Transition from Secondary to Tertiary Mathematics: Culture Shock – Mathematical Symbols, Language, and Reasoning

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Abstract Our education is marked by large-magnitude discontinuities called *transitions*, during which significant changes—which require more than just academic reconstruction—occur over a relatively short period of time. The passage from high school to university, i.e., the *secondary-to-tertiary transition*, is the subject of this chapter. A key ingredient in helping students transition successfully lies in the two-way communication between high school teachers and university instructors. The case studies we discuss illustrate what the topics of these conversations could be. For instance, Ontario high school mathematics curriculum expectations do not adequately address mathematics language and logical reasoning. However, university mathematics instructors assume that their students have experience in working with definitions, universal and existential quantifiers, in constructing simple implications, or providing counterexamples. Surprisingly, standard university textbooks that review high school material do not even have hints or guidelines about understanding mathematics language and "mathematics culture," nor do they provide examples illustrating rules of logical deduction. In another case study we investigate difficulties that students face as they navigate trough a myriad of mathematical symbols, and work with their changing, content-dependent meanings. Case studies presented in this chapter could be included into high school teachers' horizon knowledge. An ability to see and understand how mathematical ideas and reasoning develop over a longer time scale can inform teaching, and thus better prepare students for their transition to tertiary mathematics. For exactly the same reasons, these case studies should find their way into university teaching.

Keywords Secondary to tertiary transition in mathematics · Mathematics symbols · Number bias · Language of mathematics · Logical reasoning · Horizon knowledge

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Formal education, from elementary school to university, is by no means a straightforward, smooth and continuous process. Perhaps the easiest way to define a *transition* is to qualify it as referring to a large-magnitude, singular discontinuity, such as changing from having one teacher for all subjects in elementary school to teacher-specialists in high school, or finishing high school and starting university. The latter discontinuity, usually called the *secondary-tertiary transition*, is the subject of this chapter.

There is no doubt that a key ingredient in helping students transition successfully lies in the two-way communication between high school teachers and university instructors. As triggers for topics of such dialogues, we discuss four case studies—mathematical symbols, "number bias," language and culture of mathematics, and logical reasoning—which cover areas where we identified significant gaps (discontinuities) between high school and university treatments.

University mathematics courses require proficiency in navigating through a large number of *mathematical symbols*, as well as their changing, content-dependent meanings, especially when discussing applications. We use the term "*number bias*" to discuss students' expectations that numbers involved in calculations, as well as in answers to mathematics questions, are certain special types of numbers, such as integers or simple fractions. In the section on *language and culture of mathematics* we identify situations which, while routinely (and correctly) understood by mathematicians and mathematically mature students, are often a source of confusion and misconceptions for novices. Proper use of *mathematics language* and *logical reasoning* (i.e., principles of mathematical logic) are usually not covered in high school.[1] However, university mathematics instructors assume that their students are familiar with them and have experience in working with definitions, quantifiers ("for every," "there exists"), in constructing simple implications, providing counterexamples, and so on. Surprisingly, standard university textbooks (calculus and linear algebra, for instance) that review high school material do not even have hints or guidelines on understanding mathematics language, nor provide examples illustrating rules of logical deduction.

Besides outlining these themes and illustrating with specific examples, we suggest ways in which they could inform teaching practice, both in high school and in university.

Mathematical Symbols

Even something as "straightforward" and "simple" as familiarity with mathematics symbols demands time and adjustment in transition. Using our own province as an example, the Ontario grades 9–10 and grades 11–12 curriculum documents (Ontario

[1] The word "definition" does not appear even once in Ontario grades 9–10 and 11–12 curriculum documents; the word "define" appears several times, but not as a suggestion to actually write down a formal, precise mathematical statement. For instance, there is no suggestion to define the term "asymptote."

Ministry of Education, 2005, 2007) use x exclusively to denote an independent variable, and y or $f(x)$, or sometimes $g(x)$ or $h(x)$, to denote a dependent variable. Although students might be exposed to a larger variety of notation for variables in their high school classes, the deep bias toward using "standard" x and $f(x)$ notation can cause problems and difficulties in university.

