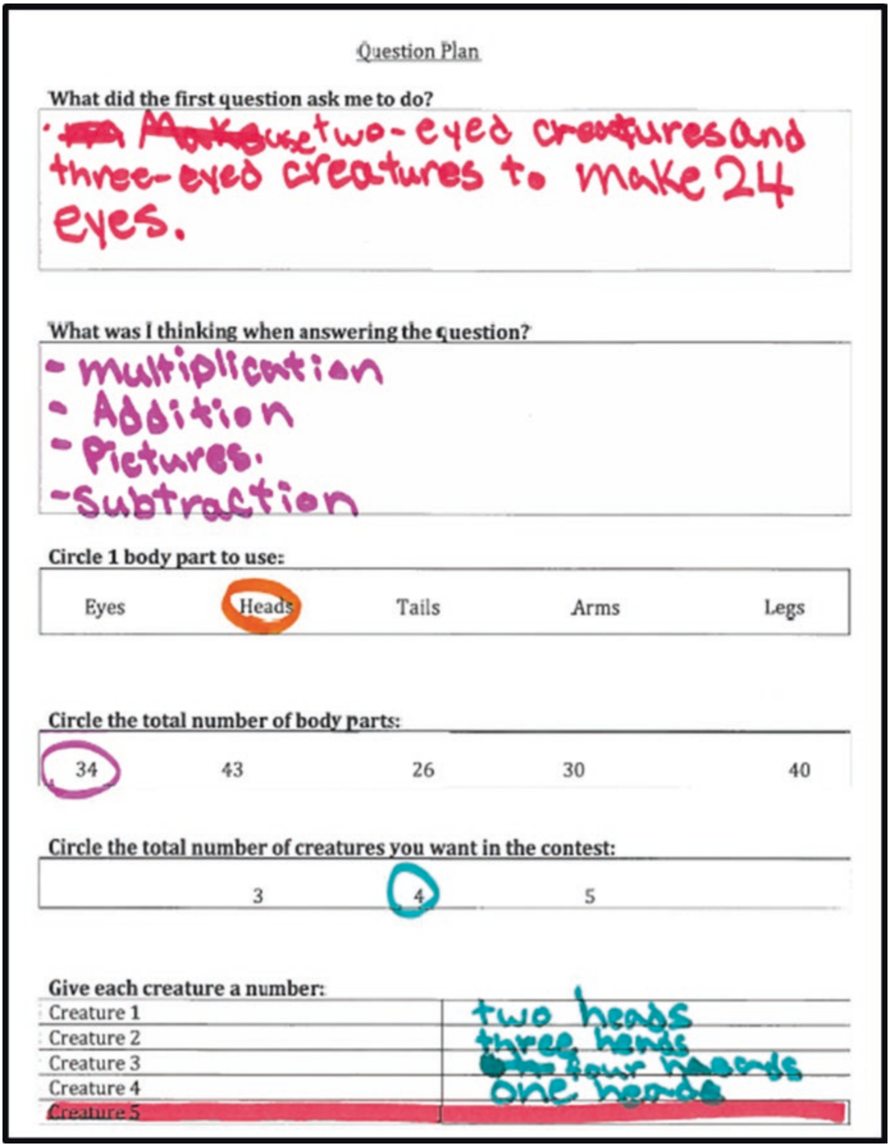
Morgan: Okay, yeah, so altogether it's six eyes?
Ho-Min: Yes.
Morgan: Okay. And then, you did 24 minus 6?
Ho-Min: Yes.
Morgan: Yeah? And you got 18 so what is, what's 18? What does that stand for?
Ho-Min: Twenty-four.
Morgan: Are those monsters or eyes?
Ho-Min: Eyes.
Morgan: They're eyes? Okay. And then you divided by two. So why'd you divide by two?
Ho-Min: Uh, this monster, eyes.
Morgan: Okay, so divide by the number of eyes he has? And so then, what does that, what does that answer give you? Nine…
Ho-Min: Monsters.
Morgan: Nine of these monsters? Very good

As evidenced above, many of Morgan's questions sought to clarify his written work; however, they often required brief responses rather than discussions of his solution strategy. In most cases, Morgan gained an understanding of Ho-Min's strategy. In others, she struggled to grasp his solution strategy and ceased her solicitation efforts.

Morgan struggled to hold Ho-Min accountable for task directions that requested he "explain his thinking" or "find all possibilities." Although these kinds of statements were emphasized throughout Morgan's task (e.g., underlining the word "all"), Ho-Min was not often held accountable to these directions. As a result, in the first problem of week 2, Ho-Min found only four of the five possible solutions. This lack of accountability was also seen in subsequent weeks when Ho-Min was not required to explain his thinking. The elimination of activities such as mathematical justifications, identification of patterns, and uncovering all possible solutions led to differences between the original intent of the task and its implementation.

Across all 4 weeks of the field experience, Morgan implemented the four instructional strategies. Her use of visual supports predominantly manifested through her heavy use of color-coded images and bolded text. Because Ho-Min was hesitant to speak, Morgan frequently checked for understanding and encouraged him to use multiple modes of communication (i.e., write or gesture) in response to the many questions she asked. Though she was often able to get Ho-Min to share his solution strategies, she frequently abandoned mathematically and linguistically demanding aspects of tasks, such as providing mathematical explanations. Thus, Ho-Min did not fully engage in the task as intended. In the following section, we examine Hannah's work with Hwa-Young on the same task.

Hannah Prior to meeting with Hwa-Young, Hannah did not modify the original task's language. She did, however, create an additional task to accompany the original Space Creatures Task (Fig. 14.4). The purpose of this additional task was to elucidate Hwa-Young's strategies used to solve the original task and extend her



Question Plan

What did the first question ask me to do?

use two-eyed creatures and three-eyed creatures to make 24 eyes.

What was I thinking when answering the question?

- multiplication
- Addition
- Pictures.
- Subtraction

Circle 1 body part to use:

Eyes	Heads	Tails	Arms	Legs

Circle the total number of body parts:

34	43	26	30	40

Circle the total number of creatures you want in the contest:

3	4	5

Give each creature a number:

Creature 1	two heads
Creature 2	three heads
Creature 3	four heads
Creature 4	one heads
Creature 5	

Fig. 14.4 Hannah's question plan

thinking by creating her own task through the use of a "question plan." Thus, Hannah planned to first enact the original task and then use the question plan as an extension, which she explained in the following excerpt.

Hannah: I just created it [the question plan]. And then, what I was thinking, hopefully by having her talk more about it, she will start to think about her own strategies… We are going to do still like aliens, but she gets to pick body parts, and she gets to pick the number of total body parts and starts off with same total of 24 eyes. Then she gets to pick how many creatures or monsters or aliens—I think it's space creatures, yeah, creatures—that they are, and she gets to decide the number of body parts each one has, so it has to be total 24 eyes. Then she can pick this one has only one eye, two eyes, three eyes, four eyes.

Interviewer: So, this is all based on whatever questions you are going to do with, so you'll give it to her first to look at, and then after fill in top part and this is kind of worksheet solving.
Hannah: This is what she gets to pick out of the number of eyes the creatures have? Pick three; then you only need to go to three, and tell me three and then she has to solve it based on the question she creates.
Interviewer: Oh, okay.
Hannah: Does that make sense?
Interviewer: So, she is making her own question?
Hannah: Yes.

Hannah began the session with a lengthy discussion of Hwa-Young's day at school. Following that conversation, Hannah began the original Space Creatures Task by asking Hwa-Young to read the task line by line. As Hwa-Young read, Hannah continually checked to see if she understood the task. When Hwa-Young arrived at a word she could not read, such as creature, Hannah would explain what the term meant, as illustrated in the transcript below.

Hannah: You know what that [creature] is?
Hwa-Young: Nope
Hannah: Okay. So, a creature is a broad term that covers multiple animals, so it could be, do you know what a monster is?
Hwa-Young: Yes.
Hannah: Do you know what an alien is?
Hwa-Young: Alien?
Hannah: Like, UFO, like Ti-Yoong Ti-Yoong *(makes sound of a UFO)*.
Hwa-Young: Oh, oh, I know.
Hannah: Okay, so monsters, aliens, bugs, animals, they are all considered creatures.
Hwa-Young: Oh.

In this excerpt Hannah drew on multimodal communication, particularly sounds and gestures, to provide Hwa-Young with additional information to understand the term creature. Throughout the experience, Hannah regularly took time to check Hwa-Young's understanding of key terms and task contexts, which she followed up with multimodal explanations as needed. For example, after explaining the terms, space creatures, and combinations, she went on as follows:

Hannah: Did you understand what the question is asking you to do?
Hwa-Young: Hmmm
Hannah: So what is the question asking you to do?

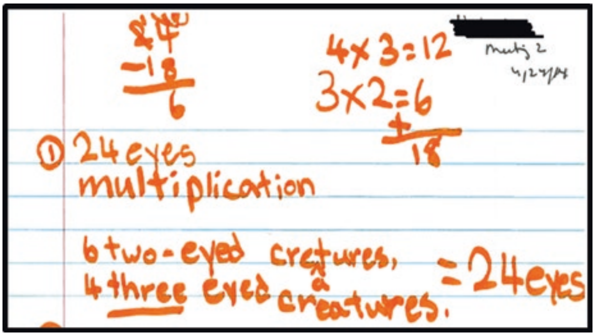


Fig. 14.5 Hwa-Young's written work on the Space Creatures Task

Hwa-Young: Like, 3-eyed space creatures and 2-eyed space creatures put together and make 24 eyes. How many creatures are 24 eyes? I think?

Hannah: I think that's an excellent way of thinking about it. So, now you know what the strategy is that you have to do this type and that type.

Of the four PSTs, Hannah spent the most time on the task set up. Much of this time was spent checking for understanding (as evidenced above) and pressing Hwa-Young to explain her thinking. Notably, these two strategies were employed prior to Hwa-Young working to solve the task. Following the extensive set up, Hannah encouraged Hwa-Young to try to solve the task.

Hannah also encouraged Hwa-Young to explain her thinking as she solved the task by asking questions as Hwa-Young spoke. For example, in the following exchange Hannah probed Hwa-Young to explain the written work she produced in Fig. 14.5.

Hannah: If you had three [two-eyed creatures], how did you know there's six eyes?

Hwa-Young: Because this is two, this was three, and so I multiply three times two because this is two.

Hannah: Can you show me that? How would you write that?

Hwa-Young: Three times two is six.

Hannah: And then you did what with this one?

Hwa-Young: Four times three is 12.

Hannah: And then what did you do?

Hwa-Young: And then, I, um, added 12 and 6, it was 18, and I subtract 24 minus 18, and it was 6.

In this excerpt, Hannah asked Hwa-Young to explain how she knew three space creatures had six eyes. Although Hwa-Young explained aloud, Hannah encouraged her to also write, which was a common way she employed multiple modes of communication, particularly when Hwa-Young faced difficulties solving a task. In the excerpt above, Hwa-Young experienced challenges relaying her thinking, and it appeared Hannah's questions were intended to clarify the calculations Hwa-

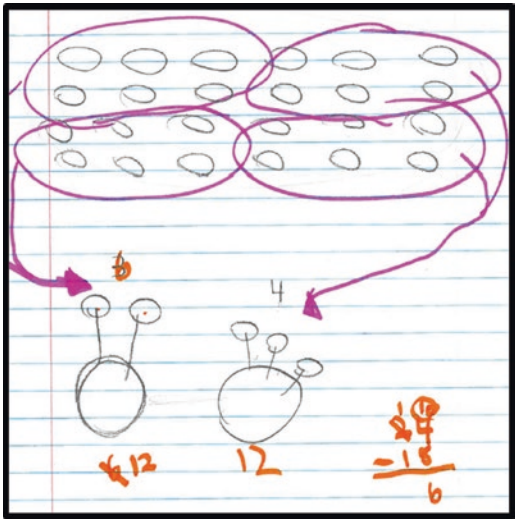


Fig. 14.6 Hwa-Young's use of Hannah's drawing of 24 eyes

Young had already completed. This is evidenced by Hannah's questions, such as "and then what did you do?"

As the meeting went on, Hwa-Young had difficulty finding multiple solutions to the task. In order to help her, Hannah drew 24 circles to represent the 24 eyes (Fig. 14.6). This use of visuals was meant to support Hwa-Young's mathematical understanding of the task. It also steered her to a particular strategy by grouping "eyes." Hwa-Young remarked that circling the eyes was "better than thinking." This suggested that she thought this strategy was easier than her original attempts that relied on finding solutions by making combinations of multiples of two and three.

Throughout the task, Hannah asked many questions and continually checked for understanding as she pressed Hwa-Young to explain her thinking. Moreover, she frequently encouraged her to write as an alternative way to communicate her thinking. Hannah also included supplemental tasks aimed to support Hwa-Young's understanding of the task language and context while helping her reflect on her solution strategies.

Summary

Across the vignettes we see the two PSTs implemented all four strategies to support their students. Both PSTs checked for understanding throughout the task, clarified directions and/or concepts, and made use of visual supports. Morgan, for instance, included images of space creatures and made use of color-coding and bolding to highlight key features of the task, whereas Hannah provided a separate question plan to guide Hwa-Young to reflect on her own thinking. In addition, although both

PSTs pressed for explanation and meaning in order to have their respective students communicate their thinking, the frequency with which they did so varied.

Although each PST utilized the four common strategies, the way in which they did so reflected the diversity of their respective ELs. Because Ho-Min was timid in his meetings with Morgan, she regularly relied on visually oriented supports and significantly altered the appearance of the task. She also encouraged Ho-Min to utilize—almost exclusively—multiple modes of communication, particularly gestures and writing. Hannah, however, generally implemented oral supports because of Hwa-Young's eagerness to talk. Although she did provide some visuals, Hannah spent significant time explaining terms and pressing Hwa-Young to explain her thinking. Such difference in strategy implementation was not unusual, given that the PSTs responded to the ELs differently and adapted their instruction as they saw appropriate.

Discussion

Our research was guided by the question: *What instructional strategies do PSTs enact when implementing cognitively demanding mathematics tasks with ELs*? Across the 4-week field experience, we found that the PSTs employed a number of strategies in an attempt to facilitate their students' mathematical learning. In particular, they made use of multiple modes of communication, included visual supports, pressed for explanation and meaning, and checked for understanding. Our findings suggest that the use of these strategies seemed to be driven by the interactions between the PSTs and ELs during the field experience. This study's findings are consistent with prior studies that found PSTs can develop strategies to support ELs during field experiences (e.g., Fernandes, 2012; Kasmer, 2013), but extends prior work by identifying specific strategies the PSTs used and the variety of ways in which they were utilized. Furthermore, given the PSTs' lack of experiences specifically focused on mathematics teaching of ELs in their teacher preparation program, findings from this study provide insight into the strategies PSTs implement without formal instruction or training on effective mathematics pedagogy for ELs.

To anticipate possible strategies the PSTs might enact, we drew on Chval and Chávez's (2012) summary of research-based instructional strategies for ELs. We found that the PSTs drew on some of these strategies despite having no explicit instruction related to the mathematics education of ELs. In particular, the PSTs' included visuals and encouraged ELs' to draw on multiple modes of communication. The PSTs also frequently employed two strategies not discussed by Chval and Chávez (2012), but are often emphasized in the broader mathematics teacher education literature (e.g., Jackson, Garrison, Wilson, Gibbons, & Shahan, 2013; National Council of Teachers of Mathematics, 2014; Stein, Smith, Henningsen, & Silver, 2009)—checking for understanding and pressing for explanation and meaning. We did not find evidence, however, of other strategies specific to ELs, such as discussing students' writing; recording key ideas, words, or concepts for later reference; or attempts to foster an environment rife with language (Chval & Chávez, 2012). Moreover, the strategies the PSTs employed were not used in conjunction with others discussed by Chval and Chávez to further develop ELs' mathematical and language understandings. For example, the PSTs included visual supports but often did not connect mathematical representations to language. Furthermore, the use of multiple modes of communication was in response to ELs' language preferences and facilitated discussion of their thinking, but the PSTs did not often discuss students' writing or record key ideas and representations. Thus, we argue that the explicit discussion of these strategies and their use with ELs would benefit PSTs' ability to effectively teach ELs. PSTs should have opportunities to enact these strategies with ELs and be supported as they learn how the strategies can be used together in more effective ways.

The PSTs' use of strategies during the field experience was largely in response to the ELs' use of language. This was most evident through their implementation of instructional strategies designed to facilitate student communication. Morgan, for example, drew on multiple modes of communication for Ho-Min, who was reserved, throughout the weeks. As a result, she and the other PSTs broadened the range of acceptable forms of communication from spoken and written modes to include drawings and gestures throughout the experience. We view this action from the perspective of the PSTs in their role as the teacher in the interaction. From this perspective, the PSTs had the power to either accept or restrict different forms of communication (Gutiérrez, 2013). Additionally, although these examples illustrate a focus on ELs' language, they also evidenced a view of English as a focal point, rather than a way to bridge between languages. We saw no evidence of PSTs using English to connect to the students' first language and vice versa. Thus, the PSTs did not draw on the students' first language as a resource to facilitate learning, an instructional strategy called for in literature (Chval & Chávez, 2012; Moschkovich, 2010). We hypothesize this stemmed from an immersion-focused mindset combined with limited personal experiences learning another language and teaching ELs. Without direct experience or concrete examples that illustrate how to draw on a student's first language, the PSTs might not employ such a strategy.

Furthermore, although the PSTs' attention to linguistic supports was well intentioned, it often resulted in the teachers taking on much of the mathematical thinking or failing to consider different student mathematical conceptions. For instance, the PSTs often removed directions that asked students to explain their thinking in order to reduce the linguistic demands of the task. By doing so, however, the students were not held accountable to engage in deep mathematical discourse, regardless of their preferred mode of communication. Similarly, in week 3, all four PSTs spent substantial time modifying the task language to increase linguistic access but failed to consider how they would provide mathematical instructional support. As a result, each of the PSTs had difficulty completing the task with the EL due to a mismatch between the ELs' and the PSTs' understanding of rounding. This situation was further compounded by a lack of common language to describe the mathematical concept (e.g., "round to" versus "round from"). These findings elucidate the challenges PSTs face when discerning whether ELs are challenged by the mathematical or linguistic aspects of the task. Moreover, these findings imply PSTs should be provided with guided experiences that facilitate the development of mathematical and linguistic learning goals for ELs while maintaining mathematical demands of tasks.

We did find some evidence that the PSTs intended to increase access to the mathematical content of the tasks. For example, they frequently checked students' understanding of the task contexts and pressed students to explain their mathematical thinking. Such strategies—while employed with good intentions—often did not uphold the cognitive demand of the task. We also found some evidence of PSTs' encouragement or support of students' use of mathematical representations. In cases where representations were used, however, they were often initiated and taken over by the PST. In the few cases where a student initiated a mathematical representation, the PSTs failed to effectively probe students' thinking and the meaning of the model. As a result, a shared mathematical meaning was not negotiated between the PST and EL, although there were instances in which they later acknowledged not knowing what the student intended. Such findings highlight the challenges that teachers face when eliciting mathematical thinking from ELs and indicate that prior experience alone is insufficient to establish such skills. These findings suggest that PSTs need direct instruction and guided experience to effectively support and elicit ELs' mathematical thinking.

There are a number of possible reasons for why the PSTs did not use many of the strategies advocated for by prior research. These reasons and our related findings carry various implications for teachers and teacher educators. For instance, the PSTs lacked formal instruction in pedagogical strategies for teaching mathematics with ELs. This was evident when Fiona learned in the first week that Jin was unfamiliar with the term "strategy"; she avoided this term in subsequent weeks rather than build his language. Her avoidance of the term ran counter to research (e.g., Khisty & Chval, 2002) that suggests teachers foster ELs' language development. As such, we argue that PSTs may benefit from instruction that highlights how to build ELs' academic language over time. For example, rather than avoiding unfamiliar terms, teachers should facilitate the transition from students' informal language to the academic language of mathematics. In the example above, Fiona could have initially connected "strategy" to phrases Jin may have known, such as *way of getting the answer*, or to situations with which he may be familiar, such as playing a board game. Then, over time, Fiona could signal how to reformulate informal to specialized language through her own talk and by encouraging Jin to use the specialized terms. She could also continue to revisit this word as it comes up so that it becomes part of Jin's repertoire.

It is also important to note that the ELs in our study were at or above grade level mathematically. ELs with different mathematical backgrounds may pose different challenges to PSTs. Finally, it seemed plausible that the PSTs were noticing particular aspects of the ELs' performance and utilizing strategies in response to those noticings. Because the PSTs had not previously engaged in experiences aimed at noticing ELs' mathematical and linguistic resources, it is not surprising they were not fully able to employ appropriate strategies for ELs. Teacher educators should consider developing PSTs' ability to notice and draw on student thinking in order to drive their use of strategies to support ELs in developing their mathematical and linguistic understandings.

Conclusion

Although we found that the PSTs' use of strategies to accommodate ELs work on cognitively demanding tasks was often insufficient to facilitate ELs' mathematical and linguistic learning, further development of these strategies in teacher preparation programs may enable future teachers to more effectively teach mathematics with ELs. The use of visuals, affordances for multiple modes of communication, frequent pressing for explanation and meaning, and checking for understanding were powerful tools when enacted effectively. Teacher educators should build on PSTs' use of these strategies and extend their repertoire to include other research-based strategies as they find avenues to remove language obstacles while maintaining the cognitive demand of mathematics tasks and eliciting ELs' mathematical thinking. Explicit instruction in, reflection on, and authentic experiences with the enactment of such strategies with ELs could help PSTs more effectively use their knowledge and skills to facilitate the development and extension of ELs' mathematical and linguistic understandings.

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Chapter 15
"How I Want to Teach the Lesson": Framing Children's Multiple Mathematical Knowledge Bases in the Analysis and Adaptation of Existing Curriculum Materials

Frances K. Harper, Corey Drake, Tonya Gau Bartell, and Eduardo Najarro

Abstract We consider how prospective elementary teachers think about multiple mathematical knowledge bases as they consider adapting existing mathematics curriculum materials to meet the needs of culturally, linguistically, and socioeconomically diverse students. Multiple mathematical knowledge bases include both children's mathematical thinking and the cultural-, home-, and community-based funds of knowledge that children inevitably bring into the classroom. We explored how prospective elementary teachers attended to multiple mathematical knowledge bases as they used a lesson analysis tool, the Curriculum Spaces Table, to evaluate existing curriculum materials and identify spaces for adapting these curriculum materials to more efficiently meet the needs of all students. Our analysis examined 47 written reflections on the analysis of and adaptations for a Grade 3 lesson. Findings showed that prospective teachers paid considerable attention to children's mathematical thinking and gave some attention to funds of knowledge, but they considered these knowledge bases largely in isolation of each other. Attention to integrating children's mathematical thinking with funds of knowledge was rare. Nonetheless, prospective teachers overwhelmingly found the Curriculum Spaces Table useful for analyzing an existing lesson and identifying spaces for adaptations to more effectively build on children's mathematical knowledge bases. Thus, we argue that a lesson analysis tool that draws attention to multiple mathematical knowledge bases offers a promising start for prospective elementary teachers' development of pedagogical practices that integrate children's mathematical thinking with diverse cultural-, home-, and community-based funds of knowledge.

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Introduction

Increasingly, mathematics teacher educators and researchers are attending to how the cultural, linguistic, and socioeconomic positionality of students impacts their mathematics learning (Zevenbergen, 2001). Such research suggests that historically underrepresented groups of students in mathematics benefit from instruction that builds on children's approaches to problem-solving and thinking about school-based mathematics (e.g., Fennema, Franke, Carpenter, & Carey, 1993; Villaseñor & Kepner, 1993) and pedagogy that draws on their cultural-, linguistic-, and community-based knowledge to support mathematics learning (e.g., Howard, 2001; Kisker et al., 2012; Ladson-Billings, 2009). Despite the promise of drawing on home- and community-based knowledge, a focus on children's mathematical thinking (e.g., problem types, solution strategies, etc.; Carpenter, Fennema, Peterson, Chiang, & Loef, 1989) overshadows considerations of students' cultural, linguistic, and socioeconomic positionality in mathematics teacher education (Aguirre et al., 2012). The failure to balance attention to children's home- and community-based knowledge with a focus on school-based mathematical thinking leaves prospective teachers (PSTs) – majority of white, female, and middle class – ill-equipped to effectively teach a racially, culturally, linguistically, and socioeconomically diverse student population (Sleeter & Milner, 2011).

