Advances in Mathematics Education

Ann Kajander · Jennifer Holm
Egan J Chernoff *Editors*

Teaching and Learning Secondary School Mathematics

Canadian Perspectives in an International Context

Springer

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Foreword

> We have proved that we will not be assimilated. We have demonstrated that our culture has a viability that cannot be suppressed. … [A]s the years go by, the circle of the Ojibway gets bigger and bigger. Canadians of all colours and religion are entering that circle. You might feel that you have roots somewhere else, but in reality, you are right here with us. – Grand Chief John Kelly (quoted in Saul 2008, p. 29)

From 2005 to 2008, I held a faculty position in the Department of Mathematics, University of Regina. While I was there, the dean of the Faculty of Science decided that everyone employed by the faculty would have to take "Aboriginal Awareness" training. I went to see the dean to try to convince her that as a Mohawk, I was already quite aware, and besides, I had taught Aboriginal Studies at the University of Toronto, I had designed the first two courses in Queen's University's Indigenous Studies program, I had spent 8 months in a Mohawk language immersion program, I had worked for 4 years at First Nations University of Canada, and so on. I felt that my time could be better spent in other pursuits.

"No," said the dean. "Everyone takes the training. You, me, everyone. No exceptions."

All right then.

When the big day arrived, the facilitators of the workshop divided us into groups of about four each to play an Aboriginal awareness quiz show, presumably designed to educate while humiliating the less aware members of the group. At stake was a set of four beaded keychains for the winning team. The dean chose to join my team.

I vowed not to hold back. If I could not spend my time productively in mathematics as I wished, I would at least spend it in the pursuit of a winning of some kind. And I felt that all my years of learning and experience should be worth something: a beaded keychain, at the very least.

I answered question after question correctly, building a comfortable lead over all the other teams. The dean seemed pleased. Everything was going well, and then we were given the question, "What percentage of the land of Saskatchewan is treaty land?"

Even though I did not know the exact answer offhand, I hit the buzzer immediately like I always did, thinking I could rely on my general knowledge and my mathematical prowess to arrive at a better answer than any of the other teams in a few seconds. Based on a memory of a map of Saskatchewan I had seen which included reserve lands, and some population numbers I had available, I answered, “One percent.”

Wrong. So very wrong. Not even close. In fact, every other team had a better answer than me.

Now it was my turn to be humiliated. The dean looked at me with confusion and concern, possibly with a vision of a beaded keychain eluding her grasp. I could only stare in surprise as the facilitators chose the team with the best answer, and then revealed the correct answer.

“The percentage of land in Saskatchewan which is treaty land is 100%.”

Ah, of course. The entirety of Saskatchewan is covered by numbered treaties, mainly Treaties 4, 5, 6, 8, and 10, but also small parts covered by Treaties 2 and 7. Every square inch of the province is governed by conscious agreement between Indigenous parties and the Crown. It was a humbling, enlightening, and educational moment for me. And despite our setback, my team recovered and we went on to win the coveted beaded keychains. Mine continues to be one of my prized possessions.

When I recounted the story to the late Elder Ken Goodwill, he said, “That is important to understand,” and then he told me another story. He said he was involved in a meeting in northern Saskatchewan, when a non-Indigenous member of the group suggested that it might be a good idea to end the treaties. Ken said that an old Indian man spoke up at that point. He said, “I guess we could talk about getting rid of the treaties. But then, where would all you white people live?”

There is a variety of lessons that could be taken from the above stories. There are two that I would like to focus on: first, Indigenous culture and issues are foundational to Canada; and second, Indigenous culture and issues continue to be a necessary part of anything Canadian, including this book, even if the Indigenous connection is not immediately apparent. John Ralston Saul writes,

> The original party, the Aboriginal, is built upon a philosophy that has interdependence at its core. This is the opposite of such European ideas as the melting pot, which was picked up by our neighbor as a way of explaining how you could get a new kind of European-style purity out of a mix of peoples. The idea of difference is central to indigenous civilization. These differences are not meant to be watertight compartments, not vessels of purity. It is all about how to create relationships that are mixed in various ways and designed to create balances. It is the idea of a complex society functioning like an equally complex family within an ever-enlarging circle. That is the Canadian model. (Saul 2008, p. 107)

With that notion of interdependence and mixed relationships in mind, I read the chapters of this book searching for connections to Indigenous cultures, issues, and peoples, and I was pleased to find a connection, sometimes strong and sometimes tenuous, in every chapter of the book. I would like to share my findings with you now.

Many of the chapters of this book are directly about Indigenous mathematics education: “Indigenous Perspectives in School Mathematics: From Intellect to Wisdom” (Aikenhead); “Drawing upon Indigenous Knowledges to Transform the Secondary Mathematics Classroom” (Lunney Borden); “Introduction to Students at

Risk: Case Studies of Often Unheard Students" (Kajander); "Considering Indigenous Perspectives and Mathematics Education: Stories of Our Experiences As Teachers and Teacher Educators" (Sterenberg and O'Connor); "My Favourite Mistakes: Experiences Teaching Cree Students in Northern Communities" (Newell); "Exploring Math Through Social Justice Context Problems" (Mamolo, Thomas, and Frankfort); and "Social Justice and the Teaching and Learning of Mathematics" (Russell) all mention Indigenous students or cultures explicitly. Many of the researchers above have produced a considerable body of work in Indigenous math and science education. It is a great pleasure to see them continue their work in this volume. Mamolo et al.'s and Russell's chapters are also about social justice more broadly, which is relevant to many other groups within our circle.

Numerous chapters in the book are about community more generally, which is of course a concern of Indigenous people: "An Unexpected Adventure" (Childs and Holm) and "Considering Both Academic and Social Domains: Increasing Student Engagement in At-Risk Classrooms" (Jao). Three other chapters emphasize the notion of students as partners and working with students rather than doing to students: "Re-framing Testing to Better Fit Within Problem Solving Classrooms: Ways to Create and Review Tests" (Rapke, Hall, & Marynowski); "Enhancing Mathematics Teaching and Learning Through Sound Assessment Practices" (Suurtamm); "Assessment: Broadening Our Conceptions to Improve Our Practice" (Pai). The chapter by Davis et al. ("Steps Toward a More Inclusive Mathematics Pedagogy") speaks to inclusivity and growing the circle. The 400-year-long tradition of Indigenous people in this country welcoming immigrants is honored in: "Learning Mathematics When Students Are New to Schooling and New to English" (Barwell, Kubota-Zarivnij, and Culotta) and in Minority students: "Success in Grade 9 Applied Mathematics Courses" (Macaulay).

Respect for culture, of one kind or another, can be found in "On Teaching and Learning Mathematics from a Cultural-Historical Perspective" (Radford, Miranda, and Lacroix); "Culturing Affect, Affective Cultures" (Roth); "Support to Thrive: Raising Resilience in Students in Secondary Schools" (Hurlington); and "Transition from Secondary to Tertiary Mathematics: Culture Shock—Mathematical Symbols, Language and Reasoning" (Burazin and Lovric), in which mathematics is responsible for a culture shock with which many Indigenous students and Indigenous people in general are familiar.

Many of the chapters of the book discuss or use some of the tools of Indigenous education: the emphasis of action over object, process over product, and verb over noun; the use of stories as powerful communication and teaching tools; working with the land, or with more abstract landscapes, or the environment more generally; the establishment of good interpersonal relationships as a foundation for learning and development; teaching by example; holism, the understanding that for health and strength we should be aiming for not only intellectual development but also physical, emotional, and spiritual development; the importance of opening our eyes and using all of our senses; and finally, the value of studying practical problems which may be relevant to our own lives. I am not claiming that those tools are absent in other traditions; after all, everyone is descended from one Indigenous people or another, all of whom may have had similar educational tools. Rather, I am suggesting that it is not a surprise that we find those tools in a volume by Canadian scholars, because the Canadian spirit is really the Indigenous spirit, as John Ralston Saul points out.

The idea of reemphasizing verbs and action in mathematics, in the manner of Indigenous thought, is quite explicit in Lisa Lunney Borden's chapter ("Drawing upon Indigenous Knowledges to Transform the Secondary Mathematics Classroom"); Lunney Borden has written extensively on the subject. Verbs, action, and process are also important topics in "Mathematical Mindsets for the Teaching and Learning of Mathematics" (Pyper) and "Digital Technology in Teaching Mathematics Competency: A Paradigm Shift" (Chorney), which discuss understanding the concept of reflection versus providing a name for it, and the sociopolitical amplification of that difference. Pai ("Assessment: Broadening Our Conceptions to Improve Our Practice") discusses the notion of assessment as a process.

The use of stories as teaching tools appears in "Powerful Stories: The Hitchhiker's Guide to the Secondary Math Curriculum Landscape" (Taylor, Lala, Ouellet, and Knebel); "Building Thinking Classrooms" (Liljedahl); "Improving Students' Approaches to Learning High School Mathematics" (McFeetors); "'Canada Is Better'—An Unexpected Reaction to the Order of Operations in Arithmetic" (Zazkis); "Culturing Affect, Affective Cultures" (Roth); and in "My Favourite Mistakes: Experiences Teaching Cree Students in Northern Communities" (Newell). Zazkis's chapter is notable for its use of a completely made-up story. The use of untrue stories and myths to approach a truth is another technique in Indigenous education.

The notion of a landscape or environment, physical or abstract, appears in "Powerful Stories: The Hitchhiker's Guide to the Secondary Mathematics Curriculum Landscape" (Taylor et al.); "Live(d) Topographies: The Emergent and Dynamical Nature of Ideas in Secondary Mathematics Classes" (Thom and Glanfield); "Bottles and Bridges: Sample Classroom Tasks Created by Beginning Teachers" (Atiya, Luca, and Kajander); "Success in Grade 9 Applied Mathematics Courses" (Macaulay); and in "'Canada Is Better'—An Unexpected Reaction to the Order of Operations in Arithmetic" (Zazkis). I would also like to note that Florence Glanfield is Indigenous; I hesitated to single her out, because other authors may have Indigenous blood or family ties of which I am unaware. For example, one study concludes, "The results indicate that, in each region [of four regions in Québec], more than half of the participants have at least one Amerindian ancestor in their genealogy" (Vézina et al. 2012, p. 99). (On the other hand, one might argue that the Indigeneity of any individual is irrelevant, if we admit that the whole culture of Canada is Indigenous.) However, in Florence's case, Indigeneity goes hand-in-hand with a long-standing commitment to Indigenous education and social justice.

The value and importance of interpersonal relationships appears in "A Teacher's View – It's a Path, Not a Gap: A Relationship-Based Approach to Mathematics and Well-Being" (Boland and Tranter); "Building Capacity in Grade 9 Mathematics: Case Studies from a Collaborative Inquiry Project in Applied level Mathematics" (McDougall and Ferguson); "Considering Both Academic and Social Domains:

Increasing Student Engagement in At-Risk Classrooms" (Jao); "Assessment: Broadening Our Conceptions to Improve Our Practice" (Pai); "Culturing Affect, Affective Cultures" (Roth); and "Support to Thrive: Raising Resilience in Students in Secondary Schools" (Hurlington). The latter chapter also discusses teaching by example, as does "My Favourite Mistakes: Experiences Teaching Cree Students in Northern Communities" (Newell).

Holistic approaches, for example, recognizing that balance and well-being is associated with the development of not only intellectual capacity but physical, emotional, and spiritual, appear in "Observing for Mathematical Proficiency in Secondary Mathematics Education" (Corrêa); "Considering Both Academic and Social Domains: Increasing Student Engagement in At-Risk Classrooms" (Jao); "Assessment: Broadening Our Conceptions to Improve Our Practice" (Pai); "Culturing Affect, Affective Cultures" (Roth); and "A Teacher's View – It's a Path, Not a Gap: A Relationship-Based Approach to Mathematics and Well-Being" (Boland and Tranter).

The Indigenous value of opening our eyes and using all our senses, "noticing," is one that I have had difficulty explaining to non-Indigenous people, who are sometimes insulted by the suggestion that they are not observing well. The issue is that non-Indigenous culture is full of theories and assumptions, mental filters that literally get in the way of seeing the world as it truly is. Saying that "the tipi is a cone" is a perfect example of a theory (solid Euclidean geometry) getting in the way of the reality of tipis, preventing one from noticing that they are much more complicated and irregular structures than cones, with a tradition that rivals that of Euclidean geometry. Another illustration of the concept is the following almost Zen-like story:

> … a Ute student was asked to determine how much his brother would have to spend on gasoline if he wanted to drive his truck from the reservation to Salt Lake City. Instead of estimating (or generalizing) a response, or attempting to calculate an answer based on the information presented in the request, the student responded quite simply: "My brother does not have a pickup" (Leap 1988, p. 176)

The idea of opening one's eyes, or "noticing," is a feature of "Reflecting on Good Mathematics Teaching: Knowing, Nurturing, Noticing" (Oesterle); "Teaching Mathematics and Developing Citizenship: How to Use Contexts to Enhance Problem-Based Learning" (Savard), which discusses authentic tasks in the Canadian context; "Teaching Probability in Junior High School Through Problem Solving: Construction and Analysis of a Probabilistic Problem" (Martin, Oliveira, and Theis); and "Promoting Students' Reasoning About Statistical Inference Through Engagement with a Problem-Based Instructional Activity Involving the Use of *TinkerPlots*© Software" (Saldanha and Thibault), which shows how statistics and statistical inference can help us understand social and political world better. The chapter by Banting, Vashchyshyn, and Chernoff ("In No Uncertain Terms: Encouraging a Critical Stance Toward Probability in School") discusses the "Ludic fallacy," the misuse of games to model real-life situations (Taleb 2010, p. 122). The use of visual and tactile senses is a feature of "Learning Algebra with Models and Reasoning" (Kajander).

Practical problems are found in "Supporting Mathematical Creativity Through Problem Solving" (Hoshino); "Problem Solving in the Secondary Classroom" (Godin); "Modelling in Secondary Mathematics Education: Moving Beyond Curve Fitting Exercises" (Caron); and "Teaching a University Bound Statistics Course" (Gardner), which references Statistics Canada's extensive collection of Indigenous statistics; and "Planning a Unit by Starting with the End in Mind: Unit and Lesson Planning" (Holm) which also mentions popcorn, a variety of corn, a food domesticated by Indigenous researchers nearly 10,000 years ago, which then spread throughout the Americas and then to the rest of the world.

The (extremely) alert reader will note that I have found an Indigenous connection to every chapter in the book, with one exception: "Encouraging Able Students: An Example of Composition of Linear Polynomials" (Barbeau). For a while I puzzled over how to connect Ed Barbeau's elegant discussion of a bit of pure mathematics to any of the Indigenous themes above, and I could not think of a way. However, earlier I promised you a connection between every chapter of the book and Indigenous cultures, issues, and peoples. The connection I finally found was truly astonishing. Over 30 years ago, Barbeau corresponded with a young Indigenous student who was interested in mathematics but struggling with problem solving at the competitive level. Barbeau sent dozens of problems to the young man, and endured his inelegant responses, patiently mentoring him. Thanks in part to Ed Barbeau's encouragement and faith in the student's potential, the young man decided to pursue a degree in mathematics and eventually completed a PhD in pure mathematics. I know the story well (though for a while today it escaped me, oddly) because that young Indigenous man was me. I would like to take this unusual opportunity to express my heartfelt gratitude to the one person who, more than any other, started me on the path which would eventually lead me to becoming a mathematician.

The lesson I take from the story above is that there should even be room in our circle for pure mathematics, with no obvious connection to Indigenous issues, or social justice issues, or culture, or whatever else, because pure mathematics can be a joy and a guiding light, something like a North Star but really more like a lighthouse because it is attainable. We do not know in advance who will be drawn in, but we can hope there will be more, many more in the years to come, and that some of them, at least, will be Indigenous. I would like to close with one final quotation from John Ralston Saul's book:

> And so I find that our education is increasingly one aimed at training loyal employees, even though the state and the corporations are increasingly disloyal. What we should be doing is quite different. It turns on our ability to rethink our education and our public expectations so that we create a non-employee, non-loyal space for citizenship. After all, a citizen is by definition loyal to the state because the state belongs to her or him. That is what frees the citizen to be boisterous, outspoken, cantankerous, and, all in all, by corporatist standards, disloyal. This is the key to the success of our democracy.
>
> The Aboriginal idea of a circle is based upon the idea of tension. We need to redesign our education to do the same. When I say it needs to be about thinking, not training, I could equally say it needs to be about engagement and aggressive debate, not about smooth expertise and passive service. (Saul 2008, p. 318)

I hope I have convinced you that Indigenous people are more than just an object of study or a problem to be solved; I hope I have convinced you that Indigenous culture and thought can contribute to enhancing mathematics education, and that Indigenous education has solutions to some of the difficulties that we, Canadians and all other human beings, face. That the question is what we, Indigenous people, have to offer to Canada, and to the world, not just what you have to offer to us.

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Edward Doolittle,

References

Leap, W. L. (1988). Assumptions and strategies guiding mathematics problem solving by Ute Indian students. In R. Cocking & J. Mestre (Eds.), *Linguistic and cultural influences on learning mathematics* (pp. 165–186). New York: Routledge.

Saul, J. R. (2008). *A fair country: Telling truths about Canada*. Toronto: Viking Canada.

Taleb, N. N. (2010). *The black swan: The impact of the highly improbable* (2nd ed.). New York: Random House.

Vézina, H., Jomphe, M., Lavoie, E., Moreau, C., & Labuda, D. (2012). L'apport des données génétiques à la mesure généalogique des origines amérindiennes des Canadiens français. *Cahiers Québécois de Démographie, 41*, 87–105.

Preface

A Northern Vision of Mathematics Education

The landscape of Canadian mathematics education has both unique and universal challenges, and this volume seeks to illuminate this diversity in an international context.

Each section of the volume includes a range of cultural perspectives: cultural both in the sense of language and ethnicity, as well as the more situated culture of the teaching context. Alternatively stated, each section includes, where possible, voices from our Indigenous cultures, French-language culture, and also from the more contextual cultures of research, mathematics education, and classroom teaching. While in some cases the chapters written by classroom teachers have been coauthored by researchers (see, for example, Parts I and II), other sections include the voices of classroom practitioners on their own. We also interpret this breadth of approach to include the transition to post-secondary education, as well as to include students at-risk, as well as able and creative students.

Canadian curricula, as is the case internationally, have been significantly changing over the past 20 years, and this change is ongoing. While the overall direction of this evolution is toward problem solving, vocal exceptions to this vision have disrupted this progression.

Many mathematics educators, perhaps by nature, are not political creatures, and all too often have quietly shaken their heads at poorly informed rhetoric and catchy headlines, which proclaim the need for a return to a traditional approach to teaching mathematics. Our volume seeks to expose such inflammatory sentiments by presenting a research-based view of the current landscape, inclusive of the challenging realities of day-to-day classroom teaching in the Canadian context.

The book is divided into six sections. Part I introduces the cultural-historical evolution and context of the Canadian landscape. Part II addresses inclusivity, including a focus on traditionally less successful learners. Part III takes a varied stance to relationships and affect in mathematics education. In Part IV, problem solving as a learning paradigm is specifically examined in our context. Part V focuses on the day-to-day reality of assessment and planning in classrooms. And lastly, Part VI focuses on enhanced content and the transition to higher level mathematics. Based on our organization of this volume, sections may be read separately, in order, or in any order.

As is the tradition of the “Advances in Mathematics Education” series, each section of the book is introduced by a preface written by an experienced Canadian mathematics educator—our Elders. The chapters are then situated in an international context via the section commentaries contributed by international scholars. Taken together, this pluralistic landscape seeks to inform secondary mathematics education both here in Canada and also, thanks to the international context, with a global perspective.

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Saskatoon, SK, Canada — Egan J Chernoff
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Contents

Part I
The Changing Landscape of Teaching and Learning Mathematics

Part I: Preface – On Changing the Landscape: Ecclesiastes Vs. Heraclitus

Rina Zazkis and Nathalie Sinclair

"What has been will be again, what has been done will be done again; there is nothing new under the sun."

— Ecclesiastes I:9.

"Nothing endures but change."

— Heraclitus.

In the summer of 2017, acknowledging Canada's 150th anniversary, the BBC featured an article, "How Canada became an education superpower". (See http://www.bbc.com/news/business-40708421) The article praised the Canadian education system, indicating that "Canada has climbed to the top tier of educational ranking." The theme of equity was a major thread describing Canadian education. The following quotes exemplify this thread:

- "Rather than a country of extremes, Canada's results show a very high average, with relatively little difference between advantaged and disadvantaged students."
- "Despite the different policies in individual provinces, there is a common commitment to an equal chance in school."
- "It is a remarkably consistent system. As well as little variation between rich and poor students, there is very little variation in results between schools, compared with the average for developed countries."
- "There is a strong sense of fairness and equal access."

It seems that the chapters in this section challenge the claims of equal chances and equal access. In particular, the chapters that focus on Indigenous ways of knowing (Aikenhead and Lunney Borden) highlight the lack of equity that has long pervaded

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in the school mathematics curriculum in Canada (and everywhere else). Interestingly, both these chapters make the argument that in order to increase equity—and increase the success rate for Indigenous students—we need to think about *changing mathematics*, and not just changing the curriculum, or the way mathematics is taught. For Aikenhead, this would involve a shift in mathematics as an intellectual activity towards mathematics as being about wisdom. For Lunney Borden, the necessary shift is more specifically in language, and, in particular, in a move away from the nounification that is common in the mathematics discourse, and towards verbs—towards movement and spatio-temporal ways of interpreting and describing the world. This move can be seen as a return to the ancient, performance-based mathematics that eventually—with the help of paper-and-pencil technology—evolved into its current depersonalised, decontextualized and detemporalised state (Balacheff 1988). While such a verbified approach may better support students' initial conceptual understanding, we wonder how they can then shift to the more objectified style of formal mathematics; and indeed, whether they must in order to participate in the mathematics-infused STEM disciplines for which we are trying to prepare them.

While the BBC article praised Canada, especially referring to the results of TIMSS, there were no voices of praise, excitement or acknowledgment of achievement from within Canada. This is a phenomenon that Rodney et al. (2016) and Chorney et al. (2016), in analyzing Canadian newspaper articles about PISA scores, have ascribed to the central metaphors used in making meanings about mathematics education in the media. These include the metaphor that mathematics education is at war (in which case, what would victory look like?) and that there are only two distinct ways of teaching school mathematics (discovery learning vs. rote learning; how ironic they are both labelled as types of 'learning'). As long as parents and the public perceive that current mathematics teaching no longer centrally involves memorisation of "basic facts", the war will be lost. In this vein, Boland and Tranter note: "Although Canadian students tend to fair well when compared internationally, some parts of Canada continue to struggle to improve math achievement results. For example, in Ontario, only 50 percent of Grade 6 students met the provincial standard in 2016/17, down from 57 percent in 2013, this despite a 60-million-dollar provincial investment in a 'renewed math strategy' (Education Quality and Accountability Office)."

For very different reasons (unrelated to TIMSS scores), in this book Radford, Miranda and Lacroix argue for the "practical need to improve teaching and learning of mathematics". They claim that the, "best teaching practices have to include the dimension of the student—the student as a social being in the making." They exemplify teaching according to a cultural-historical theory, a theory that supports the design of activities in which students "show responsibility, care, and solidarity towards the others." Their chapter highlights an ethical dimension of mathematics education, but one that is less in terms of equity than in terms of the teacher–student relationship.

The role of care is echoed in Boland and Tranter's chapter, who advocate for the importance of "strong and supportive teaching relationships" and for the need to care not just for their careers, but for their well-being. The theme of care reminded us of the work of Julie Long (2008, 2011), who used Nel Noddings' concept of care, but extended it to focus on the way in which teachers can be seen as *caring about mathematics* as well, and not just about students (and possible tensions that can emerge). We find it interesting to consider how caring about mathematics might be involved both in developing the kinds of student–teacher relationships that Radford et al. as well as Boland and Tranter refer to, and in changing mathematics in the way that both Aikenhead and Lunney Borden discuss. Might it be possible to think that "attention to mathematics IS one of the ways to care about students," as Zazkis et al. (2013, p. 209) have argued? Here, caring is not framed in terms of well-being and building relationships, but rather through attending to strategies that will promote student learning.

Within the voices of critique and dissatisfaction, there are clear calls for change. Taylor, Lala, Ouellet and Knebel are critical of the current standard curriculum, which—as the authors suggest—does not capitalise on children's intellectual capabilities and sophistication. They bring up another axiological issue in addition to that of ethics, which is aesthetics, in arguing that the current curriculum fails to evoke the kind of aesthetic responses and experiences that for many, are the *raison d'être* of mathematics. Indeed, they see mathematics as a veritable goldmine of breath-taking beauty: "we just need the courage to bring that into the classroom and (by the way) to stop worrying about whether we are preparing our kids for calculus." They call for investigations to be at the centre of mathematics curriculum, choosing content that will encourage curiosity and creativity.

Aikenhead argues for decolonizing curriculum, purging from it "non-essential Platonist content" and basing it on "a cultural belief about school mathematics" (though we found no explicit examples in his chapter for the content he terms "non-essential"). While both chapters argue for fundamental change in school mathematics curriculum, we wonder whether the ideas of Taylor et al. and those of Aikenhead are compatible. How might their approach, in which the curriculum is re-formulated in terms of "powerful stories" align with Aikenhead's criticism that the current curriculum carries a nineteenth-century hegemonic, Platonist discourse of formal mathematics? Are the powerful stories of mathematics Platonic in nature, in terms of feeding off timeless, abstract truths, or might some productive alignment be possible? Or might Taylor's notion of "powerful stories" have the potential of aligning with an Indigenous worldview?

Across the whole collection of chapters, we also read powerful stories about different research settings, as well as about the researchers themselves. For example, Radford et al. tell a story of the journey of two students describing and interpreting Cartesian graphs, and how a teacher supported this journey by triggering and challenging students' understanding. Lunney Borden's story is of the personal journey of a teacher in a new environment. While not explicitly framed as such, these stories both illustrate teaching relationships that support learning. Thom and Glanfield, using the theoretical framework of Pirie and Kieren, explore the collective of a mathematics classroom and tell a story of ideas that emerged in an algebra class and how they were supported and shaped by whole-class conversation. The authors challenge the typical unit of analysis in education research, which is the individual student or teacher, and instead attempt to conceptualise mathematics learning as a collective enterprise.

In our opening two quotations there is a signalling of dichotomy, of polar opposites that cannot be reconciled: are things always changing or are they always staying the same? Such dichotomies also emerged in our reading of the chapters, as we have tried to show in our discussion of the discourse of mathematics (Lunney Borden), of the role of caring in mathematics education (Radford et al. and Boland & Tranter) and of the kinds of stories we tell or worldviews we have about mathematics (Taylor et al. and Aikenhead).

But we can perhaps superpose the dichotomies, in a kind of quantum state where the two exist simultaneously. Change endures; and what is not new under the sun is the fact that we will always seek to change, not least in our effort to improve whichever aspects of learners' experience of mathematics we value at any given time.

Maybe Ecclesiastes can be reconciled with Heraclitus.

References

Balacheff, N. (1988). Aspects of proof in pupils' practice of school mathematics. (Pimm, D., Trans.). In D. Pimm (Ed.), *Mathematics, teachers and children* (pp. 216–235). London: Hodder and Stoughton.

Chorney, S., Ng, O., & Pimm, D. (2016). A tale of two more metaphors: Storylines about mathematics education in Canadian national media. *Canadian Journal of Science, Mathematics and Technology Education, 16*(4), 402–418.

Long, J. (2008). *Caring for students and caring for mathematical ideas in an elementary classroom* (Unpublished doctoral dissertation). Edmonton: University of Alberta.

Long, J. (2011). Labeling angles: Care, indifference and mathematical symbols. *For the Learning of Mathematics, 31*(3), 2–7.

Rodney, S., Rouleau, A., & Sinclair, N. (2016). A tale of two metaphors: Storylines about mathematics education in Canadian national media. *Canadian Journal of Science, Mathematics and Technology Education, 16*(4), 389–401.

Zazkis, R., Sinclair, N., & Liljedahl, P. (2013). *Lesson play in mathematics education: A tool for research and professional development*. Dordrecht, Springer.

Part I: Preface

Walter Whiteley

Learning and teaching mathematics and statistics lives within the wide cultural context of mathematical and statistical practices in many areas of work and of play across our cultures. We notice that mathematics and statistics are not homogeneous, even as practiced by pure and applied mathematicians and statisticians. The formal logical face (echoed as the Platonist description in Aikenhead) is not what most mathematicians and statisticians live as researchers (Burton 2004), nor is it what is typically found in our upper level post-secondary mathematics and statistics classrooms. The chapters here better match the observations that the cognitive bases of mathematical learning, even of proofs, are much more diverse and engaging (Tall et al. 2012). More broadly, the insights of books such as *The way we think* (Fouconnier and Turner 2002), connect working with multiple representations and simulations across multiple disciplines, including mathematics, with how cognitive blending connects among multiple representations.

Learning and teaching mathematics in grades 7–12 happens within the enclosing schooling landscape, with a past horizon of early years and a future school horizon for the students in post-secondary education and in activities outside of schooling. The experiences of teaching and learning mathematics and statistics along, and beyond, all those horizons are already changing. Those changes on and beyond those horizons should have major impacts on teaching and learning mathematics in grades 7–12 over the next decades. There is, however, a risk that the experiences of teachers at those horizons may be crystalized in memory and may not incorporate those emerging changes and what they have to offer as support for positive changes within grades 7–12 classrooms.

Within this changing landscape, the current high school "maths" curriculum appears relatively static, focused on developing algebra and preparing for calculus.

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This curriculum often lacks both the breadth and the coherence that rich experiences in mathematics and statistics can offer. As Taylor, Lala, Ouellet, and Knebel observe, discussions of content become a 'laundry list' of content (Whiteley and Davis 2003), the opposite of the wider contexts and the collective processes that all of the chapters in this section invite us to notice and explore. Unfortunately, the 'static' high school curriculum is often held in place by perceptions of calculus as the *only* gateway into post-secondary mathematics based studies. This view is also visible in the positions of professional bodies in programs such as Engineering, Physics and sometimes even Mathematics. If students are retained beyond the first year of post-secondary education, this narrow focus fades. Unfortunately, during this arid period at the end of high school and the beginning of post-secondary, important segments of the potential student population are pushed out through lack of interest and for lack of other skills, such as spatial reasoning (see the links under ENGAGE below).

As additional context, I highlight three ongoing developments which I anticipate will have a growing impact on teaching and learning mathematics across the landscape. Additional connections can be found with the themes in the chapters of this section.

A first example of developments across this landscape is the building recognition of the critical role of spatial reasoning. A recent book on *Early years spatial reasoning* (Davis and The Spatial Reasoning Study Group 2015) reminds us that students enter schooling with a wide range of spatial experiences, having spent their entire lives learning and exploring in 3D. Spatial reasoning strengths (or weaknesses) have a continuing impact on people's learning over our entire lives (ENGAGE n.d), including through the impact of losing spatial reasoning as we age (Possin 2010). Unfortunately, spatial reasoning is too often invisible in our mathematics classroom in the curriculum content and in the mathematical processes practiced. This absence immediately disadvantages and can exclude substantial portions of the students for whom spatial reasoning is their strength, while formal algebra and routine computations are a weakness. Surprisingly, spatial reasoning has a larger place in the Ontario Social Science, History, and Geography curriculum, across all grades, than in the Ontario Mathematics curriculum (Ontario Ministry of Education 2005a, b, 2000, 2013).

This separation of the high school curriculum from children's early cultures and from young adult visual cultures is a broad problem for inclusive teaching. It can be a filter for the early grades (e.g., the linearized number line—essential for success by grade and primarily learned in school). It continues as a filter to full access in upper level mathematics, statistics, science and engineering courses (ENGAGE). In grades 7–12, spatial reasoning can support work with graphing and manipulatives (Radford et al.) and in the verb-based and spatialized culture of "enough" and gestures described by Lunney Bordon. The absence of spatial reasoning is part of what Aikenhead observes in the formalized, Platonist presentation of "mathematics," which excludes many cultures which have strong and wide spatial practices. Among the missing cultures with richer spatial reasoning are both the current practices of professionals in engineering, physics, biology and broad interdisciplinary fields and the cultures in other high school classes in science and social science which students experience in parallel with their mathematics classes.

A second example of a changing focus is the process of simulation: following a story with a sequence of steps which can be experienced and played with, and associated computational thinking. These simulations and companion algorithms underlie key processes of *The way we think*: switching and blending among representations (Fouconnier and Turner 2002). The associated cognitive blending is shared within communities within the back and forth pathways which are highlighted in the chapter of Thom and Glanfield. These processes of simulation and algorithms are now appearing across the spectrum—in elementary grades, in post-secondary classrooms, and in the lives of students outside the mathematics curriculum. In the chapter of Radford et al., we notice the students experimenting with a simulation—something that is even richer when they have access to range-finders to test their tentative stories. A broader richer exploration can occur if they are invited to consider the graph if Pierre is holding the range-finder pointing backwards, with the same motions. This invites the students to transform the frame of reference, something that might well happen in a physics classroom. The powerful, rich stories of Taylor et al. are other ways of combining multiple representations—with implicit simulations we explore the same actions in the different representations. Without the rich stories, the teaching and learning is not drawing on the way we think!

A final example of what is underrepresented, and is likely to form a bigger part of high school classrooms, is Statistics. Already in New Zealand, all grades from 1 to 12 have a curriculum called “Mathematics and Statistics” (Ministry of Education 2014). (If you wonder how children in grades 1–4 reason with statistics—they do it with visual/spatial simulations, supported by the software iNZight.) In Ontario, Probability and Data Management is in every grade 1–8 and after a gap, in grade 12. Increasingly, post-secondary programs require some course and initial mastery in statistics. However, many high school teachers complete their preparation in mathematics and in education, feeling unprepared to support this change and teach such material. Most teachers recognize statistics is ‘different’ than what they learned in their mathematics courses or what they practice in other parts of the current curriculum. It is a critical error to assume statistical reasoning and decision making is simply applied probability—and lack of a good background is a real loss when the teachers have the opportunity to implement change.

Throughout the chapters in this section, there is a strong emphasis on learning in context, within relationships and within communities (Radford et al.; Thom & Glanfield; Boland & Tranter). There are reminders that the larger sweep of the individual pieces of mathematics could form connected stories, and rich investigations are often smaller but powerful stories (Taylor et al.) that carry us forward. As Aikenhead explores, and Lunney Borden powerfully illustrates, for Indigenous students these mismatches of their cultures and contexts with their classrooms needs to be bridged and reworked. This includes reworking the stories for the mathematics and even the words, practices and relationships that carry the learning and meaning that should be broadly recognized as appropriate for powerful mathematics. Notice that the core story within Boland’s chapter additionally describes the impact of relationships in a setting both on the boundary of elementary schooling and high school, and in a context where a majority of the students were Indigenous and at risk of not moving along within the strange (even alien) culture of high school "maths." The context is changing and the context for teaching and learning is open to further change.

Change is a long-term process. As A.J. Coleman once observed, in Switzerland one plans such change in curriculum over an arc of at least a decade. To cite another example, when the then Soviet Union was updating their geometry curriculum, the first step was to write a textbook for preparing future teachers. After a decade of preparing teachers and sending them out into the schools, it was time to roll out the curriculum and texts for the students! So different than what most teachers and students experience in Canada, when the teachers may only see the curriculum a few months before they asked to implement it, and the texts may not even arrive when the semester starts. We must all do better.

Change in schools is also collective. Even working within a static set of mathematics and statistics topics, a mutually supportive group of teachers can still integrate new ways to support students' collective and individual learning. One can spatialize the curriculum (Davis and The Spatial Reasoning Study Group 2015), and can develop student's capacities to learn well through collaborations, using multiple approaches. Building these processes and capacities may initially slow down the short-term 'progress' through topics, but research evidence is that over several years the learning accelerates and there is more time and space to explore widely and follow their passions while discovering the 'prescribed curriculum' which has also, almost incidentally, been covered. We have the opportunity of celebrating the diversity of students and the many ways students reason, connect, and communicate.

Adapting to changing landscapes, and even changing the landscapes ourselves, requires patience, hope, confidence and resilience over the long term. In turn, this requires communities of continuing support among peers, and from parents and ministries. Together, we can develop communities that build these relationships over multiple years.

References

Burton, L. (2004). *Mathematicians as enquirers: Learning about learning mathematics*. Dordrecht: Springer.

Davis, B., & The Spatial Reasoning Study Group. (2015). *Spatial reasoning in the early years: Principles, assertions, and speculations*. New York: Taylor and Francis, Routledge.

ENGAGE. (n.d.). *Spatial visual skills* [links]. Retrieved from https://www.engageengineering.org/spatial/whyitworks and https://www.nature.com/scitable/content/chapter-5-spatial-skills-18255038

Fouconnier, G., & Turner, M. (2002). *The way we think: Conceptual blending and the mind's hidden complexities*. New York: Basic Books.

Ministry of Education. (2014). *New Zealand curriculum online: Mathematics and statistics*. Retrieved from http://nzcurriculum.tki.org.nz/The-New-Zealand-Curriculum/Mathematics-and-statistics. Supporting software INZight: https://www.stat.auckland.ac.nz/~wild/iNZight/index.php

Ontario Ministry of Education. (2000). *The Ontario curriculum: Mathematics, grades 11 and 12*. Toronto: Queen's Printer for Ontario.

Ontario Ministry of Education. (2005a). *The Ontario Curriculum: Mathematics, grades 1 to 8*. Toronto: Queen's Printer for Ontario. Retrieved from http://www.edu.gov.on.ca/eng/curriculum/elementary/math18curr.pdf

Ontario Ministry of Education. (2005b). *The Ontario curriculum: Mathematics, grades 9 and 10*. Toronto: Queen's Printer for Ontario. Retrieved from http://www.edu.gov.on.ca/eng/curriculum/secondary/math910curr.pdf

Ontario Ministry of Education. (2013). Appendix C: Map, globe and graphing skills – A continuum. In The Ontario curriculum: Social studies grades 1 to 6 and history and geography grades 7 and 8 (pp. 191–198). Toronto: Queens Printer for Ontario. Retrieved from http://www.edu.gov.on.ca/eng/curriculum/elementary/sshg18curr2013.pdf

Possin, K. L. (2010). Visual spatial cognition in neurodegenerative disease. *Neurocase, 16*, 466–487 Retrieved from NIH Public Access.

Tall, D., Yevdokimov, O., Koichu, B., Whiteley, W., Kondratieva, M., & Cheng, Y. H. (2012). Cognitive development of proof. In M. de Villiers & G. Hanna (Eds.), *ICMI Study 19, Proof and Proving in Mathematics Education*.

Whiteley, W., & Davis, B. (2003). A mathematics curriculum manifesto. In E. Simmt and B. Davis, (Eds.), *Proceedings of the 2003 annual meeting of the Canadian Mathematics Education Study Group/Groupe Canadien d'Etude en Didactique des Mathématiques* (pp. 79–83). Edmonton: CMESG/GCEDM. Retrieved from http://www.cmesg.org/wp-content/uploads/2015/01/CMESG2003.pdf

Powerful Stories: The Hitchhiker's Guide to the Secondary Mathematics Curriculum Landscape

Peter Taylor, Divya Lala, Kariane Ouellet, and Stefanie Knebel

Abstract Our goals are first to capture a few significant historical moments in the changing face of secondary school mathematics, and secondly to use these to help us decide how we need to move into the future. We decided to speak in a voice that was as personal as possible and that also supported our current research interests, and that of course has conditioned our historical record. Along with that we decided that, rather than write a single piece, we would each write our own thoughts. In particular, Peter would write the central paper, and then Divya, Kariane and Stefanie would each write a reflective response in the currere style (Pinar, W. F., The method of "currere". Paper presented at the Annual Meeting of the American Educational Research Association. Washington, DC, 1975). And of course we would trade ideas at every stage.

Our focus is not on teacher education, nor is it on assessment, though these are both significant components of the shifting landscape, and definitely need continued attention, but we focus here on curriculum. Our long-term objective is to see a high school mathematics curriculum that is driven by what we call powerful stories; as such it would be richer and more sophisticated than what we have at the present and it would be a better platform for the development of "mathematical thinking."

Keywords Life · Narrative · Sophistication · Currere · Mathematical thinking · Dewey · Whitehead

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Peter

The Beginning

On April 9th 1969 at 4 pm I was standing along with a few others outside the high wall surrounding Harvard Yard throwing packets of food, bread and cheese, and juice boxes to some two to three hundred of our fellow students inside. Why was I outside and not inside? I was not sure but was certainly conscious of my inner turmoil. In just over 2 months I was scheduled to be a new father and a new PhD; it was not clear which would come first but both were calling me to be a responsible adult on one side and a revolutionary on the other and anyway why was there a difference between the two.

I lived on Everett Street, facing the wall a couple of blocks away. I was awakened at 5 am the next morning by the shouting and screaming as the Cambridge police took the students away sending 40 of them to Emergency. I went to the window but could not see anything. I looked back at Judith, 7 months along, who thankfully did not wake, and I vowed that in the many professional years ahead of me I would find another way to help steer the change that was so clearly on its way.

> And now the times are changin'.
> Look at everything that's come and gone.
> Sometimes when I play that old six-string.
> I think about you, wonder what went wrong.
> (Adams and Vallance 1984, track No. 6)

The short-lived occupation of Harvard Yard was specifically a reaction against the Harvard administration for its support of the military in the wake of the terrible war in Vietnam, but more generally it was a signal to us all that the change that was on its way would be profound and we simply did not trust the establishment to manage it properly. You see the 60s was an extraordinary decade for education, opening in the wake of the 1957 Soviet launch of Sputnik and closing with Neil Armstrong's 1969 landing on the moon. These events prompted large government investments in science, engineering and mathematics at all levels of education. And one might have asked just how was all that money to be spent?

This question was highlighted that very year with the appearance of *Teaching as a subversive activity* (Postman and Weingartner 1969) and ever since that time, Neil Postman has been one of my gurus. This early somewhat informal work opens with a catalogue of some of the leading thinkers of the day as well as some of the ways education could go wrong:

> The institution we call 'school' is what it is because we made it that way. If it is irrelevant, as Marshall McLuhan says; if it shields children from reality, as Norbert Wiener says; if it educates for obsolescence, as John Gardner says; if it does not develop intelligence, as Jerome Bruner says; if it is based on fear, as John Holt says; if it avoids the promotion of significant learning, as Carl Rogers says; if it induces alienation, as Paul Goodman says; if it punishes creativity and independence, as Edger Friedenberg says; if, in short, it is not doing what needs to be done, it can be changed; it must be changed. (p. 5)

But what exactly is this change to look like? That question has been around for a long time. The goal of this chapter is to highlight a few of the twentieth century milestones for that question and to review briefly some of the reasons that educational change is so difficult and how the situation differs at the elementary and secondary levels. I will also point to some encouraging recent progress.

I end this section with a quote from a 1995 Postman book *The end of education*. [That's a mischievous use of the word "end." One thinks right away of an ending, but in fact Postman intends (mostly) the other meaning of the word: end as goal or objective.]

> What this means is that at its best, schooling can be about how to make a life, which is quite different from how to make a living. (p. x)

I seize here on the precious word "life" and that will form the core concept for the chapter.

The First Half-Century

Alfred North Whitehead (1861–1947), philosopher and scientist, and John Dewey (1859–1952), philosopher and humanist, both wrote definitive essays on education and their ideas are needed today more than ever. For both of them, what happens in the classroom must be significant for the life of the student at that very moment. The fact of the matter is that from their time almost 100 years ago, to the present, this significance has too often been postponed to the future. Here's Whitehead (1929) commenting on the assertion that you cannot do mathematics until you have mastered the technical pieces:

> The mind is an instrument; you first sharpen it, and then use it… Now there is just enough truth in this answer to have made it live through the ages. But for all its half-truth, it embodies a radical error which bids fair to stifle the genius of the modern world… The mind is never passive; it is a perpetual activity, delicate, receptive, responsive to stimulus. You cannot postpone its life until you have sharpened it… There is only one subject-matter for education, and that is Life in all its manifestations. (p. 6)

It is ironic that this dominant idea, that students must wait till university before being confronted with real mathematics, is what is responsible for the fact that so few of them (at most 25%) have anything close to technical mastery of the discipline. This is not only an irony; it is a catastrophe as it engendered the "math wars" that for the past 25 years have pretty much sabotaged any liberal-minded attempt at school curriculum renewal.

It is certainly true that mastery of any complex procedure, whether it belongs in sports, the creative arts, or academics, requires what is often called "automaticity" and this typically requires hours of routine practice. But it is equally true that children (and adults!) love to investigate and discover things and these activities, if they clearly point towards ends that are rich and compelling, can initiate and sustain that technical practice. And by the way, humans happen to enjoy time spent in the single-minded company of routine tasks (knitting, musical scales, solving equations) that they can master, especially those that can hide unexpected variations.

For us, Whitehead's point is that the riches of mathematics need to be brought into the life of the student at that very moment. And the big question, the question that inspires this essay, is how is this to be done?

John Dewey also accepts the fact that education must prepare the student for the future and that it is not a simple matter to find the right way to do that, but he also emphasizes that one thing we must not lose is "the organic connection between education and personal experience" (Dewey 1938, p. 8).

> What, then, is the true meaning of preparation in the educational scheme? In the first place, it means that a person, young or old, gets out of his present experience all that there is in it for him at the time in which he has it. When preparation is made the controlling end, then the potentialities of the present are sacrificed to a suppositious future. When this happens, the actual preparation for the future is missed or distorted. ... We always live at the time we live and not at some other time, and only by extracting at each present time the full meaning of each present experience are we prepared for doing the same thing in the future. This is the only preparation which in the long run amounts to anything. (Dewey 1938, p. 20)

Again we ask how we do this and still prepare our students for a world that is technologically hugely more complex than the world of Whitehead and Dewey. Dewey (1938) talks about the importance of framing an intelligent theory (or perhaps a philosophy) of life experience for guiding the growth process, otherwise we are "at the mercy of every intellectual breeze that happens to blow" (p. 21). The problem remaining with us today is to translate Dewey's guiding vision into a concrete curriculum narrative.

In his chapter *The rhythm of education*, Whitehead (1929) describes at length a structure for this narrative. He lays down the three stages of education: Romance, Precision and Generalization and requires that we honour these, and further warns that if in our haste we short-change the critical first one, the second will wither and cannot deliver the ultimate fruit of education: the wisdom of the final stage.

The Past Half-Century

I now jump into the second half of the century where this same life-affirming message is found in the 1976 report on the mathematical sciences in Canada commissioned by the Science Council of Canada (Beltzner et al. 1976). The report noted an "informed opinion that the teaching of mathematics at the elementary and secondary school in Canada is unsatisfactory" and it places a good part of the responsibility on the shoulders of the mathematical community (p. 113). It proposes that the primary aim of the school mathematics curriculum should be "an understanding of what mathematics is" (p. 117) and it cites David Wheeler that "it is more useful to know how to mathematize than to know a lot of mathematics" (p. 119).

A significant outcome of this report was the inaugural meeting in 1977 of the Canadian Mathematics Education Study Group (CMESG/GCDEM), a community of mathematicians, mathematics educators, graduate students and teachers that has met annually ever since and unfailingly provides the collaboration and life-force for much of the Canada's contribution to the study of mathematics education. I will cite one such contribution and that is Whiteley and Davis (2003), a manifesto addressed to the Canadian Mathematical Society asserting that the structure of our K-12 mathematics curriculum "is an obstacle to student learning of mathematics. Over-specified and fragmented lists of expectations misrepresent what mathematics is and militate against deep and authentic engagement with the subject." (p. 83). In addition, the document referenced above all the ability to think mathematically. This is a phrase we are encountering increasingly in the literature, but my experience is that there is little attention paid to mathematical thinking in the "delivered" secondary curriculum.

Powerful Stories

My view is that a significant story, or more generally a collection of related stories that together form a significant narrative, can provide the power needed to propel our mathematics students towards a complete engagement, one that includes technical and conceptual fluency and develops mathematical thinking.

I call such stories "powerful." In literature a story is powerful if it opens the way to a significant human experience. So also in mathematics, a story is powerful if it opens the way to a significant experience of doing mathematics. The stories I look for have a sense of completeness and a natural organic connection to the framework of technical skills that supports them. They must also be accessible but nevertheless have the potential to soar. Gadanidis et al. (2016) call this a "low floor with a high ceiling" (p. 2). Finally, aesthetics plays a huge role in my selection of good stories, indeed it plays the definitive role. Much has been written about "motivation" in the learning of mathematics, and it is certainly true that different students respond positively to different types of experience, but my view is that all students have a natural response to beauty, indeed, aesthetics just might be the universal motivator. Gadanidis et al. (2016) refer to "the aesthetic that makes the experience (of mathematics) human" (p. 2). Certainly mathematicians discover early in life the deep connection between truth and beauty.

Yet when I mention the beauty of mathematics to high school graduates, they often express surprise at the connection. The reason is perhaps that beauty is not displayed as a central feature of school mathematics and thus even when it succeeds in getting into the classroom, it has little staying power. Seymour Papert (1980, as cited in Sinclair 2006) observes that "if mathematics aesthetics gets any attention in the schools, it is as an epiphenomenon, an icing on the mathematical cake, rather than the driving force which makes mathematical thinking function" (p 192). Anne Watson (Sinclair and Watson 2001) comments on this nicely:

> I had a growing disaffection with this pedestrian approach to awe and wonder in mathematics, as if there were common sites for expressing awe, like scenic viewpoints seen from a tourist bus, whose position can be recorded on the curriculum as one passes by, enroute for something else. Spontaneous appreciation of beauty and elegance in mathematics was not, for me, engendered by occasional gasps at nice results, nor by passing appeals to natural or constructed phenomena such as the patterns in sunflowers or the mathematics of tiling. (p. 39)

That's a nice phrase: "scenic viewpoints from a tourist bus." The subtext is: "that's not where I live!" This is the reason that I believe that the investigations we bring into this new curriculum model must be "significant," for example sustained narratives that connect well with the given mathematical curriculum. Otherwise they have little chance of performing as Papert's "driving force." By the way, Papert has some fascinating things to say about mathematics and aesthetics and I recount these in the *Suggestions for further reading*.

Moving Forward—The Challenges

Over the past year we have worked with rich problems of this type in different settings (workshop, classroom) with a variety of students in grades 9–12, mostly in the university preparation stream, and we have learned a lot about what they can do and what they find challenging. Some do not manage to gain a reasonable level of mastery of the methods, but certainly they are all given a significant view of the majesty of a mathematical landscape.

Our ultimate objective is to find 4 years worth of good stories at the secondary level. Certainly these stories must fit the prescribed curriculum, but in moving to a narrative focus, we find ourselves encountering a wider set of mathematical constructions, perhaps intrinsic to the story itself, perhaps raised in the inquiry process by the students themselves, and we need to be open to the pursuit of these. I am thinking here of concepts that are found in probability, geometry, logic, discrete optimization, combinatorics, game theory, stability of physical systems etc. To the extent that such topics find themselves naturally arising, they will of course need some adaptation and teacher buy-in. Most mathematics teachers at the secondary level have a reasonable mathematical background, but they tell me that it takes just about the whole term to cover the mandated technical material, so they would find it difficult to incorporate the activities I am discussing here. My own colleagues in Canadian universities are also wary of these ideas. They already find their students "unprepared" and they suggest that my model could make the situation worse.

I must say that when I look at the list of topics in, say, the Ontario Grade 10 Academic course,[1] I find it hard to believe it takes the whole 4+ months to get that

[1] See McDougall and Ferguson (Part II this volume, para. 1) for a discussion of two of the possible Ontario pathways (Academic and Applied).

done—because I have seen with my own eyes the remarkable things that grade 10 students can do in a week. I can only conclude that in the standard classroom, these students are working at half or quarter throttle. Perhaps the reason for this is that ever since their early years most (but not all) of these students have seen no compelling reason to invest serious effort into the mathematical activities they have encountered.

> There is little space in the conventional normative and normalizing classroom for wonder, for sustained engagement, for obsession, for playful bodies. But we do not seem willing to teach, or even to talk about, the very qualities that animate most mathematicians' life-work. In fact, it is not hard to make the case that precisely the opposite is being taught of mathematics: certainty instead of wonder, detachment instead of engagement, touring instead of dwelling, observing instead of obsessing, scripted performances instead of playful acts. (Davis 2001, p. 23)

There seems to be good evidence that, at least at the elementary level, kids are capable of more mathematical sophistication than the standard curriculum assumes:

> Elementary school teachers work hard to cover grade-specific math curriculum expectations, but what if this is not enough? Ginsburg (2002) suggests that "children possess greater competence and interest in mathematics than we ordinarily recognize" and that they should be challenged to understand big mathematical ideas and to "achieve the fulfilment and enjoyment of their intellectual interest" (p. 7). This position is supported by Joan Moss and her colleagues in their work with functions in Grade 4 (Moss et al. 2008). By developing a stimulating, mathematically rich context for the content that students have to learn, teachers can address grade-specific curriculum expectations while offering students the pleasure of mathematical surprise. Young students, these researchers have shown, benefit from opportunities for using imagination and sensing mathematical beauty. (Gadanidis 2012, p.1)

My view is that in secondary mathematics we are effectively telling our students that we do not think they are clever or imaginative enough to handle the ideas that we, as mathematicians, find interesting and challenging. That strikes me as a smug, elitist attitude that shortchanges the student and the subject itself. One thing I know is that we do not need to fear that our young students will let us down. They are imaginative and resourceful (and even hungry) and simply need to be given problems they can respond to and succeed at. Nor need we fear that our subject will let us down. Mathematics is a veritable goldmine of breath-taking beauty—we just need the courage to bring that into the classroom and (by the way) to stop worrying about whether we are preparing our kids for calculus.

Let me end with a story. Twenty-five years ago Mattel™ marketed a Barbie doll that said "Math class is tough." There was storm of protest and the red-faced company hastily withdrew the doll. That was totally the wrong response; mathematicians should have stood up and pointed out that tough jobs require tough tools and we are lucky that mathematics provides these. These days, trucks and heavy-duty cleansers are praised for their toughness but that is nothing compared with the task of landing a spaceship on an asteroid or designing a code that is easy to implement and hard to break. The fact of the matter is that kids really love the feeling that they are doing a tough job—all they need is some real indication that their hard work is leading to success.

Kariane

This essay for me is a giant puzzle holding many pieces whose assembly might well significantly improve the high school mathematics experience. Towards the end of our *Additional suggestions for further reading* we say: "What we need to do is let go of the imperative that all our kids need mastery of a substantial list of basic technical skills before they can climb the mathematical tree." I believe that this list is so mind numbing that it becomes difficult for anyone, let alone a high school student, to take a step back and see the greater picture. Instead of knowing how all the concepts work together, and how much beauty and power they can generate, many students end up trying to memorize each of them with the hope that it will all work out on the exam.

Reflecting on my own experience as a high school student, that certainly was the case for me. My 16-year-old self was anxious, unaware of her strengths, but certainly aware of her weaknesses—and one of these was mathematics. I was an average student until I failed miserably the first exam of Secondary 5. When I managed to get over the shock and the shame, I realized two things. The first is that if mathematics was required, it must be doable. The second is that I cannot be that dumb. Hence, success was within my reach, and through introspection, I realized that memorizing mathematics was probably not working for me. That is what I have been doing instinctively for years, but I had no fundamental understanding of any of it. Therefore, *I learned how to learn math*, and underlying that important process was a change in my motivation: I was now learning mathematics for its own sake. Middleton and Spanial (1999) would call this *mastery goals*. On the other hand, *ego goals* define success in a discipline relative to others. In terms of achievement, "students with mastery goals tend to perform better than those with ego goals regardless of the learning situation" (Middleton and Spanial 1999, p.74). This could explain my shift from underachievement to being successful in mathematics.

The question that remains is how we might replicate this for other students. Studies showed two important things regarding this question. The first is that students' intrinsic motivation (linked to mastery goals) towards mathematics decreases significantly throughout high school (Gottfried et al. 2007, p. 325). The second important lesson is that "the decline in academic intrinsic motivation is not a general developmental or ontogenetic one, nor is it inevitable" (Gottfried et al. 2001, p.10).

One way to remedy this decline is by implementing an inquiry-based classroom. By using such a model, students "are less likely to develop ego goals than are students in more traditional classrooms" (Middleton and Spanial 1999, p. 74). Moreover, they tend to believe that success in mathematics is defined by their attempts to understand and explain their thinking (Middleton and Spanial 1999, p.74). My hope is for discovery and investigation to be at the centre of the new face of mathematics in Canada.

Divya

While thinking about what to write for this chapter, I stumbled upon the article titled, "School mathematics as a special kind of mathematics" (Watson 2008). One of several responses to Watson's thoughts was Mendrick (2008) who stated three reasons for why she believed there will always be a difference between the two types of math. These were:

- School students do not get paid for doing mathematics
- They do not apply for opportunities to do it
- Even when they have an identity that is invested in being good at it, mathematics never defines them in the way that one's employment does

Mendrick's list made me think back to Peter's comments. He stated:

> My view is that in secondary mathematics we are effectively telling our students that we do not think they are clever or imaginative enough to handle the ideas that we, as mathematicians, find interesting and challenging. That strikes me as smug, elitist attitude that short-changes the student and the subject itself.

Like Peter, I also find that we are creating this "elitist attitude" with mathematics. Whether or not we feel we can accept Mendrick's three differences, I feel that they also show this attitude; that mathematicians are simply too different from their students. I find this to be a huge problem, as it can create many misconceptions for students, specifically regarding creativity and discovery in mathematics.

While I am now following a career in mathematics, in high school I fell victim to these misconceptions. Throughout my life, I have always been interested in mathematics. This was encouraged by my family, who promoted my curiosity in the subject. However, school did not. As I went through my high school years, I felt less interested in mathematics as I was constantly feeling as though I was not challenged. Therefore, like the students that Peter mentioned, I found physics more interesting, and so I decided to major in physics in university. However, my constant curiosity made me keep up with mathematics, and I eventually transferred into Applied Mathematics.[2] If it was not for the opportunities I was given and the encouragement from my family, I would never have found my place in mathematics, and seen how much there is to discover.

Finally, I want to come back to Mendrick's thoughts. Even given that school mathematics will not be exactly like the mathematics done by mathematicians, there should not be such a disconnect between the two. Just like mathematicians, students

[2] This term refers to one of the two course pathways in Ontario secondary classrooms (Applied and Academic).

should feel encouraged to use their curiosity and creativity to help them learn challenging mathematics. Perhaps then can we put an end to this elitist attitude.

Stefanie

Our research aims to enrich the learning of mathematics at the secondary level by engaging students with powerful stories. Our recent project presented grade 10 students with rich problems involving transformations and matrix multiplication (see Fig. 1). My encounter with these students clearly pointed to a lack of drive and interest towards their usual mathematical activities. The students would ask, “What is the purpose of learning this stuff? I’ll never use it again outside of this classroom!” Other students failed to connect our activities with their idea of mathematics as they would ask, “At what point will we begin to do math?” I had similar questions in my own past, as had the authors of the previous two testimonies. It is not easy to answer these questions but they certainly suggest that the learning of school mathematics could be improved.

We all strive to find a purpose in our efforts and feel there is a reason for our time spent learning. We call the activities of our project “powerful stories” because they are designed to have the power to guide the learner towards the feeling of a worthwhile purpose. This seems to be lacking in the material presented in the curriculum today.

Mehta et al. (2016) interviewed four Fields medal recipients to highlight how mathematicians view mathematics from a creative and artful perspective that is lacking in the high school curriculum. The authors suggest that the current curriculum limits students’ ability to wonder, to develop a sense of the field of mathematics and to envision themselves as mathematicians.

> These profiles [of the four mathematicians] tell us that success in mathematics comes with passion and play, and from seeking connections across fields and disciplines. They provide a very different view of mathematics—as a living, artistic, organic structure, that mathematicians actively construct in order to find truth and beauty in the world. We believe that this view of mathematics has significant implications for how we think of teaching and learning in this domain. It offers a novel and humanistic way of thinking about how to engage educators and learners in mathematical ideas. (p. 18)

We hope that by providing students with powerful stories, we will be able to help them to encounter mathematics from this perspective, be more engaged by the material, and be motivated to endure longer as they construct their own reasons for pursuing mathematics (Fig. 1).

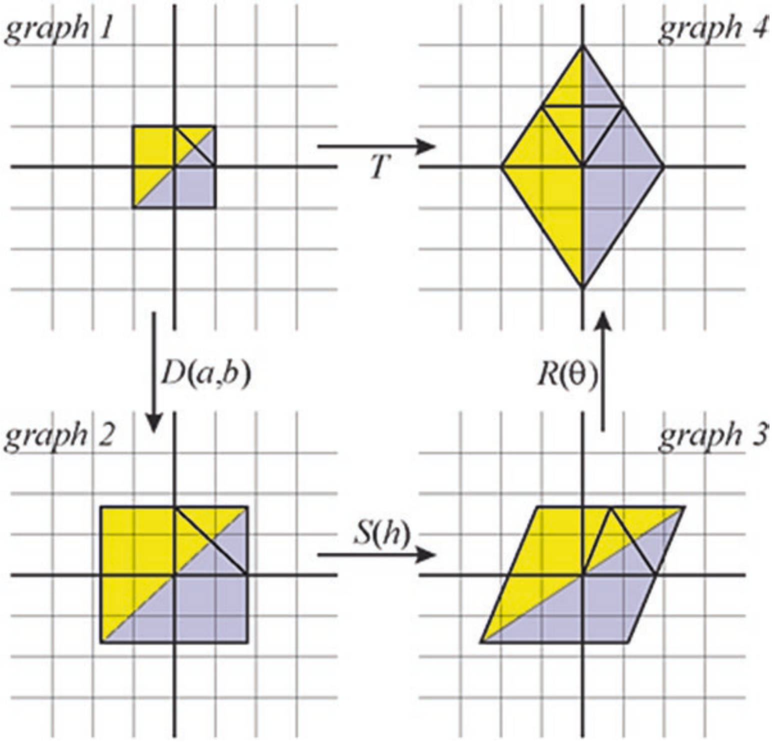

Fig. 1 Factoring transformations (www.Math9-12.ca, Transformations Example 6)
This example embraces both computational thinking (CT) and spatial reasoning (SR) and is designed for Ontario Grade 10 or 11 Academic (i.e., university mathematics/science preparation). Here T is a linear transformation, and students are required to factor it as a given composition of what we have taken as "basic" transformations—dilations D, rotations R and horizontal shears S. The task is to find the parameters a, b, h and θ. In a sense, the basic transformations act as the "primes" of the system and the problem is to "factor" T as a "product" of these primes. There are two approaches, *geometric* using coordinate geometry and triangle trig, and *algebraic* using matrix multiplication and solving equations.
At first sight this seem like a difficult problem, and it does take some focused thinking to put all the pieces together. Having done that with a few examples, one can almost see an "algorithm" emerge; it is however sophisticated, both in its geometric and algebraic form, and it would be difficult to implement without a good grasp of the basic ideas and techniques.
For this particular transformation T, the reader will notice that there is an obvious and simpler factorization as a rotation followed by a dilation. But the sequence given here—a dilation followed by a shear followed by a rotation—is not so easy to find. The answer is $a = 5/2$, $b = 12/5$, $h = 7/24$ and $\tan\theta = 4/3$.

Additional Suggestions for Further Reading

The Role of the Aesthetic in Mathematical Discovery

Much has been written about the beauty of mathematics, that for mathematicians it is a significant experience and the deep pleasure that it brings motivates them in their work. Based on this, it is a great pity that this experience is rarely to be found in the school classroom. But there is a strand of philosophical thought going back at least to Poincare that makes a stronger and more essential argument; it says that the aesthetic is a central component of the mathematical experience in that serious mathematics simply cannot be done without it. As a result, the failure to place the aesthetic experience at the centre of the mathematics classroom is more like a disaster, certainly a token of failure for folks who would construct a more sophisticated curriculum. There is quite a large literature that discusses this idea (Google: Poincare mathematics aesthetics) but I find Seymour Papert's (1980, p. 190–197) presentation of the mechanism at work here to be the clearest and the most intuitive.

Here is the idea. The mind has two components: the conscious which operates logically, and the unconscious which does not. Problems of a mathematical nature, when received, go to the conscious to be worked on and are often simply solved. But problems which are harder or less familiar, or even less well-formulated, after some preliminary analysis and a bit of struggle, are dispatched to the unconscious where they might reside for some time. During this interval, while the conscious mind is occupied with the business of living, these problems are quietly worked on but in quite a different way. The unconscious mind uses aesthetic criteria, the elegance, harmony and order found in patterns, to make value judgements, to decide what to transform and how, what to accept and what to reject. At some point, when it is "ready," it throws the results back to the conscious mind, often taking it by surprise (Poincare tells a now famous story of this happening as he was stepping onto a bus). But armed with this transformed version of the problem, the mind can now apply its prowess with logic to evaluate the configuration and hopefully move the solution forward. If this story provides a reasonable account of reality, it must follow that both aspects of mind deserve to be trained and that a child ought to receive both a logical and an aesthetic education.

Our objective here is to conclude with a mention of the many Canadian researchers who have emphasized the fundamental role played by the aesthetic in the mathematical experience and have worked to carry this idea into our secondary schools. A brief search of the literature suggests that this is a hopeless task; there is so much wonderful work, one cannot decide where to stop. I (PDT) will mention a few major influences on my own life. First mention goes to my former PhD student (shared with my colleague Bill Higginson) Nathalie Sinclair (Simon Fraser) whose wonderful book *Mathematics and beauty* (2006) argues that students are fundamentally aesthetic beings and provides examples and activities to move the curriculum in this direction. Ed Barbeau (University of Toronto) places the aesthetic experience at the centre of a rich array of problems he has developed over the years. He has argued, along with many others, for an investigative "capstone" course in the final year of high school, perhaps with the theme of optimization. Such a course could happily replace the standard functions/calculus grind and at the same time provide excursions into geometry, probability, combinatorics, and game theory—all excellent areas for the development of an aesthetic mathematical experience. Finally, I cite Walter Whiteley's (York University) persistent inspirational work building a case for the return of geometry. These personal mentions are followed by a host of others. Many of these can be captured in their work and from east to west I mention *Math circles* (Nova Scotia), and the amazing magazine publications *Accromαth* (Quebec) and *Pi in the sky* (PIMS). I find these sites remarkable for the deep, beautiful and accessible mathematics they provide. There is a cruel paradox here. These publications are rich enough to easily fuel the secondary mathematics curriculum and they would give our students an inspiring and sophisticated experience far removed from the text-books that are currently in use. But our current ideas of the nature and structure of the school mathematics curriculum would need a fundamental change.

Embracing a Sophisticated Experience in the School Mathematics Classroom

Three prominent Canadian websites that embrace the significance of rich mathematical structures in school mathematics are:

Computational Thinking in Math Education www.ctmath.ca/about/
Math for Young Children (M4YC) http://www.mathforyoungchildren.ca/.
Spatial Reasoning Study Group (SRSG). http://www.spatialresearch.org/group

The much-cited essay Lockhart's *A mathematician's lament* (2002) imagined what would have happened if music instead of mathematics had been the subject considered essential for all students to "learn." The idea here is that school mathematics would perhaps be richer and more artistic if we let go of the idea that it was so important for the future of all our students. Well, I would like to say that differently. Mathematics is as important for all our students as English and history and science and music and art and physical education, and the list could go on. What we need to do is let go of the imperative that all our kids need mastery of a substantial list of basic technical skills before they can climb into what Dan Kennedy (1995) calls the mathematical tree. I would go so far as to predict that if we effectively let go of that list, and spend our mathematics time more like they spend time in art class, the overall level of technical proficiency will actually go up rather than down.

Kennedy, D. (2007). Climbing around the tree of mathematics [Special Issue]. *Mathematics Teacher, 100*, 80–85.

Lockhart, P. (2002). *A mathematician's lament.* New York, NY: Bellevue Literary Press.

References

Adams, B., & Vallance, J. (1984). *Summer of '69* [Recorded by B. Adams]. On *Reckless.* Santa Monica: A&M Records.

Beltzner, K. P., Coleman, A. J., & Edwards, G. D. (1976). *Mathematical sciences in Canada background study No. 37*. Ottawa: Science Council of Canada.

Davis, B. (2001). Why teach mathematics to all students? *For the Learning of Mathematics, 21*, 17–24.

Dewey, J. (1938). *Experience and education*. New York: Kappa Delta Pi.

Gadanidis, G. (2012). Trigonometry in grade 3? In *What works? Research into practice* (Research monograph 42). Ottawa: The Literacy and Numeracy Secretariat.

Gadanidis, G., Borba, M., Hughes, J., & Lacerda, H. D. (2016). Designing aesthetic experiences for young mathematicians: A model for mathematics education reform. *International Journal for Research in Mathematics Education, 6*(2), 225–244.

Ginsburg, H. P. (2002). Little children, big mathematics: Learning and teaching in the pre-school. In A. D. Cockburn & E. Nardi (Eds.), Proceedings of the 26th conference of the International Group for the Psychology of Mathematics Education (Vol. 1, pp. 3–14). University of East Anglia, UK: Psychology of Mathematics Education.

Gottfried, A. E., Fleming, J. S., & Gottfried, A. W. (2001). Continuity of academic intrinsic motivation from childhood through late adolescence: A longitudinal study. *Journal of Educational Psychology, 93*(1), 3–13.

Gottfried, A. E., Marcoulides, G. A., Gottfried, A. W., Oliver, P. H., & Wright Guerin, D. (2007). Multivariate latent change modeling of developmental decline in academic intrinsic math motivation and achievement: Childhood through adolescence. *International Journal of Behavioral Development, 31*(4), 317–327. https://doi.org/10.1177/0165025407077752.

Mehta, R., Mishra, P., & Henriksen, D. (2016). Creativity in mathematics and beyond—Learning from Fields medal winners. *TechTrends, 60*(1), 14–18. https://doi.org/10.1007/s11528-015-0011-6.

Mendrick, H. (2008). What's so great about doing mathematics like a mathematician? *For the Learning of Mathematics, 28*(3), 15–16.

Middleton, J. A., & Spanial, P. A. (1999). Motivation for achievement in mathematics: Findings, generalizations, and criticisms of the research. *Journal for Research in Mathematics Education, 30*(1), 65–88.

Moss, J., Beatty, R., Barkin, S., & Shillolo, G. (2008). "What is your theory? What is your rule?" Fourth graders build an understanding of functions through patterns and generalizing problems. In C. E. Greenes & R. Rubenstein (Eds.), *Algebra and algebraic thinking in school mathematics* (pp. 155–168). Reston: National Council of Teachers of Mathematics.

Papert, S. (1980). *Mindstorms*. New York: Harvester Press.

Pinar, W. F. (1975). *The method of "currere"*. Paper presented at the annual meeting of the American Educational Research Association, Washington, DC.

Postman, N. (1995). *The end of education: Redefining the value of school*. New York: Knopf.

Postman, N., & Weingartner, C. (1969). *Teaching as a subversive activity*. New York: Dell Publishing Co., Inc.

Sinclair, N. (2006). *Mathematics and beauty*. New York: Teacher's College Press.

Sinclair, N., & Watson, A. (2001). Wonder, the rainbow and the aesthetics of rare experiences. *For the Learning of Mathematics, 21*, 39–42.

Watson, A. (2008). School mathematics as a special kind of mathematics. *For the Learning of Mathematics, 28*(3), 3–7.

Whitehead, A. N. (1929). *Aims of education*. New York: The Free Press.

Whiteley, W., & Davis, B. (2003). A mathematics curriculum manifesto. In E. Simmt & B. Davis (Eds.), *Proceedings of the 2003 annual meeting of the Canadian Mathematics Education Study Group* (pp. 79–83). Edmonton, AB: CMESG/GCEDM. Retrieved from http://www.cmesg.org/wp-content/uploads/2015/01/CMESG2003.pdf

On Teaching and Learning Mathematics from a Cultural-Historical Perspective

Luis Radford, Isaias Miranda, and Gilbert Lacroix

Abstract In this chapter, we discuss some ideas of a cultural-historical theory of the teaching and learning of mathematics. The basic ideas emerged from, and have evolved during, an ongoing long-term collaboration between researchers and teachers. Since its inception, this long-term collaboration has sought to offer an alternative to child-centred individualist educational perspectives. It endeavours to understand and foster mathematics thinking, teaching, and learning conceived of as cultural-historical phenomena. This collaboration has led to what has been termed the theory of objectification. We illustrate the basic ideas through the discussion of a classroom episode where what is at stake is the production and understanding of graphs in a grade 10 mathematics class.

Keywords Teaching and learning · Graphs · Multimodality · Gestures · Ethic · Theory of objectification

Introduction

The collaboration between researchers and teachers featured in this chapter goes back to the introduction of a new mathematics curriculum in Ontario in 1997. The revamping of the curriculum was accompanied by a new set of expectations and qualitative forms of assessment that posed a significant number of challenges for the teachers. The time was ripe for a collaboration between school boards and universities. Our collaboration with school boards started in 1998 with the constitution of a research team that included teachers from the two French school boards in Sudbury,

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Ontario, in addition to Luis Radford and a group of his prospective teachers from Laurentian University's Teacher Education Program.

At the time, mathematics education research was still heavily influenced by the principles of constructivism—and it still is, although perhaps in a more moderate measure. Constructivism exerted, indeed, a great influence on the English speaking world, particularly in North America. Its main tenet is that the student is the producer of his or her own knowledge. This tenet is an evolved formulation of the central idea behind the "student-centred" pedagogy—one of the two pedagogical paradigms that emerged with the educational reform at the dawn of the twentieth century (Radford 2014). Constructivism, as it was formulated by Cobb and his collaborators (e.g., Cobb and Yackel 1996), appealed to mathematics educators and teachers as it offered a way out of the transmissive model of direct teaching. Despite the new possibilities that such an approach brought to the fore, its theoretical formulation was not without its problems. One of them, which was particularly important to us, was the excessive emphasis on the child. The constructivism tenets move the teacher to the sideline and make learning a private and subjective enterprise. Yet, it was obvious that the learning that was unfolding before us in the classrooms where we were working was much more complex. Teachers were doing much more than simply assisting the students. They were literally organically involved in the students' learning. We felt that we needed to formulate learning in different theoretical terms and to stress, in particular, its more collective oriented nature. Another point that was essential to us and that, in our view, was oversimplified in constructivism was the question of the nature of knowledge. Since according to constructivism, knowledge is what results from the student's actions, knowledge is something subjective. For us, knowledge is not a subjective or psychological phenomenon. Knowledge is something cultural and historical. These and other considerations led us to try to rethink the question of teaching and learning.

Our purpose is not to make a detailed analysis of the differences between constructivism and the cultural-historical approach that we developed. If we mention the problems that we found in constructivism it is to provide the reader with a background of our practical actions in the classroom and our ensuing theoretical cogitations. With this chapter we seek to contribute to the general pedagogical problem related to the identification of best teaching and learning practices when these practices are considered from a cultural-historical approach. In particular, we seek to provide insights into manners in which teachers can recognize, identify, and promote collective ways of learning. Our interest is in moving pedagogical understanding beyond the traditional interpretation of learning as the reproduction of known procedures to solve familiar problems and beyond the constructivist view according to which it is the student who constructs her or his own knowledge. We hope that the data we present will provide the classroom teacher with opportunities to reflect on: (1) the active and tremendously important role teachers play in the growing of the students' mathematical understanding; (2) new courses of action to promote deep conceptual understanding based on the variety of methods that students bring to the fore when they engage in collective activity, and (3) the ethical dimension that underpins all teaching and learning.

An Example of a Mathematics Lesson: Making Sense of Cartesian Graphs

In the rest of the article we discuss a mathematics lesson and the main ideas of our approach. The mathematics lesson that we would like to discuss here comes from a grade 10 class. It is about making sense of a Cartesian graph in a technological environment based on a graphic calculator TI 83+ and a probe—a Calculator Based Ranger or CBR (a wave sending-receiving mechanism that measures the distance between itself and a target).

In previous lessons, the students became familiar with the calculator graph environment and the CBR. In these lessons, they had dealt with a fixed CBR and one moving object. In the lesson that we will discuss here, the students were provided with a graph, a drawing, and a story. The graph showed the relationship between the elapsed time (horizontal axis) and the distance between two *moving* children (vertical axis) as measured by the CBR (see Fig. 1).

Here is the story at the heart of the lesson: "Two students, Pierre and Marthe, are one metre away from each other. They start walking in a straight line. Marthe walks behind Pierre and carries a calculator plugged into a CBR. We know that their walk lasted 7 seconds. The graph obtained from the calculator and the CBR is reproduced below."