For instance, some students prefer to use x and $f(x)$ instead of a more suitable notation, such as t and $P(t)$, when studying population change. Faced with a body mass index formula (covered in a life sciences mathematics course) $BMI = \frac{m}{h^2}$ (mass divided by height squared, in SI units) students do not find it obvious that the graph of BMI as a function of m is a line through the origin with a slope of $1/h^2$. They have even more difficulty graphing BMI as a function of h. Likewise, many have problems recognizing that the function in the exponent in a cell-survival formula $S(D) = e^{-\alpha D - \beta D^2}$ is a parabola with $D = 0$ as one intercept.

As yet another example, while students do not have a problem to integrate $5x^3 + 12$, they typically find the integral $\int (At^m + B)\, dt$ which involves parameters and a "non-standard" symbol t for the independent variable, much more challenging. Further confusion is caused when implicit functions are studied, i.e., when a cognitive model of a function developed in high school needs to accommodate for the fact that the equation $5x^3 + y^2 = 10$ can be interpreted as a "usual" function $y = f(x)$, but also as a function $x = g(y)$, i.e., as a function of the independent variable y. Even further accommodation is needed in a study of functions of several variables, such as $f(x, y)$, where *both* x and y represent independent variables.

We have noticed that providing extensive opportunities to use a wide variety of symbols and notations facilitates students' learning and increases their comfort levels in our calculus classes. Based on our practice and experience, we suggest that high school teachers:

- Use the "standard" x and $f(x)$ notation in defining new terms and developing theory (thus enabling students to focus on the concepts), but then suggest a wide variety of symbols for variables and parameters in exercises, routine algebraic manipulations, as well as in problem solving activities;
- Discuss families of curves, i.e., functions whose formulas involve parameter(s), such as $y = mx$ (what happens as m changes?), $y = ax^2 + b$ (what feature of the graph is controlled by a, and what happens when we change b from positive to negative values?), or $y = \sin(ax)$ (how does the period depend on a?);
- Ask students to graph functions such as $s(D) = -\alpha D - \beta D^2$, label coordinate axes appropriately, and use terms such as "D-axis" and "D-intercept." When working on exercises related to applications, ask students to select appropriate symbols for variables and parameters (and remind them that this is common practice—for instance, illustrate with formulas from physics!).

As mentioned earlier, insisting that x represents an independent variable whereas y is used exclusively for a dependent variable could be a cause of misunderstandings and conceptual problems. When computing an inverse function, students recall that they have to "switch x and y and then solve for y." For instance, to find the inverse of $f(x)=\frac{2x-3}{x-7}$ this routine suggests that they write

$$y=\frac{2x-3}{x-7}$$

then switch x and y

$$x=\frac{2y-3}{y-7}$$

and then solve for y. Although it yields a correct answer in the end (assuming no algebraic errors are made), this method is conceptually unsound as it may not be clear to students why this works. As well, what is lost on most students is that y in $y=\frac{2x-3}{x-7}$ represents the function $f(x)$, whereas the same symbol y in the next line, $x=\frac{2y-3}{y-7}$, represents the inverse function $f^{-1}(x)$.

This routine becomes problematic when variables no longer represent abstract quantities; having to invert the degrees Fahrenheit to the degrees Celsius conversion formula

$$C=\frac{5}{9}(F-32)$$

using the process of switching the variables produces

$$F=\frac{5}{9}(C-32)$$

which is an incorrect formula (of course, one can proceed to compute C from it, and then at the end switch C and F again to obtain a correct formula; needless to say, conceptual understanding of the inverse is completely lost).

To avoid these problems, finding the inverse function routine should be rephrased as "solve for the independent variable," and followed by several worked examples[2] which use both standard and non-standard notations for the variables.

[2] Cognitive models are highly robust; even after we discuss these issues and illustrate with examples, our students will inevitably ask if they can still use what they learned in high school—i.e., "switch x and y".

"Number Bias"

We use the term *number bias* to refer to students' expectations that all numbers involved in calculations, as well as in the results (answers) are "nice".[3] A brief look at Ontario grades 9–10 and 11–12 curriculum documents (Ontario Ministry of Education 2005, 2007) reveals that there are very few places where it is suggested explicitly that students work with "non-nice" numbers, even in application problems. In the sample problem,[4] we read "The distance, d metres, travelled by a falling object in t seconds is represented by $d = 5t^2$" (instead of $d = 4.9t^2$; moreover, there is no indication that, for the given formula to hold, the vertical axis needs to point downward).

