Part IV: Commentary – Characteristics of Mathematical Challenge in Problem-Based Approach to Teaching Mathematics

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There is a strong consensus among mathematics educators, researchers and instructional designers that mathematical problem solving is among the central means—and ends—of school mathematics education. Different ideas, practices and studies in the field of mathematical problem solving are reflected in the volumes, chapters, and papers published over the course of the last 30-plus years (Felmer et al. 2016; Lester and Charles 2003; Schoenfeld 1985; Silver 1985). The problem-based approach to teaching mathematics assumes that students are presented with authentic problems that are meaningful for them, and that can be solved using mathematical tools available to them. The problem-based approach seeks to develop new mathematical knowledge and skills through solving such problems. Moreover, when solving these problems the students are assumed to develop appreciation for the power of mathematics to solve problems from different fields of life and science.

Cai (2010) argues that mathematics teaching is a system of interrelated dimensions that include the nature of classroom tasks, their content and context, the teacher's role, the classroom culture, mathematical tools, and concern for equity and accessibility. The collection of works in this section of the book demonstrates the richness and variability of problem-based approaches to teaching mathematics designated and advanced by the members of the Canadian mathematics education community.

In this response chapter, I address the nine manuscripts in this volume that are devoted to the problem-based approach to teaching and learning mathematics. Interestingly, the authors differ in their views on what constitutes good problems, the corresponding goals of mathematical instruction, teachers' role in the management of the problem solving process and the ways in which different participants of the problem solving session can be supported when solving the problems.

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Good Problems

Problem-based approach to learning unifies choices of "good problems" by the authors. All the tasks that the authors use are non-routine and directed at communication among students about the problems, and activation and development of critical reasoning and decision making. Overall, the authors tend to share the three preferences or their combinations: non-routine mathematical problems, context-based problems that require modeling, and curricular-related problems accompanied by specific explorative design.

Non-routine Mathematical Problems

For Liljedahl, good problem solving tasks are highly engaging, non-routine, collaborative tasks that encourage mathematical discussion among students as part of the problem solving process. These problems promote students' mathematical thinking ("developing a thinking classroom") and allow regulation of the level of mathematical challenge, such that the level of challenge can be fit to each student in the class. The importance of the non-curricular nature of mathematical tasks is also addressed by Hoshino, who suggests choosing problems from logic puzzles and contest questions, and by Godin, for whom "mathematics competitions are a great place to access good problems." Godin categorizes good problems as follows: investigation tasks, novel problems, real world problems and technologically aided explorations. The significance of technological tools is also addressed by Saldanha and Thibault, who describe activities that can be done with the use of *TinkerPlots*, an interactive and dynamic data exploration software that advances statistical reasoning.

Context-Based Meaningful Problems

In Savard's view, meaningful problems must guide students to make sense of mathematics within a meaningful context and require mathematical modelling of the situation described in the problem in order to solve it. When solving these problems, mathematical knowledge is needed to study the event or the phenomenon. Similarly, according to Martin, Oliveira and Theis, mathematical tasks should allow students to develop new knowledge through real mathematical activity.

Emphasis on the problems' context both as means and ends of the educational process is made by several authors. Interestingly, social justice and citizenship unify several chapters. Savard stresses the importance of the careful choice of the problem context to make solving problems intriguing to students, and argues that choosing a context associated with citizenship develops both students' citizenship skills and their mathematical knowledge and skills. Such problems encourage critical thinking, advance awareness of cultural context, and connect cultural context to the mathematical knowledge to be developed.

Russell maintains that social justice can be embraced by mathematics education in two ways: as a context for mathematical problem solving, and by teaching mathematics in socially just ways. In line with the first way, Mamolo, Thomas and Frankfort describe their experiences of incorporating social justice context problems related to the variety of topics that meet students' needs and interests into problem-based practices, thus making mathematics learning meaningful for students. In this context, mathematics serves as a tool for "understanding world issues and social trends." The authors introduce tasks that present authentic problems and allow students to choose which mathematical (and non-mathematical) tools and skills to use in solving. They also require students to discuss and defend their solutions. Mathematical exploration is an integral part of these good problems.

Explorative Design of Curricular Problems

While focusing on content-related areas (e.g., statistics in the chapter by Saldanha & Thibault, and probability in the chapter of by Martin, Oliveira, & Theis) the authors stress the importance of careful choice of appropriate didactical settings and problem solving approaches. For example, Martin, Oliveira and Theis stress the mathematical power of the combination of different approaches to solving probability problems while Saldanha and Thibault emphasize explorative technology-based learning involving dynamic and visual imagery of data, as well as the importance of encouraging students to share ideas and explain their thinking.

Martin, Oliveira, and Theis stress that good problems should engage students and allow them to discover for themselves some of the means needed for solving, using comparison, connection, and sharing of ideas.

Instructional Setting and Teachers' Roles in Monitoring Problem Solving Activity

One of the central roles of a teacher is devolution of good tasks to learners (Brousseau 1997; Steinbring 1998). When assigning cognitively demanding tasks to a particular classroom, teachers should "feel" their students, in order to ensure that the students are capable of solving the task. Moreover, development of students' mathematical reasoning is linked to the knowledgeable choice of challenging mathematical tasks and the integration of the tasks in appropriate settings (Choppin 2011).

Teachers ought to provide each and every student with learning opportunities that fit their abilities and motivate them to learn. Teachers should create an instructional setting that supports and advances the problem solving process. All the authors stress that problem solving should be appropriate to students' knowledge and problem solving capacity at the given time. Investigation, exploration and challenging ideas are the core elements of the problem solving activities in this section of the book. All the chapters in this section address these aspects with different levels of detail.