As in all our lessons, our task design is based on interrelated problems that require the students to think and discuss at deep levels of conceptualization. Furthermore, the students are encouraged to work in small groups of two or three, to suggest ideas, to try to improve the ideas of other students, to challenge the ideas of others when they see fit, and to support each other. The lesson discussed here was divided into three parts. In the first part, the students had to suggest interpretations for the graph. In the second part of the lesson they tested their interpretation using the CBR in the corridor in front of the classroom: they had to reproduce the given graph. In the third part of the lesson, the various small groups reconvened in the classroom and the teacher organized a general discussion and debate; the goal of which was to end up in a critical appraisal of ideas. We do not report here the second and third parts of the lesson. Before we present the students' strategies, we start first with a comment on knowledge about Cartesian graphs in particular, and knowledge in general.

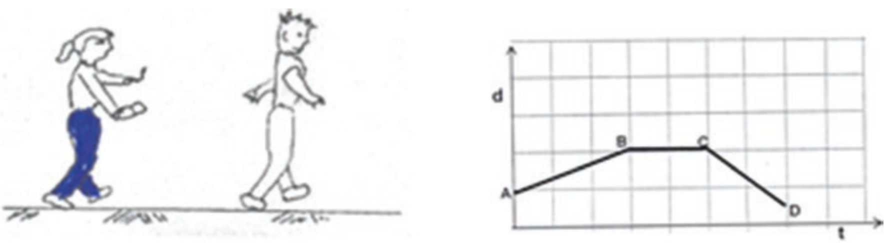

Fig. 1 Left, the drawing of Pierre and Marthe. Marthe has the calculator in her right hand and the CBR in her left hand. Right, the graph given to the students

Knowledge

A Cartesian graph is a complex mathematical representation. It serves to depict, in *specific* ways, certain relationships between things. More specifically, through a Cartesian graph, mathematicians depict a co-variational relationship between *variables*. Mathematicians use Cartesian graphs as representation or artifacts for dealing with and thinking of cultural realities in a mathematical manner. To do so, they resort to a sophisticated geometric-analytic syntax and a complex manner of conveying meanings. Now, the syntax and meanings associated with a Cartesian graph have evolved historically and have undergone a continuous process of refinement. They bear the imprint and sediments of the cognitive activity of previous generations. What this means is that the knowledge associated with Cartesian graphs is historical and cultural. But because of the highly historical refinement, the making and interpreting of Cartesian graphs are not a trivial endeavour for the students. The same can be said of the syntaxes and meanings associated with equations, functions, probabilities, and any mathematical domain.

Knowledge in general, and knowledge about Cartesian graphs in particular, offer students *potential* situated ways of thinking, acting, and experiencing the world. Knowledge exists as *potential* cultural-historical ways in which to reflect and engage in the world.

The fact that it is *potentially possible* to think of certain parts of our cultural reality in terms of Cartesian graphs or equations or probabilities, etc., does not amount to asserting that the students will end up doing so (this is, in fact, what happened in the lesson that we discuss here, as we will see below). It is here where *learning* is required.

Learning

We consider learning as a collective material, embodied, and ideational process where multiple voices and actions become entangled as teachers and students engage in mathematics classroom activity. Of course, not all entanglement of teachers' and students' voices adds up to learning. Learning is about learning *something*. In our example, it is about thinking and taking action mathematically through the use of Cartesian graphs. We need hence to define learning in more specific terms. It is here where the concept of *objectification* intervenes.

Specifically, a *process of objectification* is a social, active, creative, imaginative process through which students gradually become critically conscious of historically constituted cultural meanings and forms of thinking and action. Within this context, understanding the making and meaning of a graph, the way it conveys information, and the potentialities it carries for thinking and acting upon our world, rests on processes of objectification underpinned by one's voice, others' voices and historical voices.

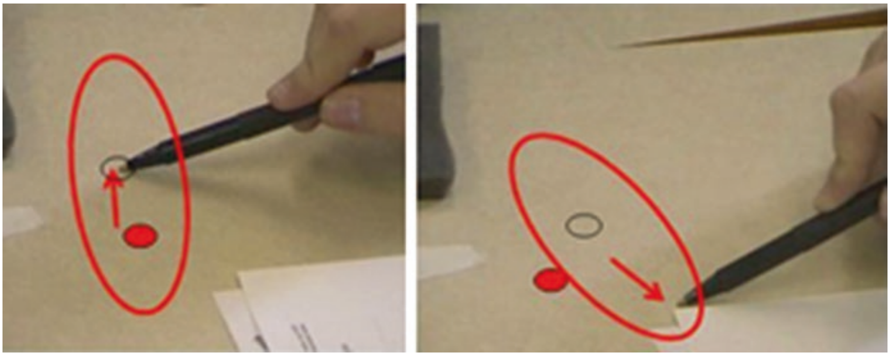

Fig. 2 Pictures 1 (left) and 2 (right). Maribel moves the pen on the desk, then stops to signify that Pierre is idle (segment BC) and Pierre returns towards Marthe

Let us turn to an excerpt from the lesson that comes from one three-student group (a previous analysis of the data was presented in Radford et al. 2008; Radford 2009). The students were Maribel (M), María (MJ) and Carla (C). The students discussed the problem for a few minutes. In Line 1 (L1), Maribel gives a summary of the group discussion:

1 M: (*Moving the pen on the desk, she says*) He [Pierre] moves away from Marthe for 3 s and then (*moving the pen further along the desk*; see Fig. 2, Picture 1), he stops, so he might have like dropped something for 2 s, and (*moving the pen back this time*; see Fig. 2, Picture 2) he returns towards Marthe.

2 C: Well, even though he moves away, but [thinking of the third segment of the graph] he returns back to…I don't know.

3 MJ: Well, if she walks with him, so, it [the graph] doesn't really make sense!

Embodiment and Sensuous Actions

The students' first interpretation rests on the idea of "absolute motion." The segments AB, BC, and CD are interpreted as Pierre moving away, stopping, and coming back. Although the students' current interpretation does not yet resonate with the expected mathematical interpretation, we can see that such an interpretation has been forged through a complex coordination of perceptual, kinaesthetic, symbolic, and verbal elements. After watching the video again and again (and many other videos too over the course of the years), we became convinced that the students' gestures are not merely redundant mechanisms of communication (e.g., Edwards et al. 2009; Nemirovsky 2003) but a key part of the process of objectification. This was observed time and again in the course of our continuous work with teachers and students. In more general terms, in the processes of objectification, recourse is made

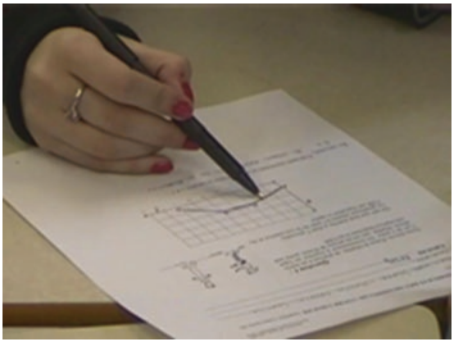

Fig. 3 MJ moves the pen from A to B, meaning Pierre's motion (L5)

to the body (e.g., kinesthetic actions, gestures), signs (e.g., mathematical symbols, graphs, written and spoken words), and artifacts of different sorts (e.g., rulers, calculators, and so on). All these signs and artifacts used in the processes of objectification we call *semiotic means of objectification*. Maribel's dynamic gesture is an example of semiotic means of objectification.

If we come back to the previous passage and investigate it in more detail, we realize that through gestures and their synchronic link with movement verbs ("to move away," "to come back"), Maribel offers an attempt at making sense of the graph. It is at the end of this episode that María reminds her group-mates that Marthe is moving too, so that, according to the current interpretation, the graph "doesn't really make sense!" Twenty seconds later, Maribel offers a refined interpretation that tries to address the issue raised by María:

4 M: Well technically, he walks faster than Marthe…right?
5 MJ: She walks with him, so it could be that […] She is walking with him, so he can walk faster than her (*She moves the pen on segment AB*; see Fig. 3). [He] stops (*pointing to points B and C*)…
6 M: No, there (*referring to the points B and C*) they are at the same distance…
7 C: (*After a silent pause, she says with disappointment*) Aaaaah!

The graph interpretation has changed: In L4, Maribel introduces the two-variable comparative expression "X walks faster than Y." In L5, María reformulates Maribel's idea in her own words while producing a more sophisticated interpretation. Indeed, L5 contains three ideas: (1) Marthe walks with Pierre, (2) Pierre walks faster than her, and (3) Pierre stops. Although improved, the interpretation, as the students realize, is not free of contradictions. Even if, at the discursive level, Marthe is said to be walking (L5), segment AB is still understood as referring to Pierre's motion (see Fig. 3). However, segment BC is interpreted not in terms of *motion* but of *distance* (L6). Moreover, it is interpreted as the distance between Pierre and Marthe. So, while segment AB is about Pierre's motion, segment BC refers to something about both children. The oddity of the interpretation leads to a tension that is voiced by Carla in Line 7 with an agonizing "Aaaaah!" The partial objectification bears an untenable incongruity.

The Ethical Dimension

In our approach, the interaction and communication between students, and between students and teachers are framed by a communitarian ethic through which students are encouraged to show openness toward others, responsibility, solidarity, care, and critical awareness. In the previous episodes, we see, indeed, that the students take responsibility in the teamwork. They listen to the other ideas, and try to improve them or challenge them. This communitarian ethic (e.g., Radford 2012) does not appear spontaneously in the classroom. Teachers nurture it by encouraging the students to engage responsibly with others (e.g., Radford 2014). For instance, if the teacher notices that one of the students is not following, he/she may intervene and ask the other students if they are making sure that everyone understands. He/she may also ask the student who may not seem to be engaging/understanding to ask for help.

The Teacher

Let us come back to the students' discussion about the graph. The students continued discussing and arrived at a new interpretation: Pierre and Marthe maintained a distance of 1 m apart throughout, but they could not agree on whether or not this interpretation was better than, or even compatible with, Maribel's interpretation (L4). Having reached an impasse, the students decided to call the teacher (T). When he arrived, María explained her idea, followed by Maribel's opposition. It is this opposition that is expressed in L8:

8 M: No, like this (*moving the pen along segment AB*) would explain why like, he goes faster, so it could be that he walks faster than her…
9 T: Then if one is walking faster than the other, will the distance between them always be the same?
10 M: No, (*while moving the pen along AB, she says*) so he moves away from the CBR and then. … What happens here (*pointing to segment BC*), like?
11 MJ: He takes a break.
12 T: So, is the CBR also moving?
13 M: Yes.

In L9, the teacher rephrases in a hypothetical form the first part of Maribel's utterance (L8) to conclude that, under the assumption that Pierre goes faster, the distance cannot be constant.

The philosophy behind our approach is to leave the students to engage with the mathematics problems as much as they can. Once they have gone as far as they can, the teacher intervenes. His/her role is not to merely assist the students. His/her role is to *engage* with the students and to try to challenge them to move the students' strategies further or to suggest other paths. The suggestion of other paths that may not have been noticed by the students is what we see in the previous excerpt.

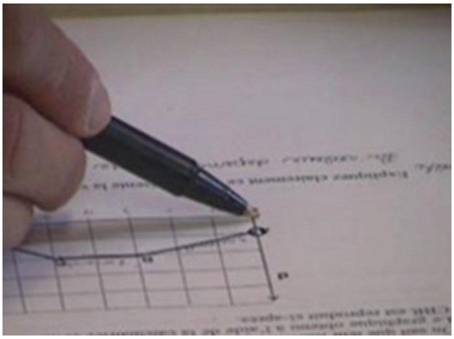

Fig. 4 The teacher moves the pen back and forth between 0 and A

Indeed, we see that the teacher's strategy helps move the students' discourse to a new conceptual level. Maribel's L10 utterance shows that the focus is no longer on relative speed but on an emergent idea of relative distance. The *gesture* is the same as María's in Fig. 3, but its conceptual content is different. However, as shown in L10, the students still have difficulties providing a coherent global interpretation of the graph. How to interpret BC within the new relative motion context? Drawing on Maribel's utterance (L10), the teacher suggests a link between Marthe and the CBR, but the idea does not pay off as expected. The teacher then tries something different:

14 T: OK. A question that might help you…A here (He writes 0 at the intersection of the axes and moves the pen along the segment 0A). What does A represent on the graph? (He moves the pen several times between 0 and A; see Fig. 4)
15 MJ: Marthe.
16 T: This here is 0? We'll only talk about the distance. OK? (He moves the pen again as in Line 14)
17 MJ: 1 metre.
18 T: It represents 1 metre, right?…1 metre in relation to what?
19 M: The CBR.
20 T: OK. So, does it represent the distance between the two persons?
21 M: So this (moving the pen along the segments) would be Pierre's movement and the CBR is 0.
22 MJ: (Interrupting) First he moves more…

Capitalizing on the emerging idea of relative distance, the teacher's strategy now becomes to call the students' attention to the meaning of a particular segment–the segment 0A. He captures their attention in three related ways: *writing* (by writing 0 and encircling the point A); *gesturing* (by moving the pen back and forth between 0 and A); and *verbally* (L14). In L15, point A is associated with Marthe. In L16, he formulates the question in a more accurate way, and takes advantage of the answer to further emphasize the idea of the relative meaning of the distance. Line 21

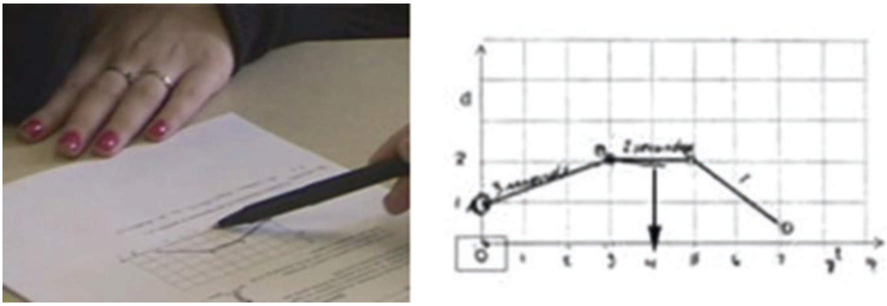

Fig. 5 Left, Maribel makes a vertical gesture that goes from BC to the time axis. Right, we have indicated this gesture by an arrow). This gesture is a generalization of the teacher's gesture (see Fig. 4)

includes the awareness that the CBR has to be taken into account, while L22 is the beginning of an attempt at incorporating the new significations into a more comprehensive account of the meaning of the graph.

The students thus entered into a new phase in their process of knowledge objectification. They continued discussing in an intense way. Here is an excerpt:

23 C: He moves away from her, he stops then comes closer.
24 M: But she follows him.… So, he goes faster than she does, after, they keep the same distance apart.

In L23, Carla still advocates for an interpretation of the graph that suggests a fragile understanding of relativity of motion. In the first part, she makes explicit reference to Marthe ("He moves away from her"), but in the second and third parts of the utterance, Marthe remains implicit. In L24, Maribel offers an explanation that overcomes this ambiguity. Even though the segment AB is expressed in terms of rapidity, the previously reached awareness of the effect of rapidity in the increment of distance makes the interpretation of BC coherent. The recapitulation of the students' efforts is made by Maribel, who, before the group starts writing their interpretation, says: "Maybe he [Pierre] was at 1 metre (*pointing to A*) and then he went faster; so now he is at a distance of 2 metres (*moving the pen in a vertical direction from BC to a point on the time axis*; see Fig. 5); and then they were constant and then (*referring to CD*) they slowed down. Would that make sense?"

The students succeeded in refining their mathematical understanding, although some edges still remained to be polished. In the interpretation of CD, Maribel did not specify in which manner they slowed down. Was it Pierre who slowed down? Was the reduction of distance the effect of Marthe increasing her speed? Was it something else? These questions were discussed in the final general classroom discussion. In writing their answer, this group, however, realized that something important was missing. Naturally, writing requires one to make explicit, and thereby make very explicit relationships that may remain implicit at the level of speech and gestures. Maribel's activity sheet contains the following answer: "Pierre moves away from Marthe by walking faster for 3 seconds. He is now 2 metres away from her. They walk at the same speed for 2 seconds. Pierre slows down for 2 seconds so he gets closer to Marthe."

Discussion

As noted in the introduction, we are interested in promoting best teaching practices that are likely to foster deep student conceptual understanding through forms of collective learning. The example discussed in the previous section shows how the interaction between students and the teacher triggered sensuous actions that are supportive of the learning process. The example shows how the teacher challenged the students' evolving understanding, pushing their conceptualization to new levels. But our interest is not only in the mathematical content. We challenge the idea that best teaching practices are about the mathematical content only. Best teaching practices have to include the dimension of the student—the student as a social being in the making. This is why we are also interested in nurturing classroom collective activities that foster an ethical dimension of solidarity, cooperation, and responsibility. In the example presented here, we see the teacher very attentive to the students' needs. The teacher is deeply engaged in the students' learning and well-being. To the commitment of the teacher, the students respond with a similar other-oriented commitment that keeps the activity unfolding. This ethical commitment appears not only on the discursive level, where the participants attend responsibly to what the others are saying, but also on the embodied level of gestures, posture or body position, and the engagement with mathematical signs.

Concluding Remarks

In this chapter, we discussed, in broad terms, an approach to the teaching and learning of mathematics that emerged out of a collaboration between a research team at Laurentian University and several French school boards in Ontario. This long-term collaboration arose out of practical needs to improve the teaching and learning of mathematics and led to the elaboration of a cultural-historical educational theory—the theory of objectification. Such a theory provides insights into the design of tasks that engage the students at deep levels of conceptualization and encourage evolved forms of human collaboration based on a communitarian ethic where students show responsibility, care, and solidarity towards the others. The theory is based on a multimodal methodology that provides room for the interpretation of the students' understanding of mathematics and the teachers' role in teaching.

In the example that we discussed here, our analysis suggests that one of the most important difficulties in understanding the graph was overcoming an interpretation based on a phenomenological reading of the segments in terms of absolute motion, and attaining one that put emphasis on relative relations. Instead of representing the state of an object in reference to a fixed point, points in the second case represented and came to signify relationships between them and a moving point. As we saw, the logic of interpreting a Cartesian representation of relative motion became progressively apparent for the students through intense discussions. The phenomenological interpretation of the graph was replaced by one centred on relative distances. Crucial in this endeavour was the teacher's intervention. The teacher was indeed able to create what in Vygotskian terms we can refer to as a successful *zone of proximal development* (Vygotsky 1978) that afforded the evolution of meanings both at the discursive and gestural levels. Thus, after his intervention, in the same way that words became more and more refined, so too did gestures: while the students' first gestures were about Pierre's motion, their last gestures were related to distances in a meaningful relational way.

The successful creation of a *zone of proximal development* was due to the teacher's ability to find a common *affective* and *conceptual* ground for the evolution of the students' meanings. The teacher brought out the students' meanings from behind, as it were, and helped them push their meanings beyond their initial locations. The coordination of words with the sequence of similar gestures and signs in the Cartesian graph (see Fig. 4) helped the students understand the meaning of the segment 0A in the context of the problem. The segment 0A entered the universe of discourse and gesture, and its length started being considered as the initial distance between Pierre and Marthe at the beginning of their walk. Without teaching the meaning directly, the teacher's interactional analysis of the meaning of segment 0A was understood and generalized by the students in a creative way (see Fig. 5).

Acknowledgments The long-term collaboration described in this chapter was initiated in 1998 and continues today. It was ensured by consecutive grants from the Social Sciences and Humanities Research Council of Canada/Le conseil de recherches en sciences humaines du Canada (SSHRC/CRSH) and various grants from the Ontario Ministry of Education.

References

Cobb, P., & Yackel, E. (1996). Constructivist, emergent, and sociocultural perspectives in the context of developmental research. *Educational Psychologist, 31*(3/4), 175–190.

Edwards, L., Radford, L., & Arzarello, F. (2009). Gestures and multimodality in the teaching and learning of mathematics. *Educational Studies in Mathematics, 70*(2), 91–215.

Nemirovsky, R. (2003). Perceptuo-motor activity and imagination in mathematics learning. In N. Pateman, B. Dougherty, & J. Zilliox (Eds.), *Proceedings of the 27th conference of the International Group for the Psychology of Mathematics Education* (Vol. 1, pp. 103–135). Honolulu: PME.

Radford, L. (2009). Signifying relative motion: Time, space and the semiotics of Cartesian graphs. In W.-M. Roth (Ed.), *Mathematical representations at the interface of the body and culture* (pp. 45–69). Charlotte: Information Age Publishers.

Radford, L. (2012). Education and the illusions of emancipation. *Educational Studies in Mathematics, 80*(1), 101–118.

Radford, L. (2014). On teachers and students: An ethical cultural-historical perspective. In P. Liljedahl, C. Nicol, S. Oesterle, & D. Allan (Eds.), *Proceedings of the joint 38th meeting of the conference of the International Group for the Psychology of Mathematics Education (PME) and the 36th North American chapter (PME-NA)* (Vol. 1, pp. 1–20). Vancouver: PME.

Radford, L., Miranda, I., & Guzmán, J. (2008). Relative motion, graphs and the heteroglossic transformation of meanings: A semiotic analysis. In O. Figueras, J. L. Cortina, S. Alatorre, T. Rojano, & A. Sepúlveda (Eds.), *Proceedings of the joint 32nd conference of the International Group for the Psychology of Mathematics Education (PME) and the 30th North American chapter (PME-NA)* (Vol. 4, pp. 161–168). Morelia: Cinvestav-UMSNH.

Vygotsky, L. S. (1978). In M. Cole, V. John-Steiner, S. Scribner, & E. Souberman (Eds.), *Mind in society: The development of higher psychological processes*. Cambridge, MA: Harvard University Press.

Indigenous Perspectives in School Mathematics: From Intellect to Wisdom

Glen Aikenhead

Abstract When some Indigenous mathematizing is taught in school mathematics, something important and strange occurs. Not only does the achievement of Indigenous students dramatically increase, but the achievement of non-Indigenous students goes up on average as well. A clash between most Indigenous students' worldviews and the worldview endemic to mathematics tends to be an obstacle for these students. For example, one fundamental epistemic difference between the Eurocentric worldview of mathematics and the Indigenous worldviews of mathematizing concerns what type of understanding is expected. On the one hand, conventional school mathematics deals with an *intellectual* tradition of understanding—only thinking with the content. And on the other hand, Indigenous mathematizing deals with a *wisdom* tradition of understanding—thinking, doing, living, and being with the content. Indigenous mathematizing is obviously richer. This worldview clash also exists but to a lesser degree for a majority of non-Indigenous students. Different worldviews reflect differences between cultures; in this case, between a student's culture and the culture of mathematics. This research finding contradicts the Platonist belief that school mathematics is culture-free, value-free, universal, decontextualized, and purely objective; a challenging obstacle for many students' success, especially for Indigenous students. Upon analysis, however, the Platonist belief turns out to be a deception, historically perpetrated in the nineteenth century when public education was being established. In Canada's era of reconciliation with Indigenous peoples, it seems most appropriate to correct the detrimental deception and revamp the nineteenth century mathematics curriculum into a twenty-first century evidence-based pragmatic curriculum.

Keywords Anti-Platonist · Culture-based · Cross-cultural · Decolonizing · Curriculum

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This chapter acknowledges school mathematics' 200 years of hegemony and highlights its hidden cultural nature. In the nineteenth century, academics and governments independently created two major obstacles that have severely depressed Indigenous students' high school graduation rates. First, academic educators defined school mathematics as being decontextualized, value-free, non-ideological, purely objective in its use, and universal in the sense of being the only legitimate way of mathematizing. Secondly, the Canadian Federal Government established Indigenous residential schools. Today in the twenty-first century, mathematics educators know how to mitigate the consequences of both obstacles, yet the secondary curriculum by and large carries on its nineteenth century function.

The Rise of Platonist School Mathematics

The nineteenth century definition of school mathematics has a historical context. What we call "mathematics" today evolved over the ages in various civilizations, from which early European mathematicians appropriated what made sense to them (Ernest 2016b). A mechanism for this appropriation, based on language-laden cognition (Kawasaki 2002), describes how a concept can have worldview presuppositions implicitly attached to it.

For example, the concept of circle in school mathematics has a cluster of Eurocentric peripheral concepts such as point and plane, as well as associated peripheral values such as intellectual purity, consistency, and objectivity. If a mathematics textbook stated, "Indigenous medicine wheels have circle properties," then the phrase "circle properties" refers to a *decontextualized* meaning of the term "circle" with its cluster of peripheral Eurocentric concepts. The textbook author has not understood, or has purposefully ignored, the contextualized subjective, holistic and spiritual peripheral concepts connected to an Indigenous meaning of circle.

Culture-based peripheral concepts get lost in translation. Thus, imagine the challenge for Indigenous speaking students when they unknowingly bring their unconscious peripheral concepts into their mathematics class that pretends to have none. Similarly, imagine the implicit ideas that were lost in translation when early European mathematicians appropriated concepts from ancient Egyptian, Hindu, Arabic and Chinese cultures (Aikenhead 2017a, section 4.4). This appropriation unconsciously stripped away ancient peripheral concepts, and unconsciously replaced them with European peripheral concepts associated with the culture of European mathematicians.

The European renaissance version of mathematics slowly found a home in elite British universities during seventeenth to eighteenth century England. In this Age of Enlightenment, mathematics had to compete for a place in Cambridge's and Oxford's stringent curricula comprised of ancient languages, religious studies, history, and the classics. A curriculum's difficulty was thought to prepare the mind for any future event, occupation, or profession (Willoughby 1967). Decontextualized abstractions ensured difficult content. Channelling Plato's dichotomy "World of

Ideas" versus "Phenomenal World" (Kawasaki 2002, p. 25), mathematics instructors divorced their ethereal abstract subject matter from the context of worldly events. This explains their "Platonist" moniker.

A Platonist belief about mathematics assumes "a static but unified body of certain knowledge. Mathematics is discovered, not invented" (Ernest 1988, website quote). This belief promotes a "doctrine that mathematical entities have real existence and that mathematical truth is independent of human thought" (Collins English Dictionary 1994). An acultural, decontextualized, value-free, non-ideological, objective school subject flourished in the competition for a place in the British elite Latin Grammar Schools, which began to teach mathematics as prerequisites for entrance to elite universities (Willoughby 1967).

The Industrial Revolution (eighteenth to nineteenth centuries) led to the establishment of the British public education system, quickly adopted in Canada. Mathematics became a core subject at a time when tensions escalated between two opposing philosophies of public education: academic elite versus practical relevance (Layton 1981) that would contextualize Platonist mathematics content in the everyday world actually experienced by students and adult citizens employed in, for example, business, manufacturing, bureaucratic and professional occupations.

Married to an absolutist philosophy, the Platonists defended their territory by eschewing practical utility and marginalizing mathematics' human features, such as its values and ideologies, plus its roles in everyday life. This stance was taken even though Platonists' knowledge could be identified with such values as generalizability and such ideologies as quantification (Corrigan et al. 2004). Ernest (1991) described what happened: "[T]he values of the absolutists [were] smuggled into mathematics, either consciously or unconsciously, through the definition of the field" (p. 259). In the end, Platonist's elitism won the battle over practical relevance.

What clever smuggling strategy did Platonists use to define school mathematics? First, they drew on a binary, "logical versus irrational," invented by "Western culture dating back to Socrates, Plato, and Aristotle" (Hall 1976, p. 213), in order to construct their own theoretical binary: "*formal mathematical discourse*" versus "*informal mathematical discourse*" (Ernest 1991, pp. 259–260). Then they arbitrarily assigned the highly abstract decontextualized aspects of mathematics to the *formal discourse* category, which would be their discipline of school mathematics. This assignment was consistent with the ancient Greek philosophy proclaiming mathematics content is to be *discovered* as abstract objects that constitute the universe, rather than being invented by humans (Aikenhead 2017a, b).

The *informal discourse* category comprised everything that would have made school mathematics a human endeavour; for example, the application of Platonist mathematics in political-societal-economic contexts (Skovsmose 2016); its presuppositions, ideologies, and values by which it operates; its history; and its preferences that guide mathematicians. Informal mathematical discourse was suppressed so effectively that most mathematics educators seem unaware of it today. Ernest (1991) characterized the Platonist's strategy as illusionary: "[A]t the heart of the absolutist neutral view of mathematics is a set of values and a cultural perspective, as well as an ideology which renders them invisible" (p. 260). In 1921, Einstein had proposed a parallel explanation, described in Aikenhead (2017a, section 4.2.1). Simply put: a surreal version of mathematizing became institutionalized; a humanistic version was suppressed.

Political-social-economic power, rather than evidence-based practice, has successfully maintained the Platonist dogma until recently for three reasons: it provides school mathematics with the highest status among school subjects; it allows other institutions to use students' success at mathematics as an unquestioned objective screening device for post-secondary education and for employers, whether or not student assessment is objective and whether or not the high school mathematics content relates to the post-secondary program or the occupation; and it guarantees that "prestige, control, authority, and power are gained by the knower" (Russell 2016, p. 75). Russell and Chernoff (2013) described this social screening function as "unethical" (p. 109).

The Platonist ideology of quantification demands that outcomes of schooling be commodified so that achievement can be assessed numerically (Ernest 2016a). This quantified worth of students, teachers, and educational jurisdictions is so simplistic it immeasurably distorts reality (Aikenhead 2017a, section 9.4). Quantification conveys a false aura of objectivity (Aikenhead 2008). Simply put, political expediency trumps quality education defined as "the human dimensions of knowing" (Ernest 2016a, p. 53). Even worse, the allocation of a government's "resources for testing is the main argument to justify math contents" in curricula (D'Ambrosio 2016, p. 33).

The Rise of Cultural School Mathematics

The Platonist belief was challenged when anthropologists discovered that in all cultures mathematical systems developed in tandem with people's everyday cultural activities (Wilder 1981). Bishop's (1988) research identified six fundamental types of mathematizing found in most major cultures: counting, locating, measuring, designing, playing, and explaining. "Mathematics, as an example of a cultural phenomenon, has a 'technological' component" (p. 146). Bishop characterized mathematics as *a symbolic technology for building a relationship between humans and their social and physical environments*.

Cultural practices are based on a group's collective worldview. A *clash* between most Indigenous students' worldviews and the worldview endemic to Platonist school mathematics tends to make mathematics foreign to many students (Aikenhead 2017a, section 3.3). The clash, for example, could be due to an epistemic dissonance caused by different expectations of learners. Conventional school mathematics expects an *intellectual* understanding by students—thinking with the content largely in analytical-deductive ways.

On the other hand, Indigenous mathematizing expects a *wisdom* understanding—thinking, doing, living, and being with a mathematizing process in a holistic way (Aikenhead and Michell 2011). To Indigenous students, their culture's mathematizing seems richer and makes common sense. Differences in expectations between an intellectual and wisdom tradition of understanding creates degrees of alienation and marginalization for most, but not all, Indigenous students entering mathematics classrooms. This worldview-based clash also exists to varying degrees for many non-Indigenous students (Aikenhead 2017a, section 9.3; Nasir et al. 2008), depending on how closely a student's worldview harmonizes with a Platonist-like worldview.

Different worldviews explain differences between cultures; such as between an Indigenous student's home culture and the culture of school mathematics with its Western or Euro-American cultural features (Aikenhead 2017a, b; Ernest 1988; Russell and Chernoff 2013). These features include, for example, an epistemology of consistency, an ontology that embraces Cartesian duality, and an axiology of objectivity; as well as Skovsmose's (2016) mathematics in action.

Ernest (1988) replaced a Platonist belief with a cultural belief by hybridizing the formal and informal discourse dichotomy into one category, a *Euro-American school mathematics* (Aikenhead 2017a, section 4.2). He characterized mathematics "as a dynamically organized structure, located in a social and cultural context" for problem solving; and a "continually expanding field of human creation and invention" (as cited in Aikenhead 2017a, p. 26).

The Political-Social Context of Reconciliation

The crucial importance of diminishing Indigenous students' cultural clashes with Platonist school mathematics becomes evident in Canada's twenty-first century era of reconciliation, which emerged in direct response to Indigenous people having endured colonial genocide (Woolford et al. 2014). Colonial genocide took the form of marginalization, violence, engineered starvation, cultural erosion, and unrelenting racism (Daschuk 2013). It continues today as neo-colonialism causing Indigenous people to suffer degrees of deprivation in education, social assistance, housing, health care, employment, and criminal justice. This is the context of teaching mathematics in today's Canadian classrooms that include Indigenous students.

One example of neo-colonialism is hearing a mathematics teacher complain, "The [Indigenous] students who come to our school have serious gaps in their education" (FNESC 2011, p. 29). By framing the issue as a lack of background knowledge, teachers implicitly fault the students. What actually happened, however, is Canada's colonization forced an "educational debt" on Indigenous students and their families (Bang and Medin 2010, p. 1023). It is this debt that the teacher is actually complaining about; a debt not caused by students. Teachers are expected to help pay it off through teaching mathematics in an anti-discriminatory way, such as teaching according to a cultural understanding of the subject, which contextualizes mathematics in both Canadian mainstream culture and local Indigenous cultures. The quoted teacher's deficit model of teaching disregarded the asset model: "being open and accepting of students' worldviews and experiences … teachers can tap into the holistic and experiential resources of students and treat these resources as assets for academic success" (Aikenhead and Michell 2011, p. 142). What began as a teacher's "objective" assessment of Indigenous students' background knowledge has turned out to be an ethical judgement over a teacher's responsibilities towards Indigenous students. Similarly, what is considered to be an objective screening process of high school students becomes a discriminatory act against one of the three founding nations of Canada; the one that originally taught the other two how to survive. Such discrimination is systemic racism to be sure (Alberta Education 2006).

What went unnoticed by the complaining teacher is the fact that the Platonist strategy to define a mathematics curriculum solely as formal mathematics discourse and to suppress its informal discourse, seriously increased the culture clash for Indigenous students; thus lowering graduation rates. A Platonist curriculum and Canadian residential schools have similar effects on high school graduation rates; albeit different degrees of racism, but systemic racism nonetheless.

Residential schools were a centre piece of colonial genocide: kidnapping children for long periods of time (TRC 2016). The Federal Government's policy to kill the Indian and save the child was severely enacted by church-run schools, from about 1834–1996. Thousands died. Those who did survive to reach high school were usually offered manual labour type of courses: a decision that prevented students from graduating from high school.

Taking responsibility to alter a deficit teaching approach to an asset approach is one way for educators to engage in reconciliation (TRC 2016). Another way is for a Ministry of Education to transform a nineteenth century Platonist curriculum into a twenty-first century curriculum based on a cultural belief about school mathematics. The transformation amounts to a shift from a narrow intellectual understanding to a broader wisdom understanding of school mathematics as cultural practices.

Implications of a Culture-Based Mathematics

Since the 1980s, research and development (R&D) projects have successfully explored ways to mitigate culture clashes between Indigenous students' home culture and the culture of Platonist curricula and conventional classrooms. Two types of R&D programs are generally evident in the literature: those drawing upon Indigenous mathematizing (e.g., ethnomathematics), and those being fully cross-cultural (illustrated below). Aikenhead (2017a, section 8) describes and critiques 10 such projects, most of which represent the first type of R&D project.

The first type investigates, on occasion, Platonist school content contextualized in some Indigenous mathematizing. When this type of instruction takes place, something unexpected occurs consistently. Not only do Indigenous students' mathematics scores rise dramatically (e.g., Lipka and Adams 2004; U.S. Congress HRSECESE 2008), but non-Indigenous students' average achievement increases noticeably (e.g., Furuto 2014; Nelson-Barber and Lipka 2008; Richards et al. 2008; Rickard 2005). Such research studies expose serious shortcomings in conventional school mathematics. Where are the research studies supporting Platonist mathematics? Is tradition a legitimate rationale, in light of this evidence-based practice?

Teaching materials must be developed to support teachers. This is accomplished within a framework of respect by collaborating with Indigenous Elders, knowledge holders, teachers, and community members; illustrated by Aikenhead (2017a, sections 6.1 & 8). Indigenous artifacts or processes are usually chosen so that mathematics educators can superimpose a Platonist concept or image onto the artifact or process, and then teach it in a mathematics lesson. A detailed language-laden cognitive model for this transformation is explained by Aikenhead (2017a, sections 4.4 & 6.3) in terms of a sequence of steps: superimposition, deconstruction and reconstruction. Student interest and engagement is heightened by using concrete Indigenous examples in mathematics classes. But some significant culture clashes still remain (Aikenhead 2017a, section 9.3). A more extensive transformation of school mathematics is required.

The second type of R&D project adds to the first type by changing Platonist school mathematics into culture-based school mathematics—Euro-American school mathematics. Curriculum content is drawn from mainstream Canadian cultural artifacts and processes having an analogic meaning in Platonist mathematics. In this context the Platonist content is taught to students. Some innovative teachers already do this to some extent. But there is more to add.

Some mathematics lessons need to include what Platonists once concealed: informal mathematical discourse; that is, certain ideologies, values, and presuppositions embedded in the culture of Euro-American mathematics and how it is used in political-social contexts (Aikenhead 2017a, sections 4.2.1 & 4.5; Skovsmose 2016). On an age-appropriate basis, teachers will make explicit this cultural nature of Platonist mathematics. Some peripheral concepts will be selected from a triad of sources: Platonist mathematics, mainstream society, and Indigenous mathematizing.

In short, Euro-American mathematics differs from, but coexists with, other culture-based mathematical knowledge systems (Bishop 1988). This means that "*cross-cultural* Euro-American mathematics" (Aikenhead 2017a, p. 42, emphasis added) will be an amalgam of formal (Platonist) and informal (cultural) mathematical discourses, plus the intermittent inclusion, to a non-tokenistic extent, of Indigenous mathematizing (the first type of R&D project). This combination effectively diminishes most culture clashes between Indigenous students' cultural self-identities and the culture of school mathematics—Euro-American mathematics. The triad combination (listed just above) illustrates that different cultures have unique ways of inventing a symbolic technology in order to build a relationship between people and their political, social, economic and physical environments.

At the same time, a twenty-first century curriculum needs to be purged of non-essential Platonist content (Aikenhead 2017a, b, sections 2.4 & 10.2). "Most secondary students" experience degrees of dissonance with the worldview endemic to a Platonist belief (Aikenhead and Elliott 2010, pp. 334–335). Mukhopadhyay and

Greer (2012) challenge the "supremacist position maintained by many mathematician educators who regard abstract mathematics as the crowning achievement of the human intellect, and school mathematics as the transmission of its products" (p. 860). Criteria for choosing curriculum content must focus on "crucial concepts" (Jorgensen 2016) that answer the perennial questions, "Why do we need to know this?", "When will I ever use this?" Political promises about a nation's competitive edge in globalized markets, or about strengthening students' critical thinking, do not stand up to scrutiny (Aikenhead 2017a, b).

For the "24 percent" of high school students living in OECD (Organisation for Economic Co-operation and Development) countries who anticipate a future in a science-related occupation (OECD 2016, p. 113), a highly challenging pre-professional, pre-calculus, culture-based pathway can be designed with a greater emphasis on *need-to-know* Platonist content, compared to a full culture-based mathematics curriculum for the 76% majority of students. A one-size-fits-all conventional curriculum does not represent twenty-first century realities (Russell 2016). Most students respond by "playing Fatima's rules" (Aikenhead 2006, p. 28) to make it appear as if meaningful learning has occurred, when it has not; only credentials have been acquired.

Implications for Teachers

Cross-cultural school mathematics involves modifying instruction. In the spirit of reconciliation, teachers and students will move back and forth between the *culture* of Euro-American school mathematics (i.e., the amalgam of Platonist content and its cultural features that include its actions in society) and the *culture* of a local Indigenous community; with an emphasis on the former. Cross-cultural Euro-American mathematics is implemented within a culturally responsive or place-based pedagogy (Aikenhead et al. 2014; Michell et al. 2008, respectively). Teachers cannot effectively begin, however, without experiencing a cultural immersion (at least 2 days to begin with) designed and run by Indigenous Elders and/or knowledge holders (Aikenhead et al. 2014). Academic workshops are simply ineffective.

Professional development must also include readings about the twenty-first century cultural understanding of the nature of mathematics, followed by self-reflection and discussions within teacher networks; all dedicated to reversing the nineteenth century Platonist indoctrination of students, teachers, and the general public. In some cases, strategies used in cult deprogramming should not be ignored because some teachers' professional identities and belief systems are being challenged. Patient, supportive, ego-centred approaches are needed. This takes time.

Teachers' transformation is a life-long journey along a path of reconciliation. The journey should begin with small innovations, and progress should be measured in years, not months. Progress is accelerated when teachers are mindful of students' diverse recurrent learning strengths (Aikenhead et al. 2014). Examples include: visual more than verbal, oral more than written, and reflective more than trial-and-error. An acquaintance with features of the local Indigenous language is equally beneficial (Aikenhead 2017a, section 6.4).

Conclusion

"There are powerful forces at work keeping cultural domination and institutional racism in place, for it serves the interests of capital and the politically powerful" (Ernest 1991, p. 268). By suppressing the cultural nature of school mathematics, and by dismissing Indigenous mathematizing as irrelevant, a Platonist belief about school mathematics works against any agenda to decolonize its curriculum. A Platonist form of racism is simply anti-reconciliation.

Because the composition of today's high school mathematics was mainly established by a narrow nineteenth century definition of the subject, it is reasonable to redefine the subject today in an evidence-based, inclusive, transparent way; and in terms of a twenty-first century cultural understanding of Euro-American mathematics. This redefinition will renew a mathematics curriculum from only offering *intellectual* understandings, to promoting *wisdom* understandings.

Many Indigenous students respond positively to cross-cultural, Euro-American school mathematics, judged by their dramatically increased achievement. Most non-Indigenous students' achievement profits as well. The result is a win-win situation for all students.

References

Aikenhead, G. S. (2006). *Science education for everyday life: Evidence-based practice*. New York: Teachers College Press.

Aikenhead, G. S. (2008). Objectivity: The opiate of the academic? *Cultural Studies of Science Education, 3*, 581–585.

Aikenhead, G. (2017a). *School mathematics for reconciliation: From a 19th to a 21st century curriculum*. Retrieved from https://www.usask.ca/education/documents/profiles/aikenhead/index.htm

Aikenhead, G. S., & Elliott, D. (2010). An emerging decolonizing science education in Canada. *Canadian Journal of Science, Mathematics and Technology Education, 10*, 321–338.

Aikenhead, G., & Michell, H. (2011). *Bridging cultures: Indigenous and scientific ways of knowing nature*. Toronto: Pearson Education Canada.

Aikenhead, G., Brokofsky, J., Bodnar, T., Clark, C.,... Strange, G. (2014). *Enhancing school science with Indigenous knowledge: What we know from teachers and research*. Saskatoon: Saskatoon Public School Division. Retrieved from http://www.amazon.ca/Enhancing-School-Science-Indigenous-Knowledge/dp/149957343X

Alberta Education. (2006). *Common curriculum framework for K-9 mathematics: Western and Northern Canadian protocol*. Edmonton: Author.

Bang, M., & Medin, D. (2010). Cultural processes in science education: Supporting the navigation of multiple epistemologies. *Science Education, 94*, 1009–1026.

Bishop, A. J. (1988). The interactions of mathematics education with culture. *Cultural Dynamics, 1*(2), 145–157.

Collins English dictionary. (1994). Glasgow: HarperCollins Publishers.

Corrigan, D., Gunstone, R., Bishop, A., & Clarke, B. (2004, July). *Values in science and mathematics education: Similarities, differences and teacher views*. A paper presented at the 35th annual meeting of the Austalasian Science Education Research Association, Armidale, NSW, Australia.

D'Ambrosio, U. (2016). Ethnomathematics: A response to the changing role of mathematics in society. In P. Ernest, B. Sriraman, & N. Ernest (Eds.), *Critical mathematics education: Theory, praxis and reality* (pp. 23–34). Charlotte: Information Age Publishing.

Daschuk, J. (2013). *Clearing the plains: Disease, politics of starvation, and the loss of aboriginal life*. Regina: University of Regina Press.

Ernest, P. (1988). *The impact of beliefs on the teaching of mathematics*. Retrieved from http://webdoc.sub.gwdg.de/edoc/e/pome/impact.htm

Ernest, P. (1991). *The philosophy of mathematics education.* London: Routledge-Falmer. Retrieved from https://p4mriunpat.files.wordpress.com/2011/10/the-philosophy-of-mathematics-education-studies-in-mathematicseducation.pdf

Ernest, P. (2016a). Mathematics education ideologies and globalization. In P. Ernest, B. Sriraman, & N. Ernest (Eds.), *Critical mathematics education: Theory, praxis and reality* (pp. 35–79). Charlotte: Information Age Publishing.

Ernest, P. (2016b). The problem of certainty in mathematics. *Educational Studies in Mathematics, 92*, 379–393.

First Nations Education Steering Committee (FNESC). (2011). *Teaching mathematics in a First Peoples context: Grades 8 and 9*. Vancouver: Author. Retrieved from http://www.fnesc.ca/wordpress/wp-content/uploads/2015/05/PUB-LFP-Math-First-Peoples-8-9-for-Web.pdf

Furuto, H. L. (2014). Pacific ethnomathematics: Pedagogy and practices in mathematics education. *Teaching Mathematics and Its Applications, 33*, 110–121.

Hall, E. T. (1976). *Beyond culture*. Toronto: Doubleday.

Jorgensen, R. (2016). The elephant in the room: Equity, social class, and mathematics. In P. Ernest, B. Sriraman, & N. Ernest (Eds.), *Critical mathematics education: Theory, praxis and reality* (pp. 127–145). Charlotte: Information Age Publishing.

Kawasaki, K. (2002). A cross-cultural comparison of English and Japanese linguistic assumptions influencing pupils' learning of science. *Canadian and International Education, 31*(1), 19–51.

Layton, D. (1981). The schooling of science in England, 1854–1939. In R. MacLeod & P. Collins (Eds.), *The parliament of science* (pp. 188–210). Northwood: Science Reviews.

Lipka, J., & Adams, B. (2004). *Culturally based math education as a way to improve Alaska Native students' math performance* (Working paper no. 20). Athens: Appalachian Center for Learning, Assessment, and Instruction in Mathematics.

Michell, H, Vizina, Y, Augustus, C., & Sawyer, J. (2008). *Learning Indigenous science from place*. Retrieved from http://iportal.usask.ca/docs/Learningindigenousscience.pdf

Mukhopadhyay, S., & Greer, G. (2012). Ethnomathematics. In J. A. Banks (Ed.), *Encyclopedia of diversity in education* (pp. 857–861). Thousand Oaks: SAGE Publication.

Nasir, N. S., Hand, V., & Taylor, E. V. (2008). Culture and mathematics in school: Boundaries between "cultural" and "domain" knowledge in the mathematics classroom and beyond. *Review of Research in Education, 32*, 187–240.

Nelson-Barber, S., & Lipka, J. (2008). Rethinking the case for culture-based curriculum: Conditions that support improved mathematics performance in diverse classrooms. In M. Brisk (Ed.), *Language, curriculum & community in teacher preparation* (pp. 99–123). Mahwah: Lawrence Erlbaum Associates.

OECD. (2016). *PISA 2015 results: Excellence and equity in education* (Vol. I). Paris: OECD Publishing. Retrieved from https://doi.org/10.1787/9789264266490-en

Richards, J., Hove, J., & Afolabi, K. (2008). *Understanding the Aboriginal/non-Aboriginal gap in student performance: Lessons from British Columbia* (Commentary No. 276). Toronto: C.D. Howe Institute.

Rickard, A. (2005). Constant perimeter, varying area: A case study of teaching and learning mathematics to design a fish rack. *Journal of American Indian Education, 44*(3), 80–100.
Russell, G. (2016). *Valued kinds of knowledge and ways of knowing in mathematics and the teaching and learning of mathematics: A worldview analysis* (Unpublished doctoral dissertation). University of Saskatchewan, Saskatoon, SK.
Russell, G. L., & Chernoff, E. J. (2013). The marginalisation of Indigenous students within school mathematics and the math wars: Seeking resolutions within ethical spaces. *Mathematics Education Research Journal, 25*, 109–127.
Skovsmose, O. (2016). Mathematics: A critical rationality? In P. Ernest, B. Sriraman, & N. Ernest (Eds.), *Critical mathematics education: Theory, praxis, and reality* (pp. 1–22). Charlotte: Information Age Publishing Inc.
Truth and Reconciliation Commission (TRC). (2016). Home page. Retrieved from http://www.trc.ca/websites/trcinstitution/index.php?p=39
U.S. Congress House of Representatives Subcommittee on Early Childhood, Elementary and Secondary Education (CHRSECESE). (2008). *Challenges facing bureau of Indian education schools in improving student achievement*. Washington, DC: U.S. Government Printing Office.
Wilder, R. L. (1981). *Mathematics as a cultural system*. Oxford: Pergamon Press.
Willoughby, S. S. (1967). *Contemporary teaching of secondary school mathematics*. New York: Wiley.
Woolford, A., Benvenuto, J., & Hinton, A. L. (Eds.). (2014). *Colonial genocide in Indigenous North America*. London: Duke University Press.

Additional Suggestions for Further Reading

Aikenhead, G. S. (2017b). Enhancing school mathematics culturally: A path of reconciliation. *Canadian Journal of Science Mathematics and Technology Education, 17*(2: Special Monograph Issue), 73–140.

Live(d) Topographies: The Emergent and Dynamical Nature of Ideas in Secondary Mathematics Classes

Jennifer S. Thom and Florence Glanfield

Abstract In this chapter, we consider the images, meanings, and metaphors provoked by the phrase in the title of the opening section of the book: The Changing Landscape. Inspired by our Indigenous and ecological sensibilities, we develop new images, meanings, and metaphors that illuminate the collective nature of secondary mathematics classes. We explore the Pirie-Kieren (1989) model in terms of how it relates to the collective level of the class. Using a video based excerpt from a secondary mathematics class, we map the observed ideas onto the Pirie-Kieren model as they emerge and evolve, moment to moment. The results reveal a dynamic ideational topography of the classroom. Through illustrative exemplars and the analyses that accompany these, we reflect on how paying attention to the ideas of the class as live(d) topographies occasion new questions for further investigation into mathematics classroom collectives.

Keywords Ideas · Ideational activity · Emergence · Dynamics · Ever-changing landscape · Live(d) topographies · Pirie-Kieren model · Collective · Secondary mathematics classroom

Introduction

When we, Florence and Jennifer, set to work on the chapter together, we were surprised to discover that our work began immediately upon reading the section title for the book: *The changing landscape of teaching and learning mathematics*. Drawn to the words "landscape" and "changing", we wondered how these terms might help us to better understand a mathematics classroom. In this chapter, we examine the words "landscape" and "changing" first as concepts and second as metaphors.

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Following this, we describe and propose how the emergence and evolution of mathematical ideas in a class can be imagined and mapped as a live(d) topography.

Given our Indigenous and ecological sensibilities, we acknowledge with gratitude, a mathematics education research collective to which we belong. The work of this collective entails the modelling of classroom dynamics (McGarvey et al. 2015). Our thinking for this chapter was inspired by this research.

(Re)Conceptualizing Meanings of Landscape and Change

Focusing first on the concept of landscape, we considered its everyday meaning; that is, "all the visible features of an area of land" (Landscape 2017). As we discussed possible ways in which the definition could be applied to mathematics teaching and learning, we moved into another conversation about the places where we grew up. The following descriptions were shared:

Florence was born and raised in the boreal forest of Northeastern Alberta. There are muskeg; a variety of trees such as spindly black spruce, jack pine, aspen, white spruce and willows; different kinds of berries such as cranberry, blueberry, and saskatoon; and many species of animals and birds. The land, once home only to Indigenous peoples such as the Cree, Dené, and Métis, is now home to settlers of all descents. As well, beneath the boreal forest lie vast deposits of oil sands and minerals. Today, there are many natural-resource development plots in this region of Alberta.

Jennifer was born, raised, and lives on Vancouver Island, British Columbia. Located at the Southwestern edge of Canada, it is the largest North American island in the Pacific Ocean. The mountains make up the backbone of the island while the network of rivers and lakes flow through each region. Salal, huckleberry, Garry Oak, Arbutus, and ancient forests of Western Red Cedar and Douglas Fir grow here. Trout, black bear, cougar, deer, elk, wolf and marmot are just some of the animals. Today, there are people from many diverse cultures. However, historically, Vancouver Island was home to 53 Indigenous cultural groups, European settlers, and the Chinese. The influence of these cultures is still seen in the land use, food, traditions, buildings, educational system, and arts.

It is interesting that while we come from the same Canadian homeland, our particular landscapes are vastly different, each characteristically unique to its location. Even more compelling, is how "all the visible features" of each region—that is, the land, water, plants, animals, weather systems and cultures—are not so much 'parts' that make up each landscape as they are the dynamic and collective expression of the interconnected patterns and recursive relations that give rise to and sustain it (Bateson 1972). For example, think about how the air, wind, and precipitation continuously shapes the visible features of the landscape of Northeastern Alberta compared to the landscape of Vancouver Island. Now consider how these ever-changing features of each region affect the way that the air currents flow or the direction from which the wind blows, or even the type of precipitation that falls from the sky. No feature of a landscape can be reduced to a single entity or separated from the whole.

Another consequence of expanding our focus beyond the visible features of landscapes involves recognition of how landscapes are constantly coemerging, coevolving, and cohering. Thus, by not assuming features to be only 'things' but also investigating how such aspects of landscapes are in fact, "distinctive features of a sphere of activity" (Landscape 2017) occasions potential for new insight. A landscape then as that which co-emerges and co-evolves is not some thing that is chang*ed* or *un*chang*ed*. Instead, as continuous interrelationships, a landscape is inherently collective, constant, and recursive events of be(com)ing. Any and all landscapes are (ever)chang*ing*.

The Classroom as Landscape and Mathematical Ideas as a Topographical Feature

Consistent with current research that conceptualizes the classroom as a learning system and metaphorizes it as a living body (e.g., McGarvey et al. 2017), we conceive the classroom as a changing landscape. As such, mathematics teaching and learning are not events attributable to individual (human) agents. Rather, as distributed activity inherent in complex systems, we understand mathematics teaching and learning to be distinctive features of activity that give rise to the classroom as landscape, thus, the classroom as a collective functions as a teaching and learning ecosystem in and of itself (Davis and Simmt 2003).

By comprehending the classroom as an ever-changing landscape, it is possible to identify other dynamic aspects within it. For example, the emergence and activity of mathematical ideas is one feature of the classroom that interests us greatly. Focusing on this particular aspect, we are curious what insights might arise when we move deeper and metaphorically conceptualize mathematical ideas and the dynamics of those ideas as live(d) topographies.

The term, *topography*, originates from the Greek word *topos* meaning "place" and *graphein* meaning "to write." Put together, topography literally translates as "the writing of place." Traditional topographical studies involve the identification of a region's surface features such as mountains and hills, slopes, bodies of water, and conditions of the terrain. Such studies serve to represent specific locations and provide environmental information for making decisions about land use (Christopherson 2002). However, in other fields of social sciences research, topographical studies examine human experiences in place. Here, the notion of place means more than material features and includes the complex and multiple ways that human knowledge and interactions shape the environment and vice versa (Walter 1988).

In this chapter, we take mathematical ideas to be a topographical feature of the classroom landscape. We consider ideas to be a critical aspect that gives rise to and thus, distinguishes the sociomaterial terrain (Katz 2001) of a classroom. Moreover, as an emergent and dynamic feature of the classroom, mathematical ideas contribute to creating the mathematics class as an "interrelated place full of its own diversity, relations, multiplicity, history, ancestry and character" (Jardine 2003, para. 3).

In the next section, we review the Pirie-Kieren model and theory. Following this, we demonstrate how we use the model to map the mathematical ideas of an algebra class. We then examine the dynamics of the ideas as they emerge and evolve as a live(d) ideational topography. In this manner, our work complements other concurrent studies within our research collective that inquire into the specific mathematics content and ideational network(s) of a classroom (e.g., McGarvey et al. 2015).

The Pirie-Kieren Model as a Map for Live(d) Ideational Topographies

To map a live(d) topography of mathematical ideas within the landscape of a classroom, we use the Pirie-Kieren model as our map. In 1989, Susan Pirie and Tom Kieren introduced the Pirie-Kieren model of a dynamical theory for the growth of mathematical understanding (see Fig. 1). The authors described mathematical

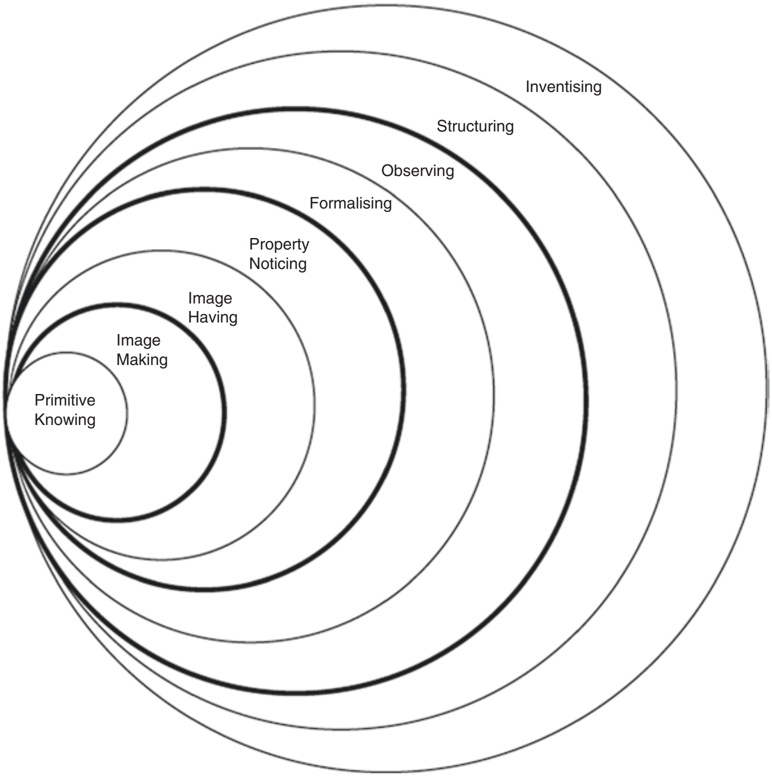

Fig. 1 The Pirie-Kieren model of a dynamical theory of the growth of mathematical understanding

understanding to be non-hierarchical and "leveled but non-linear. It is a recursive phenomenon and recursion is seen to occur when thinking moves between levels" (1989, p. 8). Similarly, we see these qualities of mathematical understanding as characteristic of the ever-changing nature of ideas in a mathematics classroom. Thom and Pirie (2006) also pointed out that sophistication of understanding is not simply having a command of formal conceptualizations but also demonstrating flexibility to move back and forth through informal and formal ways of making sense of the mathematics.

The Pirie-Kieren model consists of eight realms of understanding represented by nested but unbounded circles. Each realm characterises a particular quality of understanding, while at the same time, is fractal-like, self-similar to, and compatible with the other seven realms. Moving from the centre out, they are as follows: *Primitive Knowing, Image Making, Image Having, Property Noticing, Formalising, Observing, Structuring, and Inventising*.

Primitive Knowing is the source from which all new understanding grows and encompasses all knowledge—right, wrong or otherwise, that the learner or group brings to bear to some specific mathematics. Image Making is the action of generating distinctions about previous knowing in order to create new meaning. Importantly, "image" includes physical, verbal and mental forms of acting and expressing.

Image Having is putting an idea into action without needing to (re)create any of its conceptual meaning. Property Noticing involves making distinctions, seeing connections among images, and predicting why the qualities, effects or relationships within the given mathematics exist.

Formalising is the generation of a method or common quality from the distinguished properties that no longer depends on the specific images associated with it. Observing is the reflection on, and the coordination of, formal activity.

Structuring involves explaining or theorizing about formal observations in a logical manner. Finally, Inventising is when there is a breaking away from previous meaning(s), enabling new queries and the possibility for completely different concepts to emerge.

Two other features in the Pirie-Kieren model are *folding back* and *"don't need" boundaries*. Folding back occurs when an event at an outer level prompts the need to move inward and work more informally to 'thicken' understanding before proceeding on with the mathematics. Folding back is not a 'redoing' of what has already been; it is reintegrating and making further sense of earlier understanding. "Don't need" boundaries are indicated by the three thicker rings in the model in Fig. 1. Each of the rings is considered a "don't need" boundary because once understanding crosses one of these boundaries, it becomes matter of fact hence, any inner meanings that gave rise to the outer understanding are assumed to be known and are no longer necessary for mathematical activity to continue.

While much of the research that uses the Pirie-Kieren model focuses on assessing the understanding of the *individual*, we like other researchers, see potential in the model for exploring *collective* phenomena of a mathematics classroom (Davis and Simmt 2003; Kieren and Simmt 2002; Martin and Towers 2003, 2015; Pirie and Kieren 1994). We are interested in how the model might be used to observe and interpret the ideational activity within a mathematics classroom. By paying attention to the emergence and dynamics of ideas, we are keen to find out how new insight—theoretical and practical—might inform mathematics teaching and learning in the classroom.

An Illustrative Example of a Live(d) Ideational Topography

In this section, we analyze a 10-min video episode and corresponding transcripts from a secondary algebra lesson. The episode, Bike and Truck (NCTM 2014), involves a collective (whole class) conversation that focuses on two graphs that are on the same grid. The first graph, Graph 1, features a vehicle that moves at a constant rate over time for a fixed distance. The second graph, Graph 2, represents a vehicle that does not move at a constant rate over the same time and distance as the vehicle in Graph 1.

As we studied the video and transcripts of the lesson, we examined the gestures, utterances, explanations, and intonations of the class for the ideas made available to the collective. We then used the Pirie-Kieren model to map the ideas as well as their activity (see Figs. 2, 3, 4, 5 and 6). The resulting map, when observed in real time, renders an ever-evolving ideational topography. As such, the map makes visible the levels at which ideas emerged and the ways in which these are taken up or not taken up by the collective.

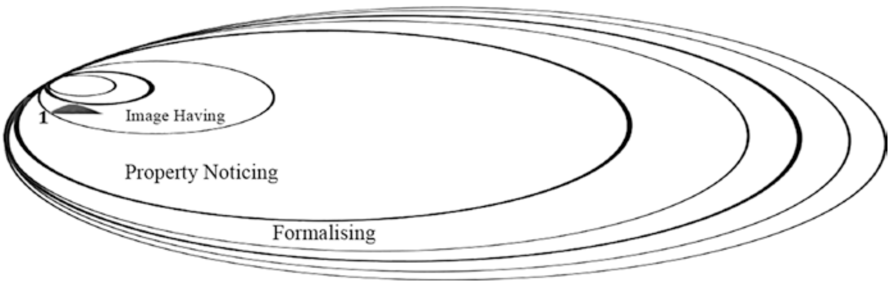

Fig. 2 Ideational topography of the mathematics classroom after 50 s of the episode

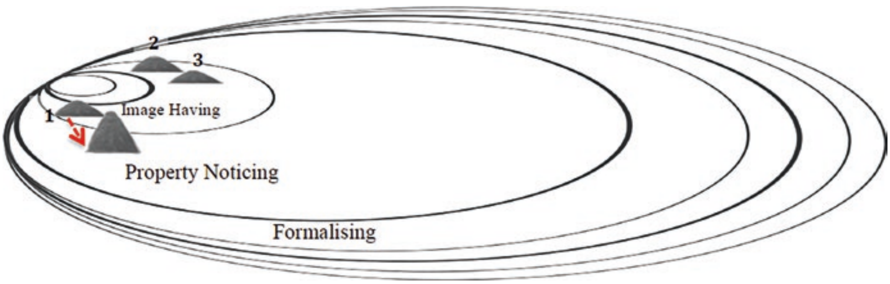

Fig. 3 Ideational topography of the mathematics classroom after 2.5 min of the episode

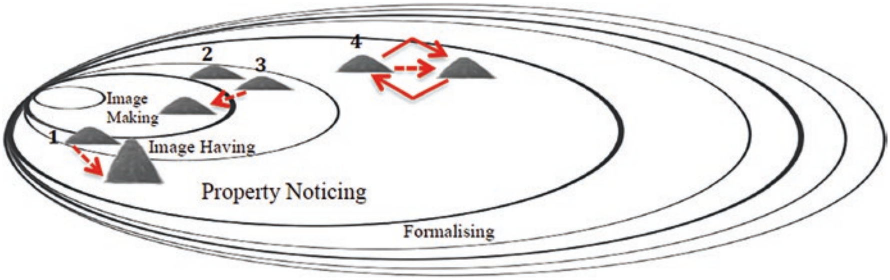

Fig. 4 Ideational topography of the mathematics classroom after 5.0 min of the episode

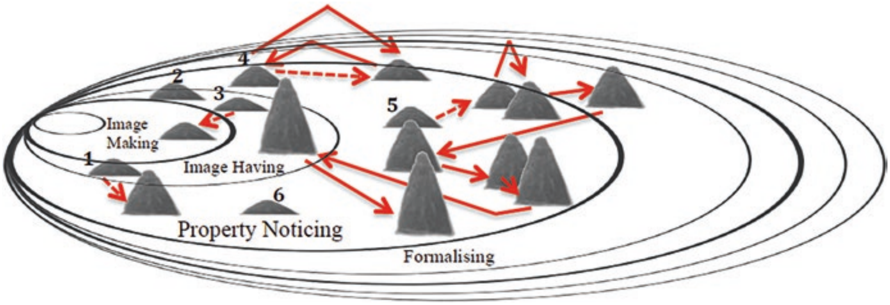

Fig. 5 Ideational topography of the mathematics classroom after 7.5 min of the episode

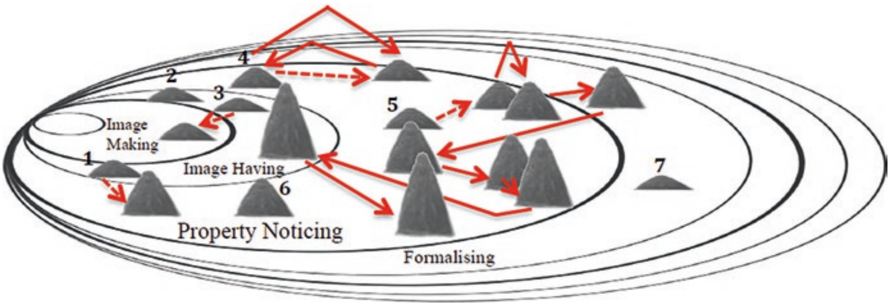

Fig. 6 Ideational topography of the mathematics classroom after 10 min of the episode

The first idea (see "1" in Fig. 2) emerges in the Image Having realm of the Pirie-Kieren model. The idea appears in the form of a conjecture about the movement of the vehicle represented in Graph 2. At the level of Image Having, there is no engagement in (re)making sense of the idea. Here, there is reference to the horizontal portion of Graph 2 and the statement that the vehicle is moving along in a straight path during this period of time.

As seen in Fig. 3, Idea 1 persists as the conjecture is discussed. When specific aspects about the distance, time, and rate of the vehicle movement in Graph 2 arise in the discussion, we observe Idea 1 move to Property Noticing. Two new ideas also appear during this time—one about constant rate (see "2" in Fig. 3) and the other about the vehicle stopping (see "3" in Fig. 3). Both emerge in Image Having early on in the discussion. Interestingly, neither idea is taken up any further during this period.

After 5 min, Idea 3 persists as it folds back to Image Making, relating the stopping of the vehicle to a flat surface. We interpret this event to signify the making of new meaning about Idea 3 in relation to Graph 2.

Further, during the 2.5 min of the episode, Idea 4 (see "4" in Fig. 4) emerges at Property Noticing. This new idea involves the comparison of the two graphs to determine which vehicle reached a particular distance first. We observe Idea 4 to continue to persist within Property Noticing. As the idea persists, terms such as domain and range are brought forth. These more formal terms move Idea 4 closer to but not across the "don't need" boundary that lies between Property Noticing and Formalising.

Two new ideas appear after 7.5 min of the episode (see "5" and "6" in Fig. 5). Idea 5 focuses on determining over what period of time the vehicle represented by Graph 2 drove fastest. This idea first arose at Property Noticing. However, as it becomes more analytic in form, Idea 5 like Idea 4 approaches the "don't need" boundary between Property Noticing and Formalising. Idea 5 then moves to Formalising when an explanation is offered for why a particular portion of Graph 2 was the steepest. Idea 5 continues but does not remain in Formalising; it folds back to Property Noticing and then further to Image Having as the class continues to focus on the reading of Graph 2. The idea then returns to Property Noticing when an explanation becomes necessary. Similar to Idea 5, Idea 6 emerges in Property Noticing. As an explanation, Idea 6 shifts attention from the vehicle represented in Graph 2 to the comparison of the vehicles represented in Graphs 1 and 2.

At the end of the 10-min episode, Idea 6 persists as the vehicles represented in Graphs 1 and 2 are compared. Additionally, we observe the suggestion of an equation as the seventh and final idea that appears as Formalising (see "7" in Fig. 6).

Reflections and New Questions

Papert (1980) argued that the richness of the environment, or in our words, the landscape of a mathematics classroom, included the powerful ideas of children. The work we have shared in this chapter attempts to make visible the ideational activity within the collective instructional practices, mathematical tasks, and teacher-student engagements of a classroom. Dynamic in real time, the resulting live(d) topography "simultaneously turns on, reveals, and specifies the [ideational] relations" (Katz 2001, pp. 720–721) that distinguish this secondary mathematics classroom as an ever-changing landscape.

As a result, the observations we made by paying attention to the activity of the ideas of this class occasion new questions for further investigation into mathematics classroom collectives. For example: Why is it that certain ideas seem to persist while other ideas appear and then get left behind? Why do some ideas arise and then become latent only to later be taken up again? What kinds of constraints enable or make difficult, the movement of an idea across different realms of the Pirie-Kieren model? In what ways do the "don't need" boundaries and folding back shape the dynamical activity of a class' ideas? And, how might the mapping of ideas inform researchers, teachers and students themselves about the richness or sophistication of a particular mathematical idea? Framed within the Pirie-Kieren model, we see the study and mapping of live(d) topographies as providing generative theoretical and metaphorical grounds for further inquiry into how the emergence and dynamics of ideas are shaping as well as being shaped by the local pedagogical terrain of the mathematics classroom.

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References

Bateson, G. (1972). *Steps to an ecology of mind*. New York: Ballantine.

Christopherson, R. W. (2002). *Geosystems: An introduction to physical geography* (4th ed.). Upper Saddle River: Prentice-Hall.

Davis, B., & Simmt, E. (2003). Understanding learning systems: Mathematics education and complexity science. *Journal for Research in Mathematics Education, 34*, 137–167.

Jardine, D. W. (2003). *On the nature of inquiry: Choosing a topic*. Retrieved from http://galileo.org/teachers/designing-learning/articles/choosing-a-topic/

Katz, C. (2001). Vagabond capitalism and the necessity of social reproduction. *Antipode, 33*(4), 709–728.

Kieren, T., & Simmt, E. (2002). Fractal filaments: A simile for observing collective mathematical understanding. In D. Mewborn, P. Sztajn, D. White, H. Wiegel, R. Bryant, & K. Nooney (Eds.), *Proceedings of the 24th annual meeting of the North American chapter of the International Group for the Psychology of Mathematics Education* (Vol. 2, pp. 865–874). Columbus: Eric Clearinghouse for Science, Mathematics, and Environmental Education.

Landscape. (2017). In *English Oxford living dictionaries*. Retrieved from https://en.oxforddictionaries.com/definition/landscape

Martin, L. C., & Towers, J. (2003). Collective mathematical understanding as an improvisational process. *International Group for the Psychology of Mathematics Education, 3*, 245–252.

Martin, L. C., & Towers, J. (2015). Growing mathematical understanding through collective image making, collective image having, and collective property noticing. *Educational Studies in Mathematics, 88*(1), 3–18.

McGarvey, L., Davis, B., Glanfield, F., Martin, L., Mgombelo, J., Proulx, J., Simmt, E., Thom, J. S., & Towers, J. (2015, November). *Collective learning: Conceptualizing the possibility in the mathematics classroom*. Discussion group at the 37th annual meeting of the North American chapter of the International Group for the Psychology of Mathematics Education, East Lansing, MI.

McGarvey, L., Glanfield, F., Mgombelo, J., & Thom, J. S. (2017, April). *Monitoring the vital signs of classroom life*. Research symposium at the 2017 National Council of Teachers of Mathematics research conference, San Antonio, TX.

National Council of Teachers of Mathematics (NCTM). (2014). *Principles to action professional learning toolkit: Bike and truck*. Retrieved from http://www.nctm.org/PtAToolkit/

Papert, S. (1980). *Mindstorms: Children, computers, and powerful ideas*. New York: Basic Books.
Pirie, S., & Kieren, T. (1989). A recursive theory of mathematical understanding. *For The Learning of Mathematics, 9*(3), 7–11.
Pirie, S., & Kieren, T. (1994). Growth in mathematical understanding: How can we characterise it and how can we represent it? *Educational Studies in Mathematics, 26*(2–3), 165–190.
Thom, J. S., & Pirie, S. E. (2006). Looking at the complexity of two young children's understanding of number. *The Journal of Mathematical Behavior, 25*(3), 185–195.
Walter, E. V. (1988). *Placeways: A theory of the human environment*. Chapel Hill: University of North Carolina Press.

Drawing Upon Indigenous Knowledges to Transform the Secondary Mathematics Classroom

Lisa Lunney Borden

Abstract In our current Canadian context, there is a considerable push to address the needs of Indigenous learners in mathematics, yet many attempts to do so by textbook publishers often result in trivializing of Indigenous knowledge. The framework shared in this article, demonstrates ways to draw upon Indigenous ways of knowing, being and doing in four key areas: (1) The need to learn from Mi'kmaw language, in particular the verb-based nature of Indigenous languages; (2) The importance of attending to value differences between Mi'kmaw concepts of mathematics and school-based mathematics, in particular ideas of 'enough', number as play, and the importance of space; (3) The importance of attending to ways of learning and knowing, approaches rooted in spatial reasoning; and (4) The significance of making ethnomathematical connections for students; in particular lessons learned from the Show Me Your Math program. Alongside the description of the framework, pedagogical suggestions are presented that demonstrate how these ideas might be enacted in practice. This work stands as an example of how educators and community members can work together relationally to decolonize mathematics by drawing upon community ways of knowing, being and doing.

Keywords Mathematics education · Indigenous education · Mi'kmaq · Verbification · Verbifying · Spatial reasoning · Decolonizing education · Ethnomathematics

I taught secondary mathematics for 10 years in We'koqma'q First Nation and during that time I learned a great deal from my students about how I needed to shift my pedagogical practices to match the ways of knowing that worked for them. I was not a perfect teacher, many things went not so well, but I remained open to questioning my practices and constantly working to do better; to be a better teacher for my

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students. Over the years I became a mathematics leader for the Mi'kmaw[1] schools in Nova Scotia that are part of Mi'kmaw Kina'matnewey[2] (MK). I shared what I was learning from my students and my own classroom experiences with other teachers within the MK organization and in public schools. These experiences informed my decision to take a leave from my teaching job to pursue doctoral studies.

When I first began my doctoral program, I thought it might be possible to develop lesson ideas and modules that would be living examples of how to transform mathematics education for Mi'kmaw students. I had visions of creating resources similar to those that had been developed amongst the Yup'ik for the Math in a Cultural Context program (University of Alaska Fairbanks n.d.). I quickly realized that this could not be the first step. It was apparent to me that before I could begin working with participants to develop Mi'kmaw mathematics materials, I needed to have a greater understanding of where the conflicts were arising for Mi'kmaw students.

The Context

For nearly 50 years, Indigenous communities in Canada have been arguing for local control of education and meaningful inclusion of Indigenous ways of knowing, being, and doing in the curriculum (Assembly of First Nations 2010). Emerging from these early calls for local control, numerous scholars have explored how mathematics and science might be taught in Indigenous contexts (Cajete 1999; Cole 1998; MacIvor 1995), and have focused attention on the tensions between Indigenous and non-Indigenous ways of knowing, being, and doing (Aikenhead 1996), the benefits of Indigenous languages in schools (Battiste 1987), and the emergence of teaching and learning from local contexts, practices, and land (Basso 1996; Garrison 1995). With this focus on the needs of Indigenous learners, many provincial/territorial ministries developed mandates to integrate Indigenous perspectives in K-12 curricula (e.g., Alberta Learning 2002; Ontario Ministry of Education 2007). With the release of the Truth and Reconciliation Report (2015), these mandates have become even more of a priority as ministries attempt to respond to the calls to action: particularly a need to develop culturally appropriate curricula. While many schools seem to have ideas of how to do this in literacy, social studies, and even science, how to best do this in mathematics still seems elusive for many.

In a recent synthesis of knowledge pertaining to Indigenous mathematics and science education in Canada, a clear focus on the importance of working relationally

[1] Mi'kmaw is used as an adjective, Mi'kmaq is used as a noun. The traditional territory of the Mi'kmaq, known as Mi'kma'ki, contains all of Nova Scotia, Prince Edward Island, parts of New Brunswick, Quebec in the Gaspé Region, and Maine. There are also many Mi'kmaw people living in Newfoundland and Labrador.