Similarly,[5] students are invited to investigate the graph of

$$f(x) = \frac{1}{x+n}$$

where n is an integer. Instead, n should have been a real number, with values such as −0.16 and 11.29 (it is somewhat unusual to insist on integers in the context of a calculus course, which is about *real numbers* and *real-valued* functions). Almost all examples of polynomials and rational functions (*ibid.*) involve integer coefficients. This is a root of the problem we witness when students in our university calculus classes have difficulties factoring expressions such as $x^2 - 0.01$, or completing the square in $a^2 - 0.52a$.

Applications are a good opportunity to work with "non-nice" numbers, and to demonstrate to our students that real-life problems demand that we use such numbers. For instance, when working with exponential functions, instead of discussing the function $y = 3x^5$, one can discuss the formula $Sk = 0.49Sp^{0.84}$ that relates the skull length to the spine length of a larger dinosaur. Or, in modeling the population of Canada, we could abandon using rounded numbers (31.6 and 33.5 million), and use thousands as units, thus working with 31,613 and 33,477 instead. As well, it is beneficial to study (and graph!) human daily temperature oscillation.

$T(t) = 36.8 + 0.34\cos\left(\frac{2\pi(t-14)}{24}\right)$ after studying an abstract function such as $f(x) = 2 + 3\cos(4t - \pi)$. In our experience, working with (many!) models and

[3] Numbers 12, 3/4 and 0.5 are viewed (declared) as "nice," however, 23/18 and 0.00102 are not considered "nice." Students sometimes refer to the latter as "unexpected."

[4] Ontario grades 11–12 Curriculum (Advanced Functions MHF4U, Understanding Rates of Change, item 1.6), page 96.

[5] Ontario grades 11–12 Curriculum (Advanced Functions MHF4U, Understanding Rates of Change, item 2.1), page 92.

applications significantly lowers the number bias levels, and modifies students' expectations of kinds of numbers their answers are supposed to contain.[6] As well, such applications give meaning and purpose to the underlying algebra.

Language and Culture of Mathematics

It is important to emphasize[7] to our students that mathematics consists of, and deals with concepts, objects, and algorithms which are precise, unambiguous, and well-defined. When we encounter something that we are not clear about, it is always a good idea to ask—what exactly is this about? What is the meaning of this word/phrase?

In other words, we need to know definitions and employ theorems, algorithms and other procedures appropriately, and with great care. We illustrate this claim in a few examples.

Consider the infinite sum

$$S = 1 - 1 + 1 - 1 + 1 - 1 + 1 - 1 + \ldots$$

Using our "finite sum" cognitive model, we "cancel" (here, subtract out) terms starting from the first term, and obtain

$$S = (1-1) + (1-1) + (1-1) + (1-1) + \ldots = 0$$

However, if we keep the first term, and cancel the remaining terms, we obtain

$$S = 1 + (-1+1) + (-1+1) + (-1+1) + (-1+1) + \ldots = 1$$

Clearly, we have a problem—what is the correct answer? It cannot be 0 and 1 at the same time. Our finite experiences and notions (adding numbers, "cancelling" numbers) do not generalize to infinite sums, and we need to know (i.e., we have to *define*) what is meant by the sum of infinitely many numbers. Once this is done, we get a clear answer—the above sum is divergent, i.e., it does not have a numeric value.

When working with prime numbers, we recall the definition:

> A prime number is a natural number that has exactly two distinct divisors: number 1 and itself.

To make sense of this definition, we need to know what natural numbers are, and what a divisor is. But on top of that, we must pay attention to the part "two *distinct*

[6] In our experience, providing ample opportunities for (carefully designed, and motivated) practice goes a long way.

[7] And keep repeating!

divisors," since it rules out the number 1 as being prime.[8] Thus, to start our list of prime numbers, we write 2, 3, 5, 7, and so on.

In our view, it does not make sense to discuss whether or not 1 is a prime number. It is not, and the definition is clear about it. What we can say to our students is that definitions are made for a reason—in this case (horizon knowledge!) the reason is to make the unique factorization theorem[9] work.