In response, the Teachers Empowered to Advance Change in Mathematics (TEACH Math) project aimed to transform elementary mathematics teacher preparation to equip new generations of teachers with powerful tools and strategies to increase student learning in mathematics in our nation's increasingly diverse public schools (http://teachmath.info/about/). More specifically, the project sought to foster equity in mathematics education by preparing PSTs to integrate children's school-based mathematical thinking and children's home- and community-based knowledge. In this chapter, we look closely at one aspect of mathematics teaching practice addressed by the project – analyzing and adapting mathematics curriculum – to understand how PSTs made sense of attending to the needs of historically underrepresented populations by integrating school-based and home- and community-based sources of mathematics knowledge and resources for learning.

Background of the Study

The study presented in this chapter was part of the larger TEACH Math project, which aimed to transform elementary mathematics teacher preparation to ensure PSTs' readiness to work with racially, culturally, linguistically, and socioeconomically diverse student populations. To do so, project members developed and iteratively refined three instructional modules for elementary mathematics methods courses designed to explicitly build teacher competencies necessary for effective mathematics teaching with diverse students. In this section, we describe the research and theoretical basis for the development of the three instructional modules and give an overview of each module.

Theorizing Multiple Mathematical Knowledge Bases

Children's Mathematical Thinking and Funds of Knowledge Theoretical perspectives from two strands of research on elementary mathematics teaching and learning guided the development and refinement of modules for elementary mathematics teacher preparation: (1) mathematics instruction guided by children's mathematical thinking and (2) mathematics instruction guided by children's cultural-, linguistic-, and community-based knowledge. First, the extensive body of research on mathematics instruction that centers children's mathematical thinking (e.g., *Cognitively Guided Instruction*, Carpenter et al., 1989) provided a basis for developing PSTs' knowledge of children's mathematical thinking in ways that change teacher beliefs and shape classroom practices. Second, research that illuminates the benefits of drawing upon the cultural-, linguistic-, and community-based knowledge of historically underrepresented groups (e.g., Ladson-Billings, 2009; Turner, Celedón-Pattichis, & Marshall, 2008) guided PSTs' development of leveraging home- and community-based knowledge in mathematics instruction. In particular, module developers drew on the theory of *funds of knowledge* (FoK) for teaching. FoK refer to the "historically accumulated and culturally developed bodies of knowledge and skills essential for household or individual functioning and well-being" (Moll, Amanti, Neff, & Gonzalez, 1992, p. 133). Using students' FoK for mathematics teaching means that classroom instruction utilizes the cultural, linguistic, and cognitive resources from home or community settings to promote students' learning of the school mathematics curriculum (Moll et al., 1992).

Multiple Mathematical Knowledge Bases These two strands of theory and research complement each other because they both focus on centering children's knowledge/thinking as resources for mathematics learning; however, one approach (children's mathematical thinking) prioritizes school-based problem-solving strategies and thinking, whereas the other (FoK) prioritizes out-of-school experiences and ways of knowing. Although both strands of research are well developed and complementary, the foci on children's mathematical thinking and FoK remain disconnected in mathematics teacher education. The TEACH Math project sought to bridge these two bodies of research by guiding preK-8 PSTs to utilize both children's mathematical thinking *and* community and cultural FoK in mathematics instruction. To do so, across three instructional modules in mathematics methods, PSTs' develop knowledge, beliefs, and dispositions specifically for supporting mathematics learning by leveraging *multiple mathematical knowledge bases* (MMKB) – the integration of children's mathematical thinking and children's FoK (Aguirre et al., 2012). By theorizing the MMKB construct and making this construct central across the three instructional modules, TEACH Math module developers hoped to balance the emphasis on children's school-based and out-of-school knowledge, thinking, and experiences as critical pieces in effective mathematics teaching for diverse student populations.

Instructional Modules for Elementary Mathematics Methods

With the goal of learning to leverage MMKB in mathematics instruction, PSTs' develop various competencies related to mathematics, children's mathematical thinking, and community and cultural FoK across three modules: (1) Case Study Module, (2) Community Exploration Module, and (3) Classroom Practices Module. In the Case Study Module, PSTs work with a case study student throughout the elementary mathematics methods course to expand their thinking about children as mathematics learners and to focus on student learning and identity. The Community Exploration Module gives PSTs an opportunity to visit and learn about local communities in order to design a mathematics lesson around community-based mathematical practices. Finally, the Classroom Practices Module guides PSTs to think about MMKB as they analyze video cases, lesson plans/curriculum, and their own teaching (for more information about each module, see http://teachmath.info). Although PSTs explore a range of mathematics teaching practices across the three modules, developing lesson plans to integrate MMKB receives considerable attention. The study described in this chapter focused on one core teaching practice related to planning for mathematics instruction: analyzing and adapting curriculum materials to integrate MMKB. Namely, we focused on one specific activity within the Classroom Practices Module to investigate how PSTs framed the idea of drawing on MMKB, specifically for historically underrepresented student groups, in the context of adapting existing curriculum materials.

Methods

Research for the TEACH Math project occurred at six university sites across the United States, with data on PSTs' work in all three modules collected from elementary mathematics methods courses at each of these sites. For this study, we analyzed data collected at one university site, a large university in the Midwest located near a small city with an increasingly diverse population. In this analysis, we used data collected from two K-5 mathematics methods courses at this university, each taught by a different co-principal investigator (PI). To investigate how PSTs framed the idea of drawing on MMKB in the context of adapting curriculum materials, we collected and analyzed PSTs' written reflections from one activity – Analyzing Curriculum Spaces – within the Classroom Practices Module. This activity is one of four in the Classroom Practices Module in which PSTs learn to analyze classroom practices through four lenses: (1) teaching, (2) learning, (3) task, and (4) power and participation. In this section, we describe the activity, our data sources, and data analysis.

Analyzing Curriculum Spaces Activity and Data Sources

In the Analyzing Curriculum Spaces activity, PSTs analyzed a lesson, *Stickers: A Base-Ten Mode*, from an existing Grade 3 mathematics curriculum, *Investigations in Number, Data, and Space* (TERC, 2008, pp. 26–33), to identify opportunities for accessing, building on, and integrating children's mathematical thinking and children's home- and community-based mathematical FoK (i.e., MMKB). This lesson focused on addition of two-digit numbers with attention to recognizing and representing the place value of each digit using multiple representations (e.g., singles and strips of 10; 100 chart; equations). PSTs analyzed the lesson with the goal of identifying opportunities for minor, feasible adjustments to existing curriculum materials; we refer to places in the curriculum where teachers can make these types of small adjustments as *curriculum spaces* (Drake et al., 2015). PSTs use a tool, the Curriculum Spaces Table, designed by the co-PIs, to facilitate their identification of curriculum spaces and to guide their adaptations of curriculum materials that create spaces for eliciting and building on children's MMKB (see http://teachmath.info/modules/classroom-practices-module/activity_2/ for a full description of the activity and for the full version of the Curriculum Spaces Table).

The Curriculum Spaces Table has three sections. In the first section, PSTs identify the central mathematical goal or ideas of the lesson. In the second section, they answer questions about the different phases of the lesson (i.e., launch, explore, summarize) and lesson peripheries (i.e., notes in the margin of the teacher's edition): (1) What makes the task(s) in each phase good and/or problematic? Consider multiple entry points, representation uses, level of cognitive demand, language supports, and alignment with lesson goal(s). (2) What are opportunities for activating or connecting to family/cultural/community knowledge in each phase of the lesson? (3) How does each phase of the lesson open spaces for making real-world connections? Do students have opportunities to make their own connections? (4) What are opportunities for students to make sense of the mathematics and develop/use their own solution strategies and approaches? (5) What kinds of spaces exist for children to share and discuss their mathematical thinking with the teacher and the class? How does the lesson create opportunities for students with varied mathematical and linguistic backgrounds and confidences to communicate their mathematical understanding? (6) Where does the mathematical authority reside in the lesson (e.g., only with teacher, only with textbook, only a few students, shared among teacher and students)? In the final section, PSTs propose possible adaptations, based on their lesson analysis, for each lesson phase and the overall lesson.

Data for this study included 47 written reflections by PSTs' on their lesson analysis and use of the Curriculum Spaces Table. In their reflections, PSTs discussed (1) strengths and limitations of the lesson, (2) spaces they identified for eliciting and building on children's MMKB, and (3) the ways in which using the Curriculum Spaces Table aided in their analysis. As an added source of validity, two authors collaboratively analyzed the same lesson using the Curriculum Spaces Table analysis tool.

Research Questions and Data Analysis

In our analysis of PSTs' reflections, we sought to understand how PSTs framed the idea of drawing on MMKB, specifically for historically underrepresented student groups, in the context of adapting curriculum materials through three research questions: (1) What features of the curriculum did PSTs attend to when thinking about making adaptations to draw on MMKB? How did PSTs evaluate those features? (2) How did PSTs draw on their personal experiences and general beliefs about mathematics teaching when thinking about adapting curriculum for MMKB? (3) How did PSTs find the Curriculum Spaces Table, a lesson analysis tool, useful for identifying possible adaptations for drawing on MMKB?

The first and fourth authors analyzed the written reflections through an iterative coding process. We began coding separately, analyzing two of the written reflections and noting themes. We compared our initial impressions and used the themes to develop an initial codebook, and we continued coding separately, comparing analyses, and revising the codebook until we produced a final (seventh) version of the codebook. Throughout our development of the codebook, we continually looked for confirming and disconfirming evidence of the identified themes (Erickson, 1986).

Using the final version, we coded the remaining written reflections together, discussing discrepancies and reaching consensus on coding. Written reflections were coded at the paragraph level (as denoted by the participant or roughly 10–15 lines) because surrounding sentences provided important context for inferring meaning. We assigned codes in four streams, which we describe in more detail here.

Coding Stream 1: Content We used the codes defined in Fig. 15.1 to identify major themes related to the content of PSTs' written reflections. These themes provided evidence of the features of the curriculum PSTs attended to when thinking about making adaptations to draw on MMKB (first part of RQ1). Codes in the first stream (Fig. 15.1) served as the primary codes for analysis, meaning that these codes were exhaustive (i.e., every paragraph received at least one code from the first stream), and codes could stand independently (i.e., were not necessarily associated with codes from the other streams). Codes were not mutually exclusive; in other words, paragraphs could (and often did) receive multiple codes from the first stream.

Coding Stream 2: Evaluation We created a second coding stream to identify themes related to PSTs' evaluation of the curriculum. This coding stream reflected

Primary Theme	Definition	Subthemes & Definitions (if applicable)
Mathematical features	Aspects of the lesson/teaching access, built on, or integrate *children's mathematical thinking*.	Sense making: Opportunities for mathematical sense making through the use of *manipulatives /visual aids, multiple representations, or multiple strategies* for solving problems
		Cognitive demand: Higher-order thinking (e.g., using procedures with connection to meaning) or lower-order thinking (e.g., memorization) (see Henningsen & Stein, 1997)
		Learning goals: Objectives for mathematics learning
Mathematical talk	Aspects of the lesson/teaching provided opportunities for students to *explain or justify mathematical work or thinking* in small groups (with/without teacher facilitation) or whole class (with teacher facilitation of discussion; see Smith & Stein, 2011). Teacher may make use of explanations/justification for *formative assessment* (i.e., eliciting student thinking for evaluative purposes to inform instruction).	
Mathematical authority	Aspects of the lesson/teaching are *primarily directed by the teacher* (e.g., teacher takes the lead in talking about mathematics) or *primarily centered around student thinking or ideas with minimal direction from the teacher* (e.g., participation-structure facilitates student engagement and discussion without direct teacher supervision).	
Learning supports	Aspects of the lesson/teaching provide supports to *facilitate student learning of mathematical content*	General Learning Supports: Considerations that will arise in essentially every classroom setting, including *scaffolding* (i.e. gradually decreasing the need for learning aids as students' comfort with language, concepts, etc. increases); *teacher questioning* (i.e. various forms of questioning recognized to support learning (e.g., Boaler & Brodie, 2004)); or *differentiation* (i.e. individualized adaptations of lessons/tasks); no reference to FoK
		Learning Supports for Diverse Learners: *give diverse learners access to the mathematical content*; particularly impo rtant when the class included culturally and linguistically diverse students or special education students; may or may not reference FoK.
Prior knowledge	Aspects of the lesson/teaching are *relevant to the students' prior knowledge*.	School-Based Knowledge: Prior knowledge that arose in a school-based setting.
		Funds of Knowledge: Prior knowledge that arose from community/family/cultural knowledge or experiences.
		Not-Specified: Source of prior knowledge is not specified
Motivation	Aspects of the lesson/teaching are *relevant or familiar to students* for the purpose of *engaging or motivating students*.	

Fig. 15.1 Coding stream 1 summary. This figure provides an overview of coding themes and their definitions for analysis of the content of written reflections

Primary Theme	Definition	Subthemes & Definitions (if applicable)
Strength	PSTs evaluated some aspect of the lesson/teaching as strong.	
Weakness	PSTs evaluated some aspect of the lesson/ teaching as weak or limited.	Too little – Weakness or limitation resulted from too little of an aspect (e.g., too little support for diverse learners)
		Too much – Weakness or limitation resulted from too much of an aspect (e.g., too much explaining/telling by the teacher)
Curriculum Space	PSTs identified an aspect of the lesson/teaching where the curriculum could be adapted. This code was used both when the PT made a specific adaptation suggestion (e.g., I would let the students try the problem on their own first.) or when the PT talked more generally about spaces in the curriculum (e.g., The lesson needs to be adapted for English Language Learners (ELLs).)	

Fig. 15.2 Coding stream 2 summary. This figure provides an overview of coding themes and their definitions for analysis of PSTs' evaluation of the lesson

three major themes in PSTs' analyses: (1) strength, (2) weakness, and (3) curriculum space (i.e., space in the curriculum materials for minor/feasible adaptations) (Fig. 15.2). Coding in this stream allowed us to examine whether PSTs framed aspects of the lesson as strong/weak and/or curriculum spaces (second part of research question 1). We linked codes in the second stream directly to codes in the first stream so that we could determine which specific aspects of the lesson/teaching PSTs evaluated in their analysis (i.e., second stream codes only occurred when they were associated with first stream codes). For example, sometimes PSTs discussed ways in which the lesson provided learning supports, and they clearly identified those learning supports as strengths of the lesson. Such a paragraph would receive a "learning supports (first stream)-strength (second stream)" code. When PSTs did not clearly evaluate lesson aspects as a strength/weakness or a curriculum space, we did not use the second coding stream. Codes in the stream were not mutually exclusive because some paragraphs included a discussion of both strengths and weaknesses of the same aspect of the lesson and because spaces for adaptations were generally identified alongside weaknesses.

Coding Stream 3: Context In order to answer the second research question, we created a third coding stream to identify what contexts, outside of the lesson plan itself (the default context), PSTs drew on as they reflected on the curriculum analysis and/or suggested adaptations. Third stream codes were independent of previous coding streams, but they were not exhaustive. In the absence of a third-level code, we inferred the PSTs' reflection was grounded in the analysis of the specific lesson.

The third coding stream contained two codes: (1) personal experience and (2) general.

When PSTs discussed their personal experiences as mathematics learners or as PSTs in their field placement classrooms, we added the "personal experience" code to the paragraph. For example, "I know in my placement, the students love going up to the board to do the problem in front of the class," indicated the PST was drawing on personal experience in their reflection. We also identified instances when PSTs made more general statements about mathematics teaching and/or learning, which we interpreted as representing their general beliefs about mathematics teaching and learning. For example:

> Every student learns differently…it is very important to understand that elementary school students can get confused very easily. So when a topic is presented to the class, the teacher needs to recognize and be aware of how the students are responding so he/she can adapt the lesson to the class.

Such statements were marked by the "general" code at the paragraph level. As previously, codes in the third stream were not mutually exclusive because PSTs could reference both their personal experience and general beliefs in the same paragraph.

Coding Stream 4: Utility In order to answer the third research question, we created a fourth coding stream to indicate when PSTs discussed their use of the Curriculum Spaces Table analysis tool. We coded paragraphs in which the PSTs described their use of the table using the code "utility." We identified paragraphs in which PSTs indicated that they found the table useful for their analysis (i.e., coded "utility-yes") and instances in which PSTs indicated that they did not find the table useful (i.e., coded "utility-no"). In these instances, fourth stream codes could occur independently of the other streams, but they were not necessarily mutually exclusive because some PSTs stated they found the analysis tool useful for some parts of the curriculum analysis but not for others. When possible, we linked fourth stream codes to the first and second coding stream. For example, if a PST described how they used the table to identify the learning goals for the lesson and stated that using the table helped them recognize that the learning goals aligned with the lesson, the paragraph received a "learning goals (first stream)-strength (second stream)-utility-yes (fourth stream)" code.

Analysis and Synthesis When coding was complete, we analyzed raw data (i.e., coded excerpts) and synthesized across written reflections to identify themes. First, we calculated the total frequency of each code for all four streams. Next, we created several tables to show code co-occurrence across streams. We examined which codes from the first stream (i.e., content codes) occurred most frequently alongside codes from the second, third, and fourth coding streams, and we considered the code co-occurrence of first stream codes with other first stream codes within the same paragraph. Finally, we investigated individual differences in PSTs' reflections for major themes by determining how many PSTs' attended to those themes in their reflections.

Findings

In this section, we share the major themes that emerged across the written reflections. We divided these themes into three categories based on our three research questions: (1) attention to and evaluation/adaption of lesson features, (2) influence of personal experiences and general beliefs about mathematics teaching, and (3) utility of the Curriculum Spaces Table analysis tool.

Attention to and Evaluation/Adaptation of Lesson Features

Overall, we found that PSTs generally paid attention to children's mathematical thinking and FoK separately in their analyses of the lesson and in their suggestions of adaptations. Attention to the needs of historically underrepresented populations of students emerged across several different themes (e.g., learning supports, motivation) related to curriculum features, but PSTs largely focused their attention on how the existing lesson and possible adaptations could support all students' mathematical learning, more generally, rather than focusing on the needs of specific groups of students. Additionally, PSTs attended to lesson features that they saw as promoting or limiting children's mathematical thinking more often than lesson features that they saw as promoting or limiting children's FoK. In fact, all 47 PSTs attended to children's mathematical thinking, but only 25 PSTs explicitly considered FoK in their reflections. Only three PSTs described the integration of children's mathematical thinking with FoK (i.e., MMKB) in their reflections. Thus, we present findings of PSTs' attention to and evaluation/adaptation of the lesson features related to children's mathematical thinking first, followed by findings related to FoK, and finally MMKB.

Attention to Children's Mathematical Thinking PSTs paid considerable attention to aspects of the lesson that accessed, built on, or integrated children's mathematical thinking. As mentioned previously, all 47 PSTs attended to children's mathematical thinking in their reflections, and this consideration represented the majority of content codes. More specifically, 61.5% of all first stream codes related to children's mathematical thinking in relation to mathematical features of the lesson that fostered mathematical sensemaking (36.5%), created opportunities for children to talk about mathematics (14.1%), and positioned mathematical authority with student or teacher (10.9%) (Table 15.1). Features of the lesson that promoted mathematical sensemaking through the use of manipulatives, multiple representations, or multiple solution strategies were particularly noteworthy, representing 25.8% of all first stream codes. In particular, the use of manipulatives was the most commonly recognized strength of the lesson across all reflections (i.e., this code occurred most frequently alongside the second stream code: "strength"). The use of multiple representations and multiple solution strategies were also evaluated as strengths of the lesson but to a lesser extent.

Table 15.1 Percent of total codes in the first stream (n = 1051) by theme across 47 written reflections

Children's mathematical thinking							
Math features		Math talk	Math authority	Learning supports		Prior knowledge	Motivation
36.5%		14.1%	10.9%	8.3%		12.1%	8.1%
Sensemaking	25.8%			General	15.7%	School 4.3%	
Cognitive demand	3.6%			Diverse	2.6%	FoK 4.9%	
Learning goals	7.1%					Not specified 2.9%	

Those PSTs who positively evaluated mathematical sensemaking in the lesson often described those lesson features as important for supporting learning (i.e., "math features" codes from the first stream frequently occurred alongside "learning supports" code also from the first coding stream). Statements such as this were common:

> Another strength of this lesson is that it allows the students to develop, and reinforce, their understanding of tens and ones through multiple mediums such as stickers, pictures, numbers, and cubes. The students are asked to physically represent the problem through the use of both cubes and a drawing. Students are then asked to turn that physical representation into a column of tens and ones and then list the total number of stickers and the equation. The use of multiple mediums promotes student learning by having the students redo the same problem multiple times.

As this example illustrates, PSTs' considerations of learning supports were generally vague, sometimes referencing the needs of various learners but rarely naming how lesson features supported the needs of specific groups of students. Although all 47 PSTs discussed general learning supports in their reflections, only 22 PSTs explicitly referenced learning supports for diverse groups of students, and those 22 PSTs did not devote much space in their reflection to this topic. References to learning supports for diverse learners generally appeared only in one paragraph and represented only 2.6% of all content codes (Table 15.1). When PSTs attended to learning supports for diverse learners, they most commonly expressed that the lesson inadequately addressed the needs of diverse learners (i.e., "learning supports: diverse" in the first stream occurred most commonly alongside "weakness" in the second stream). For example, "This lesson also limits ELLs [English Language Learners]. At no point in the lesson do I see any talk of ELLs so that's something that could limit their learning." Specific suggested adaptations for meeting the needs of diverse learners, however, were rare (i.e., "learning supports: diverse" in the first stream occurred least often alongside "curriculum spaces" from the second stream).

PSTs paid the most attention to children's mathematical thinking when suggesting adaptations (i.e., the overwhelming majority of codes linked to the second stream code, "curriculum spaces," were "math features," "math talk," and "math authority " codes (combined) from the first stream). For example, even though PSTs positively evaluated the lesson's use of manipulatives, they suggested adaptations for using them:

> The one part of the lesson I found to be the most ineffective is the use of cubes. I think the cubes are a good idea to bring to students' attention, but at a later part in the lesson. A lot of different techniques for the students used adding and subtracting the multi-digit numbers, and therefore, I think only one object of manipulative should be used during the lesson.

This excerpt shows how some PSTs viewed mathematical features aimed at student sensemaking positively, but some also suggested adaptations that would change the way that multiple representations, including manipulatives, or solution strategies were used during the lesson. PSTs were divided on whether the lesson's use of multiple representations and solution strategies was too much, thus overwhelming students, or too little, thus limiting opportunities for sensemaking.

Across reflections, PSTs agreed on two weaknesses in the lesson: too much mathematical authority residing with the teacher and too little opportunity for students to discuss mathematics as a whole class (i.e., "math authority: teacher" and "math talk: students" in the first stream occurred frequently alongside "weakness: too much" and "weakness: too little," respectively, in the second stream). In particular, PSTs recognized a need for less teacher-led mathematical sensemaking and mathematical talk:

> I would argue that [authority lies] solely [with] the teacher. The questions do not set up a discussion where students are sharing with each other. Rather, the questions set up an IRE [Initiate-Respond-Evaluate] session. The one area that was beneficial to the students was when they were invited up to the overhead to share their work, but the teacher is ultimately the authority and most likely evaluates the students that share.

Even though PSTs were specific in identifying features of the lesson that positioned the teacher as the mathematical authority, suggested adaptations were vague (based on content of excerpts with code co-occurrence of "math authority" and "math talk" from the first stream with "curriculum spaces" from the second stream). In most cases, PSTs simply noted that the lesson needed to be adapted to position students as the mathematical authorities but did not describe how they would make such adaptations. Some of the few specific adaptations suggested adding a whole-class discussion at the end of the lesson:

> The lesson provides opportunities for students to explore but the lesson could summarize more in a group discussion format … students could share their solutions and allow other students to ask questions, compare, or justify their own thinking.

Other PSTs suggested changing the sequence of the lesson as written, using the planned launch as a guide for orchestrating a more student-driven, whole-class discussion after students had time to explore the problems more freely.

Funds of Knowledge General prior knowledge (12.1%) represented one of the major themes related to how PSTs framed the idea of drawing on MMKB, in the context of adapting curriculum (Table 15.1). Attention to home- and community-based FoK (4.9%) occurred with roughly the same frequency as attention to school-based prior knowledge (4.3%) across all written reflections. Figure 15.3 provides

School Based	"The lesson connects back to Sticker Station done in second grade so it utilizes [students'] background knowledge."
FoK	"Aspects of the lesson plan that stand out as especially important for making the lesson effective in promoting students' learning are…that the lesson opens spaces for making connections to their family/cultural/community knowledge."
Non-Specified	"I do believe that it is possible to be responsive to students' thinking and background knowledge while also using the curriculum materials."

Fig. 15.3 Examples for each category of prior knowledge

example excerpts of school-based prior knowledge, FoK, and prior knowledge with source unspecified.