Liljedahl introduces 11 elements that determine the effectiveness of problem solving activities, and that encourage mathematical thought in students. These elements include starting a lesson with the task, random arrangement of small groups, use of oral instructions, defronting the classroom and answering "keep thinking" questions only. Liljedahl suggests providing students with autonomy, using hints and extensions to allow flow and "levelling to the bottom." Martin et al. suggest that one of the critical features of the problem solving setting is the problem's adaptability, that is, its level of complexity and the ease of adjusting it to students' levels of mathematical development. Another way of adjusting the instructional level is providing students with opportunities to explicitly develop various strategies for solving new problems, such as trial and error, using a model, trying a simpler problem, working backwards, and discussing their ideas in small groups (Atiya, Luca, & Kajander).

Atiya et al. focus their attention on the ways to support development of beginning teachers' proficiency and beliefs in managing problem-based instruction. They suggest gradually making classroom tasks more and more open, as students develop more strategies. An additional practice of "turning students into teachers" is suggested by Godin. Godin also acknowledges the importance of implementing a variety of settings, incorporating group work, independent work, and different combinations of the two in order to allow all students to participate actively in problem solving activities.

Mathematical Challenge As a Core Element of a Problem-Based Approach to Teaching Mathematics

Mathematical challenge, which is an interesting and motivating mathematical difficulty that a person can overcome at a particular stage (Leikin 2007, 2014), is a unifying characteristic of all the mathematical activities, tasks and problems described in this section of the volume. Here I suggest a theoretical model of a mathematical challenge embedded in a problem solving activity (Fig. 1). This model can shed light on the collection of papers observed here and suggests an additional lens for the analysis of problem solving activities suggested by the authors. The model comprises several complimentary elements, which can enhance and support each other in the creation of mathematically challenging situations. These elements include (a) intrinsic (conceptual) characteristics of mathematical problems and tasks which are in the center of mathematical activity; (b) characteristics surrounding a problem or a task such as socio-mathematical norms, instructional setting and (c) individual characteristics of the participants, such as their familiarity with the topic of the problem or their problem solving proficiency (see Fig. 1).

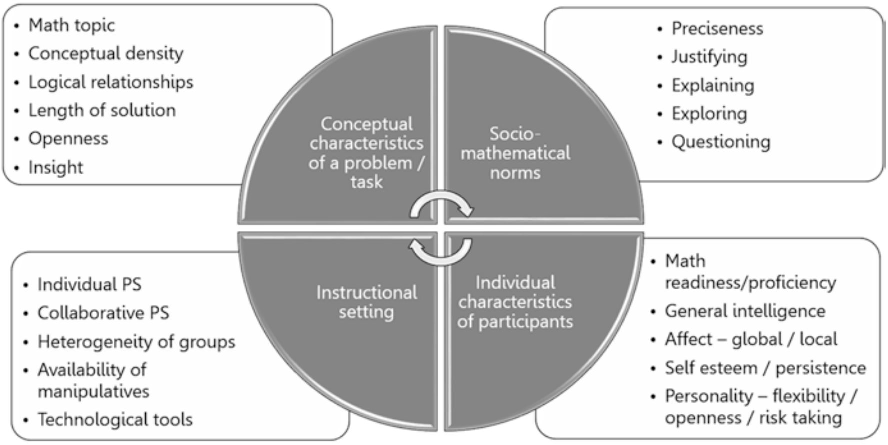

Fig. 1 Characteristics that determine mathematical challenge

The level of cognitive demand (Silver and Mesa 2011) of a particular task depends on the type and conceptual characteristics of the task, such as conceptual density, mathematical connections, and the logical relationships required for solving the problem (Leikin 2009, 2014; Silver and Zawodjewsky 1997). The openness of a problem solving task also determines its level of cognitive demand. For example, open-ended problems allow multiple answers to a problem along with critical evaluation of completeness of the set of answers (e.g., Pehkonen 1995). Open-start problems are usually multiple solution tasks (MSTs – Leikin 2007) that require solving the problem using multiple solution strategies, through activation of mental flexibility and mathematical connections. Tasks such as sorting tasks, problem posing tasks and investigation tasks are both open-start and open-end problems. Several chapters in this book section (e.g., Saldanha & Thibault and Martin, Oliveira & Theis) include excellent examples of mathematical investigation tasks. Mathematical challenge can be strengthened by the use of a non-mathematical context and the requirement to design a mathematical model that represents this context (see examples in Savard and in Martin, Oliveira & Theis). The mathematical challenge embedded in a task can be increased by socio-mathematical norms such as requirement of preciseness, explanation and justification, comparison and classification (Leikin 2014; Silver and Mesa 2011) and can be varied by instructional setting, for example, as described in the chapters by Atiya, Luca and Kajander, Liljedahl and Goding. The familiarity of the topic and associated problem solving proficiency as well as personal characteristics of the participants are additional criteria that characterize the solver rather than the problem.

The chapters in this section of the book present a variety of ideas expressed by researchers, teacher educators and mathematics teachers who share their authentic experiences and the methods that they find effective in mathematics teaching. One of the most challenging aspects of the mathematics lesson management is making the mathematical problems that students solve challenging to each student in the classroom. While different chapters refer to different components of mathematical challenge to different extents, the collective account describes a rich collection of challenging problem-based activities that can be used by teachers in their classes and by mathematics educators in teacher education settings.

Rather naturally, the practices described in these chapters can serve as an excellent playground for researchers who are interested in getting a better understanding of which approaches are more effective for the development of students' mathematical knowledge, skills and problem solving expertise along with the development of students' motivation and self-esteem in learning mathematics. The model of mathematical challenge suggested here can serve as a framework for the analysis of the quality of problem-based teaching of mathematics.

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