[2] Mi'kmaw Kina'matnewey is a collective of 12 Mi'kmaw communities in Nova Scotia who are part of a jurisdictional agreement for education with the federal government. MK communities boast an 88% graduation rate, nearly double the national average for Aboriginal children in Canada. For more on MK see http://kinu.ca/

with communities emerged as a significant theme in the work reviewed (Wiseman et al. 2017). The report emphasised the importance of honouring community knowledge as a starting point for learning and allowing mathematics and science to emerge in meaningful ways. However, as I flip through the pages of ministry approved textbooks for teaching high school mathematics, I see attempts to "infuse" Indigenous content that, sadly, are far removed from this vision. Buying a Métis flag, contrived word problems about making jingle dresses, or linear systems about building and selling Inukshuks (Davis et al. 2010) are merely an attempt to throw a coat of Indigenous paint on a pretty standard textbook question, and a rather poor attempt at that. These types of tasks show a lack of respect for Indigenous ways of knowing, being and doing and tend to further trivialize and marginalize Indigenous knowledges. My goal in this chapter is provide insight into how we can honour and learn from Indigenous knowledges to transform how we teach mathematics in the secondary classroom.

For this chapter I will draw upon a framework model for transforming mathematics education for Mi'kmaw students that I developed during my doctoral work (Lunney Borden 2010). While the model was developed while working with teachers and elders in two Mi'kmaw elementary schools, the themes that emerged have as much to say about secondary teaching as they do elementary teaching. I will draw upon my own experience as a secondary mathematics teacher to show what the implications of the model might look like in a secondary mathematics classroom.

The Framework

Using a process of *mawikinutimatimk* meaning coming together to learn together (e.g., Lunney Borden and Wagner 2013), I worked with teachers and elders in two MK community schools over a period of 1 year. We discussed the issues and complexities that arise in mathematics teaching and learning in MK schools. Through these conversations, four key areas for transformation emerged as themes: (1) the need to learn from Mi'kmaw language, (2) the importance of attending to value differences between Mi'kmaw concepts of mathematics and school-based mathematics, (3) the importance of attending to ways of learning and knowing, and (4) the significance of making ethnomathematical connections for students. At the heart of the model is meaningful personal connections for students (see Fig. 1). The various aspects of the framework are interconnected and likely the framework is incomplete; however, the ideas provide insights for beginning conversations with community members in any given context.

In my work, language seemed to be the dominant starting point and all other things could be linked back to language. Examining the Indigenous language of a given community context would provide a starting place for transforming mathematics teaching and learning. As one participant shared, even the students who come to school speaking English are not necessarily thinking in English ways,

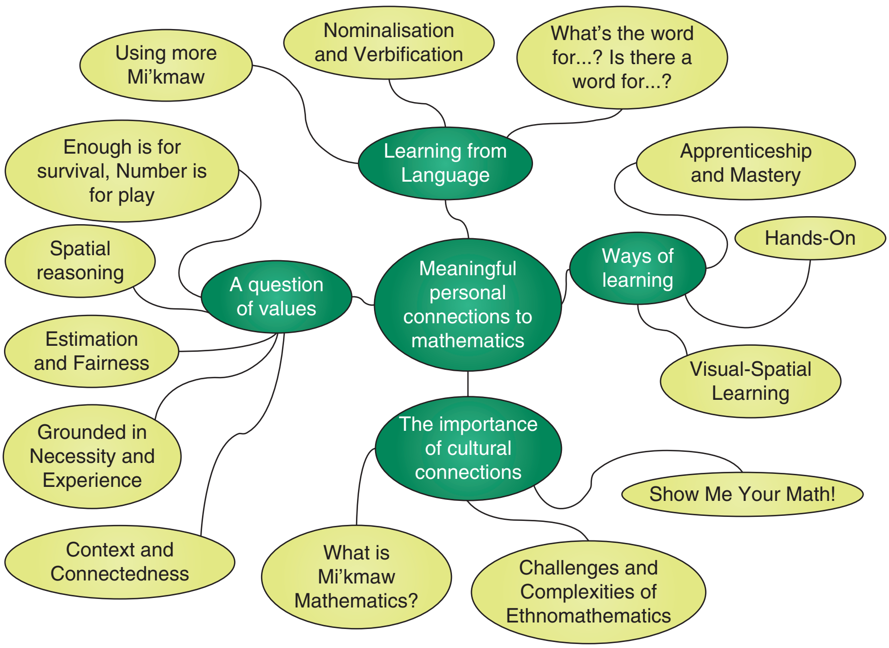


Fig. 1 From Lunney Borden 2010

rather they are using *L'nui'ta'simk* (Our peoples ways of thinking) which are embedded in Mi'kmaq language.

In working with local Indigenous communities, understanding how Indigenous languages are structured and used within the community can be highly beneficial to teachers of mathematics, recognizing of course that some communities do not have as much access to language or as many speakers as others.

In my own school, we often had conversations about language and enjoyed discussion about how to describe mathematics and science terms in Mi'kmaq. One colleague and elder, with deep language knowledge seemed to find great joy in these discussions and, as such, was always willing to entertain my inquiries. I learned a lot by asking questions such as "What is the word for…?" or "Is there a word for…?" to better understand how mathematical concepts are described in the language (e.g., Lunney Borden 2012). Gathering words that can be used to describe mathematical concepts, provides insight into concepts that may prove to be potential strengths for building a mathematics program.

Similarly, awareness of mathematical concepts that have no translation in the Indigenous language exposes the taken-for-granted assumptions that are often present in existing curricula. For example, the word flat is not commonly used in the Mi'kmaq language and no simple translation can be made, yet is frequently used in mathematics classrooms. In my doctoral work I describe a student who, when asked to say something about her cube explained that "It can sit still!" and firmly placed in on the floor rather than talking about it having a flat face. The action of sitting still is more consistent with Mi'kmaw language structures than the flatness of the face. An elder had said the same thing about the bottom of a basket to me—"It can sit still"—when I was asking her for a word for flat and suggested this as a possible context in which 'flat' might be used.

Perhaps the most important thing I learned from language was that the underlying grammar structures of the Mi'kmaq provided incredible insight into how my students were thinking about mathematical concepts. My research has shown that the prevalence of nominalisation (turning processes into nouns) in mathematics stands in direct contrast to the verb-based ways of thinking inherent in the Mi'kmaw language. Indigenous languages in Canada are verb-based and Mi'kmaq, like many Indigenous languages, contains a sense of action and motion that is not inherent in the static and fixed presentations of school mathematics.

How we speak the world is how we see the world. As Inglis (2004) has shared, "The Mi'kmaw language grammatically encodes details concerning how speakers experience the world and how a speaker and the person spoken to connect with and evidence this experience" (p. 400). Leroy Little Bear (2000) has explained this insight into worldview having argued that "language embodies the way a society thinks" (p. 78). It was this way of thinking that emerged in the stories told to me and the insights shared with me during my doctoral research. As I reflected on my own teaching experiences in light of this insight, I began to see how Mi'kmaw ways of knowing were evident in how my students talked:

> "Here, garbage this!" (Throw this in the garbage.)
> "Hey miss camera me!" (Take my picture.)
> "On the light!" (Turn the light on.)

Nouns became verbs in these sentences which is consistent with the verb based nature of Mi'kmaq. Henderson (2000) has described how Indigenous languages such as Mi'kmaq have a verb-rich structure that enables "an active relationship between the elements of a particular environment" (p.262). This sense of flux, action, motion was apparent in the way my students were also talking about mathematics and as a teacher, I began to draw upon a verb-rich approach to discussing concepts in class.

One example I like to share is the concept of slope. Slope is defined as the ratio of the change in the y-value to the change in the x-value in a linear relation, or rise over run. It reflects the steepness and direction of a linear graph. One year I had students singing "rise over run" to the tune of *Band on the Run*; though entertaining, it lacked effectiveness. At some point, perhaps out of desperation rather than design, I began asking my students questions like "How is the graph changing?" "How do I get from one point to the next point on the graph?" "Where does it start?" (Pointing to the y-axis as a place from which it goes forward and back.) Suddenly my students were having very different discussions about slopes of linear graphs. They could explain how the graph moved over and up or over and down. They began to discuss how the graph was changing and were able to explain where this "changing" was represented in the table of values and the equation. In short, using more verbs, focusing more on action and process rather than things became a more appropriate way to teach this concept—and, as it turned out, many concepts. This became a process I have described as the verbification of mathematics (Lunney Borden 2011).

Unfortunately, mathematics as taught in schools puts too much emphasis on nouns (Schleppegrell 2007). This prevalence of nominalisation (turning processes into nouns) in mathematics stands in direct contrast to the verb-based ways of thinking inherent in the Mi'kmaw language. Verbifying our discussions in mathematics class, focusing on processes and actions rather than things, creates opportunities to learn in a way that honours that action and flux of Indigenous ways of knowing. But what could verbification look like in the classroom? Let us consider some examples.

As I described above, teaching the concept of slope was a moment where I shifted my ways of talking about this concept and learned a great deal about what my students understood and could explain. The shift in questioning helped students to focus on how the graph was being formed, how it was made. This enabled students to talk about the graph with a sense of motion—"go over 2 and up 3 and over 2 and up 3" and so on. As students described how to make the graph, they were really describing the rate of change of the graph. Questions like, "what happens if we only go over 1?" enabled students to determine the slope of the graph without having to talk about the rise or run. The students were then able to connect this rate of change to how the table of values was changing and where this value appeared in the equation. Soon students were making connections across various representations of a linear function in a way that focused on how these patterns were being formed.

This notion of how graphs are formed or how they are changing can be employed in a variety of context involving patterns and functions. As I began to think about the process of verbifying my mathematics class, I reflected upon the ways in which I taught transformations of quadratic functions. Thinking now about how we ask students to describe these functions using vertical and horizontal translations, reflections, and stretches, it seems almost silly to have so many nouns to describe what is basically moving a graph around, flipping it upside down, and making it wider or skinnier. This is really about parabolas in motion, yet we nominalize this process and turn it into a series of transformations—things. Fortunately, dynamic graphing software opens up great possibilities for focusing more on motion. Students can be asked to explore what happens to the graph when various changes are made in the equation. They can examine graphs in a dynamic way that is consistent with the verbifying approach. This is not to say that they will never learn the language of transformations, but rather, they will explore concepts and once they figure it out, they can name it. Students can go from the informal exploration that is rooted in motion, to generalizations about how changing the equation changes the graph, to learning that these changes have names. When we figure it out, we name it, just like mathematicians do.

Focusing student attention on how things are made or how they are changing, brings their thoughts to process and actions. This works with functions and can also be used when exploring patterns. A typical question in mathematics class might give students a picture or description and ask them to describe the pattern rule, make a

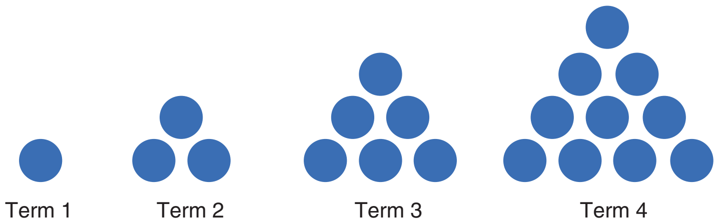


Fig. 2 Patterning example

table of values showing the input and output, identify the variables, and so on. Consider a pattern like the one given in Fig. 2 where a student might be asked to find the pattern rule and predict the number of circles in for the 10th, 50th or 100th term.

A more active approach we could take to this question is to invite students to build the pattern using concrete materials, and explain how they create the next one. This focus on how it can be built brings attention to the act of creating the pattern.

Barton (2008) has argued that

> A proper understanding of the link between language and mathematics may be the key to finally throwing off the shadow of imperialism and colonisation that continues to haunt education for indigenous groups in a modern world of international languages and global curricula. (p. 9)

Verbifying mathematics is an approach that draws upon the grammatical structures of Indigenous languages, and as such, upon the ways of knowing embedded in those languages. In my experience, this approach is not only incredibly helpful in supporting Indigenous students, it is actually a much better way to invite all students into a space of mathematical reasoning.

Values

It is also important for educators to think about how mathematical ideas are used and valued in the community context. It is important to understand how numerical and spatial reasoning emerge in the context of the community culture. I have often asked elders questions about how much or how many to determine ways in which quantity might be used within the community. These questions were almost always met with a response of *tepiaq* (enough) accompanied by a spatial gesture showing the shape of enough. These discussions with elders made evident that spatial reasoning was highly valued in the community as it pertained to matters of survival. While there are highly complex ways to count in the Mi'kmaw language, with numbers being conjugated to fit the various contexts, numerical reasoning was seen as useful in play as number does not tell you enough when it came to matters of survival. If we consider mathematics to be about examining quantity, space, and relationships (Barton 2008) then it becomes important to build a curriculum that values these concepts in a way that is consistent with, rather than in opposition to, the way these concepts are valued within the culture.

We have seen in various Show Me Your Math[3] projects over the years, how elders make use of spatial reasoning in various informal contexts such as baking or making baskets. Too often, school mathematics values numerical or quantitative reasoning over spatial reasoning and treats number as essential for young children in mathematics. This approach positions many children as incapable in mathematics even though they may have strong spatial skills. This closely links to the ways of knowing in the model which is described next.

Ways of Knowing

Language and values also influence the preferred ways of learning in any community context. It was evident in this community context that a mathematics program should provide children with opportunities to be involved in learning focused on apprenticeship with time for mastery, and hands-on engagement with concrete representations of mathematical ideas. Furthermore, building from a valuing of spatial reasoning, it was recommended that a mathematics program should place visual spatial learning approaches on equal footing with the already dominant linear-sequential approaches, providing more ways to learn so that more students can learn.

Spatial reasoning is an important part of mathematics teaching and learning at all levels. I often was mindful of finding ways to help my students to hold mathematics in their hands in my secondary mathematics classroom. This often meant finding ways to model mathematics with hands on materials such as building patterns with linking cubes, using real world experiments to collect data and create functions, or rolling cans to model trig functions. Spatial models helped my students to make sense of the mathematics they were learning. The area model in particular became an important representation for expanding and factoring and supported student learning in ways I had not expected. Many students were easily able to expand binomials or factor trinomials using algebra tiles with a focus on area (see Fig. 3).

As I progressed in my own career with the use of these models, my students gained more experience with these tools and were able to work flexibly with them. The students who had more experience with them were able to easily adapt the model to new contexts. One class in particular, when learning how to multiply two higher-order functions adapted the area model to support this operation and noted the like terms line up along the diagonal (see Fig. 4).

This same class also drew upon the area model to explore ways to rewrite quadratic functions in standard form into transformational form. Without any explicit teaching, I invited this class to consider how they might change a function such as $y = x^2 + 4x + 3$ into transformational form. I did not expect an answer, rather

[3] See http://showmeyourmath.ca/ for more on this program.

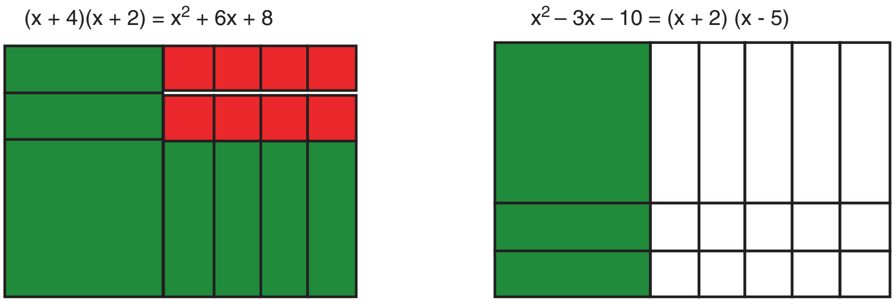
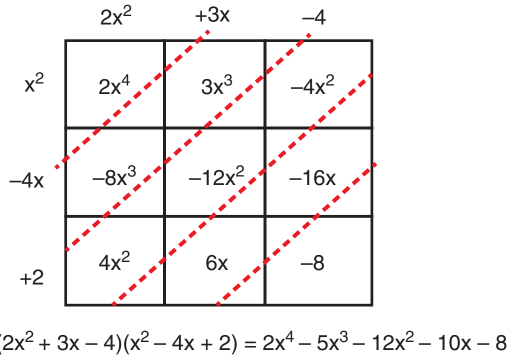


Fig. 3 Area model example

Fig. 4 Example of adapting area model for a new context



provided this as a question to think about over the weekend as we were to begin the unit on completing the square the next week. To my surprise, the students took only moments to do this. When I asked how they had gotten the answer they drew me an area model (see Fig. 5).

Creating opportunities for students to work with spatial models for various mathematical concepts, allows these students different ways to engage with ideas. I often hear secondary teachers critique the use of concrete models as something not relevant in high school, but I have seen the tremendous benefits of these models for my students and cannot emphasize enough how important it is to use these consistently across all grades.

Cultural Connections

In addition to community language, values, and ways of learning being included in a mathematics program, it is also essential to make meaningful and non-trivializing connections to the community cultural practices. This involves examining how the

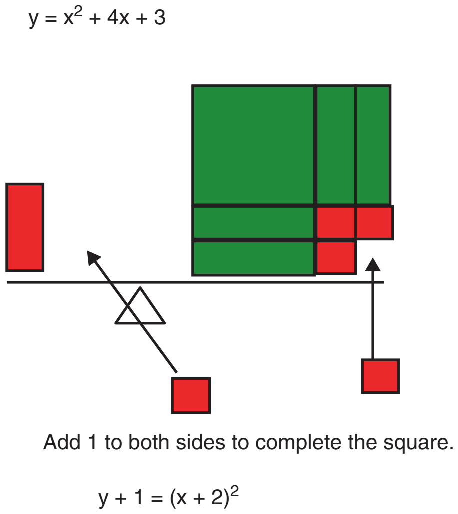


Fig. 5 Model for completing the square

school-based mathematics can be pulled in (Doolittle 2006) through identifying types of reasoning inherent in the community that can help students to make sense of the school-based mathematics. It also means creating learning experiences that help students see that mathematical reasoning is a part of their everyday lives, and has been for generations. The success of Show Me Your Math, a program that invites students to be mathematicians who investigate mathematics in their own community contexts, suggests that engaging students as researchers and authors of content is also an important component of a culturally based mathematics program (e.g., Lunney Borden and Wagner 2011; Wagner and Lunney Borden 2012).

We have seen secondary class projects that focused on making hand drums, playing the traditional game of waltes, making canoe paddles, moccasins, shawls, and mittens. We have seen students examine the thermodynamics of a sweat lodge and the structure of a wi'kwam. Each of these projects began with the cultural practice and allowed mathematics to emerge in ways that did not privilege Western knowledge over Indigenous knowledge but rather brought them together in a holistic way. Recently, I spoke with a young teacher who had herself done a Show Me Your Math project in high school and she explained to me that it was the first time that she saw that mathematics could be related to the things she liked to do and was good at. She is an excellent basket maker and was able to use her basket making skills to connect to mathematics. With Show Me your Math, Indigenous knowledges are allowed to take a central role in our mathematics classes without simply imposing western mathematics upon cultural artefacts.

Concluding Thoughts

Decolonizing mathematics for Indigenous students requires much more than a few superficial questions in textbooks. While learning from cultural practices in a more ethnomathematical sense like Show Me Your Math can provide a great space to honour Indigenous knowledges, real transformation requires that we move beyond such culturally based investigations to consider more deeply how ways of knowing, being, and doing can inform our pedagogical practices, choices of tasks, and ways of engaging in learning and teaching. The framework I have presented and the connections to practice are offered as examples of what is possible when you work relationally with community members and collectively come to understand what mathematical reasoning could mean in the context of the community. While some of these practices may be appropriate for other contexts, what is most important is that teachers and community members work together to determine the unique needs of their own context and work with promising practices that address those needs.

References

Aikenhead, G. S. (1996). Science education: Border crossing into the subculture of science. *Studies in Science Education, 27*(1), 1–52. https://doi.org/10.1080/03057269608560077.

Alberta Learning. (2002). *First Nations, Métis and Inuit education policy framework.* Edmonton: Government of Alberta. Retrieved from http://education.alberta.ca/media/164126/framework.pdf

Assembly of First Nations (AFN). (2010). *First Nations control of First Nations education*. Ottawa: Author. Retrieved from http://www.afn.ca/uploads/files/education/3._2010_july_afn_first_nations_control_of_first_nations_education_final_eng.pdf

Barton, B. (2008). *The language of mathematics: Telling mathematical tales*. New York: Springer.

Basso, K. H. (1996). *Wisdom sits in places: Landscape and language among the Western Apache*. Albuquerque: University of New Mexico Press.

Battiste, M. (1987). Mi'kmaq linguistic integrity: A case study of a Mi'kmawey school. In J. Barman, Y. Hébert, & D. McCaskill (Eds.), *Indian education in Canada: The challenge* (Vol. 2, pp. 102–125). Vancouver: UBC Press.

Cajete, G. A. (1999). *Igniting the sparkle: An Indigenous education science education model*. Skyland: Kivaki Press.

Cole, P. (1998). An academic take on 'Indigenous traditions and ecology. *Canadian Journal of Environmental Education Research, 3*(1), 100–155.

Davis, G., Ellis, D., Hope, J., Rolheiser, D., Sufrin, D., Van Bergeyk, C., Van Bergeyk, D., & Zimmer, D. (2010). *Foundations of pre-calculus mathematics 10*. Toronto: Pearson Canada.

Doolittle, E. (2006). Mathematics as medicine. In P. Liljedahl (Ed.), *Proceedings of the 2006 annual meeting of the Canadian Mathematics Education Study Group* (pp. 17–25). Burnaby: CMESG/GCEDM.

Garrison, E. (1995). Navajo scientists of the next century–Laanaa Hason. *Journal of Navajo Education, 12*(3), 3–7. Retrieved from http://mfnerc.org/wp-content/uploads/2012/11/4_Ezeife.pdf

Henderson, J. Y. (2000). Ayukpachi: Empowering aboriginal thought. In M. Battiste (Ed.), *Reclaiming Indigenous voice and vision* (pp. 248–278). Vancouver: UBC Press.

Inglis, S. (2004). 400 years of linguistic contact between the Mi'kmaq and the English and the interchange of two world views. *The Canadian Journal of Native Studies, 24*(2), 389–402.

Little Bear, L. (2000). Jagged worldviews colliding. In M. Battiste (Ed.), *Reclaiming Indigenous voice and vision* (pp. 77–85). Vancouver: UBC Press.

Lunney Borden, L. (2010). *Transforming mathematics education for Mi'kmaw students through mawikinutimatimk* (Unpublished doctoral dissertation). Fredericton: University of New Brunswick.

Lunney Borden, L. (2011). The "Verbification" of mathematics: Using the grammatical structures of Mi'kmaq to support student learning. *For the Learning of Mathematics, 31*(3), 8–13.

Lunney Borden, L. (2012). What's the word for…? Is there a word for…? How understanding Mi'kmaw language can help support Mi'kmaw learners in mathematics. *Mathematics Education Research Journal., 25*, 5. https://doi.org/10.1007/s13394-012-0042-7.

Lunney Borden, L., & Wagner, D. (2011). Qualities of respectful positioning and their connections to quality mathematics. In B. Atweh, M. Graven, W. Secada, & P. Valero (Eds.), *Mapping equity and quality in mathematics education* (pp. 379–391). New York: Springer.

Lunney Borden, L., & Wagner, D. (2013). Naming method: "This is it, maybe, but you should talk to…". In R. Jorgensen, P. Sullivan, & P. Grootenboer (Eds.), *Pedagogies to enhance learning for Indigenous students* (pp. 105–122). New York: Springer.

MacIvor, M. (1995). Redefining science education for Aboriginal students. In M. Battiste & J. Barman (Eds.), *First Nations education in Canada: The circle unfolds* (pp. 72–98). Vancouver: UBC Press.

Ontario Ministry of Education. (2007). *Ontario First Nation, Métis and Inuit education framework: Delivering quality education of Aboriginal students in Ontario's provincially funded schools*. Toronto: Government of Ontario. Retrieved from http://www.edu.gov.on.ca/eng/aboriginal/fnmiFramework.pdf

Schleppegrell, M. J. (2007). The linguistic challenges of mathematics teaching and learning: A research review. *Reading & Writing Quarterly, 23*(2), 139–159.

Truth and Reconciliation Commission (TRC). (2015). *Truth and Reconciliation Commission of Canada: Calls to action*. Winnipeg, MB: Author. Retrieved from http://www.trc.ca/websites/trcinstitution/File/2015/Findings/Calls_to_Action_English2.pdf

University of Alaska Fairbanks. (n.d.). *Math in a cultural context: Lessons learned from Yup'ik Eskimo Elders project*. Fairbanks, AK: University of Alaska Geography Program, School of Natural Resources and Natural Sciences. Retrieved from http://www.uaf.edu/mcc/

Wagner, D., & Lunney Borden, L. (2012). Aiming for equity in mathematics education research in a marginalized community: A perspective from Canada. In J. Choppin, B. Herbel-Eisenmann, D. Pimm, & D. Wagner (Eds.), *Equity in discourse for mathematics education: Theories, practices and policies* (pp. 69–87). New York: Springer.

Wiseman, D., Glanfield, F., & Lunney Borden, L. (2017). How we are coming to know: Ways in which Indigenous and non-indigenous ways of knowing, being, and doing might circulate together in mathematics and science teaching and learning. Report submitted to the Social Sciences and Humanities Research Council of Canada. Retrieved from http://showmeyourmath.ca/comingtoknow/report/

A Teacher's View – It's a Path, Not a Gap: A Relationship-Based Approach to Mathematics and Well-being

Tom Boland and David Tranter

Abstract It might appear, at first, that mathematics education and the development of well-being have little in common. However, a deeper look at the developmental nature of mathematics education reveals the inextricable connection between the two areas. An emphasis on strengthening relationships serves both mathematics education and the development of well-being. This chapter introduces the relationship-based approach to education, the eight conditions that support student success, and provides an example of how a school implemented this approach to improve mathematical proficiency, while also strengthening student well-being.

Keywords Relationship-based teaching · Well-being · Third path · Mathematics and Well-being · Mathematical achievement

A note about language: Throughout this chapter we have chosen to use "they" and "them", as opposed to she/he or her/him, as a singular, gender-neutral pronoun. This is becoming a more common practice and we see it an important to our overall theme of true relationship-based teaching.

In 1995, the World Health Organization adopted the *Statement on health promoting schools*, and called upon all countries to develop schools that implement "policies and practices that respect an individual's well-being and dignity, provide multiple opportunities for success, and acknowledge good efforts and intentions as well as personal achievements" (World Health Organization n.d., para. 5). This idea was then taken up in Canada by the Pan-Canadian Joint Consortium for School Health (2015), whose mission is "to work collaboratively across the education and health systems to support the learning, health and well-being of children and youth in school communities" (p. 4). Today, most provincial and territorial governments have made student health and well-being a central focus in education. In the

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classroom, more teachers are teaching conventional subjects, along with concepts such as self-regulation, growth mindset, and positive mental health.

The growing expectation that teachers should explicitly support student well-being may seem, at first, to have no obvious connection to mathematics. At first blush, the two subjects seem quite separate. Mathematics is a precise discipline often associated with stepwise processes in search of clarity. Well-being, on the other hand, is highly ambiguous. There are no formulas for happiness, no procedures for fulfillment. However, mathematics and well-being may have more in common than is first apparent. For example, consider the question, "What personal skills or attributes support proficiency in mathematics?" Then consider, "What skills or attributes strengthen well-being?" The answer to these two questions overlap. Qualities such as focus, persistence, creativity, self-awareness, impulse control, a willingness to take risks, and self-confidence are as important to success in mathematics, as they are for the development of well-being.

The aim of this chapter is to make the case that teaching mathematics and supporting student well-being are not independent activities. The two areas need not be taught separately. It will be argued that mathematics is learned more successfully when taught with student well-being in mind, and that well-being is strengthened when students experience success in mathematics. The connection between these two subjects is clearly not in regard to their content. In fact, a sole emphasis on content—the *what* of teaching—can perpetuate the belief that the two subjects are entirely separate. However, when the focus shifts to the *how* of teaching, the connection is revealed, as well as an approach that can serve to simultaneously improve the instruction of both mathematics and the development of well-being. Specifically, this approach—*the relationship-based approach*—emphasizes the central importance of strengthening relationships, with students, parents and caregivers, as well as colleagues and partners in education. This chapter will outline some basic tenets of the relationship-based approach and then provide a real-life example of how the approach was used by an entire school to improve mathematical proficiency, while also purposefully supporting well-being.

A Relationship-Based Approach to Mathematics

The relationship-based approach has been discussed at length elsewhere (Tranter et al. 2018). In brief, the approach contends that academic achievement and the strengthening of well-being should be done simultaneously, and that an understanding of, and emphasis on, the relational nature of teaching and learning will increase success in both areas. An important part of the relationship-based approach are the *eight conditions* that are required for students to succeed. These conditions describe in greater detail the hierarchical series of needs that create a school-based environment for all students to thrive. The conditions are:

1. *Safety*: Students need to feel emotionally safe in order to explore and learn. Some students express their emotional needs in ways that may not be direct or effective, but it is all they know. Educators should respond to student needs, rather than react to their behaviours.
2. *Regulation*: Students can become dysregulated from too much stress and too little support. Dysregulation can come in different forms and often leads to greater sensitivity to further stress. Students need regulating relationships and supportive environments.
3. *Belonging*: Belonging is a feeling that comes from all the moments of connection with others. Educators make demands on their students, which are usually experienced as disconnections. Therefore, educators should connect with their students as frequently as possible.
4. *Positivity*: Every student has unseen potential. Students are more likely to discover and express their potential when the educator/student relationship is truly positive and consistently supportive. Positive feelings lead to optimal functioning.
5. *Engagement*: All people tend to act to conserve their limited supply of mental effort. Learning is especially depleting. Organize academic tasks in order to maximize the use of mental effort, while building in multiple opportunities for students to replenish themselves.
6. *Identity*: Educators can hold unconscious biases that can advantage some students and disadvantage others. Educators should strive to remove unintended barriers to education by actively supporting the unique qualities that define the identity of each student.
7. *Mastery*: Students need regular experiences of success. Educators should strive to set tasks that are at the right level of challenge for each student. A feeling of accomplishment is essential to help motivate students to continue to learn.
8. *Meaning*: Student need to feel fulfilled by the work they do. Education is meaningful when students can see its purpose. When students understand why school is important, and experience the activities of school as relevant, they are motivated by the intrinsic value of learning.

It is worth noting that *engagement* in academic learning is the fifth condition. This suggests that students will struggle to learn unless they first feel emotionally safe, self-regulated, feel that they belong, and are part of an environment that is positive and hopeful. As well, within the context of mathematics, some other key ideas of the relationship-based approach are:

> *Put the relationship first*. Mathematics instruction should occur within strong and supportive teaching relationships. The emphasis should be first on developing and maintaining the relationship, and then on mathematics education. Students who struggle in mathematics will be more open about their struggle with a teacher they feel safe with and trust. They may be more apt to persevere when they genuinely feel that their teacher understands them and believes in them. As well, it is through a genuine teacher-student relationship that the learning needs of the student can be best identified.

Understand the student. This goes well beyond simply knowing what the student can do, and what they cannot. It also includes understanding the student's deeper conceptual awareness of math, as well as knowing their learning style, their strengths, and study habits. It should involve striving to know the student as a person, their temperament, their interests, their social circumstances, and personal struggles. All of these factors combine to influence how they learn.

Provide responsive and relevant instruction. Supporting the student to learn requires teaching that is responsive to their needs. As with other subject areas, it is important to teach the student, not the subject. The emphasis should be on what the student requires in order to move forward, to progress on their terms; not on rigid expectations that have been preselected or arbitrarily laid out in a curriculum. Most teachers would likely agree that teaching is more about facilitating learning than simply imparting information. As such, effective questioning and prompting, and the ability to differentiate instruction based on individual student needs becomes an important aspect of guiding students. Sometimes, one of the hardest things for a teacher to do is to simply be patient while a student perseveres through a period of appropriate struggle; to not jump in and give away a correct solution to a problem. To effectively teach means giving students problems that are meaningful to them, ones for which they do not have an immediate solution, and then facilitate learning while they figure it out. Then, through student-focused questioning, appropriate prompting, and support as they make generalizations, new concepts are fully acquired and procedures are effectively developed.

Student growth requires teacher growth. A thorough assessment should also include a self-assessment on the part of the teacher. They should ask themselves what mathematics knowledge they may need to acquire in order to better support the student. There is a significant difference between knowing mathematics, and knowing mathematics in a deeper and more meaningful way that supports teaching. Mathematics education requires an understanding such that teachers are able to help students make sense of different concepts and ideas. They need to truly appreciate the learning trajectories relevant to different mathematical topics so that they can move students ahead, or review previous material, in the interest of constructing new knowledge and understanding. Therefore, teachers need to embrace lifelong learning and commit to constantly improving.

Case Example

Tom Boland had the opportunity to be part of a school improvement team lead by principal Eric Frederickson at McKellar Park Central Public School[1] in Thunder Bay, Ontario, throughout the 2015/2016 and 2016/2017 school years. The school's student population is roughly 85% Indigenous, many of whom come from families that live below the low-income cut-off. Mathematics scores on standardized provincial tests were historically low. The school decided to adopt a relationship-based approach to increase mathematical proficiency, as well as student well-being. In planning the approach, mathematics education was first reframed as a developmental "journey", rather than a problem that required an immediate solution. In this way, teachers were freed from the pressure to push their students to make quick gains, and instead take more time to reflect on the needs of their students.

[1] Explicit permission was granted to use the school name and principal in discussion of the case example.

The initiative began by engaging all staff in professional learning related to teaching from a relationship-based approach. The training focused on the interconnection between relationships and learning, as well as recognizing and addressing student needs (both learning and emotional needs), along with key strategies such as thinking *relationally* about student behaviour, becoming *attuned* to students, and teaching that is *intentional* and *responsive* (Tranter et al. 2018).

Given the importance of caregiver and parent relationships, the next step was to increase the sense of connection between the school and the home. Many Indigenous parents and caregivers were directly impacted by the residential school system and, as such, often had an ambivalent relationship to education. With this in mind, the school held regular "family fun nights" where families were invited to participate in non-academic, recreational activities and meals.

Staff were provided with mathematics professional learning that focused on content knowledge as well as on teaching based on "where students were" in their learning, as opposed to teaching directly to the grade-based curriculum and offering "remedial math programs." The training required staff to become more familiar with researched-based, developmental continuums that highlighted mathematical learning trajectories, and intervention resources that targeted big mathematical concepts rather than grade-based skills and procedures.

The entire school community was also trained on the impact that abuse and neglect can have on brain development and how trauma can create challenges for students in academic settings. Teaching strategies were introduced that were considered to be "trauma-sensitive" and greater consideration was given to the impact that the physical environment of classrooms and common areas can have on some students. As a result, there was a significant effort to instill a sense of calm throughout the school. This included the use of warm colours on walls, the elimination of clutter, the softening of lighting, consistently using calm voices, and introducing calm sounds (e.g., quiet music) whenever possible. Staff were sensitized to the benefits of such environmental conditions, rather than simply being instructed to implement them. As a result, changes were made with clear awareness and a sense of purpose.

Students were also engaged in activities designed to support emotional self-regulation such as deep breathing exercises and mindfulness. These activities became a regular part of every child's school experience. Staff also made an explicit commitment to supporting the well-being of each student, as well as supporting one another. Every student in the school came to identify at least one (and in most cases, more) caring adults that they could count on. When behavioural interventions were required, staff worked to respond to the underlying needs of the behaviour, rather than simply focusing on punitive measures or compliance.

Students, caregivers, and parents soon commented on the improved school climate, often noting in particular that the school felt more inviting and friendlier. Staff expressed a renewed sense of hope and optimism. As relationships between teachers and students deepened, the need for behavioural interventions declined sharply. Problems were recognized earlier and addressed within the classroom, resulting in significantly less time away from learning for most students. Attendance and lateness improved significantly. Detentions were seldom necessary and suspensions, over a three-year period, went from 23 to 0.

Teachers used evidenced-based diagnostics to determine where each student was mathematically and used this as their starting point. Most classes had “guided math groups” (similar to guided reading groups), as a way to manage the various starting points. Student and teacher self-efficacy demonstrably improved. Mathematical interventions primarily based on strengthening conceptual understanding (e.g., intervention designed to facilitate a conceptual awareness of integer operations, as opposed to simply targeting how to multiply and divide integers) resulted in more frequent instances of student success.

As a result, the students of McKellar Elementary School demonstrated increasing improvement over time on the provincial standardized test for mathematics. Since implementing the relationship-based approach, the number of student who met the provincial mathematics standard went up by 89% in grade 3 and 21% in grade 6. These are extraordinary improvements, made more meaningful by the improvements made in virtually all aspects of student performance and behaviour, including that the students appeared to enjoy school more.

In Sum

Although Canadian students tend to fair well when compared internationally, some parts of Canada continue to struggle to improve mathematics achievement results. For example, in Ontario, only 50% of grade 6 students met the provincial standard in 2016/2017, down from 57% in 2013, this despite a 60-million-dollar provincial investment in a “renewed math strategy” (Education Quality and Accountability Office n.d.). It can be tempting to believe that students who are “behind” in mathematics need to get “caught up” simply through increased mathematics instruction. However, to do so would be to ignore the context that surrounds the student’s struggle. It can also be easy to differentiate subject areas based on their content, rather than considering the skills they have in common. Mathematics and well-being are distinct in respect to the *what* of education, but are closely related when it comes to *how* proficiency is best gained. Shifting the focus of mathematics from a largely academic pursuit, to one that is best learned through caring and responsive relationships, and by ensuring that all of the conditions for learning and development are in place, improves mathematics and student well-being at the same time. As counter-intuitive as it may initially seem to teachers, letting go of the narrow pursuit of mathematics, and instead understanding mathematics education within its relational context, can often lead to even greater gains, not just in math, but in overall growth and development.

References

Education Quality and Accountability Office. (n.d.). *Highlights of the provincial results: Assessments of reading, writing and mathematics, primary division (Grades 1–3) and junior division (Grades 4–6) English-Language students, 2016–2017*. Retrieved from http://www.eqao.com/en/assessments/results/communication-docs/provincial-report-highlights-elementary-2017.pdf

Pan-Canadian Joint Consortium for School Health. (2015). *Strategic plan of the Pan Canadian Joint Consortium for School Health 2015–2020*. Retrieved from http://www.jcsh-cces.ca/images/JCSH_strategic_Plan_2015-2020_Final.pdf

Tranter, D., Carson, L., & Boland, T. (2018). *The third path: A relationship-based approach to student well-being and achievement*. Toronto: Nelson.

What is a health promoting school?. (n.d.). *World Health Organization*. Retrieved from http://www.who.int/school_youth_health/gshi/hps/en/.

Additional Suggestions for Further Reading

Boaler, J., & Humphreys, C. (2005). *Connecting mathematical ideas: Middle school video cases to support teaching and learning*. Portsmouth: Heinemann.

Carnellor, Y. (2004). *Encouraging mathematical success for children with learning difficulties*. Victoria: Social Science Press.

Kajander, A., & Boland, T. (2014). *Mathematical models for teaching; reasoning without memorization*. Toronto: Canadian Scholars Press.

Small, M. Gr. 4-8 (2009). Gr K-3 (2010) and Gr 9–12 (2011). Big ideas from Dr. Small: Creating a comfort zone for teaching mathematics. Toronto: Nelson Education.

Third path website: http://www.thirdpath.ca/

Part I: Commentary – The Changing Landscape of Teaching and Learning Mathematics

Luckson M. Kaino

Part I of the book focuses on what is to happen or is happening inside and outside the mathematics classroom in Canada, in the effort to re-think innovative and better ways of teaching and learning mathematics. In the context of mathematics teaching, the exploration of various ways, and in particular Indigenous strategies and their integration into the school mathematics curriculum, should be considered significant towards the effectiveness of teaching and learning mathematics. For example, the artifacts available in the local environments are important tools to be used in teaching in order to bridge the gap between what is usually taught in the classroom and what exists outside the classroom, i.e. in society. Radford, Miranda and Lacroix indicate that knowledge is cultural and historical and emphasize that teaching and learning practices have to be considered from a cultural-historical approach. In this case, mathematics teachers should have knowledge to recognize, identify and use the materials around (in the environment) to promote the best ways of teaching. By using the cultural-historical approach, Radford et al. emphasize fostering deep student conceptual understanding through forms of collective learning, trying to move away from the constructivism approach which is student-centered and which the author believes tends to sideline the role of the teacher.

The forms of collective learning by Radford et al. are emphasized by Thom and Glanfield, who explain about ideational activities with instructional practices in teacher-student engagements in classroom instruction. Thom and Glanfield argue that creation of richness of the environment among teachers and students creates useful ideas to children. In this setup, ideas generated in the classroom bring powerful ideas among learners. As argued by Taylor, Lala, Ouellet, and Knebel, the ideas generated in class in the form of stories illustrate the richness of mathematics into the student's life. The latter as further argued, makes the connection between mathematics and the personal student life whereby the student is able to mathematize and

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think mathematically. Taylor et al. maintain that powerful stories open ways to reflect on student experiences in doing mathematics in real life.

Aikenhead provides a historical perspective on the rise of Platonist and Cultural schools of mathematics and explains the implications of a culture-based mathematics. In a culture-based mathematics, the teacher should not assume that learners are blank, i.e., they do not know anything about what the teacher wants to teach. Learners come from different cultural settings with some rich ideas about mathematical structures and these can be used by the teacher to teach effectively for better understanding of the material taught. Lunney Borden emphasizes the importance of good working relations with the community to develop a culturally appropriate mathematics curriculum. The emphasis is on the use of Indigenous language as a way to transform the teaching and learning of mathematics. Borden argues that language embodies the way the society thinks and the processes of doing things. Lunney Borden's argument which is in agreement with findings in literature, premises that if the artifacts available in the environment are used in the class activities they can develop knowledge of concepts that lead to the generation of mathematical rules and principles.

The above experiences by researchers gear to Boland and Tranter's brand of 'well-being' that is described by the authors to make the connection between the school and the students' parents. The parent-school relation approach of inviting parents to get involved in school mathematical social activities is intended to get parents acquainted with their students' school life. The purpose of such activities is to shape the behavior of students and improve their performance in mathematics. Such activities are conducted in conjunction with student guided group activities on their well-being.

The papers in Part I of this book are rich in enlightening various processes for learners to appreciate and develop interest not only to learn mathematics but also to improve performance in the subject. The aim here is to provide ways for long-term retention of mathematical knowledge. The articles in this part of the book would be useful to college teachers, curriculum developers, students, policy makers and interested persons in improving ways of teaching and learning mathematics.

Part II
Shifting to a Culture of Inclusion

Part II: Preface – The Particular, the Generic and the General Once More: Looking into Aspects of Canadian Mathematics Classrooms

David Pimm

While the section title focuses on inclusion, what caught my attention most when reading the chapters that constitute this section was the variety of foci and voices employed in them, not least in regard to making claims or suggestions about practices, about teachers and teaching, and about students and learning. And about various aspects of risk, of being at risk and living with a history of risk.

Apart from a historical or text-based approach, mathematics education explorations tend to fall into two broad (though not distinct) types: *studying what is* (usually while trying to avoid having the studying interact with or affect the thing studied) and *making something happen and studying that* (often with the intention or desire to alter what is). Both types are present here, often flagged as 'studies' and 'projects' respectively, and both are potentially interesting and useful. I could (but will not) give my sense of which chapters are which, and could even offer reasons as to why I see them that way. Instead, I feel it better simply to make a couple of observations across the contributing chapters and then withdraw.

There is a considerable amount of site-specific examination, though less attention was given to what extent the specificity (of focus, of location, of population, of …) mattered. Six of the chapters relate to work done in Ontario, while the Oesterle autobiographical chapter did not seem specifically rooted in British Columbia (where the author resides and teaches), and the Sterenberg and O'Connor, and Davis et al. ones engaged with contexts and settings taking place in Alberta.

'Particular,' 'generic' and 'general' are categories quite commonly discussed in mathematics and can crop up in interesting places in mathematics education too, I feel. And while they naturally come up in regard to the directly addressed material (whether specific connectives like 'than' in Barwell, Kubota-Zarivnij and Culotta, or specific students in the Kajander piece—though 'amalgam' was an interesting

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word she used that suggested the growing technique of creating composite individuals—an interesting oxymoron), I also noticed them explicitly in Davis et al.'s deliberate presentation of generic classroom question forms, as well as in McDougall and Ferguson's, Macaulay's, and Pyper's chapters which all attempt to offer direct suggestions and advice to presumed teacher-readers (something which, in passing, indirectly and gently goes against a sense of readership inclusion).

This set of distinctions also showed up in terms of variation of textual voice, with a strong 'I' voice (Oesterle's distinctive, introspective, autobiographical account), occasional (varying) 'you' voices (such as in the Sterenberg and O'Connor, Macaulay, and Pyper chapters, for example), as well as academically more familiar third-person and/or passive framing of text, as well as a 'we' one (McDougall and Ferguson) and a plural-voice one (I/we, in Childs and Holm). Each of these stylistic/voice decisions perhaps points towards a different intention with regard to the specific, generic and general in relation to address. (Though it is important to recall, as in mathematics itself with geometric diagrams – which are can be particular, generic and/or general – or determining the sign of a function across a whole interval between zeroes by examining the sign of its value at one point, that it is feasible, indeed, often productive to work on the general by means of working on the specific in the presence of others.)

The term (acronym) 'ELL' sounds like a commonality category, but, as the Barwell et al. chapter indicates, it is not, in that there is wide variation within it. My interest was caught by their account of the 'messiness' of language development and whether there was a presumed mutual independence of the triadic elements or trajectories provided. In addition, there was, for me, a far bigger question of the necessity of schooling for learning mathematics. The issue of student reticence and its potential sources added to the very human and moving particulars provided in Kajander's instances, as well as the wealth of prior experience and inexperience that, in different ways, is also addressed by Macaulay. Some links across chapters also included work on and in relation to mindsets, to habits of mind, and to Shulman's (1986) 'pedagogical content knowledge' and its ever-growing set of variants—a phenomenon which is echoed one effect of Skemp's promotion of Stieg Mellin-Olsen's distinction between 'instrumental' and 'relational' understanding that led to an explosion of attempts to differentiate types of understanding.[1]

But I am left thinking most about the Davis et al. chapter and its picking apart of certain generic classroom teacher structural/discursive moves in relation to questioning, while advising a moving back into the particular of individual student attention and understanding. One thing that was seen as StandardizEd was a sequence of questions addressed to an individual student by the teacher in a whole-class setting. Yet this too has a generic element of working on the whole class's understanding through working on an individual's understanding in a public setting, not unlike a

[1] It was brought to my attention some years ago by Stieg Mellin-Olsen, of Bergen University, that there are in current use two meanings of this word ['understanding']. These he distinguishes by calling them 'relational understanding' and 'instrumental understanding'. (Skemp 1976, p. 20).

master class, at times. Caleb Gattegno worked regularly this way. And this got me thinking hard about it once more.

References

Shulman, L. (1986). Those who understand: Knowledge growth in teaching. *Educational Researcher, 15*(2), 4–14.

Skemp, R. (1976). Relational understanding and instrumental understanding. *Mathematics Teaching, 77*, 20–26.

Steps Toward a More Inclusive Mathematics Pedagogy

Brent Davis, Jo Towers, Rohan Karpe, Michelle Drefs, Olive Chapman, and Sharon Friesen

Abstract This chapter is based on an action research project with the mathematics teaching staff of a school that serves special needs students from grades 2 to 12. Now in its fifth year, and with the aim of refining practices to ensure that all students have the opportunity to learn, the collaboration has adopted the analogy that "learning to teach differently is like learning another language." This framing has highlighted the complexities, difficulties, and strategies associated with shifting from highly familiar standardized, teacher-centered classroom practices to more authentic emphases that are consonant with reform efforts. In this chapter, we report on the power of this frame to effect transformations in teachers' beliefs and actions, while also highlighting how easily one's "native language" can seep into, undermine, and obscure efforts at transformation. By way of initial example, through action research, strategies have been implemented to ensure teachers are constantly aware of multiple learner interpretations, yet there is a persistent recurrence in classrooms of questions of the form "Can anyone tell me…?" We argue that such questions may be indicative of a mode of directive teaching that is aimed at a "representative learner," in contrast to a mode of responsive teaching that is attentive to the sense(s) that each learner might actually be making.

Keywords Learner interpretations · Sense-making · Teacher language · Inclusive practices

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Introduction

Imagine this happens in a classroom that you are invited to observe: it is mathematics time, and Robyn gathers the attention of learners who settle into their assigned seats. Robyn announces the topic of the lesson, signals a need “to remind ourselves about what we did yesterday,” directs all the children to a page from a resource containing a series of questions, and guides the learners through the questions. Each student is called on by name to contribute an answer, and each responds correctly.

With that brisk review of the previous lesson completed, Robyn announces the day’s topic as she approaches a whiteboard with a marker. She introduces a question and leads learners through a sequence of steps as she explains some of the principles at play. Once completed, she writes a second example on the board and invites students to emulate what they just watched. They attempt to do so.

Something interesting unfolds during this guided work. Noticing that several students are struggling with the same detail, Robyn calls for the group’s attention and mentions that she’s “noticed a common problem.” She walks to where the question is written on the board and places her hand to conceal a piece of it before asking, “Can anyone tell me…?”[1] A few hands go up, and Robyn points at one student who offers a correct response. Robyn repeats it as she reveals what was behind her hand. She then points at a different part of the question and asks, “Who knows…?” Several hands go up instantly and wait to be noticed. Robyn selects a learner to answer, but this time follows up with a request of the same learner to explain. As the student speaks, Robyn repeats, rephrases, and highlights aspects of the explanation. When finished, Robyn asks, “Does everyone see…?” She looks for and receives nods from a few of the students she had noticed struggling a few minutes earlier. Smiling, she directs students back to the task at hand. A few minutes later, that task is reviewed and another practice question posed. It is taken up in very much the same way. Students are then directed to their textbooks, and a set of questions for independent practice is assigned.

Our guess is that this scenario will feel familiar to most readers—even though we did not indicate grade level or topic. With much of school mathematics, it seems, such details are not always important. The teaching and learning of so many topics across so many levels can be fitted to a generic script.

In this chapter, we focus on two specific aspects of that familiar script: the interlocutory style and lesson format. On the former topic, we look specifically at the sorts of questions posed by the teacher. On the surface, they might appear intended to invite participation, to monitor comprehension, and ensure engagement. We suggest that they might actually have the opposite effect—that is, that they exclude rather than include.

We then pull the camera back to look at the familiar structure of mathematics lessons, and we offer a similar argument. As with the interlocutory style, these

[1] We have deliberately omitted the detail of the topic and problem here and in the subsequent teacher questions, for reasons that we explain in the next paragraph.

lessons might be intended to involve students—and, indeed, in the context described above there was clear evidence of behavioural compliance. However, it is not at all clear that the structure sponsors the sort of conceptual engagement for all students that we believe most mathematics teachers desire.

Context

We are currently in the fifth year of a seven-year design-based action research project with the mathematics teaching staff of a single school that spans both elementary and secondary grade levels. An explicit goal of this collaboration is to transform classroom practice in ways that support mathematical understanding, nurture healthy attitudes toward the discipline, and positively affect achievement. Our foci are simultaneously theoretical and practical, oriented by the conviction that practices are rooted in particular ways of looking at mathematics and learning—and, reciprocally, that every theory has practical entailments. Thus, for example, our collaborative group might wonder together about the enacted theories that support the practice of asking anonymous questions of the form "Can anyone tell me…?", "Who knows…?", and "Does everyone see…?"

On that point, while our account of Robyn's lesson is true to a lesson that she taught, the pedagogical trajectory is hardly unique. In fact, with only very minor variations, the account could be used to characterize many lessons we have observed with different teachers at different levels.

Our current focus on observing lesson structures and teachers' questioning strategies is actually a recent development in the project. Our initial strategy in this shared project of transforming mathematics teaching practices was to infuse cutting-edge research into the school through a course-based master's program taken by a self-selected group of 10 teachers. We hoped that infusion would occasion shifts in thinking, talking about, and enacting mathematics pedagogy. To a limited extent, it did have that effect—at least, for most of the teachers involved in the cohort. However, in spite of efforts to spread the knowledge more broadly within the school—through, for example, school-wide lesson studies led by cohort members—few mathematics lessons seemed to break from the mould depicted in our opening anecdote.

Some critical reflection on the part of all participants in the project prompted the realization that meaningful and sustained change in teaching practice would require much more than an infusion of ideas through a subset of teachers. To re-orient the work, we proposed a school-wide project of "changing the discourse," organized around an analogy between transforming one's teaching practices and learning a new language. This analogy has proven to have both interpretive and pragmatic value. On the side of interpretation, it has helped participants appreciate the complexity of changing practice. Just as speaking a new language is not a simple matter of replacing one set of words with another, shifting teaching practices involves entirely new webs of association around what it means to do, to learn, to know, and to teach mathematics. On the pragmatic side, learning to "speak the language of reform math fluently" has been greatly enabled by sharing the project across a community, dwelling in an immersive setting, and having access to "expert speakers" who can highlight inconsistencies and offer advice.

With this analogy in place, and with the expanded engagement of all mathematics teachers up to grade 8, the third and fourth years of the project have proved to be quite revealing.

Questions That Are Not Questions: Educating in Exclusionary Ways

Within the project, the two educational sensibilities under investigation—that is, analogically, the two *languages* under study—have been labeled "StandardizedEd" and "AuthenticEd." In terms more familiar within the mathematics education community, these sensibilities map onto, respectively, traditional teaching practices and reform-oriented classrooms.

We have opted for the terms *standardized* and *authentic* rather than the *traditional* and *reformed* pairing because, among project participants, there is a general agreement that some of the conceptual and theoretical commitments associated with the educational sensibilities are more readily apparent in this pair of words. For example, *standardized*, which educationists borrowed from industry, signals expectations of sameness—in outcomes, across curriculum experiences, and so on. In contrast, *authentic* tends to summon associations with personalized experiences that unfold in real situations. Thus, whereas the traditional/reform dyad often seems to focus attentions on teaching practices, within this community the standardized/authentic dyad seems to invite more nuanced, wider-ranging conversations about why teachers might be doing what they are doing.

It is beyond the scope of and space in this chapter to elaborate on the fine-grained differences between StandardizedEd and AuthenticEd. Many of the key differences have been examined at length elsewhere (Davis et al. 2015). However, for our current purposes, the following highlights are relevant. StandardizedEd is suggestive of a comfort and fluency with those approaches to schooling that focus on a one-size-fits-all program of study, age-based grade levels, uniform performance outcomes, and efficient instruction. It is underpinned by intertwined metaphoric assumptions that KNOWLEDGE IS A POSSESSABLE OBJECT, LEARNING IS THE ACQUISITION OF KNOWLEDGE, and TEACHING IS CONCERNED WITH DELIVERY OF AND/OR INSTRUCTION ON BUILDING CONTENT.[2] AuthenticEd, in contrast, draws on emerging understandings of human learning and personal development. It recognises that personal knowing is rooted in idiosyncratic personal histories and subject to a diversity of current influences. In turn, that means that the learning outcomes in any classroom

[2] We follow a convention in the cognitive science literature in the use of small caps (SMALL CAPS) to signify metaphors.

will be varied, and possibly even incompatible (i.e., authentic)—no matter how similar (i.e., standardized) the classroom experiences. Authentic education approaches thus attend both to each learner's unique history and to learners' necessarily unique interpretations. In terms of webs of association, in AuthenticEd, ideas tend to revolve around vocabularies fitted to KNOWLEDGE AS A COHERENT NETWORK OF ASSOCIATIONS, LEARNING AS CONSTRUING ASSOCIATIONS, and TEACHING AS OCCASIONING SUCH CONSTRUALS.

In each case, we understand an educational sensibility as a distinct and coherent network of associations that manifests itself in terms of a coherent vocabulary and a readily discerned set of practices. For example, in StandardizedEd, the following metaphors are among the many that are treated as synonymous to *teaching*: CLARIFYING CONTENT, CONVEYING INFORMATION, DELIVERING CONTENT, DIRECTING, INSTRUCTING, EXPLAINING, PRESENTING, and TELLING. Within AuthenticEd, teaching is more often characterized and enacted in terms of a very different cluster of metaphors: GUIDING, TRIGGERING ASSOCIATIONS, SIGNALLING CONNECTIONS, RELATING, OCCASIONING, and CHALLENGING. For this reason, reiterating our orienting analogy, occasioning a shift in educational sensibility from standardized to authentic is analogous to learning a different language.

With this backdrop in place, let us revisit Robyn's lesson. What language is likely being spoken when the actual topic of conversation (i.e., the concept being studied) does not seem to be relevant to the structure of the lesson? Or to the grade being taught? Or to the teacher and learners involved? Or, more specifically, in what language does it make sense to pose such questions as "Can anyone tell me…?", "Who knows…?", and "Does everyone see…?"

Gadamer (1990) categorized these sorts of queries as "pedagogical." Contrasted with "hermeneutic" questions (posed by someone who does not have but sincerely desires an answer) and "rhetorical" questions (posed by someone who neither knows nor expects an answer), the pedagogical question is not really a question at all—because the asker either already knows the answer or is not really interested in the answer. Such queries, he argued, only make sense in contexts such as traditional classrooms and game shows, where knowledge is treated as though frozen into object-like bits. In these contexts, the "asking" is really a thinly disguised "telling," a sort of ventriloquizing through which the questioner's articulations are made to emanate from the mouths of answerers.

The prevalence of such questions within StandardizedEd does not mean that real, hermeneutic questions are not asked. During Robyn's lesson, for example, many students posed genuine questions. But these questions came from the learners rather than the teacher, were corralled to the independent practice portion of the lesson, and seemed more concerned with getting things right than with understanding the concept at hand—that is, with ensuring that the output of the efforts was fitted to a standardized expectation.

Phrased in quite different terms, a genuine question entails a sharing of control. It is an invitation for other's participation. But there is no surrender of power with a pedagogical question. Even though the pedagogical question appears to be an invitation for someone else to speak, authority for whatever answer rests with the inquisitor. In asking "Who can tell me …?", Robyn is clearly in command. The lesson reflects a culture of management and control; objectives must be achieved in the allotted time as efficiently as possible. Deviations are unwelcome, and attending to the immediate and specific needs of each learner is a secondary concern that is pushed to the margins of the lesson. We have witnessed this quality of StandardizedEd again and again—in, for example, worries about covering the curriculum, prepping students for exams, coping with pressures from parents, and getting to the next activity in the lesson plan.

All of these constructs make perfect sense within the control-oriented, planning-obsessed, and learner-blind language of StandarizedEd. But none make sense in the language of AuthenticEd.

Ribboned Design, Rather Than Block Plan: Educating in Inclusionary Ways

One day, after observing a sequence of lessons that resembled Robyn's, one member of the research team mentioned to another that he had felt increasingly "on edge" as each lesson unfolded. The other observer confessed similar feelings. As they tried to unpack their responses, they came to quick agreement that the discomfort was due in large part to the fact that neither had been able to get reads on students' understandings. Moreover, that lack of information on student sense-making seemed to be *because of* (and not, as one might expect, *in spite of*) the questions that the teacher asked.

We took that insight to some of the teachers involved in the project, and there was ready agreement that it was an important observation. At the same time, we realized that, as a possible prompt to help people critique their own practices, it was likely inadequate—for the sorts of reasons identified a few paragraphs up. Concisely, for teachers who spoke only StandardizedEd, homing in on a single element of the work (such as the types of questions that were being asked) ran a risk of inviting excuses rather than sponsoring deep reflection. So, the group wondered, if the topic of "questioning" is too small, and the topic of "teaching" is too large, what might an appropriate chunk of classroom life be? It took some discussion, but eventually the suggestion of focusing on the format of a lesson arose.[3]

Consider the structure of Robyn's lesson: it comprised four distinct stages. First, students were primed for the day's topic with a focused review. Second, the teacher offered an example-based exposition of a new topic. Third, learners worked through other similar examples under the teacher's supervision. Finally, time was allotted for independent seatwork, during which Robyn made herself available to those

[3] Proponents of "lesson study" (e.g., Cerbin 2011; Lewis et al. 2009) have alighted on a similar insight about the granularity of focus for meaningful impact, but whereas the focus of lesson study (at least in its original form) is the iterative refinement of specific lesson content, in ours the focus is on examining the structures of teaching.

needing focused help. These four chunks will be familiar to almost anyone who has attended a modern school, which is to be expected. They have served as the core elements of a standardized lesson across many generations.

So how might one describe a lesson in AuthenticEd?

We will bypass the several months of our iterative discussion-observation-revision process, and leap straight to the current state of our answer to that question. Our first point of agreement, informed by research into cognitive load theory (van Merriënboer and Sweller 2005) and variation theory (Marton and Booth 1997), was that the image of the lesson needed to shift from *blocks* of activity to *ribbons* of activity, guided by the following principle:

- RIBBONING: Whenever encountering a potentially novel discernment that is necessary to a concept, probe student understandings.

That is, RIBBONING is a deliberate interruption of the notion of standardized, one-size-fits-all explanation. It arises in the realization that mathematical concepts and operations typically involve many features, and missing any one of them will derail a learner. In terms of teaching practice, RIBBONING entails a shift in questioning practice. It makes little sense in this approach to ask "Can anyone tell me…?" since the point of ribboning is to get frequent and direct feedback from each student on his or her understandings.

Our initial thought, some months ago, was that a focus on RIBBONING might be sufficient to shift practice substantially. It did for some but did not for many. In fact, several teachers readily subsumed it into their earlier practices. That is, they asked more questions, but those questions tended to be in StandardizedEd, of the variety that Robyn asked in our opening anecdote. Our second principle was introduced in response to this tendency:

- MONITORING: Ensure that every student is able and obligated to provide feedback that is audible/visible to the teacher for each ribboned query.

In terms of the theoretical and research basis of this principle, we link MONITORING to the literature on metacognition and self-directed learning. In brief, it is a teaching act that compels every learner to make their thinking explicit at every juncture of the lesson—driven by the coupled convictions that more powerful learning happens when the teacher has fine-grained insights into student understanding and learners themselves have frequent opportunity to make their thinking explicit. Additionally, in this frame, divergent interpretations can potentially become rich sites for exploration and elaboration. That is, the point of MONITORING is not simply to ensure everyone can answer each question; it is also an opportunity for the teacher to become more familiar with the interpretive breadth within the classroom community. Underlying these notions is the radical constructivist insight that learning cannot be determined by teaching, since it is not about ingesting a truth but about construing coherent sense out of one's unique set of experiences. On all these counts, this conception of monitoring makes much more sense in AuthenticEd than it does in StandardizedEd.

As with our preliminary thought about RIBBONING, when we collectively invented this mode of MONITORING, our assumption was that it would be the magic bullet. Surely a teacher who is asking frequent, genuine questions and then ensuring feedback from every learner could not help but teach authentically. And, once again, we were confronted by the distorting power of StandardizedEd. As we observed teachers become more adept at and more comfortable with RIBBONING–MONITORING, we noticed a new issue: in a manner reflective of a typical standardized lesson, most continued with the planned trajectory of their lessons, regardless of the sorts of answers they were getting to the questions they were asking. A third practical emphasis was thus introduced:

- ADAPTING: Revise/devise/improvise tasks, explanations, and other engagements to fit with demonstrated understandings.

As with MONITORING, ADAPTING is a notion that makes much more sense in AuthenticEd than in StandardizedEd. Anchored to the mastery learning and the teaching-as-improvisation (e.g., Shem-Tov 2011) literatures, the principle here is that it makes little sense to press on until an adequate understanding of the notion at hand is demonstrated. If students are unable to answer questions in one lesson ribbon, it does not make sense to introduce the next one.

Teachers, of course, did not dispute the sense of this suggestion. However, it was heavily criticized as largely impracticable. Issues with coverage, time constraints, and the inevitability of “the kid who does not get it no matter what you do” were raised. These concerns have not gone away, and we do not expect them to vanish any time soon. Even when one starts to become functional in a new language, it takes much, much longer to start to think in that language. In terms of classroom practice, all the teachers have demonstrated some level of success with the AuthenticEd RIBBONING–MONITORING–ADAPTING cycle, but their concerns, criticisms, and frustrations continue to be overwhelmingly expressed in the language of StandardizedEd.

In particular, one aspect of standardized teaching practices revealed itself as especially powerful and resilient at this stage. Several teachers noted that, in their efforts to structure lessons around fine ribbons rather than course blocks, very often they and their students were able to glide through topics with few glitches … but with no coherent understandings by the end of the lesson sequence. A grade 7 lesson on “order of operations” stood out as a case in point. Rather than simply imposing the BEDMAS rule,[4] as she had in previous years, this time Elaine examined the topic by systematically looking at what happened when different orders were followed, aiming at generating principles that might be used to settle on a final ordering rather than simply imposing what would otherwise look like a set of arbitrary rules. Students followed each micro-step of the analysis, but when it came time to consolidate them into a general guideline for computation, the “concept” appeared to exist as a set of disconnected fragments rather than a coherent idea. The percep-

[4] BEDMAS (Brackets, Exponents, Division & Multiplication, Addition & Subtraction) is one of many acronyms used to help learners remember the order of operations for basic arithmetic and algebraic manipulations.

tion was that this fragmentation was a consequence of the RIBBONING emphasis, and so the research group proposed a fourth point of emphasis:

- CONNECTING: Move between "part" and "whole" when ribboning to ensure that learners do not lose sight of the concept under study.

This emphasis is perhaps less a strategy and more the overarching imperative that weaves itself through the previous three emphases. The core of this emphasis is that teachers' capacities to communicate—that is, to make sense of student articulations and to frame things in manners accessible to students—are less about precise vocabulary and more about (1) the robustness and flexibility of their own mathematical understandings, anchored to commitments to attend to the usually tacit instantiations that enable those understandings (Davis and Renert 2014) and (2) the extent to which they appreciate (and act on the appreciation) that people make their own sense of things. The former enables conversations; the latter compels them.

Rephrased, like each element of the RIBBONING–MONITORING–ADAPTING–CONNECTING arc, this one is motivated by reform mathematics' commitment to the development of relational understanding (Skemp 1976)—that is, to developing a robust yet flexible network of coherences that can withstand or that can be adapted when subjected to tests of its viability.

Keeping It Real: Recognizing the Inevitable Hybridity of Teaching Practices

To re-iterate what we are up to in this research, we recognize that the project of transforming teaching practice is immensely difficult, in large part not because it is hard to identify ineffectual and indefensible approaches, but because of the invisible, entrenched, and broadly enacted webs of association that afford those approaches their coherence. Kelly (2010) makes the point more poetically:

> ideas never stand alone. They come woven in a web of auxiliary ideas, consequential notions, supporting concepts, foundational assumptions, side effects, and logical consequences and a cascade of subsequent possibilities. Ideas fly in flocks. To hold one idea in mind means to hold a cloud of them. (pp. 44–45)

And so, for example, it is a mistake to think that Robyn's teaching might be meaningfully reformed if she embraced a different mode of questioning—simply because that mode is not an isolated practice. It flies in a flock with other practices and associated beliefs. Removing or altering a member of this flock is unlikely to have much impact on that flock's character or trajectory.

That is not to say that we do not need to pay attention to specific practices. On the contrary, in our ongoing efforts to design an effective strategy to support teachers' professional growth, we are compelled to reconcile the profound limitations of human consciousness and the vastness of flocks of ideas that might need to be changed. One of the major findings of cognitive psychology over the last century is that the human mind can only juggle a few details at any given moment—and, pragmatically, this limitation means that a teacher who is working to make practice more effective really cannot handle much more than a single focus, such as asking more genuine questions.

It is for precisely these reasons that we have chosen our core analogy that changing teaching practice is like learning a new language and developed our four-emphasis strategy of RIBBONING–MONITORING–ADAPTING–CONNECTING. Following on a half-century of cognitive science research into analogical/associative reasoning, we are aware that figurative devices such as analogy, metaphor, metonymy, and image can "smuggle" much into one's thinking. These devices operate by triggering webs of association beneath the surface of conscious awareness. Similarly, each of the elements in our four-emphasis cycle is simultaneously a manageable piece of practical advice and a distillation of a great deal of theoretical and empirical work. Each is fitted to the constraints of consciousness, but designed in acknowledgement of the vastness of ideational systems.

All that said, we would be risking a lie if we were to close this chapter by intimating that we think we have figured things out and that all signs are pointed toward smooth sailing for mathematics teaching practice in this school. Such impressions reflect neither our goals nor our current realities. On the contrary, we are interested in emergent issues and aiming to offer teachers "just in time" challenges. To that end, it is fair to say that every participant in the project is grappling with one or another personalized challenge.

Across these labours, it is also fair to say that most members of our collaborative team find assurance in our core analogy, aware that this work is going to take time, it can only happen in community, and that it is difficult. On the last point, anyone who has experienced the demands and frustrations associated with learning a new language will have a sense of what participants are facing. To complicate matters, it is important to recognize that StandardizedEd continues to be the overwhelmingly dominant language of modern schooling. Its power and resilience are enabled by the fact that no other flock of educational associations could be better fitted to the modern culture of productivity and commerce. Hence, no matter how attentive one might be to the differences between standardized education's TEACHING-AS-INSTRUCTING and authentic education's TEACHING-AS-ENGAGING, it would be unreasonable to expect that teachers could ever avoid StandardizedEd. That's certainly what participants are experiencing in this project. Even the most impressive emerging moments of authentic teaching are actually hybrids in which authentic imperatives are injected into the routines of a predominantly standardized teaching regime.

However, our mention of that inevitable hybridity is not an acknowledgement of limitation. Rather, for us, it is the very focus of the work. A powerful side of productive tension arises where a commitment to authentic classroom practices meets an awareness of the pervasiveness of the artifacts of StandardizedEd—manifest in learning objectives, evaluation tools, classroom resources, lesson structures, school architectures, and so on. Teaching authentically does not entail ignoring these structures, but it does require a recognition that the core assumption of StandardizedEd is untenable for twenty-first-century classrooms. No matter how precisely articulated or rigidly regulated, learning can never be determined (even by StandardizedEd's structures), which makes it difficult to pre-specify what authentic mathematics teaching should look like. On this note, we were asked by a reviewer of our chapter to include a second vignette that would parallel the vignette with which we opened our chapter—one that, rather than describing a StandardizedEd lesson, would exemplify an AuthenticEd lesson and that might be instructive, for example to a novice teacher. We gave this request careful thought, but decided not to attempt to include such a vignette. When introducing a new flock of ideas, a singular vignette, even though intended as illustrative, has the tendency to coalesce attention on specific teaching strategies or teacher behaviours that are then taken to be prescriptive. Our intent here has been to set out the conditions that make authentic teaching possible, rather than to prescribe techniques for teaching. Authentic teaching involves attunement to and anticipation of emergent possibilities of learning moments for all learners. It calls for a readiness to harness the power of the collective learner as well as the individual, throughout the lesson, in order to surface a range of possible views on what is being learned through activity. To engage authentically means to be both inclusive and deliberate in pedagogy—inclusive, in that one intentionally prioritizes the voices that constitute the collective, and deliberate, in that one purposefully welcomes so-called deviations from a lesson timeline to engage in critical conversations with learners. Engaging authentically also calls for listening to students' responses to understand the kinds of coherences being construed, and creatively responding in ways that foster students' agency and interest in relation to a topic or subject matter.

References

Cerbin, W. (2011). *Lesson study: Using classroom inquiry to improve teaching and learning in higher education*. Sterling: Stylus Publishing.

Davis, B., & Renert, M. (2014). *The math teachers know: Profound understanding of emergent mathematics*. New York: Routledge.

Davis, B., Sumara, D., & Luce-Kapler, R. (2015). *Engaging minds: Cultures of education and practices of teaching* (3rd ed.). New York: Routledge.

Gadamer, H.-G. (1990). *Truth and method*. New York: Continuum.

Kelly, K. (2010). *What technology wants*. New York: Penguin.

Lewis, C. C., Perry, R. R., & Hurd, J. (2009). Improving mathematics instruction through lesson study: A theoretical model and North American case. *Journal of Mathematics Teacher Education, 12*(4), 285–304. https://doi.org/10.1007/s10857-009-9102-7.

Marton, F., & Booth, S. A. (1997). *Learning and awareness*. New York: Psychology Press.

Shem-Tov, N. (2011). Improvisational teaching as mode of knowing. *The Journal of Aesthetic Education, 45*(3), 103–113.

Skemp, R. (1976). Relational understanding and instrumental understanding. *Mathematics Teaching, 3*, 20–26.

van Merriënboer, J. J., & Sweller, J. (2005). Cognitive load theory and complex learning: Recent developments and future directions. *Educational Psychology Review, 17*(2), 147.

Learning Mathematics When Students Are New to Schooling and New to English

Richard Barwell, Kathryn Kubota-Zarivnij, and Debby Culotta

Abstract Teaching mathematics to students who are new to schooling and new to English, such as, in particular, refugee arrivals in Canada, is particularly challenging. In this chapter, we describe some aspects of our work with a group of teachers at a Catholic school in Toronto, in which we collaborated to find ways to support the many such students in the school. The chapter includes an overview of key ELL mathematics learning principles arising from research, mathematics classroom learning examples that illustrate these principles, and discussion of school-wide strategies. Examples include activities designed to support both mathematics and language learning in relation to the topics of multiplicative and proportional reasoning, and money.

Keywords Refugees · English language learners

Introduction

In the second half of the twentieth century, Canada accepted refugees fleeing from several conflicts, including people escaping from Hungary, Uganda, Chile, Vietnam and Kosovo. More recently, Iraqis and Syrians have come to Canada to escape the violent conflicts in their homelands and neighbouring countries. The arrival of children in such circumstances presents challenges for education systems in general and for teachers in particular. Despite the recurring nature of migration and the significance of

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migration to Canadian society, there is little research or guidance for teachers on how to develop effectively the mathematics learning of refugee and immigrant students.

How do students learn mathematics when they do not speak English and have very limited school experience? What would a whole-school approach for improving student learning and achievement in mathematics look like in a school with about 75% of the students requiring specialized English Language Learners (ELL) instruction?

In this chapter, we describe some aspects of our work with a group of teachers at a school in Toronto, in which we collaborated to find ways to support the many students in the school who were new to schooling and new to English. More specifically, in this chapter, we

- outline some key ELL mathematics learning principles arising from research;
- describe some mathematics classroom learning examples that illustrate ELL mathematics learning principles in the design of learning tasks, lessons and mathematics instructional strategies;
- examine a few school-wide strategies used to develop ELL learners' mathematical understanding and language production.

The observations, findings and suggestions about ELL students' learning of mathematics presented in this chapter were derived from the collaborative, mathematical work of classroom, special education and ELL teachers in a Toronto Catholic elementary school. Approximately 75% of the 700 students at the school are from the Middle East and speak Assyrian and/or Arabic. These students are new to Canada and do not speak English. For many of them, ranging from kindergarten to grade 8, this school is their first experience of formal education. In addition, about 20% of the students are from Nigeria and Ghana, and many of these students are also refugees with limited schooling. The remaining 5% of the students represent a smattering of various ethnicities and cultures. In Ontario, grades 7 and 8 are considered part of "Intermediate Senior" certification; that is, teaching mathematics in these grades is a component of secondary level mathematics teacher certification. While this chapter mostly focuses on ELL students in grades 7 and 8, many of the points made also apply to senior grades and our examples cover a wider range of grades.

In this school, a whole-school approach to mathematics professional learning involves every teacher, including ELL teachers, grouped by division (pre-primary, primary, junior and intermediate), who participate in monthly mathematics study group sessions, along with the district school board's mathematics program coordinator, mathematics consultant, mathematics coach and school principal. The professional learning group in the school has been studying "mathematics for teaching" (Thames and Ball 2010; Davis and Renert 2014) for several years to improve the precision and depth of their mathematics pedagogical content knowledge, with an emphasis on the analysis and use of mathematics learning trajectories (Clements and Sarama 2014; Sztajn et al. 2012) in order to co-construct effective instructional strategies that improve ELL students' learning of mathematics. As well, this group aimed to better understand ELL student readiness for learning mathematics (e.g., mathematics content, cultural background and experiences, learning skills) and to become familiar with and apply research findings about promising ELL mathematics learning and teaching practices. In 2015, the group aimed to study specifically the structure of mathematics language development in relation to student learning of mathematics concepts. Consequently, the members of the mathematics study group sought out a mathematics education researcher who specializes in ELL mathematics learners to work with them during their study group sessions.

The study group sessions focused on the teachers' questions and dilemmas, such as

- How can I improve my intermediate students' communication of their mathematics understanding more effectively?
- How do we better monitor ELL student learning, track progress and develop next steps in a timely manner?
- What does ELL students' mathematical thinking tell us? What are our next steps?
- How do we include ELL students with no prior mathematics knowledge (brand new to the country, never been to school before) throughout a mathematics lesson?
- What mathematics content do we teach all ELL students who have not been to school at all or for several years prior to the classroom?