Few PSTs identified prior knowledge (combined) as a limitation in their lesson analysis (combined "prior knowledge" codes from the first stream rarely occurred alongside "weakness" codes from second stream); however, when PSTs did identify prior knowledge as a weakness, they overwhelmingly focused on FoK (i.e., when "prior knowledge: FoK" from the first stream co-occurred with a second stream code that code was "weakness"). For example:

> I thought the lesson was limited at making a connection to cultural, community, and/or family knowledge. It was great that the lesson used stickers which students were familiar with from the sticker station store in 2nd grade. This would hook student's interest and motivate them for the lesson.

As the above excerpt illustrates, school-based prior knowledge was rarely identified as a weakness in the analysis of this lesson. Moreover, this quote illustrates how PSTs who identified FoK as a weakness identified the need for adaptations related to FoK, but few suggested adaptations for integrating FoK. In fact, only nine PSTs suggested how to adapt the lesson to integrate FoK. Specifically, they suggested changing the context of the problem to build on FoK. For example:

> The teacher could have had students create their own problems using something from [students'] culture, home, or of interest that could be bundled in a set of 10 or used individually; for example, my case study student was very interested in basketball and played it at home with family often. He could create a problem having one team of basketball players represent the 10's place and an individual basketball player representing the 1's place. They could then share these problems in partners or with the whole class. I feel if students could connect with the problem better, it would better promote students learning.

Although this PST gave a specific example for possible adaptations (i.e., basketball), most PSTs referenced changing the context but did not identify a specific context that would draw on FoK in their reflections.

As the previous two excerpts show, attention to student interest and motivation in the lesson plan was common throughout the reflections (i.e., a major code: 8.1%, Table 15.1), and discussions of student interest and motivation frequently occurred alongside considerations of FoK (i.e., "motivation" and "prior knowledge: FoK" codes, both from the first stream, frequently occurred together in paragraphs). PSTs identified aspects of the lesson that attended to motivation in both strong and limited ways (i.e., alongside both the "strength" and "weakness" codes from the second stream), and they suggested adaptations toward integrating FoK and other sources of prior knowledge to increase student motivation and learning, as illustrated by these previous two excerpts (i.e., "prior knowledge" codes frequently occurred alongside "curriculum spaces" code).

MMKB Attention to integrating children's mathematical thinking with FoK (i.e., MMKB) was rare (only three PSTs attended explicitly to MMKB), but again, these rare instances are important because they provide insight into PSTs' potential sensemaking of this construct. More specifically, these few PSTs (three of the nine who suggested adapting the context of the problem) discussed MMKB as they considered ways of adapting the use of manipulatives to integrate FoK. For example:

> The teacher [could] make story problems up about the students and their interest. For example, instead of saying the cubes are just cubes, say they are lumps of sugar that 'Tom' has to feed to his horses. (Given there's a Tom in the classroom who has horses to feed at home.) The kids will see that this problem has a meaning more so than just seeing the cubes as cubes.

Because PSTs overwhelming saw the use of manipulatives as building on and supporting children's mathematical thinking in this lesson, their attempts to adapt these manipulatives to integrate FoK represented the most explicit attention to MMKB across reflections.

Influence of Personal Experiences and General Beliefs About Mathematics Teaching

PSTs primarily grounded their reflections in their analysis of the specific lesson (as directed in the assignment description); however, PSTs also drew on their personal experiences and general beliefs about mathematics teaching when reflecting on their lesson analysis and adaptations. Almost all of the PSTs (45 of 47) included some reference to their general beliefs about mathematics teaching in at least one paragraph of their reflection. Connections to personal experiences – either as mathematics learners themselves or as PSTs working with students in their field placement classrooms – were less common (26 of 47 PSTs included references to their personal experiences). When PSTs drew on their personal experiences, they most commonly did so to elaborate on their evaluation of general learning supports and mathematical features of tasks (i.e., "personal experience" code from third stream occurred most commonly alongside "learning supports" and "math features" codes from the first stream). In particular, PSTs described how manipulatives, multiple strategies, and multiple representations supported student learning in their field placement classrooms, more generally, and during their student thinking interviews done as part of the Case Study Module (see "Background of the Study" for overview of module). For example, this PST justified her positive evaluation of the lesson based on her assessment of her case study students' needs:

> [This lesson] also encourages students to visually represent the problem using a picture of strips and single stickers or through the use of lines or sticks and small dots. I believe that this is very beneficial for promoting the students' learning because it encourages them to use the solution strategy that works best for them. I believe that this would especially be beneficial for students who do not have a fully developed understanding of place value, such as my case study student Janette. I think that having multiple visual representations of the stickers, whether it is through cubes, pictures, or lines and dots, would help her to understand that there are ten singles in a strip.

Less frequently, PSTs drew on experiences with students in their field placement classrooms to evaluate the lesson or suggest adaptions for connecting to students' prior knowledge, including FoK. (i.e., "personal experience" code from the third stream occurred less frequently alongside "prior knowledge" codes from the first stream). For example, one PST noted, "My case study student was very interested in basketball and played it at home with family often," when suggesting an adaptation to the problem context of the lesson (see p. 253 for full excerpt from this PST's reflection).

A focus on learning supports equally permeated PSTs' general statements about mathematics teaching and learning (i.e., "general" from the third stream occurred most frequently alongside "learning supports" from the first stream). Additionally, prior knowledge, and FoK in particular, garnered considerably more attention than other topics, including mathematical features of tasks, in PSTs' general statements about teaching (i.e., "prior knowledge" codes from the first stream were the second most frequently linked codes to "general" codes from the third stream). These general statements often focused on what teachers would or should do to meet the needs of diverse learners. For example,

> I think that teachers need to analyze the curriculum materials and manipulate the lessons to include more student led discussions and instruction…When being provided with a curriculum it is not necessarily tailored for every classroom of students and after knowing one's specific students the lesson can be altered in ways to reach all students in the classroom. For example, if I had a classroom of primarily ELL students then I would make sure the task is authentic to these students by connecting it to their community or family, and also by spending more time reviewing the language used, allow them to work in partners first, and find ways for them to contribute in the discussion.

This statement illustrates how the PST was making sense of incorporating FoK in future teaching. General statements, such as this one, about providing learning supports, shifting mathematical authority to students, or connecting to prior knowledge were common across reflections. Although some PSTs named specific groups of underrepresented students, most commonly ELLs, most PSTs focused broadly on all students.

Utility of the Curriculum Spaces Table

PSTs overwhelmingly stated that they found the Curriculum Spaces Table useful for analyzing existing curriculum materials and identifying spaces for integrating MMKB toward supporting the needs of all students (i.e., 43 of 47 reflections included "utility-yes" codes from the fourth stream). For example, the PST quoted previously concluded by reflecting on the utility of the table:

> By using the curriculum analysis [table] I was able to get an idea of a lesson on a topic that I wanted to cover and come up with alterations to the curriculum lesson. Although I would not want to teach the lesson as it is written, I think that looking at the original resource gave me more ideas on how I want to teach the lesson.

Many PSTs similarly expressed that existing curriculum materials can provide valuable resources for mathematics teaching and learning, but adaptations are necessary for supporting the learning of all students. Moreover, they considered the table useful for identifying possible adaptations to different aspects of the lesson. In particular, PSTs found the Curriculum Spaces Table helpful for evaluating/adapting the following lesson features (i.e., the following codes from the first stream occurred alongside "utility-yes" from the fourth stream): learning supports (most common but rarely focused on learning supports for diverse learners), mathematics features of tasks (second most common), and prior knowledge (third most common but rarely focused on FoK specifically).

Only four PSTs offered critiques of the Curriculum Spaces Table (i.e., "utility-no" from the fourth stream only occurred four times). Two PSTs said they found the table useful for analyzing existing curriculum materials, but they did not find it as useful in planning their own lessons:

> Personally, during my lesson planning process I only find the front page [i.e., questions 1-3; see methods] of the curriculum analysis helpful … Then once I am done with a portion of my plan, I begin to look through the rest of the spaces just to get an idea of what I still need to add or change in my plan.

The other two PSTs stated that they found the structure of the table difficult to work with because the existing curriculum materials did not provide enough information to complete the table, but one suggested the tool would be better suited for teacher-created lessons:

> I spent too much time trying to fit the "right answer" into the cells of the chart instead of focusing on the content of the lesson [when analyzing existing curriculum materials]. Using something like this chart when writing my own lesson plans would be helpful though. It is much easier to cover all – or most – of the topics with my own words and ideas before writing my lessons to make sure all of the important areas are covered. But using it to analyze someone else's work was just overwhelming and frustrating. I eventually stopped filling in the cells and just started making notes in the margins.

As this quote illustrates, even when PSTs had critiques of the analysis tool itself, their statements suggested that they appreciated the importance of attending to different aspects (e.g., learning supports, prior knowledge) of mathematics lessons.

Discussion and Conclusion

Our findings suggest that the Curriculum Spaces Table analysis tool may offer a promising start for PSTs, who are early in their professional development as teachers, to develop equitable mathematics teaching practices. Elsewhere, our TEACH Math colleagues and other mathematics teacher educators have argued that awareness and openness to multicultural- and community-based approaches to mathematics represent an important first step toward developing more sophisticated equitable mathematics teaching practices, such as incorporating MMKB meaningfully into instruction (Turner et al., 2011; White, Murray, & Brunaud-Vega, 2012). In particular, our findings showed how PSTs found the Curriculum Spaces Table analysis tool was especially useful for drawing attention to and creating awareness of a need to adapt, extend, or build on learning supports, mathematics features of tasks, and connections to prior knowledge within existing curriculum materials. Although most suggested adaptations for connecting children’s MMKB in the lesson lacked specific details – when considering adaptions for diverse students, integrating FoK, positioning students as mathematical authorities, etc. – PSTs’ initial attempts to make these connections suggest that they are developing the key practices along a trajectory toward engaging children with MMKB (Turner et al., 2011). Moreover, general statements about teaching, which focused on what teachers would or should do to meet the needs of diverse learners, suggest an emerging openness and commitment to multicultural- and community-based approaches to mathematics (White et al., 2012).

Here, we discuss what our findings suggest about PSTs’ development of initial practices for integrating MMKB into instruction. In particular, we reflect on the promising ways that roughly half of PSTs’ attended to racially, culturally, linguistically, and socioeconomically diverse students’ learning (22 of 47 PSTs) and attempted to connect to children’s cultural- and community-based FoK (25 of 47 PSTs). We focus on PSTs’ attention to diverse students and attempt to connect to FoK because equitable mathematics teaching demands instruction that responds to the specific needs of racially, culturally, linguistically, and socioeconomically diverse students and integration of the cultural and community resources they bring for mathematics learning (Martin, 2003) and because our findings illustrate a need to continue challenging the overemphasis on children’s mathematical thinking in teacher preparation. We consider implications for using the Curriculum Spaces Table analysis tool in mathematics teacher education to increase, even more, the number of PSTs who develop the initial attention, awareness, openness, and commitments necessary for equitable mathematics teaching.

Building on Experiences with Children's Mathematical Thinking to Shift Focus to Integrating FoK

PSTs' tendency to focus on children's mathematical thinking, even when analysis tools attempt to direct their attention to other considerations, has been observed elsewhere (Aguirre, Zavala, & Katanyoutanant, 2012). The overwhelming emphasis on children's mathematical thinking shown in our findings also makes sense given that (1) the use of manipulatives and multiple representations is a prominent feature of the *Stickers* lesson (TERC, 2008) and (2) PSTs' reflections indicated that they had experience working with students or observing teachers working with students using manipulatives and/or multiple representations in their field placement classrooms. In light of the lesson focus and their personal experiences, PSTs naturally paid considerable attention to manipulatives/multiple representations as both a strength of the lesson and as a possible space for adapting, extending, or building on the existing lesson features to support learning about place value and two-digit addition.

We see this attention to and evaluation of using manipulatives/multiple representations as promising because some PSTs built on this particular mathematical feature of the lesson in attempts to connect to children's mathematical thinking and FoK. Most initial attempts to connect to children's MMKB represented emergent connections (i.e., superficial in nature; Turner et al., 2011), such as changing the context of problem, thus changing the manipulatives from stickers to something (usually unnamed) more relevant to students' cultural-, community-, or home-based experiences. Some attempts to connect to children's MMKB, however, reflected more meaningful connections (Turner et al., 2011), which PSTs made by drawing on either their actual experiences with students in their field placement classrooms (e.g., "My case study student was very interested in basketball and played it at home with family often") or the experiences of hypothetical students (e.g., "There's a Tom in the classroom who has horses to feed at home"). This finding suggests that familiarity and experience with a prominent mathematical feature of the lesson (i.e., manipulatives) provided an entry point for PSTs to imagine ways to integrate children's mathematical thinking and FoK.

Making sense of and connecting children's cultural-, home-, and community-based FoK to mathematics instruction is a complex practice (Civil, 2007). Lessening the need to adapt a lesson to elicit and support children's mathematical thinking – because opportunities for eliciting and building on children's mathematical thinking through manipulatives/multiple representations already existed within the lesson and because PSTs had already thought about these opportunities at other times – could free up PSTs' to focus on the less familiar and, arguably, more challenging task of leveraging FoK. This has implications for the types of lessons that mathematics teacher educators select for analysis using the Curriculum Spaces Table tool and for the types of experiences they help create in field placement classrooms. Mathematics teacher educators could strategically select lessons for analysis that effectively support children's mathematical thinking with specific mathematical features of the tasks, which are already familiar to PSTs, as a way of shifting more attention to cultural- and community-based FoK in lesson analysis and adaptation.

Creating Opportunities for More Experience with and Attention to Diverse Cultures and Communities

In the promising examples from our findings when PSTs attended to FoK, they rarely framed their analysis or suggested adaptations in terms of specific groups of students who are underrepresented in mathematics. Instead, they focused either more generally on supporting learning of all students or on specific individuals in their field placement. Our findings showed how references to FoK were framed more vaguely in terms of “culture” or “community” and motivating students. The complete absence of attention to supporting learning along race, ethnicity, gender, or class differences throughout PSTs’ reflections is perhaps unsurprising for several reasons: (1) the majority of our PST population comes from a white, suburban middle-class background, which often leaves them with little contact with people outside their dominant cultural group, particularly the underrepresented students they intend to teach (Sleeter & Milner, 2011); (2) even as former mathematics teachers who worked with diverse K-12 student populations, we (the first and fourth authors) found it challenging to suggest specific adaptations to connect FoK and children’s mathematical thinking in this lesson without imagining a specific classroom context with which we were intimately familiar; (3) reference to diverse learners in the Curriculum Spaces Table analysis tool is mostly general (i.e., “family/cultural/community knowledge; “real-world connections”), with the exception of a reference to “language supports.”

Research on integrating equity and social justice into teacher education suggests that PSTs need experiences with racially, culturally, linguistically, and socioeconomically diverse students in order to move from conceptual (e.g., awareness, attention) to more practical pedagogical tools (e.g., making connections, integrating MMKB into instruction) (McDonald, 2005). Even though some of the PSTs in this study worked with diverse student populations in their field placement classrooms, they did not name specific students’ backgrounds, with the exception of ELLs, when reflecting on their personal experiences in the context of analyzing and adapting this lesson. Thus, we propose that our findings have implications for the ways that mathematics teacher educators support PSTs to make sense of their limited experiences with diverse students during teacher preparation and into their teaching career to support efforts to analyze/evaluate lessons toward integrating MMKB into instruction.

PSTs overwhelmingly expressed that they found the Curriculum Spaces Table useful because it provided guidance on what to look for in their lesson analysis, and their fidelity to the analysis tool showed in our findings. Every topic explicitly named in the Curriculum Analysis Table (e.g., cognitive demand, lesson goal) emerged as a primary theme in our analysis. We posit that we observed a tendency for PSTs to name one specific group of underrepresented students – ELLs – because a question on the Curriculum Spaces Table directly prompted them to consider "language supports." Providing more explicit prompts as part of the Curriculum Analysis Table questions that ask PSTs to consider race, ethnicity, gender, and class diversity might offer one way of supporting PSTs to unpack ideas about relevance and culture.

Given the difficulties we (the first and fourth authors) faced in our own lesson analysis/adaptations, explicit prompts asking PSTs to consider race, ethnicity, gender, and class will likely be insufficient for supporting meaningful connections to diverse children's MMKB. We found that we were only able to craft specific adaptations to integrate MMKB once we imagined a context in which we had previously taught. Because PSTs were already finding ways to draw on their field placement experiences, mathematics teacher educators might consider asking PSTs to suggest adaptations specifically for their field placement classroom, if those classrooms include diverse groups of students. Having a specific context in mind, combined with more explicit prompts to consider race, ethnicity, gender, language, and class diversity, while analyzing the lesson might encourage PSTs to draw more strategically on their personal experiences to craft specific lesson adaptations for historically marginalized students.

In addition, PSTs will likely need additional supports to connect to their personal experiences with diverse students or to consider explicit prompts about race, ethnicity, gender, language, and class in ways that challenge deficit views and support openness to multicultural approaches in mathematics (e.g., White et al., 2012). In our findings and in other research (e.g., Barton & Tan, 2009), PSTs cited a variety of reasons for attending to FoK and diverse learners, including motivating students, increasing access to the content, and changing the content itself. Connecting lesson analysis/adaptions to other activities designed to build self-awareness of cultural differences, to reveal and challenge stereotypes and deficit thinking, and to frame cultural differences as assets for mathematics learning (e.g., Jackson, Appelgate, Seiler, Sheth, & Nadolny, 2016; White et al., 2012) could help ensure that PSTs' attention to and initial attempts to connect to diverse groups of children's MMKB foster practices and dispositions for the ongoing development of equitable teaching.

We view the Curriculum Spaces Table analysis tool as a promising way that PSTs can begin to think about adapting existing curriculum materials to meet the needs of diverse groups of students by integrating MMKB into instruction. By starting from existing curriculum materials, efforts to meet the needs of diverse groups of students can seem more feasible in day-to-day instruction. We recognize, however, that there is room within teacher preparation to improve the way that PSTs use this analysis tool to think more specifically about groups of historically marginalized students in mathematics and to add more emphasis to cultural- and community-based FoK as they imagine how they want to teach the lesson. We hope that our findings offer some insight into how the Curriculum Space Table analysis tool might support development of initial practices and continue to support teachers' ongoing development of equitable teaching that integrates MMKB into instruction.

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Chapter 16
Seeing Mathematics Through Different Eyes: An Equitable Approach to Use with Prospective Teachers

Christa Jackson, Cynthia E. Taylor, and Kelley Buchheister

Abstract Teacher educators need to prepare prospective teachers by encouraging them to critically examine their current beliefs about the teaching and learning of mathematics while also providing opportunities for prospective teachers to develop an equity-centered orientation. Attending to these practices in teacher preparation programs may help prospective teachers observe actions that occur in classrooms and determine effective strategies that provide the opportunity to enhance all students' access to high-quality mathematics instruction. As mathematics teacher educators, we must recognize what prospective teachers attend to as they direct their attention to various classroom events and how they relate the events to broader principles of teaching and learning. In this chapter, we investigate what prospective teachers attend to in a classroom vignette of a student who is above grade level in mathematics and exhibits disruptive behavior during instruction. Keeping everything constant in the vignette except the student's race and sex, we examined prospective teachers' responses when the student was an African American male, White male, African American female, and White female. By attending specifically to race and sex, we explored whether prospective teachers demonstrated (1) an equity-centered orientation toward mathematics instruction or (2) deficit views of students based on race, sex, or the intersection of the two. Using a constant comparative method, the data were coded and analyzed using the equity noticing framework. The results indicate that prospective teachers are beginning to attend to cultural influences and their responses reveal differences not only between races but also between males and females.

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Introduction

Tell me to what you pay attention and I will tell you who you are. (José Ortega y Gasset)

The National Council of Teachers of Mathematics (NCTM) asserts that in order to teach in an equitable manner, teachers and schools must maintain "high expectations and strong support for all students" (2000, p. 11), meaning mathematics teachers must provide opportunities for students to learn challenging mathematics regardless of their students' "personal characteristics, backgrounds, or physical challenges" (p. 12). For the past three decades, mathematics educators have conceptualized what it means to teach mathematics for equity (Gutiérrez, 2002; Gutstein, 2003; Hart, 2003; Matthews, 2005), from gender studies (e.g., Forgasz & Rivera, 2012) to the considerations of students of color (e.g., Battey, 2013; Diversity in Mathematics Education Center for Learning and Teaching, 2007), socioeconomically disadvantaged students (e.g., Lubienski, 2002), and students who struggle mathematically (e.g., Allsopp, Kyger, & Lovin, 2007). Only in recent years, though, have mathematics teacher educators documented efforts to prepare prospective teachers (PSTs) to teach mathematics while considering matters of equity (e.g., Bartell, 2010; Freitas, 2008; Wager, 2014).

Although there is an emerging body of research on equitable mathematics instruction (e.g., Leonard & Martin, 2013; Pinnow & Chval, 2014; Vomvoridi-Ivanović & Chval, 2014), there is more to learn about supporting PSTs as they work to implement pedagogical practices that promote and support the mathematical learning of all students (Wager, 2014). In fact, "very few teacher education programs have successfully tackled the challenging task of preparing teachers to meet the needs of diverse populations" (Watson, Charner-Laird, Kirkpatrick, Szczesiul, & Gordon, 2006, p. 396). To address issues of equity and support mathematical learning for all students, it is imperative for teachers to integrate aspects of the students' culture and language within their instruction (NCTM, 2014). Unfortunately, many PSTs do not know how to make these necessary connections, especially with students who are different from their own culture and background (Futrell, Gomez, & Bedden, 2003; Turner, et al., 2012). Consequently, within the current educational system, students from non-dominant backgrounds are often denied equitable opportunities to learn (Wager, 2014), and maintaining the status quo will only continue to "widen the gap between teachers and children in schools" (Sleeter, 2001, p. 96).

Currently, about 49% of the US public school population is comprised of culturally and racially diverse groups such as African Americans, Asian-Americans, Latinas, and Native Americans (Hussar & Bailey, 2016). By 2023, researchers have predicted a "minority-majority flip" for students of color with White students' enrollment declining to 45% (Hussar & Bailey, 2016). By 2050, students of color are predicted to represent approximately 62% of the school-age population (National Center of Educational Statistics, 2010). As the population of public school students continues to become more and more diverse, it is critical that teachers' instructional practices purposefully align with students' funds of knowledge when teaching mathematical concepts because the thoughtful integration of community contexts, resources, and language promotes connections between students' lived experiences and the mathematical content addressed in the classroom setting, thus contributing to meaningful, high-quality instruction (NCTM, 2014).

To further complicate existing disparities in mathematics education, many mathematics teachers and PSTs dismiss issues of equity as relevant factors in the mathematics classroom because they view mathematics as a universal, culture-free subject (Jackson & Jong, 2017; Rousseau & Tate, 2003). There is a growing body of mathematics education researchers, however, who understand that mathematics and mathematical knowledge are neither universal nor culturally neutral but are situated in a sociocultural framework (Turner & Drake, 2016; Ukpokodu, 2011). Specifically, Gay (2000) argues that if we "decontextualiz[e] teaching and learning from the ethnicities and cultures of students [it] minimizes the chances that their achievement potential will ever be fully realized" (p. 23). To move PSTs beyond viewing mathematics as a "neutral" subject, work in teacher education programs must purposefully address this point of view by reinforcing how mathematics "reflects particular cultural and sociopolitical contexts, and ways of knowing" (Turner & Drake, 2016, p. 32). More specifically, mathematics teacher educators must implement pedagogical practices that not only encourage PSTs to reflect upon their perceptions of the subject area, but it is also essential that mathematics teacher educators promote meaningful learning experiences by emphasizing explicit strategies for connecting mathematical concepts to students' lived experiences.

Reducing the opportunity gap in mathematics education is possible by transitioning to an equity-centered paradigm. Teaching mathematics from an equity stance requires teachers to understand that students from diverse backgrounds come into the mathematics classroom with different worldviews. Teachers must be willing to move beyond teaching mathematics from a Eurocentric viewpoint and impart equitable pedagogical practices by building relationships, setting high expectations, and helping students maintain their identities (Ladson-Billings, 1995; Matthews, 2005). According to Irvine (2003), in order for all students to be successful, teachers must make connections "between subject-area content and their students' existing mental schemes, prior knowledge, and cultural perspectives" (p. 47). Furthermore, teachers need to "value the cultural and lived experiences of all children … [and emphasize] the belief that all children possess strong intellectual capacity and bring a wealth of informal, out-of-school knowledge to the teaching and learning process" (Lemons-Smith, 2008, p. 913). Although mathematics education researchers recognize what needs to be done to teach mathematics to diverse populations of students (e.g., Leonard & Martin, 2013; Pinnow & Chval, 2014; Vomvoridi-Ivanović and Chval 2014), mathematics methods courses must better prepare PSTs for their role in creating opportunities that provide students with equitable access for learning high-quality mathematics (Aguirre, et al., 2012). Hand (2012) documents the importance of the relationship between teachers' instructional practice and their dispositions toward equity. Therefore, it is imperative for mathematics teacher educators to prepare PSTs by promoting an equity-centered paradigm, which begins by implementing activities in mathematics methods courses that encourage PSTs to critically examine their current beliefs supplemented by discussions that explicitly address the elements of teaching mathematics through an equitable lens.