The fact that 0.99999… = 1 becomes clear once the infinite decimal representation of a number is given its precise[10] meaning as an infinite sum of numbers

$$0.99999\ldots = \frac{9}{10} + \frac{9}{100} + \frac{9}{1000} + \frac{9}{10{,}000} + \frac{9}{100{,}000} + \ldots$$

and when the definition of the sum of a series[11] is employed. In our view, discussions about the "last digit" (or the absence of one) in the expression 0.99999… are not worth much.

With time and through exposure, we learn that, although mathematics language is mostly clear and unambiguous, there are exceptions.

The most common exception is related to the use of the indefinite article "a." In some cases, such as in the statement "a differentiable function is continuous," the article "a" means "any," or "all," and thus represents a universal quantifier. Likewise, in the statement "for a real number x, the graph of the function $y = e^x$ lies above the x-axis," the article "a" means "for all."

In some cases, however, "a" refers to an existential quantifier. For instance, in the sentence "find a prime number between 10 and 1000," we interpret the article "a" as meaning that we need to find *one* (i.e., *any*) prime number between 10 and 1000, but not all of them. That this is a real issue, can be seen from students' replies to the true/false question.[12]

If a function has a critical point at c, then it has an extreme value at c.

A student, interpreting "a" as "some" (existential quantifier), will say that the statement is true (and indeed, for *some* functions, it is true). However, an instructor, interpreting "a" as a universal quantifier (as is common practice) will mark student's answer as incorrect. The roots of students' beliefs that an example can constitute a proof could easily be traced back to this misinterpretation.

Together with learning mathematics and its language, we also need to become familiar with its *culture*. The indefinite article case illustrates one aspect of it. There are many others, and as illustrations, we mention a few.

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[8] Of course, there are different ways to phrase the definition, but the meaning is always the same.

[9] Every natural number greater than one is a prime number, or can be written in a *unique* way as a product of prime factors. If 1 were a prime number, then uniqueness would be lost; for instance, $24 = 2^3 \cdot 3$, but also $24 = 1 \cdot 2^3 \cdot 3$. $24 = 1^5 \cdot 2^3 \cdot 3$, and so on.

[10] And only possible.

[11] In this case, the sum of a geometric series.

[12] Identify the statement as true or false.

Although the domain is part of the definition of a function, we do not write it explicitly in all situations. For instance, a common question

Find the derivative of the function $y = \sqrt{x^2 - 3x + 2}$.

usually comes without "for all x for which it is defined." Likewise, we ask students to find vertical asymptotes of the function

$$f(x) = \frac{2x+4}{x^2-4}$$

without adding "defined for all $x \neq -2, 2$." We need to clearly communicate to our students that the assumptions are always there, even when we do not write them out explicitly. A root cause of students' erroneous work with theorems (using a conclusion of a theorem without checking assumptions) might be related to this issue.

Ignoring assumptions leads to all kinds of errors. For instance, when solving the equation $x^2 = 7x$ students routinely divide both sides by x, and, forgetting that at that step the assumption $x \neq 0$ has been made, obtain a single solution $x = 7$. Or, asked to compute the composition $f \circ g$ where $f(x) = \ln x$ and $g(x) = \ln(\cos x)$, they routinely calculate $(f \circ g)(x) = f(g(x)) = f(\ln(\cos x)) = \ln(\ln(\cos x))$ without realizing that the composition makes no sense: the range of $g(x) = \ln(\cos x)$ consists of zero and negative numbers; the assumption for the composition[13] $f \circ g$ to be defined does not hold, and thus the composition does not exist![14]

An example of an imprecise mathematics statement is a common question such as

Where (for which x values) is the function $f(x) = x^2 + 3$ increasing?

Of course, we mark the answer (0, 1) as incorrect and $(0, \infty)$ as correct, because we[15] expect the question to be understood as:

Identify the largest interval of real numbers on which the function $f(x) = x^2 + 3$ is increasing.

A common calculus question

Find all x where the function $f(x) = x^{-1/3}$ is not continuous

is a cause of confusion: what numbers x do we consider—those which are in the domain of $f(x)$, or all real numbers? By checking the answer ($x = 0$) we realize that it is the latter.[16] Often, we ask students to "find the limit …" even when the answer to the question is that the limit does not exist.