In this chapter, then, we offer an account of some aspects of these teachers' mathematical work.

What Does Research Say About Supporting ELLs in Mathematics?

While there is little research on mathematics teaching and refugees, there is rather more work on ELLs, including recent immigrants and new arrivals (e.g., Barwell 2009; Barwell et al. 2016; Moschkovich 2010). Based on the discussions of this school-based mathematics study group, the following key ideas, synthesized from this literature, seemed most relevant to the participants' questions.

First, research shows that, in the right conditions, children can learn and be successful learning mathematics in a second or additional language (e.g., Barwell 2009, pp. 3–6 for a general discussion). Research in bilingual education has shown that it takes several years to develop academic language skills in a second or additional language, and that developing these academic language skills is important for academic performance (e.g., Cummins 2000). This finding has been extended to mathematics learning specifically. For example, Clarkson's (e.g., 2007) research with immigrants to Australia showed that students who developed academic language skills in either English, or their home language, or both, performed just as well as children who only spoke English, and some performed better. Meanwhile, long-standing research into mathematics taught in French immersion programs in Canada also suggests that learning mathematics in a second language does not necessarily lead to lower performance and can be linked with higher performance (Turnbull et al. 2001; Lapkin et al. 2003; Swain and Lapkin 2005). The important point to take from these research findings is that ELLs can, in principle, succeed: it may take time, and the right conditions need to be created, but success is possible.

Second, the language of mathematics, in any language, is complex. While vocabulary is perhaps the most salient feature of mathematical language, for students and teachers, in some ways it is also the least problematic. Consider the following example from an Ontario grade 9 provincial mathematics assessment (EQAO 2014):

The following is the formula for the area of a circle: $A = \pi r^2$. If the radius of a circle is 1.25 cm, which of the following is closest to its area?

- 15. 4 cm^2
- 7. 9 cm^2
- 4. 9 cm^2
- 3. 9 cm^2

Clearly to understand this question, students need to know what a formula is or what a circle is. There are, however, several other important language demands in mathematics apart from vocabulary (e.g., Moschkovich 2013). Logical connectives, such as *and*, *because*, *if*, and *or* are important in constructing mathematical arguments and can be challenging for ELLs (e.g., Dawe 1983), perhaps because they involve the co-ordination of syntax and logic. In the above test item, the word *if* is an important part of the question.

Mathematics also involves the use of some particular sentence genres, including definitions, conjectures, questions and explanations. The Ontario grade 8 mathematics curriculum, for example, suggests that students should be able to explain "why a square with an area of 20 cm^2 does not have a whole-number side length" (Ontario Ministry of Education 2005, p. 112). Constructing an explanation involves more than knowing why; it involves using language in a particular, mathematical way. Mathematical language demands also include the common use of *deixis*—words that 'point.' In the test item shown above, for example, the words "the following" are used twice, but refer to different things on each occasion. Mathematical language demands also include the interpretation of the organization of mathematical texts, which may include symbolic expressions, graphs, diagrams, and so on. In the above test question, students need to realize that the symbol *A* is conventionally used to indicate area, and hence the *A* in the formula, the word *area* in the question and the four possible solutions are all referring to the same thing.

Third, researchers in second language education have long argued that language production (i.e., speaking and writing) is just as important as hearing or reading for the learning process. Swain (2000), for example, argues that, among other things, language production prompts deeper, more focused processing of the target language. She also argues that meaningful dialogue is an effective way to expose students to speaking in a second or additional language. In mathematics education, there is supporting evidence that this principle applies to ELLs in mathematics classrooms. Both Khisty (1995) and Moschkovich (1999) have shown how ELLs need to participate actively in meaningful mathematical discussion in order to develop proficiency in the language of mathematics in English, as well as in mathematics. In Khisty's (1995) and Moschkovich's (1999) studies, meaningful dialogue was facilitated by mathematics teachers, through strategies that included using multiple ways to talk about a particular mathematical idea, revoicing students' ideas using more conventional mathematical language, and ensuring that the focus of discussion remains on the mathematical ideas, rather than on linguistic details. Indeed, there is a well-documented tension in second language mathematics classrooms between attention to mathematical content and attention to mathematical language (Barwell 2012), although both seem to be important.

Some researchers have proposed models for the development of mathematical language in a second language, in which students' mathematical language develops along three trajectories (e.g., Clarkson 2009; Setati and Adler 2001). These trajectories are from first language to second language, from everyday language to mathematical language, and from spoken language to written language. The proposed models assume that students can move along various combinations of these three trajectories, such as developing everyday ways to talk about mathematics in the first language, then everyday ways to talk about mathematics in English, then developing mathematical language in English, and finally developing written mathematical English. In reality, students' language development in mathematics is likely to be messier, with everyday and mathematical language developing in spoken and written forms, in English and possibly in students' home languages, simultaneously.

Finally, research has shown that bilingual students make use of many communicative resources to participate in mathematical meaning making (e.g., Barwell 2005; Moschkovich 2009; Planas 2014; Setati 2005). These resources include

- their own experiences of the world;
- the different languages they may know;
- their knowledge of mathematics and mathematical language;
- concrete materials, graphs, diagrams;
- deixis (e.g., this one, that one);
- other students' ideas and interpretations;
- multiple representations (e.g., models, symbols, gestures, writing);
- different types of talk (e.g., expository, exploratory).

This list underlines the wide variety of ways in which ELLs make sense of and make meaning in mathematics, and through which they can develop mathematical language. These resources offer teachers something to work with, even when working with students who are new to English.

Based on this brief review of research relevant for the questions raised by the teacher study group, the following principles for teaching mathematics with ELLs can be proposed. According to age and over time, teachers can

- ensure students have the opportunity to talk and write mathematical language;
- include and address mathematical language objectives alongside mathematics objectives as part of planning;
- combine language learning and mathematics learning in the same activity;
- work with colleagues who have expertise in language learning and teaching.

In working with ELLs, teachers can discuss students' mathematical thinking with them using the following strategies:

- listening to make sense;
- posing questions to provoke further thinking;
- revoicing using more mathematical language (written or spoken);
- drawing attention to important mathematical ideas;
- drawing attention to important mathematical language (visually, verbally, …).

Based on these research-based ideas, the teachers in the intermediate division mathematics study group conceived specific lesson design and teaching strategies to use in their classrooms. These strategies are discussed in the next section.

Examples from Teachers' Work

Each year there is an influx of ELL student enrollment at the school. Most of the ELL students are Middle Eastern refugees who have not been schooled at all or, at least, not for several years prior to school enrollment. The district school-board policy requires all students to be enrolled in age-appropriate grades, and consequently most of the ELL student refugees have significant difficulty engaging with the age-appropriate mathematics curriculum. Therefore, the teachers have set the standard that ELL students must develop specific foundational mathematics knowledge during their first two years at this school.

Guided by the grade 9 and 10 mathematics curriculum, the teachers (with suggestions from their local secondary school mathematics teachers) identified several mathematics concepts and strategies that every student needed to understand and practise. Using the grade-by-grade Ontario mathematics curriculum (Ontario Ministry of Education 2005) as a curriculum trajectory, specific mathematics concepts in each grade were identified as learning goals that every student must attain (i.e., number and operations, equivalence, multiplicative and proportional relationships). Based on their observations of the ELL students' mathematical learning readiness and needs, the teachers emphasized student learning of (1) mental mathematics strategies for addition, subtraction and multiplication; (2) equivalence of numeric expressions (addition, subtraction and/or multiplication); and/or (3) proportional reasoning across the different strands of mathematics. Such learning goals prompted the teachers to aim towards collectively identifying and organizing mathematics content relative to those three identified areas, so that from kindergarten to grade 8 mathematical coherence would be constructed in relation to precise mathematics scaffolding as a differentiated learning practice.

The teacher study groups began their mathematics professional learning by examining the mathematical details of different definitions of proportional reasoning in order to discern key concepts, models and mathematical contexts. An example of their notes are as follows:

- Small (2013) stated that "proportional reasoning involves the deliberate use of multiplicative relationships to compare quantities and to predict the value of one quantity based on the values of another."
- According to Lesh, Post and Behr (1988), "proportional reasoning is a form of mathematical reasoning that involves a sense of co-variation and of multiple comparisons, and the ability to mentally store and process several pieces of information. Proportional reasoning is very much concerned with inference and prediction and involves both qualitative and quantitative methods of thought" (p. 93).
- "The essence of proportional reasoning is the consideration of number in relative terms, rather than absolute terms. … Proportional reasoning involves thinking about relationships and making comparisons of quantities or values" (Ontario Ministry of Education 2012, p. 3).

From these proportional reasoning definitions, the teachers selected several key concepts and terms that would inform the focus of the language and mathematics learning goals, such as: reasoning, multiplicative relationships, compare, quantities, predict, value. For example, mathematics learning goals included the multiplicative relationships of quantity, such as double (2 times), triple, halve, and twice, within measurement contexts (linear, capacity, mass, money, time) by using objects for direct comparison (e.g., half as long, double the hour, a dime [10 cents] is double the value of a nickel [5 cents] or $\$0.05 \times 2 = \$0.10$). These goals included multiple representations of measurements, using concrete materials, labelled diagrams, models, several symbols and English words.

The language learning goals were relative to the mathematics proportional reasoning learning goals. For example, language learning goals focused on differentiating the language structure for terms like *than* (comparative) and *then* (sequential), as well as the use of patterned phrases and sentences that could organize students' communication of their mathematical reasoning systematically and for student accessibility. Language learning goals also emphasized students' co-construction of patterned phrases and sentences, which distinguished student explanations, questions, hypotheses and summary statements (pseudo definitions). As well, mathematics vocabulary was used for the purpose of concisely distinguishing mathematical concepts from one another.

Proportional Reasoning Using Spatial Reasoning and Linear Measurement

Using the research findings from the Math for Young Children (M4YC) project (Moss et al. 2015), the kindergarten teachers in the group emphasized the significance of developing students' spatial reasoning skills at every opportunity.

Fig. 1 Table image

The teachers hypothesized that ELL students' development of mathematical concepts and mathematical language would be accessible within a concrete measurement context. What problems about this table (see Fig. 1) could be posed to develop ELL students' mathematical thinking and language?

Using the non-standard, uniform lengths of the small and large rod, the teachers brainstormed problems like

- How many small rods would it take to measure the length of 1 side of the table?
- How many small rods are needed to be the same length as 1 large rod?
- How many large rods would it take to measure the lengths of 4 sides of the table?
- How many small rods would it take to measure the lengths of 4 sides of the table? How do you know?
- What is the relationship between the small rod and the large rods?

The teachers anticipated several key details to develop the "language of mathematics," in terms of

- mathematics concepts—length using non-standard uniform units, multiplicative comparison (e.g., half, double of a length, 2 times as much as), rectangle, side length of a rectangle (e.g., 2 pairs of same size lengths), perimeter of a rectangle (e.g., 2 pairs of opposite side lengths are the same, so that the perimeter of a rectangle is the sum of 2 pairs of opposite side lengths);

- mathematics vocabulary—shorter, longer;
- representational model—length of bar;
- relational patterned statement—"If … then …", greater than, less than.

When the teachers realized that a learning trajectory linear measure could be described in terms of transitions from qualitative, to additive, to multiplicative reasoning, they refined their questions to be specific and intentional about the kinds of mathematical thinking the students could use to build from previous learning discussions:

- Qualitative (Attribute)—"Tell me something about the large rod and the small rod" (e.g., The small rod is shorter than the large rod. The large rod is longer than the small rod;
- Additive—"How many small rods makes the same length as one large rod?" (e.g., two short rods is the same length as one large rod);
- Multiplicative—"Compare the small and large rods. If you have two large rods, then how many short rods do you need?" (e.g., the small rod is half as long as the large rod, the large rod is double the length of the small rod).

In this case, the problems the teachers created and revised reflected their introductory use of mathematics vocabulary (e.g., shorter, longer) for the purpose of prompting the ELL students to describe comparative length measurement attributes, using the logical connective "than." The lengths of different objects in the classroom were directly compared with finger gestures showing difference in lengths between two objects, in order to consolidate the meaning of shorter and longer. The students in grades 3 to 8 understood that both objects needed to be aligned on one end and then the length at the other end showed the difference in the length. The structure of comparative sentences, like "The small rod is shorter than the larger rod" and "The larger rod is longer than the small rod" were used with gestures as the students were comparing the lengths of different objects in the classroom. These observations focus on the use of logical connectives (than) to form their observation and explanations using their direct linear measures.

The teachers' subsequent questions (e.g., "How many small rods would it take to measure the length of one side of the table?"), with gestures inferring the comparison of lengths, prompted the students to take a more active response and directly compare the lengths of the rods. They cut out paper models directly from the diagram of the small and large rod and physically compared the lengths by folding the larger rod and superimposing the small rod over it. The ELL students happily reported through gesturing with their fingers the quantities 2 and 1, which was complemented with other students' words, such as that it took "2 small rods to make 1 large rod" and that "the small rod is half of the large rod." These explanations include the use of logical connectives (and, because, then) to communicate the sequence of actions they took for their direct linear measurement.

Thus, the introductory proportional reasoning concepts, double and half within a concrete measurement context, provided the ELL students with greater awareness of the relationship between double and half, as well as differentiating between the explanatory statements that use "than" for comparison and "then" for sequencing of mathematical observations and explanations. The students' use of these patterned statements is leveraged and further developed in the next example.

Proportional Reasoning Using Canadian Coins

Because the ELL students are also new to Canadian culture, the teachers anticipated that student learning of proportional reasoning would be more meaningful and better understood if it was situated within the functional aspects of the students' daily lives. Learning about Canadian currency was of great interest to the students. The following example was developed for use with ELL students in grades 3 to grade 8. The Ontario mathematics curriculum requires students to work with money up to total amounts of $5.00 using all the different coins. In the school, the students exchange coins, like toonies (two dollars, $2 or $2.00), loonies (one dollar, $1 or $1.00) and quarters (25 cents, 25¢ or $0.25) in order to make purchases during $2 pizza luncheon, popcorn snack sales during school dances and school baked goods sales.

During a mathematics study group session, the teachers co-constructed a bansho (board-writing) plan, which is reproduced below. It is a record of the teachers' preparation for and anticipation of the students' co-constructed mathematics and language learning during a lesson, which is focused on determining money amounts in relation to the multiplicative relationship between coin values. Also, enfolded in the lesson is the relationship between arrays of objects (coins), repeated addition of objects and the equivalent multiplication expression. The mathematical annotations on the bansho (board writing) plan are organized to make explicit students' co-constructed ideas within the structure of patterned number equations, which can be later leveraged when grade 7 and 8 students develop relationships between the structure of number equations and algebraic equations for linear relations.

Look at Fig. 2. (Note that at the time of this study, one cent coins were still in use. The coins have since been discontinued, but cents are still used in calculations, so one cent units can still be represented with other unit chips.) What do you notice about the teachers' thinking about the "language of mathematics?"

This bansho (board-writing) plan (see Fig. 2) represents the anticipated student responses to the lesson problem, "How many nickels would it take to have 75¢?" as well as the key mathematical details that are co-constructed by the students as a result of class discussion (i.e., explanation, analysis, questions, comments), about the student solutions.

Japanese bansho has been "interpreted and adapted so that it complements the Ontario curriculum's emphasis on teaching and learning mathematics through problem solving and supports the current exploration of collaborative approaches to knowledge building in the classroom" (OME 2011, p. 2). Bansho (board writing) is

- a mathematics instructional strategy that makes explicit students' mathematical thinking and provokes students' collective knowledge production through strategically coordinated discussion, organization and mathematical annotation of students' solutions to a lesson problem

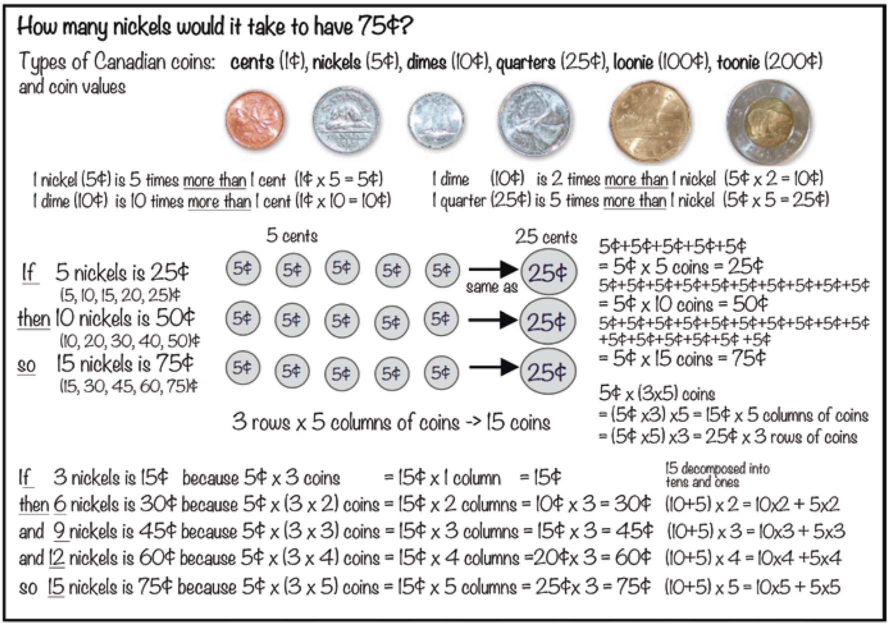

Fig. 2 Coins bansho (board-writing) example

- an assessment for (and as) learning strategy that enables the teacher and students to discern the range and relationships between mathematical ideas, strategies and models of representation
- a classroom artifact that is constructed collectively by the teacher and students in order to display the mathematical relationships derived from students' solutions; it can be organized and used as a mathematics learning landscape or as a mathematics anchor chart. (OME 2011, p. 2)

The teacher's bansho (board-writing) plan focused on integrating several number concepts and relationships, using the measurement context of money, such as

- coin images in relation to coin names
- coin names in relation to coin values
- coin values in relation to other coin values
- skip counting to repeated addition to determine a sum
- grouping to multiplication arrays to determine a product
- the functional use of the associative property for multiplication and the distributive property for multiplication.

The ELL students were aware that the Canadian penny coin (1¢) was no longer in circulation as of February 4, 2013. The teachers believed it was necessary to include it as a unit coin from which multiplicative relationships among the coins nickel (5¢), dime (10¢), quarter (25¢), loonie (100¢ or $1) and toonie (200¢ or $2) could be explored. ELL students were very interested in understanding the multiplicative relationships between cents and dollars, as they could apply that knowledge directly into their everyday school experiences (e.g., purchasing pizza lunch, paying for school trips) and shopping at the local grocery store and shopping malls.

The teachers emphasized the coordination of students' mathematical talk for learning mathematics and for learning the language of mathematics. Their lesson plan showed that they believed that the "language of mathematics" involves more than mathematics vocabulary (e.g., naming coins, such as nickels and quarters). It includes the use of logical connectives (e.g., and, because, if, then, so), which provides a structure for making explicit multiplicative relationships between the number of coins and coin values using patterned mathematical statements and equations. The number of coins were modelled in groups and arrays, which required the students to interpret the text and graphics of the types and number of coins, coin values and money amounts. Different sentence structures were used repeatedly to show logical mathematical arguments using patterned statements, to represent equivalency between repeated addition and multiplication expressions, as well as the introduction to the grades 7 and 8 concept of a linear relation or constant rate of change (e.g., as the number of columns increases by 1, then the number of nickels increases by 3 or the amount of money increases by 15¢).

Thus, in the design of this work on proportionality, the teachers found ways to integrate:

- mathematics learning goals relating to money, number and multiplicative relations;
- mathematical process goals relating to organising and explaining thinking;
- and language learning goals, including vocabulary and sentence structures for explanations.

The regular use of patterned 'if – then – because' sentences, for example, should support students simultaneously to develop mathematical reasoning and the language of mathematical reasoning.

Discussion

The context for mathematics teaching at this Toronto school is extremely challenging. Teaching mathematics to heterogeneous groups of students, including students who have attended the school from kindergarten alongside refugees who have only recently arrived in Canada, are beginning learners of English and have little or no prior schooling, is demanding. Teachers need to attend to the mathematics learning needs of all these students. The research literature offers some valuable insights relating to teaching mathematics to ELLs. There is much less literature, however, on teaching mathematics to students with little prior schooling, or on teaching mathematics to refugees, let alone teaching mathematics in the context of all three of these challenges.

Nevertheless, the teachers from the mathematics study group used the key points identified in this chapter to analyze their mathematics teaching practices and develop concrete strategies that would improve the support of the ELL students in their mathematics classes. These strategies included

- linking mathematics and language learning objectives;
- identifying a range of linguistic features, including vocabulary, logical connectives, and sentence structures, as relevant for a particular mathematics unit;
- using sentence patterns to support the development of explanations, definitions, etc.;
- including and linking different representations of mathematical concepts and related mathematical language structure using concrete materials, labelled diagrams, models, several different symbols and the written word;
- planning language and mathematics learning trajectories within bansho (boardwriting) preparation.

As a result, the mathematics study group was able to make some progress towards addressing their original questions. This work was not without its tensions. For the teachers and the principal, it is undoubtedly overwhelming to have 90–95% of the student population at some stage of ELL development. Because this Catholic elementary school has an ongoing stream of ELL learner registrations throughout the year (usually of refugee status), the teachers experienced difficulty in creating timely next steps instruction for each ELL learner relative to the current mathematics learning goals set for the whole class.

Tracking ELL student learning and progress is a priority at the school, with regards to classroom assessments, board standardized tests at grades 2, 5 and 7, as well as EQAO provincial standardized assessments. Based on the teacher's inference of the school's provincial assessment scores from 2012–2015, there has been improvement over the last 4 years (i.e., from 15% to 27% for grade 6 students who are achieving at level 3 or 4). Yet, the teachers were disheartened that the reported student scores continued to be significantly below the provincial average. However, the grade 3 students had shown greater achievement increases over the same three-year period (i.e., 22% to 37% are achieving at level 3 or 4). Also, they noted a consistent achievement score differential between the grade 3 and grade 6 students. The local school improvement team (ELL and divisional teacher representative) analyzed the EQAO mathematics student scores further and reported that every ELL student showed improvement from grade 3 to 6. So, what accounts for the lower EQAO achievement on the junior mathematics assessment?

At this school, learning mathematics through problem solving is about all students co-constructing mathematical ideas, methods and strategies by explaining their methods and asking their classmates questions for clarification and mathematical details. In particular for the ELL students, the teachers reported that it would take about 3 to 6 months for them to actively engage in the mathematics class by solving lesson problems (individually or in pairs), responding during class discussions and building on other students' ideas with comments and questions. The teachers inferred that ELL students' minimal oral participation is not solely attributed to their minimal use of the English language, as Middle Eastern refugee children have learned to mistrust any institution. Several teachers recounted how younger ELL students stood silent and still; an older ELL student explained, “Our parents tell us to be quiet and say nothing when asked questions.”

During the first few months of school, ELL students’ mathematical communication is limited to head nodding, pointing at the board-writing, the use of symbols for simple calculations. Because bansho (board-writing) provides a visual structure for students, much of the ELL students’ processing of the mathematics learning is only evident through their note-taking. Teacher emphasis of particular parts of the bansho (board-writing) in relation to ELL students’ note-taking focuses students on particular parts of their notes. ELL students’ classmates often interpret what is written on the board, not while note-taking, but when the class is finished and the other students have left. It is common for early-stage ELL students to go to the chalkboard and listen to ELL classmates with more advanced English proficiency explain each part of the board-writing (mathematics terms, explanations, labeled diagrams, student solutions to one lesson problem) and ask them questions to check for their classmate’s understanding of the written record of the class’s mathematical discussion. It is at this time that ELL students are orally communicating their mathematical understanding. Throughout a lesson, it is common to see some ELL students read aloud their notes, while other ELL students shake their head, say nothing, divert their gaze to their desk and/or put their head down on their desk.

This work raises important questions for research. The questions and challenges identified by teachers at this Catholic elementary school in many cases have not been adequately addressed by researchers. There is clearly a need for substantial research on teaching mathematics to refugee children, who have often made long and arduous journeys, lived in multiple unstable situations, who may have been traumatized and who may continue to suffer from psychological problems. These students, more often than not, have had little or no formal schooling prior to their attendance at an urban school, like this Catholic elementary school. For many of these children, attending a school like the one described provides them with a safe, stable, routine and a caring environment. Mathematics has a place in this environment. The work we have described shows that success is possible, but much remains to be done.

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References

Barwell, R. (2005). Working on arithmetic word problems when English is an additional language. *British Educational Research Journal, 31*(3), 329–348.

Barwell, R. (2009). *Mathematics in multilingual classrooms: Global perspectives*. Bristol: Multilingual Matters.

Barwell, R. (2012). Heteroglossia in multilingual mathematics classrooms. In H. Forgasz & F. Rivera (Eds.), *Towards equity in mathematics education: Gender, culture and diversity* (pp. 315–332). Heidelberg: Springer.

Barwell, R., Clarkson, P., Halai, A., Kazima, M., Moschkovich, J., Planas, N., Setati Phakeng, M., Valero, P., & Villavicencio Ubillús, M. (Eds.). (2016). *Mathematics education and language diversity: The 21st ICMI study*. Cham: Springer.

Clarkson, P. C. (2007). Australian Vietnamese students learning mathematics: High ability bilinguals and their use of their languages. *Educational Studies in Mathematics, 64*(2), 191–215.

Clarkson, P. C. (2009). Mathematics teaching in Australian multilingual classrooms: Developing an approach to the use of classroom languages. In R. Barwell (Ed.), *Multilingualism in mathematics classrooms: Global perspectives* (pp. 145–160). Bristol: Multilingual Matters.

Clements, D., & Sarama, J. (2014). *Learning and teaching early math: The learning trajectories approach* (2nd ed.). New York: Routledge.

Cummins, J. (2000). *Language, power and pedagogy: Bilingual children in the crossfire*. Clevedon: Multilingual Matters.

Davis, B., & Renert, M. (2014). *The math teachers know: Profound understanding of emergent mathematics*. New York: Routledge.

Dawe, L. (1983). Bilingualism and mathematical reasoning in English as a second language. *Educational Studies in Mathematics, 14*(4), 325–353.

Education Quality and Accountability Office (EQAO). (2014). *Grade 9 assessment of mathematics 2014: Released assessment questions*. Toronto: Queen's Printer for Ontario.

Khisty, L. (1995). Making inequality: Issues of language and meanings in mathematics teaching with Hispanic students. In W. G. Secada, E. Fennema, & L. B. Adajian (Eds.), *New directions for equity in mathematics education* (pp. 279–297). New York: Cambridge University Press.

Lapkin, S., Hart, D., & Turnbull, M. (2003). Grade 6 French immersion students' performance on large-scale reading, writing, and mathematics tests: Building explanations. *Alberta Journal of Educational Research, 49*(1), 6–23.

Lesh, R., Post, T., & Behr, M. (1988). Proportional reasoning. In J. Hiebert & M. Behr (Eds.), *Number concepts and operations in the middle grades* (pp. 93–118). Reston: Lawrence Erlbaum & National Council of Teachers of Mathematics.

Moschkovich, J. N. (1999). Supporting the participation of English language learners in mathematical discussions. *For the Learning of Mathematics, 19*(1), 11–19.

Moschkovich, J. N. (2009). How language and graphs support conversation in a bilingual mathematics classroom. In R. Barwell (Ed.), *Multilingualism in mathematics classrooms: Global perspectives* (pp. 78–96). Bristol: Multilingual Matters.

Moschkovich, J. N. (Ed.). (2010). *Language and mathematics education: Multiple perspectives and directions for research*. Charlotte: Information Age Publishing.

Moschkovich, J. N. (2013). Principles and guidelines for equitable mathematics teaching practices and materials for English language learners. *Journal of Urban Mathematics Education, 6*(1), 45–57.

Moss, J., Hawes, Z., Naqvi, S., & Caswell, B. (2015). Adapting Japanese Lesson Study to enhance the teaching and learning of geometry and spatial reasoning in early years classrooms: A case study. *ZDM Mathematics Education, 47*, 377–390.

Ontario Ministry of Education. (2005). *The Ontario curriculum, Grades 1–8: Mathematics*. Toronto: Queen's Printer for Ontario. Retrieved from http://www.edu.gov.on.ca/eng/curriculum/elementary/math18curr.pdf

Ontario Ministry of Education. (2011). *Bansho (board-writing)*. Toronto: Queen's Printer for Ontario.

Ontario Ministry of Education. (2012). *Paying attention to proportional reasoning K-12*. Toronto: Queen's Printer for Ontario.

Planas, N. (2014). One speaker, two languages: Learning opportunities in the mathematics classroom. *Educational Studies in Mathematics, 87*(1), 51–66.

Setati, M. (2005). Teaching mathematics in a primary multilingual classroom. *Journal for Research in Mathematics Education, 36*(5), 447–466.

Setati, M., & Adler, J. (2001). Between languages and discourses: Code switching practices in primary classrooms in South Africa. *Educational Studies in Mathematics, 43*, 243–269.

Small, M. (2013). *Powerpoint presentation*. Toronto, ON.

Swain, M. (2000). The output hypothesis and beyond: Mediating acquisition through collaborative dialogue. In J. Lantolf (Ed.), *Sociocultural theory and second language learning* (pp. 97–114). Oxford: Oxford University Press.

Swain, M., & Lapkin, S. (2005). The evolving socio-political context of immersion education in Canada: Some implications for program development. *International Journal of Applied Linguistics, 15*(2), 169–186.

Sztajn, P., Confrey, J., Wilson, P. H., & Edgington, C. (2012). Learning trajectory based instruction: Toward a theory of teaching. *Educational Researcher, 41*(5), 147–156.

Thames, M. H., & Ball, D. L. (2010). What math knowledge does teaching require? *Teaching Children Mathematics, 17*(4), 22–229.

Turnbull, M., Lapkin, S., & Hart, D. (2001). Grade 3 immersion students' performance in literacy and mathematics: Province-wide results from Ontario (1998–99). *The Canadian Modern Language Review/La revue canadienne des langues vivantes, 58*(1), 9–26.

Introduction to Students at Risk: Case Studies of Often Unheard Students

Ann Kajander

Abstract Students deemed at risk in mathematics, as chosen by their classroom teachers, were studied in this series of case studies of grades 7, 8 and 9 students. A researcher shadowed each student individually during several mathematics periods per week over a full semester, in order to allow a relationship with each participant to develop. A wide range of different student characteristics, as well as teacher behaviours, was observed across the cases. Not only was it found that many students were disenfranchised with mathematics, a broad range of other issues were observed. These included reading difficulties, attendance issues, home life challenges, low self-esteem, and extreme shyness. The relatively long duration of the study and the one-on-one nature of the intervention allowed the researcher to develop a relationship with the participants, which may have contributed to the willingness of the students to share their thoughts and feelings in a way that would not have been possible with the classroom teacher working alone. Differences in teacher style were also found, from more traditional, to a teacher who deeply believed in reform-based learning, but was highly challenged by the environment to the point of reverting at times to more traditional practices.

Keywords Students at risk · Mathematics teaching · Secondary · Struggling students in mathematics · Case studies of mathematics students · Self-concept and mathematics

It is often the case that prospective teachers of mathematics choose teaching because their own school mathematical experiences were positive ones. Yet typically, both in practica and early in their career, new teachers may find themselves assigned to non-university bound mathematics classes, such as the Applied[1] stream in Ontario.

[1] See McDougall and Ferguson (Part II this volume, para. 1) for a discussion of two of the possible Ontario pathways (Academic and Applied).

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The extreme angst, fear, apathy and insecurity felt by many of the students in such classrooms (Balfanz et al. 2006; Hannula 2006) are simply beyond the experience base of many more able mathematics students—including those who choose to become mathematics teachers. Here, case studies of students in three different such intermediate classrooms, who were deemed by their teachers to be "at risk" in mathematics, are presented. These case studies may be very eye-opening for those who have mostly experienced success in their own mathematics learning. The contribution of this study is its in-depth description of the participants, who, through learning to trust the researcher, shared their fears and struggles with respect to the learning of mathematics. The level of pain experienced by each participant in their mathematics class was far greater than we ever imagined.

Context

This study took place about 10 years ago in Northwestern Ontario, and while slightly dated by now, nevertheless provides rich and thick examples of classrooms typical to the time many current teacher candidates were themselves students in intermediate (grades 7 to 9) classrooms. The study came about in the context of a study of professional learning groups (Kajander and Mason 2007), in which teachers in one such group expressed the desire to learn more about students at risk, and willingly offered up their classrooms for such a study. Funding was obtained, and an initial group of about ten students in various schools and classrooms were identified by their teachers who were in the above professional learning group. A graduate student was chosen, permissions obtained, and visits to each of the student participants were set up. The graduate student researcher, a young female student who we felt would be minimally intimidating to the participants, sat one on one with each participant for two to three full mathematics periods for an entire semester. Data sources included detailed researcher field notes, student work samples, and student attendance and grade records. This extended time commitment allowed the participants to develop a relationship of trust with the researcher, sharing with her personal information not typical of a formal research study.

Participants

Data collection as noted above involved about ten student participants. Given the large number of classroom visits, up to 40 per participant, a large volume of data resulted, and some commonalities in traits were observed; thus, six illustrative cases were chosen on which to focus in more depth. The three cases presented here are an amalgam of these six cases which were selected for publication at the end of the study (Kajander et al. 2008). For the sake of brevity, the three students described to follow are drawn from these cases, encompassing their characteristics. One example each from a grade 7, 9 and Grade 9 Applied (the non-university mathematics/science stream in Ontario) class is provided here.

Results

Brian

Brian was a quiet and well behaved student in a grade 7 classroom. The teaching environment was a traditional one, in which discussion was not particularly encouraged. Lessons were provided formally by the teacher. When the teacher asked a question during a lesson, wrong student responses were treated with comments such as "does anyone have a better answer?" After each lesson, students were to work on their own on assigned work, individually at their desks. During each day's mathematics seatwork, students were allowed to ask a maximum of two questions: one of a classmate, and one to the teacher.

Brian was deemed by the teacher to be falling behind in mathematics, so she assigned him to work on some remedial mathematics workbooks of about grade 5 and 6 levels. He was required to do this while sitting at a desk alone in the hall. Occasionally the teacher would collect Brian's work and grade it, and that was his main feedback from the teacher. Of course his working in the hall alone meant he was missing the current day's grade 7 lesson.

When our study began, the researcher observed a few class lessons, and then joined Brian at his hallway desk. When Brian appeared stuck by a question, she would ask him what he found difficult. It became clear very soon that Brian's main difficulty was with reading the question. For example, Brain struggled with a question which he read as asking for the perimeter of a "potato"; once he was able to understand it was actually a "patio" he was to measure, he was able to complete the question independently and correctly.

While Brian did have some other characteristics which may have added to his difficulties, such as a tendency to messiness and lack of organisation, his overall mathematics performance improved greatly once he was receiving assistance with his reading.

Diane

Diane was a grade 8 student in a school with an open concept design, which thus had noisy classrooms. Her teacher believed strongly in the importance of real world contexts in learning, group work, use of manipulatives and so on, and in fact had been moved to this particular school to work with the students at this school, which was viewed as a challenging one, as he was acknowledged as a strong teacher.

The teacher tried very hard to engage the students in contextual tasks that might interest them. However, the students were very distracted by the noise level from other classrooms. When a lesson which required the use of manipulatives was used, the students were very disruptive and used the manipulatives in other ways, such as to make "guns." Finally, in frustration, the teacher put the materials away, causing the students working on the task to become angry as they could now no longer complete it as written. An interesting side note to this case is that as the term progressed, the teacher became more and more dispirited. Originally brought to the school as a skilled teacher to deal with this group of students, he resorted more and more to traditional lessons, without the use of materials. His enthusiasm and energy were observed to significantly flag during the year, and by the end of the semester he appeared very discouraged.

In this school, as well as in this class, there was a large percentage of self-identified Indigenous students in the classroom (up to about one third overall), and Diane was one of these. Diane sat at the back, and rarely spoke during lessons. She was extremely reticent when it came to any sort of group work and would hang back from it. If she had a question when working on assigned work, or did not understand, she simply stopped working. Sometimes she stopped even after writing down a correct answer. When asked why by the researcher she responded that "it's probably wrong." Diane simply refused to go up to the teacher's desk, either to pick up materials or manipulatives, collect or submit work, or to ask a question. When something needed to be picked up she would whisper to the researcher, "can you go?" Her fear of approaching the teacher seemed to stem from past trauma when asking teachers for help. She explained to the researcher that teachers had gotten angry with her in the past, and she had simply shut down. Whether real or imagined, this fear was very real to her. She was convinced the teacher did not want to talk with her as she was "so bad at math," and she was sure he would get angry.

Diane's struggles may have stemmed as well from things other than her reticence to participate. Her attendance was very poor and on average she missed about a third of the classes due to absence or lateness. Sometimes she was required to stay home to babysit her siblings. She had also moved a great many times over the last number of years, often attending two different schools during a given school year or being away from school altogether for months.

The only way Diane seemed to do any substantial work was when the researcher sat next to her and encouraged her. She did not always need help with the mathematics; sometimes simple words of encouragement were all she needed to complete a question. Yet she was very reticent to speak with the teacher when he approached her, and would absolutely not approach him herself. We did not at any time witness any behavior from the teacher that seemed to cause this response; rather, we had the sense that it stemmed from past experiences in mathematics.

Encouraged individually by the researcher, Diane did begin to do a bit of mathematics work, and showed some progress. She did not own a pencil case or basic supplies, and appeared happy to receive these as a gift from the researcher. We had a sense that this relationship between the two of them contributed to her progress. On the last day of the study, I witnessed Diane's farewell to the researcher, who had by now become her helper and friend. “Who will help me now?” she whispered plaintively, with tears in her eyes. It was the saddest moment I have ever felt during a research project, and touches me to this day.

Susan

Susan was in a Grade 9 Applied mathematics course, an Ontario course for non-university mathematics or science bound students. The teacher appeared to care deeply about her students, however her practice was typically traditional. Students sat at individual desks in rows. The teacher taught a formal lesson at the beginning of each class, which went on for quite a while, often at least 40 min. It was obviously carefully prepared, and neatly presented, yet students’ attention drifted after about 10 min. The students’ off task behavior was carefully timed to take place when the teacher was not looking at them—we had the sense as observers that the teacher simply had no idea what was going on when her back was turned.

As soon as the teacher-directed portion of the lesson was over, Susan asked to leave to “go to the washroom.” She usually left the room at least once, if not twice, after the lesson for extended periods. When she did sit with the researcher, she often appeared distracted and disinterested, and often asked wildly off-task questions such as “do you think animals can get high?”

Nearly all the assigned homework tasks and questions were highly abstract, and Susan displayed almost no interest in the material. One question which involved a perimeter of a room had her slightly more interested. In a subsequent question, she had to find the area of a right angle triangle inside a rectangle. She could not remember the formula for triangle area, but she came up with the idea on her own that she could take half of the rectangle area. Other instances which suggested a grasp of the material indicated that her issues were more to do with motivation and interest than ability. When the researcher suggested that she might need her mathematics credits to graduate from high school, Susan responded that she planned to collect unemployment insurance as a way to get by in life.

Samples of student work in the classroom illustrate the decontextualized nature of much of the work, which often involved terms and definitions rather than problems which might be relevant to the students. For example, see Fig. 1.

In the provided example, note that the feedback given was not descriptive but rather judgemental and evaluative—the “thinking cap” stamp marked by the teacher on the bottom right of the paper might suggest that the student should simply try harder. Formative feedback was not evident in the sample. Note also the very decontextualized and formal nature of the questions.

As when working with Diane, the researcher provided encouragement, appropriate questions, and gentle nudges to Susan to stay on task. At the beginning of the project, Susan had failing grades, but by the end had improved enough and engaged in enough work to earn an overall pass in the course. She remained relatively unmotivated throughout however.

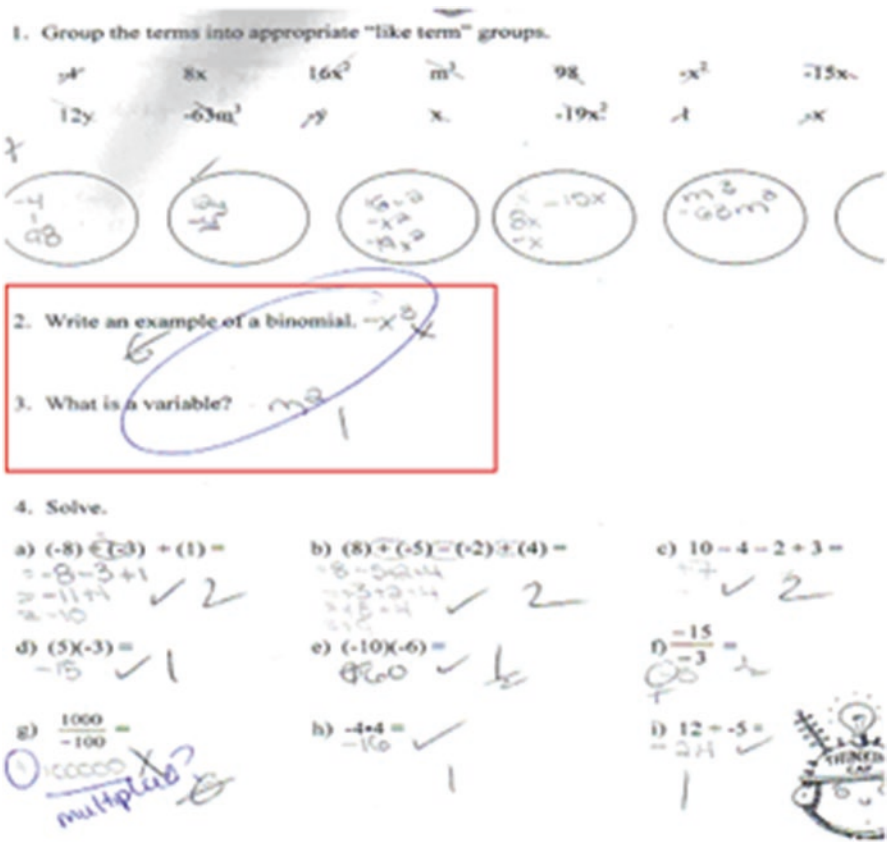

Fig. 1 Sample of graded student work. (Note the "Thinking Cap" stamp on the bottom right corner)

Discussion and Conclusions

The case studies presented here provide a backdrop for some of the subsequent chapters in this volume, such as those by Macaulay (Part II) and Jao (Part III), whose studies share ways of supporting struggling students more effectively. Keeping in mind some of the characteristics of the students at risk in the current chapter might help to motivate and ground the further readings on supporting student learning with more reticent mathematics learners.

In a subsequent paper reporting on the current case study work, we summarized some of the issues as follows:

> The students in the study demonstrated varying characteristics and levels of mathematical understanding. However, most were significantly disengaged from the mathematics classroom activities; attendance and attention span were problems for some. All appeared shy, unmotivated, and/or hesitant to ask questions. All appeared to be significantly lacking in

> self-confidence. In some cases, these underlying issues and frustrations had manifested themselves into substantial behaviour problems: they were completely off task. (Kajander et al. 2008, p. 1058)

Readers are encouraged to seek solutions to these observations in subsequent chapters, as well as in their own classroom teaching experiences. It may be important to remember the response of the grade 8 teacher in this study however, and how he became more and more discouraged: working with students at risk can indeed be emotionally challenging for teachers. It may help to remember that, for such students, the behaviours described may have developed in response to years' worth of bad experiences in mathematics. Teachers can hardly expect to undo this all at once. Further, it must be understood that many of these students' difficulties were broader than simply having gaps in their mathematical knowledge.

Although I had worked in such classrooms myself as a teacher prior to engaging in this study, the particular design of this study allowed me a window into these students' feelings, the likes of which I had not seen before. The depth of their pain and suffering in mathematics class, as well as for some in other aspects of their lives, was an eye-opener.

Later chapters in this volume report on more recent work in grade 9 classrooms which has strong potential to improve our understanding of best practices for such disengaged students. The importance of such efforts cannot be over-estimated.

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References

Balfanz, R., Mac Iver, D. J., & Byrnes, V. (2006). The implementation and impact of evidence-based mathematics reforms in high-poverty middle schools: A multi-site, multi-year study. *Journal for Research in Mathematics Education, 37*(1), 33–64.

Hannula, M. (2006). Motivation in mathematics: Goals reflected in emotions. *Educational Studies in Mathematics, 63*(2), 165–178.

Kajander, A., & Mason, R. (2007). Examining teacher growth in professional learning groups for in-service teachers of mathematics. *Canadian Journal of Mathematics, Science and Technology Education, 7*(4), 417–438.

Kajander, A., Zuke, C., & Walton, G. (2008). Teaching unheard voices: Students at risk in mathematics. *Canadian Journal of Education, 31*(4), 1039–1064.

Additional Suggestions for Further Reading

Kajander, A., & Zuke, C. (2008). Listening for their voices: A study of intermediate students at risk in mathematics. *delta K, 45*(2), 21–24.

Building Capacity in Grade 9 Mathematics: Case Studies from a Collaborative Inquiry Project in Applied Level Mathematics

Douglas McDougall and Sue Ferguson

Abstract In this chapter, we report on the results of a 4-year professional development projects focusing on improving teaching and learning for grade 9 students in Ontario. The participants were administrators, department heads and teachers from four secondary schools out of ten involved in a mathematics professional development project. At each school, an implementation team was responsible for identifying the mathematics strands to be investigated, share resources, discuss and implement teaching practices, collaborate on assessment (particularly moderated marking), and co-teaching/peer coaching. The university research team visited the participating schools to support their Mathematics Implementation Team twice per year. We describe four case studies to illustrate the various ways in which these participants improved the provincial large-scale mathematics assessment test scores from 80 to 256%. This study shows that professional development is more effective when administrators, department heads and teachers participate cohesively with university partners to improve mathematics learning in schools.

Keywords Middle school mathematics · Teacher professional development · Teacher inquiry · School improvement

This chapter describes a 4-year professional development project that involved 10 schools, 10 school administrators, and 60 teachers of Grade 9 Applied level mathematics across two Ontario school boards. In the Ontario grade 9 and 10 curricula, Academic level courses are the most rigorous, focusing on abstract concepts in mathematics, while Applied level courses are less rigorous, emphasing practical, concrete applications of concepts. This study yielded statistically significant increases in students' year-over-year scores on Ontario's annual Education Quality and Accountability Office (EQAO) large-scale mathematics assessments. Over the

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4-year professional development project, on average, the percentage of students at Level 3 (meeting the Provincial standard) or 4 (exceeding the Provincial standard) more than doubled. Students, who were previously underperforming relative to Provincial EQAO averages, were now performing at or close to Provincial norms.

With some variation in the degree of improvement among schools, the Collaborative Inquiry Project was, overall, successful in improving teaching as well as student learning in Grade 9 Applied mathematics. This result is significant as research has indicated that 85% of students who fail a mathematics course in grade 9 eventually drop out of school in the Toronto District School Board (Brown 2008). Students, normally considered at-risk for failing, can be provided with a mathematics program that will help their chance at success (Kajander et al. 2008). Recent provincial test data suggest that half as many students in Ontario Applied level classrooms are reaching the provincial standard compared with the number of students doing so in Academic courses.

The Collaborative Inquiry Project research demonstrated that a collaborative teacher professional learning community is essential for supporting improved student learning and further suggested that the social contexts of the professional development, such as workshops, online communication, and assistance by mathematics implementation teams, are central to the sustainability of professional development initiatives has also supported the importance of community and social context in professional development in Ontario.

Literature Review

Studies have linked student academic success to the knowledge and pedagogical skills of their teachers (Baumert et al. 2010; Darling-Hammond and Sykes 1999; Firmender et al. 2014; Fullan et al. 2006; Glazerman et al. 2009; Hattie 2009; Nye et al. 2004; Slavin and Lake 2008; Wilson et al. 2001). Teaching is an extraordinarily complex profession, requiring a mastery of foundational content knowledge (Ball et al. 2008; Mitchell et al. 2014), as well as a sophisticated repertoire of instructional strategies, and the ability to flexibly tailor practices to particular situations and the needs of individual learners (Bransford et al. 2005; Cole and Knowles 2000; Turner-Bisset 2001). It is not surprising, therefore, that teacher professional development is the focus of a great deal of research.

Where traditional "one shot" sessions, delivered primarily through lecture or other transmission-oriented methods, have been found to be largely ineffective (Arbaugh 2003; Gojmerac and Cherubini 2012; Hattie 2009; Warren-Little 1999), successful interventions typically employ a combination of some or all of the following elements: job-embedded learning, collaborative (peer) inquiry, attention to and tracking student performance, institutional and administrative support, the provision of time and other resources, and a commitment to continuous, sustained, and intensive engagement in professional development initiatives (Avalos 2011;

Darling-Hammond et al. 2009; Guskey 2000; Hawley and Valli 1999; Holm 2014; Jao 2013; Jao and McDougall 2015; Kajander and Mason 2007; Suurtaam and Vezina 2010). While there is a lack of agreement regarding the relative merits and efficacy of particular methods (e.g., action research vs. lesson study), there does appear to be a growing consensus that effective professional development requires a skilful interweaving of the above-mentioned factors (Berliner 2005; Fullan 2001, 2005; Guskey 1995, 2000, 2003; Hammerness et al. 2005; Hawley and Valli 1999; Jao and McDougall 2016; Lieberman and Wilkins 2006; McDougall and Jao 2009; McDougall et al. 2010).

Recent innovations in professional learning for teachers have included a wide variety of collaborative approaches to professional development. Collaborative teacher development has been shown to be successful in enhancing teacher learning as well as driving improvements in student learning (Holm 2014; Jao and McDougall 2015; Kazemi and Franke 2004; Little 2003; Nelson and Slavit 2008; Slavit and Nelson 2010; Soine and Lumpe 2014). A vehicle for creating community and reducing isolation, collaborative, school-wide professional development has been found to be effective in creating a sense of shared community (Darling-Hammond et al. 2009; Huffman et al. 2003). For teachers, being part of an implementation team is important, especially when the team includes strong administrative support (Egadawatte et al. 2011; Macaulay 2015; McDougall and Jao 2009; McDougall et al. 2010). With respect to mathematics, the focus of the current study, research has shown that teachers who believe that they are part of a capable instructional team support higher student achievement on mandated assessments of mathematics performance (Goddard 2001; Macaulay 2015; Ross and Gray 2005).

A series of case studies were used to create a conceptual framework called the "Ten Dimensions of Mathematics Education." The Ten Dimensions were developed to help elementary and secondary school teachers better understand the many elements of effective mathematics instruction, and to identify areas of improvement (McDougall et al. 2006). The framework has also been helpful for teachers to improve the performance of special needs students (Egadawatte et al. 2011; McDougall and Jao 2009; McDougall et al. 2010). Framed within a social constructivist perspective (Bartlett and Burton 2007; Cobb and Yackel 1996; Copsey-Haydey et al. 2010), the research conducted in Grade 9 Applied level mathematics helped us to better understand how teachers work in collaborative inquiry teams and how they integrated their new experiences into existing knowledge structures.

Method

The research study was a professional development project involving 10 schools, 10 school administrators, and 60 teachers of Grade 9 Applied level mathematics across two large urban Ontario school districts. The data was primarily collected through interviews

Table 1 EQAO score increases for schools A and B

School	EQAO prior to study	EQAO year one	EQAO year two	Percentage increase
A	9	16	32	256
B	14	30	35	150

with teachers and administrators via two sets of school visits, four full-day workshops, and an online wiki for shared resources.

The schools were selected based on ranking in the bottom one-third on the EQAO scores for the Grade 9 Applied level classes in the two boards. A critical component of the Collaborative Teacher Inquiry Project was the establishment of a Mathematics Implementation Team at each participating school. Comprised of an administrator, the department head of mathematics or curriculum leader, and three to five participating teachers, the team was responsible for identifying the mathematics strands that would be investigated, sharing resources, discussing and implementing teaching practices in mathematics, collaborating on assessment (particularly moderated marking), and co-teaching/peer coaching. Twice per year, the research team visited participating schools to support their Mathematics Implementation Team.

The participants attended four in-service sessions per year at the Ontario Institute for Studies in Education (OISE), completed personal reflections about their participation and learning in these sessions, completed the Beliefs and Attitudes Survey (McDougall 2004), and were interviewed by one of the researchers. The Beliefs and Attitudes survey was used to identify the areas of improvement from the Ten Dimensions of Mathematics Education (McDougall 2004).

Findings

We will describe two schools in the 2-year professional development project, followed by a description of 2 of the 4-year schools. We have selected the four schools that initially had EQAO scores where fewer than 20% of the students were at provincial standard to illustrate the implementation strategies that they used as well as the overall impact of the project (Table 1).

School A

School A is situated in northwest Toronto. At the time of the study, the school had a culturally diverse population of approximately 750 students, representing an array of cultures, backgrounds, and languages. School A offered courses at both Academic and Applied levels, as well as two special programs: a cross-curricular enriched program in science, mathematics, and computer science, and a specialized arts program.

Implementation Strategies

The implementation team adopted the Ontario Ministry of Education Targeted Implementation and Planning Supports for Revised Mathematics (TIPS4RM) resources and materials. The TIPS4RM resources were downloaded from the EduGAINS website, which hosts the Ontario Ministry of Education-developed resources that support Kindergarten to grade 12 policies and programs to improve teaching and learning. Their lesson plans incorporated more student-centred approaches to teaching through cooperative learning and group work, and included rich performance tasks such as creating a children's storybook to describe distance-time graphs, and investigating relationships between variables by performing experiments. The department invested in and utilized technology including: an interactive whiteboard (Mobi View), response systems (clickers), and online simulations (Gizmos) for learning. One team member was very comfortable with technology and provided a motivating and supportive presence for the others.

The team conducted half-day numeracy meetings every 5 weeks to discuss teaching, assessment, and evaluation strategies, as well as to collaborate on lesson planning, unit planning, and additional projects, such as field trips. The teachers also collaborated informally and a result of both formal and informal collaboration, teaching methods, strategies, and resources were shared and discussed to a much greater extent than before.

Impact of the Project

The teachers reported that incidents of problem behaviour and student-teacher conflicts in their Grade 9 Applied level mathematics classes declined since the implementation of the teaching strategies discussed during the workshops. Where teachers placed a greater emphasis on assessment for learning and increased the role of observation and conversation in the classroom, more opportunities for mathematics talk occurred between teachers and students, as well as among students, increasing learning engagement.

The learning environment improved greatly with the incorporation of technology for instructional and assessment purposes. Utilizing an interactive whiteboard in conjunction with Internet resources, such as consumer websites and online simulations, made learning more dynamic and relevant to students' everyday lives. The incorporation of clickers into lessons provided students and teachers with immediate feedback that informed student learning and teaching practices. Using manipulatives (for example, 3D relational solids) to investigate the volume of three-dimensional solids grounded student learning in real-life tactile objects. Finally, focusing on building healthy relationships and rapport with and between students created a safe and positive learning environment where learning was a common objective.

The team noted that some students with a prior history of poor attendance did not improve in spite of changes made. They also noted that the sharing of reform teaching approaches and strategies with teachers new to the Grade 9 Applied course were not always effective. In the future, the team plans to use demonstration classrooms to model new approaches and strategies to teachers new to the Grade 9 Applied mathematics course. School A will continue to use technology and new instructional practices in mathematics classes and expand those strategies to other classes; however, the team is seeking ways to provide extra time for staff training in the use of technology. As an additional incentive for all teachers to become more comfortable using technology, an online learning community (Moodle) is being created for Grade 9 Applied classes as well as other mathematics courses. As an outcome, the percentage of students who attained the grade 9 mathematics credit grew from 64% to 84%.

School B

School B is a mid-sized high school located in northeast Toronto. Fifty-four percent of their students spoke a primary language other than English, and 12% of the student body had been in Canada for fewer than 5 years.

Implementation Strategies

The Implementation Team determined the need for extra practice materials that were missing from the TIPS4RM resources. The teachers in School B decided to create a student workbook that contained the TIPS4RM materials, resources from the project wiki, as well as other helpful resources they either developed or located. In an effort to give their students a sense of confidence, they decided to begin the year with a unit on measurement, as most students find this one of the easiest strands. Another strategy was the use of tablet computers daily in class. The course materials were loaded on the tablets and teachers used the projector to illustrate the work.

The teachers incorporated practice questions found on the EQAO website into classes regularly throughout the semester. Additional EQAO review sessions were held for students after school on a regular basis.

Impact of the Project

The teachers reported that their lesson structure became more standardized on the three-part lesson plan. They also used a variety of instructional strategies in the classroom, particularly the use of cooperative learning strategies. The students worked in pairs and small groups on a regular basis in class. The teachers made a

Table 2 EQAO score increases for schools C and D

School	EQAO prior to study	EQAO year one	EQAO year two	EQAO year three	EQAO year four	Percentage increase
C	12	10	19	27	31	158
D	20	23	37	31	36	80

strong effort to explain the learning goals for the lesson at the beginning of class and to consolidate knowledge at the end of the class.

Before the project, the collaboration amongst teachers was low, only connecting formally once or twice per month. During the project, the teachers met and collaborated almost daily. They rotated the responsibility to prepare common assessments such as tests, quizzes, and rich tasks.

There was an increase of positive interactions between students and teachers, as the students preferred having a workbook geared to the course expectations, rather than a traditional textbook. The students were able to make a connection with the content of the lessons that were presented via the tablet over text written on the blackboard. The students also recorded more examples from the tablet than they had previously from the blackboard. This increased student achievement improved student-teacher relationships.

The team also reported better class attendance. Students increasingly initiated the organizing of solutions to mathematics problems without being directed. Other teachers within the department who did not teach Grade 9 Applied level students noticed how frequently the Grade 9 Applied level teachers were communicating. This had the positive effect of increasing discussion about mathematics teaching within the department as a whole. In addition, resources were created for Grade 10 Applied level teachers to use at the beginning of their course to make links to the Grade 9 Applied level course. The percentage of students who attained the grade 9 mathematics credit grew from 54% to 77% (Table 2).

School C

School C is located in the northwest area of Toronto, and had 1100 adolescent students and 1400 adult students.

Implementation Strategies

The teachers were focused on the use of learning goals in the planning and implementation stages of the project. The teachers ensured that the students were aware of the learning goals at the start of the unit, and were familiar with the success criteria. This was done by sharing rubrics with the students at the beginning of the unit so that students were aware of the expectations when completing assignments. The teachers used Learning Goals Exit Cards, as well as descriptive feedback on assignments.

A differentiated learning strategy was used by the teachers to help their weak students understand fully, at their own pace, the requirements while providing more difficult questions for stronger students. The learning goals, whether they were easy or difficult, helped give students an opportunity for success.

The grade 9 implementation team was able to share and demonstrate a variety of effective instructional strategies as a result of team teaching. A Math Coach supported the co-teaching process, and teachers were able to discuss and practice a wide range of successful strategies used by other schools. Both weak and strong students were monitored through the use of a Data Wall.

One member of the implementation team ran an after-school numeracy class for grade 9 mathematics students to attend for extra support. The school also organized a two-week summer program for incoming grade 9 students to bolster literacy and numeracy skills.

Impact of the Project

The school C Grade 9 Applied mathematics implementation team increased the amount of collaboration amongst teachers. All of the teachers were enthusiastic and open to suggestions. There was an increase in the rapport between teacher and student. The teachers developed a classroom culture in which students were not afraid to take risks. As a result, students were not resistant to answer questions, suggest solutions or share their work.

The project provided an excellent example to other teachers in the department about ways in which collaboration can increase the effectiveness of classroom teachers when delivering the course curriculum. For example, the English department was brought into the Math Department's Professional Learning Cycle. Consequently, English teachers adopted some of the same instructional strategies.

The teachers found a number of opportunities for collaboration. Those areas included lesson planning creating opportunities to observe each other teach and debrief their practice. Time was also given to focus on evidence-based instructional strategies and track students' achievement on a Data Wall. All Grade 9 Applied mathematics teachers were expected to use TIPS4M materials, investigations, and activities in class.

There were some challenges over the four years the school was involved in this project. Some teachers found it difficult to change their teaching practice to the extent that students became engaged collaboratively during the mathematics period.

Turnover in the Grade 9 Applied teaching team also affected consistency of approach. In addition, the percentage of students who attained the grade 9 mathematics credit was about 86%.

School D

School D is a mid-sized collegiate in the eastern area of Toronto. Twenty percent of grade 9 students had been in Canada for fewer than 5 years, and 130 had an IEP or were supported by Academic Resources. Approximately 45% of the grade 9 students were transferred from grade 8 without successfully completing grade 8 math.

Implementation Strategies

There were a number of implementation strategies that were used by the teachers. As an outcome of an OISE Professional Development session, the teachers changed the course sequence in order to improve results in geometry and algebra. The teachers now start with Measurement at the beginning of the course to encourage a more balanced approach. This is particularly helpful to those students having difficulty with number sense and numeration.

They also incorporated the use of more manipulatives and technology (TI'Nspire calculators, SMART board and clickers). There were full department sessions on the use of TI'Nspire calculators. There were two full-day SMART board training for the entire mathematics department plus an additional day on clickers and use of the document camera. The teachers compiled SMART board-ready activities for each other to access on specific topics facilitating the use of SMART board technology in class. They also actively participated in OISE sessions on instructional strategies, student tasks, summative evaluations, and the use of technology.

The grade 9 implementation team also participated in co-teaching opportunities with other schools and hosted several co-teaching demonstration sessions by the school district program department. The culture within the department has changed to one where there is constant conferencing and frequent communication about instruction for Grade 9 Applied and all other courses.

The teachers focused on assessment practices. They provided on-going student feedback through both formative and summative assessment and student conferencing. The use of weekly quizzes provided students with timely feedback that was descriptive and could inform the teachers about students understanding, allowing teachers to revisit concepts as needed. The teachers and department head collaborated constantly with other Grade 9 Applied level mathematics teachers—daily communication of topics and pacing, same assessments used—including a weekly quiz on Friday.

The teachers created rich tasks for each unit and shared them with one another. They embedded EQAO materials throughout the course and in the student workbook. They used the three-part lesson planning format, as well as collaborative planning sessions, to enhance the teaching practices. The department promoted a cross-curricular approach to numeracy by creating an in-school Numeracy Day with a motivating guest speaker.

Impact of the Project

There were a number of significant impacts of the project on the teachers, the mathematics department, and the school. The Grade 9 Applied level team constantly conferred on the Grade 9 Applied course, with daily communication about pacing, assessment, and instruction.

The grade 9 teachers found that their students were more engaged after experiencing increased success. This ceated a positive learning environment, with active student participation. There was a decrease in "math phobia" and students were willing to approach their teacher for help and clarification. This was partly the result of increased access and use of technology and manipulatives by all Grade 9 Applied mathematics teachers. In addition, the percentage of students who attained the grade 9 mathematics credit was about 86%.

We did notice a decrease in the scores during the third year of the study. In this year, the department head and two of the experienced Grade 9 Applied level teachers stopped teaching the course and were replaced by two new teachers. We feel that there should be some continuity amongst the teachers so that the knowledge can be shared in the team.

Summary

The Collaborative Teacher Inquiry Project focused on the improvement of instructional strategies for teachers of Grade 9 Applied level mathematics courses, with the overall goal of improving student achievement and engagement in mathematics. An additional purpose was to investigate collaborative inquiry as a professional development strategy for both experienced and inexperienced secondary school teachers. The project provided teachers with the opportunity to refine their instructional practices through inquiry and collaboration with colleagues both in and outside their schools.

Teachers found that collaboration with other departments was both necessary and helpful. The special education departments assisted with practice EQAO exams and support for students with IEPs, while, in some schools, other subject departments found ways to infuse mathematics into their courses. Overall, the teachers felt that both internal and external professional development and collaborative effort made this project successful.

Consistency of staffing was a challenge in many of the schools. Teachers were concerned that staffing changes would affect the efficacy of a successful team. Some teams wanted to ensure that the team remained together for another year to further implement the changes required for the course, while others wanted to continue their collaboration as a group and redirect their efforts towards another course and/or grade level. Teachers felt that keeping the team intact was preferable in order to sustain growth; however, when staff changes were inevitable, many were able to bring new colleagues up to speed during the course of the project.

The most noticeable change was the extent to which teachers took responsibility for student learning by the end of the project. Many of the teachers found that their students experienced greater success and developed confidence in their mathematics studies as the teachers changed the teaching-learning dynamic in the classroom. They found that increased student engagement and better achievement led to improved attendance and in class participation. The non-traditional delivery of the curriculum made the most positive outcome for the students as well as the teachers.

The primary benefits of the project were that the teachers and administrators collaboratively investigated, discussed and implemented evidence-based teaching and assessment strategies and techniques, which culminated in improvement in the achievement of students enrolled in Grade 9 Applied level mathematics. The teachers and administrators learned about strategies and resources that proved successful for teachers in other schools and in other school boards. Teachers found ways to meet the challenges that they faced in their school environment and to grow as reflective practitioners. As a result of teachers working collaboratively to improve credit attainment, EQAO scores also rose—significantly, in most schools.

There will always be students with such challenging circumstances such as gaps in knowledge of basic mathematics concepts, students with learning disabilities, immigrant students learning English, and students living in poverty with the ramifications these issues often bring. Despite the diversity in their students' backgrounds, the teachers in this project had a significant and positive effect on the outcomes of many grade 9 students.

For more papers on these projects, please see (Egadawatte et al. 2011; Jao and McDougall 2015; Jao and McDougall 2015; McDougall et al. 2010, 2013; Stoilescu et al. 2015). Other recent work (Holm 2014; Holm and Kajander 2015; Macaulay 2015).

References

Arbaugh, F. (2003). Study groups as a form of professional development for secondary mathematics teachers. *Journal of Mathematics Teacher Education, 6*, 139–163.

Avalos, B. (2011). Teacher professional development in teaching and teacher education over ten years. *Teaching and Teacher Education, 27*(1), 10–20.

Ball, D. L., Thames, M. H., & Phelps, G. (2008). Content knowledge for teaching: What makes it special? *Journal of Teacher Education, 59*(5), 389–407.

Bartlett, S., & Burton, D. (2007). *Introduction to education studies* (2nd ed.). London: Sage.

Baumert, J., Kunter, M., Blum, W., Brunner, M., Voss, T., Jordan, A., et al. (2010). Teachers' mathematical knowledge, cognitive activation in the classroom, and student progress. *American Educational Research Journal, 47*(1), 133–180.

Berliner, D. C. (2005). The near impossibility of testing for teacher quality. *Journal of Teacher Education, 56*(3), 205–213.

Bransford, J., Darling-Hammond, L., & LePage, P. (2005). Introduction. In L. Darling-Hammond & J. Bransford (Eds.), *Preparing teachers for a changing world: What teachers should learn and be able to do* (pp. 201–231). San Francisco: Jossey-Bass.

Brown, R. S. (2008). *The grade 9 cohort of fall 2002: A five-year cohort study, 2002–2007*. Technical report to Toronto District School Board. Toronto District School Board, Toronto, ON.

Cobb, P., & Yackel, E. (1996). Constructivist, emergent, and sociocultural perspectives in the context of developmental research. *Educational Psychologist, 31*(3/4), 175–190.

Cole, A., & Knowles, G. (2000). *Researching teaching: Exploring teacher development through reflexive inquiry*. Toronto: Allyn and Bacon.

Copsey-Haydey, D., Zakaluk, B. L., & Shaw, S. (2010). The changing face of content area teaching. *Journal of Applied Research on Learning, 3*(3), 1–29.

Darling-Hammond, L., & Sykes, G. (1999). *Teaching as the learning profession*. San Francisco: Jossey-Bass.

Darling-Hammond, L., Wei, R. C., Andree, A., Richardson, N., & Orphanos, S. (2009). State of the profession: Study measures status of professional development. *Journal of Staff Development, 30*(2), 42–50.

Egadawatte, G., McDougall, D. E., & Stoilescu, D. (2011). The effects of teacher collaboration in Grade 9 Applied mathematics. *Educational Research for Policy and Practice, 10*(3), 189–209.

Firmender, J. M., Gavin, M. K., & McCoach, D. B. (2014). Examining the relationship between teachers' instructional practices and students' mathematics achievement. *Journal of Advanced Academics, 25*(3), 214–236.

Fullan, M. (2001). *The new meaning of educational change* (3rd ed.). New York: Teachers College Press.

Fullan, M. (2005). Leadership and Sustainability. Thousand Oaks, CA: Corwin Press.

Fullan, M., Hill, P., & Crevola, C. (2006). *Breakthrough*. Newbury Park: Corwin Press.

Glazerman, S., McKie, A., & Carey, N. (2009). *An evaluation of the teacher advancement program (TAP) in Chicago: Year one impact report*. Washington, DC: Mathematical Policy Research.

Goddard, R. D. (2001). Collective efficacy: A neglected construct in the study of schools and student achievement. *Journal of Educational Psychology, 93*(3), 467–476.

Gojmerac, A., & Cherubini, L. (2012). Connecting for greater purpose: An examination of a district professional learning community. *Journal of Teaching and Learning, 8*(1), 43–58.

Guskey, T. (1995). Professional development in education: In search of the optimal mix. In T. Guskey & M. Huberman (Eds.), *Professional development in education: New paradigms and practices* (pp. 114–132). New York: Teachers College Press.

Guskey, T. (2000). Does it make a difference? Evaluating professional development. *Educational Leadership, 59*(6), 45–51.

Guskey, T. (2003). What makes professional development effective? *Phi Delta Kappan, 84*(10), 748–750.

Hammerness, K., Darling-Hammond, L., Bransford, J., Berliner, D., Cochran-Smith, M., McDonald, M., & Zeichner, K. (2005). How teachers learn and develop. In L. Darling-Hammond & J. Bransford (Eds.), *Preparing teachers for a changing world: What teachers should learn and be able to do* (pp. 258–289). San Francisco: Jossey-Bass.

Hattie, J. A. C. (2009). *Visible learning: A synthesis of 800+ meta-analyses on achievement*. Oxford: Routledge.

Hawley, W., & Valli, L. (1999). The essentials of effective professional development: A new consensus. In L. Darling-Hammond & G. Sykes (Eds.), *Teaching as the learning profession: Handbook of policy and practice* (pp. 127–150). San Francisco: Jossey-Bass.

Holm, J. (2014). *Improving mathematics teaching through professional learning groups*. Unpublished doctoral dissertation, Lakehead University, Thunder Bay, ON.

Holm, J., & Kajander, A. (2015). Lessons learned about effective professional development: Two contrasting case studies. *International Journal of Education in Mathematics, Science and Technology, 3*(4), 262–274.

Huffman, D., Thomas, K., & Lawrenz, F. (2003). Relationship between professional development, teachers' instructional practices, and the achievement of students in science and mathematics. *School Science and Mathematics, 103*(8), 378–387.

Jao, L. (2013). Peer coaching as a model for professional development in the elementary mathematics context: Challenges, needs and rewards. *Policy Futures in Education, 11*(3), 290–297.

Jao, L., & McDougall, D. (2015). The collaborative teacher inquiry project: A purposeful professional development initiative. *Canadian Journal of Education, 38*(1). Retrieved from http://journals.sfu.ca/cje/index.php/cje-rce/article/view/1600.

Jao, L., & McDougall, D. (2016). Moving beyond the barriers: Supporting meaningful teacher collaboration to improve secondary school mathematics. *Teacher Development, 20*(4), 557–573.

Kajander, A., & Mason, R. (2007). Examining teacher growth in professional learning groups for in-service teachers of mathematics. *Canadian Journal of Mathematics, Science and Technology Education, 7*(4), 417–438.

Kajander, A., Zuke, C., & Walton, G. (2008). Teaching unheard voices: Students at risk in mathematics. *Canadian Journal of Education, 31*(4), 1039–1064.

Kazemi, E., & Franke, M. L. (2004). Teacher learning in mathematics: Using student work to promote collective inquiry. *Journal of Mathematics Teacher Education, 7*, 203–235.

Lieberman, J., & Wilkins, E. (2006). The professional development pathways model: From policy to practice. *Kappa Delta Pi Record, 42*(3), 124.

Little, J. W. (2003). Inside teacher community: Representations of classroom practice. *Teachers College Record, 105*(6), 913–945.

Macaulay, A. V. M. (2015). *Effective practices in Grade 9 Applied mathematics*. Unpublished doctoral dissertation, Ontario Institute for Studies in Education, University of Toronto, Toronto, ON.

McDougall, D. (2004). *PRIME school mathematics improvement: Leadership handbook*. Toronto: Nelson.

McDougall, D. E., & Jao, L. (2009). *Report on the Grade 9 Applied math project for the learning consortium*. Report for Toronto District School Board, Toronto Catholic District School Board, Peel District School Board and York Region District School Board. Ontario Institute for Studies in Education, Toronto, ON.

McDougall, D., Ross, J. A., & Ben Jaafar, S. (2006). *PRIME ten dimensions of mathematics education: Research study*. Toronto: Thompson Nelson.

McDougall, D. E., Jao, L., Maguire, K., Stoilescu, D., & Egodawatte, G. (2010). *Collaborative teacher inquiry project: Grade 9 Applied mathematics*. Report for Toronto District School Board and Toronto Catholic District School Board. Ontario Institute for Studies in Education, Toronto, ON.

McDougall, D. E., Yan, X. H., Kam, M., Wang, Z. Y., Kwan, K., & Ferguson, S. (2013). Collaborative teacher inquiry project: Grade 8 mathematics. Report for Toronto District School Board. Ontario Institute for Studies in Education, Toronto, ON.

Mitchell, R., Charalambous, C., & Hill, H. (2014). Examining the task and knowledge demands needed to teach with representations. *Journal of Mathematics Teacher Education, 17*, 37–60.
Nelson, T., & Slavit, D. (2008). Supported teacher collaboration inquiry. *Teacher Education Quarterly, 35*(1), 99–116.
Nye, B., Konstantopoulos, S., & Hedges, L. V. (2004). How large are teacher effects? *Educational Evaluation and Policy Analysis, 26*(3), 237–257.
Ross, J. A., & Gray, P. (2005). School leadership and student achievement: The mediating effects of teacher beliefs. *Canadian Journal of Education, 29*(3), 798–822.
Slavin, R. E., & Lake, C. (2008). Effective programs in elementary mathematics: A best-evidence synthesis. *Review of Educational Research, 78*(3), 427–515.
Slavit, D., & Nelson, T. H. (2010). Collaborative teacher inquiry as a tool for building theory on the development and use of rich mathematical tasks. *Journal of Mathematics Teacher Education, 13*, 201–221.
Soine, K. M., & Lumpe, A. (2014). Measuring characteristics of teacher professional development. *Teacher Development, 18*(3), 303–333.
Stoilescu, D., McDougall, D., & Egadawatte, G. (2015). Teachers' views of the challenges of teaching Grade 9 Applied mathematics in Toronto schools. *Educational Research for Policy and Practice, 15*(2), 83–97.
Suurtaam, C., & Vezina, N. (2010). Transforming pedagogical practice in mathematics: Moving from telling to listening. *International Journal for Mathematics Teaching and Learning*.
Turner-Bisset, R. (2001). *Expert teaching: Knowledge and pedagogy to lead the profession*. London: David Fulton Publishers.
Warren-Little, J. (1999). Organizing schools for teacher learning. In L. Darling-Hammond & G. Sykes (Eds.), *Teaching as the learning profession: Handbook of policy and practice* (pp. 51–60). San Francisco: Jossey-Bass.
Wilson, S. M., Floden, R. E., & Ferrini-Mundy, J. (2001). Teacher preparation research: Current knowledge, gaps, and recommendations. A research report prepared for the U.S. Department of Education. Center for the Study of Teaching and Policy, Seattle, WA.

Success in Grade 9 Applied Mathematics Courses

Alison Macaulay

Abstract Streaming—the practice of sorting students into ability groups or streams—is a common practice in many jurisdictions around the world, and often favoured for mathematics classrooms. Research has established, however, that streaming leads to lower outcomes for those students who are placed in the lowest streams (or tracks). This paper begins with a discussion of the literature on streaming, highlighting the issues that contribute to the disparity in student achievement.

The paper then moves in to a discussion of streaming in the province of Ontario, Canada where grade 9 students who take the lower—or Applied mathematics course—are more likely to not reach the provincial standard on the provincial assessment than they are to reach it. The paper highlights findings from case study research of four Ontario schools that have bucked this trend and can boast strong, or unusual, performance for all of their grade 9 mathematics students, regardless of course selection. The research is distilled into ten recommendations for Applied mathematics classroom settings.

This paper offers practical advice for teachers who aim to create mathematics learning environments where all students can thrive.

Keywords Applied mathematics · Streaming · Tracking · Effective mathematics teaching

This chapter describes some of the outcomes of recent research around best practices in non-university stream classes at the grade 9 level. These best practices are described based on data collected from Ontario schools which showed strong scores, or unusual growth, on provincial assessment scores in Grade 9 Applied mathematics.[1] Based on this research, recommendations for teachers are described.