The purpose of this chapter is to examine what PSTs attend to in a classroom vignette of a student who is above grade level in mathematics and exhibits disruptive behavior during instruction. More specifically, the research question underlying this study is what do PSTs notice when the student in the vignette is an African American male, White male, African American female, and White female? By attending specifically to race and gender in this study, we can explore whether PSTs demonstrate (a) an equity-centered orientation toward mathematics instruction or (b) deficit views of students based on race, gender, or the intersection of the two. The focus on what PSTs attend to in classroom situations and a purposeful analysis of how PSTs interpret teachers' actions provides a mechanism that contributes to the field in two major ways. First, the empirical data adds to the research by identifying what is most salient to PSTs in the classroom and how what PSTs attend to intersects with the equity-centered paradigm. Secondly, with this knowledge, mathematics teacher educators can begin valuable work to disrupt existing inequities in students' learning experiences by opening critical dialogue that highlights how our perceptions and actions can either enhance or hinder students' access to high-quality, rigorous mathematics instruction.

Equity Noticing Framework

Gutiérrez (2007, 2009, 2013) argues that equity includes four dimensions: *access* (i.e., resources that provide students an opportunity to learn rigorous mathematics), *achievement* (i.e., students' outcomes, students' scores on standardized and non-standardized mathematics assessments, students' participation), *identity* (i.e., drawing on students' cultural frame to see themselves and the broader society in mathematics), and *power* (i.e., social transformations such as voice, who is being privileged, and using mathematics to read and change the world). To examine how PSTs perceive classroom situations and identify what they notice – or attend to – during the teaching and learning process, we integrated Gutiérrez's (2013) four dimensions of equity within van Es and Sherin's (2002) three main components of noticing – attention, interpretation, and implementation/instructional decisions (see Fig. 16.1). The structure of the Equity Noticing Framework provides a lens for researchers to investigate how each dimension of equity is or is not evident within each level of noticing.

Classroom episodes are complex; it is inevitable that individuals choose, consciously or subconsciously, what they notice, or attend to, and use the interpretations of these events to make instructional decisions (Sherin, Russ, Sherin, & Colestock, 2009). Teachers may attend to equitable issues in the classroom as a process or as a product (Gutiérrez, 2002; Martin, 2003; Rousseau & Tate, 2003). Seeing equity as a process means treating all students equally, without regard to race, ethnicity, or economic background. On the other hand, seeing equity as a product means differentiating instruction based upon students' needs; implementing equitable approaches that are respectful of students' ethnic, racial, and eco-

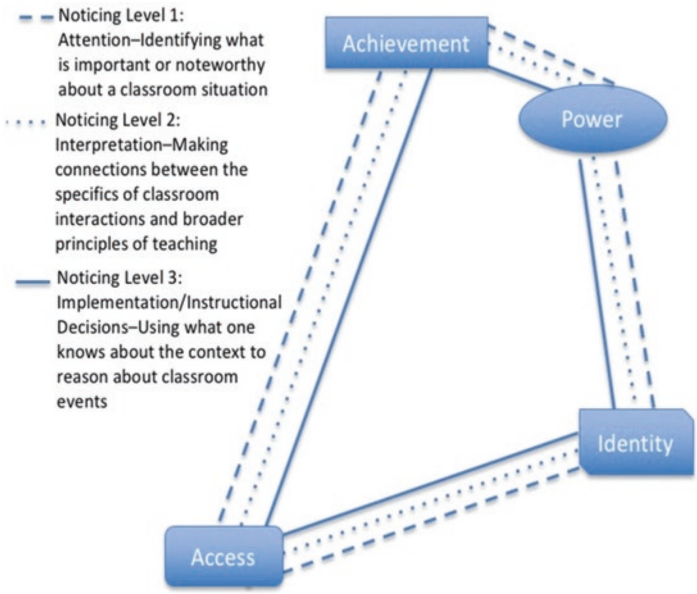


Fig. 16.1 The equity noticing framework

nomic background; and promoting equal learning outcomes. According to Gutiérrez (2013), when teachers have equitable practices, the power of predicting students' outcomes (e.g., participation, achievement, ability) is not based on students' background, race, class, or gender but in how students are positioned and the power they are given in the mathematics classroom. In other words, "[students are] both greatly influenced by and greatly influence the taken-for-granted rules and institutions in mathematics education" (Gutiérrez, 2013, p. 46). This approach to teacher education emphasizes the importance of being responsive to students from diverse backgrounds by developing a mindset in teachers that attends to preK-12 students' cultural capital (Averill et al., 2009; McCulloch, Marshall, & DeCuir-Gunby, 2009).

In the classroom, many things happen simultaneously, and the teacher must decide what deserves immediate consideration (van Es & Sherin, 2006). Noticing is a culturally shaped perception, which may include preferences and biases. Ball (2011) claims the noticing required in teaching is highly specialized because teachers must be attuned to notice things such as existing knowledge, potential difficulties, and learning trajectories, as well as how cultural capital contributes to students' interpretations. Therefore, it is vital that teachers acknowledge and effectively address how cultural experiences influence their perceptions and interpretations of students and classroom events. van Es and Sherin (2002) argue there are three main components of noticing: (1) identifying what is important about a situation (attention), (2) making connections between the specifics of classroom interactions and the broader principles of teaching and learning they represent (interpretation), and (3) using what one knows about the context to reason about classroom events (implementation/instructional decisions).

The first level of noticing (attention) involves the ability to focus one's attention on what is significant. In other words, teachers decide what deserves immediate consideration (van Es & Sherin, 2006) while simultaneously regulating what dimensions of equity – access, achievement, identity, power – are valued in their practice. Although what is noticed in a classroom episode might not be understood as an explicitly reasoned choice, the second level of noticing (interpretation) also underlines the role of active reflection, which involves using knowledge of one's context to reason about events that occur. During the reflection process, the teacher makes connections between specific events and the broader principles they represent. For example, when encouraged to extrapolate from the specific to the general, teachers form connections between the particular instances they observed and the broader pedagogical issues such events may represent (van Es & Sherin, 2010). Consequently, it is vital to examine how PSTs perceive students and interpret classroom events with respect to the four dimensions of equity (i.e., access, achievement, identity, and power).

The third level of noticing (implementation/instructional decisions) includes teachers using knowledge about their students, curriculum, and school context to reason about events that occur while simultaneously attending to the four dimensions of equity as teachers make instructional decisions. For example, when teachers provide students with opportunities to engage in high-quality, appropriate tasks, students are given access to learn rigorous mathematics and feel a sense of empowerment as they develop the strategies and skills necessary to use mathematics to change the world. Moreover, students' identities are shaped by agentive opportunities to participate. It is the role of the teacher to develop experiences and guide discussions so all students, particularly those from non-dominant backgrounds, have opportunities to develop strong mathematical identities. Conducting an empirical study that uses the equity noticing framework to focus on what PSTs attend to in classroom situations can add to this research and is critical to disrupt existing inequities in students' learning experiences.

Method

In this chapter, we describe an activity used in our elementary mathematics methods course that was designed to encourage PSTs to face existing (and often hidden) bias in order to alter unproductive beliefs and consequently broaden their ways of seeing. We view this as an essential activity to help PSTs (a) develop an awareness of equity, (b) define what equity means in classroom instruction, and (c) implement equity practices within the mathematics classroom. We examined what PSTs noticed about students' mathematical thinking and its relation to culture, home, and community.

PSTs from four different universities in the southeast, Midwest, and northeast sections of the United States participated in the study. All participants were within 1 year of student teaching. The demographics of the PSTs reflected demographic patterns of elementary education majors at our universities and included 92.2% White females, 3.7% White males, 0.3% Latino, 1.4% Asian females, and 2.7% African American females. These demographics continue to support the existing data that a less diverse population of PSTs is facing an ever-increasing diverse population of students in our public school systems (Hussar & Bailey, 2016).

The Vignette

To access PSTs' thoughts and ideas on issues related to equity in the mathematics classroom, we used classroom vignettes to identify what they noticed. Each vignette represented a different scenario that potentially challenged students' equitable access to high-quality mathematics. The five authentic topics focused on the following teachers: (a) one who exhibits gender bias for class participation, (b) one with preconceived biases that cause him/her to withdraw from students who are from different backgrounds than himself/herself, (c) an instructor who does not take time to develop relationships with his/her racially diverse students, (d) one who recommends a new English learner with limited English proficiency for special education services without adequately assessing the student's content knowledge, and (e) a teacher who is frustrated with a student who is above grade level in mathematics and exhibits disruptive classroom behavior. For this chapter, we describe what the PSTs attended to in their responses to the vignette that involved Eric, an African American male, who was above grade level in mathematics but was disruptive during mathematics instruction (see Fig. 16.2) because the case provided the most information regarding how PSTs attended to students' actions and mathematical thinking and its relation to culture, home, and community.

Case: Eric

Eric, an African American third grader, is a good-looking nine-year old boy who was retained in first grade. He is below grade level in reading and language, but above grade level in mathematics. He has been removed from his home because of abuse and is being raised by a caring grandmother. During mathematics lessons, Eric harasses the other children. He closes their books while they are working, knocks their books and pencils off their desks, writes on their papers, and crumples up their work. He does his best to disrupt the lessons by making gestures and deliberately making noises. Ms. Maben, his teacher, is frustrated.

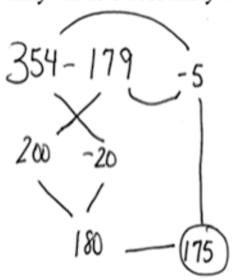
Reflection Questions:

1. If you were the teacher in this situation, how would you respond?
2. What are the implications for elementary teachers?
3. Additional comments or thoughts?

Fig. 16.2 Vignette of Eric and corresponding reflection questions

Case 3: Eric

Eric, an African American third grader, is a good-looking nine-year old boy who was retained in first grade. He is below grade level in reading and language, but above grade level in mathematics. He has been removed from his home because of abuse and is being raised by a caring grandmother. During mathematics lessons, Eric harasses the other children. He closes their books while they are working, knocks their books and pencils off their desks, writes on their papers, and crumples up their work. He does his best to disrupt the lessons by making gestures and deliberately making noises. When he solves mathematics problems, he always solves them a different way than how Ms. Maben, his teacher, taught. For example, Eric was given the following problem: Ricky has 354 markers. He gave 179 to Tommy. How many markers does Ricky have? To solve the problem Eric did the following:



Ms. Maben is extremely frustrated with Eric.

1. What are your thoughts on the above scenario?
2. If you were the teacher in this situation, how would you respond?
3. What was Eric's mathematical thinking?
4. Additional comments or thoughts?

Fig. 16.3 Modified Eric[a] vignette

After conducting an initial analysis of PSTs' responses on the Eric vignette, we wondered if we would have the same results if Eric was White and if Eric was changed to a female, Erica, who was White or African American. Consequently, we administered a modified version of the Eric vignette to our students the following semester. In the modified version, we included an example of the student's strategy for solving a mathematics problem to further emphasize and provide PSTs an opportunity to examine a student's mathematical thinking, and we varied the demographics of the student by race (African American or White) and sex (male [Eric] or female [Erica]) while keeping all other characteristics and descriptions constant (see Fig. 16.3). We noticed a dichotomy of what the PSTs perceived between Eric[a]'s mathematical understanding and his/her disruptive behavior during a mathematics lesson.

Data Collection and Data Analysis

To access the PSTs' thoughts and ideas on issues related to equity in the mathematics classroom, we used classroom vignettes that represented situations that potentially challenged students' equitable access to high-quality mathematics.

Data Collection The PSTs (n = 180) were asked to respond to the five vignettes described earlier that focused on authentic topics for pedagogical discussions related to teachers facing equity issues. To help ensure all PSTs fully participated in the class discussion, each PST independently read and responded to corresponding reflection questions for each vignette. Using a modified version of "jigsaw," a cooperative learning structure (McGuire, 2006), the PSTs were divided into five groups, where each group was randomly assigned a specific vignette and asked to discuss their responses, which were audio-recorded and transcribed. The PSTs recorded their thoughts to the guiding questions that accompanied their case on chart paper, and an expert was selected from each small group to share their results with other members in the class. Everyone from each group, except the expert, rotated from vignette to vignette and listened to the "expert" report the group's analysis of the situation before the PSTs shared their individual thoughts pertaining to the events in the vignette. At the conclusion of the discussion, the PSTs recorded their thoughts and ideas on post-it notes that were placed on chart paper. Once the PSTs finished rotating to each of the other four groups, they returned to their original group where the expert shared what he/she learned from the discussions with the other groups. Finally, the PSTs shared their thoughts and ideas on each case within a large group discussion.

In order to accurately draw conclusions on whether the PSTs' responses were based on Eric's race (i.e., African American), we modified the vignette and randomly assigned it to our PSTs. The PSTs (n = 114) at three universities were randomly assigned three vignettes (i.e., White Eric, White Erica, African American Erica) as comparison cases to the vignette with African American Eric, which was previously administered. The PSTs individually responded to the prompts and discussed their responses in a small group setting with those who were assigned the same student – White Eric, White Erica, or African American Erica. The PSTs recorded their thoughts and ideas that were most salient on a chart paper, which was shared with the class. During a whole group discussion, the PSTs identified the similarities and noted key differences within and across the responses to each vignette. Thus, the data collected for this study included PSTs' individual responses to the prompts, collective chart paper responses, and class discussion.

Data Analysis To analyze the data for evidence of the three levels of noticing across the four dimensions of equity, we (i.e., each author) independently coded the data and wrote analytical memos. We met throughout the process to verify that the coding was consistent and to resolve any differences, first through refining the code

Table 16.1 Coding dictionary

	Access	Achievement	Identity	Power
Noticing level 1: *attention*	The task was to find the difference between 179 and 354	"He is above grade level in mathematics." "The child has the correct answer"	"Why does it matter specifically that he is white and good-looking?"	"Erica harasses the other children"
Noticing level 2: *interpretation*	"If he is truly above grade level in mathematics, and he is acting out during the math lessons, she needs to give him harder assignments"	"I personally think that her way of thinking is crazy yet intriguing. She subtracted each place value and then took the remaining numbers and subtracted those. Erica took 300 − 100 to get the 200; she took 50 − 70 to get −20; and she took 4 − 9 to get −5. After those steps, she took the 200 − 20 to get 180 and then 180 − 5 to get a final answer of 175"	"Erica has a mathematical mind and is capable of expanding her knowledge much further at this time." "Erica has some serious problems. It sounds like it starts at home and continues over to school"	"If Erica is held accountable for new and more challenging learning, she may sustain from causing more trouble in the classroom." "This took me about 10 min to figure out, which is why it should not be recommended for students to use"
Noticing level 3: *implementation/ instructional decisions*	"I would try to provide work for Erica that provides a challenge for her. I would also see if there was a more advanced math class that she could switch to"	"I would make sure I am not attacking him but point out that his disruption is robbing his classmates of a chance to learn, as well as himself." "I would put Eric in a place by himself, so he cannot harass anyone"	"I would have Erica be my special math scientist. I would have her help students when she is finished. She can be like my STA (student teacher assistant)"	"If I wanted her to do the problem the way I taught it, then I would sit down with her and ask her how she solved it and then tell her that I would like it to be done the way I taught it"

Note: All of the codes in each cell are from the data except noticing level 1/access. There was not any data for this cell, so we included a representative code

definitions and then through discussions of the coding until agreement was reached (see Table 16.1 for coding dictionary). We then quantified the number of "referring responses" (total $n = 632$), which represented the number of references the PSTs made for each coding category (i.e., noticing level 1: attention/access).

Results

In this study, we examined what PSTs noticed and attended to when the student in the classroom vignette was an African American male, White male, African American female, and White female. In this section, we discuss what the PSTs attended to within each of the three levels of noticing and across the four dimensions of equity. In addition, we not only identify what PSTs noticed in relation to Eric[a], but we also describe how these findings were similar or varied depending on the race and sex of the student.

Overview

When looking across the data, the findings demonstrated how the PSTs' responses corresponded to the three levels of noticing and how the percentage of total responses focused on each dimension of equity (see Table 16.2).

The first level of noticing – attention – had the fewest references, with only 8.2% of the responses, while about 39% of the PSTs' responses were in level 2 noticing – interpretation. The majority of the responses from the PSTs focused on their instructional decisions, with over half of the references corresponding to level 3 noticing – implementation. While these data are interesting, it is not surprising given that one of the prompts specifically asked the PSTs "what would you do if you were Ms. Maben?" Consequently, these results also serve as a continued reminder

Table 16.2 Prospective teachers' level of noticing within the dimensions of equity

	Access	Achievement	Identity	Power	**TOTAL**
Noticing level 1: attention	*0* *0%* *0%*	*40* *19.0%* *76.9%*	*1* *0.9%* *1.9%*	*11* *5.3%* *21.2%*	**52** *8.2%*
Noticing level 2: interpretation	*21* *20.2%* *8.4%*	*119* *56.7%* *47.8%*	*26* *23.2%* *10.4%*	*83* *40.3%* *33.3%*	**249** *39.4%*
Noticing level 3: implementation/ instructional decisions	*83* *79.8%* *25.1%*	*51* *24.3%* *15.4%*	*85* *75.9%* *25.7%*	*112* *54.4%* *33.8%*	**331** *52.4%*
TOTAL	**104** *16.5%*	**210** *33.2%*	**112** *17.7%*	**206** *32.6%*	**632**

Note: Purple percentages indicate percent of the dimensions of equity, while the blue percentages indicate the percent of the levels of noticing

Table 16.3 Noticing level 1: attention – what is noteworthy?

	Access	Achievement	Identity	Power	TOTAL	TOTAL
African American ERIC	*0* *0%* *0%*	*9* *22.5%* *81.8%*	*0* *0%* *0%*	*2* *18.2%* *18.2%*	***11*** *21.2%*	***39*** *75%*
African American ERICA	*0* *0%* *0%*	*19* *47.5%* *67.9%*	*0* *0%* *0%*	*9* *81.8%* *32.1%*	***28*** *53.8%*	
White ERIC	*0* *0%* *0%*	*2* *5%* *66.7%*	*1* *100%* *33.3%*	*0* *0%* *0%*	***3*** *5.8%*	***13*** *25%*
White ERICA	*0* *0%* *0%*	*10* *25%* *100%*	*0* *0%* *0%*	*0* *0%* *0%*	***10*** *19.2%*	
TOTAL	***0*** *0%*	***40*** *76.9%*	***1*** *1.9%*	***11*** *21.2%*	**52**	

Note: Purple percentages indicate percent of the dimensions of equity, while the red percentages indicate the percent of the student's demographics

that the tasks teacher educators provide PSTs – and the questions that are posed – may implicitly, or purposefully, guide what PSTs attend to and subsequently learn.

Across the three levels of noticing, the majority of the references corresponded to achievement (33.2%) and power (32.6%). This result implies that PSTs are focused on student outcomes, such as the correctness of a student's mathematical response, as well as the opportunities for, and the hindrances to, classroom participation. Moreover, these data suggest PSTs seem to view the role of the classroom teacher as one who maintains order and control, which often reflects privilege, and in some cases stifles the student's voice in the classroom. Throughout the total responses, the PSTs' attention to developing tasks (access) and reflecting on drawing upon Eric[a]'s identity was less prominent.

In the subsequent sections, we provide a detailed explanation of each level of noticing and discuss how the four dimensions of equity are addressed. Specifically, we describe the disparities that emerged across race and sex to answer the research question underlying this study: What do prospective teachers notice and attend to when the student in the vignette is an African American male, White male, African American female, and White female?

Noticing Level 1: Attention – What Is Noteworthy?

The majority of references (76.9%) in the first level of noticing are related to achievement (see Table 16.3), which seems to imply that PSTs mainly focused on outcomes and answers. While references to power within this level occurred less often (21.2%), the responses were generally related to classroom management and behavioral issues (e.g., "Erica harasses the other children"). And, of the 52 total references at noticing level 1, only one of the PSTs' responses corresponded to identity (1.9%), while none referenced access.

Achievement Of the 40 references that corresponded to achievement, 37 stated either Eric[a] was above grade level or had the correct answer. Furthermore, only in the responses for Erica did the PSTs indicate the student had the correct answer, whereas when PSTs discussed retention and limited reading skills, these responses were only in reference to African American Eric. Moreover, when closely examining discrepancies between races, 70% of the PSTs' references in achievement were for the African American students. While this may seem arbitrary, it leads us to question if this finding implies that PSTs are more inclined to notice and attend to achievement for African American students when they were identified as high achievers in mathematics. It appears that PSTs expect mathematical success for White males (5% of the achievement responses); therefore, the fact a White male was above grade level in mathematics seems "normal" to PSTs, whereas the success was more "noteworthy" for all of the other students.

Identity The only identity reference at this level was for White Eric, and the PST stated, "Why does it matter that he is white and good-looking?" By asking this question, the PST dismisses the socialization of race and the treatment of students based on looks. It has been documented that nicer-looking people are treated more favorably than those who are not (Olson & Marshuetz, 2005), and often the attractive student's misbehaviors are overlooked, or they are given a lighter consequence.

Power References attending to privilege, position, control, and voice comprised 21.2% of the responses. Examining the disaggregated responses more closely reveals that all of the power references were for African American Eric[a]. In these responses, the PSTs attended to and recorded the description that the student was harassing other students or was disruptive by throwing paper and closing books.

Noticing Level 2: Interpretation – Connections to Principles of Teaching and Learning

At the noticing level 2 – interpretation – more than 70% of the references were for the African American students (see Table 16.4), and within each race the number of references for females was about twice as much as the males. Females also dominated the responses in achievement with 73.1% of the references. In addition, within this level of noticing, the references within achievement (47.8%) were the highest, followed by power (33.3%). This result implies PSTs were mainly focusing their responses on outcomes and interpreting the vignette through the lens of a dominant voice or control in the classroom, particularly for African American students.

Table 16.4 Noticing level 2: interpretation – connections to principles of teaching and learning

	Access	Achievement	Identity	Power	TOTAL	TOTAL
African American ERIC	*5* *23.8%* *7.1%*	*24* *20.2%* *34.3%*	*7* *26.9%* *10%*	*34* *41%* *48.6%*	***70*** *28.1%*	***178*** *71.5%*
African American ERICA	*11* *52.4%* *10.2%*	*58* *48.7%* *53.7%*	*15* *57.7%* *13.9%*	*24* *28.9%* *22.2%*	***108*** *43.4%*	
White ERIC	*2* *9.5%* *8.7%*	*8* *6.7%* *34.8%*	*2* *7.7%* *8.7%*	*11* *13.3%* *47.8%*	***23*** *9.2%*	***71*** *28.5%*
White ERICA	*3* *14.3%* *6.3%*	*29* *24.4%* *60.4%*	*2* *7.7%* *4.2%*	*14* *16.9%* *29.2%*	***48*** *19.3%*	
TOTAL	***21*** *8.4%*	***119*** *47.8%*	***26*** *10.4%*	***83*** *33.3%*	**249**	

Note: Purple percentages indicate percent of the dimensions of equity, while the red percentages indicate the percent of the student's demographics

Access Within access, 95.2% of the references indicated that the PSTs perceived the student was not challenged, and 75% of these responses were in reference to the African American students. For example, one PST noted, "It sounds like [African American] Eric is not being pushed hard enough and there might be an issue with the expectations of him." Similarly, another PST stated, "I believe the reason he [White Eric] is being disruptive is because the math lessons are probably too simple or easy for him." While the responses for Eric focused on why he was disruptive, responses centering on Ms. Maben's frustration, which inhibits student's access to rigorous mathematical tasks, were predominantly for the African American students.

Achievement Approximately 30% of the PSTs responses stated Eric[a] was bored, which may hinder the student's participation, as one of the reasons why he/she was causing disruptions. For example, a PST suggested, "[African American] Eric is very smart in the subject of mathematics so acting out may be caused by boredom." Another PST offered, "[African American Erica] is bored and that is why she only bothers other students during math class."

In relation to Eric[a]'s mathematical thinking, some of the PSTs understood the student's thinking and articulated how the student solved the problem. For example, a PST commented [African American] Erica used the "place value method" to solve the problem. The PST went on to explain the student's understanding:

> Subtracting 5 from 179 give 174 making it easier to compare to 354. Giving a −5. Looking at the ones & tens place it is easy to see that −20 is the difference between 54 and 74. Next comparing the hundreds place you can see that 300 − 100 is 200. Erica is left with 200, −20, and −5 which once added [is] 175.