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[13] The range of g is contained in the domain of f.

[14] Not all is lost—if we look at the formula for the composition and ask what the domain is, we'll figure it out.

[15] Teachers, instructors, and all others familiar with mathematics culture (i.e., "math nature").

[16] This particular situation is left vague in many calculus textbooks.

Although these (and many other) situations present no problems for teachers, instructors, or experts, who may be aware of the inherent embedded assumptions, they could be (and are!) quite confusing to novices. In some cases, it is easy to avoid confusion (for instance, by rephrasing the limit question as "find the limit or else say that it does not exist"); however, in general, as they are encountered, such situations have to be clearly identified and their precise meaning revealed.

Logical Reasoning

Since the word "theorem" does not appear in the grades 9–10 and 11–12 Ontario curriculum documents, it is safe to assume that the logical structure of a theorem (namely, the implication[17]) is not at all discussed in high school, at least in Ontario.[18] A theorem consists of one or more statements which constitute assumption(s), and of one or more statements which form its conclusion(s); its logical structure is expressed in the English language as "if assumption(s) then conclusion(s)."

Once we verify that all assumptions are true for a given problem, we draw the conclusions. The most common belief that students hold is that if an assumption in a theorem is not true, then the conclusion is not true either. It is easy to show that this is not so: consider the theorem

If the last digit of a number N is 4, then N is even.

Clearly, if the last digit of N is not 4 (assumption not satisfied), N could still be even (say, $N = 18$), i.e., the conclusion of the theorem still holds.[19]

It is clear that the reverse of the above theorem, i.e., the statement

If N is even, then the last digit of N is 4.

is not true. However, when this same construction is cloaked in abstract context, things are no longer as obvious. The best evidence is the theorem

If the series $\sum_{i=1}^{\infty} a_i$ *is convergent, then* $\lim_{i\to\infty} a_i = 0$.

which is often reversed to, and used in the incorrect form

If $\lim_{i\to\infty} a_i = 0$, *then the series* $\sum_{i=1}^{\infty} a_i$ *is convergent.*

We have noticed that students understand logic better if we discuss an "obvious" logical structure first ("obvious" meaning in an easy-to-understand, familiar context), and then apply it in the abstract situation, as we have done with the if-then statement above.

[17] Or equivalence ("if an only if").

[18] However, university mathematics instructors routinely assume that students are familiar with it.

[19] But the theorem does not apply.

Consider another example. Students are asked to determine whether the following statement is true or false, and to justify their answer:

For every natural number n, the number $n^2 + n + 41$ is prime

A common difficulty is the strategy, as students are not sure how to prove that their answer (true or false) is correct. It helps to consider a simple statement, such as "Every dog in Ontario is black" and ask students to articulate what would it take to prove that this statement is true (we have to check that every single dog in Ontario is black) or false (we need to find one dog in Ontario which is not black). Armed with this understanding, students are now more confident that by saying that when $n = 41$ the number $n^2 + n + 41$ has a factor of 41 and thus they have proved that the given statement is false.

An appropriate suggestion to help students overcome the difficulties illustrated here and to improve their mathematical reasoning skills is to use writing assignments, in particular expository[20] and excogitative[21] types of writing. DeDieu and Lovric (2018) are exploring ways in which students benefit from having to write in the context of a differential equations course. Burazin and Lovric (2015) suggest that working on, and keeping an archive of one's mathematical work (including narratives, of course) in the form of a learning portfolio can further enhance student learning.

Conclusion

We presented a few cases of the *culture shock* situations that students experience in transition to university mathematics. With time and through teachers' and students' active involvement, these transitional issues can be minimized.

Case studies presented here could be included into high school teachers' horizon knowledge. An ability to see and understand how mathematical ideas and reasoning develop over a longer time scale can inform teaching (and lesson planning), and thus better prepare students for the transition to tertiary mathematics. For exactly the same reasons, such case studies should find their way into university teaching.

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[20] Use narratives to describe and explain a mathematical idea, theorem, or definition.

[21] Carefully, and in detail, explain the reasoning in a mathematical argument or in an algorithm.

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Suggestions for Further Reading

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