[1] See McDougall and Ferguson (Part II this volume, para. 1) for a discussion of two of the possible Ontario pathways (Academic and Applied).

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The practice of separating grade 9 students into different levels of courses, referred to as streaming, will be discussed first.

Streaming

In many jurisdictions, students will be streamed at some point during their secondary schooling, usually at the grade 9 or 10 level. This process involves sorting and grouping students into different courses, based on their perceived ability, for the purpose of instruction (Oakes 1985). The typical rationale for streaming is an efficiency argument (Van Houtte 2004)—presumably when students are placed in homogeneous classes or groupings, teachers can adapt the materials, level, and pace of instruction to better meet the needs and cognitive level of individual students. This thinking implies a fixed mindset towards mathematics learning and a belief that students have relatively static levels of ability and should therefore be taught accordingly (Boaler et al. 2000). It is worth noting here that mathematics teachers are more likely to support streaming than are teachers from any other discipline (Talbert 1995). In Ontario, for example, for students to switch from the Applied (non-university mathematics/science stream) pathway to the Academic (university) pathway, they must take a transfer course. No other discipline has this requirement. This fact, in and of itself, is worthy of reflection.

Although streaming mathematics courses is prevalent in Canadian and North American secondary schools, the practice is not supported by research. In fact, researchers have demonstrated that when they control for ability level and socioeconomic status, being in the top stream accelerates achievement and being in the low stream significantly reduces achievement, especially for mathematics (EQAO 2012; Gamoran and Berends 1987; Hamlin and Cameron 2015; Slavin 1990). Furthermore, the achievement between students in the high and low streams becomes more and more unequal over time (Gamoran 2002), resulting in gaps that inevitably widen as students progress through the grades. In the province of Ontario, for example, there is a solid decade of provincial assessment data that shows students in the higher stream of grade 9 mathematics are twice as likely as their counterparts in the lower stream to reach the level of achievement that the Ontario Ministry of Education has set as "the provincial standard" on the provincial assessment, which is equivalent to a "B." There is also a solid base of evidence that demonstrates poor, working-class, and minority students are disproportionately labeled as slow learners in elementary schools and assigned to the lowest streams in secondary schools (People for Education 2013). For example, there are about four times as many students with special needs in the Applied stream of grade 9 mathematics in Ontario. To make matters worse, there is a third and even "lower" stream in Ontario, which is exempted from the provincial assessment altogether. This indicates that by default, most at-risk students are streamed into the "lower" and less academic streams, making them especially vulnerable to under-achieving in mathematics.

In Ontario, the curriculum is structured around pathways, which are linked to post-secondary destinations. The Academic courses have been designed to prepare students for university, while the Applied courses have been designed for students who plan to go to college or directly to the workplace. Perhaps because of a bias that most teachers have (being university-educated themselves), the Applied course is often viewed as being less rigorous, and "basic." This is certainly not the intent of the curriculum, but nonetheless, students often get labelled as Applied kids and often students are counselled to "move down to Applied" if they show any sign of struggle in the Academic course.

Many researchers have shed light on why it is that streaming—whatever you call it, or how you package it—actually derails student performance. They have found, quite simply, that students in the lower streams have less opportunity to learn than their peers in higher streams. For example, Oakes (1982, 1986) established that students in high stream classes have a more rigorous curriculum, higher quality instruction, and lessons that engage higher-level thinking skills. Moreover, teachers place more emphasis on reasoning and inquiry skills in the more academic streams. In contrast, instruction in lower stream classrooms has been found to be more fragmented with an emphasis on isolated bits of information, instead of sustained inquiry (Hattie 2002). As such, students in lower stream settings are more likely to be subject to drill-and-practice activities that focus on memorization. This emphasis arises because there is often a perception amongst mathematics teachers that students cannot engage in problem solving and higher order thinking until they have "the basics" mastered. The inquiry focus of Ontario's curriculum, for example, is often relegated to "Problem Solving Fridays" or End of Unit tasks, instead of being the mainstay of teaching that the curriculum calls for.

Gamoran, Nystrand, Berends and LePore (1995) found that questioning patterns differ significantly in the different streams. For example, students in lower stream classes will answer five times more multiple-choice, true/false, and fill-in-the-blank style questions than those in higher streams (Gamoran and Mare 1989). Consequently, these students have much lower expectations placed on them and they are not expected to be critical thinkers (Callahan 2005). They are very likely, therefore, to spend their time reading textbooks and filling in worksheets (Gamoran et al. 1995). This lack of opportunity to learn challenging mathematics contributes to the gap in performance between streams (Balfanz and Byrnes 2006). This situation also becomes an issue of institutionalized expectations, or lack of them, the consequence of which is a demoralizing and demotivating setting for the children who end up in the lowest streams (Rubin 2008).

As might be expected, studies have also suggested that streaming has a negative effect on the attitudes, self-esteem, and motivation of students that are placed in the lower-ability groups (Berry et al. 2002; Callahan 2005). Students internalize labels, become alienated and develop anti-school attitudes that put them at risk of delinquency, dropping out, and other social problems (Ireson et al. 2002; Slavin 1990).

What Can Teachers Do?

In view of this evidence, it can reasonably be argued that the very nature of streaming can set up teachers, and their students, for low outcomes and levels of success. Notwithstanding this fact, many secondary school teachers will find themselves working within a streamed environment at some stage of their career. The question, then, is what teachers can do to optimize teaching and learning in low stream settings.

Through my own research, I conducted case studies of Ontario schools that have been extremely successful on the provincial assessment for grade 9 mathematics, in both the Applied and Academic courses. Specifically, my research was concerned with discerning the practices that are effective in supporting student achievement and success in the Applied level course. I have distilled my findings into ten powerful and promising practices that appear to have supported high levels of mathematics learning for students in low stream environments.

1. Have and hold high expectations for students in Applied classrooms.

As was discussed, sorting and sifting students into streams assumes that there are students that are more and less able to undertake study in the discipline. A by-product of this approach is that teachers, and even students, develop mindsets about what students are and are not capable of, depending on the stream in which they are placed. In some classrooms, students in lower streams are assigned less complex and low-demand tasks because the assumption is that they are not capable of higher level thinking. In the high performing schools that I studied, I found that quite the opposite was true of the classrooms that I visited. These schools were chosen as case studies because they consistently—over 5 years—performed above the provincial average, for both the Academic and Applied courses on the provincial mathematics assessment. Over this time, these schools also had a performance gap between the two courses that was smaller than the provincial gap. Given the scope of my research, I did not study low achieving schools, so I cannot comment on what may or may not be happening in those environments. What I am able to report, however, is what was common to four schools that have had outstanding success with provincial assessment results in grade 9 mathematics.

In order to determine what kind of thinking was being required of students in the high achieving, and lower-streamed, classrooms, I used a taxonomy to analyze the level of work that the students had been assigned during my classroom visits. This taxonomy, developed for The International Mathematics and Science Study (TIMSS), distinguishes the cognitive dimensions of a task by specifying the thinking processes that are needed to successfully complete it:

> The first domain, *knowing*, covers the facts, concepts, and procedures students need to know, while the second, *applying*, focuses on the ability of students to apply knowledge and conceptual understanding to solve problems or answer questions. The third domain, *reasoning*, goes beyond the solution of routine problems to encompass unfamiliar situations, complex contexts, and multi-step problems. (Grønmo et al. 2013, p. 24)

Fig. 1 Cognitive Skills. From "TIMSS 2015 Assessment Frameworks," by TIMSS & PIRLS, http://timssandpirls.bc.edu/timss2015/downloads/T15_Frameworks_Full_Book.pdf, pp. 25–27. Copyright 2013 by the International Association for the Evaluation of Educational Achievement

Cognitive Skill	Associated Verbs
Knowing	Recall Recognize Classify/Order Compute Retrieve Measure
Applying	Determine Represent/Model Implement
Reasoning	Analyze Integrate/Synthesize Evaluate Draw Conclusions Generalize Justify

These dimensions are further articulated by verbs that can be associated with mathematical tasks, as outlined in Fig. 1. I found that without exception, the level of work assigned in the case study classrooms reached the highest cognitive level of reasoning. The students were being asked to do more than carry out mathematical procedures; they were asked to apply them in novel contexts and then reason about the results.

It is also worth mentioning here that oftentimes teachers, with all of the best intentions, will scaffold more complex tasks for students that they perceive to have weak abilities. The problem with this is that by overly scaffolding these kinds of tasks, the thinking is actually being done for the students. If you think of the brain as a muscle, then it actually needs to be exercised in order to grow. If students are never given the opportunity to think, then they will not expand their capacity to think. Saying this, it is important that teachers set their students up for success by creating the conditions that will help them to engage in the thinking and subsequent learning.

An effective strategy to engage students in thinking and problem solving is to be open to a wide variety of approaches. In my study, teachers reported that students in Applied classrooms are less formulaic in their thinking and approach problems more creatively. It is very critical to play to this strength by accepting a wide variety of strategies and methods, even if they do not "look pretty" or follow conventional formats. This is actually more helpful to students in the long run because they will be better equipped to solve problems intuitively, instead of relying on formulas that they may or may not remember correctly.

The descriptions of the remaining practices will provide more direction on how to best support thinking mathematics classrooms.

2. Build confidence and efficacy for students.

Typically, students in the lower streams have lower levels of confidence and efficacy when it comes to mathematics. At the very least, the nature of the streaming process has signaled to them that they are not capable of higher levels of mathematics. My research of low stream classrooms revealed that teachers in these settings find many of their students to be disenfranchised, and even traumatized, by their prior experience of mathematics. These students express strong sentiments about not liking mathematics, not being good at mathematics, and not seeing how mathematics matters to them. For many, their history with mathematics education has not been very positive. Many of them have experienced mathematics as working in isolation on drill and practice activities to build their skills. As such, the teachers reported that one of their first goals was to help students to repair their relationship with mathematics and the damage caused by the perceived stereotype of what it says about you if you are a student in Applied mathematics. To do so, they worked to quickly foster a feeling of success and comfort in the classroom. An important strategy was to begin the course in areas that students traditionally do well in, such as measurement or geometry. This got students off to a strong start in the course and helped to build their confidence and efficacy—their belief that they were capable of doing mathematics.

It is also important to value the learning and strengths that students bring to the classroom. Recognizing that the students are not blank slates is imperative, and so too is activating their prior knowledge so that they understand what they are learning now is simply building on what they already know. It is always a good idea for teachers to peruse the prescribed curriculum for the grade that precedes the one they are teaching. This will help them to understand the mathematical content and skills that students should already have been exposed to. This, in turn, will provide insight into how new learning might be anchored by prior knowledge and experience. By way of example, one of the expectations in the Ontario Grade 9 Applied mathematics course is that students "construct table of values and graphs to represent linear relations derived from descriptions of realistic situations" (Ontario Ministry of Education 2005, p. 42). If teachers look at the curriculum that precedes grade 9, they will see that students actually began recording patterns on a table of values in grade 5 and plotting them graphically using ordered pairs in grade 6. By grade 7, students represent and describe linear growing patterns algebraically and in grade 8 they use algebraic equations to describe linear patterns. Therefore, to treat this expectation as brand new learning can be a great dis-service, and even monotonous, to the students. Using diagnostic tasks is a great way for teachers to see who has mastered certain skills, who might need support, etc. for the upcoming learning.

3. Capitalize on the social nature of adolescents.

Research has demonstrated that learning and making sense of mathematics is a social enterprise (Kilpatrick et al. 2001; Newman and Holzman 1996; Sfard et al. 1998; Spillane 2000). Therefore, using collaborative grouping in Applied classrooms is an important strategy, especially given the social nature of adolescents. Working within these supportive structures, students can together investigate mathematical concepts and solve mathematical problems. In collaborative groups, students become resources for one another's learning, allowing individuals to go beyond what they might be able to do on their own.

It is important to recognize that students in lower streams might not necessarily have experience in working this way during mathematics class. Many of them may be more accustomed to working alone, doing different mathematics than the rest of their classmates. Some of them might not be comfortable with sharing their thinking with others because they may not have a history of being called upon to do so. As such, it is important to support students in working collaboratively with one another. This involves making the classroom a supportive space where students know that it is okay to make mistakes and in fact, learn from doing so. Getting students comfortable to work in these ways will require persistence and support on the part of the teacher. A good strategy that I have observed to get students to work collaboratively is “Think—Pair—Share.” Here, after assigning a task or problem, the teacher gives students a couple of minutes of individual think time to reflect and strategize. Then, students are paired with a partner to share their thinking. This sharing gives all students an opportunity to rehearse and refine the articulation of their thinking. From here, students can then be assigned to larger groupings, if desired. With this approach, all students will come to class discussions with their own ideas, or are the very least, an idea from their partner.

The teachers that I observed through my research also embedded clear accountability structures. They would precisely articulate their expectations for the students: e.g., “There are ten minutes left and then I want to hear from each group what you found out.”

4. Use a variety of resources that engage students in active and hands-on learning.

In the classrooms that I studied, teachers did not limit the learning experience for their students to a textbook. Instead, they used a variety of resources that both met the needs and interests of their students and provided opportunities for active, hands-on, and experiential learning. A popular resource was the Ministry of Education’s TIPS for Revised Mathematics, or TIPS4RM, which is freely available at www.edugains.ca on the mathematics homepage. This resource provides three-part lessons designed to address the expectations outlined in the Ontario mathematics curriculum. These lessons can be used as is, or modified by the teacher to meet the needs of his or her students.

I found that the teachers also offered open access to mathematical thinking tools such as manipulatives, calculators, and pencils, and expected students to use them to show and explain their thinking. They also made widespread use of instructional technologies, such as interactive whiteboards, that help students to conceptualize and connect mathematical ideas.

The teacher talk around the use of manipulatives and technology positioned both as being tools for thinking, which can actually help the students to think through a problem. In essence, these tools allow students to engage in “doing mathematics” in the way that mathematicians would (OME 2005). The general sentiment was that these tools are especially important in Applied classrooms because the courses have been designed to be very “hands on” and appeal to the concept that students learn by doing. In this sense, manipulatives and other concrete materials can act both as a hook and a support to doing the mathematics. The importance of meaning-making

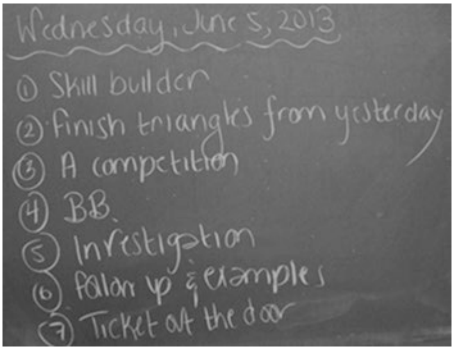

Fig. 2 Sample 75 minute lesson agenda

needs to be underscored here. Oftentimes teaching in lower streams will default to a skills approach and a focus on "the basics." When students learn skills in isolation and out of context, they are hard pressed to use those skills appropriately in any meaningful way. In order to learn to think mathematically, students need to do more than rehearse someone else's mathematics. They must be engaged in the mathematical enterprise, which involves problem solving, making conjectures, reasoning, reflecting, connecting ideas and communicating thinking.

It is also important to point out that there is not a long history of manipulative use in secondary schools. Teachers in these settings will often forgo their use altogether (Kajander and Zuke 2007; Suurtamm and Graves 2007). In this sense, new teaching graduates have an important role in trail-blazing innovative ways of learning for both students and teachers. Support for the use of manipulatives can be found on Ontario's Edugains website at http://www.edugains.ca/newsite/math/manipulative_use.html.

5. Maintain a rigorous pace.

In the case study classrooms that I studied, I was struck by the rigorous nature of the lessons that I observed. In all cases, the teachers had chunked their lessons into 10–15-min learning episodes with a short mid-lesson break where students could get up, move around, and re-focus their energies. Figure 2 illustrates the agenda for one such lesson.

This lesson design is in fact supported by brain science (Sousa 2006). Neuroscientists have discovered that our working memory is where we build, take apart, and rework ideas that will eventually be discarded or put into our long-term memory. Researchers have established that working memory is capable of handling only a few items at a time. This implies that depth of learning over breadth of learning should be considered in lesson design.

Brain research has also established that a newly learned idea is likely to fade from working memory and be discarded unless something else is done with it. Any new learning, therefore, is best retained when students have adequate opportunity to re-process it. Therefore, different experiences within a lesson will reinforce new learning, increasing the chance that it will be put into long term memory.

Researchers have also determined that the capacity to process new learning is also time bound and is about 10 to 20 min for the adolescent learner. This means that an adolescent can process an item in working memory for 10 to 20 min before fatigue or boredom sets in. In order for the adolescent to continue to focus, there needs to be a change in how he or she is dealing with the item. In teaching terms, this means the need for different learning experiences within the same lesson, as illustrated by the different activities outlined in Fig. 2.

6. Provide a rich learning environment in Applied classrooms.

A rich learning environment must attend to the emotional, as well as academic needs of the students. As previously discussed, it is important that the classroom be a supportive space that is respectful of all learners. It is important that all thinking is valued and that all students feel that they have a voice. Again, this may require persistence on the part of the teacher who may, for example, need to help students understand how to respectfully disagree with one another by offering an opposing line of thinking. For more on how to build a “Math Talk” community, refer to Bruce’s research monograph (2007) on student interaction in the mathematics classroom.

Expectations should be set high and clearly communicated to students in applied classrooms. The use of a lesson agenda, as illustrated in Fig. 2, is a great way to inform students of what will be happening during the lesson and also signals high expectations. Providing frequent prompts is also helpful: e.g., “In five minutes I want to hear from each group what strategy you used to solve the problem.”

It is also important that the classroom space reflect that this is a place of learning. One observation that I have made in Ontario schools is that there is often very little posted in secondary classrooms. Teachers tell me that this is because they regularly have to share classroom spaces; teachers will not necessarily have their own classrooms, and instead move from room to room throughout the day. This practice is not very supportive of students, however, especially when they are coming from very rich classroom spaces in elementary school. Having established this, the case study classroom spaces that I visited for my study were not typical.

In these classrooms, a clear account of the mathematics content that had been covered during the course was evident by just looking at the walls. There were a variety of teacher, student, and co-created visuals including charts and word walls, that provided both an anchor to and record of student learning. These records of learning can be extremely helpful to students who may have poor organizational and note taking skills because they can refer back to them when needed or prompted. Having student work posted is also beneficial because it allows students to see the variety of ways in which others approached a problem.

7. Skill building in context.

A common practice in mathematics classrooms is to begin the school year or semester with a review of material that was covered in the previous grade or course. Some teachers will devote several weeks to this review. Teachers in my study did not favour this practice. Instead, they preferred to work review of skills into their lessons, on an as needed basis. Figure 2 demonstrates, for example, how one teacher began her lesson with a skill building activity. On this particular day, students practiced the skill of mentally multiplying. For example, 18×6 is the same as $2 \times 9 \times 6$ or 2×54, which is equal to 108. This skill would come in handy later in the lesson when students were conducting an investigation of the sum of the interior angles of a polygon [$S = (n-2) \times 180°$]. In this way, practicing the skill was purposeful, relevant, and seamless to instruction.

8. Provide samples of what good work looks like and engage students in self- and peer-assessment.

There is more and more research that demonstrates that self-regulation and the monitoring of one's own learning has a huge impact on student achievement (OME 2010). When students understand the criteria for success, they are better positioned to actually be successful. Therefore, developing, or co-constructing success criteria can be an important strategy in Applied classrooms. It is also important to help students monitor their own progress in meeting the criteria by having them reflect on their own work to assess their progress. Providing models of good work can facilitate this process. Similarly, when students help peers to assess their work, they become more adept at articulating the criteria and operationalizing it in their own work.

When getting started, a teacher may want to look to outside sources for examples of criteria and student exemplars, such as those based on Ontario's provincial assessment, available at http://www.eqao.com/en/assessments/grade-9-math/Pages/example-assessment-materials-2015.aspx. Over time, though, teachers should strive to collect their own student samples, based on tasks that can be used again in subsequent years.

9. Provide students with frequent, oral, and descriptive feedback.

In my research of Applied classrooms, I heard repeatedly from the teachers that it is important to monitor the progress of each and every student and to connect with students on an individual basis to provide them with oral and descriptive feedback that can move their learning forward. Teachers would accomplish this in a variety of ways. For example, many of the teachers used some kind of exit strategy such as a "Ticket out the door" where students would independently answer a question related to the day's lesson. This allowed teachers to immediately target those students who may be having difficulty by providing remediation during the next lesson, or facilitating peer support by pairing someone who was struggling with a concept with someone who had mastered it.

A really important strategy for all of the case study teachers was to monitor students when they were at work during the classroom activities. As students were involved in investigations, for instance, the teacher would move about the room engaging in conversations and observations of students as they were at work. Interacting with students in this way gives teachers a much better sense of what students are thinking than can be surmised by simply looking at a piece of written work. It is during these kinds of interactions that the teacher can gather more qualitative descriptions of what it is that students can do, where they struggle, and what might be next steps for their learning.

10. Foster productive dispositions around mathematics by sharing the wonder and beauty of the discipline.

As was discussed previously, oftentimes students in Applied settings have been traumatized by their experience of mathematics. Couple this with the damage caused by the stereotype associated with lower stream classes, and it is not hard to understand why students in Applied classrooms may not come to the class with the most positive of attitudes. It is very important to be mindful of this and to understand that an essential part of the work with these students will be to help them to build a positive relationship with mathematics and to begin to see themselves as capable and competent. The strategies discussed thus far will help.

Sharing the love and joy of mathematics is also imperative. Mathematics is an elegant, creative, and beautiful enterprise and too often students do not witness this in their experience of school mathematics. Bringing interesting mathematics puzzles, anecdotes, and stories of mathematical interest and application to the students helps them to develop a more robust appreciation of what mathematics is, the often-compelling history behind it, and the importance of it to daily life and living. Enthusiasm is contagious and when teachers are truly passionate about their discipline, students perk up and take notice.

An example might be the illustration of how the Fibonacci sequence and Golden Ratio, are reflected in nature. Doing a simple internet search will result in many examples that can be shared with students, such as flower petals; pinecones; fruits and vegetables such as apples, cauliflower, and pineapples; tree branches; galaxies, animal bodies; and hurricanes. Good sources for these kinds of materials are the Illuminations page on the NCTM website at illuminations@nctm.org or the enriching mathematics activities found on the NRICH website at http://nrich.maths.org/frontpage. Investigating famous mathematics thinkers and writers such as Martin Gardner will also yield many great mathematical ideas to share with students.

In closing, it is vital that teachers think very carefully about the context in which he or she is teaching. An important part of this requires understanding the learner and what we can do to best support them in our teaching practice.

References

Balfanz, R., & Byrnes, V. (2006). Closing the mathematics achievement gap in high-poverty middle schools: Enablers and constraints. *Journal of Education for Students Placed at Risk, 11*(2), 143–159.

Berry, M., White, H., & Foster, P. (2002). Streaming reviewed: Some reflections from an inner-city, multi-ethnic primary school. In O. McNamara (Ed.), *Becoming an evidence-based practitioner: A framework for teacher-researchers* (pp. 141–151). New York: Routledge Falmer.

Boaler, J., Wiliam, D., & Brown, M. (2000). Students' experiences of ability grouping—Disaffection, polarisation, and the construction of failure. *British Educational Research Journal, 26*(5), 631–648.

Bruce, C. D. (2007). Student interaction in the math classroom: Stealing ideas or building understanding. *What Works? Research into Practice. Research Monograph # 1*. Retrieved from http://www.edu.gov.on.ca/eng/literacynumeracy/inspire/research/Bruce.pdf.
Callahan, R. (2005). Tracking and high school English learners: Limiting opportunity to learn. *American Educational Research Journal, 42*, 305–328.
Education Quality and Accountability Office. (2012). *An analysis of questionnaire and contextual data for grade 9 students in the Academic and Applied mathematics courses*. Toronto: EQAO Retrieved from http://www.eqao.com/Research/pdf/E/Analysis_Questionnaire_ContextualDataG9_en.pdf.
Gamoran, A. (2002). *Standards, inequality, and ability grouping in schools* (CES Briefing No. 25). Edinburgh: Centre for Educational Sociology, University of Edinburgh.
Gamoran, A., & Berends, M. (1987). The effects of stratification in secondary schools: Synthesis of survey and ethnographic research. *Review of Educational Research, 57*(4), 415–435.
Gamoran, A., & Mare, R. D. (1989). Secondary school tracking and educational inequality: Compensation, reinforcement or neutrality? *American Journal of Sociology, 94*(3), 1146–1183.
Gamoran, A., Nystrand, M., Berends, M., & LePore, P. C. (1995). An organizational analysis of the effects of ability grouping. *American Educational Research Journal, 32*(4), 687–715.
Grønmo, L. S., Lindquist, M., Arora, A., & Mullis, I. V. S. (2013). TIMSS 2015 mathematics framework. In I. V. S. Mullis & M. O. Martin (Eds.), *TIMSS 2015 assessment frameworks* (pp. 11–27). Chestnut Hill: TIMSS & PIRLS International Study Center, Boston College.
Hamlin, D., & Cameron, D. (2015). *Applied or academic: High impact decisions for Ontario students*. Toronto: People for Education.
Hattie, J. A. C. (2002). Classroom composition and peer effects. *International Journal of Educational Research, 37*, 449–481.
Ireson, J., Hallam, S., Hack, S., Clark, H., & Plewis, I. (2002). Ability grouping in English secondary schools: Effects on attainment in English, mathematics and science. *Educational Research and Evaluation: An International Journal on Theory and Practice, 8*(3), 299–318.
Kajander, A., & Zuke, C. (2007). *Factors contributing to success for intermediate students of mathematics: The needs of the teachers and the characteristics and needs of students at-risk*. Thunder Bay: Northern Ontario Educational Leaders.
Kilpatrick, J., Swafford, J., & Findell, B. (Eds.). (2001). *Adding it up: Helping children learn mathematics*. Washington, DC: National Academy Press.
Newman, F., & Holzman, L. (1996). *Unscientific psychology*. Westport: Praeger.
Oakes, J. (1982). The reproduction of inequity: The content of secondary school tracking. *Urban Review, 14*(2), 107–120.
Oakes, J. (1985). *Keeping track: How schools structure inequality*. New Haven: Yale University Press.
Oakes, J. (1986). Keeping track. Part 1: The policy and practice of curriculum inequality. *The Phi Delta Kappan, 68*(1), 12–18.
Ontario Ministry of Education. (2005). *The Ontario curriculum grades 9 and 10: Mathematics* (Rev ed.). Toronto: Queen's Printer for Ontario. Retrieved from http://www.edu.gov.on.ca/eng/curriculum/secondary/math910curr.pdf
Ontario Ministry of Education. (2010). *Growing success: Assessment, evaluation and reporting in Ontario schools: First edition, covering grades 1 to 12*. Toronto: Queen's Printer for Ontario.
People for Education. (2013). The trouble with course choices in Ontario high schools. *Should low income = high Applied?* Retrieved from http://www.peopleforeducation.ca/wp-content/uploads/2013/04/trouble-with-course-choices-in-high-school-2013.pdf.
Rubin, B. (2008). Detracking in context: How local constructions of ability complicate equity-geared reform. *Teachers College Record, 110*(3), 646–699.
Sfard, A., Nesher, P., Streefland, L., Cobb, P., & Mason, J. (1998). Learning mathematics through conversation: Is it as good as they say? *For the Learning of Mathematics, 18*(1), 41–51.
Slavin, R. E. (1990). Achievement effects of ability grouping in secondary schools: A best evidence synthesis. *Review of Educational Research, 60*(3), 471–499.

Spillane, J. P. (2000). Cognition and policy implementation: District policymakers and the reform of mathematics education. *Cognition and Instruction, 18*(2), 141–179.
Sousa, D. A. (2006). *How the brain learns mathematics*. Thousand Oaks: Corwin Press.
Suurtamm, C., & Graves, B. (2007). *Curriculum implementation intermediate math: Research report executive summary*. Ottawa: University of Ottawa, Faculty of Education.
Talbert, J. E. (1995). Boundaries of teachers' professional communities in U. S. High Schools: Power and precariousness of the Subject Department. In L. Siskin & J. W. Little (Eds.), *The subjects in question: Departmental organization and the high school* (pp. 68–94). New York: Teachers College Press.
Van Houtte, M. (2004). Tracking effects on school achievement: A quantitative explanation in terms of the academic culture of school staff. *American Journal of Education, 110*(4), 354–388.

Additional Suggestions for Further Reading

Macaulay, A. (2015). *Effective practices in Grade 9 Applied mathematics*. Unpublished doctoral dissertation, Ontario Institute for Studies in Education. Toronto: University of Toronto. Retrieved from https://tspace.library.utoronto.ca/handle/1807/71461.
Ontario Ministry of Education. (2016). *Edugains: Math home page*. Retrieved from http://www.edugains.ca/newsite/math/index.html.

Reflecting on Good Mathematics Teaching: Knowing, Nurturing, Noticing

Susan Oesterle

The more you know, the more you know you don't know. – Aristotle

Abstract What does it mean to be a "good" mathematics teacher? Ultimately, the answer to this complex question depends on the goals of mathematics education and on one's beliefs about teaching and learning. This chapter describes the evolution of one mathematics teacher's views as practice, research, and changing curricula have influenced her understanding. She reflects on how attention to three particular aspects can support effective mathematics teaching: *knowing*, not merely the mathematics content, but developing mathematics-knowledge-for-teaching (Shulman 1986; Ball, Thames and Phelps 2008); *noticing*, as applied to the subject matter, the students, and oneself as teacher (Mason 1998, 2002); and *nurturing* habits of mind (Cuoco, Goldenberg and Mark 1996; Lim and Selden 2009), in particular fostering a growth mindset in students (Dweck 2006; Boaler 2015). She also advocates for the importance of mathematics teachers holding a growth mindset with respect to their own teaching.

Keywords Teacher effectiveness · Mathematics-knowledge-for-teaching · Teacher noticing · Mathematical habits of mind · Growth mindset · Reflective practice

My entire career I have strived to be a good mathematics teacher. As a student, I enjoyed mathematics tremendously—I loved the structure, the beauty, the utility, and the challenge. I also enjoyed teaching—explaining ideas to others and feeling satisfaction as they experienced their "aha" moments. Becoming a mathematics teacher was a natural step for me. In the early days of my teacher education, my

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naïve view was that these attributes were all I would need to be good at it: a love for and knowledge of the mathematics that I would teach, and an ability to communicate or explain the mathematical ideas to others. While these were a good start, they were just the tip of the iceberg, with so much more to be found below the surface. Over the years, through a combination of experience and academic research, I have come to appreciate just how much more there is to the craft of teaching mathematics. In particular, there is more to know than the mathematics itself, and more to nurturing than sharing my love of math.

What Is "Good" Mathematics Teaching?

Not surprisingly, when I started to teach, I taught the way that I had been taught. My own high school mathematics classes had followed a very predictable, structured format:

1. possibly some review of previous topics;
2. introduction of a new topic by the teacher at the board which included definitions, key ideas, and worked examples; followed by
3. seat work for the students, typically a list of exercises, assigned from the text book.

Depending on the teacher (or the day), the seat work was sometimes done independently or sometimes collaboratively with others seated nearby. This is what I was used to. This is what I liked as a student. And this is what was expected of me as a student teacher.

What did it mean to me to be a "good" mathematics teacher at that time? I felt satisfied that I was doing a "good" job if my explanations and demonstrations were clear enough so that the students could experience success in answering the assigned questions, both in the short term (in the classroom that day) and in the longer term (on an in-class test). If the students had fun with the assigned questions, that was ideal, but it was a bonus.

On the surface, there was nothing wrong with this—but the problem was that it was very superficial. Over time I began to wonder: was I teaching mathematics or was I teaching students how to memorise and follow procedures? Certainly the approach that I (and many others) were taking allowed students to be successful (i.e., to pass) their mathematics course through memorisation and regurgitation. Years later as a post-secondary instructor, time and time again I would encounter students who had done well in high school but were unable to use and apply rules and processes in new contexts; or students who, unsuccessfully, still tried to memorise procedures and apply them blindly in situations where deeper analysis was needed. These students had a difficult time adjusting to college-level mathematics. At the same time, other students did make that transition successfully—having been taught definitions, rules and procedures, at some point they learned to use this knowledge flexibly, to apply it appropriately in novel contexts, and to build on it.

Was it how they were taught that made the difference? Was it natural ability? What was I doing (or not doing) to promote that kind of mathematics learning? What did I need to know and do to be a "good" mathematics teacher? Indeed, what is "good" mathematics teaching?

As with all interesting questions, there are no simple nor definitive answers. One's understanding of what constitutes "good" mathematics teaching will depend heavily on one's beliefs, particularly beliefs about mathematics and about teaching and learning. Space precludes a detailed discussion of the various perspectives and their implications, however I will touch on some perspectives that I have found useful in framing my ever-evolving understanding.

With respect to mathematics, I have become sensitive to a distinction between mathematics and school mathematics, where the latter can sometimes be conceived of as a finite body of knowledge, consisting of facts, procedures and methods for solving particular problems. School mathematics can also include what might be called "every-day mathematics": practical numeric and geometric content and skills that are intended to support real-life problem solving and decision making. However, the mathematics I experienced as a graduate student went beyond this—it included unsolved problems, brought out deep and powerful connections between concepts, and demanded both rigorous justification as well as flexibility, creativity and insight. My understanding of "good" mathematics teaching, even at the pre-college level, has come to include approaches that convey a fuller scope of what mathematics is, beyond the definitions and the routines, to include genuine problem solving, and further, to provide at least a glimpse of the structure, the beauty and the mysteries of the subject of mathematics. Of course most of our students will not pursue more advanced mathematics, but there is much to be gained by opening their eyes to the possibilities.

With respect to teaching and learning, my initial approach when I started teaching was very much based on a transmission model: the view that knowledge can be transferred from the mind of the teacher to the mind of the student. Since that time my perspective has been deeply enriched by other models, including *constructivism* (e.g., Pirie and Kieren 1992) which recognises that the learner constructs his/her own knowledge, using prior knowledge and understandings as a scaffold for new information and new connections. Further, *social constructivism* highlights the role of community and context in the construction of knowledge, while the *emergent perspective* (Cobb and Yackel 1996) incorporates both cognitive and social factors in its analysis of learning. There are many other theories, of course, which can contribute to our understanding of teaching and learning. In turn, this understanding should inform and shape the pedagogical choices we make in our efforts to be "good" mathematics teachers.

My personal shift toward a more emergent view means that teaching mathematics is no longer merely about covering the content and crafting clear explanations; it is about creating and providing classroom experiences and activities that provide an opportunity for learners to construct their own understandings and build a solid scaffold for future mathematics learning. To be a "good" mathematics teacher, one needs to be "effective" and the "effect" should be (at least in part) a long-term positive change in learners that not only increases their knowledge but gives them a stronger base upon which to build. What does it take to be such a teacher?

What Does a "Good" Mathematics Teacher Need to Know?

Knowing the mathematics is essential, but *just* knowing the mathematics (the definitions, the facts, the processes) that are being taught is not enough. In his seminal article, Shulman (1986) called this *subject content knowledge*, but distinguished it from knowledge of the curriculum and resources (curricular knowledge), and most importantly, from *pedagogical content knowledge* (PCK). PCK specifically addresses "the ways of representing and formulating the subject that make it comprehensible to others" (p. 9). Ball, Thames, and Phelps (2008) further parsed PCK to highlight and distinguish between "knowledge of content and students" (KCS) and "knowledge of content and teaching" (KCT) (p. 389). With KCS, the teacher knows and understands the relationship between the students and the mathematics. He/she can anticipate students' difficulties, is aware of common misconceptions, and knows what will interest and motivate them. With KCT, the teacher is able to sequence content and examples effectively, can call on and apply a variety of representations as needed, and can respond to student comments and questions appropriately in-the-moment.

Another useful description of what mathematics teachers need to know was presented by Ma (1999), in her comparative study of Chinese and US teachers. She identified a "profound understanding of fundamental mathematics" (PUFM) (p. xxiv) as the distinguishing characteristic that allowed the Chinese teachers to consistently out-perform the American teachers on specific teaching tasks. This included specific mathematics content knowledge, but went beyond to include "awareness of the conceptual structure and basic attitudes of mathematics inherent in elementary mathematics and the ability to provide a foundation for that conceptual structure and instil those basic attitudes in students" (p. 124).

Simply knowing the mathematics is not enough—to be effective, teachers need a deep and connected knowledge of mathematics, of mathematics and teaching, and of mathematics and their students. We need to know more than the content—and just as importantly, we need to teach more than the content.

More than Content—Nurturing Habits of Mind

Ma's (1999) description of PUFM talks about "basic attitudes of mathematics" (p. xxiv). Of course, as teachers we cannot forget about the affective component. We want our students to have positive attitudes towards mathematics, to have self-confidence, to appreciate its power and its beauty and ideally to enjoy it. But there is more to this than one might think: certainly more than simply setting students up for success, engaging in aesthetically pleasing activities or playing games in math class. Over the last two decades there has been increasing attention drawn to the notion of "mathematical habits of mind."

One of the first references to this in the mathematics education literature was in 1996, by Cuoco, Goldenberg, and Mark. In it, their motivation was not the affective aspects of mathematics teaching, rather, they were questioning the appropriateness of defining the mathematics curriculum (as has been the custom) around specific lists of topics. They pointed out the impossibility of predicting exactly what mathematics future graduates will need, given how quickly our modern world is changing. They argued that it would make more sense to build a curriculum around "mathematical habits of mind", by which they meant the authentic ways of engaging with mathematics characteristic of mathematicians.

> A curriculum organized around habits of mind tries to close the gap between what the users and makers of mathematics do and what they say….[It] lets students in on the process of creating, inventing, conjecturing and experimenting…. It is a curriculum that encourages false starts, calculations, experiments, and special cases. (p. 376)

In their characterisation, they identified instantiating with examples, generalizing/abstracting, thinking in terms of functions, mixing deduction and experiment, and "pushing the language" as examples. Since that time many others have developed these ideas further. Lim and Selden (2009) provide a good overview. Informed by these further elaborations, I have come to see mathematical habits of mind as encompassing "those ways of thinking and those inclinations and beliefs about how to think that are typically useful and productive in the exploration, creation, and use of mathematics" (Oesterle et al. 2016, p. 54). This has both a cognitive and an affective aspect.

Cognitive aspects of mathematical habits of mind include: noticing patterns, making and investigating mathematical conjectures, and developing and evaluating mathematical arguments and proofs (cf., NCTM 2000). Ultimately these highlight the importance of sense-making. Affective aspects include curiosity, a sense of wonder, perseverance, willingness to play and take risks, and a positive attitude towards making mistakes. The cognitive and affective are closely related. Perseverance and self-efficacy are supported by possessing strategies for dealing with being "stuck" when tackling a difficult problem. Mason, Burton, and Stacey (2010) identify two particular strategies, "specialising" (reducing a problem to a simpler case) or "generalising" (looking for connections to related problems), both of which are typical and productive approaches employed in solving problems.

A teaching approach that fosters mathematical habits of mind aims to provide students with strategies for solving problems and learning mathematics that are transferable across content topics and contexts. These habits of mind should support students in building the desired deep connected understanding of the mathematics content and open their experience to *authentic* mathematics. While opportunities to engage in mathematics the way that mathematicians do (e.g., questioning, conjecturing, persisting, verifying, proving) can be found within common school mathematics topics, occasional extra-curricular digressions can help promote mathematical habits of mind. For example, activities involving fractals or topics in number theory can provide rich ground for exploration, offering insights into the beauty and nature of mathematics. Unsolved problems in number theory, in particular, are accessible to younger students and can help debunk pervasive views that mathematics knowledge is finite and complete.

Such digressions are easier to justify when the formal curriculum explicitly identifies development of mathematical habits of mind as an objective. This seems to be increasingly the case in modern curricula. As far back as 2000, in their *Principles and standards for school mathematics,* the National Council for Teachers of Mathematics (NCTM 2000) made a distinction between the mathematics content that students are expected to learn ("Content Standards") and what they called "Process Standards", which included:

- problem solving: including "ways of thinking, habits of persistence and curiosity, and confidence in unfamiliar situations" (p. 52);
- reasoning and proof: including noticing "patterns, structure, or regularities in both real-world situations and symbolic objects" (p. 56), asking "if those patterns are accidental or if they occur for a reason" (p. 56), making and investigating mathematical conjectures, and developing and evaluating mathematical arguments and proofs;
- communication: including learning to be "clear and convincing" (p. 60), and developing "a language for expressing mathematical ideas and an appreciation of the need for precision" (p. 60); and
- representations: including acquiring "a set of tools that significantly expand their capacity to think mathematically" (p. 67), and to be able to "use representations to model and interpret physical, social, and mathematical phenomena" (p. 70).

The Western and Northern Canadian Protocol (2006) mathematics curriculum, cooperatively developed by seven provinces and territories incorporated some of these ideas, providing a list and description of more general proficiencies that students were meant to develop (e.g., communication, reasoning, visualisation) alongside learning the particular content outlined in pages of Key Performance Indicators (KPIs). However, the list of proficiencies appeared only in the front-matter to the curriculum package, and as a result were missed by most teachers, who went straight to the KPIs. The tradition of content-focussed instruction was and is difficult to upset. Recently, in British Columbia, a new curriculum has been implemented, which raises the profile of the more general proficiencies even further. In this version, mathematical habits of mind have been pulled to the forefront (BC Ministry of Education 2013, 2016).

This has implications in terms of classroom teaching. To facilitate developing mathematical habits of mind, "good" mathematics teaching will require providing opportunities for students to:

- make sense of mathematics: to explore, to make and challenge conjectures, to notice what is the same and what is different, to visualize;
- reason and prove: to use logic and reasoning based on shared assumptions and understandings to justify thinking, both to themselves and to others;
- explore multiple solutions and representations. (Oesterle et al. 2016, p. 59)

Allowing opportunities for students to work with peers provides a natural situation that requires communication and clarification of definitions, notation, and conceptual understanding. At the same time, this can present challenges: students can lead each other astray, certain students can dominate discussions to the detriment of others, fear of making mistakes can paralyse progress. Teachers need to be aware of and help to build classroom norms (Yackel and Rasmussen 2002) that will support collaborative learning and thinking. Liljedahl (2012) suggests specific strategies that can facilitate this, including making relatively simple changes to classroom layout. He also makes recommendations for effective grouping strategies and approaches to handling student questions.

From my own observations, successful navigation of the transition from high school to college/university mathematics would be facilitated by some particular habits of mind, including

- expecting the mathematics to make sense, rather than simply accepting a procedure or method that appears to be arbitrary;
- willingness to begin on a problem, even when the solution path isn't evident (linked to confidence and persistence);
- having a sense of when to ask for help or talk to someone or switch tracks—while persistence is important, getting locked in to an unproductive approach for too long can be very ineffective and inefficient; and
- having an ability to move between the concrete and the abstract and be aware of this shift.

Particular approaches that may facilitate these include, providing mathematical experiences that allow for sense-building, allowing opportunities for non-routine problem solving, and explicit teaching of metacognition.

Nurturing a "Growth Mindset"

Hand in hand with helping our students develop mathematical habits of mind, and in fact essential to it, is supporting them in adopting a 'growth mindset' (Boaler 2015; Dweck 2006) with respect to the learning of mathematics. Students with a growth mindset see doing well in mathematics as the result of hard work and effort, as opposed to those with a more fixed mindset, who believe that either you have a gift for mathematics or you don't. Mindset plays an important role in students' response to making mistakes and also in perseverance. With a fixed mindset, errors are signals that you are not good at math, and that persistence is futile. With a growth mindset, making mistakes is an opportunity for learning and a signal that you need to think further/deeper or try harder.

There are some simple things that teachers can do to help students develop or keep a growth mindset. Starting with a growth mindset oneself is helpful, as is attending to how one issues praise. Saying "Aren't you smart?" gives credit to natural ability, while "Good work!" puts the focus on the effort. Sharing alternative solution strategies can demonstrate that there are different valid ways to solve problems. This can be facilitated by simply asking the right types of questions, ones "that allow for multiple solution paths" (Jacobbe and Millman 2009, p. 299). Explicitly modelling and discussing reflection and metacognition, productive habits of mind in their own right (Mason et al. 2010), gives students opportunities to experience and notice positive results of their own and others' choices and efforts.

Becoming a "Good" Mathematics Teacher

As important as supporting a growth mindset in students with respect to their learning of mathematics, is having a growth mindset with respect to one's own teaching of mathematics. Kilpatrick, Swafford and Findel (2001) call this a "productive disposition": perceiving oneself as in control of one's own learning, seeing oneself as a lifelong learner whose learning is "generative", built through the study of curriculum materials, from analyzing practice and from interactions with students (p. 384). The emphasis is less on *being* a "good" mathematics teacher, and more on *becoming* a "good" mathematics teacher—a life-long endeavour.

One simple yet helpful notion I have encountered with respect to supporting my own reflective practice is that of "noticing". The importance of noticing is not a new idea—considerations of noticing go back to Dewey, although more recently "teacher noticing" has become an area of education research in its own right (e.g., Sherin et al. 2011). John Mason's *Researching your own practice: The discipline of noticing* (2002) offers a practical guide to enhancing one's sensitivity to noticing in order to be aware of opportunities, make informed choices, and transform one's teaching practice over time.

In the context of the mathematics classroom, there are three (not mutually exclusive) key areas that I have found to be particularly useful to attend to: the students, the mathematics, and the students-and-the-mathematics. The first almost goes without saying—the students themselves pull one's attention. To respond appropriately, to provide opportunities for learning, it is essential to notice their actions and interactions, their needs in the moment. Noticing what motivates them, what frustrates them, what challenges them and inspires them, and then responding to those observations through changes in practice can contribute to building one's Knowledge of Content and Students.

Noticing the mathematics also seems to be inevitable, but it can be taken for granted. Concepts and methods learned decades before can become routine and automatic. We can forget what made a concept difficult to grasp or why a particular notation might be confusing. The challenge in teaching mathematics is to notice this, and to be able to "unpack" (Ball and Bass 2000) the ideas for new learners. To do this one needs to slow down and re-notice that which has become taken for granted. More than this, it can be enlightening to stay alert and open to seeing new connections, even within mathematics content that one thinks one knows well. It has never ceased to amaze me how I can continue to learn new mathematics, often as a result of a student question. Attending to the mathematics, looking for the connections, pushing for one's own deeper understanding of the concepts helps build the "profound understanding of mathematics" identified by Ma (1999).

Noticing the students-and-the-mathematics is about being aware of what students are or are not attending to. They may not see what the teacher sees, or it may not mean the same thing to them. An example of this that has been studied is the use of the equals sign. Students often interpret it to mean "calculate" rather than as a symbol establishing equivalence (Knuth et al. 2006). This leads to difficulties in completing open sentences like: $6 + 3 = \square + 4$ (e.g., writing 9 as the "solution"), and contributes to later difficulties in distinguishing between expressions and equations. In algebra, when asked to factor an expression, students will sometimes go too far, writing $x^2 + 4x + 3 = (x + 3)(x + 1) = x = -3$ or -1, expecting that the equals sign demands a numerical response. Being or becoming aware of misconceptions and attending to what students know and what prior knowledge they are drawing on to understand new concepts builds Knowledge of Content and Teaching (KCT), supporting better choices for how to introduce and develop mathematical ideas.

Rising to the Challenge

Over the years, my understanding of what constitutes "good" mathematics teaching has changed considerably. It has been shaped through reflection on my own practice and observations of my students, but it has also been reinforced by new curricula and mathematics education research. This new understanding has demanded changes in my practice—but with change comes challenge. It can be difficult to give up teaching the way we were taught, especially if it worked for us. For experienced teachers, it can be difficult to alter the way we have been teaching, especially if it seems to "work" for at least some significant portion of our students. But generations of students who become adults who "hate math" and pervasive avoidance of mathematics in North America, suggests that what we have been doing is not "working."

Despite the difficulties, it is critical that we as mathematics teachers take steps to rise to the challenge. This means continuing to learn mathematics, to build our own deep and connected understanding of the subject, but also to build our knowledge of mathematics-and-students and our knowledge of mathematics-and-teaching to inform our pedagogical choices. It means keeping the big picture in mind, nurturing mathematical habits of mind and a growth mindset in our students. It means 'noticing' the mathematics, the students, the students-and-the mathematics, and of course, noticing our own practice in order to make positive change. Resources to support teachers abound—there is not shortage of advice, books, professional development opportunities, and research studies. So much so, that it can seem overwhelming at times. The key is to remember, above all, that "becoming" a good mathematics teacher is a life-long endeavour and that even the smallest steps toward the ultimate goal of building students' capacity to learn, use, and appreciate mathematics is a step in the right direction.

References

Ball, D. L., & Bass, H. (2000). Interweaving content and pedagogy in teaching and learning to teach: Knowing and using mathematics. In J. Boaler (Ed.), *Multiple perspectives on the teaching and learning of mathematics* (pp. 83–104). Westport: Ablex.

Ball, D. L., Thames, M. H., & Phelps, G. (2008). Content knowledge for teaching: What makes it special? *Journal of Teacher Education, 59*(5), 389–407.

Boaler, J. (2015). *Mathematical mindsets: Unleashing students' potential through creative math, inspiring messages and innovative teaching*. San Francisco: Wiley.

British Columbia Ministry of Education. (2013). *Transforming curriculum and assessment – Mathematics*. Retrieved from https://curriculum.gov.bc.ca/curriculum/Mathematics

British Columbia Ministry of Education. (2016). *Mathematics: Introduction*. Retrieved from https://curriculum.gov.bc.ca/curriculum/mathematics/introduction

Cobb, P., & Yackel, E. (1996). Constructivist, emergent, and sociocultural perspectives in the context of developmental research. *Educational Psychologist, 31*, 175–190.

Cuoco, A., Goldenberg, E. P., & Mark, J. (1996). Habits of mind: An organizing principle for a mathematics curriculum. *Journal of Mathematical Behavior, 14*(4), 375–402.

Dweck, C. (2006). *Mindset: The new psychology of success*. New York: Ballantine Books.

Jacobbe, T., & Millman, R. S. (2009). Mathematical habits of the mind for preservice teachers. *School Science and Mathematics, 109*(5), 298–302.

Kilpatrick, J., Swafford, J., & Findell, B. (Eds.). (2001). *Adding it up: Helping children learn mathematics*. Washington, DC: National Academies Press.

Knuth, E. J., Stephens, A. C., McNeil, N. M., & Alibali, M. W. (2006). Does understanding the equal sign matter? Evidence from solving equations. *Journal for Research in Mathematics Education, 36*, 297–312.

Liljedahl, P. (2012). Building thinking classrooms: Conditions for problem solving. In P. Felmer, E. Pekhonen, & J. Kilpatrick (Eds.), *Posing and solving mathematical problems: Advances and new perspectives* (pp. 361–386). New York: Springer.

Lim, K., & Selden, A. (2009). Mathematical habits of mind. In S. L. Swars, D. W. Stinson, & S. Lemons-Smith (Eds.), *Proceedings of the 31st annual meeting of the North American chapter of the International Group for the Psychology of Mathematics Education* (Vol. 5, pp. 1576–1583). Atlanta: Georgia State University.

Ma, L. (1999). *Knowing and teaching elementary mathematics: Teachers' understanding of fundamental mathematics in China and the United States*. Mahwah: Lawrence Erlbaum Associates.

Mason, J. (1998). Enabling teachers to be real teachers: Necessary levels of awareness and structure of attention. *Journal of Mathematics Teacher Education, 1*, 243–267.

Mason, J. (2002). *Researching your own practice: The discipline of noticing*. New York: Routledge.

Mason, J., Burton, L., & Stacey, K. (2010). *Thinking mathematically* (2nd ed.). London: Pearson.

National Council of Teachers of Mathematics. (2000). *Principles and standards of school mathematics*. Reston: NCTM.

Oesterle, S., Olson, M., Hwang, S., Slovin, H., Olson, J., Murawska, J., et al. (2016). Mathematical habits of mind. In L. C. Hart, S. Oesterle, S. Swars Auslander, & A. Kajander (Eds.), *The mathematics education of elementary teachers: Issues and strategies for content courses* (pp. 51–85). Charlotte: Information Age Publishing.

Pirie, S., & Kieren, T. (1992). Creating constructivist environments and constructing creative mathematics. *Educational Studies in Mathematics, 23*, 505–528.

Shulman, L. S. (1986). Those who understand: Knowledge growth in teaching. *Educational Researcher, 15*(2), 4–14.

Sherin, M. G., Jacobs, V. R., & Philipp, R. A. (Eds.). (2011). *Mathematics teacher noticing: Seeing through teachers' eyes*. New York: Routledge.

The Western & Northern Canadian Protocol. (2006). *The common curriculum framework for K-9 mathematics*. Retrieved from https://www.wncp.ca/media/38765/ccfkto9.pdf

Yackel, E., & Rasmussen, C. (2002). Beliefs and norms in the mathematics classroom. In G. Leder, E. Pehkonen, & G. Törner (Eds.), *Beliefs: A hidden variable in mathematics education?* (pp. 313–330). London: Kluwer Academic Publishing.

Additional Suggestions for Further Reading

Hewitt, D. (1999). Arbitrary and necessary part I: A way of viewing the mathematics curriculum. *For the Learning of Mathematics, 19*(3), 2–9.

Mathematical Mindsets for the Teaching and Learning of Mathematics

Jamie S. Pyper

Abstract The purpose of this chapter is to invite pre-service teachers to think more broadly, and more closely, at what knowing mathematics could be, and about mathematical mindsets and mathematical habits of mind. A mindset is a way of thinking, or a way of organizing your thoughts into a coherent manner. In the secondary school classroom, mathematical thinking is present in the teachers' lessons, and is the desired student outcome. While describing and explaining what makes up one's mathematical thinking may be challenging, it is necessary in the process of the teaching and learning of mathematics. What does it mean to know mathematics, and what underlies mathematical thinking? This chapter provides examples of what it might mean to know mathematics, and explores mathematical mindsets and how knowing mathematics can be supported by mathematical habits of mind.

Keywords Pre-service teachers · Mathematics · Secondary · Mindsets · Habits of mind · Mathematical thinking · Teaching · Learning

The purpose of this chapter is to invite pre-service teachers to think more broadly, and more closely, at what knowing mathematics could be, and about mathematical mindsets and mathematical habits of mind. Not only is this valuable for teachers' professional practice, but also for insight into students' thinking—how students can know mathematics, how students develop a mathematical mindset, and how mathematical habits of mind are important to students' learning.

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Thinking Mathematically and Mathematical Mindsets

A mindset is a way of thinking, or a way of organizing your thoughts into a coherent manner. A mindset is also a way of thinking about your talents and abilities. For example, if you are considering your talents and abilities with respect to mathematics, you are likely thinking about a mathematical mindset. There are strategies, structures, and tactics, as well as general rules about conventions and analogy that help us to think and then to problem solve mathematically. Many people who work with mathematics appreciate the facts that have been constructed over time and the imagination often required to pull everything together—knowledge, problem solving, reasoning, communication—to solve something mathematical (see Skemp 1971, for more reading on schemas and the organization of knowledge).

Teachers of mathematics come to mathematics classrooms from a diversity of mathematical backgrounds (e.g., statistics, engineering, geometry, financial applied mathematics). It is not unreasonable to appreciate that as a group, teachers 'do' math, or rather, think through the mathematics of a particular problem in individually diverse ways. In the classroom, teachers are the most visible and present role model helping students develop a mathematical mindset and the associated automaticity and fluency (e.g., mathematical habits of mind) of calculation, algebraic simplification, or problem solving. Teachers can overtly present mathematical thinking when they talk aloud while writing notes, or talk through their problem solving steps and decisions as they work through a problem or come up with a solution. However, teachers' mathematical thinking is often hidden and unspoken, or left to be deciphered by students after classwork is done and class notes are reviewed later in school or at home.

How are secondary school mathematics teachers' mathematical mindsets different from their students' mathematical mindsets? That is a very good question, and one that should perhaps be at the forefront of lesson planning. Thinking about the teaching of mathematics concepts and how individually these concepts lead to the development of more complex and interconnected concepts provides a jumping off point to understand what "knowing" mathematics is all about. I put quotation marks around "knowing" because there is a particular way to consider 'knowing' mathematics, and it is not necessarily tied to the common definitions and understanding of terms such as "knowledge" with which we are familiar in curriculum documents.

What Does It Mean to Know Mathematics? What Does It Mean to Do Mathematics?

Knowing and doing are not necessarily synonyms. Perhaps "knowing" is less about "knowing the mathematics" and more about knowing to act in a particular way, with appropriate tools and strategies, calling upon necessary concepts and skills at a particular moment in time. The mathematics one *knows* can be the necessary concepts and skills, but the *doing* may be the performance of mathematical action. Consider then, mathematics as a verb rather than a noun; during mathematical problem solving what actions and knowledge does one call upon, what mathematical mindsets are drawn upon, and what habits of mind automatically appear?

Some pertinent and relevant frameworks for how we perform mathematically will be valuable to understand how "knowing" mathematics can be more like action than passivity. To consider mathematics as a verb rather than a noun, we can examine the words used to describe mathematics. For example, in the latest Ontario mathematics curriculum Achievement Categories have been added to the way the curriculum is described. Knowledge is one of four Achievement Categories. The other Achievement Categories are Application, Thinking, and Communication. An implication of Knowledge being identified as one Achievement Category of the mathematics to be learned is that the authors of the curriculum feel knowing something is only part of the story, or one way to think of the mathematics to be learned. Clearly, the authors of the curriculum also feel learners should be able to apply mathematical knowledge and skills "to make connections within and between various contexts" (i.e., in familiar contexts, or new contexts), use "critical and creative thinking skills and/or processes," and communicate and "convey meaning through various forms" (Ontario Ministry of Education 2005, pp. 20–21). The verb forms of these Achievement Categories are to know, to apply, to think, and to communicate, however, these verb forms do not seem to be particularly informative of how to do the mathematics. For that, we can turn to the Harvard Balanced Assessment Project (Harvard 1995).

The Harvard Balanced Assessment Project was started in 1993 with a purpose to develop innovative mathematics assessment tasks—that means, to create mathematics tasks (or problems) and develop a method to assess the mathematics used in the tasks. In the process of deciding how to assess mathematical thinking presented in the solutions, the team also examined the nature of the mathematics they were trying to assess. The team came up with two big ideas to describe the mathematics they wanted students to be experiencing and learning, and to think about assessing mathematics: (a) rather than a content-process sense of mathematics to know and an answer to find, there is an object-action sense of mathematics concepts and skills to work with and actions to be taken with these concepts and skills, and (b) a general framework for how people do mathematics, the actions of mathematics, which some call the Harvard verbs of mathematical inquiry. These verbs are, hypothesising, modeling/formulating, transforming/manipulating, inferring/drawing conclusions, and communicating (Harvard 1995).

Take a moment to think about a mathematics question you might see in a textbook: perhaps it has some words to describe a context, or a bit of a story. Often the first action would be to guess, or *hypothesise*, where the solution will end—with a number, an expression, a proof, etc. The next action would be to *model* the situation numerically or graphically, which might lead to *formulating* an algebraic expression or equation. The next action might be to *transform* the equation so it is more easily solved for the unknown, or substitute some values and *manipulate* an expression. Once some work or calculations have been completed, an *inference* or *conclusion* about the correctness of the answer or the fit of the answer to the question occurs, and the whole process is *communicated* in some way: for example, writing a solution or a proof, or explaining it verbally to a peer. While these familiar mathematical actions are often the usual process, they are not always performed linearly. Sometimes these actions are performed recursively; sometimes these actions are performed in small chunks or cycles of one or two actions. I have used a puzzle image to help remember that this can be a rather dynamic process (see Fig. 1). Thinking of mathematics from the perspective of being a verb rather than a noun transforms the sense of *knowing* or *doing* mathematics into an active and inquisitive activity that might require tools such as computer technology, graphing calculators, or manipulative materials, and might be a collaborative effort of working together.

Now let's look at another way to describe and explain what could be happening when someone *does* mathematics. For this, I use the work from John Mason (e.g., Mason and Spence 1999) and his conceptualization of "knowing-to." The big idea behind "knowing-to" is that people need to know to act before anything can be accomplished. However, knowing one needs to act requires "knowing" other things as well. The following illustration in Fig. 2 represents Mason and Spence's graphic conceptualization of "knowing-to." Notice all the ways of knowing presented in the image.

Knowing-to is the overarching sense of knowing to act to complete a task, such as solving a system of linear equations. Knowing-to act requires one *knows-about*

Fig. 1 Harvard verbs of mathematical inquiry as a puzzle image

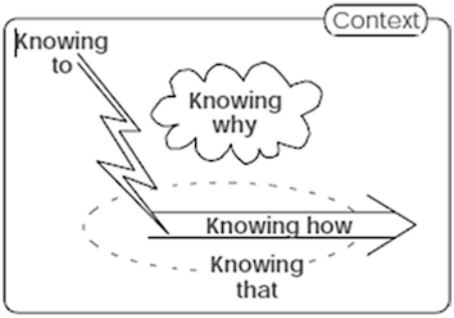


Fig. 2 From Mason and Spence (1999, p. 145), illustrating the relationship between knowing-about and knowing-to

the reasons for acting (*knowing why*), knows the facts relevant to the situation (*knowing that*), and knows a procedure that might/could work (*knowing how*). Knowing-to is the ultimate goal:

> once the moment of knowing-to takes place, knowing-how takes over to exploit the fresh idea; knowing-that forms the ground, the base energy upon which all else depends and on which actions depend; knowing-why provides an overview and sense of direction that supports connection and modification if difficulties arise en route. (Mason and Spence 1999, p. 146)

I suggest here, that success is more about having a mathematical mindset as described in this chapter, and less about imitation and regurgitation of solutions to familiar questions. Hence, the development of a mathematical mindset is important. So, how does one go about developing a mathematical mindset? Developing a mathematical mindset will require learning skills, those behaviours of planning or hypothesising, action or manipulation and transformation, observation, inference, and reflection that constitute knowing-to act without having to be given a solution pattern to follow. Developing a mathematical mindset will require persistence when the solution does not appear quickly, the independence and inter-dependence of individual and teamwork when necessary and appropriate, and the judgement to know-to act at any given moment. To develop a mathematical mindset, one must have learning skills, or what can be called habits of mind.

Habits of Mind

Costa and Kallick (2008) explored the emerging conceptions of intelligence and how these conceptions have been changing. Most importantly, intelligence is being better understood as a changing state of mind and that the growth of intelligence is a reflection of self-regulation and metacognitive skills (see Dweck 2006 for more on growth mindsets). Costa and Kallick (2008) present habits of mind as a "set of behaviours that discipline intellectual processes" (p. 12), and "a pattern of intellectual behaviours that leads to productive actions" (p. 16). Table 1 lists their 16 habits of mind.

Table 1 Sixteen habits of mind (Costa and Kallick 2008, pp. 18–39)

Persisting
Managing impulsivity
Listening with understanding and empathy
Thinking flexibly
Thinking about thinking (metacognition)
Striving for accuracy
Questioning and posing problems
Applying past knowledge to new situations
Thinking and communicating with clarity and precision
Gathering data through all senses
Creating, imagining, innovating
Responding with wonderment and awe
Taking responsible risks
Finding humour
Thinking interdependently
Remaining open to continuous learning

Table 1 lists general habits of mind, developed with the purpose of providing concrete behaviours that can be identified, noticed, and cultivated to help students become more disciplined in their approach to thinking and learning, and to get into the habit of behaving intelligently. These habits of mind act as learning-skills and what is desired is an automaticity, or fluency, of use. If these habits of mind can be drawn upon automatically, that is, to become habits, then learning occurs more readily.

It is not a far reach to imagine that automaticity of habits of mind and the associated increase in learning leads to other automaticity, such as algebraic and algorithmic fluency—for example, the ability to multiply numbers quickly, to calculate with fractions quickly, simplify algebraic expressions quickly, or to solve equations quickly. It is beneficial for learners to have some automaticity in their work, to count on some habits to get them through the long calculations in complex processes. For example, when finding a solution to a moving object problem in a senior grade of mathematics, it is very helpful to automatically recall the necessary differentiation rules and be mechanically proficient to get to the point of the problem, which was to answer the question posed in a particular situation. However, this is just the knowing-that and knowing-how of mathematics. What about the knowing-why and knowing-to aspects of thinking mathematically? Is it possible to have automaticity, fluency, or habits of mind for mathematical thinking like the Harvard verbs of mathematical inquiry?

Hypothesising, modeling/formulating, transforming/manipulating, inferring/drawing conclusions, and communicating are very concrete, and visible performed mathematical actions evident in mathematics solutions. These can be habitualized into automatic actions. In general, for any mathematical problem solving context, could there be behaviours of mathematical thinking to support the development of mathematical mindsets?

Cuoco, Goldenberg, and Mark (1996) suggest eight mathematical habits of mind: pattern sniffing, experimenting, tinkering, inventing, describing, visualizing, conjecturing, and guessing. Pattern sniffing is about searching for and being able to recognize regularity as well as the excitement of finding hidden patterns. This is very much like the realization that the commutative property works in many situations but not in subtraction. Experimenting draws on a sense of play that comes from making different hypotheses or conjectures and following each path of thinking for a while to see where it leads. Not every path must come to a successful endpoint, sometimes the experimenting suggests paths not to follow. Experimenting is closely related to tinkering—the process of taking things apart and putting them back together. For example, if a solution requires a number of steps, does a different order of those steps result in the same solution?

"Tinkering with existing machines leads to expertise at building new ones" (Cuoco et al. 1996, p. 380). Inventing is a mathematical habit of mind that has provided the field of mathematics with wonderful new and different ways of understanding the world around us, for example, the development of non-Euclidean geometry, and fractal geometry. As a language with symbols and grammatical structures, describing mathematics requires precision and accuracy of notation, expressing oneself coherently and presenting a convincing argument, in both written and oral manners. Visualizing the multiplication of two binomials as the multiplication of the length and width for the area of a rectangle, or visualizing place-value as a mental strategy for multiplying two-digit numbers, or visualizing three-dimensional space while working a solution on a flat piece of paper are examples of powerful mathematical habits of mind. Conjecturing and guessing are the two remaining mathematical habits of mind but they are not the same. Conjecturing is more about predicting outcomes based on available evidence; guessing requires one to think of something "out of the blue," which often in conjunction with other mathematical habits of mind can lead to new insight on a challenging or familiar problem.

Putting It All Together

Notice the parallel between these behaviours, the mathematical habits of mind, and the actions of the Harvard verbs of mathematical inquiry. These mathematical habits of mind describe how one can think, as well as, how one can act. For example, a habit of experimenting as an automatic behaviour when confronted with a mathematical problem can be made visible as an action of trying various approaches, strategies, and/or tools. Habits of mind can be leveraged for both behaviours and automaticity of action for the development of a mathematical mindset.

To conclude this chapter, I present playing games as a means to integrate thinking mathematically and the development of mathematical mindsets through general and mathematical habits of mind. Playing games requires everyone to know that there are rules to any game and the enjoyment of playing a game comes from following the rules. As long as everyone is willing to play a game (a mindset), the rules will be followed. This is most obvious when someone tries to cheat in a game—that cheater is called out loudly and vigorously by fellow players (knowing-why). Playing games requires knowledge of the strategies used by oneself (manipulating, transforming) and the other payers and the interactions of those strategies (communication). Many games require players to listen to others (communication) and simultaneously think about past knowledge (inferring, habits of mind) and consider new situations, to work collectively, individually, or even make temporary alliances as necessary in order to win the game (habits of mind). Playing games requires behaviours of habitualized actions for fast and effective game play, as well as the behaviours required for learning and improving the play of the game.

Automaticity of actions and the development of the technical skills of game play and game strategies are necessary for speed, accuracy, and efficient play. Think about the ability to play complex board games such as chess, or backgammon that require knowledge of board piece movements (knowing-that), and long- and short-term strategies (knowing-how, pattern sniffing). There is a persistence (habits of mind) to following the rules of a game. Players must manage their impulsivity and play fairly with each other, and take turns (habits of mind). Some games demand creativity, and many games involve an element of risk (habits of mind). The habit of mind of finding humour may not be present in all games, but games are clearly not played again if they are not fun—humour is understood as playing fair, enjoying winning and losing, and playing with good sportsmanship.

Let's move to a mathematics classroom context and example, and apply the principles of mathematical mindsets, mathematical habits of mind, and game play. Consider the skill of algebraic manipulation in the process of solving multi-step linear equations, such as solving for 'x' in $3(x-5) + 9 = 12$. The ultimate end-result is a solution that might look like Fig. 3. To the dismay of many mathematics teachers, the necessary thinking of each step in the algorithm often seems elusive for some students. This may be because many students do not perceive they will require such mathematical thinking and ability in their future lives, and these

Fig. 3 Algebraic manipulation to solve a multi-step linear equation

$$3(x-5)+9=12$$

$$3(x-5)+9-9=12-9$$

$$3(x-5)=3$$

$$\frac{3(x-5)}{3}=\frac{3}{3}$$

$$x-5=1$$

$$x-5+5=1+5$$

$$x=6$$

thoughts get in the way of learning the formal algebraic process presented in the solution. As students often say, "I am never going to use this in my life, why do I have to learn it now?"

I am a teacher and not a fortune teller, so I do not have the ability to foretell whom of these students will or will not need such algebraic facility and fluency in their future. However, considering mathematics for mathematics sake, solving equations is a skill that underlies so many other mathematics concepts, as well as learning in many different areas, such as accounting, business, retail, the trades, etc. Additionally, and pragmatically, this mathematical skill of solving equations is a curriculum expectation that will appear on the final exam and in future mathematics courses. So how does the teacher get students to learn to solve equations?

What if students' game habits of mind were leveraged? What if learning mathematical habits of mind piggy-backed on game habits of mind? The following is a possible unit set in a grade 9 mathematics course. It starts with a hook, a story, something to distract students' attention away from their expectations of being in a mathematics class. Something or someone currently culturally relevant works very well, such as a current YouTube street magician. "Have you seen them do this!?" "Pick a number between one and ten; multiply it by 2; add 3; tell me your answer," then the street magician tells the original number. Teachers know this is just memory work, and moving backwards through the calculations. However, for students, there can be an element of power that they can do a 'trick' and be 'smart' with mathematics once they realize the process. The next step though, is to leverage the possible game aspect and the habits of game play to move students from relying only on mental abilities to a needed outcome of writing formal algebraic solutions on paper.

The unit is presented as a developing process of learning the game and gradually increasing the complexity of the game. The following is the unit, loosely scheduled as a day-by-day plan.

Day 1. "What if there is a game board for this trick? Maybe it would look like this…" (see Fig. 4).

The rules of the game initially involve (a) writing the game board, and (b) filling in the board spaces. This can be practiced by playing the game with peers and completing the game board each time. A completed game board could look like this—the prompts from the lead player are typed at the top of the game board and then follow the arrows when writing on the game board. See Fig. 5.

Day 2. The next phase of the game is to

(a) write the game board,
(b) fill in the board spaces,
(c) decide on a letter to use instead of "?" that we have been putting in the "Pick a number" box so far. The usual letter we seem to use for something unknown is "x", and this seems reasonable to students in this game, and
(d) write the answer "x = …" to the right of the game board.

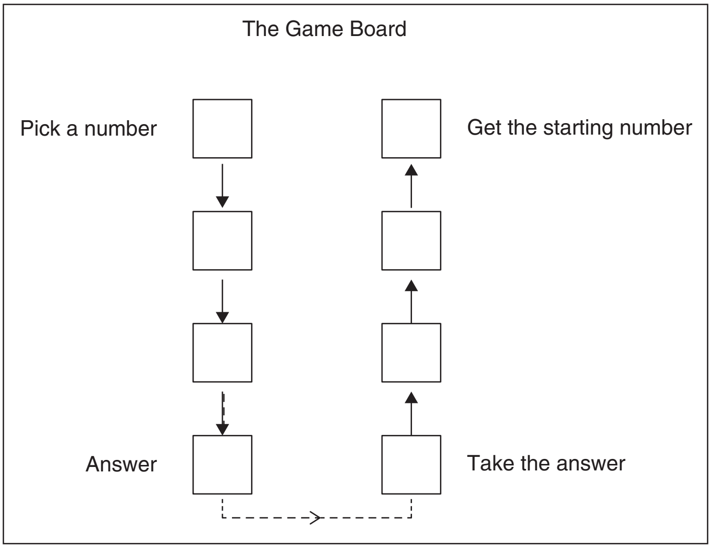


Fig. 4 The game board

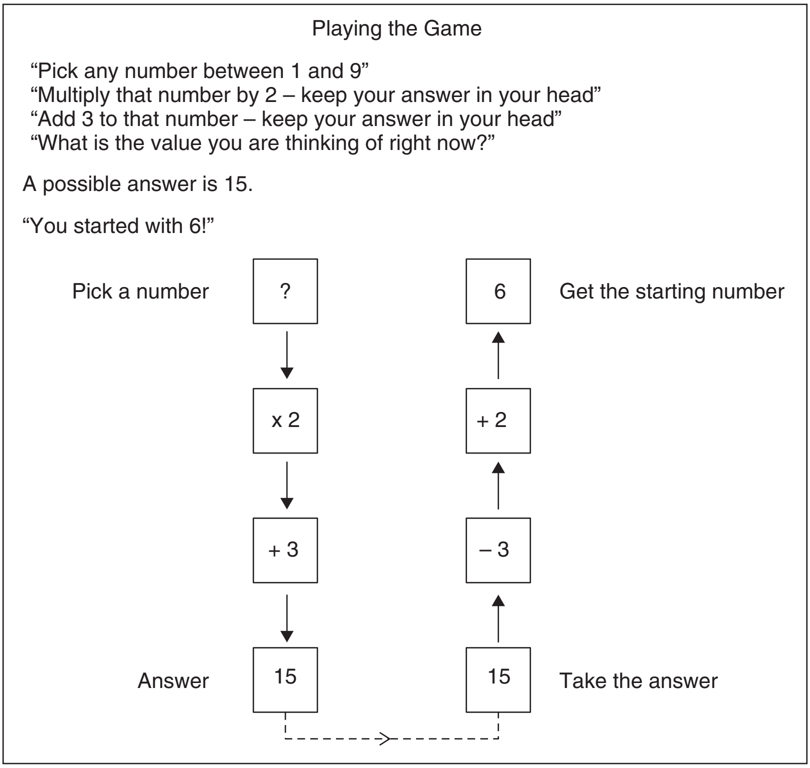


Fig. 5 A completed game board

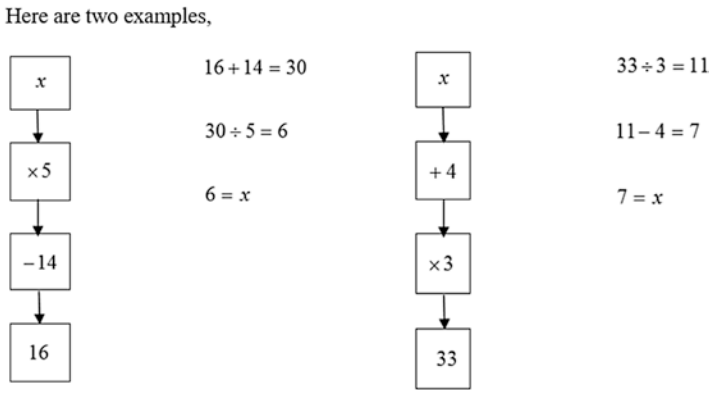


Fig. 6 Two examples for Day 3

Day 3. The next phase of the game is to

(a) write the game board,
(b) fill in the board spaces,
(c) write each calculation on the side of the game board as the step is done.

See Fig. 6. Notice the right column of boxes of the game board (moving up the page) have been replaced by the sentences of each calculation (moving down the page).

Day 4. Finally, writing the street magician's verbal prompts on the board in order creates an equation; if students have not realized it yet, they see they have been solving equations. This realization turns into another feeling of power because students now know they CAN solve equations, that solving equations is just going backwards through the calculations, and that the order of the calculations spoken by the magician, follows the order of operations students already know.
Notice, all the steps necessary for an algebraic solution appear beside the game board, as they appear in the "More formal method" on the right in Fig. 7.

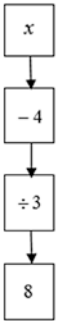
Because a game format was used, following the rules of a game is an ingrained habit of mind. The last step of the game is always to write "x = …" the numerical result.

Where Are the Mathematical Mindset, and the Mathematical Habits of Mind?