Unfortunately, 45.9% of the PSTs' references revealed they did not understand the student's mathematical strategy for solving the subtraction problem. A PST articulated,

> I am unsure how she arrived at the solution. It is correct which also confuses me because I am uncertain how she got to the right answer. I understand how she got the (−5) but then I am unclear where the 200 and −20 come from.

Another PST stated, "I can't tell what her logic is with the math. I can't tell the beginning steps of her system." It is interesting to note that some of the PSTs interpreted African American Erica's mathematical thinking as an attempt to avoid the mathematics. None of the responses for African American Eric related to interpreting his mathematical thinking. Instead, the PSTs suggested African American Eric had a learning disability in mathematics and should be held back a grade level.

Identity Over 80% of the references for identity were for the African American students with a considerable focus on the difficult home life. For African American Erica, 11 of the 20 responses addressed her difficult home situation. For example, "She [African American Erica] has a lot of emotions and is looking for a way to solve them." On the other hand, it is interesting to note that two references identified African American Erica as one with a "mathematical mind." Moreover, out of the three references regarding the size of the student, two indicated African American Eric would be "much bigger" than the other students and interpreted his identity as that of a bully. For example, one PST stated, "I would talk to him about being a bully to other students."

Power About 70% of the references in power corresponded to the African American students. Moreover, these responses were more negative when compared to the responses for the White students. For example, a PST explained, "[African American] Eric would be a great student that could really benefit from being held back this year so that he could grow and learn to practice age appropriate behaviors and gain the reading skills he needs to move on." This PST's comment completely disregarded the fact that Eric was above grade level in mathematics. For White Eric, a PST commented, "Once he is engaged in the lessons, the disruptive behavior will stop."

The PSTs (45.7% of the referring responses) primarily focused on the need for the student to have attention and or control in the classroom. One PST stated, "I think [White] Erica is doing extra steps and going around the problem for extra attention. She knows the teacher is going to spend time talking to her if she does the problem wrong." In addition, the PSTs only indicated that African American Erica should be "held responsible for the actions."

It was reassuring, however, that 14 responses explicitly identified the need to empower or bring voice to Eric[a] by making adjustments to Ms. Maben's teaching. One PST mentioned, "there could be judgments about his [African American Eric] race that are unfair, just because he is being disruptive." While there were no explicit implications for instruction listed, these highlight that a small number of PSTs are

Table 16.5 Noticing level 3: implementation actions such as instructional or pedagogical strategies

	Access	Achievement	Identity	Power	TOTAL	TOTAL
African American ERIC	*39* *47%* *26.5%*	*22* *43.1%* *15%*	*44* *51.8%* *30%*	*42* *37.5%* *28.6%*	***147*** *44.4%*	***251*** *75.8%*
African American ERICA	*30* *36.1%* *28.8%*	*17* *33.3%* *16.3%*	*22* *25.9%* *21.2%*	*35* *31.3%* *33.7%*	***104*** *31.4%*	
White ERIC	*9* *10.8%* *25.7%*	*6* *11.8%* *17.1%*	*6* *7%* *17.1%*	*14* *12.5%* *40%*	***35*** *10.6%*	***80*** *24.2%*
White ERICA	*5* *6%* *11.1%*	*6* *11.8%* *13.3%*	*13* *15.3%* *28.9%*	*21* *18.8%* *46.7%*	***45*** *13.6%*	
TOTAL	***83*** *25.1%*	***51*** *15.4 %*	***85*** *25.7%*	***112*** *33.8%*	**331**	

Note: Purple percentages indicate percent of the dimensions of equity, while the red percentages indicate the percent of the student's demographics

beginning to interpret the situation as an area that can be altered by conscious pedagogical actions that address the needs of the students.

Noticing Level 3: Implementation Actions Such as Instructional or Pedagogical Strategies

The third level of noticing had the largest number (331/632, or 52.4%) of total references (see Table 16.5). In this level, the PSTs focused mainly on actions that would reflect power, control, and voice in the classroom (33.8%) and emphasis on pedagogical decisions that correspond to identity (25.7%) and access (25.1%).

Access Instructional actions that corresponded to access were those that related to tasks, resources, or tools that provide Eric[a] with opportunities to learn challenging and rigorous mathematics. Some of these actions benefited and challenged the student, while others potentially inhibited the student's access to high-quality and appropriately challenging mathematics. One PST stated the job of the classroom teacher is to "make sure that a student is being challenged and engaged on a level that is appropriate for them." Noticing and supporting the need to select appropriately challenging tasks for Eric[a] indicated that the PSTs did recognize increasing rigor, and stimulating the student's thinking was a viable strategy to quell the issues related to behavior. In fact, one PST wrote, "I would provide [African American Erica] with more challenging math concepts to deepen her thinking, hoping that will curve some of the disruptions." Another PST stated, "[African American] Eric needs to have access to more difficult problems to deepen his thinking."

While some PSTs stated they would give the student challenging problems, other PSTs commented they would give Eric more work, essentially "busy work." Responses referencing "busy work" seemed to overlook the possibility that Eric was not receiving appropriate assignments, and instead of reflecting on and exploring opportunities to stimulate the student's thinking through high-quality tasks that appropriately and effectively challenged Eric, the PSTs merely indicated the need to increase the quantity of the work. Decisions such as these serve as additional obstacles in providing access to high-quality mathematics instruction. While most of the references (66.3%) within access indicate actions that would benefit the student, such as providing more challenging work or including the student in advanced classes or grade levels, the remaining references either ignored Eric[a]'s mathematical needs (i.e., focusing on tutoring for literacy) or suggested providing one-on-one practice to make sure Eric[a] understood the mathematics.

Achievement Approximately 32% of the PSTs' responses referred to addressing classroom management to further support the student's participation and outcomes. While classroom management was a component of each of the vignettes, actions such as separating the student or isolating Eric[a] from the rest of the mathematical community were more commonly referenced for the African American students. In fact, of the 51 responses in achievement, 27 of these were specific references to removing African American Eric[a] from the group (17 males and 10 females). One PST expressed that African American Eric's behavior was such an obstacle that he should not continue with his peers to the next grade level. This is alarming because although Eric was noted in the vignette to be above grade level in mathematics, PSTs inhibited his participation in appropriate mathematical opportunities by disregarding his academic aptitude and instead focused solely on his behavior.

In looking at differences among males and females, the only times the PSTs referenced they would have the student share their thinking was in reference to Erica, and this occurred twice as much for African American Erica because the PSTs wanted to ensure she had the correct understanding of the problem. One PST suggested, "I would have [African American] Erica share her strategies with the rest of the class since they are so different."

Identity In the third level of noticing, the PSTs identified positive roles and positions for the student that would encourage him or her to identify as a contributing member of the classroom community. While not all of the suggested roles and responsibilities corresponded to Eric[a]'s identified success in mathematics, the responses did encourage the student to take a more leadership role in the classroom. For example, one PST expressed,

> I would make [African American] Eric the classroom leader and give him jobs to do around the classroom like having him turn the lights on and off and having him sharpen the pencils. This will make him feel more like he is a part of the classroom and hopefully it will make him more of a helper than someone who disrupts other students' work.

While references to African American Eric were the only ones that reflected roles and responsibilities that did not relate specifically to mathematics, references for African American Eric[a]'s leadership roles, like those for White Eric[a], regularly did involve offering roles to teach peers such as "pairing her up with a lower level math student," "allowing him to be the teacher," or inviting her "to be my special math scientist."

It was reassuring that the PSTs noticed the need to develop students' identities as mathematicians and positive members of the classroom culture by connecting with the student and providing systems of support. For example, one PST claimed, "students just want love and attention. This is why I think it is so important to create a caring community in the classroom. I truly believe that when students trust and feel loved and accepted, they will perform much better in school." Another PST stated, "I think the issues could be solved by just letting Erica know there is someone who cares." The PSTs also noticed the former abuse in Eric[a]'s past and recognized the student's stressful living situations can negatively impact a student and may be exhibited through defiant or inappropriate behavior. Consequently, several PSTs saw the need to create a place in which Eric would feel safe and secure. One PST articulated,

> I think that the student is acting up because of his home life and I have seen this many times. Yes, you should have clear, set rules in class, but as an educator you should take the time to see where the student is coming from and maybe provide resources such as counseling.

Despite the PSTs' reference to develop a caring classroom environment for all students, the PSTs identified the need to have culturally relevant pedagogy for all of the students except White Eric. A larger number of these connections were made for White Erica – by mostly White female PSTs.

Power Several of the PSTs' responses focused on positive pedagogical practices such as mentoring or personally connecting with the student. For example, family discussions with the grandmother to glean ideas that would support Eric[a] had the greatest number of references within power, and providing a positive role model/mentor was more focused on African American Eric. Moreover, recognizing the issues at home and considering referrals to the guidance counselor were also viewed as considering Eric[a]'s needs. Finally, putting a reward system in place that would provide positive reinforcement through praise and tokens was suggested by several PSTs. In each of these cases, these references were made more often for African American Eric[a], than for the White counterpart.

Despite these more positive interventions and instructional strategies, a large number of the PSTs' responses focused on "support" systems that could potentially disempower the student and cause Eric[a] to lose his/her voice in the classroom. Actions such as discrediting the student's voice by not allowing Eric[a] to solve the problem in the way that is both mathematically correct and makes sense to him/her were discouraging. For example, one PST explained that White Erica was only using this strategy to be defiant and more difficult. Responses like this are quite alarming because mathematics teacher educators not only want to prepare PSTs with a deep conceptual understanding of the mathematics but to also be prepared to accept multiple strategies as viable approaches to solving mathematical problems.

More worrisome is the disproportional distribution of these less beneficial practices for males and African American students. For example, none of the PSTs responses for African American Eric encouraged him to justify his mathematical thinking. In addition, the PSTs argued that the mathematical strategy Erica (both African American and White) used should align with the teacher's.

Discussion

We used the equity noticing framework to examine what PSTs attended to in a classroom vignette and how these observations, interpretations, and instructional decisions compared across race and sex. Because of the dichotomy the PSTs perceived of Eric[a], his/her mathematical knowledge and the disruptive behavior exhibited during the mathematics lesson, the responses to the vignette provided an opportunity to reveal hidden biases in PSTs' noticing.

The results of this study tell a very interesting tale of what PSTs noticed across the four dimensions of equity both within and among each level of noticing. Some of the most telling results included references to stereotypes within and across both race and sex. For example, when examining the referring responses by sex, a larger percentage of references were made when Erica shared her mathematical thinking, had the correct answer, and completed the problem as the teacher instructed, whereas a greater percentage of references for Eric corresponded to giving him "busy work" that was not cognitively demanding.

When examining disparities across race, the largest number of references across the four dimensions of equity and three levels of noticing is related to the African American students. For example, the PSTs attended to the African American students' achievement across all levels of noticing. This is particularly interesting because several references often indicated the PSTs' desire to remove the African American student from the classroom, which could result in the student falling behind due to lack of instruction. On the other hand, the PSTs' responses indicated they would remove White Eric[a] from his/her group but keep him/her in the classroom. The PSTs were more cognizant of and had more negative consequences for African American Eric[a]'s disruptive behavior. It was alarming that the PSTs stated African American Eric should be held back a grade level to learn appropriate behaviors.

Although the PSTs shared culturally relevant practices for the African American students, they were more adept to suggest culturally relevant strategies for White Erica. It was interesting that none of the responses related to culturally relevant pedagogy were for White Eric. This appears to indicate the PSTs did not think they needed to make explicit connections to White Eric's culture, which may be because White males are considered dominant in the Western society. Therefore, because of the privilege and dominance that exist for White males, cultural relevance is unnecessary because the connections are already being made for him.

Furthermore, the PSTs referenced all the students should be given a leadership role in the mathematics classroom. The leadership roles ascribed to African American Eric, however, were unrelated to mathematics, while the suggested leadership roles for the White students were directly and explicitly related to mathematics and mathematics instruction.

Finally, the PSTs alluded to the importance of the students having role models. This was more evident for African American Eric, which may indicate the stereotypical view that PSTs contend African American families lack male role models in the home. It is also interesting the PSTs did not state White Erica needed a role model or mentor. This could be because the PSTs may have viewed themselves as White Erica's role model since the majority of the PSTs were White females.

The results of this study provide an initial glimpse into what PSTs attend to, and through a comparative analysis of the demographic variation in the provided Eric[a] vignette, we documented how PSTs perceived issues of equity when a single classroom episode was represented where only the race and sex of the student were changed. Several results found in the study indicate distinct discrepancies – not only between races but also between males and females – which can be detrimental for particular populations if not explicitly and carefully addressed in teacher education programs.

The results from this study provide implications for mathematics teacher educators in that PSTs must have additional opportunities to discuss the impact of students' cultural identity and perceptions of themselves within the mathematics community as well as the resources, tasks, and tools that provide access to learn mathematics. Teachers cannot attend to everything that occurs in a classroom; thus they must make choices – whether consciously or subconsciously – about what they notice. In several responses, the PSTs directly and explicitly identified the student as the sole source of the problem or disruption that was occurring in the mathematics classroom without critically reflecting on how the teacher's actions or home events potentially contributed to Eric[a]'s outbursts. From these responses, it was evident that a large proportion of the PSTs first noticed the behaviors – even without explicitly identifying them in their written responses – and wanted to "deal with" Eric[a]'s actions. While this attribution may be expected of PSTs, it is important that mathematics teacher educators help PSTs focus on what they are and are not attending to in their instruction that may cause students' disruptive behavior.

The data from Eric[a]'s vignette provides an opportunity for mathematics teacher educators to reflect on how equity issues are addressed in their mathematics methods courses. Moreover, the results may prompt teacher educators to consider how to design tasks and activities that purposefully and explicitly address equity in mathematics education by supporting productive actions, while simultaneously bringing awareness to and challenging PSTs' stereotypes, hidden biases, and unproductive beliefs about students from diverse backgrounds. In order to create opportunities that foreground an equity-centered approach to mathematics teaching and learning, we, as mathematics teacher educators, must recognize what PSTs attend to as they direct their attention to various classroom events and how what they notice relates the events to broader principles of teaching and learning.

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Chapter 17
Using Concept Maps in Teacher Education: Building Connections Among Multiple Mathematical Knowledge Bases and Assessing Mathematical Understanding

Lynette DeAun Guzmán

Abstract A role of teacher education programs is to provide support for prospective teachers to develop professional skills that are specific to and required for teaching. Mathematics teacher educators may provide opportunities for prospective teachers to recognize and validate children's many ways of knowing mathematics, which is especially powerful for addressing the needs of students who are marginalized in mathematics classrooms. This study investigates how 20 prospective elementary teachers in a mathematics methods course made connections among children's multiple mathematical knowledge bases in their thinking about assessing children's understanding of fractions. This paper focuses on using concept-mapping tasks as a research tool and a pedagogical tool in supporting prospective teachers in addressing the needs of marginalized students through building stronger connections among concepts related to children's multiple mathematical knowledge bases, teaching practices, and mathematics content. Findings suggest that prospective teachers made more connections to children's multiple mathematical knowledge bases in their end of semester concept maps; however, these connections used more general language than specific examples of children's multiple mathematical knowledge bases. One potential entry point to better support these connections involves emphasizing high-level tasks as a concept related to teaching practices that connects to both children's mathematical thinking and children's lives and experiences. Considerations and implications for teacher education include a focus on the role of assessment to serve the needs of all students, especially those who have been historically marginalized in the mathematics classroom, and explicit attention to whose multiple mathematical knowledge bases are represented in concept maps.

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Introduction

Equity and social justice efforts in education and educational research are multifaceted. In mathematics education, some researchers have explored such efforts through supporting teachers to engage in equitable practices that recognize students' identities and knowledges (e.g., Aguirre, Mayfield-Ingram, & Martin, 2013). Mathematics teacher educators may provide opportunities for prospective teachers to recognize and validate children's many ways of knowing mathematics, which is especially powerful for addressing the needs of students historically marginalized in the mathematics classroom by narrow perspectives about mathematics teaching and learning (Aguirre, 2009; Gutiérrez, 2013).

One approach to support this goal in mathematics education research emphasizes the importance of eliciting and building on children's mathematical thinking in teaching mathematics by attending to students' multiple solution strategies (e.g., Carpenter, Fennema, Peterson, Chiang, & Loef, 1989; Jacobs, Lamb, & Philipp, 2010). Extending this focus on children's mathematical thinking, more recent research calls for incorporating children's home and community-based *funds of knowledge*—termed by Vélez-Ibáñez (1988) to refer to an array of historical and cultural knowledges and skills—in mathematics teaching to support student learning (e.g., Aguirre et al., 2012).

When teachers consider both children's mathematical thinking and children's home and community-based mathematical funds of knowledge, they ultimately have more resources to draw upon and inform their teaching practice to design meaningful classroom experiences that incorporate their students' knowledges and experiences both inside and outside the mathematics classroom (Aguirre et al., 2012; Vélez-Ibáñez & Greenberg, 1992). There is limited research, however, on how prospective teachers connect these two practices—assessing children's understanding of mathematics and incorporating children's mathematical thinking (Aguirre et al., 2012)—in the classroom. This work seeks to further understand how prospective teachers make sense of both of these practices together as part of a coherent effort to engage equity-oriented practices in mathematics education.

The purpose of this study is to explore prospective elementary teachers' (PSTs') thinking about connections between children's multiple mathematical knowledge bases and assessing children's mathematical understanding. In this chapter, I address the following research question: In what ways do prospective elementary teachers link concepts related to children's multiple mathematical knowledge bases with the root concept of *assessing children's understanding of fractions* during concept-mapping activities?

Literature

As González, Andrade, Civil, and Moll (2001) pointed out, humans, as sociocultural beings, are inseparable from their social worlds. Drawing on a diversity of histories and cultures can provide windows for some students to learn about people in the world and mirrors for some students to see themselves as people who engage with mathematics. Therefore, ways of knowing mathematics cannot be reduced to individual traits. Conceptualizing ways of knowing mathematics, Nasir, Hand, and Taylor (2008) argued that mathematics knowledge is fundamentally linked to cultural practices. Many scholars have also highlighted this argument through their studies of out-of-school mathematical practices (e.g., Carraher, Carraher, & Schliemann, 1985; Nasir, 2002; Saxe, 1988; Taylor, 2009). Researchers have also pointed to evidence that indicates all children, regardless of demographic groups, participate in multiple forms of mathematical play outside of the mathematics classroom (e.g., Parks, 2015). Mathematics, from these perspectives, is not limited to a formalized body of knowledge. Instead, this scholarship validates multiple practices and ways of meaning making involving mathematical ideas.

Mathematics teacher educators could provide opportunities for PSTs to highlight children's multiple ways of knowing mathematics through teaching practices that foreground commitments to equity in mathematics education, such as focusing on children's multiple mathematical knowledge bases (e.g., Aguirre et al., 2012; Turner et al., 2012) or culturally responsive assessment of children's learning. In the following sections, I briefly discuss this literature in order to situate an equity-oriented focus of this work in mathematics teacher education. Lastly, I examine research that uses concept mapping as a tool in teacher education—one that might function as both a learning tool for PST reflection and an evaluation tool for teacher educators.

Children's Multiple Mathematical Knowledge Bases

As noted above, mathematics education research has documented the importance of eliciting and building on children's mathematical thinking in teaching mathematics (e.g., Carpenter, et al., 1989; Jacobs et al., 2010). More recent research also calls for incorporating students' home and community-based mathematical *funds of knowledge*—defined as "historically accumulated and culturally developed bodies of knowledge and skills essential for household or individual functioning and well-being" (Moll, Amanti, Neff, & Gonzalez, 1992, p. 133)—in mathematics teaching to support student learning of mathematics with a broader range of resources (Aguirre et al., 2012; Turner et al., 2012). For example, Aguirre and colleagues (2012) call for drawing on children's *multiple mathematical knowledge bases (MMKB)* in mathematics instruction, where MMKB includes children's mathematical thinking and children's community, linguistic, and cultural funds of knowledge.

Considering children's MMKB provides more resources for teachers to draw upon to inform their instructional practice while aligning with a sociocultural perspective of teaching and learning mathematics. In turn, mathematics teacher educators might encourage and support PSTs to make connections among children's MMKB by foregrounding what students do both inside and outside their classroom communities. By building stronger connections to children's MMKB in mathematics instruction, teachers may support their students' learning in powerful ways by connecting students' lives and experiences to their mathematics classroom community.

Connecting children's MMKB in instructional practice might also support PSTs' engagement in mathematical practice. Recommendations in *The Mathematical Education of Teachers II* (Conference Board of the Mathematical Sciences, 2012) suggest that PSTs should have opportunities to engage in Common Core State Standards for Mathematical Practice (CCSSM, 2010) and "to *mathematize* situations by focusing on the mathematical aspects of a situation and formulating them in mathematical terms" (Conference Board of the Mathematical Sciences, 2012, p. 33). In particular, *MET II* connects mathematizing situations to Standard for Mathematical Practice 4: "Model with mathematics" (CCSSM, 2010, p. 7). These recommendations suggest that teacher educators should support PSTs' learning and teaching of mathematics while expanding their capacity to support children's engagement in mathematical practice and learning of mathematics.

Culturally Responsive Teaching and Assessment of Children's Learning

Culturally responsive teaching may be generally identified as "using cultural characteristics, experiences, and perspectives of ethnically diverse students" (Gay, 2002, p. 106) as resources in teaching practice. Culturally responsive teaching promotes ideas for *inclusive teaching*, which involves the "ways in which pedagogy, curricula and assessment are designed and delivered to engage students in learning that is meaningful, relevant and accessible to all" (Hockings, 2010).

Castagno and Brayboy (2008) claim that culturally responsive teaching provides multiple entry points and opportunities to learn for students and also encourages multiple forms of assessment for students, which maintains an equitable way for all students to exhibit their competence. As Moschkovich (2007) convincingly showed in her research about bilingual mathematics learners, using a sociocultural perspective toward assessment of student learning can help uncover student competencies and support teachers in considering how to build upon those competencies in mathematics teaching. This focus on competencies moves away from all-too-common deficit views of students' mathematical understandings (Gutiérrez, 2008).

Highlighting students' competencies is a productive way to frame formative assessment practices. Many scholars view *formative assessment* as a continual process in which adjustment in instruction is necessary to shape future instruction for positive gains in learning (Black & Wiliam, 2009; Popham, 2006; Shepard, 2005). Shepard (2005) noted that the purpose of formative assessment is to improve instruction "as teachers interact with students during the course of instruction" (p. 5). The short-cycle nature of formative assessments distinguishes it from benchmark or interim assessments because results may be meaningfully used for adjustments in how teachers teach and in how students approach their own learning

(Popham, 2006, p. 86). I hypothesize that mathematics teachers, then, through their various interactions with students, could draw on students' MMKB when designing and implementing a variety of formative assessments.

Formative assessment is not, of course, the only assessment in mathematics classrooms; standardized tests remain ever present. Considering children's MMKB also applies in this context. Solano-Flores and Nelson-Barber (2001), for example, argued that *cultural validity*, which refers to the "sociocultural influences that shape student thinking" (p. 555), is a key component to accommodate student diversity in standardized test design and implementation. Sociocultural influences include students' inherent cultural values, beliefs, experiences, discourses, and epistemologies. Full accommodation and sensitivity to students' sociocultural influences requires consideration at the beginning of test design and cannot be accounted for merely through language translation at the end of the test design process (Solano-Flores & Nelson-Barber, 2001). Designing assessments that do not marginalize or disadvantage students of a particular culture or language fluency is important for equitable learning opportunities (Lyon, 2013; Solano-Flores & Trumbull, 2003).

Using Concept Maps as a Research Tool in Teacher Education

Novak and Cañas (2007) defined *concept maps* as "graphical tools for organizing and representing relationships between concepts indicated by a connecting line linking two concepts" (p. 29). The research literature on concept mapping in teacher education has provided evidence that its use as a research tool is valid and robust (e.g., Beyerbach, 1988; Beyerbach & Smith, 1990; Hough, O'Rode, Terman, & Weissglass, 2007; Koc, 2012; Miller et al., 2009; Morine-Dershimer, 1993; Varghese, 2009; Wallace & Mintzes, 1990; Williams, 1998). For example, Miller and colleagues (2009) examined concept maps as a research tool by studying pre- and post-concept maps of 251 prospective and practicing teachers. The authors used a concept map scoring method (Novak & Gowin, 1984) and found that participants' concept map scores distinguished expert to novice levels in conceptual understanding and growth over time.

Scholars have also used concept map artifacts to examine teacher knowledge about mathematics content (e.g., Hough et al., 2007; Varghese, 2009; Williams, 1998) and knowledge about teaching skills (e.g., Koc, 2012; Miller et al., 2009; Morine-Dershimer, 1993). Beyerbach (1988), for example, found that PSTs reported that concept mapping is a useful and educative task that helps them reflect on the ways in which their understandings of topics change.

I follow Hough and colleagues (2007) to engage concept maps as a tool to examine changes in PSTs' connections and concepts related to a main idea: assessing children's learning of mathematics. In particular, I draw on these authors' methods to engage quantitative and qualitative analyses of the content in PSTs' concept maps. Additionally, I combine these analyses with PSTs' reflections on their growth in understanding about assessing children's learning of mathematics over the period of a semester-long elementary mathematics methods course. The purpose of this study is to explore PSTs' thinking about connections between children's MMKB and assessing children's mathematical understanding as equity-oriented practices in mathematics education.

Method

This study was conducted within the context of a larger research project, TEACH Math, that produced modules designed to support PSTs' integration of children's MMKB in their mathematics instruction (www.teachmath.info). Participants included 20 PSTs enrolled in an elementary mathematics methods course using these modules. All PSTs in this course were members of a specialized cohort program designed to prepare educators qualified to work in school communities where resources are most limited. PSTs in this cohort all took designated sections of required teacher education courses together during their bachelor's degree study and also engaged in field placements at local urban area schools. Participating PSTs reflected typical trends in terms of gender, race, class, and age—predominantly White, middle-class, females in their early twenties (Green, 2010). All PSTs created concept map artifacts and reflections during their regular class meeting time. The methods course instructor allowed me to conduct research during this part of the class and was not present during the facilitation of research activities. All names that appear are pseudonyms. I use "they" as a singular pronoun throughout this chapter to acknowledge that I did not ask participants for their preferred pronouns.

Data Generation

Given the intricacies of teaching practice, I selected concept mapping as a tool that provides PSTs with an opportunity to create a representation of their thinking. I used concept-mapping activities to examine evidence of how PSTs might link concepts in their maps together and what PSTs said in their written reflections about their maps. Based on work by Hough et al. (2007), we generated maps at the beginning (pilot activity) and at the end of the methods course.

Pilot Activity

I facilitated the beginning of semester concept map activity as a pilot to inform the design of the end of semester concept-mapping activities. First, I introduced and explained parts of a concept map to the whole group: *root*, which is the main concept on the map; *concept*, which is one idea depicted on a map by a circle or box; and *link*, which is a connecting line between two concepts (Fig. 17.1). During this

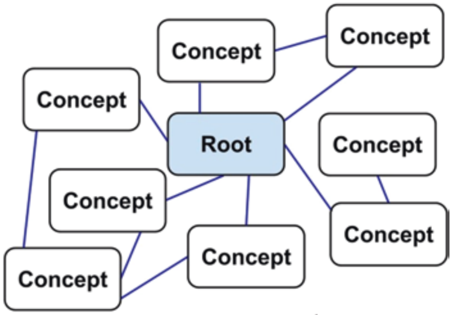

Fig. 17.1 Parts of a concept map with root, concepts, and links

introduction, I instructed PSTs to think about our concept map activity with a network structure—containing cycles and web-like interconnection—to encourage making connections among any related concepts on their maps.

After the introduction, I provided PSTs an opportunity to individually practice constructing a concept map with *fractions* as the root concept. I chose “fractions” as the root for this exercise because it was a major topic in the mathematics methods course in which they were enrolled. Then, I prompted PSTs to create a concept map to represent their knowledge about *assessing children’s understanding of fractions* with that main concept as the root. PSTs had the option to draw the map entirely by hand or to write single concepts on provided sticky notes to place on a sheet of paper. After creating concept maps, I asked PSTs to specifically think about content in their elementary mathematics methods course, such as *funds of knowledge*, and discuss potential concepts they might add to their maps. Finally, I asked each group to create a list of concepts that they would add to their maps after thinking about and having these discussions with their group members. They added items such as *real-world applications*, *open questions*, and *culturally relevant problems.*

Mathematics concepts were quite prevalent across concept maps in the pilot activity. Evidence also suggested PSTs had knowledge of concepts related to children’s MMKB, such as in the added items listed above. I used what I learned from this pilot activity to design the end of semester concept-mapping activities, which are the data analyzed and presented in this chapter.

End of Semester Concept Maps

Data sources for end of semester activities included individual concept maps, researcher field notes, and individual written reflections. During the last class meeting of the semester, I revisited concept map terminology (e.g., root, concept) and the concept map activity with the whole group (e.g., create a concept map for assessing children’s understanding of fractions). In this activity, I used the same prompt from the pilot activity for PSTs to create a concept map to represent their knowledge about *assessing children’s understanding of fractions* with that main concept as the root.

Edited Concept Maps

After drawing an initial map, I asked PSTs to have a focused discussion in their small groups about evidence of connections among children's MMKB in their concept maps. After a few minutes, I interrupted small group discussions to direct attention to specific concepts related to children's MMKB that I provided on a sheet of paper:

- Children's mathematical thinking
- Problem-solving strategies
- Making sense of students' mathematical ideas
- Students' personal experiences
- Students' interests and activities
- Students' home and community knowledge bases
- Regular routines
- Places in community
- Funds of knowledge (e.g., cultural, community, and linguistic resources)
- [Blank spaces for PSTs to fill in their own ideas]

I compiled this list of concepts based on my review of course materials (syllabus, assignments, and weekly course PowerPoint presentations) to match the language used within the course context. My purpose in providing a list was twofold: (1) for PSTs to engage in discussion about children's MMKB with familiar language from their methods course and (2) as a resource for PSTs to then make decisions in linking these ideas within their end of semester concept maps. I asked each group to review the list and determine if any of these concepts, including any concepts not on this list that they came up with as a group, connected to their concept maps. I also asked each group to discuss how they would change their concept maps to include any of the provided concepts. I designed this activity in two parts with different colored pens to distinguish and collect data for both parts (before and after focused group discussion). PSTs individually revisited their concept map to make any changes with a different colored pen. Finally, PSTs individually wrote a brief reflection (50–150 words) to a prompt about what *they* saw as "evidence" of children's MMKB in their map. Beyond connections, I wanted to also provide a space for PSTs to share with me their thoughts on what they saw as children's MMKB within their concept maps.

Data Analysis

Drawing on Hough et al. (2007), I used content analysis techniques to examine links and concepts within PSTs' concept maps by specifically looking for evidence of children's MMKB in their maps. I first coded all concepts in the maps using three initial categories based on my research question: *mathematical concepts*, *assessment*, and *children's MMKB* (see Table 17.1). I started analysis by coding five maps

Table 17.1 Final coding scheme with examples

Category	Definition	Examples
Mathematical concepts	Includes all concepts related to fractions or number and operations, more broadly	Numerator; reciprocal; common denominator
Representations and tools	Includes representations such as number lines, manipulatives, and examples of fractions	Number lines; pie charts; ½
Teaching practices	Includes examples of and concepts related to assessment, activities, tasks, and planning	High-level tasks; formal assessment; differentiation
Children's mathematical thinking	Includes examples and concepts related to students' prior knowledge, solution strategies, common understandings and misconceptions	Problem-solving strategies; seeing students' thinking; misconceptions
Children's lives and experiences	Includes examples and concepts related to children's funds of knowledge: linguistic, community, home, and cultural knowledge	Funds of knowledge; culturally relevant; relate to students' interests

at a time with this initial coding scheme, identifying any concepts that did not seem to fit well in any of the categories. Through multiple iterations, I refined my coding scheme to include emergent categories based on patterns from my analysis.

Mathematical concepts remained a category, *representations and tools* emerged as a new category, and I broadened the initial *assessment* category to include all concepts related to *teaching practices*—examples of and concepts related to assessment, activities, tasks, and planning. I decided to split the *children's MMKB* category (the shaded region in Table 17.1) to make a distinction between *children's mathematical thinking* and *children's lives and experiences* because I wanted to examine how PSTs represented both of these ideas in their maps. Finally, I used the PSTs' reflections as another source of data to examine how they saw evidence of connections to children's MMKB in their concept maps. After I analyzed each concept map with my coding scheme, I analyzed each reflection corresponding to each concept map to confirm some of the examples that PSTs saw as children's mathematical thinking or children's lives and experiences. I used these PST reflections to guide and frame the interpretations that follow.

Findings

In this section, I present examples of how PSTs linked concepts related to children's MMKB with the root concept of assessing children's understanding of fractions in their end of semester concept maps and reflections. Findings suggest that PSTs made connections to children's MMKB in their concept maps; however, these connections used more general language as opposed to specific examples of children's MMKB. Furthermore, these connections did not draw attention to historically marginalized groups. Additionally, I discuss high-level tasks as a frequently linked concept in PSTs' maps and reflections.

Linking to (and Across) Children's MMKB in Concept Maps

Across all 20 PSTs, there were 349 concepts in the end of semester maps before the discussion about children's MMKB and editing process. Fourteen PSTs linked concepts related to children's MMKB in their initial concept maps before the editing process. I identified 41 concepts (approximately 12%) related to children's MMKB across the concept maps of these 14 PSTs. Of these 41 concepts, I coded 32 concepts as children's mathematical thinking (e.g., numerator, reciprocal, common denominator) and 9 concepts as children's lives and experiences (e.g., funds of knowledge, culturally relevant, relate to students' interests).

In comparison, examining the edited end of semester maps, there were a total of 448 concepts across the group, and I identified 131 concepts (approximately 29%) related to children's MMKB. All 20 PSTs linked concepts related to children's MMKB in their edited concept maps, and 2 PSTs only included concepts related to children's mathematical thinking and not children's lives and experiences.

Unsurprisingly, facilitating a concept-mapping activity in two parts with probing for children's MMKB increased their inclusion in the maps. Thus, in my content analysis, I specifically investigated where PSTs linked these concepts in their edited maps. Across the group, over half of all links (52%) connected concepts related to children's MMKB to concepts I coded as *teaching practices* (i.e., examples of and concepts related to assessment, activities, tasks, and planning). In particular, 7 of the 20 PSTs made a direct link between children's MMKB and *high-level tasks*, which was the most commonly linked concept across all maps connecting to children's MMKB. For example, in Fig. 17.2 a PST connected *using high-level tasks to solve problems* with a link to *interests and activities* as a concept related to children's MMKB. Only 8% of the links connected concepts related to children's MMKB to concepts I coded as mathematical concepts.

Although PSTs made connections to children's MMKB, these connections used broad language and did not specifically attend to marginalized students. For example, Fig. 17.2 shows Harper's edited end of semester map with added concepts (in red pen written directly on the paper, not the sticky notes) *acknowledging strategies*, *interests and activities*, *students' home life*, *students' community*, and *personal experiences*. I selected Harper's map as a typical example from the group of PSTs. It is unclear from Harper's concept map what these concepts mean with regard to mathematics and with connecting children to mathematics.

In Harper's reflection about the concept map process, they wrote that they saw evidence of concepts related to children's mathematical thinking before editing the map, such as *number talks*, *seeing students' thinking*, and *high-level tasks* [using high-level tasks to solve problems]. After editing the map, Harper added more concepts related to children's lives and experiences:

> At first, the only discussion I had about mathematical thinking was in describing how we can use number talks to see students thinking, and allowing them to explore different strategies with *high-level tasks* [emphasis added]. Once I edited the map, I added things about students' home life, community, personal experiences, etc.

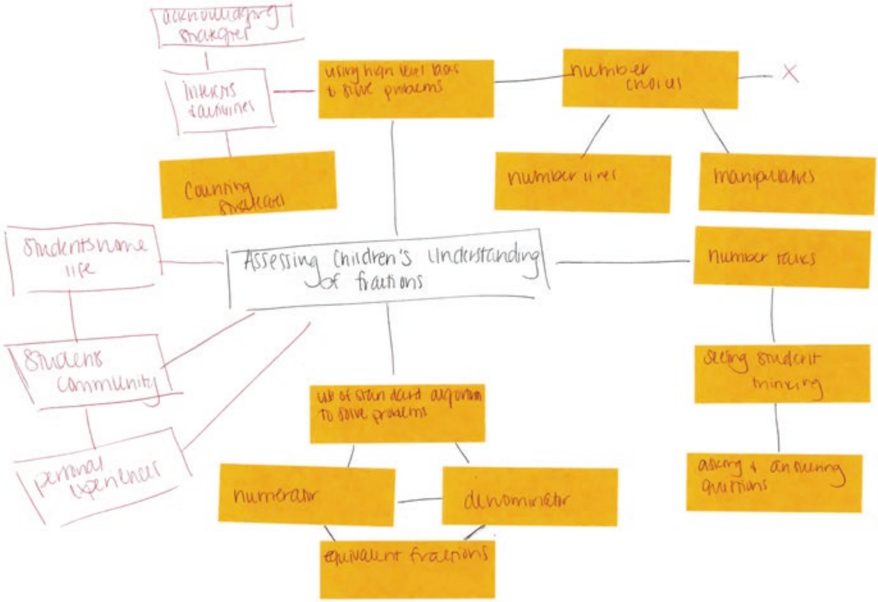

Fig. 17.2 Harper's edited end of semester concept map

Harper does mention a shift to focusing on students' funds of knowledge only after the discussion and editing process. Other PSTs mentioned thinking about funds of knowledge after discussion, which may be related to multiple ways of thinking about assessment and its role in teaching and learning mathematics. Harper initially highlighted eliciting students' mathematical thinking about fractions through number talks, an activity in which students participate in 15-minute conversations about computation problems to communicate their mathematical thinking (e.g., Parrish, 2010). Harper also described how high-level tasks may provide opportunities for students to explore different problem-solving strategies, which is also closely connected to students' mathematical thinking. In the following section, I provide examples of how PSTs reflected on how they thought high-level tasks were related to children's MMKB.

High-Level Tasks *as an Entry Point*

As previously stated, seven PSTs (35%) made a direct link between children's MMKB and *high-level tasks* to assess fraction knowledge. Avery, for example, marked *high-level task* as a concept directly connected to *assessing children's understanding of fractions* with a cluster of concepts also connected to it (Fig. 17.3). This concept was part of Avery's map before the editing process. After editing the map, Avery added *problem-solving strategies* to this cluster of concepts, which I

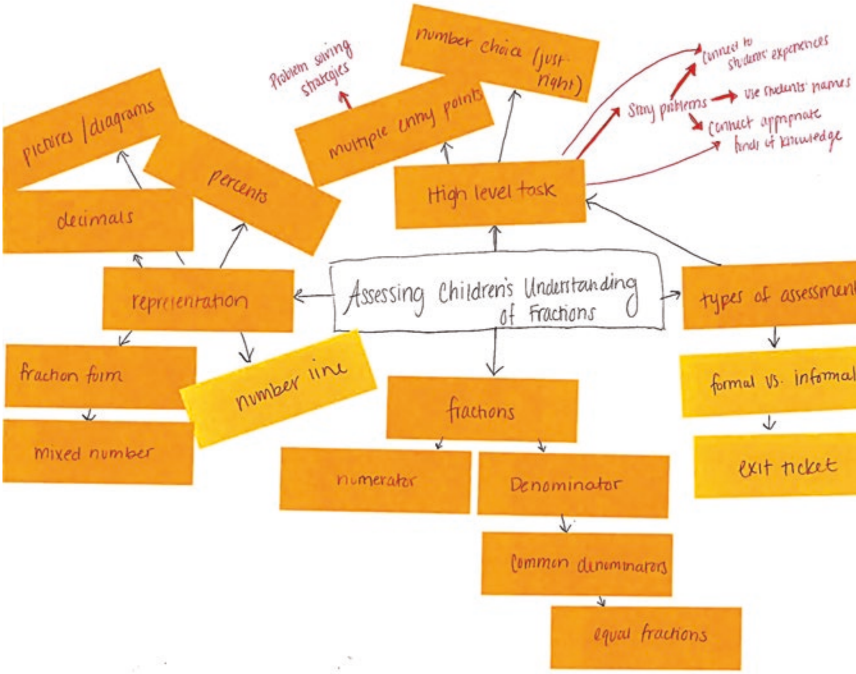

Fig. 17.3 Avery's edited end of semester concept map

coded as a concept related to children's mathematical thinking. Avery also added *connect to students' experiences*, *use students' names*, and *connect appropriate funds of knowledge* to this cluster, which I coded as concepts related to children's lives and experiences.

In the end of semester reflection, Avery wrote about seeing evidence of connecting children's MMKB to assessing children's mathematical understanding by using high-level tasks relevant to students:

> I made the biggest connection between developing *high-level tasks* [emphasis added] with relating students' experiences and funds of knowledge. When assessing students and how they think we need to make sure that all students relate to the problem and can understand the context or background of a problem. The material needs to be relevant to every child so that they can one day use their knowledge in the real world. Even something as small as changing the names in a story problem will increase student interest and motivation.

While Avery primarily elaborates on their thoughts about why connecting mathematics to children is important, Avery makes a connection to creating high-level tasks in this professional work. Avery also noted that it is important to acknowledge students' relationship to mathematical problems, including the problem context and background. Part of this relationship may be related to student motivation, but Avery focused on the potential utility of mathematics in students' lives outside of the mathematics classroom. It is worth noting, though, the specific example that Avery

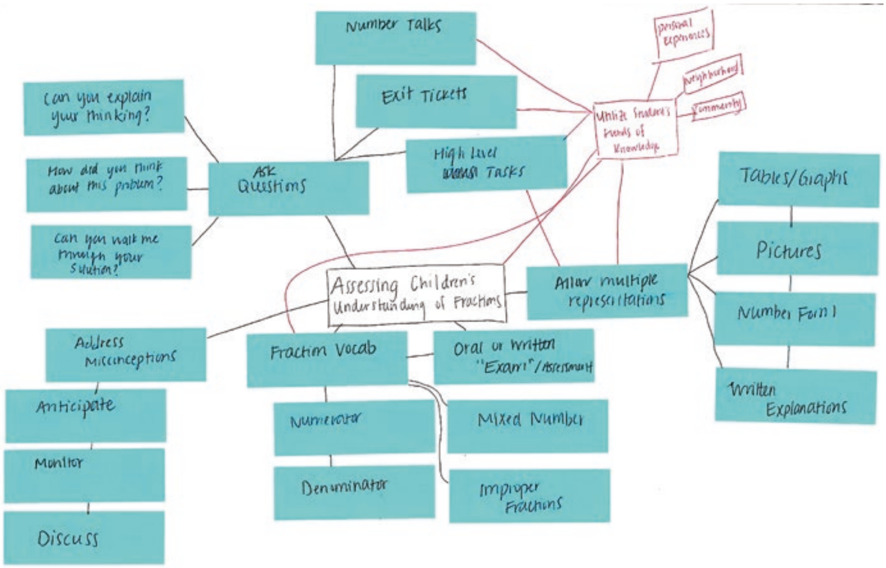

Fig. 17.4 Morgan's edited end of semester concept map

used of changing students' names in word problems as a space to further explore critical examinations of power and identity in mathematics curriculum materials. That is, PSTs might critically examine messages in word problems about how to do math, what math is for, and who it applies to (Felton, 2010), constructions of gendered subjectivity with mathematics (Hottinger, 2016), and representations of groups of people by race, class, gender, and ability (Sleeter & Grant, 1991).

Another example (Fig. 17.4) shows Morgan's end of semester concept map with connections to high-level tasks. In this edited map, *utilize students' funds of knowledge* is the lead concept of an added cluster directly connected to *assessing children's understanding of fractions*, *fraction vocab*, *number talks*, *exit tickets*, *high-level tasks*, and *allow multiple representations*. Additionally, Morgan added a direct link between *high-level tasks* and *allow multiple representations* in the edited map. In Morgan's reflection, they explained that *high-level tasks* and *multiple representations* connected to concepts related to children's MMKB:

> Creating *high-level tasks* [emphasis added] that provide students with multiple entry points into problems allowing them to think and use strategies that makes sense to them will tap into students' prior knowledge of what they already know and what strategies they are comfortable using. Allowing students to use *multiple representations* [emphasis added] also connects to their experiences of what types of strategies they have used in school before and what representations they prefer to use.

Morgan's reflection speaks to connections between high-level tasks and a range of students' mathematical knowledge and strategies or children's mathematical thinking. It is not clear, however, in what ways Morgan is making connections between high-level tasks and students' funds of knowledge with attention to community, linguistic, and cultural knowledges. This example demonstrates, however, how concept mapping, together with reflection activities, may have potential to provide spaces for PSTs to interrogate and clarify their thinking about teaching practices.

How Are We Thinking About Assessment?

In the end of semester reflections, 19 of the 20 PSTs wrote about what they saw as evidence of children's mathematical thinking and children's lives and experiences in their end of semester concept maps. For example, one PST, Robin, wrote in their end of semester reflection that "part of assessing involves *applying the students' funds of knowledge, experiences, and making math culturally relevant to them* [emphasis added]" and also noted how "[the concept map activity] was interesting to see how I view assessing students in comparison to their lives and experiences."

Additionally, Casey noted the lack of concepts related to children's lives and experiences in their map before the editing process. In Casey's reflection, they mentioned a primary focus on students' prerequisite knowledge and skills to learn fractions:

> When thinking about assessment I did not immediately think about things such as fund of knowledge. I thought about what types of skills my students would need in order to understand fractions. *As an afterthought* [emphasis added], things such as funds of knowledge are crucial when touching on these skills. They should apply to every skill.

Casey's reflection hints at the importance of funds of knowledge in understanding fractions but does not elaborate on specific reasons for why funds of knowledge matters for understanding fractions as a mathematical concept. In thinking about assessment, Casey described funds of knowledge concepts as more of an afterthought but, with discussion among peers, determined that these concepts are actually important for practice and added these concepts to their map during the editing process.

Taylor described a similar experience from the concept-mapping activities. After thinking about children's lives and experiences connected to assessing children's understandings during the small group discussion, Taylor added a chunk with lead concept *cultural relevance* connected to both assessment and content concepts:

> Once I started to think about students' lives and experiences in connection to assessing I realized that in my original map I did not include this. When I was able to add to the map I added an entire new concept of cultural relevance that I believe connects to all of the other ideas I already had on my map. I connected it to informal and formal assessments along with connecting it to the content that is being assessed, fractions.

Taken together, these examples from Casey and Taylor suggest a space for PSTs and teacher educators to explore connections between assessment and children's funds of knowledge and to strengthen these connections as more salient in our thinking about teaching and learning mathematics than just an afterthought.

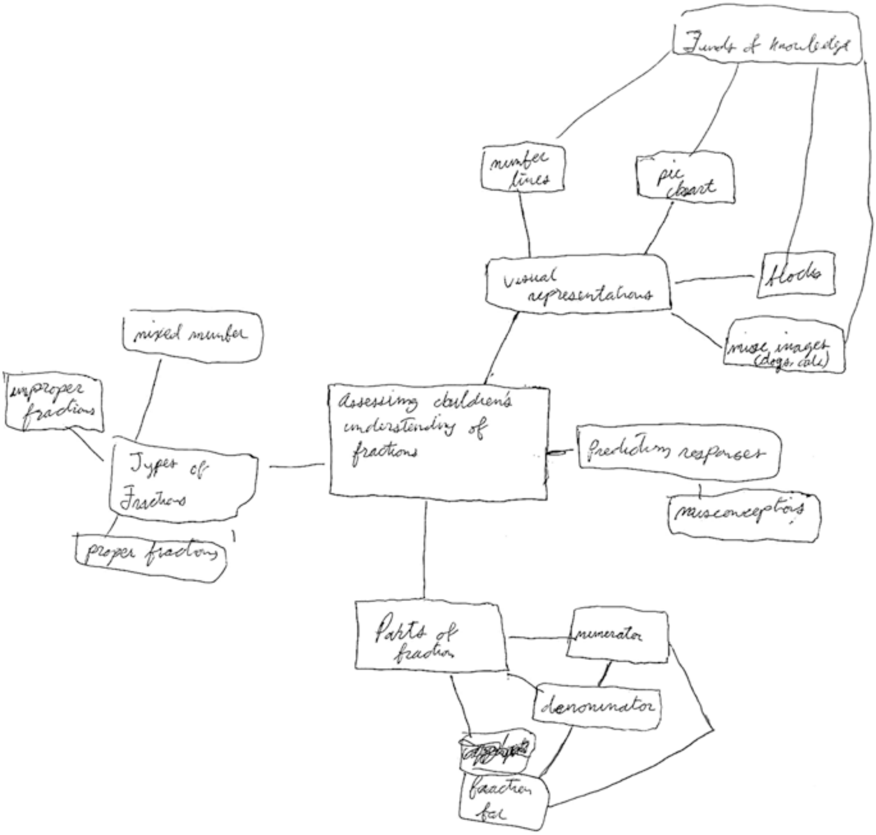

Fig. 17.5 Parker's edited end of semester concept map

Out of all the PSTs' reflections, Parker was the only participant who stated in their reflection that they did not include concepts related to children's MMKB in the end of semester concept map (Fig. 17.5).