The mathematical mindset is evident through two lenses, the Harvard verbs of mathematical inquiry, and Mason and Spense's knowing-to perspective. For example, hypothesising exists for both the person calling the game and the person

Example: Solve $\frac{x-4}{3}=8$

Game Solution:



$$\frac{x-4}{3}=8$$
$$8\times 3 = 24$$
$$24+4=28$$
$$28 = x$$

More formal method

$$\frac{x-4}{3}=8$$
$$3\left(\frac{x-4}{3}\right)=3(8)$$
$$(x-4)=24$$
$$x-4+4=24+4$$
$$x=28$$

Fig. 7 Game board with corresponding formal algebraic solution

listening to the game or reading the solution—what could the answer be? Does the answer make sense? Modeling and formulating exists in the creation of a game board—writing the game as it is called by the first person as an equation, and then writing the thought processes to arrive at the unknown number as mathematical expressions on paper. Transforming and manipulating exists as the writing of the game progresses towards the formal algebraic solution to solve linear equations. Inferring and drawing conclusions exists as a reflection moment reaching back to the hypothesis—is the answer correct for the game? And finally, communicating exists in various formats—verbal and written. Written communication may also be left to the student's comfort level and teacher's discretion of which written format of a game board to use: as a modified game board (the left side of Fig. 7), or as the more formal algebraic solution (the right side of Fig. 7).

Alternatively, knowing-that is the mathematics of the order of operations to find the number the other person started with in the game. Knowing-how is the use of the order of operations backwards to solve the game, and forwards to write the verbal instructions down onto paper as an equation. Knowing-why is the understanding that order matters when performing calculations in the context of a grade 9 mathematics solving linear equations curriculum expectation. This understanding gives students the personal feeling of knowing-to solve linear equations, and possibly have the tools to solve almost any equation with some ease.

Game behaviours are great support for the development of mathematical habits of mind. For example, there are patterns to experiment with and tinker with as students play the game described here with more than two operations, or with larger ranges of numbers. Conjecturing and guessing naturally emerge as students invent game play that uses numbers that fall afoul of division rules (e.g., not everything is easily divisible by three if the game calls for "multiply by three," especially if calculations are still being performed mentally rather than with a calculator). Visualizing the solving of equations mentally, and then as a game, and then as a formal algebraic process on paper provides numerous opportunities for describing the process of solving equations as well as explaining what is really going on with the numbers as a solution is determined.

In Conclusion

Underlying this chapter has been the appreciation of mathematical mindsets that are present in the diversity of who people are and where their mathematics knowledge has come from. As you read the chapter, you might have reflected back on your own mathematical thinking as this chapter described what "doing mathematics" could be, and explored understanding mathematical thinking from the perspective of mathematical mindsets and the mathematical habits of mind one calls upon in the process of learning and doing mathematics.

Teachers are encouraged to continue to critically reflect upon their mathematical mindset and mathematical habits of mind, as well as their general habits of mind. These skills, behaviours, and knowledge will be invaluable in professional practice.

Additionally, imagine the diversity of thinking that exists for those students who have not had the same amount or level of mathematical experience as a teacher—how do students think through a problem, or decide on the necessary concept or skill to use in the next step of the solution? Imagine the potential for developing students' mathematical mindsets through general and mathematical habits of mind.

References

Costa, A. L., & Kallick, B. (Eds.). (2008). *Learning and leading with habits of mind: 16 essential characteristics for success*. Alexandria: ASCD.

Cuoco, A., Goldenberg, E. P., & Mark, J. (1996). Habits of mind: An organizing principle for mathematics curricula. *Journal of Mathematical Behaviour, 15*, 375–402.

Dweck, C. (2006). *Mindset: The new psychology of success*. New York: Ballantine Books.

Harvard. (1995). *Harvard balanced assessment project*. Retrieved from http://hgse.balancedassessment.org/amuse.html.

Mason, J., & Spence, M. (1999). Beyond mere knowledge of mathematics: The importance of knowing-to act in the moment. *Educational Studies in Mathematics, 38*, 135–161.

Ontario Ministry of Education. (2005). *The Ontario curriculum: Grades 9 and 10 mathematics* (Rev ed.). Toronto: Queen's Printer for Ontario. Retrieved from http://www.edu.gov.on.ca/eng/curriculum/secondary/math910curr.pdf

Skemp, R. R. (1971). *The psychology of learning mathematics*. Bungay Suffolk: The Chaucer Press.

Considering Indigenous Perspectives and Mathematics Education: Stories of Our Experiences As Teachers and Teacher Educators

Gladys Sterenberg and Kevin O'Connor

Abstract In a Canadian context where calls to action are made in response to the *Truth and Reconciliation Commission* and where the *United Nations Declaration on the Rights of Indigenous Peoples* have been recently adopted, it becomes important for teachers to engage in the decolonization of education. Both documents acknowledge the need to rebuild relationships with First Nations, Métis, and Inuit peoples and specifically call for the development of strategies to eliminate the educational gaps between Indigenous and Non-Indigenous Canadians, the development of culturally appropriate curricula, and the incorporation of principles that respect and honour treaty relationships. In an effort to respond to these calls to action, significant program changes in school mathematics have been made across Canada that encourage teachers to consider Indigenous perspectives. These initiatives challenge pre-service teachers to shift their ways of knowing that are typically based on a Euro-Western perspective. This chapter presents the process we took as teachers and teacher educators to enact Indigenous perspectives within our educational contexts. Exemplars from research on culturally-responsive education and learning from place are offered as ways of integrating Indigenous perspectives and mathematics. It is our belief that all students benefit from curricula that incorporates Indigenous knowledges and our hope that the teaching and learning of mathematics can support these calls to action.

Keywords Indigenous education · Mathematics education · Culturally-responsive education · Learning from place · Pre-service teachers

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Introduction

Current curriculum changes encourage pre-service teachers to incorporate Indigenous perspectives into their mathematics classrooms. However, many pre-service and in-service teachers grapple with how to respond in respectful and authentic ways. As teachers and teacher educators, we have experienced shifts in our own ways of thinking as we responded to these changes. In this chapter, we provide a brief summary of policy and curricular initiatives in a Canadian context and present stories of our experiences to help pre-service teachers who are beginning to consider links between Indigenous perspectives and mathematics. Resources for further reading are listed at the end of this chapter to help you extend your own understanding.

In a Canadian context where calls to action are made in response to the *Truth and Reconciliation Commission* (Truth and Reconciliation Commission of Canada 2012) and where the *United Nations Declaration on the Rights of Indigenous Peoples* (United Nations 2008) has been recently adopted, it becomes important for teachers to engage in the decolonization of education. Both documents acknowledge the need to rebuild relationships with First Nations, Métis, and Inuit peoples and these have important implications for teachers.

In an effort to address this need, significant program changes in school mathematics have been made across Canada that encourage teachers to consider Indigenous perspectives. For example, the *Common curriculum framework for K-9 mathematics: Western and Northern Canadian protocol* (Alberta Education 2006) that informs programs of studies in British Columbia, Alberta, Saskatchewan, Manitoba, Northwest Territories, Yukon Territory, and Nunavut now includes Indigenous perspectives as broad statements are made about teaching and learning: "Teachers need to understand the diversity of cultures and experiences of [Indigenous] students," and "A variety of teaching and assessment strategies is required to build upon the diverse knowledge, cultures, communication styles, skills, attitudes, experiences and learning styles of [Indigenous] students" (p. 3). Similarly, the *Ontario curriculum, grades 1–8: Mathematics* (Ontario Ministry of Education 2005) states:

> Learning activities and resources used to implement the curriculum should … enable students to become more sensitive to the diverse cultures and perceptions of others, including Aboriginal peoples. For example, activities can be designed to relate concepts in geometry or patterning to the arches and tile work often found in Asian architecture or to the patterns used in [Indigenous] basketry design. (p. 28)

While mathematics program documents for Quebec (Gouvernement du Québec, 2001) and the Atlantic provinces (Atlantic Provinces Education Foundation, 1996) do not specifically identify a focus on Indigenous perspectives, they include a consideration of the cultural diversity of all students. Presumably this could include Indigenous students.

These initiatives challenge pre-service teachers to shift their ways of knowing that are typically based on a Euro-Western perspective. As teachers and teacher educators, we have considered possibilities of enacting an Indigenous perspective within our own educational contexts. Here are our stories of this journey in hopes that they will provide examples of such possibilities.

Gladys' Story

My consideration of different ways of knowing began during an undergraduate class where I listened to the story of a Blackfoot elder. I was profoundly jolted when she asked us to consider what it would be like to live in a community without children. Thus began my own quest to better understand our treaty history and my role within this context. During my teaching career, I encountered some Indigenous students and during my graduate learning, I became friends with colleagues who were Indigenous. When I became a professor, I was invited to work alongside colleagues who were teaching in First Nation schools. My learning began within these relationships as I struggled to make sense of who I was as a white teacher engaged in teaching and research within Indigenous settings (Donald et al. 2012). I began reading articles and books written by Indigenous researchers and teachers to help me understand how I might integrate Indigenous Knowledges into my teaching of mathematics.

In order to integrate Indigenous Knowledges and mathematics curriculum in respectful and appropriate ways, I needed to develop a deeper understanding of Indigenous Knowledges. In the Blackfoot context in which I live and work, balance and harmony with the environment are recognized as part of the knowledge system (Bastien 2004; Cajete 1994, 2000; Peat 2002). Battiste (2002) links Indigenous Knowledges to particular "landscapes, landforms, and biomes where ceremonies are properly held, stories properly recited, medicines properly gathered, and transfers of knowledge properly authenticated" (p. 13). Little Bear (2000) describes the land as integral to the Native American mind.

As I grew to better understand Indigenous ways of knowing, I became interested in how mathematics could be reframed. I now believe that mathematics can be defined and understood in many different ways. To help you gain a better understanding of these various perspectives, I draw parallels and distinctions between views of mathematics and Ogawa's (1995) perspectives of science. Drawing on Ogawa's proposal of three subcategories of science of interest to educators, I find it useful to consider Indigenous mathematics, Western mathematics, and Personal mathematics. Indigenous mathematics refers to the mathematics in a particular culture that reflects a collective worldview. Examples of Indigenous mathematics could include Australian mathematics (Watson and Chambers 1989), Japanese Wasan mathematics (Aikenhead 2017), Polynesian mathematics (Ball 2013), or Māori mathematics (Barton et al. 1998). Ogawa describes Western modern science as "a collective rational perceiving of reality, which is shared and authorized by the scientific community" (p. 589). Rather than focusing on natural phenomena, "Western modern science pertains to a Cartesian materialistic world in which humans are seen in reductionistic and mechanistic terms" (p. 589). Similarly, Western mathematics could be described as Eurocentric, focused on Platonist values (Aikenhead 2017; Bishop 1990; Ernest 1989), and one that emphasizes abstraction. Personal mathematics is unique to each person and involves personal observations or explanations of the world. Like Ogawa, I recognize a relativistic perspective of mathematics and note that worldviews and knowledge systems are deeply connected.

Kevin's Story

My understanding of different ways of knowing was influenced by the magical times my father would recite poetry to me by Robert Service. These poems had elements of frontierness, adversity, Indigenous culture, and the power of the natural environment and the north. Subconsciously, this had a tremendous impact upon my life, as I arrived in Whitehorse in my '71 Volkswagon van with my Bachelor of Education diploma fresh off the presses. While teaching in the Yukon Territory, I had the opportunity to teach many students of different cultures and backgrounds. With that cross-cultural exposure came the challenge to effectively relay the curriculum in a meaningful and an engaging manner to meet the wide range of learning needs, cultural and social conditions.

As I acknowledge the influence colonial forces and Western-style practices in educational institutions have had upon Indigenous identities, I began my search for a teaching path with the recognition that I am a product of those colonial forces, whether I am cognisant or not, and bring an identity and a certain way of thinking that may not always be consistent with Indigenous worldviews. I refer to my position as "non-Indigenous teacher." While I may be White, I am Canadian, of Irish descent, a proud Quebecer, English Montrealer and Northerner with a specific past and identity that aligns me with much more than "whiteness," as I believe it is the people, communities and environments that surround us that create a sense of place in which we define our identity. This principle is what guides the practice of place-based education.

Experiential and place-based programs of study encourage students through field activities, a multiple perspective curriculum and active exposure to social, cultural and political issues to venture 'outside the box' of the conventional education system. Students are asked to develop beliefs that are based on their own critical assessment; differing opinions and ways of thinking are encouraged. Through student-centered learning initiatives and cooperative work, learners develop a cognition that values multiple perspectives (O'Connor 2009).

The notion of interconnectedness and the understanding of the relation between things, which is a key component to experiential learning and some Indigenous thought, becomes a necessary component of curriculum design. Many experiential and place-based programs are developed around the principle of integration, in which people are able to learn more effectively when they are able see things in relation to other things.

I have always been at odds with traditional Western forms of knowledge. It separates areas called mathematics from those areas referred to as art and spirituality or religion. I have a common understanding of the interconnectedness of all things. With respect to mathematics education, I believe forms of knowledge are integrated and need to be taught in such a form. Knowledge is relational; it is shared with all living things. This has fuelled my passion for constructivist learning and its application in school programs. I believe this theory of learning that is in accordance with place-based and experiential initiatives best serves students in the process of learning about themselves and their place in the natural world. Place-based learning is developed with the notion that things thrive because of the web of interconnectedness between an individual and the community and between the community and nature. Also, everything we do affects everything else around us.

Exemplars

When we reflect on our stories, we are able to identify some of the qualities that we believe are necessary to have as teachers who are beginning to consider how to integrate mathematics and Indigenous perspectives. Forming relationships with Indigenous communities is imperative and when engaging in this process, Kevin suggests that pre-service teachers need to be respectful as guests in the community; need to understand the web of interconnectedness among individuals, the community, and nature; and need to engage in reflective thought and critical analysis of their actions while teaching (O'Connor 2006). Gladys notes that establishing meaningful and trusting relationships with Indigenous teachers and students takes time, energy, thoughtfulness, and an ethical commitment to attend to the particularities of the context (Donald et al. 2012).

Throughout our experiences of teaching and learning, we acknowledge that we are non-Indigenous. Yet we have stories that include an Indigenous context and lived experiences with Indigenous people. We are also cognisant that we are products of the colonial forces that have traditionally dismissed Indigenous ways of knowing in exchange for the dominant Western paradigm. Throughout our journeys to better understand our role in integrating Indigenous Knowledges and mathematics, we have been guided by two approaches, namely culturally-responsive education and learning from place.

Culturally-Responsive Education

In North America, teachers, knowledge holders, and researchers in Indigenous communities have considered various ways of integrating Indigenous Knowledges and mathematics curriculum from culturally responsive perspectives. One such example is *Math in a Cultural Context*, a culturally based mathematics curriculum for Yup'ik students in Alaska (Lipka 1994; Lipka et al. 2005). It was developed jointly in collaboration with Yup'ik elders, teachers, schools, and communities; what makes this curriculum unique is the emphasis on starting from the elders' knowledge.

The *Transformative education for Aboriginal math and science learning* at the University of British Columbia supports the work of Nicol and Archibald (2009) who focus on creating and living culturally responsive mathematics education in both rural and urban settings and consider how we can use community, culture, and place as inspirations for mathematics. Nicol and Archibald consistently emphasize the importance of forming strong relationships with Indigenous communities.

The *Show Me Your Math* project in Atlantic Canada was developed by Lunney Borden, Wagner, and Johnson (in press) and Elders of Mi'kmaw communities (see Lunney Borden, this volume). This yearly fair brings together members of the Indigenous community as students demonstrate how mathematics and Mi'kmaw ways of knowing intersect. Their research considers how teachers can learn from students' Indigenous language and from cross-cultural understandings.

These examples of how culturally responsive education can be enacted in specific cultural contexts offer insights when considering the integration of Indigenous Knowledges and school mathematics. In these projects, accumulated generational knowledges of living in a particular place are incorporated into mathematics curriculum through traditional stories, activities such as star gazing and the study of patterns used on clothing. Through our own informal experiences of engaging in culturally-responsive education, we have come to understand the importance of designing mathematical activities situated in places relevant to the experiences of the children in the community, of including elders in the process, and of attending to content knowledge, contextual knowledge, and pedagogical knowledge.

Learning from Place

Researchers in North America are beginning to better understand student learning from place. When Gladys began her research and teaching within the framework of learning from place, she met with elders from the Blackfoot nation (Sterenberg and Hogue 2011). This informed her work with high school students (Sterenberg 2013) where she worked in partnership with teachers in a First Nation community where they took the students on two field trips to sacred Blackfoot sites, one of which was to the Big Rock. The elder who was with them began their visit with an offering and stories of Napi, a Blackfoot trickster. Students engaged in using trigonometry, similar triangles, Pythagorean Theorem and Global Positioning Systems to describe the place. The teacher summarized her thoughts on the impact of learning from place:

> I believe that integrating the curriculum and finding math in the land made students feel more connected to their land and community, definitely pride of who they were, concern for the care and treatment of traditional sites, certain of their place and belonging in the Blackfoot Territory and culture, and in their place in Canada, safe to express themselves

mathematically, confident in their knowledge and skills in math, and engaged with math with a positive attitude. I believe that they felt like math was more human and more social than they had previously thought and that that was an acceptable way to see it; and that their ways of knowing are valid and can carry them into any subject. (Sterenberg 2013, p. 104)

Conclusion

Though our experiences, we have come to understand that Western and Indigenous mathematics can be viewed as having complementary strengths. Recognizing the strengths of each view will maximize mathematical learning for all students. According to researchers, a cross-cultural approach to mathematics represents a much needed paradigm shift (Aikenhead and Huntley 1999; Antone 2000; Corbiere 2000; Corsiglia and Snively 1995; Davidson 2002; Ezeife 2002; Lewis and Aikenhead 2001). Integrating Indigenous and Western knowledge systems holds generative possibilities for mathematics education.

Through our own study of multiple perspectives of mathematical thinking and knowing, we have come to know mathematics and ourselves in a different way. Calls to action made in response to the *Truth and Reconciliation Commission* (Truth and Reconciliation Commission of Canada 2012) include the development of strategies to eliminate the educational gaps between Indigenous and Non- Indigenous Canadians, the development of culturally appropriate curricula, and the incorporation of principles that respect and honour treaty relationships. It is our belief that all students benefit from curricula that incorporates Indigenous knowledges and our hope that the teaching and learning of mathematics can support these calls to action.

References

Aikenhead, G. (2017). Enhancing school mathematics culturally: A path of reconciliation. *Canadian Journal of Science, Mathematics and Technology Education, 17*(2), 73–140. https://doi.org/10.1080/14926756.2017.1308043.

Aikenhead, G., & Huntley, B. (1999). Students learning Western and Aboriginal science. *Canadian Journal of Native Education, 23*(2), 159–175.

Alberta Education. (2006). *Common curriculum framework for K-9 mathematics: Western and Northern Canadian protocol*. Edmonton: Alberta Education.

Atlantic Provinces Education Foundation. (1996). *Foundation for the Atlantic Canada mathematics curriculum*. Halifax: Atlantic Provinces Education Foundation.

Antone, E. M. (2000). Empowering Aboriginal voice in Aboriginal education. *The Canadian Journal of Native Education, 24*(2), 92–101.

Ball, P. (2013, December 16). Polynesian people used binary numbers 600 years ago. *Nature*. Retrieved from https://www.nature.com/news/polynesian-people-used-binary-numbers-600-years-ago-1.14380

Barton, B., Fairhall, U., & Trinick, T. (1998). Tikanga Reo Tātai: Issues in the development of a Māori mathematics register. *For the Learning of Mathematics, 18*(1), 3–9.

Bastien, B. (2004). *Blackfoot ways of knowing*. Calgary: University of Calgary Press.

Battiste, M. (2002). *Indigenous knowledge and pedagogy in First Nations education: A literature review with recommendations*. Ottawa: National Working Group on Education and Indian and Northern Affairs Canada.

Bishop, A. J. (1990). *Western mathematics: The secret weapon of cultural imperialism*. Thousand Oaks: Sage.

Cajete, G. (1994). *Look to the mountain: An ecology of Indigenous education*. Durango: Kivaki Press.

Cajete, G. (2000). *Native science: Natural laws of interdependence*. Santa Fe: Clearlight Publishers.

Corbiere, A. I. (2000). Reconciling epistemological orientations: Toward a wholistic Nishaabe (Ojibwe/Odawa/Potowatomi) education. *Canadian Journal of Native Education, 24*(2), 113–119.

Corsiglia, J., & Snively, G. (1995). Global lessons from traditional science of long-resident peoples. In G. Snively & A. MacKinnon (Eds.), *Thinking globally about mathematics and science education* (pp. 5–46). Vancouver: Centre for the Study of Curriculum and Instruction, University of British Columbia.

Davidson, D. M. (2002). Teaching mathematics to American Indian students: A cultural approach. In J. E. Hankes & G. R. Fast (Eds.), *Changing the faces of mathematics: Perspectives on Indigenous people of North America* (pp. 19–24). Reston: National Council of Teachers of Mathematics.

Donald, D., Glanfield, F., & Sterenberg, G. (2012). Living ethically within conflicts of colonial authority and relationality. *Journal of the Canadian Association for Curriculum Studies, 10*(1), 53–76.

Ernest, P. (1989). The impact of beliefs on the teaching of mathematics. In P. Ernest (Ed.), *Mathematics teaching: The state of the art* (pp. 249–254). London: Falmer Press.

Ezeife, A. N. (2002). Mathematics and culture nexus: The interactions of culture and mathematics in an Aboriginal classroom. *International Education Journal, 3*(3), 176–187.

Gouvernement du Québec. (2001). *Québec education program: Preschool education and elementary education*. Québec: Ministère de l'Éducation.

Lewis, B. F., & Aikenhead, G. S. (2001). Introduction: Shifting perspectives from universalism to cross-culturalism. *Education, 85*(1), 3–5.

Little Bear, L. (2000). Foreword to Native science: Natural laws of interdependence. In G. Cajete (Ed.), *Native science: Natural laws of interdependence* (pp. ix–xii). Santa Fe: Clearlight Publishers.

Lipka, J. (1994). Culturally negotiated schooling: Toward a Yup'ik mathematics. *Journal of American Indian Education, 33*(3), 14–30.

Lipka, J., Hogan, M. P., Webster, J. P., Yanez, E., Adams, B., Clark, S., & Lacy, D. (2005). Math in a cultural context: Two case studies of a successful culturally based math project. *Anthropology and Education Quarterly, 36*(4), 367–385.

Lunney Borden, L., Wagner, D., & Johnson, N. (in press). Show me your math: Mi'Kmaw community members explore mathematics. In C. Nicol, S. Dawson, J. Archibald, & F. Glanfield (Eds.), *Living culturally responsive mathematics curriculum and pedagogy: Making a difference with/in Indigenous communities*. Rotterdam: Sense Publications.

Nicol, C., & Archibald, J. (2009, April). *Living culturally responsive education and research: Teaching and learning with/in Indigenous communities*. Paper presented at the annual American Education Research Association conference, San Diego, CA.

O'Connor, K. B. (2006). *Investigations into Indigenous research and education through an experiential and place-based lens*. Unpublished master's thesis, McGill University, Montreal, QC.

O'Connor, K. (2009). *Puzzles rather than answers: Co-constructing a pedagogy of experiential, place-based and critical learning in Indigenous education*. Unpublished doctoral thesis, McGill University, Montreal, QC.

Ogawa, M. (1995). Science education in a multiscience perspective. *Science Education, 79*(5), 583–593.

Ontario Ministry of Education. (2005). *Ontario curriculum, grades 1–8: Mathematics*. Toronto: Queen's Printer. Retrieved from http://www.edu.gov.on.ca/eng/curriculum/elementary/math18curr.pdf

Peat, F. D. (2002). *Blackfoot physics*. York Beach: Red Wheel/Weiser.

Sterenberg, G. (2013). Learning Indigenous and Western mathematics from place. *Mathematics Education Research Journal, 25*(1), 91–108.

Sterenberg, G., & Hogue, M. (2011). Reconsidering approaches to Aboriginal science and mathematics education. *Alberta Journal of Educational Research, 57*(1), 1–15.

Truth and Reconciliation Commission of Canada. (2012). *Truth and Reconciliation Commission: Calls to action*. Retrieved from http://www.trc.ca/websites/trcinstitution/File/2015/Findings/Calls_to_Action_English2.pdf

United Nations. (2008). *United Nations declaration on the rights of Indigenous peoples*. Retrieved from http://www.un.org/esa/socdev/unpfii/documents/DRIPS_en.pdf

Watson, H., & Chambers, D. W. (1989). *Singing the land, signing the land*. Geelong: Deakin University Press.

Additional Suggestions for Further Reading

Alaska Native Knowledge Network. (n.d.). *Curriculum resources*. Retrieved from http://www.ankn.uaf.edu/

Alberta Education. (2016). *Walking together*. Retrieved from http://www.learnalberta.ca/content/aswt/index.html#/home

Sterenberg, G., Barrett, L., Blood, N., Glanfield, F., Lunney Borden, L., McDonnell, T., Nicol, C., & Weston, H. (2010). To become wise to the world around us: Multiple perspectives of relating Indigenous knowledges and mathematics education. *delta-K, 47*(2), 21–29.

A Teacher's View – An Unexpected Adventure

Jarron Childs and Jennifer Holm

Abstract This chapter examines a current secondary school teacher's journey of being the Department Chair in a Northwestern Ontario school. His journey is highlighted by different insights that were gained from working with other teachers with different beliefs about how to effectively teach mathematics. In fulfilling his role as Department Chair, many reforms and initiatives were taken with his teachers, leading to different reactions from the other teachers in his mathematics department. The chapter highlights the stories of this "unexpected" journey while supporting it with research about professional development and the potential challenges of working with other teachers with strong (and differing) beliefs about teaching and learning mathematics. The hope is that this chapter will inform teachers in a similar role, facing similar challenges, in gaining some clarity on how to enact changes within secondary schools in order support the learning of secondary students in mathematics.

Keywords Beliefs · Professional development · Department chair · Secondary teachers

In our experiences, teachers working together has been an important aspect of the profession. Sometimes these collaborations are teacher driven and organic, and other times they are ministry, school board, or principal directed. Each type of collaboration brings its own positives as well as challenges, but collaboration is something that teachers need to learn to navigate and often accept. Many studies in current research examine teachers working together to make changes and improve their own teaching practices and how these collaborations can be effective professional development (i.e., Brahier and Schäffner 2004; Hord 2009; Vescio et al. 2008).

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Vangrieken, Meredith, Packer, and Kyndt (2017) provided a review of what they termed "teacher communities" to look at the effectiveness of teachers working together. This term places professional learning groups (PLCs) and communities of practice (CoP) under a single umbrella term. Shulman and Shulman (2004) note that a community that a teacher belongs to can have an impact on the beliefs and practices of that teacher. This chapter focuses on the firsthand account of a secondary school mathematics teacher (Jarron) as he worked with colleagues on supporting and developing practices that would ultimately support the students they taught. Following the account, the observations are linked to literature studies wherever possible in order to provide a structure for some of the underlying ideas of teachers working together. The purpose of this chapter is not meant to define the types of teacher collaboration, but simply to highlight the complexities and some of the lessons learned along the way that can create a culture of teacher collaboration.

Jarron's Adventure

I have been teaching mathematics for over 16 years, 14 of those as the department chair. Needless to say there have been many observations, reflections and "a-ha" moments along the way. It certainly has not been a smooth ride, but few things are. The following is a collection of some of those observations. Perhaps they will be useful to others as signposts in their own respective journey.

Observation #1: Changing Personal Practices

It is safe to say that the majority of new teachers "teach the way they were taught," at least in the beginning. This is neither a good thing nor a bad thing, it is just what seems to happen when the overwhelming amount of job expectations crash home on a new teacher. Figuring out what to teach and how quickly, classroom management, the political circus, etc. can be daunting, not to mention that new teachers seldom get a timetable or courses that would minimize some of those challenges. Teaching in several departments, or even schools, in the most difficult behavioural classes, and the "sink or swim" mentality of colleagues can lead to more job stresses than are perhaps necessary. So when everything is quite chaotic, it is completely understandable that new teachers, and even experienced teachers, rely on the method of teaching that they remember, are confident in, and after all, got them through the course as a student.

It is with this thought in mind, that we take our first major step forward. The lessons that were good for us as individuals leave a deep imprint; an imprint that when challenged by something different, say an alternative pedagogical approach, can lead to a very defensive and fixed mindset—cognitive dissonance. From my own experience, I taught like the teachers of my youth. I ridiculed and denied the use of manipulatives, specifically algebra tiles, for years as a waste of time and not connected enough to traditional algebra. I taught trigonometry for a year before I actually figured out what the ratios really represented. I have made mistakes in bunches. It is safe to say, I was trying to be a good teacher, but not really succeeding.

There are two moments when it all changed. The first is when I bought my first house. Teaching was no longer the "possible" career choice; it was THE career choice, because I had bills to pay and a major financial responsibility. The second happened one semester later, when I became the department chair. I was no longer responsible for just my own professional growth; I now had to help others with their growth as well. These two events may seem common to some people, but they were paradigm shifting for me. From that day forward, my own teaching practices became more dynamic, more open-minded, and more energetic. For me, I cannot ask anyone to do a job or a task that I am not prepared to do myself. I cannot ask people to use algebra tiles and other manipulatives unless I am prepared to; so I dove in and explored. The result was that I am now a huge algebra tile user, and it has expanded my practice into numerous other topics and strategies. I have even gone as far as making video lessons using algebra tiles for mobile (internet) use.

I share this story with you because I was very fortunate to have a distinct point in my career that changed my path. A lump of clay has limitless potential, but if it is never shaped, molded, fired and glazed, it remains a lump of clay and not a precious vase.

Observation #2: Traffic Light Analogy for Professional Development

As a department chair, I have worked with colleagues within my own department, from other departments, other schools, other boards, and other provinces. After a while, you begin to recognize colleagues that have undergone the same experience you have and are vibrant and brimming with energy and excitement for their job. These people are affectionately called "Greens," because they are ready to GO on any new idea or initiative, they take little motivation, and even less supervision or prompting. The opposite of green is "Red." These folks have either never had their moment of refinement, or they had it and have been broken or worn down along the way. They are difficult to move out of their set ways, teach the same way every year, and contribute little in terms of energy or enthusiasm to the school or department. "Yellows" are in the middle, they can be swayed and will try new things, but it takes a lot of prompting and reminding. Yellows do not oppose or resist but do not move too far on their own. The important point to remember about the colours is that they are a descriptor of the person's career location and not their personality. It is quite possible to have a colleague be a "red" at meetings and professional development, but then be your basketball-coaching partner or weekend running mate. Career mindset and social likeability are very separate concepts. A "red" at a meeting is frustrating and can impede the process, but this does not imply they are a bad person. Also, the colour can change depending on the topic or given activity; a "green" usually stays "green," but "yellows" and "reds" can change quite quickly and without advanced warning. All and all, it makes for a very dynamic and challenging environment to try and lead and guide as the department chair. I share these thoughts because they serve as vital background information for the journey we embarked on as a department over the past few years.

Observation #3: "The Plan"

I was asked if my Grade 11 UC (university bound) class would like to be part of a trial run of an entrance test for a provincial college. The test would be 70 multiple choice questions done over two periods on various numeracy concepts (arithmetic, percent, ratios, fractions, etc.) with the one caveat being that no calculators were allowed. The reward for class participation was a honourarium being paid to the department to be used for mathematics resources, texts, etc.

I felt it would be a good experience for my students and the extra cash for the department would be an added bonus, but before I let them try the test I felt it necessary to ensure that their non-calculator skills were refreshed and ready to be used. I remember the question that started the journey we are on, it seemed simple enough for a grade 11 class: 14 × 16. Out of a class of 23 students, only one student came up with the correct answer (224). I was shell shocked. We spent the next few days reviewing multiplication and division strategies before attempting the test. Needless to say, 70 multiple choice questions with no calculator was not an uplifting experience for the students, but it was for me. I took the news back to our department PLC (Professional Learning Committee) and it started dialogue on how we can ensure that students that graduate from our institution have the mental math skills that are expected of them in the work world, but no longer hold a prominent place in the curriculum (elementary or secondary).

When I shared my experience with my department, all of the different colours, Green, Yellow and Red had insights into what to do—to me this was a huge success already. We had buy in from everyone on the need to do something, the how and what we were going to do would take a bit more time to iron out.

In the end, we developed a strategy we affectionately named "The Plan." Each block would last about 2 weeks, would start with a pre-assessment, have several short (10 min) mini lessons on a very specific topic, say adding, and would culminate with a post assessment to gauge how much progress was made. All work would be calculator free, written calculations or mental math would be acceptable but no technology, multiplication tables or outside assistance. We hoped to hit the following topics during the semester: addition, subtraction, multiplication, division, percent, fractions, ratios, but quickly found out that was too ambitious. In the end, we settled on addition, subtraction, multiplication and division; with each topic being scaled up

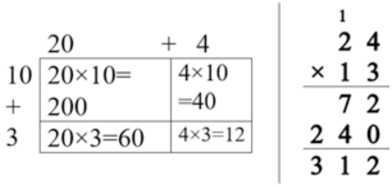

Fig. 1 Solving 24 X 13. The area model is on the left and the "traditional" method is on the right. (Please see Kajander, Part VI, this volume for creating area models with algebra tiles.)

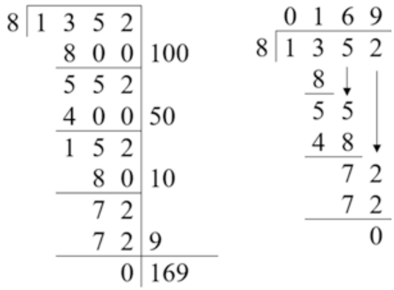

Fig. 2 Solving 1354 ÷ 8. Repetitive subtraction is on the left and long division is shown on the right

in difficulty for the corresponding grades, i.e., Grade 12U addition problems would be significantly more challenging than a Grade 9 Applied[1] level.

As the chair of the department and supposed curriculum leader, I felt it my responsibility to seek out the current teaching methods being used at all schooling levels and bring that information back to share with the department. I found it to be a very interesting and enlightening journey, and I was excited to share some of the "new" methods with my colleagues, which is where we hit our first major road block. The aforementioned "teach the way I was taught" and the "Red" resistance to change all leapt to the forefront. The hot topics were the area model for multiplication versus the "old school" column model of multiplying (see Fig. 1 for an example) and dividing numbers using repetitive subtraction versus long division (see Fig. 2 for an example). The discussion spanned several PLC meetings, even getting to the point where we would ambush fellow colleagues who were not in the department to see what they remembered of multiplication and division from their youth, how they would solve a problem we gave them, and then countering their technique with an alternate strategy to see if they liked the new one more. In the end we did not achieve consensus immediately amongst ourselves, but we did expand our overall knowledge of the topic and albeit somewhat reluctantly expanded the teaching repertoire of most of the department.

[1] See McDougall and Ferguson (Part II this volume, para. 1) for a discussion of two of the possible Ontario pathways (Academic and Applied). Grade 12U are courses in the university mathematics or science pathway in Ontario schools.

We are now in our second full year of working on "The Plan," and it has met with varying levels of success. Fitting in the extra teaching time has been a challenge with all department members. It seems so simple a concept, use 10 minutes a day, three times a week to teach some mini lessons, but with our densely packed curriculum it is easy to lose sight of trying to fit those mini lessons in.

Observation #4: Dealing with Change

The only thing constant in this life is change. As we move into our second year of "The Plan" we still have hurdles for timing but our dynamics in terms of green, yellow and red lights within the department has changed dramatically.

The challenges of a small school mean staffing can change quite radically and quite abruptly. From one term to the next, we moved from having the full complement of red, yellow and green, to now only having yellows and greens. Needless to say the conversations, goals and energy at meetings has changed dramatically. While I am enjoying the new dynamics and I am grateful for the ease at which we currently meet, I am also aware that next semester it could be very different. It is in the more positive times that it becomes necessary to establish department norms and expectations that everyone can buy in on. This gives precedent and structure for when things become less harmonious.

Observation #5: Mathematics Initiatives

If what you are doing is built on solid evidence and student need, stick with it. We started this journey 2 years ago because we recognized the need for improved student skills, specifically in the mental math realm. We had evidence to support our decision, we gathered different strategies and began to implement them immediately. Recently our board began implementing initiatives to improve mental math strategies and more skill based activities in the elementary level. The need we recognized early on, was validated by the board's recognition of the same need, only we had a head start on the process. The key is always the evidence. We live in a world where decisions must be supported with facts, especially decisions that may go against the grain. Non-calculator skill based practice is a polarizing subject in the mathematics realm at this time. The inquiry model versus skill-based learning has stakeholders at all levels expressing their opinion. Personally I believe a balance between both inquiry and skills is needed for success. The skills need to be in place to allow for meaningful inquiry, but the inquiry gives the purpose on why we need to learn the skills. The two concepts feed each other, not just in mathematics but science as well.

In summary, these are observations from my career to date. They are just that, observations, not rules or absolutes. Hopefully these observations will lend some insight into your own career, the career of your peers, or will be at the very least an entertaining insight from the education realm.

Links to Literature

Observation #1: Changing Personal Practices

According to Festinger's theory (1957), at a core level, people want thoughts, attitudes, and beliefs to be "balanced." When dissonance occurs, then there is a desire to make changes to one's thoughts, attitudes and/or beliefs to restore the balance (Heider 1946). In Jarron's example, his dissonance came from outside forces. In order to retain balance, he made changes in his practice so that balance was restored as he found success in using the new practices that he was bringing into his classroom. In essence these experiences causing imbalance allowed for a shift in his beliefs about teaching secondary mathematics. Philipp (2007) notes that "beliefs might be thought of as lenses that affect one's view of some aspect of the world or as dispositions toward action" (p. 259). Some research suggests that efforts to change teacher practices in mathematics have failed partially because of not accounting for the beliefs of the teachers impacted (Grant et al. 1994; Handal and Herrington 2003). In a department head role or working with a colleague, there could be a variety of differing beliefs at play in a single school. Finding a way to cause dissonance with the beliefs of the teachers can be a vehicle for change. Gregoire (2003) adds that these changes in beliefs also need to be accompanied with a way to increase teacher efficacy in the new methods in order for changes to be effective.

Observation #2: Traffic Light Analogy for Professional Development

Research has indicated that team community and collegial support is important for making changes in teacher practices to develop strong teachers (Graham 2007). In looking at those on the stoplight, the "green" individuals are going to be those colleagues who are inspiring to work with and bring new ideas and will reflect on practices with you, but they do not necessarily need the teacher community support. The "yellow" teachers on the other hand will need the teacher community as a vital portion of their growth and change in order to keep them motivated and moving forward. McNeal and Simon (2000) argue that "norms and practices do not change simply by virtue of the teacher using his [sic] authority to assert the new set of rules accompanied by student compliance" (p. 506). Instead teachers need experiences that have them analyze or question their own beliefs and practices (Grant et al.

1994). By relating this back to the stoplight analogy, the "red" teachers will be set in their beliefs and not easy to change. In Holm and Kajander (2015), one of the teachers could have been described as a "red" teacher and it was not until an experience that forced him to confront what he thought he knew about teaching mathematics that he began to think about other ways to teach. In thinking about teacher communities, not only paying attention to beliefs, but also how entrenched they are in those beliefs becomes an important consideration in working together. The "green" teachers would potentially be more fluid and flexible in their beliefs and open to new possibilities. In the end, teacher communities need to prioritize changing practices while creating a place for teachers to engage in learning in order to benefit the students (Vescio et al. 2008).

Observations #3 and 5: "The Plan" and Mathematics Initiatives

These types of collaborative groups are great for professional development since there is also easier access to the relevant knowledge since it is contained in the school itself (Webb et al. 2009). In 2004, DuFour noted that "the best staff development happens in the workplace rather than in a workshop" (p. 63). This can be exemplified in Jarron's observations made about the test being given to the students and the resulting departmental changes. A workshop could have told the group of teachers what to do, but bringing the group together to pool resources and try ideas was effective in making a real difference.

One caveat here in these observations is to note that the teacher community did not just make "The Plan" and leave it be, they took more of an action research approach to the change. "Action research aims to design inquiry and build knowledge for use in the service of action to solve practical problems" (Punch 2009, p. 136). These ideas fit into the work of DuFour and Eaker (1998) who defined professional learning communities by including elements of action research: action orientation and experimentation, continuous improvement, and results orientation. As seen with Jarron's observations, it is this cycle of continuous improvement with a focus on results and then trying out new methods that drives practice by refining strategies for teachers' classrooms. By focusing on this cycle together, the teacher group created a shared personal practice that would include newer strategies that would ultimately benefit their students.

These observations align with the research call for teacher communities to be based in testing and reflecting on research-based strategies (Eaker 2002; Eaker et al. 2002; Hord and Sommers 2008; Gojmerac and Cherubini 2012) and teachers looking for ways to improve their personal practices (DuFour and Eaker 1998). Aligning with the observations provided by Jarron, the teacher group collaboration combined teacher needs and beliefs based on their classroom observations with the theories of researchers, as Bednarz et al. (2007) has noted is important. The Ontario Ministry of Education (2007) encourages a focus on results by keeping student learning at the centre of the teacher discussions through teachers who are reflective of their teaching and consider student achievement when making instructional decisions. This can be seen clearly in Observation #5. The push in the group was to keep students at the front of the decisions about what to keep working with and what would need to be changed. Ultimately the teachers have seen firsthand that working together is leading to improvements in student achievement which has also been noted in the research (i.e., Vescio et al. 2008).

In the end, teacher communities can serve an important role in the developing practices of teachers and the learning of students. Research has shown that working collaboratively is important to the field and has an impact on teacher retention (Webb et al. 2009). As seen in the observations provided by Jarron, these groups necessarily need to focus on the needs of their own classrooms, but also push to make changes and then test those changes to examine the results. The beliefs of the teachers involved in the teacher community can also have an impact by either forcing them to stay still ("red") or leaving them open to make changes and work on their own ("green"). The need for a catalyst moment to be an impetus to change was also highlighted by the observations. As Jarron believes, it is important that teachers "Prepare the child for the path, not the path for the child."

References

Bednarz, N., Maheux, J., & Barry, S. (2007). Context-based professional development in mathematics education: Analysis of a collaborative research process aimed at supporting it. In T. Lamberg & L. R. Wiest (Eds.), *Proceedings of the 29th annual meeting of the North American chapter of the international group for the psychology of mathematics education* (pp. 788–795). Stateline: University of Nevada, Reno.

Brahier, D. J., & Schäffner, M. (2004). The effects of a study-group process on the implementation of reform in mathematics education. *School Science and Mathematics, 104*(4), 170–178.

DuFour, R., & Eaker, R. (1998). *Professional learning communities at work: Best practices for enhancing student achievement*. Bloomington: National Educational Service.

DuFour, R. (2004). The best staff development is in the workplace, not in a workshop. *National Staff Development Council, 25*(2), 63–64.

Eaker, R. (2002). Cultural shifts: Transforming schools into professional learning communities. In R. Eaker, R. DuFour, & R. Burnette (Eds.), *Getting started: Reculturing schools to become professional learning communities* (pp. 9–29). Bloomington: National Educational Service.

Eaker, R., DuFour, R., & Burnette, R. (2002). Getting started: A conceptual framework for creating a professional learning community. In R. Eaker, R. DuFour, & R. Burnette (Eds.), *Getting started: Reculturing schools to become professional learning communities* (pp. 1–8). Bloomington: National Educational Service.

Festinger, L. (1957). *A theory of cognitive dissonance*. Evanston: Row, Peterson.

Gojmerac, A., & Cherubini, L. (2012). Connecting for greater purpose: An examination of a district professional learning community. *Journal of Teaching and Learning, 8*(1), 43–58.

Graham, P. (2007). Improving teacher effectiveness through structured collaboration: A case study of a professional learning community. *RMLE Online, 31*(1), 1–17.

Grant, T. J., Hiebert, J., & Wearne, D. (1994). *Teachers' beliefs and their responses to reform-minded instruction in elementary mathematics*. In Paper presented at the annual meeting of the American Educational Research Association, New Orleans. Retrieved from ERIC database online. (No. ED376170)

Gregoire, M. (2003). Is it a challenge or a threat? A dual-process model of teachers' cognition and appraisal processed during conceptual change. *Educational Psychology Review, 15*(2), 147–179.

Handal, B., & Herrington, A. (2003). Mathematics teachers' beliefs and curriculum reform. *Mathematics Education Research Journal, 15*(1), 59-69.

Heider, F. (1946). Attitudes and cognitive organization. *The Journal of Psychology, 21*(1), 107–112.

Holm, J., & Kajander, A. (2015). Lessons learned about effective professional development: Two contrasting case studies. *International Journal of Education in Mathematics, Science, and Technology, 3*(4), 262–274.

Hord, S. M., & Sommers, W. A. (2008). *Leading professional learning communities: Voices from research and practice*. Thousand Oaks: Corwin Press.

Hord, S. M. (2009). Professional learning communities: Educators work together toward a shared purpose—Improved student learning. *Journal of Staff Development, 30*(1), 40–43.

McNeal, B., & Simon, M. A. (2000). Mathematics culture clash: Negotiating new classroom norms with prospective teachers. *The Journal of Mathematical Behavior, 18*(4), 475–509.

Ontario Ministry of Education. (2007). *Professional learning communities: A model for Ontario schools* (Secretariat: Special Edition #3). Toronto: Ontario Ministry of Education. Retrieved from http://www.edu.gov.on.ca/eng/literacynumeracy/inspire/research/PLC.pdf

Philipp, R. A. (2007). Mathematics teachers' beliefs and affect. In F. Lester Jr. (Ed.), *Second handbook of research on mathematics teaching and learning* (pp. 257–315). Charlotte: Information Age Publishing.

Punch, K. F. (2009). *Introduction to research methods in education*. Thousand Oaks: SAGE Publications, Inc.

Shulman, L. S., & Shulman, J. H. (2004). How and what teachers learn: A shifting perspective. *Journal of Curriculum Studies, 36*(2), 257–271.

Vangrieken, K., Meredith, C., Packer, T., & Kyndt, E. (2017). Teacher communities as a context for professional development: A systematic review. *Teaching and Teacher Education, 61*, 47–59.

Vescio, V., Ross, D., & Adams, A. (2008). A review of research on the impact on professional learning communities on teaching practice and student learning. *Teaching and Teacher Education: An International Journal of Research and Studies, 24*(1), 80–91.

Webb, R., Vulliamy, G., Sarja, A., Hämäläinen, S., & Poikonen, P. (2009). Professional learning communities and teacher well-being? A comparative analysis of primary schools in England and Finland. *Oxford Review of Education, 35*(3), 405–422.

Part II: Commentary – Shifting to a Culture of Inclusion

Kim Beswick

Introduction

The chapters that comprise this section are essentially about what "good" mathematics teaching is and how teachers come to practice "good" mathematics teaching. Such teaching is often characterised as student-centred or reform-oriented and contrasted with traditional or teacher-centred pedagogy, although Davis et al. suggest the alternate and rather broader descriptors, StandardizedEd and AuthenticEd. Inclusion requires a focus on the needs of all learners and hence is necessarily student-centred but such a stance alone does not prescribe pedagogy with any degree of specificity. That is a subtler matter and one that relates to aims of the teacher and the of education system and society in which she/he works.

It is unsurprising that the teaching advocated for the learners who are the focus of the chapters in this section is the kind of teaching that we know is effective for all learners but I begin this commentary with some remarks about what "good" mathematics teaching might be and the ways in which this relates to inclusive practice. I then make some comments about the role of professional learning and particularly the promotion of strategies as a means of promulgating inclusive teaching of mathematics, and on the notion of a culture of inclusion. Together, these remarks form a backdrop against which I comment briefly on the individual chapters. I finish with a reminder of the complexity of teachers' careers and cumulative nature of the learning.

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Inclusive Mathematics Teaching

Mathematics educators have articulated goals for school mathematics in terms of what it means to learn and do mathematics; to think mathematically. Notably, Kilpatrick et al. (2001) described the goal of school mathematics as the development of mathematical proficiency. Versions or subsets of the five strands that they saw as comprising mathematical proficiency—conceptual understanding, procedural fluency, strategic competence, adaptive reasoning, and productive disposition—feature in, for example, the mathematics curricula of Australia (ACARA 2017), Ontario (Ontario Ministry of Education 2005a, b), and Singapore (Singapore Ministry of Education 2012). Although the messages about what it means to learn and do mathematics, that is, to be mathematically proficient, captured in the process strands of these curriculum documents can be undermined by other government agendas such as standardised testing regimes (e.g., Australia's National Assessment Program: Literacy and Numeracy), they provide some guidance for teachers about what it is they are mandated to achieve, and from this pedagogy should follow. It could be argued that any pedagogy that enables all students to develop mathematical proficiency is "good" and inclusive pedagogy. The degree to which teachers aim to develop their students' mathematical proficiency is related to their understanding of that concept and the extent to which they agree that it is the aim of school mathematics.

Hersh (1986) suggested a reciprocal relationship between teachers' beliefs about the nature of mathematics and the way that they teach it:

> One's conception of what mathematics is affects one's conception of how it should be presented. One's manner of presenting it is an indication of what one believes to be most essential in it …The issue, then, is not, "What is the best way to teach?" but, "What is mathematics really all about?" (p. 13)

Mathematics teachers' beliefs about what it is that they are trying to teach are foundational to how they go about teaching it, but it is also true that teachers may have differing contexts in mind when they consider what they are aiming to achieve. Many teachers will be thinking about mathematics in terms of school mathematics as defined by curriculum content (Beswick 2012) rather than mathematical proficiency. They may also answer the question differently according to the students that they have in mind or in front of them. The "most essential" elements of mathematics for students deemed low attainers is more likely to be automatic recall of basic facts, and facility with life related (although often contrived) calculations than it is to be understanding powerful mathematical ideas or developing the capacity to reason or solve problems (Beswick 2017). Even teachers who endorse mathematical proficiency of the sort described by Kilpatrick et al. (2001) may place conscious or unconscious caveats around that goal. They believe mathematical proficiency is important BUT: these students need to be ready for the external examination or standardised test; students must master "the basics" first; or it is not realistic for these students. We know that mathematics teachers tend to believe that mathematical ability is innate (Boaler and Sengupta-Irving 2016); a belief that is almost certainly linked to the resistance, uniquely staunch among mathematics teachers compared to their colleagues teaching other subjects, to teaching mathematics in mixed-ability groups (Horn 2006) and one that militates against mathematics teachers believing that they can influence student learning (Boaler and Sengupta-Irving 2016).

In addition to the beliefs of individual teachers, the culture that teachers create in their classrooms cannot be insulated from the broader culture in which they are situated because teachers are necessarily members of the societies in which they live and work. In many Western societies, including Canada, the increasing focus on education and particularly education in Science, Technology, Engineering and Mathematics (the so-called STEM disciplines), as a driver of economic growth it is unlikely that the nature of proof or the beauty of mathematical structures rate highly in most teachers' minds when thinking about what is most important in what they are trying to achieve as teachers of mathematics. The extent to which various categories of learners are marginalised or deemed to be low attainers or experiencing difficulties in learning mathematics can thus be seen as socially, culturally and politically constructed (Broderick et al. 2012). We know that not all students are afforded the same opportunities to learn (Schmidt et al. 2015). Differences accrue from a range of factors that combine to constitute varying levels of socioeducational advantage and not all of the contributing factors can be influenced directly by schools or teachers. Nevertheless, some factors, like the opportunity to learn defined in the Program for International Assessment (PISA) as exposure to content (Schmidt et al. 2015), can be. Teachers have a responsibility to work to mitigate the impacts of socioeducational disadvantage as well as disability.

Teacher Development

Nespor (1987) pointed out that teachers change their practice when they see a need to change and have an alternative available that they regard as feasible. Although more recent research has not contradicted this formulation it is apparent that, even when Nespor's conditions are in place, change is usually neither a simple nor easy process. Reid and Zack (2010), for example, drew attention to the vulnerability of teachers engaged in changing their practice and to the importance of emotional engagement in the process. Pui-Wah (2008) implicitly incorporated both Nespor's (1987) preconditions for change and Reid and Zack's (2010) observations about teachers engaged in change in her concept of meta-learning ability. From her case-studies of kindergarten teachers Pui-Wah (2008) identified the capacity to identify and confront problems in current practice, and persistence and determination in seeking alternative approaches as essential teacher qualities that enable change. Confronting problems connotes the emotional work involved in Nespor's (1987) idea of seeing a need to change, and persistence and determination convey an active approach to finding a plausible alternative paradigm. Pui-Wah (2008) challenged developmental models of teacher change grounded in studies of novice and expert teachers that, simplistically interpreted, suggest that experience and growth go hand-in hand.

The social context of teacher learning has increasingly been recognised, largely based upon work on communities of practice (e.g., Wenger 1998). More recently the importance of collaboration to the success of professional learning communities has been stressed (Jäppinen et al. 2016). Related to the emphasis on the social context of teacher learning is the notion of culture, comprising the values and norms shared by members of a collective (e.g., a class, school, or education system). MacNeil et al. (2009) stressed the importance of the principal to the culture of a school. Analogously teachers are crucial to establishing and maintaining the culture of their classes. Teacher development is supported in a culture in which "beliefs, values, attitudes, expectations, ideas and behaviours" are aligned with the desired change (MacNeil et al. 2009, p. 74). An individual teacher's shift toward inclusive practice is thus more likely to occur and to be sustained when the beliefs and values that underpin that shift are shared by colleagues, the school leadership, and ideally by society. In spite of the importance of culture, relatively little research has investigated the nature and development of culture among mathematics teachers, and although much of the literature about culture that can be found in relation to leadership applies, groups of teachers engaged in teaching mathematics share in a particular epistemic culture related to the subject they teach (Knor-Cetina 1999).

Alongside this social emphasis is the longstanding acknowledgement that learning is an individual process in which the life and work experiences of individual teachers interact (Connelly and Clandinin 1985). Professional learning or development is essentially biographical and hence "filled with plateaus, discontinuities, regressions, spurts and dead ends" (Huberman 1995, p. 196). A particular professional development event, program or project is just one more experience and hence although conventional wisdom concerning effective professional development emphasises the importance of long term engagement there is evidence that one of events occurring at an opportune time can have significant impact (Beswick et al. 2017).

In spite of the unpredictability of teachers' development that arises from the complexity outlined above, professional development programs and projects are designed to influence teachers in some particular direction and/or to foster some change in pedagogy that is deemed desirable. Different programs are based on different frameworks intended to encapsulate the change that is hoped for. These can take the form of sets of principles (e.g., Muir, 2008), lists of practices or strategies (e.g., Sullivan 2011), or contrasting paradigms (e.g., Boaler and Sengupta-Irving 2016). While most have merit they are necessarily incomplete having been compiled or selected based on various judgements about priorities and the perceived needs and interests of participating teachers. In the course of their careers teachers may encounter many such formulations of desirable practice. Questions arise as to how teachers make sense of the latest encapsulation of good practice in relation to their own experience and current knowledge, and the community in which they are currently working.

Comments on Individual Chapters

Six of the chapters warrant their place in a section ostensibly about inclusive cultures in mathematics classrooms by being situated in a context involving one or other group of learners considered marginalised by "usual" mathematics teaching: special needs learners (Davis et al.), English language learners (Barwell, Kubota-Zarinnij, & Culotta), disengaged low attainers (Kajander; McDougall & Ferguson), and Indigenous learners (Sterenberg & O'Connor). Inclusion requires attention to all learners as individuals: their thinking, life-experiences to date, and particular needs. By considering learners who might be considered "difficult cases" we can learn much that is of value for all learners. The danger is that we restrict our attention to cases where business as usual is clearly not satisfactory and fail to apply the learning more broadly. The remaining four chapters (Childs & Holm; Oesterle; Pyper; Sterenberg and O'Connor), chronicle authors' personal stories of learning to teach mathematics. Sterenberg and O'Connor siuate their reflections in the context of Canadian policy and curriculum initiatives.

Davis et al. describe four strategies that emerged from their work with teachers working with special needs learners that draw attention to particular aspects of teaching that are important and situate these with the notions of StandardizedEd versus AuthenticEd. Each of these are underpinned by deep seated metaphors of beliefs that drive teaching and explain the difficulty of shifting practice. Teaching is an outworking not only of what the teacher believe mathematics is (Hersh 1986) but also of what is understood about the nature of knowledge, and what it means to learn and to teach. The chief value of this chapter is in drawing attention to these things for, although the strategies identified have resonance with many other mathematics classrooms and certainly not only those for students identified as having special needs, it is the underpinning principles and the process of arriving at the strategies in such a way that the teachers own them that is powerful and most likely to be transferable in the sense of underpinning future work with other groups of teachers. The comparison of learning to teach differently with learning another language recognises the difficulty of not just changing particular actions but of shifting one's paradigm and nicely explains the ease with which teachers slip back into familiar ways of operating. As Davis et al. state, teachers, like language learners, need to have many opportunities to produce the new pedagogy, often imperfectly, in order eventually to achieve fluency in it.

The importance of producing language is taken up by Barwell et al. in a literal sense. Learners of mathematics need to produce mathematics just as English language learners need opportunities to write and speak English. In the context of mathematics classrooms, they need opportunities to produce English communication of mathematics. Although the context in which the work reported by Barwell et al. was conducted was challenging in the extreme, the questions posed and strategies that emerged could usefully be asked and applied in any context.

Kajander highlights the need to attend to individuals, and the power of doing so, and also to the need to be cognisant of students' affective states. Disengagement can be a quiet phenomenon that that does not necessarily attract the teacher's attention; it can place students at risk of being ignored. As Davis et al. illustrated students can be engaged in trying to get answers; that is in the performance of school mathematics but disengaged from meaningful learning. One of the students that Kajander describes, as well as the classes discussed by McDougall and Ferguson and by Macaulay, was in the Year 9 Applied mathematics stream that appears to be usual in Ontario. Macaulay reviews literature that shows that students in lower ability streams have reduced opportunities to learn mathematics as a result of less rigorous curriculum, lower academic expectations and poorer quality teaching. None of the authors challenge the association of applications of mathematics with lower ability (actually prior attainment). Attempting to develop inclusive practice within what is an inherently exclusionary structure strikes me as an attempt to staunch an arterial bleed with a tiny sticking plaster. Nevertheless, mathematics teachers whose power does not extend to structural change, are compelled to attempt inclusive pedagogy in these settings and it is testament to their determination and that of their learners that positive change can be made. At least as moving as the stories of the three students is that of Diane's teacher retreating from non-traditional teaching in the face of classroom management difficulties. In classrooms constituted of learners who have been labelled by their assignment to the class as incapable, it is surprising how infrequently the accumulated frustration and insult do not manifest in rebellion.

For McDougall and Ferguson positive change was defined as improvement of standardised test scores for Year 9 Applied mathematics students. The improvements achieved, from a low base, by most schools are impressive and worthwhile even though arguably more, or at least equally important goals related to mathematical proficiency and students affect may not have been furthered in the process. McDougall and Ferguson illustrate how a relatively large scale project can be customised by individual schools and although there is a tendency to equate student-centred pedagogy with specific practices like group work, the authors' observation that at the end of the project teachers tended to take greater responsibility for student learning suggests deeper change might have occurred.

After discussing the ways in which streaming contributes to lower attainment Macaulay describes four Ontario schools that achieved higher than expected results for year 9 students. She summarised the approaches used as ten recommendations for teachers of the Applied stream. None of these are surprising and would be sound recommendations for teachers of any students. Macaulay's research was focussed on practicalities but it would have been interesting to have explored the beliefs of the teachers of the Applied students in these schools and the extent to and ways in which school cultures may have supported their efforts. Although the achievement gap between students in the Academic and Applied streams was less than that for the province overall, there still was a gap. This points to a problem ameliorated rather than solved. Although teachers have limited influence beyond their classrooms, researchers in these contexts have an obligation to be mindful of the broader correlates of achievement gaps. For example, socio-economic status is related to students' learning both directly as well as through reduced opportunity to learn (Schmidt et al. 2015). One of the negative impacts of standardised testing and associated achievement norms is that it normalises gaps related to such things as SES or subject stream.

Oesterle summarises what she has learned from her teaching career in terms of knowing, nurturing and noticing. The knowledge referred to aligns with various categories distinguished in the mathematics education literature. Nurturing refers to students' mathematical "habits of mind." It encompasses the strands of mathematical proficiency and Oesterle makes the link between these, particularly productive disposition, and developing a growth mindset. She also acknowledges the importance of teachers having a growth mindset in relation to their own learning—a point made by Kilpatrick et al. (2001) but often overlooked. Noticing, as described by Oesterle, amounts to paying attention to students and their learning. The description of good teaching provided in this chapter amounts to current orthodoxy in mathematics education, drawing upon a variety of influential ideas. Pyper provides similarly mainstream advice to pre-service teachers, focussing particularly on mathematical habits of minds and taking a dynamic view of what it means to know mathematics. The question that Oesterle's account raises in my mind but that she does not answer is why is it that she has developed an orientation to her teaching of constantly seeking to improve whereas other teachers do not? What is it that makes the difference? Similarly, how has Pyper come to his views? Is the experience of becoming a teacher educator, as Oesterle and Pyper have, necessary or sufficient and if it is, what is it about that experience that makes it potent?

For Childs (Childs & Holm), the pivotal experience seems to have been assuming responsibility for mathematics teaching in his school, rather than becoming a mathematics educator. Childs and Holm outline a series of "observations" that they see as leading to their emphasis on collaboration for teacher learning, and of establishing a school culture in which that is the norm. Is it the experience of being or feeling responsible for other teachers' learning, either as a mathematics teacher educator or as a head of department in a school, that triggers growth? Might such experiences stimulate an intensity or quality of reflection on one's views that is rare in the context of everyday school teaching?

Sterenberg and O'Connor also reflect on their journeys toward culturally responsive teaching. They describe how they have come to see Western and Indigenous mathematics as complementary. Theirs is a story of learning to respect all learners that, common to all of the accounts of inclusive practice in this section, has implications for mathematics teaching for all learners. Their stories are different but include encounters with Indigenous culture and learners. What is it about these encounters that was powerful and how can teachers who have not had similar experiences develop similar awareness and sensitivity?

What is the research contribution of personal stories of development such as those of Oesterle; Pyper; Childs and Holm; and Sterenberg and O'Connor, and Kajander's account of her encounters with individual students? First, they illustrate Huberman's (1995) point about the unique and non-linear nature of teachers' careers and remind us that even though we devise programs and processes based on research and that these may appear to be broadly effective in whatever way we have defined that, individual teachers will, like learners in any context, respond in idiosyncratic ways. Not all schools, for example, showed improvements in standardised test scores despite participating in the same program (McDougall & Ferguson). Second, and relatedly, they can remind us that there is a lot going on in teachers' work lives. When teachers reflect on the trajectory of their career a range of largely unplanned experiences and opportunities are interpreted as a coherent sequence that led them to where they currently are (Huberman 1995). Those who work with teachers need to be aware that the impact of their input into the experience of individual teachers cannot be predictable. Nevertheless, lessons can be learned from commonalities in individual stories, including from the questions that they raise. They can also highlight particular aspects of practice in particularly powerful ways such as Kajander's reminder to attend to the emotional connotations of teachers' beliefs as a means of helping teachers to reflect on their beliefs and their implications for students, or Sterenberg and O'Connor's foregrounding of respect and cultural appropriateness.

Concluding Remarks

The overall impression gained from considering this collection of chapters is that it is not the particular recommendations that are used to frame teachers' efforts to change practice or that arise from their work, but rather it is the experience of the process of change; working with colleagues (including in unstructured and informal ways) to generate principles, lists of strategies, or processes that are meaningful to them in their context, that is powerful. There is no definitive formulation of inclusive pedagogy that can be transferred from one context to another, although differing formulations have commonalities. We can, however, learn from the processes by which others learned and were supported to learn. We make progress by distilling general principles, not to tell others to implement, but to underpin our own work with colleagues.

Regardless of whatever progress has been made in understanding how shifts in pedagogy occur, the models encapsulated in these chapters in both the descriptions of projects and accounts of personal journeys can be understood in terms of Huberman's (1995) open collective development cycle. Although conceived as a collaborative model in formal professional development contexts, the model is readily applicable to an individual who has normal informal access to colleagues and external sources of information perhaps in the form of literature or colleagues in other schools or universities. It can also be considered to operate over a longer time period than the duration of most professional development projects and repeated in different ways throughout a teacher's career.

In this model conceptual inputs (e.g., from university researchers, informal encounters colleagues, or personal reading) give rise to the sharing of experiences in a collaborative context or perhaps internally as an individual becomes conscious of particular elements of his/her experience. Further inputs lead to the development of new methods and experimentation with them. This process may be supported by outside experts or colleagues, or driven by personal attention to the data provided by students. Huberman (1995) presents the next part of the cycle as exchanges of data with colleagues or external experts who may assist with analysis and provide further inputs leading to further experimentation. Further external inputs may occur before the new method is either applied or abandoned and the cycle repeated. The process can be seen as continuous across an individual teacher's career including as that teacher moves between schools or, as was the case for several of the authors, into pre-service teacher education contexts. The learning about how to create inclusive classroom cultures is in the process of becoming an inclusive teacher/educator.

References

Australian Curriculum, Assessment and Reporting Authority. (2017). *Australian curriculum: Mathematics F-10*. Retrieved from http://v7-5.australiancurriculum.edu.au/mathematics/curriculum/f-10?layout=2#page=1

Beswick, K. (2012). Teachers' beliefs about school mathematics and mathematicians' mathematics and their relationship to practice. *Educational Studies in Mathematics, 79*(1), 127–147.

Beswick, K. (2017). Raising attainment: What might we learn from teachers' beliefs about their best and worst mathematics students? In C. Andrà, D. Brunetto, E. Levenson, & P. Liljedahl (Eds.), *Teaching and learning in math classrooms. Emerging themes in affect-related research: Teachers' beliefs, students' engagement and social interaction* (pp. 95–106). New York: Springer.

Beswick, K., Fraser, S., & Crowley, S. (2017). "The best ever": Mathematics teachers' perceptions of quality professional learning. In B. Kaur, W. K. Ho, T. L. Toh, & B. H. Choy (Eds.), *Proceedings of the 41st conference of the International Group for the Psychology of Mathematics Education* (Vol. 2, pp. 169–176). Singapore: PME.

Boaler, J., & Sengupta-Irving, T. (2016). The many colors of algebra: The impact of equity focused teaching upon student learning and engagement. *The Journal of Mathematical Behavior, 41*, 179–190.

Broderick, A. A., Hawkins, G., Henze, S., Mirasol-Spath, C., Pollack-Berkovits, R., Clune, H. P., & Steel, C. (2012). Teacher counternarratives: Transgressing and 'restorying' disability in education. *International Journal of Inclusive Education, 16*(8), 825–842.

Connelly, F. M., & Clandinin, J. (1985). Personal practical knowledge and the modes of knowing: Relevance for teaching and learning. In E. Eisner (Ed.), *Learning and teaching the ways ofkKnowing* (pp. 174–198). Chicago: National Society for the Study of Education.

Horn, I. S. (2006). Lessons learned from detracked mathematics departments. *Theory Into Practice, 45*(1), 72–81.

Hersh, R. (1986). Some proposals for reviving the philosophy of mathematics. In T. Tymoczko (Ed.), *New directions in the philosophy of mathematics* (pp. 9–28). Boston: Birkhäuser.

Huberman, M. (1995). Professional careers and professional development: Some intersections. In T. R. Guskey & M. Huberman (Eds.), *Professional development in education: New paradigms and practices* (pp. 193–224). New York: Teachers College Press.

Jäppinen, A.-K., Leclerc, M., & Tubin, D. (2016). Collaborativeness as the core of professional learning communities beyond culture and context: Evidence from Canada, Finland, and Israel. *School Effectiveness and School Improvement, 27*(3), 315–332.

Kilpatrick, J., Swafford, J., & Findell, B. (Eds.). (2001). *Adding it up: Helping children learn mathematics*. Washington, DC: National Academy Press.

Knorr-Cetina, K. (1999). Epistemic cultures: How the sciences make knowledge. Cambridge, MA: Harvard University Press.

MacNeil, A. J., Prater, D. L., & Busch, S. (2009). The effects of school culture and climate on student achievement. *International Journal of Leadership in Education, 12*(1), 73–84.

Muir, T. (2008). Principles of practice and teacher actions: Influences on effective teaching of numeracy. *Mathematics Education Research Journal, 20*(3), 78–101.

Nespor, J. (1987). The role of beliefs in the practice of teaching. *Journal of Curriculum Studies, 19*(4), 317–328.

Ontario Ministry of Education. (2005a). *The Ontario curriculum: Grades 1–8: Mathematics*. Toronto: Queen's Printer for Ontario. Retrieved from http://www.edu.gov.on.ca/eng/curriculum/elementary/math18curr.pdf

Ontario Ministry of Education. (2005b). *The Ontario curriculum: Grades 9 and 10: Mathematics*. Toronto: Queen's Printer for Ontario. Retrieved from http://www.edu.gov.on.ca/eng/curriculum/secondary/math910curr.pdf

Pui-wah, D. C. (2008). Meta-learning ability: A crucial component for the professional development of teachers in a changing context. *Teacher Development, 12*(1), 85–95.

Reid, D., & Zack, V. (2010). Observing the process of mathematics teacher change – Part 1. *Journal of Mathematics Teacher Education, 13*, 371–374.

Schmidt, W. H., Burroughs, N. A., Zoido, P., & Houang, R. T. (2015). The role of schooling in perpetuating educational equality: An international perspective. *Educational Researcher, 44*(7), 371–386.

Singapore Ministry of Education. (2012). *Mathematics syllabus: Primary one to five*. Retrieved from https://www.moe.gov.sg/docs/default-source/document/education/syllabuses/sciences/files/primary_mathematics_syllabus_pri1_to_pri5.pdf

Sullivan, P. (2011). *Australian education review teaching mathematics: Using research-informed strategies*. Camberwell: Australian Council for Educational Research.

Wenger, E. (1998). *Communities of practice: Learning, meaning and identity*. New York: Cambridge University Press.

Part III
Fostering Relationships

Part III: Preface – Building Relationships Through an Ethic of Care: Insights from the Work of Julie Long

Elaine Simmt

In my reflection on the papers from "Fostering relationships," I raise the concept of care – care as an ethical stance in teaching. "An ethic of care emphasizes the particular, as carers recognize and respond to individuals, as well as attending to context" (Long 2008, p. 116). I raise this concept of care for two reasons: firstly, because care is a theme that emerged for me from the papers, as I read about the ways in which particular teachers interact with learners to enhance their educational experiences; and secondly, I raise it to insert into this volume the contributions of a Canadian scholar who had much to contribute but too little time to make those contributions. In my reading each author in this section points to relationships among teachers and learners that are fostered through care.

Canadian scholar and mathematics teacher educator Julie Long[1] explored the notion of care within the context of teaching children mathematics. She explains her orientation to care.

> While my use of the words care and caring includes both an emotional aspect and an intellectual aspect, its main focus is an ethical stance. An ethic of care is a way of living, of teaching and learning, of considering dilemmas, and of working with others. It goes beyond helping, beyond being nice or polite. It requires intellectual work and is associated with emotional risk, just as studying mathematics involves work and risk. (Long 2008, p. 24)

In her work she asked an important question: How does a teacher care for students and mathematical ideas? In Long's response she uses conceptions of caring actions initially explored by Nel Noddings: profound attention and reciprocity. She extends Noddings' work by nuancing the notion of proximity (pp. 49–50). These caring actions, Long asserts, are found in mathematics classrooms where the mathematics

[1] Dr. Julie Long passed away at the age of 42 in March 2016. Julie was an assistant professor of elementary education at the University of Alberta.

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teacher focuses on sense making, takes seriously student questions, accepts mistakes as opportunities for learning, helping students learn to deal with their frustration, and by encouraging students to make conjectures. (Importantly, she includes caring for mathematics as an element of this ethic.) Underpinning all of these actions she asserts is profound attention; listening being one form of profound attention. “If you care you must listen” (student cited in Long 2008, p. 51). From listening, Long notes reciprocity can emerge with teacher and learner responding to one another. The back and forth of give and respond contributes to the caring relationship (p. 58). Finally, she explains how proximity is about the physical, emotional and intellectual closeness needed for the teacher and students to relate with each other and mathematics. With these concepts in mind, I turn to the chapters in this section.

As I read each chapter I considered the relationships being described and I began to connect to the conceptions of care that Long wrote about. Roth’s paper begins by describing a teacher who attends to both the intellectual and affective dimensions of a student’s cognition. Hurlington’s and Jao’s analyses focus on the relationships between the intellectual and social needs of the learner. Chorney’s study examines the teacher’s actions and interactions with learners within the constraints of conforming to external and systemic demands for assessment. Newell, reflects on his actions as an inexperienced teacher as he attempted to build relationships with students when he arrived as the “new teacher” to their schools.

Roth (as both teacher and researcher) reflects on his own teaching experiences with students who disengaged from participating in mathematics classes to address the importance of the intellectual and the affective dimensions of cognition. He comments on how his own negative experiences with mathematics impacted him emotionally. However, it is those experiences that he believes led to the empathy he had for his learners: a student who was seen as “slow” by others, a disengaged student, and a class of “slow learners.” He suggests that although he was able to work with these learners because of his empathy for them, it took him many years to identify and connect with the theoretical work that explained the complicit relationship between intellect and affect when they are “[t]aken together with the practical aspects of life” (Roth, this volume). As Roth explains, “both are integral parts of a human relation; and the relations with other persons are the genetic origins of all higher psychological functions and personality.” Long too makes the connection of the emotional and intellectual aspects of learning to the caring relationship. I believe Roth demonstrates another aspect of this ethic of care in his work as a researcher by trying to understand his teaching relationships and actions through theorizing them. Long posits that the teacher’s ethic of care must include learners and mathematics, I would like to suggest that as researchers there must be care for the researched and care for the educational theories we explore.

Turning to the chapter by Hurlington, where he explores building student resiliency, provides another example of focused teacher attention. He claims, “[t]he first principles of all resilience enriching classroom environments comes back to developing meaningful relationships with student.” In this study, the teacher’s attention (much like a talent scout) is focused on the strengths and talents of learners. By paying attention to and acknowledging student strengths, talents and existing knowledge the teacher can act to enhance student resilience. Long would agree that by teachers focusing on learners' strengths rather than weaknesses and deficiencies teachers demonstrate an ethic of care.

Jao reports that teachers who focus on reinforcing student strengths have positive effects on learner engagement in at-risk classrooms. The teachers in Jao's study endeavoured to enhance student engagement in the mathematics class by focusing on social interaction in the classroom among students and with the teacher, as well as interaction with concrete materials from which the learners could make meaning of mathematical concepts. The teachers in this study demonstrated an ethic of care through their invitations to learners to participate in small groups and by providing academic and social support for the learners' development. To be more specific, Long's (2008) notion of proximity (closeness) is manifested in the choices the teacher makes to have students work with each other in small groups. In Long's study the she noted that "[t]heir proximity [of teachers and students] creates a physical, personal, and intellectual intimacy as Karen [the teacher] works to be close to them, to who they are, and to what they might be thinking" p. 68). Jao reports the students in the at-risk classrooms became more engaged as they experienced these kinds of caring actions.

Chorney's paper provides an illustration of a teacher whose actions may be seen as a counter example to the ones explicated above. In this study, learners were provided with digital materials that resulted in deep exploration of the mathematics under consideration—at least as witnessed by the researcher. However, the depth of the learners' meaning making "went unappreciated [by the teacher] because of the institutional socialization of the teacher to evaluate the students quantitatively based on content rather than qualitatively based on critical thinking" (Chorney, this volume, abstract). In the paper, Chorney describes the teacher's interaction with a pair of learners as one in which only cursory attention was paid to the learners meaning making because the teacher was only concerned with them expressing the expected answer. The teacher did not linger in close proximity to the learners (Long 2008) observing what they were doing, nor did the teacher listen to them with a hermeneutic ear (Davis 1996). A teacher listening for a particular answer fails to create a space of interaction from which a caring teacher-learner relationship becomes a site of meaning making, for both teacher and learner. Chorney's paper contributes to the critique of the hegemony of summative assessment that is focused on correct answers, and the negative interaction pattern between teacher and student that impedes the caring act of listening for student meaning making.

The account and reflection by Newell speaks to the complexities of building relationships when teachers and students not only do not know each other, but when the context too is unfamiliar to the teacher; in this case the situation involves a novice teacher in an unfamiliar community. Newell describes some experiences from his early years of teaching and then discuss the lessons he learnt about listening. A lesson that Long's work suggests is critical to an ethic of care.

Together the group of papers in this section provide insight into caring relationships between teacher and learners (and mathematics). I close this commentary wondering how can teacher educators begin to focus pre-service teachers' attention on an ethic of care. One possibility is to use Julie Long's descriptive and detailed narratives as well as the theoretical tools she developed to examine pedagogical relationships and actions. A close study of her work provides valuable insights into teaching with an ethic of care.

References

Davis, B. (1996). *Toward a sound alternative*. New York: Routledge.

Long, J. (2008). *Caring for students and caring for mathematical ideas in an elementary classroom.* Unpublished doctoral dissertation, University of Alberta, Edmonton.