Parker's reflection explained that they saw no evidence of concepts related to children's MMKB in the map because of the way they were thinking about assessing and about using funds of knowledge in mathematics teaching:

> I think there is no evidence because I thought of assessing in the pedantic sense. I thought that experiences of the children would go more along with the *actual teaching of concepts* [emphasis added] … I could add funds of knowledge to the concepts that I stated as being a part of a formal assessment.

I highlight Parker's reflection as an interesting case of being the only PST who stated they did not see children's MMKB in their map. While Parker was the only PST who stated this in their reflection, they are not alone in that opinion. I heard multiple PSTs voice similar thoughts from my field notes of small group discussions where PSTs talked about whether funds of knowledge were more directly related to the actions of teaching mathematics rather than assessment, which could follow the act of teaching. These discussions point to another opportunity for using concept mapping together with reflection activities to provide spaces for PSTs to interrogate and clarify their thinking about teaching practices.

Discussion

In this study, I explored how PSTs linked concepts related to children's MMKB with assessing children's mathematical understanding during concept-mapping activities. Broad language was common across the group of concept maps, which may be related to how PSTs engaged with the concept-mapping activity having a focus on big ideas and less emphasis on specific attention to how race, class, and culture interact with teaching practices that center equity efforts for historically marginalized groups. To be clear, I am not claiming that these PSTs are not making connections between children's MMKB and school mathematics. I am only pointing out that the concepts and links in their maps do not explicitly attend to marginalized groups of students, even though the PSTs in my study are part of a specialized cohort that strives to address issues of inequity in mathematics education. While it is possible that these PSTs have marginalized groups of students in mind, I make a point to recognize the ways that teacher educators may work to make the invisible more visible through invitations to attend to race, class, and culture in our thinking about teaching.

Concept-mapping activities may provide a space for PSTs and teacher educators to further explore these ideas in addressing the needs of marginalized students by exploring the nuances and complexities of teaching mathematics. From PSTs' concept maps, *high-level tasks* might be a possible entry point to build and strengthen connections between children's MMKB and assessing children's understanding of mathematics. In reflections, PSTs in this study commented on how they saw high-level tasks connected to children's MMKB, both in drawing on a range of students' mathematical strategies and in efforts to make mathematics more relevant to students' lived experiences.

Each PST did not use the same terms in the same ways from what they wrote and foregrounded in their reflections; while this might be framed as a problem, I see it as a valuable learning opportunity that is available throughout the semester. Teacher educators may support PSTs to critically reflect on their thinking about children's MMKB and the complexities involved with learning and teaching mathematics. For example, whose funds of knowledge are represented in our maps: all students, students with similar backgrounds and identity markers as us, or students with different backgrounds than us? Additionally, how do (or could) we see funds of knowledge as intricately connected to mathematical concepts?

Finally, I provided examples that suggest how PSTs may not initially consider children's MMKB as useful in *assessing children's understanding of mathematics* based on their own understandings of assessment. A possible explanation for this perspective is that although PSTs adapted existing tasks and curriculum materials, they might have limited experiences in designing assessments at this point in their program. One PST told me during the whole group discussion that they have experience adapting problems to align with students' needs, although they have not created summative assessments in the course. Consequently, this evidence made me wonder about how PSTs made sense of the phrase "*assessing children's understanding*" as the root of this concept map activity, and ultimately, how their understandings influenced the construction of their concept maps. Finally, how could these different understandings of assessment constrain and support our efforts to support the needs of marginalized students in mathematics classrooms? These questions may also provide learning opportunities in mathematics methods courses.

Implications and Directions for Future Research

Mathematics teacher educators must support PSTs to make explicit and stronger connections among concepts related to children's mathematical thinking, children's lives and experiences, and assessing children's understanding of mathematics. Concept-mapping activities could provide insight into how PSTs might link children's MMKB to specific mathematics content, assessing children's understanding, or other teaching practices. One entry point to better support these connections may be emphasizing *high-level tasks* as a concept related to teaching practices that connects to *both* children's mathematical thinking and children's lives and experiences.

From my analysis of concept maps and brief reflections, I found evidence that the PSTs in this study made connections between high-level tasks and children's mathematical thinking with more connections to children's funds of knowledge after focused discussions. Another implication for teacher educators is to be cognizant of a possible perception that funds of knowledge, including children's lives and experiences, are not used or useful in assessing children's understanding of mathematics. I recommend a stronger focus on assessment as one of many teaching practices and urge critical examination of the role of assessment for equitable teaching practices that serve the needs of all students, especially those who are historically marginalized in the mathematics classroom. Furthermore, concept mapping, together with reflection activities, may provide spaces for PSTs to interrogate and clarify their thinking about equity-oriented teaching practices for mathematics education.

Building on this work for further research, I would be interested in gathering more information about how PSTs make sense of the concepts related to *children's MMKB*. More specifically, I would like to examine *whose* MMKB are represented in these concept maps—in other words, examining who has capital and of what kind (e.g., Yosso, 2006). Are particular racial or ethnic groups of students in mind when we (both teachers and teacher educators) use the phrases *funds of knowledge* or *MMKB*? That is, do we construct funds of knowledge as something that is only important for students of color? Where do our understandings of MMKB come from (e.g., course readings and activities, prior experiences with children)? As teacher educators, what contexts and people do we use as exemplars in our mathematics methods courses, especially in supporting teachers in addressing the needs of marginalized students? These questions also have connections to philosophical conversations and orientations to how we construct mathematics, which is most often seen as a body of knowledge that is inherently factual, logical, and separate from other subjects (Ernest, 1991). How might we disrupt this perspective on mathematics through our work with children's MMKB?

As both a research and pedagogical tool, I found concept-mapping activities useful as a method to examine and capture potential changes over time. PSTs actively created these concept maps to communicate their thinking through a meaningful representation. Teacher educators and PSTs may also use concept-mapping activities to focus on explicit connections and enter critical discussions that tease out nuances and challenges in our efforts to disrupt and minimize inequities in mathematics education. This work provides implications for supporting PSTs in recognizing their attention to race, class, culture, power, and social justice to address the needs of marginalized students through their practice. Attention to equity work must be intentional and not an afterthought. As teacher educators, we must support PSTs in their growth by pushing for more specificity and criticality. Concept-mapping activities can serve as a starting point to think about big ideas; however, comments such as "we need to make sure that *all students* relate to the problem" may still maintain inequities by not interrogating power or actively disrupting larger systems of oppression, such as white supremacy (Aguirre et al., 2017; Larnell, Bullock, & Jett, 2016; Martin, 2015; Martin, Gholson, & Leonard, 2010).

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Chapter 18
Using Ethnomathematics Perspective to Widen the Vision of Mathematics Teacher Education Curriculum

Nirmala Naresh and Lisa Kasmer

Abstract In this chapter, we draw upon our collective professional and practical knowledge bases to describe our efforts in fostering connections between ethnomathematics theory and practice. The theoretical field of ethnomathematics empowered us to envision a dynamic and equity-oriented teacher education curriculum. The enactment of this curriculum enabled us to create spaces wherein prospective teachers came to bear on the significance of teaching mathematics to promote a more just and democratic agenda. Prospective teachers explored and promoted meaningful connections between mathematics content, context, culture, and society in order to promote the development of empathy, consideration, and the skills necessary to appreciate and educate all learners, in particular those that are marginalized because of their social, cultural, economic, and political backgrounds.

Introduction

> Academic mathematics education has failed for the majority of the people. This failure is due in part to the conventional portrayal of mathematics as a prized body of knowledge that is the property of an elite group of people. (Millroy, 1992, p. 50)

Traditionally, mathematics education has mostly been associated with the K–12 institutional context. The problem is that mathematics in school settings is mostly perceived and presented as an elite body of knowledge stripped of its rich social, cultural, and historical connections. It is far removed from the "lives and ways of living of the social majorities in the world" (Fasheh, 2000, p. 5). We contest this view and argue for countering a narrow vision of mathematics that confines it to the school.

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When thinking about mathematics, seldom does one consider culture, context, diversity, history, or politics (D'Ambrosio, 1985; Powell & Frankenstein, 1997; Zaslavsky, 2002). Traditionally mathematics has been taught from a narrow perspective, and this has resulted in the omission of "significant contributions of most cultures and other groupings of people" from academic mathematics curricula, and this exclusion can "have the consequence of devaluing and disrespecting many students' cultural backgrounds" (Mukhopadhyay, Powell, & Frankenstein, 2009, p. 65).

One way to address this issue is through teacher education, using it as a space to transform a traditional mathematics curriculum and thereby to challenge many prospective teachers' traditionally held perceptions of mathematics and its pedagogy. The constructs of a culturally responsive mathematics education (CRME) (Gay, 2000) and a critical ethnomathematics curriculum (CEC) (Mukhopadhyay et al., 2009) lend themselves well to this cause. For a truly transformative mathematics education, it is imperative that we continue to engage with and support teachers who are in a strong position to realize the goals of a CRME in K–12 mathematics classrooms. In this chapter, we describe ways in which we used tenets of a CEC within two teacher education courses to promote an awareness of the importance of attending to the social and cultural dimensions of mathematics education.

Positionality

An educator's pedagogical decisions and actions are greatly informed and influenced by their personal, practical, and professional experiences (Kitchen, 2009). In light of this, we situate this paper in the broader context of our personal and professional backgrounds. Having been born and raised in a metropolitan city in a developing and formerly colonized nation, the first author's school and college mathematics learning experiences were rooted in an imported curriculum that was devoid of any connections to the local context and culture. In retrospect, there was also a noticeable paucity of attention to the significance of and connections to culture, language, and context in the second author's undergraduate and graduate learning experiences. Such experience perpetuates among many learners a narrow perception of mathematics that limits their ability to perceive and present mathematics as a human activity.

Through our collective mathematical learning and teaching experiences, we have witnessed firsthand how educational institutions used mathematics as a "subtle weapon" to "monopoliz[e] what constitutes knowledge" (Fasheh, 2012, p. 94). Our perceptions and beliefs about mathematics and its pedagogy prevailed until we began a self-study (Arizona Group, 1998) focused on ethnomathematics and its pedagogy, which proved both an immersive and transformative learning experience during which we became ardent enthusiasts of ethnomathematics and its theoretical and practical implications. Embracing an ethnomathematics perspective has opened our eyes to a world of mathematics that exists and flourishes outside the rigid boundaries of academia. Beyond that, it has enabled us to develop a renewed perception of mathematics and its pedagogy, its status in the society, and the role that it can play in the advancement of mankind. We have come to realize that mathematics is a human endeavor and developed a "respect for the 'other' and the intellectual achievements of all" (Mukhopadhyay et al., 2009, p. 4). This perspective has deeply impacted both our teaching and research and has pushed us to re-envision the ways in which we enact these practices.

Theoretical Underpinnings

The research field of ethnomathematics serves as a broader theoretical base on which a critical mathematics curriculum rests. Ethnomathematics represents a view of mathematical thinking that incorporates the ways in which culture and mathematics are intertwined (D'Ambrosio, 1985). Definitions of the term ethnomathematics range from very specific, such as "the mathematics which is practiced among identifiable cultural groups such as national, tribal societies, labor groups, children of certain age brackets, and professional classes" (D'Ambrosio, 1985, p. 45), to very broad: "Ethnomathematics is a research programme of the way in which cultural groups understand, articulate and use the concepts and practices which we describe as mathematical, whether or not the cultural group has a concept of mathematics" (Barton, 1996, p. 214). A grave concern with the exclusion of marginalized groups such as women, and racial and ethnic minorities from the mainstream mathematics curriculum resulted in researchers studying the ethnomathematics of these groups. Thus, issues of social change, equity, and justice have served as a foundation for the development of ethnomathematics.

Broadly speaking, we can classify ethnomathematics research into four strands (Vithal & Skovsmose, 1997): (1) a study of *non-Western mathematics*, which emphasized contributions from researchers who challenged the traditionally told history of mathematics (e.g., Joseph, 1987; Selin, 2000); (2) an exploration of *multicultural mathematics*, through which researchers analyzed mathematics associated with little-known exotic cultures and groups of people (e.g., Ascher, 2002; Gerdes, 2010; Zaslavsky, 1998); and (3) documentation of *everyday mathematics*, which highlighted the mathematical practices of participants in everyday settings (e.g., Carraher, Carraher, & Schliemann, 1987; Lave, 1988, Millroy, 1992). The fourth strand articulates the implications of ethnomathematics for school mathematics education (e.g., Brenner & Moschkovich, 2002). Though the terms *ethnomathematics*, *everyday mathematics*, and *multicultural mathematics* have been interpreted in more than one way by researchers, they are guided by the belief that the role of culture is significant in the learning and teaching of mathematics.

Mukhopadhyay and colleagues expand the scope of ethnomathematics to include mainstream academic mathematics and school mathematics. From this perspective "ethnomathematics is not proposed, as is often believed, as an alternative to either academic mathematics or school mathematics" but could coexist and thrive within the realms of academia (p. 70). We adopt this approach and in this chapter explicate connections between various strands of ethnomathematics to address this question:

How can we use the tenets of ethnomathematics to develop an increased awareness among educators of the significance of a culturally responsive mathematics education and thereby address the needs of all students, in particular those that are marginalized in the mathematics classroom?

Marginalized students are those students who are considered to have low socio-economic status and/or students whose cultural and linguistic backgrounds are different than their White peers (Garcia & Guerra, 2004). A traditional mathematics curriculum offers teachers very little scope to understand, appreciate, and view these students and their backgrounds as rich sources of mathematical knowledge. Thus, many teachers treat “such students as blank slates, ignoring potential new strategies, conceptual understandings or unique algorithms that they could offer in a U.S. mathematics classroom” (Gutiérrez, 2008 p. 361).

We propose that an ethnomathematics curriculum with a critical perspective will help address this issue and generate a meaningful dialogue centered on a CRME (Gay, 2000), aimed to empower and transform all learners. The key tenets of a CRME include (1) helping students connect academic mathematics to other forms of mathematics, (2) connecting school mathematics to the sociocultural-ethnical aspects of home culture, (3) enabling teachers with equitable pedagogical practices that cater to all learners, and (4) allowing both students and teachers to acknowledge and celebrate their own and each other’s cultural background (Gay, 2000). A CEC draws on the principles of a CRME and addresses the challenges that it poses to a traditional mathematics curriculum. It challenges the “Eurocentric narrative,” confronts “what counts as knowledge in school mathematics,” and attends to the disconnect between “mathematics education and social and political change” (Mukhopadhyay et al., 2009, p. 72). In order to accomplish these goals, it is necessary to design and enact a curriculum that will offer the maximum scope to empower all learners by broadening their perspective of mathematics and its pedagogy. We draw upon our collective, professional, and practical knowledge to describe ways in which we used key aspects of a CEC to support preservice teachers (PSTs) in making meaningful connections between mathematics content, context, culture, and society.

Self-Study Using Narrative Inquiry Methods

We share a common commitment to improving our own practice as mathematics teacher educators and in particular as ethnomathematics researchers and practitioners. To honor this, we engaged in a self-study with an emphasis on collaborative learning. This approach helped focus the lens on our “own selves” and enabled us to maintain focus on a more “diverse variety of selves.” Within the self-study, we have used narrative inquiry (Creswell, 1998; Kitchen, 2009) to organize, present, and analyze the qualitative data. In choosing this method, we “adopt a particular view of experience as phenomenon under study” (Connelly & Clandinin, 2006, p. 375). In the present context, we draw from our practical and professional experiences specific to two content courses for PSTs (phenomenon under study), immerse ourselves in these lived experiences, and narrate our stories. The stories were extracted from two distinct professional experiences aimed to challenge and broaden PST views of mathematics and teaching through (1) content-focused coursework offered in an on-campus setting and (2) an immersive early field experience facilitated in a non-Western developing country.

The two teacher education courses shaped the contextual setting, and the PSTs enrolled in these courses are an integral part of the study. In our narratives, we consciously chose and presented data that would best help us address the central goal of this chapter and involve the "selves" who we wanted to know more about. The selves include both MTEs (us) and PSTs (our students). PST-contributed data include their solutions to mathematical tasks (SMT), course project artifacts (CPA), written reflections (WR), and summaries of their field experiences with middle school students (FE). MTE-contributed data include positionality statements (PS) and personal philosophies of teaching (PPT), course artifacts (CA), summaries of our interactions with PSTs (SI), collaborative reflective journals (CRJ), and a shared understanding and analyses of PST-contributed data (UA). Our narratives are structured around a chronology of events, including (a) course design and enactment, (b) documentation of data, and (c) our analyses and interpretation of data in relation to the chosen theoretical domain. In our narratives, we consciously choose and present data and episodes from our professional and practical experiences that will best help us attend to the central theme of this paper.

University A: Content-Focused Coursework

The contextual setting for University A is a mathematics content course, *Patterns and Structures through Inquiry* (*PSI*), for PSTs. This is a three credit-hour (38 contact hours) course for PSTs pursuing licensure to teach middle school mathematics. PSTs typically enrolled in this capstone course in their fourth year at the university. Prerequisites for this course include successful completion of at least 9 hours of mathematics education courses that address topics such as numbers and operations, algebra, geometry, technology, and the history of mathematics.

PSI Course: Goals, Content, and Structure

This course is designed to foster critical thinking, engage PSTs in solving complex problems and with other learners, and facilitate PSTs' communication of their mathematical ideas. The first author chose *ethnomathematics* as the focus theme. To this end, the key course goals were to support PSTs' examination, from sociocultural, critical, and political standpoints, of the evolution of mathematical ideas. The PSI course was designed in line with the principles highlighted in Presmeg's (1998) graduate course on ethnomathematics. Presmeg's course required participants to (a) adopt a sociocultural and a critical approach to view mathematics and its pedagogy; (b) acknowledge and understand that different cultural groups and people in many walks of life do, perceive, and explain mathematics; and (c) complete a course project to design and accentuate an ethnomathematical activity specific to their unique cultural background. The first author, having had an opportunity to participate in such a course, saw firsthand a practical application of ethnomathematics theory in an MTE course and its transformative effect on the participants. As an ethnomathematics researcher and practitioner, when provided an opportunity to teach a capstone course, the first author readily embraced and adopted the principles of Presmeg's ethnomathematics course in the design and enactment of the PSI course.

In the PSI course, PSTs engaged in problem-solving activities in small groups and shared their work with peers through formal and informal presentations. To establish specific connections to the principles of ethnomathematics and a CRME, participants were asked to think deeply about the types of tasks and activities that are usually presented in a traditional mathematics curriculum by comparing and contrasting such tasks with those that were presented in the PSI course. For example, while focusing on the algebra strand, particularly on pattern recognition and analysis, three tasks chosen from different cultural and historical contexts were presented to the PSTs: (a) the classic rice and the chessboard problem, (b) analyzing Kolam patterns of South Indian women (Siromoney, 1978), and (c) Sona patterns of the Chokwe people (Gerdes, 1997). In their written solutions to such tasks, PSTs were asked to attend to and explore the connections between a given mathematical task and the local and global contexts that framed the tasks (SMTs). Each week, PSTs were asked to describe and reflect on their perceptions of mathematics and articulate their beliefs on what constitutes a mathematical activity (WRs). The first author met each participant once a week to discuss coursework, learn more about their views on mathematics and the teaching of mathematics, and their emergent views on the role of culture in the teaching and learning of mathematics based on which we wrote my SIs. Course activities can be broadly classified into two major categories: (a) content explorations (emphasis: local/global contexts and cultures) and (b) course project investigations (emphasis: connections between everyday mathematics and academic mathematics). Below, we highlight two activities drawn from course artifacts.

Activity 1: Teaching Geometry Using Cultural Artifacts

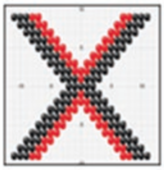
For content analysis, participants (re)examined mathematical topics that typically lie outside the focus of the traditional K–8 mathematics curriculum. The tasks were chosen to enable PSTs to make explicit connections to the sociocultural, historical, and political dimensions of the evolution of mathematics. As part of content explorations on geometry, the following activities were planned and offered: (a) creating a timeline for the evolution of geometry (Lumpkin, 1997), (b) investigating the Sona geometry of Chokwe tribes (Gerdes, 1997), and (c) exploring geometry and technology through Native American cultural artifacts (Eglash, 2002).

A bead loom, which is used by many ethnic groups, including many Native Americans, can be viewed as a pattern based on a Cartesian coordinate system and can be used to explore transformational geometry. We used the cultural artifacts (bead patterns) with our PSTs to support them in thinking deeply about the Native American cultural context and the mathematical content implicit in the artifacts. Using Virtual Bead Loom software (Eglash, 2002), PSTs replicated bead loom patterns while at the same time making connections to their knowledge of symmetry and transformations. To ensure a successful enactment of the lesson, we suggested the following steps: (1) visit https://csdt.rpi.edu/culture/legacy/index2.html, identify and select the Virtual Bead loom tool, (2) read the cultural background section to better understand the significance of the bead loom in Native American culture, (3) read the "how-to" overview to understand the function and purpose of the different bead loom tools (point, line, line iteration, triangle iteration, rectangle iteration) and how they work, (4) choose a bead loom pattern and use appropriate tools to replicate this pattern, and (5) write a brief summary describing the steps that you used to replicate this pattern. Two sample PST responses are highlighted here (SMTs, Figs. 18.1 and 18.2).

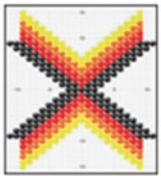
Virtual Beadloom

I chose the following design for my pattern

Using linear iteration, horizontal and vertical reflections I created a black X through the center. First, I chose my starting point as (0,0). Then I chose a line of 3 beads and iterated the line moving down and left for 10 rows. Then I reflected the same image down, left, and right to complete the pattern.



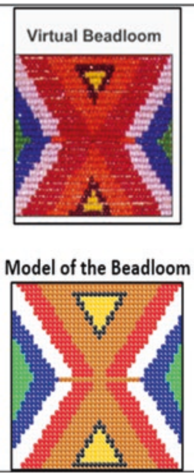
I was able to somewhat easily iterate the red beads for the top and bottom but am struggling to do it for the left and right. I can't get it to line up in the row I want it to go in.



I decided it may be easier to work vertically at first and get all of those parts finished before trying to figure out the horizontal pieces.

Working on the horizontal pieces, I've found that I can still iterate in the positive and negative y direction; before I was trying to do it in the x direction and it was getting messed up.

Fig. 18.1 Technology-based construction of a bead loom



- In order to replicate the image I started by creating two large red triangles that were reflections of one another.
- For example, when I decided to make the vertices (0,0), (-20,20), and (-20,-20), I realized that the (0,0) coordinate remained the same; but since the reflection occurred over the Y-axis the X-coordinate would both now need to become positive thus creating the other two vertices as (20,20) and (20, -20) while the y-coordinate remained the same.
- I repeated this process starting with the red triangle and then a white triangle and then reflected it over the Y-axis. Then I created a blue triangle and a green triangle and reflected them over the Y axis as well.
- Once that portion of the beadloom was created I had to perform reflections over the X-axis. I started with the orange/brown triangles and used (0,0), (-12, 20) and (12, 20). To perform a reflection, my new coordinates have to be (0,0) (-12, -20) and (12, -20). In this situation, the X-coordinates remained the same and the Y-coordinates changed. I repeated this process again to create the smaller yellow triangle.

Fig. 18.2 Technology-based creation of a bead loom

Activity 2: Teaching Geometry Using a Sociocultural Art Form

All PSTs completed a course project requiring them to identify and investigate a personally meaningful practice, highlighting the fact that mathematics is a human activity. Each participant developed a mathematical activity and engaged in both content and pedagogical explorations. PSTs explored in depth the mathematical ideas inherent in the activity, and they drew upon their mathematical understandings, designed a mathematics lesson, and completed a field experience with a group of middle school students. Here is an example drawn from one PST's (Sandy) course project (CPA, FE). We chose this project to specifically highlight Sandy's transformation and growth through her participation in this course. At the beginning of the semester, when asked to describe her cultural background, Sandy noted, "I am just from here... so I don't think that I have culture like you do... So I don't know how to find a meaningful cultural/everyday activity" (WR). Upon further elaboration, we came to realize that Sandy associated the term "culture" with the other, a person from a different race, ethnicity, or nationality. Gradually, through her participation in the course activities and continued discussions with her peers, she came to realize that every individual possesses a unique cultural background informed by their lived experiences.

Sandy developed a mathematical activity based on her "family art," cross-stitching. Her late grandmother had taught her the art, and Sandy dedicated this project to her. Sandy used this cultural art form to enhance her own understanding of cross-stitching and graphing on a Cartesian plane using GeoGebra. Her project report included two cross-stitched designs (University A logo and a five-pointed star) and a geometry task based on GeoGebra that she eagerly and proudly shared with her peers. During her field experience (FE), Sandy also shared her designs and the activity with a group of middle school students. It is through these artifacts that the middle school students were able to perceive connections between the art of cross-stitching and mathematics. The students created colorful and intricate patterns on a Cartesian grid and uncovered many connections between their newly created art forms and geometry. Below (Fig. 18.3) is a snapshot of Sandy's description of the mathematics of cross-stitching.

Here (Fig. 18.4) is Sandy's (re)creation of a five-pointed star on GeoGebra. She presented the art form to her peers and engaged in geometrical investigations on coordinate geometry, congruence, similarity, transformations, area, and perimeter.

During our class discussion, many PSTs lamented about the paucity of curricular resources that could be used to teach mathematics using a sociocultural lens. Exposure to and engagement in the ethnomathematics of the Chokwe tribes, South Indian women, nonacademic professionals, and Warlpiri tribe (Ascher, 2002) mitigated part of this concern and enabled PSTs to appreciate the contributions of socially marginalized groups to the evolution of mathematical ideas. From an analysis of PST reflections, we noticed that many PSTs saw a firsthand situation where multicultural contexts became alive in a mathematical classroom and wanted to find additional resources that may be available and were excited to incorporate such activities in their own classroom. As one PST summarized,

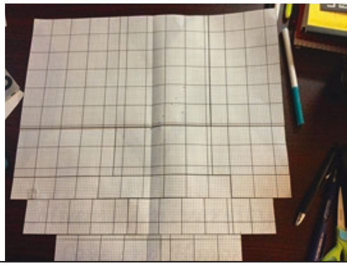
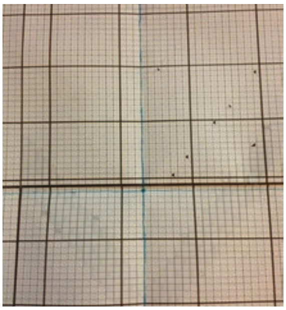
When you are cross-stitching, you are basically plotting points on a quadrant plane. But instead of having the origin marked out for you and the x and y axis labeled and visible, you have to find it yourself and know where it falls within the pattern and on your fabric (as seen in the picture below).	
When starting a new pattern, you always want to start in the middle (origin) and either start off stitching above the middle line (x-axis into quadrants I and II) or work your way below the middle line (x-axis into quadrants III and IV). As you can see in the picture, the origin is not always defined and will have to found by the students themselves, along with the axis. Every box in the pattern is the same thing as being a coordinate point on the quadrant plane. You have to know how many spaces to the left of right of the y-axis to find the x-value and then how many spaces above or below the x-axis to find the y-value.	

Fig. 18.3 Sandy's description of the mathematics of cross-stitching

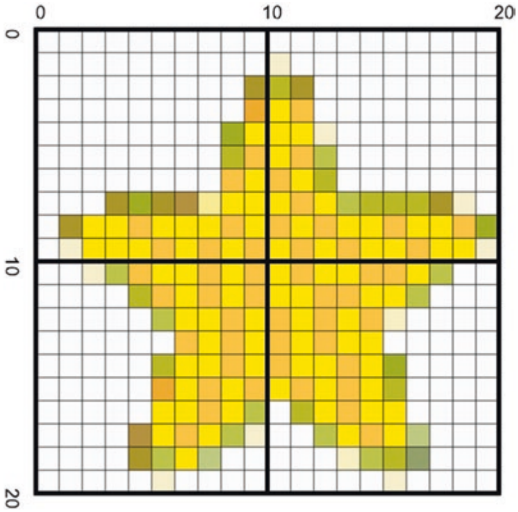


Fig. 18.4 Technology-based creation of a five-pointed star

> I have Native American heritage myself and this type of activity makes me think of my own lineage and the legacy left to me by my ancestors – Moreover, I feel greatly empowered in knowing that I can use this legacy to teaching mathematics and modeling hand in hand.

Many PSTs noted that they gained a greater appreciation for and better understanding of the role of other cultures "within the study of [school] mathematics" (Zaslavsky, 2002, p. 66). Furthermore, they were able to engage in a "two-way dialogue in which [the different forms of] knowledge (community knowledge, school knowledge) and their associated values are brought into the open for scrutiny" (Civil, 2002, p. 146). In the next section, we present a narrative of a course that provided PSTs in University B, an immersive ethnomathematics experience in a non-Western developing nation.

University B: Early Field Experience in a Non-Western Country

Currently, teaching in the USA remains a homogeneous profession; the majority of teachers and PSTs are of European-American descent; PSTs tend to be cross-culturally inexperienced, live within 100 miles of where they were born, and desire to teach in schools similar to those they attended, with fewer than 10% hoping to teach in either an urban or multicultural setting (Cushner & Mahon, 2009). Heyl and McCarthy (2003) suggest that experiences in international education have the potential to be the most influential factor in developing cultural awareness in PST education. They contend, "A key role for higher education institutions must be to graduate future K-12 teachers who think globally, have international experience, demonstrate foreign language competence, and are able to incorporate a global dimension into their teaching" (2003, p. 3). In order to gain a more global perspective, an early field experience was developed at University B to address the PSTs' inexperience relative to a CRME.

Setting and Background

At this institution, mathematics majors seeking teaching certification are able to fulfill their senior capstone requirement through a month-long field experience in Arusha, Tanzania (the fieldwork is completed in 1 month, while the coursework is completed over 8 months, 2 months prior to travel, and 6 months post travel). These PSTs are a representative subset of the students enrolled as mathematics education majors at the university. That is, the PSTs are typically White, middle class, juniors, and seniors [3rd and 4th year university students] who have completed their mathematics content course requirements. They begin their student teaching within a year after returning from Tanzania.

Prior to departure, students attended six orientation meetings in which they read and discussed articles about Tanzanian culture, history, and education and learned basic Kiswahili in preparation for the program. A Kenyan faculty member also attended many of the orientation sessions to respond to questions from the PSTs and assist in the Kiswahili instruction. During their month-long experience in Tanzania, the PSTs taught mathematics in elementary and secondary English medium schools for 60–75 hours. (English is the language of instruction, even though English is not the native language in Tanzania.) The PSTs were responsible for developing the lessons, teaching the lessons, and designing assessments, often without textbooks, materials, or resources. The teaching day concluded with each PST individually conferencing with the university faculty members, followed by a whole group reflection. During this individual conference, PSTs analyzed and reflected on the mathematics lessons they taught and what struggles that both they and their students encountered. It should be noted that the university faculty assisted the PSTs with mathematical content and classroom management recommendations when needed but did not offer suggestions pertaining to negotiating teaching mathematics; the PSTs navigated these issues on their own or with one another. Data were generated and collected from these conferences and teaching episodes. Field notes and audio recordings taken by faculty of these discussions (CPA), observations of classroom visits (CPA), and written documentation of instructional decisions and reflections (WWR) provided a rich source of data to capture PSTs' evolving attention to the principles of CRME. The PSTs responded to reflection prompts posed by the faculty at the conclusion of each teaching day. Prompts included questions such as "What did you do today in your teaching that you hadn't planned to do?", "What surprised you today in terms of your teaching and your students", "What was something you learned today that you didn't know before", "What did you notice is the same in your classroom here (in Tanzania) as classrooms in the U.S", and "Why do you think... ". Next, we provide examples of how this experience addressed some of the key tenets of a CRME.

Using Kangas to Teach Parallel and Perpendicular Lines

Over the course of the month-long experience, the PSTs had numerous opportunities to infuse African culture and mathematical context. At the onset of this experience, however, the PSTs, in an attempt to contextualize the mathematics they were teaching, provided contexts which were unfamiliar or irrelevant to the Tanzanian students. For example, one PST taught a lesson about parallel and perpendicular lines. She provided the context of roads "that run parallel to one another" and roads "that are perpendicular to each other." It was evident to the PST that the students were confused by her explanation, and she attempted to provide another context that did not provide clarification for her students. When she debriefed with her peers at the conclusion of the day, she came to realization that her attempt to contextualize the mathematics was a meaningless (and Western) context. That is, most streets in Arusha do not run parallel or perpendicular; rather, roads were built based on availability of long stretches of land. In order to make the concepts of parallel and perpendicular lines meaningful to her students, she decided that it would be necessary to provide a context that was familiar and relevant to her students. She accomplished this by situating parallel and perpendicular lines with kanga fabric. (Kangas are brightly colored, rectangular fabrics, often with geometric patterns that African women wear as dresses, skirts, and headpieces.) Using a familiar context to the students, they were easily able to see the meaning she was trying to convey of parallel and perpendicular lines. The PST then realized the power of situating her instruction within a culturally relevant context. In this way, the PST connected school mathematics to the sociocultural-ethnical aspects of her students' Tanzanian culture.

Teaching Fractions with Ugali

For another PST, teaching fractions became a struggle as she attempted to demonstrate equivalent fractions. While the context she provided was a familiar one to her students (measuring ingredients to make ugali, a local dish), the vocabulary she used was problematic. That is, terms such as *renaming*, *numerator*, *denominator*, and *equivalent* that she did not know the Kiswahili equivalence for were difficult for her to explain to her students. In order to acknowledge and celebrate the students' cultural background and facilitate understanding, she decided it would be important to infuse Kiswahili terms when appropriate. For students who struggle with understanding the language of instruction, *code-switching* is an important instructional strategy teachers use to assist understanding the meaning of the concepts conveyed. In this way, the PST also practiced equitable pedagogical practices, as she realized the language barrier was impeding some of the students' abilities to understand what was being taught.

Hisabati Katika Kazi (Mathematics at Work)

During the lesson planning session one evening, many of the PSTs decided to create a lesson based on the mathematics adults used at work and home. Unbeknownst to the PSTs, they were attempting to connect the academic mathematics to other forms of mathematics. They were surprised by this connection and remarked that they didn't realize they were even attempting to infuse culture into mathematics. In this lesson, the PSTs asked their students to interview an adult to gather information about the mathematics they use at work. The PSTs' motivation for this assignment was to help their students see the value of mathematics outside of the classroom. The PSTs were not explicitly considering the principles of a CRME; rather, this was merely an assignment they created, "as an activity to do, that we think they will enjoy and help them think about math in another context." When debriefing their lessons, the PSTs began to understand and realize the significance of providing opportunities for their students to connect academic mathematics to other forms of mathematics, a central tenet of a CRME. When the students shared their interview responses (e.g., "My mother is a seamstress and she said she uses a lot of math in her work; measuring fabric, figuring out how much to charge, and what her costs are for making a dress," "My brother drives a cab. He needs to know how much to make people pay for a route so he doesn't short himself"), the children were intrigued and amazed at how important mathematics is outside of the school setting. For the PSTs, they came to the realization that "this was a very powerful lesson, and we didn't even know it!"

Return of the Kangas: Transformations and Kangas

Using kangas also proved a useful tool to teach transformations. The PSTs came to understand the importance of situating the mathematics in a meaningful context, as evidenced in their reflections (WWRs) and our analysis of class discussions (CPA, CRJ), and decided to deviate from the textbooks and create lessons that utilized kangas. Below is an example of the introduction to the lesson the students created about translations (Fig. 18.5). The PSTs took photographs of kangas from a local market which highlighted the aspects of transformations they were teaching. By recognizing the significance of the kanga in the Tanzanian culture, the PSTs were able to make valuable connections to the students while celebrating an important part of their culture.

Mathematics in the Street

The PSTs also had opportunities to interact with various Tanzanians outside of the school setting and gain a broader perspective and appreciation of the importance of a non-Western view of mathematics. For example, they came in daily contact with the local touts (street peddlers), who attempted to persuade them to purchase their wares. Most of the local touts do not have a formal education past primary school.

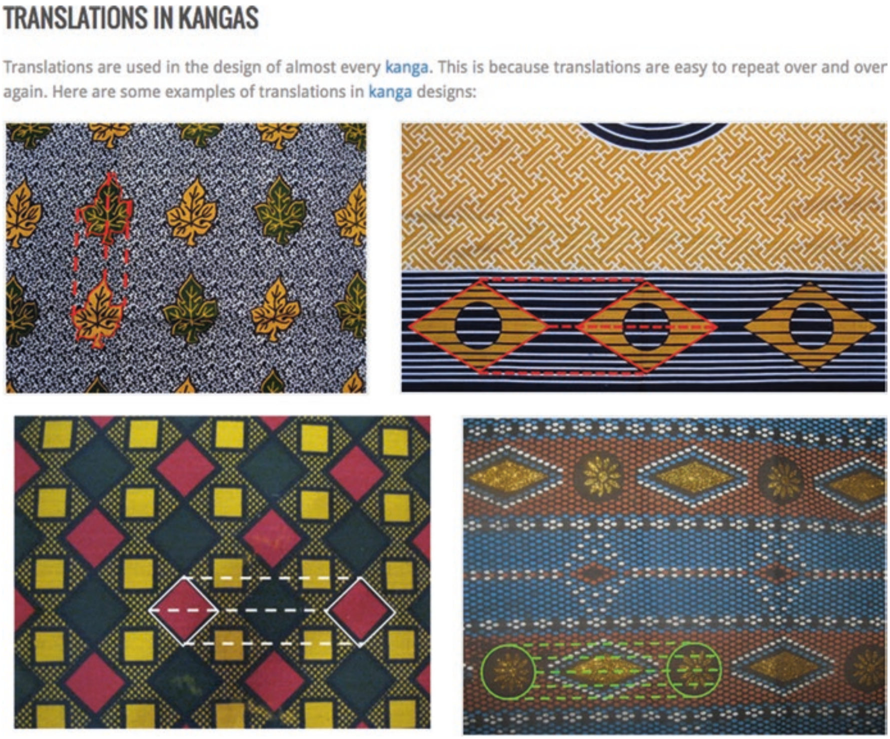


Fig. 18.5 Geometric transformations in kangas

Their ability to make transactions in their heads, however, both amazed and impressed the PSTs. Many of the PSTs remarked that the touts often performed complicated problems related to ratio and proportion with ease. They were also surprised at how they calculated costs and change without a calculator or paper and pencil. One PST remarked, “I am in awe of how easily the touts do mathematics in their head, they are better at this than I am with a major in mathematics.” During a whole group discussion, the PSTs came to the conclusion that “maybe we should think about math differently, it doesn’t always have to be the way we think of math.” This opportunity enabled PSTs to conceive, with the help of local citizens, an alternative view of doing mathematics and acknowledge the way these individuals come to know and understand mathematics.

Reflections: Continuing the Self-Study

As proponents of ethnomathematics, we believe that the central purpose of mathematics education is “to contribute to the development of our collective world, in the direction of more justice” (Mukhopadhyay et al., 2009, p. 75). For quite some time now, we have realized that "the record [dominant perspective] is wrong, and [realized] how that wrong record is culturally disrespectful" (Mukhopadhyay et al., 2009, p. 74) and thus we are better positioned to understand the need and the potential for engaging PSTs in mathematical activities that offer a counter narrative to the dominant perspective.

Thus, in both courses, it was a key priority for us to support PSTs in broadening their perspectives of mathematics and its pedagogy so that they can better attend to the needs of all of their students. The narratives we presented above, and the embedded examples, illustrate ways in which we provided opportunities for PSTs to think deeply about the constructs of ethnomathematics and a CRME. We provided many opportunities for PSTs to acknowledge that much of the so-called Western mathematics "originated in the ad hoc practices and solutions to problems developed by small groups in particular societies" (Katz, 2003, p. 557) and that traditionally told histories of mathematics have neglected the contributions from the non-European cultures and have presented a *Eurocentric view* of mathematics (Joseph, 2000).

In both courses, we introduced PSTs to the social constructions of mathematics through course readings that highlighted the contributions of people who were not necessarily from the mainstream academia. For too long, academic mathematics has presented our students with only European, so-called "refined," version of mathematics, only taught to regurgitate whatever processed form to which we have limited academic mathematics. We wanted PSTs to think deeply about community knowledge and draw upon funds of knowledge (Moll, Amanti, Neff, & Gonzalez, 1992) that existed in rich cultural settings. Thus, we encouraged them to look for connections to mathematics in their own cultural backgrounds and the immediate social backgrounds. PSTs learned about, and engaged with, the mathematics inherent in the sociocultural activities of groups that they barely knew.

It is our hope that, in light of their exposure to the mathematical activities outlined in the two courses, this group of PSTs will be willing to look for and acknowledge other forms of mathematics that exist outside the realms of the academia. We have presented one example through Sandy's project. In her final course reflection journal, she noted: "I see that I have been oblivious to [their] expertise. Upon reflection, I see that I have been oblivious to [their] expertise because I truly did not believe that [they] can be authentic sources of knowledge." We believe that Sandy and many PSTs who took these courses have begun to notice how individuals from many walks of life can become co-constructors of mathematical knowledge. In their final course reflections, many PSTs remarked that they will continue to be mindful of the different connections and ideas that students might bring into the classroom and the importance of connecting mathematics to a culturally relevant context. As a result, we hope that these PSTs will begin to embrace the meaning of learning as "a truthful collaboration in which all parties come both as learners and as resource" (Civil, 1998, p. 7).

In retrospect, we acknowledge that striving to develop and implement a mathematics content course on CRME or ethnomathematics is no easy task. Often times there are neither context, nor content, for such a course in a traditional teacher education program, and at times some resistance from students occurs (Naresh & Poling, 2015). Nevertheless, we believe that if we failed to produce meaningful dialogue, or if it is absent, change cannot occur, and we cannot provide a transformative learning experience for our students. In order to prepare culturally responsive mathematics teachers, it is necessary to "[deconstruct] the aura of incontestability and status that now surrounds mathematics, and certain notions about what constitutes quality instruction" (Gay, 2000, p. 194).

During class discussions and field experiences our PSTs examined questions such as: What is mathematics? If and how is informal mathematics different from formal mathematics? How are mathematical ideas exemplified in the activities of just plain folks? What is the role and purpose of mathematics in their everyday lives? From a dominant Eurocentric view, which forms of mathematics are acknowledged and valued? Why might this be the case? What can we do about this? Many of the course activities were geared toward helping them think deeper about plausible responses to such questions. We hope that, in their quest for deeper answers to these questions, PSTs will further broaden their perceptions of mathematics and its teaching.

Our self-study is specifically focused on explicating ways in which we can use the principles of ethnomathematics to question, understand, and reevaluate the meaning of mathematics and its pedagogy. MTEs' efforts to address practice-oriented problems "may rarely result in tidy answers... when viewed through the lens of self-study" (Berry, 2009, p. 1312). However, this lens enabled us to explicitly model key components of CRME to our PSTs.

Moving forward, we will continue to engage in this self-study and find ways to purposely infuse the principles of ethnomathematics and a CRME into all of the mathematics education courses we teach. In future, we hope to develop a framework to share with our mathematics education colleagues, so they can understand the importance and impact of threading culturally responsive mathematics teaching into all of the courses they teach. As a community of learners, we have come to realize that it is important to "empower students, through broadening, not narrowing their knowledge of mathematics; through inspiring their participation and creativity in contributing to the development of mathematical knowledge, and, for teachers, through the creation of a culturally responsive mathematics teaching" (Mukhopadhyay et al., 2009, p. 72).

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Chapter 19
A Commentary on Supporting Teachers in Addressing the Needs of Marginalized Students

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Abstract This chapter presents a commentary on the pieces in the book related to supporting teachers in addressing the needs of marginalized students.

The chapters in this section are set in the context of university course work in mathematics education for prospective teachers (PSTs). In the case of three of the chapters (Chaps. 15, 16, and 17), the context was mathematics methods courses. In one chapter (Chap. 14), the study was set in the context of an intensive field experience focused on work with a single English learner. The fifth chapter (Chap. 18) included two studies, one of which was conducted as part of a mathematics content course and the other of which was conducted during an intensive field experience in an African country. The variety of settings, as well as the variety of activities that form the bases of the data collection, allow us the opportunity to engage with many ways of supporting PSTs to work with and address the needs of marginalized students. Through the activities detailed in the studies, we see PSTs engaging in what the literature has shown are needed experiences with racially, culturally, linguistically, and socioeconomically diverse students (McDonald, 2005).

As is the case in the USA more generally, the PST participants in these studies were by and large White, middle-class, monolingual women (Goldring, Gray, & Bitterman, 2013). The studies share a common goal of engaging their participants in activities that afford them the opportunity to engage in some way with, or with issues concerning, marginalized students. Furthermore, ideas for engaging these PSTs from largely dominant backgrounds in developing more equitable instructional practices are presented in a variety of ways in these studies. Nonetheless, there are commonalities across some of the studies. These include (a) the use of whole class discussion to broaden PSTs' understanding of attending to biases they bring to analysis of classroom practices (Chap. 16) and ways in which they need to include out-of-school knowledge in their assessments (Chap. 17), (b) attention to the needs of English learners both domestically and in an international setting

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(Chap. 14 and one study in Chap. 18), and (c) the use of specific tools for focusing PSTs on improving their equitable instructional practices: a Curriculum Spaces Analysis Tool (Chap. 15), vignettes (Chap. 16), and concept mapping (Chap. 17). After a brief overview of each chapter, I will return to a discussion of the commonalities identified across the studies.

The authors of Chap. 14 examined instructional strategies employed by PSTs when working with a single English learner. The authors were interested in which strategies PSTs would "intuitively" employ since they had had no previous coursework in pedagogies for working explicitly with English learners. The authors used a framework developed by Chval and Chávez (2012), who describe effective strategies for working with English learners; they examined to what extent their participants used these strategies in their work with the English learners with whom they were working.

As part of the Teach Math project (Turner et al., 2011), the authors of Chap. 15, engaged PSTs in an activity focused on adapting curriculum materials to reflect children's mathematical thinking and the out-of-school knowledge and experiences that students might bring to the elementary classroom in an attempt to "open spaces" in the curriculum to accommodate this knowledge and these experiences. As well as looking at the features of the curriculum that PSTs attended to and how their personal experiences impacted their thinking, the authors examine the extent to which a tool developed by the Teach Math project (the Curriculum Spaces Table) supported PSTs in their curriculum analysis.

In the study described in Chap. 16, PSTs were afforded the opportunity to examine a vignette of a student with demonstrated competence in mathematics who had also been labeled as disruptive in the mathematics classroom by the teacher. Small groups of PSTs examined the same vignette with each small group being given a different student demographic: African-American male, African-American female, White male, and White female. The study examined whether instructional suggestions as to how to support the student varied depending on the race and gender of the student.

In Chap. 17 the author examined, through concept mapping, how PSTs link various concepts to the root idea of "assessing children's understanding of fractions." The hope was that PSTs would link the various multiple mathematical knowledge bases (MMKB) that students draw on in their mathematical thinking to this assessment of fractional knowledge.

Chapter 18 presents two studies which are conceptually linked through the use of a critical ethnomathematics curriculum intended to support PSTs in attending to social and cultural dimensions of learning. In the first study, as part of a mathematics content course, PSTs studied mathematics content through engagement with activities drawn from a variety of cultural and historical contexts. In the second study, an intensive field experience in an international setting, PSTs confronted cultural knowledge bases that were required for accessing particular academic content as well as the challenges involved in learning mathematical content in a language that is not the home language.

In two studies (Chaps. 16 and 17), authors found that whole class discussions that followed initial attempts to engage in the activity were an effective tool that supported PSTs in revising their thinking in some important way. In Chap. 16 this discussion raised PSTs' awareness of biases they were bringing to their analysis; in the case of Chap. 17, PSTs were supported in thinking about out-of-school funds of knowledge that might be related to understandings of fractions.

Needs of English learners were examined in two studies (Chap. 14 and the second study in Chap. 18). Through their work with the students in these cases, PSTs became aware of particular needs of English learners in the mathematics classroom that they might not have encountered if not afforded the opportunity to work with this population.

Effective tools for use by PSTs in developing their equitable practice in mathematics were introduced in several of the studies (Chaps. 15, 16, and 17). In Chap. 15 authors employed the Curriculum Spaces Analysis Tool developed by the Teach Math project (Drake et al., 2015) to focus PSTs on how they can adapt curriculum to meet the needs of marginalized students whose knowledge bases might not be present in the written curriculum. The authors of Chap. 16 used vignettes as a tool to focus PSTs' attention on particular biases they might be bringing to analysis of classroom practice and to make suggestions for supporting students from particular non-dominant backgrounds. In Chap. 17, concept maps were used as a tool for examining the incorporation of various MMKBs in assessing children's fractional knowledge.

The greater understandings of the needs of marginalized students that PSTs were able to develop across the multiple experiences in university classes support them in developing more equitable instructional practices that can in turn support the academic achievement of some of our most vulnerable students.

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