Culturing Affect, Affective Cultures

Wolff-Michael Roth

Abstract Psychological theories of learning employed in mathematics education separate intellect and affect. As a result, if affect (emotions) enters investigations of mathematical thinking and understanding at all, it is considered as an outside force or condition that generally diminishes cognition. This, however, is a hidden form of Cartesianism that separates the ideal-mental from the sensual-bodily aspects of being in the world. A very different approach was proposed in the notes of the Russian psychologist L. S. Vygotsky near his death. In the Spinozist-Marxian take of these writings, we find a "unity/identity of intellectual and affective processes" no longer "divorced from the full vitality of life, from the motives, interests, and inclinations of the thinking individual" (Vygotsky LS. The collected works of L. S. Vygotsky, vol. 1: Problems of general psychology. Springer, New York, 1987, p. 50). Educators therefore can foster mathematical thinking and understanding only when they address not only the intellectual side, but affect in intellect. In this chapter, I start with the seeds of the Spinozist-Marxian take found in Vygotsky's last writings and develop them into a post-constructivist account of affect in intellect, which constitutes the foundation of an approach that cultures affect—in the senses of cultivating and making affect a cultural feature—and thereby leads to affective cultures of mathematics education that inherently foster thinking and understanding.

Keywords Concrete human psychology · Emotion · Vygotsky · Mead · Post-constructivist epistemology

Psychological theories of learning employed in mathematics education, including constructivism, separate intellect and affect. As a result, if affect (emotions) enters investigations of mathematical thinking and understanding at all, it is considered as an outside force or condition that generally diminishes cognition. This, however, is

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a hidden form of Cartesianism that separates the supersensual (ideal, mental) from the sensual (bodily) aspects of being. A very different approach was proposed by the Russian psychologist L. S. Vygotsky, sometimes called the "Mozart of psychology." In personal notes written near his death, he complained about the traditional approaches to understanding knowing and learning that equally were taken (and are so to the present day) by scientific and interpretive psychology. He wrote that in these theories "thinking itself became the thinker of thought," an autonomous stream of thought (Vygotsky 1987, p. 50). In the Spinozist-Marxian spirit marking his last writings, he sought to reestablish the unity/identity of intellectual and affective aspects of psychological phenomena. But whereas many readers might expect that this unity/identity be established within the classroom alone, such expectation would be unwarranted. The unity/identity that Vygotsky sought extended beyond the individual's activity in any one classroom. He would have qualified much of mathematics education research today as "divorced from the *full vitality of life*, from the motives, interests, and inclinations of the thinking individual" (Vygotsky 1987, p. 50, emphasis added). For fear of losing the fundamental message in this paragraph, consider the following experiences that have shaped my life as a teacher not only when they happened but also many years later when the past caught up with me in my life as a university professor writing about affect in the mathematics classroom life.

Affect and Mathematics Teaching—An Auto/Ethnographic[1] Grounding

In 2012, I received an email from Paul, a student I had taught mathematics some 30 years earlier. After presenting himself in the attempt to help me remember him, Paul wrote: "You seemed to understand that even though my marks in most classes were barely at the passing grade, I had potential, and that the marks were not a reflection of my potential. Most teachers I had assumed I was 'stupid,' you made me realize they were wrong." He also noted, "As far as teachers go, I thank you for being one of the most positive influences on me."

Paul grew up in an isolated village on the lower north shore of the St. Lawrence River (province of Quebec), connected by dirt road to a few other villages—none more than a few hundred souls, if that much; the entire part of the country is accessible only by boat or plane. The village students desiring to go beyond ninth-grade had to attend boarding school 1000 miles away from home. I remember having been asked by his parents to come over to their house to talk about Paul's future. Even though Paul had not done well in most of his classes other than those that I taught

[1] Auto/ethnography is one of several research methods (apprenticeship being another one) whereby the researchers investigate cultural patterns in their own practices (e.g., Roth 2005). The key to the success of the method is not to wallow in one's own feelings but to approach actions as rigorously as if they were those of someone else.

him (mathematics, science, and physical education/arts), I did indeed encourage him and his parents to continue schooling. I had liked the kid and had done everything I could to help him with his insecurities, which also came out in physical education and arts classes that I was teaching. Why had they chosen me, an outsider to the village rather than the school principal, who was from that village or any other teacher with longstanding roots in the community? The parents had called me over, they said, because they knew that over 90% of the students returned from the fall term to spend Christmas at home, never returning to the distant high school where they tended to fail miserably. The parents did not want Paul to have to go through the experience of attending high school far away without the support of his family. But Paul, like all others whom I had taught mathematics, succeeded (which I attributed to the fact that the students had covered part or all of tenth-grade mathematics by ninth-grade). He finished high school and *Cégep*,[2] started doing "residential construction and renovation and then went into commercial and industrial." He had found his calling. Over the years, he developed a strong understanding of all trades; and most recently he moved into supervision, and management of projects, including the installation of a large oil well system in Alberta.

In the preceding sketch, readers can see the importance of the affective dimension in teaching generally. Affect is not just some abstract thing out there but a characteristic of the relation we all have with other human beings. Some sociologists consider the affect constituting and communicated in concrete relations with others the glue that holds together society as a whole (Collins 2004). It was in our relation that Paul realized that he was not stupid as he considered other teachers to think about him (perhaps because of the grades); indeed, it was the relation *itself* that was the genetic origin of this realization. Such a relation and the realization that comes with it is not reducible to an intellectual act but is affective through and through. It was that relation where Paul felt that the other teachers were wrong; and it was out of the feeling that came with the relation that he dared making the jump to the distant boarding school and subsequently into *Cégep*. In this particular instance, it was not just in mathematics—and the other classes—that was influenced by the affective dimensions of our teacher–student relation but also his school life more generally. Although other teachers said or seemed to be saying that he was "dumb," his feeling of our relation mitigated that judgment for him. In fact, going to finish high school meant having to leave behind the family, the close-knit community, and a specific culture and to exchange these for city life. This is a tremendous emotional step to take for a 14-year-old who had never seen more than his own and a neighboring village.

I had started teaching in that very village after having graduated with a master's degree in physics and without any teacher education training. The lack of jobs during an economic crisis led me to apply for a teaching position in the Canadian north. One of the precepts that guided my teaching since the instant that I first stepped into a classroom career arose from the experience of having had to repeat fifth-grade,

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[2] *Cégep* is a publicly funded post-secondary and college education system where students get general degrees required for university admission or technical training.

and most importantly because I had flunked mathematics. As in Paul's case, teachers thought that I was a dumb village kid who had come to the city to go to grammar school. Like Paul, I eventually did well. I even took mathematics as a major in college and as a minor during my master's degree; and, a few years after teaching Paul, I completed a doctoral minor in statistics. But it was that early experience of failing a grade that later made me attentive not only to the intellectual needs of my students but also—something I really understood only many years later—their affective (emotional) needs. Because I had failed, I could truly empathize—a verb formed from the prefix *em-*, a derivation of *in-*, and *-pathy*, a word particle forming nouns denoting feeling and being affected. It literally means to feel the feelings of another. I was feeling for and with Paul; and it was with and out of this feeling that I related to Paul. But we can only feel what another feels if we have felt it before and if we take the perspective of another on our own feelings (Mead 1972); otherwise we only grasp the feelings intellectually, which is not the same as feeling them. But I knew what Paul and students like him felt because I had felt it before.

The recent exchange with Paul also brought up another memory: that of Earl, one of his classmates. Earl had been in a similar situation. He had already repeated seventh-grade and now, in eighth-grade, he was considered a "bit slow." Because of the large range of abilities in my classes—which in any single class ranged from first- to tenth-grade reading ability—I had the students work in three- to four-member groups with approximately the same needs and capacities. Each group wrote up a weekly contract that stated what it would be doing and learning.

One Monday morning, Earl approached me saying that he did not feel like doing mathematics. I asked, "So what do you want to do instead?" "Read my novel." I told him to go ahead but not to disturb the others in his study group. And so it happened. Earl read while his peers continued fulfilling their contract.

The next morning, Earl approached me again.

– Sir, I don't feel like doing math, he said.
– What do you feel like doing?
– Read my novel.
– Okay, go ahead. Same rule as yesterday: no bothering others.

Tuesday passed the same way as Monday, Earl sitting by himself reading his novel while his peers worked on their contract. Wednesday morning, we repeated the exchange; and we played the same little drama on Thursday and on Friday. But there was a change at the end of the week. After class, as students were getting ready to leave for the weekend, Earl came to me.

– Sir, thanks for letting me read the novel. I just realized that I am way behind the others.
– Yea, so?
– I promise you, Earl replied, in 3 weeks I will have caught up with them.
– Okay, that's a deal.

On the following Monday morning, Earl began doing mathematics without any ado. He did so all week. Then, a week later, he not only had caught up with the others, but he was indeed leading the group. Throughout the year, there was never another instance of that kind. Earl had become the leader of his group and its work schedule.

The year passed by and I had forgotten about the episode when Earl approached me on the last day of school. “Sir,” he says, “you know, I love mathematics.” “That’s great.” “And you know why?” “No, why?” “I knew all year that I do not have to do mathematics whenever I do not feel like doing it.”

In this instance, affect, too is written all over the story. Earl did not feel like doing mathematics. In contrast to what happens in most schools and in most classes, he was not forced to do it—not for an entire week. Whatever it was, the feeling of not wanting to do mathematics did not return for the remainder of the year. In part it had to do with *knowing* that he did not have to do mathematics if he did not feel like. He was and felt in control over his activity, which allowed him to do mathematics when he was emotionally prepared—rather than at the arbitrary time and for the arbitrary duration when the school had scheduled mathematics. As a result, he not only did mathematics but also started liking the subject matter. These were not just words but he showed it in his deeds by becoming the leader of his group. At the end of the year, the group had covered at mastery level (achievement levels above 80%) about one-third of the curriculum of the following year. In this case we also observe the role of affect for the whole person, where there is a hierarchy of needs at any one point in time. During the week when the episode occurred, reading his novel was higher up in the hierarchy of his school-related needs; and those needs pertaining to school were only part of the sensible contexture (Schütz 1932) of all of his life-process and personal needs.

The importance of students’ affective needs, though I did not realize it in this way at the time, became more evident during subsequent years of teaching in a Newfoundland town. I was homeroom teacher to a class of “slow learners,” that is, the students from the bottom achievement quartile (bottom 25%) of all ninth-grade students in that high school. Some of the students already had repeated one or two, and one young woman even 3 years; and they stayed in school (frequently being absent) because the governmental child allowances were contingent on their presence in school. During the non-teaching time that I was required to spend with them in the classroom, the male students often challenged me to arm wrestling matches; and while we were competing, the female students were leaning on us and had their arms on and around our shoulders. Male and female students made me feel like a father or uncle to them. I sensed they were seeking (and getting) an affective relation that they did not get at home or elsewhere. It felt like they were seeking (and perhaps in need of) the warmth of a human relation. At the time—or, depending on the subject matter, separately with them individually—I was told what I initially thought was the domain of the guidance counselor (whom the students did not trust) or parents (with whom they did not feel comfortable talking about their problems). It was here that I came to realize the importance of considering students as real people, for whom mathematics (or science, or school) constituted only a fraction of their intellectual and affective life-processes. Having to go to court because caught with a case of beer, being pregnant, or being subject (or witness) to an abusive relation among their own parents all were so much more important than factoring a polynomial; and these other aspects of life were all-consuming. Indeed, all of these relations constitute the personality of an individual (Mead 1972; Vygotsky 1989). It is thus important for researchers and mathematics teachers alike to take into account the whole person rather than the individuals reduced to their "mathematics identity."

For the longest part of my career as a teacher and then as a research professor, the epistemologies I knew were only concerned with the intellectual aspects of learning—and this is especially the case with the latest educational ideology, constructivism. If there are emotions at all in that theory, then it is considered to be an outside force, something that affects—generally diminishes—(rational) thinking. In (radical, social) constructivist theory, mathematics students and teachers are said to construct their emotions or they are asked about their emotions and all researchers thereby get is emotion talk not emotion itself. It was only when I delved more deeply into the work of L. S. Vygotsky, especially his work of the last 18 months of his life, that I found a theory that was overcoming the body–mind (psychophysical) problem and with it a way of approaching the unity/identity of intellect and affect. Taken together with the practical aspects of life, both are integral parts of a human relation; and the relations with other persons are the genetic origins of all higher psychological functions and personality (Mead 1972; Vygotsky 1989). The consequences of this approach are clear: educators foster mathematical thinking and understanding only when they indeed address not only the intellectual side, but affect in intellect. In his later work, Vygotsky was deeply influenced by two books: B. Spinoza's *Ethics* and K. Marx and F. Engels' *The German ideology*. In the following, I show with empirical materials from a mathematics class the Spinozist-Marxian seeds that Vygotsky was sowing in his last writings. I develop the seeds into a post-constructivist account of affect in intellect, which constitutes the foundation of an approach that cultures affect—in the senses of cultivating and making affect a cultural feature (e.g., Collins 2004)—and thereby leads to affective cultures of mathematics education that inherently foster mathematical thinking and understanding.

Affect in the Mathematics Classroom: A Cultural-Historical Perspective

The works of Vygotsky have been taken up increasingly in the Anglo-Saxon scholarly literature over the past several decades. There are two main problems that have impeded an appropriate take-up of these works: (a) many Russian terms have been inappropriately translated, thereby falsifying what was actually written, including the use of the adjective "mental" rather than the "psychical" or "psychological" that are found in the original and translations into other languages; and (b) the radical reversal of Vygotsky's theoretical position near the end of his life, which remained misunderstood in his last texts and became more widely available only with the recent publication from his personal notebooks (e.g., Vygotsky 2010). As a result of the poor, truncated, and considerably alterated translations into English, readers of Vygotsky in this language were not familiar with the fact that he always wrote about the person and the psyche as a whole rather than about mind (brain) and mental development. The psychological development he was writing about inherently includes the affective side of personality. Even though Vygotsky was interested in achieving a holistic psychology, he realized near the end of his life that much of his theory had been reproducing the Cartesian split. He found in the works of Spinoza a starting point for theorizing the unity/identity of body and mind, and therefore the unity/identity of intellect and affect. In the works of Marx, who was deeply influenced by the Spinozist and materialist philosopher L. Feuerbach, Vygotsky found a way of thinking about the person holistically; and, at the end of the last book that he actually finished, *Thinking and speech* (Vygotsky 1987), he quotes Marx concerning the materiality of consciousness: “language *is* the real consciousness, which also exists in practice for other people, and only therefore also for exists myself” (Marx and Engels 1978, p. 30). The quotations in the opening paragraph of this chapter derive from the first chapter of Vygotsky’s book and are nearly identical to phrases that appear in the preface of The German Ideology (Marx and Engels 1978), a book that was first published during the last year of Vygotsky’s life. The editors state that in the *Ideology*, “Marx and Engels explain the nature and function of thinking, the intellectual needs, interests, inclinations and feelings of humans”; importantly, these authors show that “the decisive cause of their change and development are grounded in the material life of society” (p. x).

The following episode takes us into a fourth-grade mathematics classroom in northern Ontario, where the two regular teachers in this French immersion classroom have gotten together with a mathematics educator from the local university to implement a pre-algebra curriculum (Radford and Roth 2011). The students are presented with the story of a girl, who received for her birthday a piggybank containing $6. She decided to save $3 each week. The students, working in small groups, are asked to (a) model what the girl is doing using goblets and different colored chips and (b) fill a table of values, where the partially filled second and third row call for additive (3+6, 3+3+6, etc.) and multiplicative representations of the piggybank contents (1×3+6, 2×3+6, etc.). One of the cameras follows a group of students (Aurélie, Thérèse, Mario) where trouble is emerging when they begin trying to fill up the table of values. Aurélie has already pounded repeatedly on the desk, has thrown herself against the backrest of the chair, and no longer is trying. Thérèse gives the appearance to succeed such that others ask her what she is doing; but there is little interaction with the other two students. Mario has tried but eventually raises his hand to call the teacher (Jeanne), who eventually starts working with him. She has made him reread the story; and they already have worked through the composition of the $9 in the first goblet representing the piggybank at the end of Week 1. They already have had a couple of tries at decomposing the piggybank contents of the second week when they get to the lesson fragment of interest here (see Fig. 1).

Fig. 1 In this fragment from an algebra lesson, the teacher (right) assists students generally and Mario specifically in moving from a physical model of the story about saving money in a piggy-bank to its presentation in a table of values. From left to right are Aurélie, Thérèse, Mario, and Jeanne

The lesson fragment is replete with manifestations of affect. Even in Jeanne's voice and speech contents, impatience is notable enough for Mario to react to it. They have already spent repeated cycles of decomposing the contents of the first and second week when she asks Mario how much there should already be in the piggybank (i.e., at the beginning of the second week), the reply states an unexpected 12 as answer. We can see that Mario is affected, and his intonation manifests exasperation as he invites Jeanne to look, pointing with both hands to the table of values and goblets. In the third panel (Fig. 1), while Jeanne apparently makes another attempt at taking him through the reasoning, Mario's head is supported by his hand, his eyes staring at the table of values as if in resignation. Once Jeanne's saying has ended, Mario states that he is not understanding, and the sentence fragment "that makes–" may well have been the beginning of "that makes no sense." We observe affect manifesting itself: in Mario's exasperation suggesting "wh*at*" and in Aurélie's pounding the desk with her right hand (final panel, Fig. 1).

Affect appears all over the episode. We do not need to hear what the participants say *about* how they feel. We *see*, *hear*, and *sense* (*feel*) exasperation and frustration rather than having to *intellectually* grasp it. The participants do not have to *interpret* the situation. They live *in* their mutual affects without having to think about "what is the other trying to communicate?" There is impatience that manifests itself in Jeanne's voice, *while* she is asking a mathematical content-related question. Affect here is in the concept: "It's composed of wh*at*?" As soon as we watch the videotape, affect is there for us. We see here what Vygotsky realized near the end of his life in a set of notes entitled "The Lightening Bolts of Spinoza's Thought": "affect [is] in perception" and "affect [is] in concept" (Vygotsky 2010, p. 92). The concept of affect, therefore, is not reducible to something metaphysical (supersensual). Instead, "the concept of affect is an active state" (p. 92). Affects (emotions) are written all over human movements and gestures, and, thus, all over a situation (Mead 1972). That is, there is a unity/identity of a scientific concept and the physical phenomenon of affect, just as Spinoza had theorized it to be and just as a Spinozist-Marxian conception would have it.

It is easily seen that communication has both material-sensual and intellectual-supersensual dimensions. These two characteristics cannot be taken apart. This is another one of the key points in the fundamental revision Vygotsky brings about during the final months of his short life. Sound itself is physical, but in its use as part of exchanges among humans, it also has a supersensual dimension: there is an intellectual component. The problem he identified—both in psychology and linguistics—is the splitting of the two characteristics of the sounds that come from our mouths, as if they were elements, independent components from which the larger phenomenon can be built (Vygotsky 1987). But it is apparent in and from the present example that the (intellectual) sense of the word is irremediably tied to the bodily sense (hearing) and thereby to affect. Thus, the sound-word is the relation between two people, here Jeanne and Mario, precisely because it is part of the physical world (Mead 1972). This becomes even more evident in the following analysis of a small part of the original episode, with pauses included (see Fig. 2). The transcription takes into account that responding to something another person says or does means

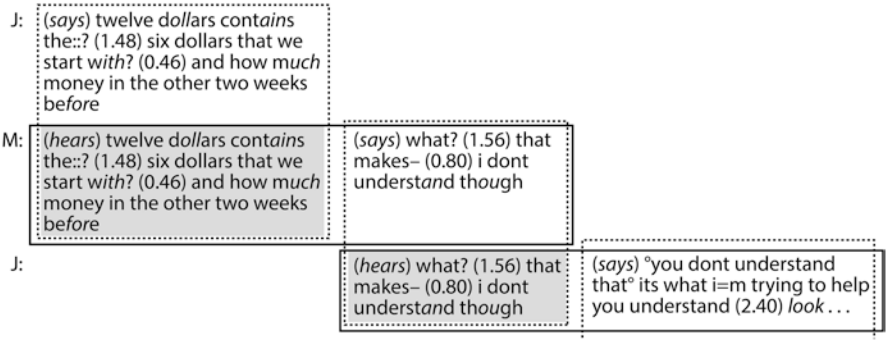

Fig. 2 This transcription fragment involving the teacher Jeanne and the student Mario also contains what the recipient hears

actively attending, and it is the transformation of the original act and the additions on the part of the respondent that has the reply as its outcome. That reply therefore cannot be attributed to the speaker because it essentially is the transformation of what another speaker has said to which the current speaker adds; it is only the second part of an inherently *social* act that the first speaker initiated (Mead 1972).

The transcription highlights two different dimensions that are integrated into a single whole (Roth 2017). On the one hand, the transcription captures what happens simultaneously (arranged vertically), which makes salient that the event includes not only talking but also actively receiving (grey). On the other hand, a participant first actively attends (hears) and then replies (horizontal arrangement). We denote the entire event as *responding*, which includes *actively receiving* and *replying* as its two constitutive parts. *Actively receiving* has two characteristics: it is active but also passive-receptive. If someone speaks but we do not actively attend to it, we might eventually say, "Sorry, what did you say?" Thus, to hear and act upon what another says, we have to be active: we attend. But we, as any recipient, are affected: we are given and receive something initially unknown to us. Recipients do not know what is coming at them while actively attending to another. Consider the fragment spoken in slow motion, beginning with: "t w e l v e d o l l a r s…." As the sounds unfurl from the mouth of Jeanne, they simultaneously ring in Mario's ears. When Jeanne arrives at "t w e," Mario already is being affected but he cannot know by *what*. He cannot grasp what is coming at him until the Saying has ended and the Said of the statement is available in its entirety. In actively attending to another, the recipient (Mario) makes himself vulnerable. Unsurprisingly then students feel insulted, made fun of, belittled, and affected in other unexpected and undesired ways: precisely because they open up and attend to the other (importantly, to the mathematics teacher). If a student is scared to approach her mathematics teacher, as some mathematics education studies have reported in the past (e.g., Walshaw 1999), this is a manifestation of affect forewarning the student that there may be another unpleasant encounter, should she dare asking for help.

The transcription (Fig. 2) also helps us understand that thinking requires going beyond the individual brain case, where the "constructions" are said to be located. Vygotsky (1987), immediately prior to noting that psychological research was theorizing thinking to be an autonomous force, invites his readers to conduct "*a causal genetic analysis of thinking and speech*" (p. 49). For an example, consider Mario. Because he actively attends to Jeanne's Saying, his thinking (which transforms the Saying) is affected by something coming at him from the outside (i.e., "t w e …"). In turn, he acts and changes his environment. Specifically, his reply acts on and thereby affects Jeanne, something outside of Mario. The effect of his reply is available to him only in Jeanne's subsequent actions. That is, any thinking extends beyond the mere words he produces and right into the perception of the effect these words have on the other. The process of thinking attributable to him therefore constitutes an arc: *from* outside, *to* inside, and *to* outside again (Il'enkov 1977). The causal origin of thinking is outside, and the effects of thinking are outside again. Mind here exists in the coming and going of an exchange relation with another person (Mead 1972). This allows us to understand the implications of Vygotsky's work derived by a philosopher of philosophy: "*the very existence of the mind is possible only at the borderline where there is a continual coming and going of one into the other*, at their dynamic interface" (Mikhailov 2001, p. 20). This coming and going is not merely intellectual. Recipients of communication are affected. The episodes show that such coming and going is affective through and through—which, when "neutral," may be hard to detect (e.g., Collins 2004). Thought and affect are but characteristics of that single process of the mutual generation of the *self* and *other*.

Speakers do not and cannot fully grasp[3] what they are saying or doing until after having seen its effect (Vygotsky 1987). This is so because any speaker only initiates a social act, which is completed in and through the reply (Mead 1972). There is an effect because whatever Jeanne says actually is affecting Mario. But Jeanne is not in control over how Mario is affected and how he completes the social act (communicating). This social act is affecting Mario not merely cognitively by means of the content of her speech. It is affecting Mario literally: physically and emotionally. Mario is affected by the saying that is coming at him and that he cannot initially grasp. He is affected a second time by the reply that his own reply has brought forth. Thus, when Jeanne says, "You don't understand that" and then elaborates, "it's what I am trying to help you understand," then it also can be heard as a complaint. Mario does not understand *even though* Jeanne is trying to explain it to him. Readers who do not see this second affectation immediately may want to consider what would have happened had Jeanne intended to make a joke but had Mario announced to be insulted. Now Jeanne would have been a perpetrator not because she intended to be one but because she would have been made to be one. Every reader will have had sufficient life experience to know that Jeanne now would be in a position where she had to explain that no, she did not intend insulting him and that she only meant to

[3] Unless they simply reproduce by heart an existing text, which nobody does in everyday conversation.

joke. Whatever Mario had said before and thereby affected Jeanne, her reply would have been turned into an insult ex post facto and without anything she could have done against it. But it then becomes a fact of life that she has to deal with.

The preceding shows that Jeanne indeed is exposing herself in speaking. In speaking, she has taken a risk because she cannot know how she will have affected (future perfect sense) those who actively attend to her Saying. When Mario questions with exasperation “what?”, he says that he does not understand, then this can be heard as a negative evaluation of an attempt at helping him. Aurélie’s pounding the desk with her fist also is a reply to and a negative evaluation of what has just been said. Although speech generally is recipient-designed, that is, speakers have among others an in-order-to motivation to be understood, Jeanne has not spoken in a manner that allows understanding; the impossibility to understand is a fact that is manifested in and can be obtained from the students’ reactions. The students thereby also are (implicitly) teaching Jeanne about the inadequacy of her earlier attempt at explaining what to do and how to understand the task.

Mario and Jeanne eventually get to the point where, as their relation, the contents of the cells in the third column are produced. The core of that achievement is produced over a few turns in which affect also is inscribed (Fig. 3).

The fragment (Fig. 3) shows how Mario comes to fill the cell in the second row (the first containing the number of the week) and third column of the table of values. He does not do it on his own but in reply to the preceding utterances. It is not Jeanne who gets the answer and the table filled, for Mario provides these. The outcome is the result of their exchange, in which Jeanne and Mario are irreducible parts. More so, it is *as* the exchange that the outcome exists. A few minutes later, after the teacher has left and after Mario has completed the table of values, he announces, “Me, I now understand.” Tracing backwards, we see that this newfound understanding

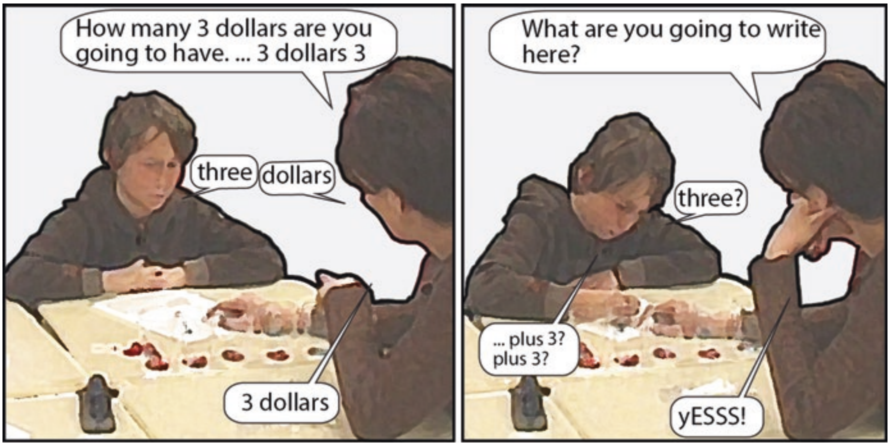

Fig. 3 When Mario eventually says with satisfaction that he now understands (how to do the task), the genetic origin of this new capacity will have been the exchange relation depicted in this fragment concerned with the contents of the piggybank at the end of the third week

of how to do the translation from the physical model (goblets) to the more abstract mathematical representations in additive and multiplicative form *was* this relation first. Here the emphasis is on was, for the understanding did not somehow mysteriously exist *in* the relation. This relation exists in the form of a material connection between them whereby the sound-words literally create a resonance. That relation *is* the algebraic formulation of the piggybank (goblet) contents.

Again, affect is written all over this fragment. First, the replies do not come from a computer; they are not the results of some disaffected thinking machine (mind). In each of the three instances of "three dollars," the intonation is rising, as it tends to do in questions. That is, "three dollars" is not just a constative statement. Jeanne (and, vicariously, any observer) does not have to interpret what Mario says but she hears him ask a question, "Is it three dollars?" That is, Mario is doing two things simultaneously: making a statement but doing so tentatively, and, in being uncertain, also asking whether the statement is correct. Uncertainty and tentativeness are affective qualities, and these are produced together with the semantic qualities of the two-word combinations. As a result, each of these three instances is tinged by intellect and affect, and they are practical as well, because they affect their recipients. One and the same sound has affective (sensible, practical) and intellectual (supersensible) qualities, and that sound, resonating in the mouth of one and in the ear of the other, belongs to both; being in resonance is their relation. There are not two elements—i.e., the material and the intellectual (semantic)—not two parts put together to compose a word in the way chemical compounds are made from elements. Instead, there is only one phenomenon, sound, which has both sensible (material practical, affective) and supersensible (intellectual) qualities (Vygotsky 1987). Vygotsky was right in criticizing linguistics and psychology. His *concrete human psychology*—in the same way as Mead's (1972) social behaviorist psychology—gets us back to the fullness of life as we live it out in our classrooms, in real affectively charged relations with other human beings.

The fragment also exhibits the important role of the teacher, a role that present-day computers will not be able to fill (even if these can beat human players at the game of Go or chess). Jeanne already knows how to do tasks of that kind. That is, the higher psychological function already exists in what she can do (her practice). But doing the task here is spread over their relation so that we observe a "*renewed division into two of what had been fused in one*…the experimental unfolding of a higher process…into a *small drama*" (Vygotsky 1989, p. 58). That is, how to articulate the piggybank (goblet) content in algebraic terms exists as the relation, as a small drama. This algebraization exists as a social drama, and even though Jeanne and later Mario play out this small drama on their own, it is social nevertheless (cf., Mead 1972). This fact is most clearly seen in the *renewed* unfolding over two persons what was and will again be fused into one. Importantly, as all drama, is affectively colored. When Jeanne left the group, Mario began acting upon himself; and when he sees the results and knows them to be correct, he also knows that he understands. We can now see that Vygotsky (1989) was right when he wrote: "the means of acting upon oneself is *first* a means of acting on others and the action of others on one's personality" (p. 56).

The Jeanne–Mario relation is affective—each participant affecting, and being affected by, the other merely by speaking to, thus acting on, the other. The relation is also affective-emotional, as seen, in the uncertain, tentative, and perhaps anxious production of the reply, which then is received with apparent enthusiasm and satisfaction. That satisfaction is not merely a characteristic of Jeanne; as integral part of the sound, it also rings in Mario's ear and thereby is affecting him. We hear that satisfaction again when Mario announces to have achieved understanding. Similarly, when the words from Mario's mouth manifest frustration and perhaps discouragement, those same sounds ring in Jeanne's ears and thereby are affecting her. In part, there is also frustration manifest in her voice, though it disappears again when she makes yet another attempt at helping Mario. The preceding episode shows that Aurélie, too, is affected. Thus, not only Mario's words and actions but also Aurélie's pounding of the desk manifest frustration. That is, even though Jeanne clearly is oriented and talking to Mario, Aurélie is affected, acting upon Jeanne's explanation simultaneously with her peer.

Successful Teaching by Culturing Affect

This chapter is about more than just a little addition to mainstream educational psychology for the mathematics educator. As the preceding section shows, affect is not just something to be thought of as existing within the body of individuals. Instead, affect is written all over social situations. It exists in the form of resonances, an important dimension of which is available in and communicated by means of speech (intonation, rhythm) but which manifests itself in other visible bodily rhythms as well (Roth 2011). We can therefore speak of affective (classroom) cultures, and these can be cultured. This recognition is precisely what led to the title of this chapter: the culturing of affect leads to affective cultures. I now understand in a new way my early experiences of teaching, some of which I present above. I now understand that students were willing to go to the limits of their capacities because there was an affective classroom culture. It allowed students to take risks (including not doing mathematics for a week) without fear of failure, and, of course, the more they succeeded while taking risks, the better they got at succeeding even when and perhaps because of taking risks.

In the classroom example, I had to leave out everything else from the lives of Mario, Aurélie, or Jeanne.[4] To understand their participation and affective relations to the task and others, however, requires us to take a look at the whole persons (Radford and Roth 2011). What persons do and how they do it is a function not merely of the specific activity but of the place that *this* activity has in the life of the person as a whole. Why this is so is not apparent from traditional learning theories; but it is apparent from the cultural-historical approach, where *personality* is understood as the "totality of societal relations" (Vygotsky 1989, p. 68). It is the social

[4] Another reason is that the original research did not collect such data.

structure of these relations that determine the place of individual psychological functions, including those that relate to the contents and processes of mathematical practices. Most importantly, “the dynamic of the personality is drama” (p. 67). That is, each encounter with another person—mine with Paul or Earl, Jeanne’s with Mario or Aurélie—is a relation that contributes to the formation of personality. In writing about personality formation, I not only highlight that students are becoming—loving, fearing, or hating mathematics; able or not able to algebraize some everyday situation. I also intend emphasizing that we, teachers, are continuously becoming—getting better at teaching, being fulfilled by our contributions to the becoming of students, or, in the worst-case scenario, we may be burning out. All of this becoming is not an abstraction but the very essence of our being with others, for our affective, sense-giving relations toward one another are relations generative of ourselves.

The descriptions and analyses in this chapter suggest that the classroom is a place that affects the teacher, who is integral to and a constitutive part of this affective culture. Teachers are not machines, but affective human beings in relations with others that inherently are affective. Students expose themselves to the risks: when they listen, attempt a task, and participate in relations. But the classroom also is a risky place for the teacher because they expose themselves and thereby make themselves vulnerable. Such vulnerability on the part of teachers and students alike are possible in an affective culture. We teachers can do a lot to culture affect.

Some readers may now want to ask me for some recipe how to achieve an affective culture. I do not have recipes, such as those offered by present-day quacks claiming to have simple solutions, but I know a good starting point. Sociologists of emotion recognize that every interaction has ritual form, and these interaction rituals are emotion transformers (Collins 2004). When interaction participants focus on a common thing (i.e., object/motive) and are aware of one another’s focus, “they become caught up in each other’s emotions” (p. 108). Important here is the “common thing,” which in the examples from my own teaching existed in the learning contracts that the students established with me and over which they had control. We had these learning contracts in common. In the class of Jeanne, Mario was trying to take up the object/motive without knowing exactly what it was that he had to do. I showed how the two got “caught up in each other’s emotions.” So he and Jeanne still shared an object/motive, and it was that common orientation hat also contributed to the ultimate success and the related affect. The same was not the case for Aurélie. As teachers, we have to work on creating conditions that offer possibilities to have common things so that students like Aurélie stay with mathematics.

References

Collins, R. (2004). *Interaction ritual chains*. Princeton: Princeton University Press.
Il’enkov, E. V. (1977). *Dialectical logic: Essays on its history and theory*. Moscow: Progress.
Marx, K., & Engels, F. (1978). *Werke band 3* [Works vol. 3]. Berlin: Dietz.
Mead, G. H. (1972). *Mind, self, and society: From the standpoint of a social behaviorist*. Chicago: University of Chicago Press.

Mikhailov, F. T. (2001). The "other within" for the psychologist. *Journal of Russian and East European Psychology, 39*(1), 6–31.
Radford, L., & Roth, W.-M. (2011). Intercorporeality and ethical commitment: An activity perspective: An activity perspective on classroom interaction. *Educational Studies in Mathematics, 77*, 227–245.
Roth, W.-M. (2005). *Auto/biography and auto/ethnography: Praxis of research method*. Rotterdam: Sense Publishers.
Roth, W.-M. (2011). *Geometry as objective science in elementary classrooms: Mathematics in the flesh*. New York: Routledge.
Roth, W.-M. (2017). *The mathematics of mathematics: Thinking with the late, Spinozist Vygotsky*. Rotterdam: Sense Publishers.
Schütz, A. (1932). *Der sinnhafte Aufbau der sozialen Welt* [The phenomenology of the social world]. Vienna: Julius Springer.
Vygotsky, L. S. (1987). *The collected works of L. S. Vygotsky, Vol. 1: Problems of general psychology*. New York: Springer.
Vygotsky, L. S. (1989). Concrete human psychology. *Soviet Psychology, 27*(2), 53–77.
Vygotsky, L. S. (2010). Two fragments of personal notes by L. S. Vygotsky from the Vygotsky family archive. *Journal of Russian and East European Psychology, 48*(1), 91–96.
Walshaw, M. (1999). *Paradox, partiality and promise: A politics for girls in school mathematics*. Unpublished doctoral dissertation. Massey University, Palmerston North, New Zealand.

Additional Suggestions for Further Reading

Roth, W. -M., & Radford, L. (2011). *A cultural-historical perspective on mathematics teaching and learning*. Rotterdam: Sense Publishers.
Yasnitsky, A., & van der Veer, R. (2016). *Revisionist revolution in Vygotsky studies: The state of the art*. East Sussex: Routledge.

Support to Thrive: Raising Resilience in Students in Secondary Schools

Kingsley Hurlington

Abstract Resilience refers to the capacity to return to good mental health after being exposed to challenging experiences or environments. Teachers are well positioned to observe and provide support for children and thereby bolster their resilience. Through the development of relationships and mentorships, teachers of mathematics can positively affect the development of children by focusing on and leveraging their strengths to overcome challenges. These ecological constructions of resilience afford a greater responsibility to schools and other environments to support the processes of development with children.

Keywords Ecological resilience · Educational resilience · School · Adolescence · Teachers

Introduction

Often in education, teachers rely on deficit approaches to student success, and this may be particularly true in mathematics (Henderson and Milstein 2003). This approach has merit: we can determine what students do not yet know, cannot yet do, or do not yet believe, and facilitate the development of the knowledge, skills and attitudes essential to support their growth. Intuitively, this is a time-honoured approach: once we identify their weaknesses we can develop their strengths. To add contemporary flair, we may even ask students to identify their own weaknesses and begin the building process from there. Deficit-focused approaches to student growth can feel satisfying and effective because it is about capturing and redressing what is not there. Furthermore, deficit-focus approaches are particularly easy for teachers to track students' progress. However, these approaches do not work for all students. A focus on strengths rather than weaknesses and on creating learning environments where strengths are celebrated can enhance student learning. This paper explores

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insights I have gathered from resilience research and considers how strength-based approaches can be used to ensure classrooms are supportive environments for student growth.

Considering Support: A School Case Study

I was working at a school recently where students were asked to provide information about themselves in a social survey about the supports they have in their lives. In particular, students were asked to reflect on the supports that allow them to be successful at school. The results were analyzed and, using z-scores, a group of students who acknowledged that they were struggling were identified as outliers. To better understand who the outliers were, students who had existing and identified special needs (e.g., special education, recent immigrants with English language needs, reluctant learners and alternative education) were removed from the list as they are already provided additional support at school. The remaining list was comprised of students who were generally academically successful but who had identified themselves as having lower than desirable levels of supports in their lives; they openly acknowledged their need for more support at home and at school. When teachers were subsequently asked to review the list of outlier students, the most common response from them was one of disbelief: "She is doing really well in my class! They should not be on this list! There must be a mistake. This student doesn't need any additional support. He is a good student who never gets in trouble and pays attention in class." This moment of educational cognitive dysphoria should give all educators a reason to pause: students self-identified that they were struggling and in need of assistance and their teachers rejected their self-identification and request for assistance. This is, in part, caused by the underlying principle that teachers often use to frame their concept of success: a strong academic score. In effect, this is a type of positivity bias. The question that it forces us to ask is if a strong academic performance in a class is not enough, what do students actually need to feel success at school and can we (as teachers) give it to them?

What Is Resilience?

Fundamentally, teaching is about robustly connecting new information to well-established prior learning. If we are beginning from what students already know and concepts they already own, rather than focusing on their deficits or weaknesses, we become focused on their strengths and abilities. This is the foundation of a resilience framework for educational success. Resilience is a strength-based approach to learning and growth. The term resilience has its etymological roots in the Latin word *resilre* which means "to jump back, recoil." Its current usage has been adapted from material science where it is used to capture the property of materials to return to their previous shapes after being distorted by external forces (Condly 2006; Luthar et al. 2000). In social science literature, it has come to be defined as a "class of phenomena characterized by good outcomes in spite of serious threats to adaptation or development" (Masten 2001, p. 228).

Early conceptions posited that resilience was a purely psychological phenomena borne out of an individual's psychic abilities. For example, individuals who did not take themselves too seriously, were able to laugh at their mistakes, and who had a strong sense of personal identity would manifest higher levels of resilience. The initial work done by Werner and Smith (1982) with children in orphanages in Hawaii demonstrated the importance of these internal strengths. Their seminal longitudinal study indicated that most of these seriously disadvantaged children matured into positive prosocial contributing members of society. The simple hope was that if researchers could distil the personal characteristics that allowed individuals to overcome challenges and create the conditions to replicate and instill those characteristics in others, they could facilitate success widely in society. Additional research indicated that there were social factors that could facilitate resilience characteristics in youth (Greene 2002; Luthar 2003; Walsh 2002). These social conditions, known as protective factors, included elements such as growing up in a stable healthy and financially advantaged family. Masten (2001) noted that "the great surprise of resilience research is the ordinariness of the phenomena. Resilience appears to be a common phenomena that results in most cases from the basic human adaptational systems" (p. 221). The general success of individuals as manifested through resilience, could be understood as "ordinary magic" (Masten 2001, p. 227).

Contemporary conceptions of resilience have grown to be more nuanced and sensitive to personal lived experiences. Amongst these newer approaches to understanding the phenomena of resilience is an ecological sensibility. Ungar (2004) argued that "ecological approaches to the study of risk and resilience are informed by Systems Theory and emphasize predictable relationships between risk and protective factors, circular causality and transactional process that foster resilience" (p. 342). In the light of the work of Urie Bronfenbrenner (1979) on ecological systems theory, resilience can be understood as growth environments that bolster an individual's growth mindset. The environment of support cannot be ignored in exploring resilience—it influences personal resilience factors (e.g., Fraser and Galinsky 1997). A postmodern and ecologically sensitive definition of resilience is offered by Ungar (2015) as follows: "In the context of exposure to significant adversity, resilience is both the capacity of individuals to navigate their way to the psychological, social, cultural and physical resources that sustain their wellbeing, and their capacity individually and collectively to negotiate for those resources to be provided in culturally meaningful ways" (para. 2).

How Do Schools Affect Resilience?

At its core, ecological resilience is about the development of relationships. School is a central developmental and social environment where relationships are key for many students—if for no other reason than the amount of time that students spend there. A healthy school community allows students to experience a sense of belonging. In educational contexts, Goodenow (1993) has defined that sense of belonging as “the extent to which students feel personally accepted, respected, included and supported in the school environment” (p. 80). Thus a range of studies (Brown et al. 2001; Garmezy 1991; Henderson and Milstein 2003; Katz 1997; Masten and Obradovic 2006; Moore and Lippman 2005; Rutter 1979; Thomsen 2002) has indicated the important role that schools play in supporting resilience development.

Donnon and Hammond’s (2007) work on the Youth Resiliency Framework indicates that there are 11 resiliency factors that are essential for youth. These factors integrate the seminal concepts of developmental strengths and assets (e.g., Benson 2007). One of these factors is commitment to learning at school which involves three developmental strengths: achievement, school engagement and homework. A second factor, however, is *school culture* which involves school boundaries (clear rules and expectations), school bonding (safety at school), caring climate (collegiality and caring environment) and high expectations (appropriate goals and personal excellence). In essence, this indicates that school culture plays a greater role in bolstering student resilience than the academic aspects of school. While caution must be used in extrapolating these types of data, anecdotally, teachers acknowledge that these school level data map directly into classroom settings as classrooms are a microcosm of the larger school community. Classroom environments have as much to do with resilience supports as do academic progress and success (Henderson and Milstein 2003). Students who experience school as a supportive environment do better in and beyond school than those who do not. Furthermore, classrooms are learning collaboratives in which positive relationships to peers and positive peer influence play a role (Donnon and Hammond 2007). To be clear: learning is about relationships and resilience is about relationships. A classroom environment in which positive relationships are nurtured and encouraged will provide a fertile learning foundation, an environment that encourages openness and vulnerability and a place where students are encouraged to do their best. These conditions will only exist when the relationships in the classroom are positive and affirming.

Culturally Meaningful Classrooms

An important element of the resilience definition (posited by Ungar above) is that of cultural appropriateness. Well intentioned teachers have often offered what they themselves considered to be excellent support that was not well received or appreciated by their students. This can happen in situations where the support is not culturally appropriate. For example, offering extra help to students at lunch time requires teachers to altruistically give up a part of their well-deserved lunch break or preparation time. When students do not show up for that extra support that can be frustrating. However, some students head home during their lunch break to provide support to their families by looking after younger siblings or aging grandparents, to give a break to parents who are caregivers, to participate in important religious rites or to get the physical activity they need in order to face the rest of the school day amongst a myriad other reasons. Students might need and want the academic support but may be unable to accept it in the form in which it was offered. Scheduling that academic support at another time (perhaps before or after school) or in another way (online rather than in person) might work better for some students. Without question, there must be realistic limits to the number of options for support that an individual teacher can offer, yet, if the opportunities offered fail to meet the cultural needs of students, then it cannot have its intended effect.

The entire exploration of classroom culture is beyond the scope of this article; however, it warrants attention for any educator who hopes to provide an environment that bolsters resilience in students. Programs like TRIBES TLC have sought to provide the parameters for these enriched classroom environments (e.g., Benard 2005). The first principles of all resilience enriching classroom environments comes back to developing meaningful relationships with students. Healthy relationships require open and clear communication. The currency of relationships is vulnerability (e.g., Brown 2012) which demands teachers to be willing to take concrete steps to learn from their students about how to support them. Teachers who inquire of their students (and their families) about which supports they need and how to best deliver those supports will generally find eager engagement and willing partnerships. Where once this type of inquiry would have required many hours of personal contact, today's technology allows teachers to rapidly facilitate contact with students and parents to seek guidance about appropriate supports. Are students completing their homework each night or are they attending to other family responsibilities? Is there someone at home who understands mathematics well enough to assist with assignments? Is mathematics treated as a scary subject that you either "get" or not, or as an unimportant triviality ("you are never going to use this stuff in the real world")? These issues can only be recognized and addressed when teachers engage in conversation with students and become aware of their home lives and contexts. If modern classrooms demand anything, it is customization where students are not essentialized and treated as identical buckets to be filled but rather, as William Butler Yates famously suggests, fires needing to be lit in unique and specialized ways.

There are many examples that illustrate the importance of supportive learning communities. A particularly fascinating anecdotal scenario is that of skater culture (i.e., recreational skate board users). Learning how to passively ride a skateboard is relatively easy; learning how to master tricks on the skateboard is particularly arduous. Many skaters would characterize themselves as marginalized youth who do not identify with positive school experiences. They often find school as an environment where they do not feel engaged. However, it is not that skaters cannot be motivated and engaged as they clearly display these characteristics in relation to skateboarding. Getting good at skateboarding requires hours of practice, tenacious repetition and an invariable acceptance of getting hurt. Skaters frequently suffer injury as they develop their skills. Despite this frustration and risk of injury they persist in learning new skills and perfecting essential ones. Their resolve is driven not only by a personal sense of accomplishment and self-esteem but also by the culture of the skate park. At the skate park, there is a physical environment that provides obstacles that encourage users to attempt new challenges and to approach barriers with new skills. The physical environment matters to the development of growth. The skate park also provides an essential relational environment supported by meaningful relationships. New skaters are supported by experienced ones. They pick them up when they fall, they encourage them when they fail, and they give advice and suggestions. Intermediate skaters who succeed in landing new challenging tricks are celebrated by the community. People clap, cheer and whistle when a great trick has been completed. When they are injured, while they may be laughed at or teased, they are offered a helping hand and are celebrated for attempting a new difficult skill. Finally, advanced skaters are revered and honoured while providing tutorials and master classes for their peers. They are video recorded and the community may gather to evaluate and assess the technical aspects of their tricks. Your life outside of skating is never questioned—if you can skate then you are afforded respect. The skate park provides an insight into strength-based resilience-focused environments. Here in this youth-centred, youth-organized and youth-created culture, learning is rich, performance-based and self-sustaining. It is an environment that respects attempts, honours errors and problem solving, and challenges stagnation. It is a rich growth environment.

Safe and Sound: Resilience Approaches in Mathematics

A classroom environment that encourages the development of resilience characteristics in students is one in which students are comfortable with the teacher, their peers and themselves (Masten and Coatsworth 1998; Thomsen 2002). Counterintuitively, students do not have to be comfortable with the course content. When students are feeling supported in the environment they are willing to take more risks which encourages deep learning. A safe learning environment will encourage the development of good coping skills and meaningful adaptability. These are particularly important characteristics for the enhancement of inquiry, critical thinking and problem solving. Inquiry, critical thinking and problem solving all require risk taking. Drawing from the skater culture example, the elements of risk taking are, at minimum, influenced by environmental factors. All learning demands risk taking which fuels growth mindset. Students need to be convinced to step beyond “I can’t do it” to “I can’t do it yet, but I am willing to try and with help, eventually I know I will.”

In an earlier chapter in Part II (Kajander) in this volume provides case study examples of students identified as at-risk in mathematics by their teachers. The students in these cases demonstrate varying levels of dislike, fear, low self-confidence, apathy, and general angst related to mathematics. Attendance in some cases is very poor. Students were observed to be passing notes, asking for washroom passes, or demonstrating other types of disengagement during mathematics classes. The teachers generally responded by attempting to provide mathematics-specific feedback, or by cajoling students to try harder. See Fig. 1 for the "Thinking Cap" stamp provided as feedback on student work (Kajander, Part II, this volume). Resilience theory suggest that these students can be better supported by creating a strength-focused growth-centred environment.

The need for resilience-bolstering environments are critical in mathematics classrooms because mathematics requires an environment where willingness to try is celebrated as a strength. The mathematics classroom can mirror the growth mindset richness of a skate park when students feel safe taking risks, using unconventional approaches to problem solving and exploring real world issues.

A common maxim amongst resilience focused practitioners is that they are "talent scouts." A talent scout is someone who looks for and identifies strengths in individuals first and foremost. They are not naïve—it is not that they fail to recognize that individuals have weaknesses, they have simply chosen to focus on strengths. Strengths can leverage the limitations of weaknesses. Teachers have the ability to reverse the typical construction of the learning environment where successful students are accepted in the classroom. Instead, the classroom comes to accept all students by recognizing their strengths which allows them to be successful. Focusing on strengths puts growth mindset at the centre of learning (Boaler and Dweck 2016). By honouring and operationalizing strengths, students learn not to be controlled by their weaknesses—real or perceived. Teachers, classrooms and schools can attend to this by applying established resilience principles.

Bonnie Benard's 2004 seminal meta-analysis of resilience research yielded a praxis focused book entitled *Resilience: What we have learned*. In it she summarizes what research indicates as the three most important elements of resilience practice: caring relationships, high expectations and opportunities for meaningful participation. Each of these elements of resilience practice can be applied to the mathematics learning environment.

Caring Relationships in the Mathematics Classroom

Students who have difficulty succeeding at school do not fail to make gains because they cannot learn; the far more likely reason is that they do not feel that school is an inclusive caring environment for them. As mentioned earlier, resilience is about strengthening relationships. Everything that can be done to support prosocial bonding needs to be done. This is accomplished by taking time to learn about students: their lived experiences, their awarenesses, their skills and abilities, their learning strengths, character strengths, learning preferences and interests. In this way, teachers can leverage this knowledge to great effect in the selection of the examples and case studies that they use in guiding students' growth. Since mathematics is everywhere and permeates every aspect of life, there are countless ways to approach content. Successful teachers have been doing this very effectively for aeons: students who are car obsessed get car-related problems to solve, while students who love cooking have opportunities to solve problems related to that. I once knew a teacher who ensured that every student was assigned the role of hero in mathematics word problems throughout the year focusing questions to match personalized areas of interest for students each time. Teachers can demonstrate caring relationships in very simple ways including introducing themselves to students and their parents at the start of the course, encouraging the idea of continued support through extra help sessions (including schedules and contact information), providing overviews of foundational concepts essential to success in this course, reinforcing a "Yes, you can!" growth mindset and direction for getting help beyond the teacher (including peer mentorship). Furthermore, elsewhere (Hurlington 2010) I have advocated for the importance of encouraging students to get to know each other and to rely on each other as academic supports. The permutations of possible working groups in a classroom of 30 provides students many opportunities to learn from each other. Beyond strengthening their academic abilities, the capacity for youth to recognize strengths in others is an important life skill. When mathematics is treated as a subject in which getting the right answer is more important than critical thinking and discussion, relationships can be ignored with limited detriment. Conversely, when mathematics is understood to be about patterns, beauty and thought, then conversation and discussion are essential. Conversation, discussion and debate cannot be accomplished in an environment devoid of caring relationships.

It is worth noting that a mathematics learning environment in which caring relationships are central, risk taking is celebrated, and individual interests are acknowledged will enrich the depth of problem solving. Students at all levels of academic performance will be willing and able to explore more challenging problems to solve. As Atiya, Luca and Kajander (Part IV, this volume) suggest, weaker mathematics students do not need to be deprived of rich learning opportunities; however, they will only be able to approach such problem solving when they are confident that they will be supported. Their tolerance for abstract concepts is a function of the trust they have in the teacher's interest in them as individuals. Thus mathematics teachers must become "talent scouts" who seek to identify and celebrate the strengths their students. This extends from knowing what students lives are like to individual feedback provided on assignments and class work. Teachers can acknowledge what aspects of a multistep computational solution was done correctly before indicating where errors were made. Additionally, when peer-to-peer relationships are meaningful and positive, students can discuss problem solving approaches in a collaborative way. For example, a placemat activity in which a problem is posed and each student is asked to propose and record an approach for solving that problem can allow a greater level of discourse in a safer small group environment.

High Expectations in the Mathematics Classroom

It is easy to assume that students have high expectations for their own academic performance but this is not always true. If we learn from role models, students who are accustomed to relationships and environments where very little to nothing is expected of them, may come to believe that they simply cannot succeed. In the environment of caring relationships it becomes possible to disrupt negative messages and beliefs about self-worth and success. In a caring environment, everyone should have the highest expectations for academic growth. Teachers who know their students can work with them to establish realistic growth-oriented goals for success. High expectations are not only for academic growth but for social growth as well. As such, an environment of high expectations is one in which clear, consistent, collaboratively-created and constructive boundaries are taught, modeled and maintained. Students who are offered the opportunity to co-create rules will have an easier time defending and respecting the rules. Goals need to follow a spectrum of challenge from readily accessible to stretch achievements. All the while teachers, parents and peers form a cheering squad to celebrate incremental successes. Furthermore, it is important to note that high expectations for academic performance cannot fail to recognize the importance of mistakes and errors. As Boaler and Dweck (2016) indicate, making mathematical mistakes in a friendly caring environment will offer insight and growth to the entire learning community. When students are held to high expectations in classroom environments that they do not perceive as protected by caring relationships, their instinct is to hide their mistakes to protect themselves from derision. Should a student's growth mindset be supported by an environment buttressed by caring relationships, they will be far more willing to take the risks necessary for collaborative collegial critical learning.

Environments of high expectation are not instinctive for many students but as we learn from the skater example, growth can only be realized with meaningful opportunities to practice skills. This practice is not simply mechanical repetition. Some researchers (e.g., Foster 2013), have advocated techniques for developing procedural fluency in mathematics which involves a balance of practice and performance of concepts rather than an obsession with computational algorithms. Foster's use of connected algebraic expressions (in which students are provided a foundational algebraic framework where multiple solutions are derived) demonstrates an opportunity for students to develop skills in a collaborative, mastery learning approach. These approaches allow students to recognize the importance of having discussions with peers which results in better awareness of approaches of problem solving. Ultimately, students will recognize that they can solve problems in many ways which solidifies high expectations not only in terms of correct answers but robustness of thinking. This approach explores problem solving as puzzles which adds a degree of excitement while demanding excellence. Teachers can plan activities and assessments in such a way that students have the opportunity to improve on previous performances and track their own successes. Such procedural fluency encourages growth mindset, which underpins a sense of high expectation for mathematical proficiency.

Opportunities for Meaningful Participation in the Mathematics Classroom

The frequent cry of students in moments of educational frustration is "when am I ever going to use this again!" In mathematics, students must see that there are practical applications for learning and refining their knowledge and skills. Everyone needs to participate in activities that provide them opportunities for meaningful participation. Which practitioner (teacher or otherwise) relishes the opportunity to participate in meetings where the result is a foregone conclusion or the decision making authority lies multiple bureaucratic or political layers away? For students, engagement in such activities is no less soul destroying than it is for anyone else! Learning is about engaging imagination and the need for a mathematical imagination comes before the need for abstract computational proficiency. Students need to participate in situations where they can genuinely feel that they can influence the end result or where they can affect positive change. Meaningful participation means "that youth are encouraged to make the world a laboratory for exploring the ramifications of their positive actions" (Hurlington 2010, p. 4). Within classrooms, students need to know that their voice is being heard and their experiences are being honoured. Assignments or projects that engage students in authentic problem solving (while maintaining a culturally appropriate frame) are often the most fascinating and challenging because they engage imaginations. Again, offering students the opportunity to choose problems from their own lives that they can solve using their skills can lend a strong sense of legitimacy. Furthermore, in a throwback to bucolic scenes of the single room schoolhouse that concurrently offered learning for children from grade 1 to 8, there can be deeply meaningful opportunities for participation for students who are helping to support their younger colleagues. Much has been made of the value of peer tutoring both for the tutor and tutee while taking pressure off the teacher. In a caring environment, students can be trusted to provide support to each other in respectful and authentic ways.

Rather than focusing on individual mathematical skills that need to be perfected, teachers can work with students to determine individual culturally appropriate real-world problems that can be solved. In such cases, students develop skills as they need them to solve actual problems. This approach removes a level of abstraction which allows students to see the usefulness of particular mathematical procedures and approaches. Furthermore, students are given the ability to affect the direction of the content or approaches that are used in the class. For example, students exploring correlations can collect data from their lived experiences to demonstrate the strengths of relationships and then discuss potential explanations with their peers. For many students, this would feel like a more authentic approach to mathematics which would defy arguments against meaningful applicability.

Conclusion: School as a Refuge

In the introduction, we considered a school where students were asked about their awareness of their ecological resilience. A group of students previously hidden from view were suddenly illuminated by their answers to the resilience inquiry. They did not have the support they needed from home, school, peers and family that would lead to optimal growth. In every classroom and school, there are similar students—those who are completing acceptable academic work, who keep to themselves and who avoid antisocial antics but are feeling unsupported. According to resilience research, academic success does not provide a complete view of success in or beyond school. In order for students to flourish, they need healthy relationships and prosocial environments that demand them to accept positive growth as critical to a successful and meaningful life.

When it is recognized that students have complex lives beyond their academics, teachers can have a profound impact on those lives. Students can have great lives beyond their academics; they can have great academics in spite of their very challenging lives—our responsibility is not to overlook either. As talent scouts, it becomes the duty of the teacher to recognize strengths as strengths and every student as an individual worth building a meaningful relationship with; however, a talent scout also listens carefully to what students are and are not saying about themselves and the supports they have. Positivity bias can lead teachers to look past significant indicators of risk, challenges and frustrations because students are performing well academically. Students with high anxiety can present as high academic performers while struggling to maintain a healthy sense of self and of good mental coping. This is particularly true for mathematics learning environments where students may feel elevated levels of anxiety on a daily basis.

When it is at its most effective, school operates as a refuge. Many students rely on the school community to provide them an environment where they can be accepted, respected and directed. Some educational professionals may bristle at this suggestion arguing that school ought not to be expected to exact so high a societal function. Whether we desire this responsibility or not, the reality is that school may be one of the only environments where youth are esteemed for their strengths rather than denigrated for their weaknesses. The easy functional deficit-oriented views of youth can never elevate or inspire them to greatness. Strength-based views, on the other hand, recognize what students have, who they are and what they believe as an initial platform for growth. Resilience does not ignore weaknesses or inabilities; it simply demands that they are never treated as the defining characteristics of one's life. This idea is communicated plainly by Mark Katz (2016) in his aptly named book *Children who fail at school but succeed at life: Lessons from lives well-lived*: "There's never anything so wrong with us that what's right with us can't fix" (p. 236).

From a design perspective, every teacher—every day—has the ability to construct a classroom environment that provides a strong platform for ecological resilience for students. Beyond the seemingly quotidian tasks of lesson planning and assessment, teachers need to interrogate their interactions with students at all levels. Teachers who take the time to metacognate on creating caring relationships, setting high expectations and providing opportunities for meaningful participation will provide strength-focused environments that can effectively bolster resilience in their students to allow them to weather the times of personal and academic stress.

In conclusion, a final thought is warranted regarding what teachers and other educational practitioners believe about themselves and their own growth. Teachers who no longer believe that their students have much to teach them are at risk of professional brittleness. Such teachers need only to immerse themselves in new learning (of almost any kind) to push their professional growth and deepen their sense of empathy for the complexities and challenges their students are facing. In effect, this type of continual learning educates teachers to the importance of hope and optimism (e.g., Brown 2012) as a function of growth and development. Resilience development is a continual human process that applies to all people. When teachers acknowledge their ongoing growth, it provides students with a model to mirror and admire. Teachers are lifelong learners, mentors and models. In the words of author Robert Fulghum, “Don’t worry that children never listen to you; worry that they are always watching you” (as cited in Zimmer 2003, p. 182). Students need to see that teachers are just as willing to engage in their own professional and private learning and growth, that they have their own strengths and weaknesses and that they not only create supportive relationships but benefit from them as well. Students need to know that teachers grow in strength-focused environments—if for no other reason than to convince them that today is a good day to begin the life-long journey of building a grander sense of growth.

References

Benard, B. (2004). *Resiliency: What we have learned*. San Francisco: WestEd.

Benard, B. (2005). *What is it about Tribes?* (1st ed.). Windsor: CenterSource Systems.

Benson, P. L. (2007). Developmental assets: An overview of theory, research, and practice. In R. K. Silbereisen & R. M. Lerner (Eds.), *Approaches to positive youth development* (pp. 33–58). Thousand Oaks: Sage.

Boaler, J., & Dweck, C. (2016). *Mathematical mindsets: Unleashing students' potential through creative math* (1st ed.). Mississauga: Wiley.

Brofenbrenner, U. (1979). *The ecology of human development: Experiments by nature and design*. Cambridge, MA: Harvard University Press.

Brown, B. (2012). *Daring greatly: How the courage to be vulnerable transforms the way we live, love, parent, and lead*. New York: Gotham Books.

Brown, J. H., D'Emidio-Caston, M., & Benard, B. (2001). *Resilience education*. Thousand Oaks: Corwin Press.

Condly, S. J. (2006). Resilience in children: A review of literature with implications for education. *Urban Education, 41*(3), 211–236. https://doi.org/10.1177/0042085906287902.

Donnon, T., & Hammond, W. (2007). A psychometric assessment of the self-reported youth resiliency: Assessing developmental strengths questionnaire. *Psychological Reports, 100*, 963–978.

Foster, C. (2013). Mathematical études: Embedding opportunities for developing procedural fluency within rich mathematical contexts. *International Journal of Mathematical Education in Science and Technology, 44*, 765–774.

Fraser, M. W., & Galinsky, M. J. (1997). Toward a resilience-based model of practice. In M. W. Fraser (Ed.), *Risk and resilience in childhood: An ecological perspective* (pp. 265–276). Washington, DC: NASW Press.

Garmezy, N. (1991). Resiliency and vulnerability to adverse developmental outcomes associated with poverty. *American Behavioral Scientist, 34*(4), 416–430. https://doi.org/10.1177/0002764291034004003.

Goodenow, C. (1993). *The psychological sense of school membership among adolescents: Scale development and educational correlates. Psychology in the Schools, 30*(1), 79–90. Retrieved from https://doi.org/10.1002/1520-6807(199301)30:1<79::aid-pits2310300113>3.0.co;2-x

Greene, R. R. (Ed.). (2002). *Resiliency: An integrated approach to practice, policy and research*. Washington, DC: NASW Press.

Henderson, N., & Milstein, M. (2003). *Resiliency in schools: Making it happen for students and educators: Updated edition* (1st ed.). Thousand Oaks: Corwin Press, Inc.

Hurlington, K. (2010, February). Bolstering resilience in students: Teachers as protective factors [research monograph #25]. *What Works? Research into Practice*. Retrieved from http://www.edu.gov.on.ca/eng/literacynumeracy/inspire/research/WW_bolstering_students.pdf

Katz, M. (1997). *On playing a poor hand well: Insights from the lives of those who have overcome childhood risks and adversities*. New York: Norton.

Katz, M. (2016). *Children who fail at school but succeed at life: Lessons from lives well-lived* (1st ed.). New York: W. W. Norton & Company.

Luthar, S. (Ed.). (2003). *Resilience and vulnerability: Adaptation in the context of childhood adversities*. Cambridge: Cambridge University Press.

Luthar, S. S., Cicchetti, D., & Becker, B. (2000). The construct of resilience: A critical evaluation and guidelines for future work. *Child Development, 71*(3), 543–562. https://doi.org/10.1111/1467-8624.00164.

Masten, A. S. (2001). Ordinary magic: Resilience processes in development. *American Psychologist, 56*(3), 227–238. https://doi.org/10.1037/0003-066x.56.3.227.

Masten, A. S., & Coatsworth, J. D. (1998). The development of competence in favorable and unfavorable environments: Lessons from research on successful children. *American Psychologist, 53*(2), 205–220. https://doi.org/10.1037//0003-066x.53.2.205.

Masten, A. S., & Obradovic, J. (2006). Competence and resilience in development. *Annals of the New York Academy of Sciences, 1094*(1), 13–27. https://doi.org/10.1196/annals.1376.003.

Moore, K. A., & Lippman, L. (Eds.). (2005). *What do children need to flourish?: Conceptualizing and measuring indicators of positive development*. New York: Springer.

Rutter, M. (1979). Protective factors in children's responses to stress and disadvantage. In M. W. Kent & J. E. Rolf (Eds.), *Primary prevention of psychopathology: Social competence in children* (Vol. 3, pp. 49–74). Hanover: University Press of New England.

Thomsen, K. (2002). *Building resilient students* (1st ed.). Thousand Oaks: Corwin Press.

Ungar, M. (2004). A constructionist discourse on resilience: Multiple contexts, multiple realities among at-risk children and youth. *Youth and Society, 35*(3), 341–365.

Ungar, M. (2015). *What is resilience?* Retrieved from http://resilienceresearch.org/

Walsh, F. (2002). A family resilience framework: Innovative practice applications. *Family Relations, 51*(2), 130–137. https://doi.org/10.1111/j.1741-3729.2002.00130.x.

Werner, E. E., & Smith, R. S. (1982). *Vulnerable but invincible: A longitudinal study of resilient children and youth*. New York: McGraw-Hill.

Zimmer, B. J. (2003). *Reflections for tending the sacred garden: Embracing the art of slowing down*. Lincoln: IUniverse.

Additional Suggestions for Further Reading

Ungar, M. (2004). Nurturing hidden resilience in troubled youth. Toronto: Buffalo.
Ungar, M. (2014). *I still love you: Nine things troubled kids need from their parents* (1st ed.). Toronto: Dundurn.

Digital Technology in Teaching Mathematical Competency: A Paradigm Shift

Sean Chorney

Abstract I analyze a simple classroom situation in which students in a mathematics classroom use a computer application to engage in a mathematical activity. The technology allowed the students to engage in deeper mathematical thinking than was previously possible, but this depth went unappreciated because of the institutional socialization of the teacher to evaluate the students quantitatively based on content rather than qualitatively based on critical thinking. New digital technology has made possible a reconceptualization of what it means to do and to learn mathematics. These changes in technologies, however, do not align with the expected requirements of teaching. This chapter looks at evaluation of mathematical knowledge by a teacher as a sociopolitical issue, for new developments in digital technology have made some perennial challenges, such as evaluation, more pronounced.

Keywords Sociopolitical · Content · Competency

Introduction

I was a high school teacher for 20 years, and in retrospect I realize that my working conception of how mathematics is learned was informed more by external forces than my own thinking, beliefs, wishes, and training. I was being led by curriculum guides, examinations, departmental norms, and the need to report out. It may be common to think of these things as insignificant to the actual teaching of mathematics, but they are more influential than one might think.

The school administration and provincial government establish expectations in the general practice of learning and teaching. Teachers are required to conform to the standards determined by curriculum writers and administrators. Teachers have to report out in the form of grades. The practice of giving marks and reporting out is widespread all over the world and has a long history. It has positioned the teacher

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to not only teach but also to assess student learning. The assessor role affects what the teacher sees. And in the act of quantifying student learning, mathematics, which has the potential of being deep and rich, often becomes shallow and flat.

In this chapter I explores a situation where two students engage in a rich mathematical activity using digital technology but the teacher sees something different. The teacher frames the situation so that the activity aligns with his evaluative expectations. There have been two recent notable changes in mathematics education in recent years: students are now using digital technology more than in the past and teachers are supposed to put more value on general mathematical skills ("curriculum competencies" in addition to "content"). However, there is a tension when these changes are met with unchanged institutional priorities. Mathematical competency is compromised when teachers are expected to give out grades. To understand students' activities in mathematics in a meaningful way takes time that is cut short if one is constantly evaluating them for the purposes of reporting out. And the stimulation of competency by new technology is making starker the shortcomings of a focus on grades. This chapter sees mathematical practice as a social endeavor but also, in the spirit of Guttierez's (2013) notion of the sociopolitical, argues that top-down institutional influence affects the teaching and learning of mathematics.

What Are We to Value in Mathematics Teaching?

The Curriculum

In British Columbia (BC), there has been a restructuring of the curriculum (BC Ministry of Education 2015). The K–9 curriculum was changed to reflect a twenty-first century approach to learning. (The revised grades 10–12 have partially been implemented with a full implementation occurring in September, 2018). In each school subject, there are now two objectives: mastering content and developing curricular competency. For mathematics, content is such things as knowing how to solve one-variable linear equations or factor a trinomial, and knowing that opposite, reciprocal, and inverse all mean different things. The curricular competencies are what we might call the dispositions of the mathematician or the kinds of things mathematicians do. Examples of this outcome might include reasoning, estimating reasonably, communicating, and justifying. These two expectations can be seen as a distinction of knowledge and skills.

When I have asked teachers what they want their students to walk away with from classes, they almost always speak about the sorts of things found on the competency list rather than the content list. They want their students to be active, thinking citizens who can use mathematics to help them make decisions. So if we ask what the practice of mathematics is, we can ask whether mathematics is content and proper technique, such as remembering and implementing the quadratic formula, or practicing mathematical competencies such as conjecturing, justifying, and testing.

It is reasonable to say that mathematics in a classroom should include both content and competency. In the episode outlined below we see content being valued, but not competency.

The Digital Technology

Digital technology gives students the opportunity to act like mathematicians (Sinclair et al. 2016). When given a problem in a mathematically rich digital environment, students are able to do things they could not before. They can now engage with continuous and variable mathematical objects. Mathematical expression shifts to a visual-dynamic system of expression (Rotman 2008). In a traditional classroom of paper, whiteboards, overhead projectors, and textbooks, most things remain static. The images and diagrams of the textbook do not move. Though the numbers and symbols students draw on their paper do go through a process of construction, once set on paper they do not move. There is the textbook, the exercise book, the teacher, the upcoming test, and the students' questions; and these are all parts that we hope connect in sophisticated ways. Attempting to forge relations between all these things is what the traditional classroom looked like.

Digital technology can offer more. Ironically, although digital technology is based on discrete (non-analog) bits of information, it presents visual and audible data in a way that is closer to the continuous flow of real life. There are certain mathematical ideas, such as continuity (dragging a point) which cannot be expressed in words, but which can be shown through actions on a computer. *The Geometer's Sketchpad* (Jackiw 2001), the digital technology used in this episode, offers ways to drag geometrical objects. This is not possible in traditional environments. So digital technology offers new and unique opportunities for students to engage in a high quality mathematics.

Digital technology can also offer students new environments; for example, the black box sketch. In a black box sketch, a mathematical relationship is hidden and the only way to reveal it is to move things around on the screen to see what happens. The results of a motion will often highlight the properties of the mathematical relationship, which in turn aids in identifying the relationship itself. With movement, the student is able to observe and hypothesize. In the black box sketch described in this episode there were only two points on the screen: one labeled A and one labeled B. When A is dragged, B moves in a mathematical path.

The Students and the Task

The teacher in this classroom was open-minded and modern in trying a digital technology. He was not averse to trying new things. He had used *The Geometer's Sketchpad* in a recent Master's program and now was trying some of the activities he had experimented with in the program. But while the technology was not part of traditional practice, he still had to mark and evaluate in the traditional way. The result was a clash of worlds. On the one hand, the students were engaged in a rich environment where they were able to act like mathematicians, but on the other hand, the teacher was in a position that did not match the new environment. The teacher moved around the classroom evaluating student activity and constantly checking to see whether it was achieving what he wanted: the identification of a mathematical concept that had been taught the previous year, but without the technology.

This raises a question about the use of digital technology. What kinds of teaching outcomes might there be in having students work with it? Is providing a rich digital environment such as *The Geometer's Sketchpad* about providing an opportunity for students to explore and conjecture? Or is it about being able to better master content? Although the ideal answer here would be both, I would argue that it should be more about the former than the latter. And this is because the computer offers a variety of opportunities that were not traditionally available. Stimulating competency is one of their comparative advantages. Computers provide a deep and dense environment, one with a lot to consider and act upon. In this environment, problems and situations may be more difficult to reproduce. This creates a new challenge because there is no simple, clear-cut way how to evaluate student exploration. Evaluating students' understanding of and practice using a mathematical concept is often presented in a repeated form, such as a factoring worksheet. When evidence is needed for a percentage or letter grade, parents and administrators are satisfied that it was based on a repeatable kind of problem. Its fairness is less open to dispute. Opening up the mathematics classroom to more generalized skills gives rise to the problem of evaluating where repetition is harder to bring about.

The episode presented in this chapter is based on data that resulted from a research project that looked at the use of digital technology integrated into the mathematics classroom. It was chosen because it exemplifies the tension between perceptions of teacher and students. I (the researcher) had set up a video camera to film two students while I directly observed the classroom activity. Most of my data came from the video recorder, which captured the sounds and sights of the girls interacting with the computer, each other, and the teacher. Because of the discrepancy between what the teacher and the students saw and did, I requested an interview with the teacher and with the students to try to understand what each was thinking and doing after the activity. The students were interviewed together. The teacher brought in a black-box sketch for the students to work on. He chose the concept of reflection, which had been formally introduced the previous year. Reflection exists when the distance from a reflection line is preserved in movement and also points on a reflection line are invariant (Ng and Sinclair 2015).

There were 29 students in a grade 10 classroom. Students were paired up and assigned to work on a set of black-box sketches. The two girls had used *The Geometer's Sketchpad* once before, but had never used a black-box activity. They took turns moving the mouse and exploring the environment. On the teacher's recommendation, they had chosen the command 'trace,' which meant that all their dragging on the screen would leave a trace. Both girls were thoughtful in their movements, as one can see from watching the movements of their mouse along with their head movements.

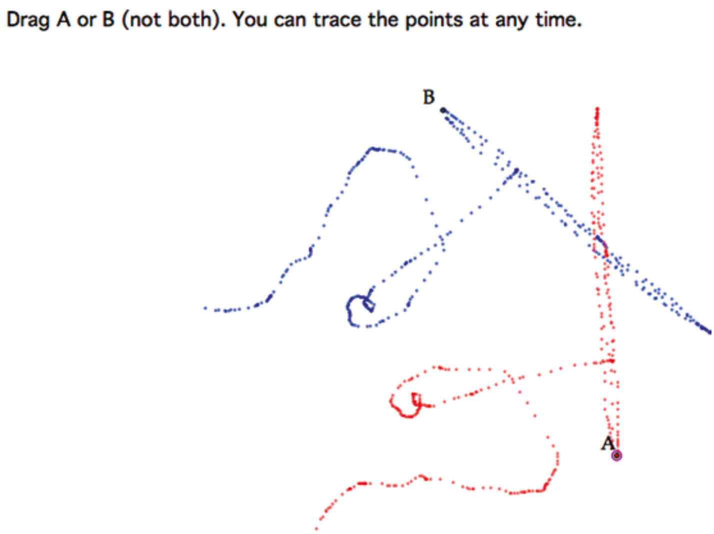

Fig. 1 Retracing lines

Once, when Vanessa had her hand on the mouse and was dragging point A, she asked out loud, "What happens if I drag over here?" Vanessa says this as she drags A slowly and with deliberation across a path she had just created. Here is an example of fulfilling the curricular competency to "Use tools or technology to explore and create patterns and relationships, and test conjectures" (BC Ministry of Education 2015). When Jessica was exploring with the mouse, she retraced the lines she had just drawn. Looking for consistency, she was presumably looking to see whether the identified relationship consistently held true (Fig. 1). As mentioned earlier, this is a way to determine consistency. While it might be a very likely conclusion that the same thing will happen if you travel on the same path as before, it is the testing of such a conjecture that makes it mathematically rich. Communicating and reasoning are evident in this comment from Jessica: "If it goes up, it goes across." When she said this, she moved her point A in one direction while point B went in another direction.

Vanessa seemed to have understood what the relationship was because she started helping Jessica with some of her dragging and moved her hand along the 'invisible' line of reflection (Fig. 2).

After Jessica moved point A upward again, she asked Vanessa, "You said this one, right?" as she pointed at her vertical line. Vanessa replied, "No, no, no." Communication and explaining are apparent here. Finally Vanessa said to Jessica, "Can I try something?" She has a conjecture. Vanessa cleared the screen and drew wavy lines that intersect (Fig. 3).

Jessica said, "woahhhh," took the mouse, and drew a line through the waves, instantiating the reflection line. As she dragged, her head was angled in the same way as the reflection line (Fig. 4). The teacher came over, watched for a few moments, and looked at the image on the screen. He asked Jessica if she had determined the relationship. Jessica was silent; and when the teacher asked again, she said, "I'm trying to think of how I can word it correctly."

Fig. 2 Pointing at hidden reflection line

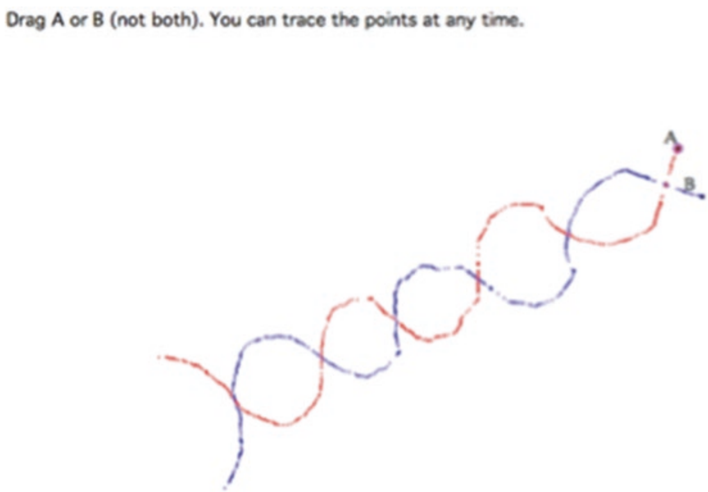

Fig. 3 Instantiating the reflection line

Fig. 4 Drawing the reflection line

In this episode, the mathematical activity was not hidden from the camera. It seems quite clear that the girls were exploring. The evidence for this is that they were initially dragging the point A around the screen. There is evidence that they were conjecturing when Vanessa asked, “Wait can I try something?” To conjecture is to propose an idea and a follow-up would be to check whether this idea is true when tested. The students were given the opportunity to do exactly that. And although neither girl verbally articulated what they were thinking, they presented it on the screen. That they were investigating is indicated by the question “What happens if I drag here?” They were creative when Vanessa’s curving lines outlined and drew attention to the reflection line. They were communicating both through words and through pointing and gesturing. They were debugging by erasing their traces. In so many ways they were being thoughtful, reflective, and inquiring. The digital technology and the black box activity constructed their experience as rich and sophisticated.

The Teacher

The teacher came by three times, in between working with other students, and asked the girls if they had figured out the relationship. The girls did not answer the first two times. We don’t know the reasons for not answering, but it is not that important. What is important is that the teacher, in his haste, moved on when the girls could not give him the answer he was looking for. When he came by for the third and final time, the curves from Vanessa’s drawing were visible, but the teacher was still asking the same question: can you name the relationship? He spent a bit more time with the girls this time, but when he looked at the screen and saw Jessica holding the mouse, he asked if she could explain the relationship for what he saw on the screen. She responded, “I’m trying to think of how I can word it correctly.” Even that response alone hints that she has engaged in the activity in a deeper way than simply being able to name it. She is valuing the response to such an extent that she cannot apply words to the phenomenon. Yet the teacher, eager to move on from this activity, was just hoping to hear the word “reflection.”

Later, in the teacher’s interview, he said, “I asked the students to figure out the relationship. It is a grade 9 topic. I thought it’d be easy, but many students could not state the relationship as reflection.” Later in the interview, the teacher shared that he thought the activity was not very helpful because it did not live up to his hopes that students would able to identify the relationship sooner. He also hinted that he had wanted to use this activity as review, as an opportunity to recall a topic from the previous grade. But it turned out not to be as simple as he had hoped.

As I reflect on this episode and what the teacher said, I also go back to my own experiences and think of the many times I did not have time or resources to spend looking to see what students actually achieved or created. This is not a commentary on how hard teachers work or how little time they have to cover the curriculum. It has more to do with the pressure they are under to value signs of learning that allow them to give a report for each student. This requirement leads to a failure in the teaching of mathematics. What is the right way to assess or evaluate students? What are the right signs? Students are engaging in mathematics in sophisticated ways, but can teachers evaluate them in terms of their own preoccupation with countable outcomes?

Although McGarvey (2014), a professor from the University of Alberta, is addressing early learning of mathematics, her call to teachers is relevant here too: "the challenge for an educator is to be aware of the potential mathematical objects that could arise with and through objects, understand children's learning in relation to those objects, and also be attuned to new possibilities for acting in the space" (p. 10). Content (as opposed to competency) can be easily assessed if presented simply and evaluated based on correct answers. But curricular competencies of reasoning, justifying, and communicating are much more difficult to see—and, if seen, report on.

Mathematical thinking has often been associated with cognitive psychology (Sternberg and Zhang, 2001). That perspective suggests knowledge is something either present or not present. In our episode, the teacher came to the conclusion that Vanessa and Jessica did not know the mathematical relationship because they did not say "reflection." In the teacher's concluding that the girls did not know the concept, he is implicitly expressing the idea that mathematics consists of the final answers or in being able to clearly state a word that summarizes all the previous processes. Much of this problem emerges from the administrative, parental, and curriculum pressure that come with the teacher's role. One overriding concern is the need to produce a letter grade. The desire to evaluate on some objective basis is not unreasonable, but the potential consequences of seeking to fulfill that desire are not always understood. When the teacher sees the inability to use the term "reflection" as an index of a student's inadequacy, he is implicitly acceding to extrinsic pressures on how to teach mathematics. It is extrinsic in the sense that it arises not from a concern with mathematics, but a concern with processing. It is tempting to conclude that the girls did not really understand the mathematics, but this temptation should be resisted as we can see in this episode that the students were engaged deeply in the phenomenon of reflection. They also expressed many competencies of the mathematics curriculum, but simply could not name the phenomenon.

One might ask how one ought to characterize a mathematical object: by its conventional name, or the name points to. The position of the teacher in a social structure makes him susceptible to external pressure from administration, curriculum, and parents. These pressures direct the teacher's expectations and actions, such as in asking questions. As demonstrated by this episode, by having the teacher continue to ask them to name the phenomenon, the students are guided to see mathematics only in terms of correct answers. Had the students made the connection with the term they were taught in grade 9, thinking "aha, *reflection*—what we studied last year"—the activity and the exploring may have been ignored.

With the activity outlined above, it is important to reconfigure what a teacher sees and acknowledge that the absence of the answer is not tantamount to failure because there are other ways a concept can be interacted with. Reflection is dynamic by its very nature. While there are many static images of reflection, it is not possible to determine whether something is a true reflection and not just two identical looking objects until something moves. By analogy, one sees in a movie a person standing still in front of what appears to be a mirror with their reflection in it. Then they move but the "reflection" does not move. It is then you know you are not seeing a reflection. It is similar for the mathematical concept that metaphorically uses the name reflection. Without being able to move something and determine whether properties remain invariant, naming the mathematical relationship is premature. Expecting a particular answer to a mathematical relationship may ignore the necessary stage of checking.

Reanimating Mathematics

The interaction of teacher and student outlined in the episode is the result of the teachers being a part of an institutional hierarchy: one with a certain incentive structure. The resulting expectations of the teacher do not fit with the rich activity of the girls. There is a sort of injustice.[1] The varied student experience of mathematics made possible by technology is not getting its due from the teacher. In order for it to do so, expectations that fit with the non-content side of the BC standards need to be developed. I draw on this episode to communicate the need rethink the mathematics classroom situation. This is necessary in order to make a break from tradition and to acknowledge something we may be unaware of. More often than not, "school mathematics serves to obscure things that otherwise were previously visible to [students]" (Guttierez 2013, p. 48). Vanessa and Jessica were both engaging with a mathematical concept, yet were not aware of the richness of their work. The teacher was not there to validate what they were experiencing and encourage them to proceed. The girls were doing mathematics, but the teacher reported out that they were not. Dylan William (2003) indicates that adopting a particular definition of mathematics is as much a moral decision as an epistemological enterprise. The questions teachers ask convey the kind of mathematics they value. If content, which is easier to assess than skills, is stressed by a teacher, students will come to see mathematics as a body of knowledge rather than an attitude or an approach. The point here is not just that individual teachers need to see more and be more aware, and it is not that there needs to be a change in the content of the curriculum. The discrepancy between student activity and the reporting on that activity means that at the institutional level we must find a way to encourage the development and exercise of curriculum competency and find a way to appreciate it in the reporting. This may require more subjective grading. Regardless, we must find grading and valuing methods that fit what we want to achieve in education rather having our education truncated to allow for ease in reporting.

[1] Justinian, *Digest*, 1.1.10 (AD 530–533): *Iustitia est. constans et perpetua voluntas ius suum cuique tribuendi* ([The virtue of] justice is the constant and perpetual will to render to each his due.) See also first sentence of Justinian, *Institutes* (AD 535).

Boylan (2007) states that "by highlighting features such as authority, diversity and conflict, [it is] suggest[ed] that mathematics classrooms …be seen as communities of political practice" (p. 1). The issue in this episode is expressed in the difference between the students' and the teacher's perspective. While it may be argued that the problem of the differing views was based on a breakdown of communication, I suggest that since the teaching and learning of mathematics was affected it becomes a sociopolitical issue (Gutierrez, 2013). Gutierrez argues that it is imperative to be aware of how and when invisible power structures of the institution determines who is successful and who is disadvantaged through the very act of its policies. She writes, "Taking the sociopolitical turn means deconstructing the taken-for granted rules and modes of operating and making the familiar seem strange, not as a kind of intellectual exercise, but as a means to open up possibilities for something new" (p. 56).

Technology changes what students can do, and what they do affects what they can see and understand. During my time as a high school teacher, I too would probably have interpreted the girls' lack of a response as simply not knowing the material. However, if I had realized that mathematics is not a black-and-white matter and had had a fuller understanding of what mathematical knowledge consists in, I may well have taken note of other signs of progress in the student's work. Such awareness would have fostered a more constructive engagement with the students.

I draw on this episode not to argue that questions oriented toward grading should be completely avoided. Rather, I want to emphasize that the burdens that teachers now work under discourage them from appreciating signs of competency, in particular the wonderful new ways that are possible with new technology. Mathematics is capable of being even more interesting than it already is—and interesting to students who might not see its attractions as currently taught.

References

Boylan, M. (2007). Teacher questioning in communities of political practice. *Philosophy of Mathematics Education Journal,* 20. Retrieved from http://socialsciences.exeter.ac.uk/education/research/centres/stem/publications/pmej/pome20/index.htm

British Columbia Ministry of Education. (2015). *Building student success – BC's new curriculum.* Victoria: Queen's Printer. Retrieved from https://curriculum.gov.bc.ca/curriculum/mathematics/9

Gutierrez, R. (2013). The sociopolitical turn in mathematics education. *Journal for Research in Mathematics Education, 44*(1), 37–68.

Jackiw, N. (2001). *The Geometer's Sketchpad*. Emeryville: Key Curriculum Press.

McGarvey, L. M. (2014). Intentional play-spaces for teaching/learning mathematics with young children. *For the Learning of Mathematics, 34*(2), 9–11.

Ng, O., & Sinclair, N. (2015). Young children reasoning about symmetry in a dynamic geometry environment. *ZDM – The International Journal on Mathematics Education, 51*(3), 421–434.

Rotman, B. (2008). *Becoming beside ourselves: The alphabet, ghosts, and distributed human beings*. Durham: Duke University Press.

Sinclair, N., Bartolini Bussi, M., de Villiers, M., Jones, K., Kortenkamp, U., Leung, A., & Owens, K. (2016). Recent research on geometry education: An ICME13 survey team report. *ZDM – The International Journal on Mathematics Education, 48*(5), 691–719.
Sternberg, R., & Zhang, L. F. (Eds.). (2001). *Perspectives on thinking, learning, and cognitive styles*. London: Lawrence Erlbaum.
William, D. (2003). Constructing difference: Assessment in mathematics education. In L. Burton (Ed.), *Which way social justice in mathematics education?* (pp. 189–207). Westport: Praeger.

Considering Both Academic and Social Domains: Increasing Student Engagement in At-Risk Classrooms

Limin Jao

Abstract This chapter describes the teaching practices that three mathematics teachers in at-risk contexts used to increase student engagement and enhance student learning. Qualitative data were collected in the form of teacher interviews, classroom observations and teacher journals. Findings show that these teachers considered aspects of both social (e.g., creating a classroom community and developing a teacher-student relationship) and academic (e.g., using technology, manipulatives, group work and student-centered activities) domains of student engagement in their teaching, but to varying degrees and with different emphases. All teachers noted that these strategies also appeal to their students' characteristics as early adolescents.

Keywords Adolescence · At-risk students · Mathematics education · Student engagement · Teacher practices

Student engagement has direct implications on student retention and academic success (Lowe et al. 2010; Mergendoller et al. 1988). Students who are engaged with school will find the experience rewarding and enjoyable (Marks 2000); whereas, disengagement leads to rebellion, disruptive behaviour, academic disinterest, and failure (Hand 2010). Student engagement is particularly fragile during adolescence, a developmental period in which youth undergo many physical and emotional changes (Archambault et al. 2009). To support early secondary school students, teachers may consider components within two domains for student engagement: social and academic. Following a brief summary of some of the relevant literature, three case studies of teachers who used different emphases to support student engagement will be presented. The cases are all situated in Ontario's Grade 9 Applied[1] level classrooms. In the province of Ontario, students in grades 9 and 10 are generally

[1] See McDougall and Ferguson (Part II this volume, para. 1) for a discussion of two of the possible Ontario pathways (Academic and Applied).

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streamed into two levels: Applied and Academic. Students in the Applied level stream tend to transition into the workplace or to college programs after completing their secondary diploma; whereas, students in the Academic stream tend to seek entry into university programs. Research has shown that Applied level classrooms are contexts often associated with poor engagement (Brown 2008; King et al. 2005).

Social Engagement

Often, adolescents are self-aware and fragile (Archambault et al. 2009; Carnegie Council on Adolescent Development 1989). In this tentative stage of development, teachers should act to reassure their adolescent students and encourage them to persevere. Teachers should develop students' self-confidence, personal identity, and autonomy (Archambault et al. 2009). Adolescent students need to feel like they belong. Students who have a sense of membership in their learning community are more likely to make an effort in their learning and persevere through academic challenges that they may face along the way (Solomon et al. 2000). Schools need to foster an environment in which students feel both invited and supported. In addition to developing student-teacher relationships, teachers can create a classroom-learning environment that fosters a sense of community (Hargreaves et al. 1996).

Teachers' support can have an impact on students' feelings of belonging and engagement (Libbey 2004). Rosenfeld et al. (2000) found that middle and secondary school students who felt that their teachers cared about them were more engaged. Additionally, if the teacher conveys his or her high expectations of the student, this can positively influence student engagement and achievement (Rubie-Davies, 2015).

Within their classrooms, teachers need to create an environment in which their adolescent students can thrive. These environments should be equally inviting to students and in these positive social-emotional spaces, teachers can further incorporate specific teaching strategies that encourage student engagement.

Academic Engagement

Canadian classrooms are growing in diversity (Hutchinson 2016; Statistics Canada 2013). In order to meet the needs of all students, teachers must differentiate their instruction and this can be done in many different ways. Some students may learn better in smaller groups while their peers may learn better through independent work. Some require scaffolding to help them understand a concept, while others do not. Some students are visual learners while others are auditory. In classrooms where teachers do not vary their teaching approach, students' interest may wane resulting in an increased risk of academic failure (Karp and Voltz 2000).

Research has shown that teachers in more engaging classrooms use numerous research-based practices. This includes teachers connecting material to prior knowledge, making learning challenging and relevant, using appropriate student tasks, and keeping cultural and technological conditions consistent (Luke et al. 2003;

Pressley et al. 2003). The use of rich learning tasks provides teachers with an opportunity to use many of these desirable practices. Tasks are "rich" if they are challenging for students (Raphael et al. 2008). These meaningful activities should integrate mathematics curriculum as well as allow students to focus on a broad range of ideas rather than smaller, discrete concepts (Sullivan et al. 2009). Additionally, the tasks should be open-ended to allow for students to explore rather than direct students on a prescribed pathway of learning. Teachers should also use tasks and activities that are matched to their students' interests. By making mathematics learning relevant to the students' lives, the students can become more invested in their learning (Bobis et al. 2011).

Another way that teachers can make mathematics relevant to adolescent learners is through the use of technology (Gee 2003). Students use technology in their daily lives, and removing this component of their lifestyle would detract from a school's attempt to provide a welcoming space to students. Adolescents use computers, the Internet, iPods, gaming systems, and cellular phones, among others. If teachers can incorporate these tools into their teaching, student engagement will be easier to maintain. Technology can have benefits on student achievement by enhancing student understanding (National Council of Teachers of Mathematics 2000). Examples of technology that are commonly used in the mathematics classroom include: (graphing) calculators, interactive whiteboards (e.g., SMARTBoard technology), virtual manipulatives (e.g., ExploreLearning's Gizmos), and immediate response devices (e.g., i > clickers).

Raphael et al. (2008) suggested that it is not the type of practice that teachers use to stimulate student engagement but rather the quantity of practices. They asserted that the sheer diversity of practices ensures that students are engaged. Based on personal preference, classroom resources, and contextual considerations, teachers will choose to implement their own complement of teaching practices. Not only should teachers personalize the strategies that they use to increase student engagement within the academic domain, considerations within the social domain will also be unique to individual teachers.

Recent Research

To follow, a study which investigated the practices used by Grade 9 Applied level mathematics teachers to increase student engagement is described. To represent the diversity and individualization of how teachers enact their approach to increase student engagement, I share the cases of three teachers: Benjamin, Mathieu, and Nadia. All are teachers at secondary schools in a large urban city in Southern Ontario. The teachers were participating in the Collaborative Teacher Inquiry Project (Jao and McDougall 2015, 2016) and thus, engaging in professional development sessions and were actively focusing on improving their instructional strategies for the Grade 9 Applied Mathematics course. I used interviews,[2] classroom observations, and

[2] In the following sections, participant quotes are referred to by participant name and the interview number (e.g., B1: Benjamin, interview 1).

teacher journals to explore the teachers' beliefs and practices. Findings suggest that these teachers considered aspects of both social and academic domains of student engagement in their teaching but to varying degrees and with different emphases. All names of schools and teachers are pseudonyms.

The Case of Benjamin – A Balanced Approach

Benjamin showed a relatively balanced focus in his teaching. He created a learning environment that supported his students' social needs (social domain) and used teaching strategies to improve student achievement (academic domain). This balance could be attributed to his desire to integrate new teaching strategies and improve upon those that he already used. Through professional development opportunities and in working with colleagues, Benjamin sought out new student tasks and activities and was keen to discuss adaptations and modifications to existing strategies.

In the following quote, Benjamin discussed one student whom he described as going from disengaged at the beginning of the semester to being fully engaged by the end:

> I have one student who, at the start of this semester, I thought would be a non-attender, because that is what he is in most of his other classes. But he is engaged by a number of the strategies that we have used and he does not miss class ever. [T]here is certainly a proudness [sic] when I see him in the hallway and he says, "How is it going?" I think that in providing engagement and just giving him some sort of validity in the classroom has been huge. (B1)

Benjamin's description of this student demonstrated that he considered factors in both the social and academic domains for student engagement. Within the social domain, Benjamin mentioned that there has been an increase in this student's self-confidence. Benjamin encouraged his students and often validated them by reinforcing their strengths. In the academic domain, Benjamin noted that the inclusion of a variety of teaching strategies had a positive effect on this particular student. Again, as evidenced by an improved attendance record, Benjamin believed that the strategies that he used in class were compelling enough for this student to make the decision to attend class more often. In addition to creating a community that supported self-conscious students and using a variety of teaching strategies, Benjamin considered social factors such as developing relationships with his students and providing opportunities for students to develop relationships with one another.

Specific academic factors considered by Benjamin included using small group learning to meet individual's needs and support the learning community; and using the Targeted Implementation and Planning Supports for Revised Mathematics (TIPS4RM) for Grades 9 Applied (Ontario Ministry of Education 2005) and technology to provide opportunities for student-centered learning, rich learning tasks, and allowing for interactive learning contexts. Lesson plans found in the TIPS4RM resource integrate multiple components of mathematics curricula and a variety of learning experiences for students. In this research-based resource, a

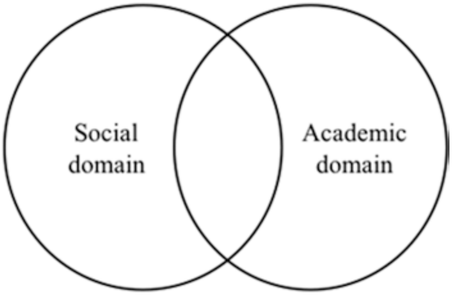

Fig. 1 Interaction of Benjamin's factors for student engagement. This figure illustrates the interaction between the factors considered by Benjamin to increase student engagement

student-centered, inquiry-based approach is valued and conceptual understanding is prioritized over procedures and algorithms.

Benjamin personally valued building a community as a learning environment, thus ensuring that he created a positive, supportive and inclusive space for his students. These social and academic domains intersected via specific strategies that support the notion of community (e.g., cooperative learning). Similar to the work of Sullivan et al. (2006), all of the factors demonstrated by Benjamin's practice took into consideration the characteristics of the early adolescent, especially developing positive relationships with peers and the influence that they had on one another. Figure 1 is a visual representation of the interaction between the factors considered by Benjamin to increase student engagement.

The Case of Mathieu – A Focus on the Academic Domain

Mathieu focused on the academic domain of student engagement. As a result of a new initiative at his school to invigorate the Grade 9 Applied Mathematics program, Mathieu implemented the TIPS4RM resources and a variety of technologies into his teaching. Although initially skeptical of how much this approach would impact the students, after seeing his students' attitudes towards mathematics change and seeing the increase in student engagement, Mathieu continued to use these teaching approaches. Mathieu described his current approach as a way to increase student engagement:

> In my class, we use the new methods, which are working in pairs, student interaction and group work. The kids are always engaged and do the richer type problems where they have to collaborate, work together, try to come up with a solution so it is mostly student driven. So the teacher is more or less just a facilitator who gives directions. We try to get the kids engaged every day. So they are doing stuff. Hands-on. So they are not sitting there being bored. (M2)

Through Mathieu's description, we can see his focus on factors within the academic domain. These teaching strategies allowed students to become interested in learning the material and supported their developing mathematics understanding. At the same time, Mathieu realized that, to best implement these academic practices into his teaching, he needed to step back from his formerly authoritarian approach in the

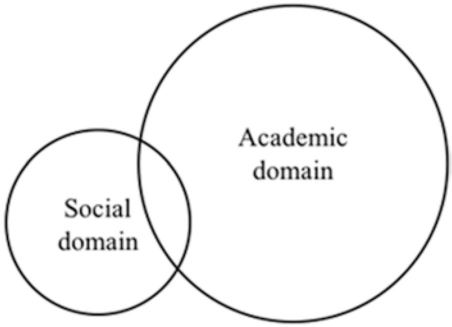

Fig. 2 Interaction of Mathieu's factors for student engagement. This figure illustrates the interaction between the factors considered by Mathieu to increase student engagement

classroom. Although this approach did not align with his laid-back character, he saw that his students were used to this environment from other mathematics contexts and he found it easy to maintain.

With these new student-centered approaches during which students often engaged in cooperative learning, Mathieu's role changed as he had to let the students work together and construct their own knowledge. This immediately appealed to Mathieu in that he did not need to maintain control of the learning situation and thus could sit back, not worry about having something happening at all times, and just support the students when they needed him.

As a result, Mathieu began to consider the factors within the social domain of student engagement, for example, by creating an environment in the mathematics classroom that was similar to the students' regular environment so that the students felt comfortable in the mathematics classroom and so that it could be a community in which they could belong. Thus, the emphasis on belonging to a community as discussed by Solomon et al. (2000) was only realized after Mathieu implemented academic factors to create a student-centered environment. This environment demonstrated to Mathieu the benefits for a community on his students' social needs. The relaxed classroom environment that Mathieu created allowed students to feel a sense of independence. Eccles et al. (1991) stated that independence is appealing to early adolescent learners. Figure 2 is a visual representation of the interaction between the factors considered by Mathieu to increase student engagement.

The Case of Nadia – A Focus on the Social Domain

For Nadia, student engagement extended beyond students being interested in the mathematics content. Nadia explained that, especially for Applied level students, students showed their engagement on a personal level. If students were engaged, they would connect with their peers and their teacher on a personal level. Even after students completed the mathematics work for the class, Nadia said that engaged students would continue to linger. Engaged students will socialize with their peers and develop a stronger relationship with the teacher. Nadia said that she could tell if a student was engaged based on their communication in the classroom. During the

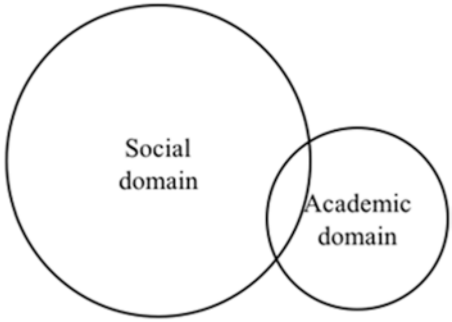

Fig. 3 Interaction of Nadia's factors for student engagement. This figure illustrates the interaction between the factors considered by Nadia to increase student engagement

lesson portion of the class, Nadia said that a student was engaged if there was active communication (verbal and non-verbal). Students would remain engaged throughout the class regardless of whether or not mathematics learning was happening. Nadia explained what student engagement meant to her:

> When I think about student engagement, all I can think about is [students'] communication with [the teacher]. Talking to you. It could be about their work. That is my hope, usually, but it is not always the case and sometimes it is just about their life, or religion, the way they think about the religion or so on. Sometimes it is just about their life…with the Grade 9 Applied [stream], they become more personal. (N1)

Nadia had limited experience with the Grade 9 Applied course prior to participating in the Collaborative Teacher Inquiry Project. Although Nadia expressed that she was confident with her content knowledge for the course, she shared that her comfort level with the teaching strategies and approaches best received by the Grade 9 Applied Mathematics students was still in development. Nadia explained that these approaches were different from those that she used with the Academic stream and upper year courses. Although Nadia was eager to integrate new approaches into her teaching (e.g., manipulatives), she shared that she needed time to process how to use the strategies herself as a learner before being able to consider how to present and facilitate their use in the classroom. Of this, she said:

> Everything is new. I cannot do everything. I need my own time to do well. I need to be comfortable and to be comfortable, I need to be able to put in my own time. As a teacher, you need to be comfortable first. And it has to be your style. (N2)

In part, due to her hesitation to fully implement various academic approaches, Nadia's consideration of the social domain for her students' engagement shone. As Murray (2009) described, a trusting relationship between a teacher and student has positive benefits on student engagement. Nadia's personality and obvious care of her students was the foundation of the relationship that she developed with her students. Additionally, as Nadia cultivated her arsenal of teaching strategies, these new teaching practices were often guided by her being able to strengthen the student-teacher relationship and to provide additional support to these learners who may have wavering self-confidence. Archambault et al. (2009) also indicated that increasing student self-confidence may support student engagement. Figure 3 is a visual representation of the interaction between the factors considered by Nadia to increase student engagement.

Final Thoughts and Considerations for Teacher Education

The cases of Benjamin, Mathieu, and Nadia show that these three Grade 9 Applied Mathematics teachers considered factors of social and academic domains for student engagement. Within these domains, the teachers were cognizant of characteristics of the adolescent learner. Specifically within the social domain, key factors considered by all teachers included developing students' self-confidence, creating a sense of community and belonging, developing relationships (student to student, and student to teacher). Within the academic domain, teachers use a variety of factors including issues of program planning, meeting individual needs, programs to support student learning, cooperative learning, student-centered approaches, student mathematical communication, student tasks, and technology.

The teachers shared that developing student self-confidence is a critical component of increasing student engagement for early adolescent learners. Each of the three teachers' practices showed evidence that developing student self-confidence is a factor within the social domain for student engagement. Additionally, the teachers fostered developing student self-confidence within their teaching practices targeted for academic engagement.

Student engagement can be examined through two domains (social and academic). These domains, and the factors found within, are not independent. Mathieu's original focus solely on the academic domain for student engagement and incorporating investigative activities and rich tasks shifted Mathieu's classroom towards a student-centered model. This model encouraged classroom discussions amongst peers and took control away from Mathieu. The increased peer interaction, initially intended for academic gains, was also socially beneficial to the students. For Nadia, the strong teacher-student relationships that she developed to support her students within the social domain also had academic benefits. As a result of the personal support and encouragement from Nadia, her students felt that they too could be academically successful. The rich tasks found in the TIPS4RM resource had multiple benefits across academic factors. For example, Benjamin used these student-centered tasks to allow his students to learn across a range of concepts as well as to help support construction of knowledge.

Some teachers may focus on one domain more than the other as a means to increase student engagement. Additionally, teachers may choose to prioritize one domain over another as a result of their personal comfort with that domain. Nadia's teaching practice, for example, focused on the social domain as a result of her developing comfort with new academic instructional strategies such as technology and manipulatives. Mathieu, on the other hand, was comfortable with technology and personally valued a hands-on and interactive approach to learning. Thus, Mathieu focused on developing these academic factors in his teaching as a means to increase student engagement highlighting the impact of teacher self-efficacy on practice (Bruce and Ross 2008) and the natural tendency for teachers to individualize their practices (Siegel 2005).

These findings have implications on our teacher education practices. The cases of Benjamin, Mathieu, and Nadia show varied ways in which teachers can attend to academic and social domains for student engagement and some challenges they faced in expanding their approach to teaching. Thus, it may be of value to reflect on current models and priorities for professional development to better support teachers seeking to develop their teaching practice to meet the needs of at-risk students. Firstly, professional development initiatives encouraging teachers to implement new or "best" practices teachers should be thoughtfully examined. Is the professional development providing enough opportunity to allow teachers to be fully comfortable with the approaches? Schools and school boards should be encouraged to reconsider the traditional, one-day workshops that are still consistently used for professional development as they are limited in the resources and experiences needed for teachers to implement the presented ideas into their practice (Stein et al. 1999). In contrast, professional development initiatives where teachers can engage in collaborative models such as co-teaching and peer coaching to follow up on new strategies learned may counter these challenges (Jao 2013). Secondly, a more concerted effort to acknowledge both the social and academic domains for student engagement must be made. Professional development tends to focus on the academic domain, yet unless a student's social needs are met, academic approaches will not be as effective (Fredricks et al. 2004; Li and Lerner 2011). Without recognizing the positive potential benefits of both domains, it will be a challenge to keep at-risk students interested and engaged in their education.

References

Archambault, I., Janosz, M., Morizot, J., & Pagani, L. (2009). Adolescent behavioural, affective, and cognitive engagement in school: Relationship to dropout. *Journal of School Health, 79*(9), 408–415.

Bobis, J., Anderson, J., Martin, A., & Way, J. (2011). A model for mathematics instruction to enhance student motivation and engagement. In D. Brahier (Ed.), *Motivation and disposition: Pathways to learning mathematics*, *National Council of Teachers of Mathematics Seventy-third Yearbook* (pp. 31–42). Reston: NCTM.

Brown, R. S. (2008). *The grade 9 cohort of fall 2002: A five-year cohort study, 2002–2007*. Toronto: Toronto District School Board.

Bruce, C., & Ross, J. A. (2008). A model for increasing reform implementation and teacher efficacy: Teacher peer coaching in grades 3 and 6 mathematics. *Canadian Journal of Education, 31*(2), 346–370.

Carnegie Council on Adolescent Development. (1989). *Turning points: Preparing American youth for the 21st century*. Washington, DC: Carnegie Council on Adolescent Development.

Eccles, J., Lord, S., & Midgley, C. (1991). What are we doing to early adolescents? The impact of educational contexts on early adolescents. *American Journal of Education, 99*(4), 521–542.

Fredricks, J. A., Blumenfeld, P. C., & Paris, A. H. (2004). School engagement: Potential of the concept, state of the evidence. *Review of Educational Research, 74*, 59–109.

Gee, J. P. (2003). *What video games have to teach us about learning and literacy*. New York: Palgrave Macmillan.

Hand, V. M. (2010). The co-construction of opposition in a low-track mathematics classroom. *American Educational Research Journal, 47*(1), 97–132.
Hargreaves, A., Earl, I., & Ryan, J. (1996). *Schooling for change: Reinventing education for early adolescents*. Bristol: Falmer.
Hutchinson, N. L. (2016). *Inclusion of exceptional learners in Canadian schools: A practical handbook for teachers* (5th ed.). Toronto: Pearson Education Canada.
Jao, L. (2013). Peer coaching as a model for professional development in the elementary mathematics context: Challenges, needs and rewards. *Policy Futures in Education, 11*(3), 290–297.
Jao, L., & McDougall, D. (2015). The collaborative teacher inquiry project: A purposeful professional development initiative. *Canadian Journal of Education, 38*(1), 1–22. Retrieved from http://journals.sfu.ca/cje/index.php/cje-rce/article/view/1600
Jao, L., & McDougall, D. (2016). Moving beyond the barriers: Supporting meaningful teacher collaboration to improve secondary school mathematics. *Teacher Development, 20*(4), 557–573.
Karp, K. S., & Voltz, D. L. (2000). Weaving mathematical instructional strategies into inclusive settings. *Intervention in School and Clinic, 35*(4), 206–215.
King, A. J. C., Warren, W. K., Boyer, J. C., & Chin, P. (2005). *Double cohort study phase 4 report for the Ontario Ministry of Education*. Kingston: Queen's University.
Li, Y., & Lerner, R. M. (2011). Trajectories of school engagement during adolescence: Implications for grades, depression, delinquency, and substance use. *Developmental Psychology, 47*, 233–247.
Libbey, H. P. (2004). Measuring student relationships to school: Attachment, bonding, connectedness, and engagement. *Journal of School Health, 74*(7), 274–283.
Lowe, K., Lee, L., Schibeci, R., Cummings, R., Phillips, R., & Lake, D. (2010). Learning objects and engagement of students in Australian and New Zealand schools. *British Journal of Educational Technology, 41*(2), 227–241.
Luke, A., Elkins, J., Weir, K., Land, R., Carrington, V., Dole, S., Pendergast, D., Kapitzke, C., van Kraayenoord, C., Moni, K., McIntosh, A., Mayer, D., Bahr, M., Hunter, L., Chadbourne, R., Bean, T., Alverman, D., & Steven, L. (2003). *Beyond the middle: A report about literacy and numeracy development of target group students in the middle years of schools*. Canberra: Commonwealth Department of Education, Science and Training.
Marks, H. M. (2000). Student engagement in instructional activity: Patterns in the elementary, middle, and high school years. *American Educational Research Journal, 37*(1), 153–184.
Mergendoller, J. R., Marchman, V. A., Mitman, A. L., & Packer, M. J. (1988). Task demands and accountability in middle-grade science classes. *Elementary School Journal, 88*(3), 251–265.
Murray, C. (2009). Parent and teacher relationships as predictors of school engagement and functioning among low-income urban youth. *Journal of Early Adolescence, 29*(3), 376–404.
National Council of Teachers of Mathematics. (2000). *Principles and standards for school mathematics*. Reston: Author.
Ontario Ministry of Education. (2005). *TIPS4RM for grades 7, 8, 9 applied and 10 applied math*. Toronto: Queen's Printer for Ontario.
Pressley, M., Dolezal, S. E., Raphael, L. M., Mohan, L., Roehrig, A. D., & Bogner, K. (2003). *Motivating primary-grade students*. New York: Guilford.
Raphael, L. M., Pressley, M., & Mohan, L. (2008). Engaging instruction in middle school classrooms: An observational study of nine teachers. *The Elementary School Journal, 109*(1), 61–81.
Rosenfeld, L. B., Richman, J. M., & Bowen, G. L. (2000). Social support networks and school outcomes: The centrality of the teacher. *Child and Adolescent Social Work Journal, 17*(3), 205–226.
Rubie-Davies, C. M. (2015). *Becoming a high expectation teacher: Raising the bar*. London, UK: Routledge.
Siegel, C. (2005). Implementing a research-based model of cooperative learning. *Journal of Educational Research, 98*(6), 339–349.
Solomon, D., Battistich, V., Watson, M., Schaps, E., & Lewis, C. (2000). A six-district study of educational change: Direct and mediated effects of the Child Development Project. *Social Psychology of Education, 41*, 3–51.

Statistics Canada. (2013). *Immigration and ethnocultural diversity in Canada*, *DHHS Catalogue no. 99–010-X2011001*. Ottawa: Author.
Stein, M. K., Smith, M. S., & Silver, E. (1999). The development of professional developers: Learning to assist teachers in new settings in new ways. *Harvard Educational Review, 69*(3), 237–270.
Sullivan, P., Tobias, S., & McDonough, A. (2006). Perhaps the decision of some students not to engage in learning mathematics in school is deliberate. *Educational Studies in Mathematics, 62*(1), 81–99.
Sullivan, P., Mornane, A., Prain, V., Campbell, C., Deed, C., Drane, S., Faulkner, M., McDonough, A., & Smith, C. (2009). Junior secondary students' perceptions of influences on their engagement with schools. *Australian Journal of Education, 53*(2), 176–191.

Additional Suggestions for Further Reading

Miller, J. P. (2010). *Whole child education*. Toronto: University of Toronto Press.
Stocker, D. (2017). *Maththatmatters 2: A teacher resource linking math and social justice*. Toronto: Between the Lines.

A Teacher's View – My Favourite Mistakes: Experiences Teaching Cree Students in Northern Quebec

Michael Newell

Abstract The experiences of a recently graduated (non-Indigenous) teacher are told in this first person account of 5 years of teaching in Canada's far north. Although pre-service teachers often take courses meant to prepare them for working with Indigenous populations, this chapter reminds readers that much more than initial knowledge is involved. While relationship-building remains an important goal, these stories of first-hand experiences portray this process as more subtle, complex, and gradual than originally thought. Shared in the chapter are personal experiences and anecdotes, changes made based on some insights from various sources both in and out of school, a description and examples of attempts to teach students some of the mathematics curriculum from the Cree School Board, and finally a synopsis of the gratitude and humility felt by the author for the experience.

Keywords Indigenous education · Mathematics and Indigenous communities · Mathematics in northern communities · Northern schools · Novice teachers' experiences in the north

We call it "pulling a punch." Remember the expression, "He doesn't pull any punches," meaning your supervisor is very frank with you. He does not soften his blows when he verbally rebukes you. In this case, my student Gary had pulled his punch, completely. The knuckles on his clenched hand tickled those tiny, white facial hairs just below the bottom of my eye socket. He had stopped his fist from connecting, from driving into my face. At that moment, I recalled that my Special Education professor's assistant's question, "What did the teacher do to provoke that?" I heard this question during a conversation a year prior. It was regarding a report that a student had physically assaulted their teacher somewhere in Canada.

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The purpose of this chapter is to share my experience as a recently graduated and inexperienced teacher, who began their teaching career in Canada's north.

While I had taken some courses which, on paper, might have prepared me, my reality was far different. I needed to better understand the context where I was teaching. My education to that point was important; however, I was unaware of the how the community would perceive my behaviour. I did not understand how to establish mutual trust and respect. I did not realize how to conceive of my students' needs: not only in education but in their general lives in the village. Lastly, it terms of teaching math, I did not appreciate the difficulties that students could experience in trying to learn a subject like math.

After a period of almost 5 years teaching students in three different communities, a reader might expect that someone like myself would have some very solid advice for those who are interested in teaching there. However, the sum of my experience to date still includes a humbling reticence. And honestly, I find this reserve is an asset. Life in Cree communities is an amalgam of history, politics, culture, language, etc. and much of it convoluted by various aspects of colonialism and Cree people trying to find themselves between a minimum of two very different worlds. In so many of the conversations that I have had with different people, many who are more knowledgeable than me about life among the Cree, I continually hear the catch phrase, "It's complicated." I do not want to be glib or trite. Hardly. I am trying to share an understanding to which I have given a great deal of thought. Hence, I can share some of my experiences that shaped my limited understanding to date. These are not all my experiences. The mistakes I made in this article mainly occurred when I taught in one northern community. However, my sharing may offer the reader a point of departure for reflection, conversation and further exploration. I will share how I came to teach in Cree communities. I will explain further my experience with my student Gary whom I described above and how that situation devolved. Then I will share some changes that I made based on some insights I had from various sources. Finally, I will offer a description of my attempts to teach my students some of the mathematics curriculum from the school board. I will end with an understanding of the gratitude and humility that I feel from my experience.

I have avoided naming any of the individuals because I find that many of the people I know there would not be comfortable being named in an article for publication.

My Journey

I received my Bachelor of Education degree from the Lakehead University Faculty of Education on Friday, May 18th, 2012. Twelve weeks later, on Friday, August 10th, my 16-year-old son Merrick and I pulled out of our driveway in Toronto, Ontario, Canada. For the next 2 days, we drove some 1400 km (870 miles) to a Cree community on the eastern coast of James Bay in northern Quebec. I would teach a Secondary 2 (grade 8) class at a school in a village of less than a thousand people. Four years earlier, Merrick and I had read an article in the Wheels section of The

Saturday Star entitled, *Solo to James Bay-and back again* (Wheels 2008). The author had driven north in a BMW X6 from Toronto to James Bay. We were excited to make the trip together. This day we drove into our dream in my 2006 Saturn Ion Sedan. Supposedly I had prepared for this experience at Lakehead with course number 4416-FC called *Aboriginal education* with Professor Dolores Wawia. As well, I had completed a workshop at OISE (Ontario Institute for Studies in Education of the University of Toronto) called *Beginning to teach in an Indigenous way*. I had no idea what I was in for. I will provide some stories of my experiences. They are offered as illustrative rather than representative. They may provide the reader with a glimpse of the possible challenges one might experience there.

I never anticipated what would happen after Gary was escorted to the vice principal's office. His mother came into the room. I joined a meeting with the principal, vice principal, the student and his mother. In the crowded little office, the mother was visibly upset.

She glared at me. "What were the three swear words?"

The rule in my class was that students were sent to the *Student Support Room* after their third recorded swear word of the day. (In the Student Support Room, an on-duty teacher answers any questions and offers support to students in an unused classroom after they are taken out of class for disruptive behaviour. Students have a quieter, less distracting space and the teacher's support for completing their work.) The schoolwide rule was that a student is sent to the *Student Support Room* after they swear once in class. For myself, students' swearing was either the familiar expression in response to some pain or discomfort (sh*t!); or a crude way to emphasize an opinion (that f***in' jerk!), or a test behaviour used by some students to explore just how much they could disrupt the classroom (sh*t, f***, sh*t, f***, sh*t, f***!). My difficulty was that I did not experience foul language here in a Cree village as I had experienced in southern Ontario. Outside of school, parents, professionals, even teachers used this vocabulary. Not necessarily in the classroom, but I soon gathered that this was familiar diction in families and in the community. It did not have the social impact that I am accustomed to. Similarly, English swears are almost meaningless among French speaking Quebecers. I began to see myself as a stranger in the community, who was using a fabricated norm from a very different culture. I arbitrarily forced this onto these students. That did not mean that foul language was not used to disrupt classes and undermine a teacher's authority. It was. However, for me to make a meaningful response, I felt I had to make an adjustment. Eventually I changed the rule in my classroom. A student would go to the *Student Support Room* after three, instead of one swear in class. So, each student received two warnings before they received a consequence. That's how we had arrived at the question, "What were the three swear words," that Gary's mother had yelled at me. Whatever the words were, I shared them with her, however she kept crying and insisting that I was at fault. I do not recall how the meeting ended, but while we were leaving the office the tone felt as if the issue was not settled. The principal had not stated how she understood the situation. We left the room and the student returned home with his parent instead of finishing the day. I was told later that the principal had agreed with the mother. She had understood that my choice was racially motivated.

Later, I learned that this student had been suspended from other schools. When these incidents had occurred in the past, the mother brought her son to another community and so another school, to avoid the suspension. On a Parent/Teacher Night I had had a conversation with this mother regarding her son. She had struck me then as very intelligent and reasonable. She agreed with me that her son could have a very disruptive impact on a classroom. At that time, I understood that we might work together, parent and teacher, to help him focus his behaviour in a more constructive manner. Still, her son did not respond to my efforts to engage him. After the classroom incident, this parent-teacher understanding changed. She now felt her son's situation was the result of my sinister motives. As it happened, the principal agreed with her.

Let me share how complicated this situation was. Although the vice principal persuaded the principal to suspend the boy, it meant that the principal was taking a personal risk. She is Cree and was born into the community. There were community members who were upset about the decision. I had seen this community pressure before when a colleague of mine was accused of hitting a student. For my colleague's science class, he had organized a "hot water to ice" demonstration on a winter day when the outside temperature was below −40°C. An interesting example of this experiment is found at https://www.youtube.com/watch?v=2LFtYUUXJlE. A student stood on a small desk so that he could reach the open window above him. He intended to throw some very hot water out of the classroom window into the intensely cold air outside and see it instantly change into powered ice crystals before it reached the ground. However, the safety of this situation was threatened by a second student who was agitating the desk that on which the first student was standing. He might fall, and possibly spill some freshly boiled water onto himself and the class looking on. When the delinquent student did not response to the teacher's requests to stop, he patted him on the head to get his attention.

The student immediately strode to the administration to report that he had been struck by his teacher. As it happened, my student Gary's smart phone video of the incident eventually exonerated my colleague, yet some in the community were resentful over the decision to support the teacher from out of town instead of the student from the village. Months later community members let me know that there were still resentments regarding the favourable resolution for the teacher. The situation could be revisited from a few other perspectives; however, the point is that a teacher who is not well known by a community can easily be misunderstood, receive no benefit from any doubt and offend a community. Furthermore, school administrators who choose to support such an "outsider" risk losing their support from parents and community leaders when they make such a decision. The principal's situation was problematic for her in this instance and later in my own.

If a person might take this information as a reason to condemn Indigenous communities across Canada, they might want to recall that most Indigenous people are aware of a long-standing history of abuse at the hands of Canadian bureaucrats and officials such as teachers. One need only read the history of Duncan Campbell Scott (n.d.): the history of his work carrying out the Canadian federal policy of forcibly sending Indigenous children to Residential Schools far from their homes (Hanson 2009), and all of the problems that policy has made (Joseph 2015) to gain a basic understanding of the history of this Canadian tragedy. When I heard the personal stories of individuals and family members who bore this burden over the generations I responded this way. I understand that a great hurt, a grief, a wound of shame and self-reproach is held by Indigenous peoples in Canada. Furthermore, the Indian Act withheld the tools that families might have used to recover from that ongoing catastrophe were denied them. In Residential Schools, children were forced to live away from their families, their language, culture and their traditions. In its place, they lived in an atmosphere of cruelty, deprivation and abuse. Gary's mother coped with the consequences of her boy's difficulties at school by moving him to an alternate school, where he (she) might avoid those consequences. I am sure that more than one reader has noted that choice was less than adequate parenting. I do not disagree. Only I have seen this familial pattern many times over in northern Indigenous communities. The Residential School intended to "kill the Indian in the child" (as cited in Farber 2017, para. 3). That meant the parenting skills that Gary's grandparents intended to convey to his mother were not there when she needed them. They knew the punishment, disrespect and cruelty of Church run schools and not the love, forgiveness and parental guidance that was traditional to Cree ways of living. As Gary's teacher, I attempted as best I could, to stand inside my classroom, to accept that these were his parents' circumstances and to accept that there was little I could do but to take this incident into consideration in my endeavours to grow as an effective teacher in this situation.

I had arrived in the community with a vision that I would enthusiastically reach out to the parents of my students so that I might include them, their knowledge of and connection with their children as a way to support each of my students. I had learned how to make bread from my father and intended to make an impression with that very skill. I sent a letter home with each student requesting the opportunity to drop by to each of parent's home with my freshly baked loaf of bread. Then I could ask parents and guardians what my students' personal preferences were, how they liked to learn, what were good strategies to help their student with issues such as shyness. I was trying overtly and directly to initiate a supportive rapport with these parents. In the end, even though I got some lovely complements regarding my bread, I did not really develop the rapport for which I had hoped. The parents who agreed to my request and invited me into their homes shared superficial things regarding their children that I already knew. Some seemed hesitant and others even stressed, overtly uncomfortable with my presence in their home. After a month or more of this, one student was even adamant that I was trying to seduce his mother and refused to give her the letter. Other families, I heard, had surmised that I was trying to make a covert inspection of their homes so I might judge the quality of their home and family. Furthermore, I had intended that each student would attend the meeting with their parents. None of the students did so. Some may have been at home however they refused to join the conversation. Usually, they were out with their friends. The entire project did not work out as I expected.

In retrospect, I look back on that time and see myself rushing into peoples' homes, asking my questions, yet never finding a comfortable rapport with my students' parents. I did find that having a rapport with parents was important. Many times, my requests for help with their child, brought about a small but effective change in attitude. One that I could not have cajoled from a student.

New Directions

My formula for arriving at an effective rapport in the community now involves a different set of physics. An important factor is attraction rather than promotion. Today, community members need to discover me from a distance. Word gets around. In the meantime, I listen to people and hear what they are trying to tell me. Many times, they are just trying to make me laugh. It is a wonderful feeling to discover how funny most community members are in most of their interactions.

Another quality that I have learned to acknowledge is how guarded people can behave with an individual who is not an accepted member of their community such as myself. There was once when one of my classes was discussing various responses to bullying. One of the options was to "Talk to your teacher." I asked the class if anyone would be willing to talk to me about it if they were to experience a bullying incident. Maybe there were one or two hands raised. More likely it was none. I asked the class, "Is there a reason that it's so difficult to share that kind of information with me?" Their answer was almost unanimous, "You're too new!"

Wary is normal. It is a kind of cautious, guarded approach that I must never understand as an overt dislike of someone like myself even though on occasion it is expressed exactly as that. Instead it is a matter of self-care. It is more heightened than I am accustomed. I now recognize it and accept it. For example, one of my classroom management techniques is an on-the-spot mini-conference. I will ask a child who is having difficulty behaving appropriately in class to step just outside to the classroom door. There is relative privacy there. I ask them what is going on that they need to behave so disruptively. At times, a student may tell me the details of something that is bothering them. Sometimes they will only acknowledge that they are "having a bad day." When I ask if it is something that they can tell me about, they may provide details regarding their situation to help me understand and support them better. Often, they refuse, even to recognize my question. There is a familiar shrug and, "I don't know."

"Okay Gary," I said, "but please remember I am here to support you as your teacher. If we both know you are struggling today, we can find a way to help you get through class without all this difficulty. Is there something going on, say at home?"

"Yes."

"Can you tell me about it?"

"No."

"Can you talk to Maisy (a trusted school counselor) about it?"

"No."

"Okay, I am going to go easy on you for now Gary, but I want you to keep the Maisy option in mind, okay?"

"Okay."

"So, let's go back into class, you'll only have to answer the first two questions instead of all four. I can help you. However anymore in class disruptions and I expect you to speak with Maisy, understood?"

"Yes."
"Nice goin'."

My new set of physics was that rapport develops largely with trips around the sun. That is, every year that passes and I have shown my students, their parents and the community the kind of respect that they deserve, our trust will grow. From my hasty foray into my first community, armed with my fresh loaves of bread, I learned to just let people see me. I do not necessarily recommend this for others, however I eventually kept my curtains open most of the time. This is partly because I need a lot of sunshine to maintain my positive state of mind. However, my idea was to let local people see me in my home. I dressed appropriately for that kind of exposure. I went about my time at home in full view of anyone who glanced in. People did not stare. Apart from passing glances, they paid no attention at all to me. But it was clear that I had nothing to hide. I was exactly as I portrayed myself. I made my meals and cleaned my house. I played my guitar and entertained friends. In one community, I read in a chair that sat right beside a window that was feet away from one to the paths that walkers used. On occasion, I would almost jump out of my seat when some of my students were walking past my window on their way home from hockey practice. They said, "Hello" very suddenly and loudly, breaking my concentration on whatever I was reading and frightened me. The prank never failed to get a laugh from all of us.

Whereas once, I was "too new," eventually students and some parents tried to ensure that I would teach their children the following year. According to periodical anonymous class evaluations, students described me as "funny' and "fair," however also as "strict," "boring," and "too perfect." Towards the end of the year, that same class asked me, "Are you my teacher next year?" "Will you be here next year?" I have several degrees, diplomas and certificates to support my position as a professional teacher. I count these as "nice to haves" now. Today, time in the community is a concrete, hard won commodity and especially if I have found a way to work with my students so that it visibly improved my class attendance. That commodity is measured in trips around the sun.

Let me elaborate. Firstly, as most teachers experience in different parts of the world, students initially test teachers when they begin the school year. They want to have a concrete sense of what the teacher's *actual* instead of *spoken* boundaries are and how they will maintain them. My experience in these Cree communities is that this process is exaggerated. The first year a teacher is in a community, students are aware that the teacher has no rapport with their parents and relatives. Furthermore, I understand there is a resentment towards teachers who come for a short time (1 or 2 years) and then leave. This means there are students who believe they must take advantage of the situation. There is more than the usual disrespectful and disruptive behaviour in the new teacher's class. Students usually attempt to gain attention from and impress their classmates and peers. "Such is the stuff that great stories are made." I have heard stories of classes who attempted to make a teacher cry and some of their "successes." I am told by families in these Cree communities that they often see teachers come and go to a point that they are hurt and become defensive about participating in a rapport with other short-term teachers. On the other hand, the teachers who are perennial members of these communities tend to experience a familiar affinity with most community members, especially with students and their parents. Although they do experience their own difficulties with certain students, these are either resolved with a conversation or some contact with the family, or there are deeper problems present that other school board professionals must help with. The first time I returned to a community for my second year, I experienced some of this more familiar rapport. I had learned to approach community members in a less forward manner than the year before and I found that students were more willing to take direction from me.

One of the reasons that a steadiness of rapport made sense to me was that experienced teachers adjusted for the needs they perceived in their students. Many children had a history of what my first teaching mentor called "sadness." She told me that "our students experience too much sadness in their lives outside of school as it is." She did not want to bring them into more with her teaching. Let me offer some examples. During the time that my student Gary was in my class, an older brother of his was taken into police custody and put on trial for murder. Following my mentor's lead, I purposely did not broach this subject. I wanted my class to be a place where he and his classmates were away from the trauma of that event. I am not sure if this was a mistake or not. It might have been helpful to have asked if there was anything I could do to help, and to show him some understanding and concern. However, I followed his lead and did not broach the topic. However, in the conference with his mother and the school administrators, I understood her apparent hysteria was partly due to her trauma from her other older son's mistake and the consequences that she, her son and their family were experiencing.

I mentioned earlier that a boy in one class refused to deliver my home visit request to his mother because he believed that I was trying to seduce his mother. His parents were divorced and I was unaware of any subsequent relationships that his mother had had that might have interfered with his rapport with her. Still, it made sense that he experienced some difficulty with his mother so that he may have thought of me as a threat to his own connection with his mother. I never did meet his mother and would not have known her if she had stood in front of me. He remained adamant for the school year. I let it go.

Lastly, I was struck by how strongly community members were affected by death of individuals in the village. This was especially so for the young who died. I was told of one boy who had died attempting rescue his friends in a river. Another, who had experienced a freak accident on a ski mobile. In the middle of a school day, I even learned that one of my students had taken his own life. These communities are very close. Tragedies, whether by family dysfunction, natural causes or accidents, affected students (community members) almost in silent, unspoken ways. So that as a teacher, it is important for me to take this into account in my planning. I offer these examples, because they may give the reader a sense of how I understood my mentor's meaning of her term, sadness.

I am sure that many teachers over the world have experiences like this. Even though I am not alone, I do feel that this awareness is meaningful and useful in the context of that setting. When I started my first Moral and Religious Education (MRE) class for a group of Secondary 4 (grade 10) students I launched into the class with a topic regarding Indigenous Canadians that was prevalent in my mind: Residential Schools. In my own class on Indigenous teaching at Lakehead, I had learned a great deal about the Federal Canadian policy of forcing Native Canadian children to go to school in what is called the Residential Schools system. I was outraged and indignant over this history of my country because we Canadians claimed a kind of international moral authority. I wanted to encourage these students to express their anger and indignation regarding this history too, even though we had hardly spent time getting to know each other. They treated me with either curiosity, indifference or distain. Typical to many students, there was an ambivalence about spending any time learning with me, especially when it came to matters of Cree history and experience. "I can't learn anything about Cree people from some white guy from Toronto!" I was told by one very bright and outspoken student.

However, more to the point, I had asked them to think about "sad" things. This was a first foray into realizing that the children I taught already were dealing with far too much tragedy than most children are asked to cope with. I recall in a conversation with one principal regarding a girl in a different school. She, on the one hand, could be very engaged and on the other hand, would respond to the small consequences she received for her disruptive behaviour with first an absolute refusal to cooperate and then long utterances of grief with her head down on her desk crying inconsolably. On many occasions, she would joke with me regarding blatantly sexual matters. She refused to accept my directions that such exchanges were not appropriate for any teacher to participate in with their students. My principal explained to me that in the case of this child, she may have been exposed to some blatant sexual impropriety in her home, possibly including her own participation. She was 10 years old. She had gone in and out of court ordered foster homes. My role was to help her manage her way through her school days as best I could and to rely on support staff when it was too difficult for her to remain in class. I wanted to hug such children yet I knew that such a gesture was too intrusive to be consoling. Instead, I had to offer them a place where they could feel safe and their wounds were not shameful. That was task enough.

There were students who sometimes came to me to inform me about the abuse that they were experiencing in their homes at the hands of their parents. I discussed this with the administration and only once actually called Youth Protection Services. The idea that I share here is that on the day in my first Moral and Religious Education (MRE) class, my expedition into the topic of my students' grandparents' experiences in Residential Schools was my introduction to the ongoing impact of Residential Schools, and the abuse wrought then that still impacts families today. Certainly, there are communities who are beginning to deal with it more and more explicitly. However, as a teacher I should understand, that I must give great care to difficult topics. My manner of often speaking plainly and frankly about the difficulties of life needs to be weighed with the needs of some of the students who were in my classes, *especially when I have not really developed a workable rapport with them*.

A community member gave me some help. I sat many times with this Cree man. He was unusual because he had avoided the forced deprivation that came with attendance at a Residential School. Instead, he had lived in the bush with his extended family, avoiding the shame that his peers had known. My role in these conversations involved active listening, that is, listening and asking questions to clarify my understanding as we proceeded. Even if I offered a different opinion, I used it within the context of the conversation, less to make a point but more to clarify a something I was struggling to comprehend. He occasionally shared how he had experienced his early education. For example, he said his mother had suggested to him that he sit near the circle of adult and elder men in his community and listen to their conversations. He could hear the experiences and stories that were meaningful to these men. He could gather information that was meaningful to himself and related to the world where he lived. By sitting quietly, he showed these men his respect. By listening and later asking questions of his family members, he could clarify his learning of the traditional ways of his people and how adults lived.

Later, I had a similar experience. I had flown into another Cree community one summer to interview for a teaching job. On the morning of the interview, I sat at a table in the restaurant of my hotel to have my breakfast. Different men walked into the dining hall and sat at this same table with a coffee or some breakfast until the table was full. It turned out, this was an event for senior men of the community that took place regularly at the very table where I sat. I was not asked to leave. I became quietly fascinated and I followed the advice of my friend's mother, listening to these men's conversation. I could not understand the conversation in Cree. However, enough of it was in English that I heard these men speak of their lives and their times. There was a great deal of humour, so when I was asked a question by one of these men, I was comfortable to answer. They learned where I was from and what my business was in town. There was not a lot of advice. I mostly recall laughter. My main memory was how honoured I felt to have been included in this circle and to have learned as much about their community as I had in a very short time. Although it was not identical to my friend's experiences as a boy, I had witnessed the aspect of traditional Cree education that he had shared with me.

Another attribute of Cree education was the phrase, "Watch and learn." Often when I attended a traditional Cree event such as a Goose dance or a sweat lodge ceremony, I asked many different questions so that I might understand what I was witnessing and experiencing. For some, my inquisitiveness was unwelcome. I was told, "Watch and learn." At the goose dance, it came my turn to dance solo around the Goose Lodge. I imitated the far more proficient dancers as best I could from what I had seen. Let's say my efforts were appreciated.

I soon learned that teaching is often less a matter of words and more a matter of actions. "You join us hunting Cariboo to learn to hunt Cariboo"; "You watch me back up a trailer with my pickup to learn how to do that"; "You watch me negotiate to learn how to negotiate." To learn something, it made sense better to watch and then imitate. Words were almost a blunt, ineffective instrument compared to modeling and imitation. One very experienced teacher told me that he could settle an unruly class by asking them to take down notes. He would write out the history that his students were expected to know, and they would copy it in their notebooks. It fit the imitation model and made it safe for students to complete their classwork. There were no mistakes, awkward feelings, or difficulties of that nature. This information suggested to me that I had to make this uncomfortable exercise as safe and glitch free as possible to encourage students' confidence. I decided to try "watch and learn" in my class.

Mathematics

This brings us to teaching mathematics. Thus far I have explained that I developed a complicated understanding of the needs of many of my students. Let me show you a lesson plan for helping my students begin to confront the daunting task of learning division. It is a scaffolded lesson that attempts to expand the students' proximal areas of development. The title at the top of the lesson plan is, "*Division using Subtraction and Comparing to Multiplication*" (see Appendix). The format of the lesson plan is a template provided by the school board that I used to fabricate an example of what I might have used in January of 2013 during my first-year teaching for the school board. My diagnostic evaluation revealed that almost the entire class was not able to complete division problems. The very topic was a source of anxiety for some students. I decided that I must make it familiar by first drawing upon their comfort with their subtraction skills. Then I could help them to compare their results to their growing understanding of multiplication using area models. If area models did not work, a student could use other manipulatives, however some students struggled to focus when manipulatives were in the classroom. Experienced teachers may have experienced students initially responding to manipulatives as a toy and distraction, either using them as building blocks or throwing them at their friends. This issue was more persistent in this classroom than in others I had experienced, so I was frugal using them. The point is that it is helpful to have a variety of methods for students to interact with a problem and as much as area models were useful and less disruptive, other options were sometimes necessary for individual students.

In section "New directions" of the lesson plan, called "*Categories of Instructional Strategies*," note that the box called, "*Cooperative Learning*" is highlighted. Students were expected to work with one or two partners to solve these problems if they were comfortable. My experiences in teacher education suggested an approach called "reform," or "problem-based learning" (National Council of Teachers of Mathematics, 2000). Such an approach requires students to be actively involved in a task: exploring, discussing, asking questions, and defending their ideas. I was trying to get the students more accustomed to this style of learning.

In section "Mathematics", called "*Core Learning*," you may notice the words emphasized, "**Watch and Learn**!" The presentation format here would be a "Think Aloud" where I show students how I solve a problem and I expect them to imitate me in the same way that I was told to do when I was learning traditional Cree ceremonies. Also in section "Mathematics", you see three sections displayed almost as three pictures. They compare to the seven pages of worksheets entitled "Division, Subtraction and Multiplication." The point of these worksheets is to offer practice and to minimize any appearance of something intimidating. The problems are deliberately simple and do not involve remainders. The first problem in each of the sections is solved as a way for students to compare and imitate. The pages of area models are deliberately made large and take up a lot of space to help students feel confidence. Students may go through the problems with the teacher if they need to. They are expected to first finish the subtraction exercise at the start and then turn to the area model of the problem they are solving as shown consecutively on the rest of the pages. As the students move through the three sections, with the teacher, they will take on more responsibility to make the area model.

- The first is made for them.
- The second they will outline the area from a larger area model by choosing the height and width themselves.
- In the third they will get an empty rectangle and choose the height and width, then draw in the area model on their own.

You may be uncomfortable at just how easy this is for Secondary 2 (grade 8) students. To be clear, this is an aspect of the scaffolding. The strategy is to win the students' confidence first and guide them towards eventually understanding the reciprocal nature of multiplication and division. In another community, I often listened to a teacher friend who taught high school mathematics to Secondary 3, 4 & 5 (grades 9, 10 & 11). He insisted on several occasions that I ensure that my students (his future students) understand the operation of multiplication as the repeated addition of a number a specific number of times and with a known and predictable sum or product. He and other teachers throughout the school board have students who learn and complete their high school graduation requirements. The school board is working and succeeding at increasing the number of students who do fulfill those requirements. There is still a culture that makes this difficult.

Different families will have different approaches towards school and academic achievement. Some of this may depend on some of the issues that I have already shared such as the functionality of a family. However, it goes beyond this. I found that priorities are often different where I was teaching than where I went to school in Massachusetts or Ontario. Sports, outdoor activities such as hunting, fishing, trapping, and just "being on the land," as well as a person's general state of mind seem to influence attendance and participation at school. Parents take children out of school to hockey, broomball, baseball, and basketball tournaments in a manner that *does* impact their schooling. Parents often take their children "in the bush" when they go there. I was told by many community members that experience on the land is held in very high regard by most Cree people. It is not questioned as a reason to take students out of school, however like the sports, it does have an impact on academic results. I have given homework to students who were going with their family into the bush, however it was not always done and even if it were completed it would certainly not have replaced a student's participation in class with their peers. These cultural differences are very significant to these students' learning, but may be foreign ideas to curriculum designers.

Lastly, a concept that I found unfamiliar to me but was cited as a frequent reason to delay school work came in phrases like, "I don't feel like it." In all the three communities that I lived in, I heard this presented to me as valid reason to delay work. To a certain extent, I came to respect it because it could have merit given a student's circumstances. However, I have described how students dealt with low levels of confidence and unusual anxiety over their schoolwork. Their state of mind influenced their attendance and participation at school. Some parents have reported to me that they have allowed a student to stay home for just this reason. Certainly, these circumstances motivate me to engage students in their studies as best I can so that they do "feel like" coming to school. However, this situation together with circumstances like taking students to sports events or into the bush gives me the impression that school and academic achievement is less of a priority—or a different priority—amongst the Cree with whom I worked. In fact, I theorize that changing such priorities (that is doing something even though one does not feel motivated) is almost a way of denying or losing one's identity. The ability to listen to one's personal motivation on any given day is accepted as valid. Does one surrender this common understanding to conform to an alien norm that only makes sense in circumstances that are not understood as natural and familiar? Still, the point is that academic achievement, although lauded amongst the Cree when it happens, is not made such a priority as I have seen in most of North America, Europe and the Middle East where I have travelled. That does impact the level of academic attainment that one sees in Cree territory. I have watched my students struggle almost between two worlds: one Cree and the other post modern Western culture. I have only met a few who seem to have come to adequate terms with it. I do not envy them the problem.

Concluding Thoughts

I left Toronto in August 2012 knowing that I would learn something, however not knowing exactly what it would be. Sitting here in Toronto in August of 2017, I am different person. I came to love the students, their families and community members who I grew to know. I learned a massive amount of compassion and appreciation for people who are like me and not like me. I would say it is mostly the laughter that I recall with students, villagers and the other staff with whom I worked. I once saw a video of some Indigenous communities inviting people like myself to come and teach in their communities. An elder in the community spoke suggesting that when an individual comes to their village, it is best to come with a humble attitude: one that is open to hearing, learning and understanding about who these people are and how they live. I can only repeat her wisdom. When I arrived in my third Cree village, I had already spent several years teaching and participating in two other Cree communities. I arrived with an attitude that I already knew. What did I already know? Now, I am not sure, but my approach made it more difficult to become the part of the community that I thought I wanted to be in. I recall asking people how I could join any of the sweat lodges. One person would tell me to ask another and then that individual would suggest some other person, until I never participated in a sweat lodge. I experienced two sides of participating in a community: one with my humility and one without.

If you choose to teach in the Cree communities, I suggest that you first establish firm workable relationships with your colleagues, and while you do that attend the feasts, the sports events, enjoy the hunting and fishing as much as you are willing to and even try your hands with the local grandmothers at traditional crafts. Participate in the Pow Wow's, sweat lodges, and other traditional ceremonies and always arrive with a mind as open as you can make it. There will likely be birthday parties and gatherings that are more familiar to you amongst your colleagues. These are important too. I recommend that you plan a longer stay than just one or 2 years. You will make it past your difficult first year and learn many ways to teach and to reach students. One of the especially wonderful things about teaching for the school board was collaborating with my colleagues. We needed each other's support. Colleagues offered us empathy and compassion for our struggles. They had fantastic ideas for us to gain some success where we struggled. And we had the same for them. For me, it was a wonderful adventure.

Appendix

Division using Subtraction and Compare to Multiplication

TODAY'S *LESSON* OBJECTIVE(S)
By the end of today students will (know, understand or be able to do)… perform division using the operation of subtraction and compare division to multiplication.

Date: Thurs 24 Jan/ 2013 **Segment #:**

Days spent on GVC segment

1	2	3	4	5	6	7	8	9

① WARM-UP ROUTINE ____ minutes

LOOK Back	LOOK at Yourself	LOOK Ahead
What did we learn yesterday? Before? Division is backwards Multiplication	*What do we know about what we are learning today?* We've got subtraction. Did you know you can use subtraction to do division?	*We're going to practice this for a while so that students are comfortable with operation, then we'll move into remainders. Eventually, we'll be looking at fractions.*

② CATEGORIES OF INSTRUCTIONAL STRATEGIES—Target at least 1 during today's lesson

Setting Objectives & Providing Feedback	Reinforcing Effort & Providing Feedback	Cooperative Learning	Cues, Questions & Advance Organizers	Nonlinguistic Representations	Summarizing & Note Taking	Homework & Practice	Identifying Similarities & Differences	Generating & Testing Hypothesis

③ CORE LEARNING ____ minutes

● Natural Numbers ○ Statistics ○ Probability ○ Geometry ○ Measurement

Watch and learn!

Students will subtract the divisor from the dividend and compare it an area model.

What is 16/8?

Watch 16 −8 (1) 8

Then 8 (2) −8 0

Complete the first four problems on page one together with the four area models on pages two and three.

Students will notice how the divisor and the the dividend compares to the height and width of the area model. This compares to multiplication.

What is 16/8?

$16 \div 8 = 2$

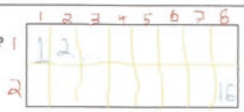
$2 \times 8 = 16$

Watch 16 −8 8

Then 8 −8 0

Complete the second row of problems on page 1 (5 - 8) together with the area models on pages four and five.

What is 16/8?



Now complete the last row of problems on page one together with the area models found on pages 6 -7.

Students will make their own representative area model.

Resources: Worksheet attached, manipulatives,

Evaluation: Self-evaluation, exit slip

④ CLOSURE ____ minutes (*summarize learning, review progress towards objective of the day, reflect, self-evaluate, etc.*)

So, in this lesson we saw that we can divided a number by another number just by subtracting that number from the first number, only we soon discovered that it's much easier to think of it almost like multiplication but in reverse. Using an area model, just like multiplication, we can quickly count the number of we need to get our answer, the dividend.

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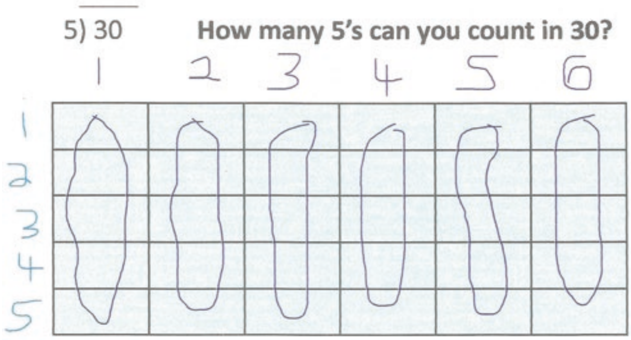
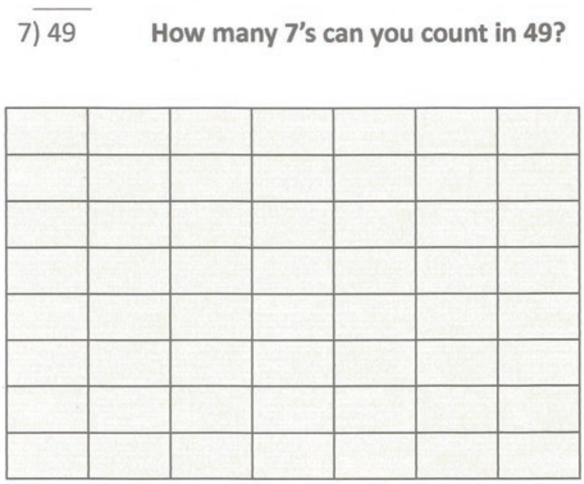
Division, Subtraction and Multiplication

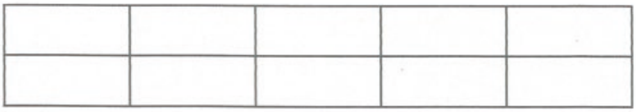
Name ____________________ Date __________

6 5)30 -8 1 25 -5 2 20 -5 3 15 -5 4 10 -5 5 5 -5 6 0	7)49	2)10	9)27
3)27	5)15	4)36	8)32
7)35	6)42	6)24	3)18

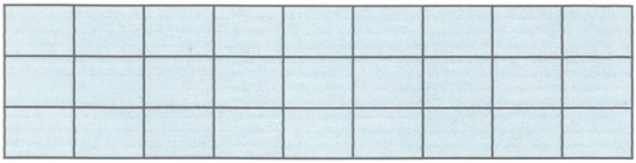

5) 30
How many 5's can you count in 30?
1 2 3 4 5 6
1
2
3
4
5


7) 49
How many 7's can you count in 49?

2) 10 **How many 2's can you count in 10?**



9) 27 **How many 9's can you count in 27?**



$3\overline{)27}$ **Outline the Appropriate Box**

1	2	3	4	5	6	7	8	9	
10	11	12	13	14	15	16	17	18	
19	20	21	22	23	24	25	26	27	

How many 3's can you count in 27?

$5\overline{)15}$ **Outline the Appropriate Box**

How many 5's can you count in 15?

4) 36

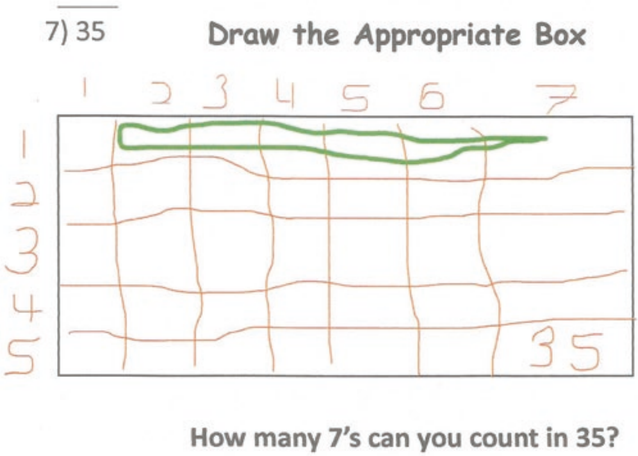
Outline the Appropriate Box

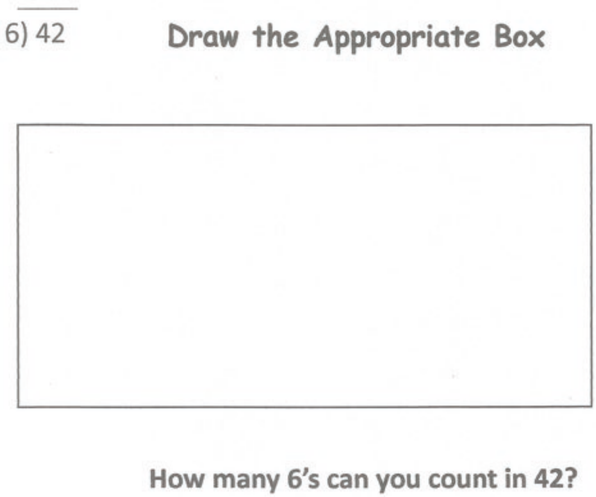
How many 4's can you count in 36?

8) 32

Outline the Appropriate Box

How many 8's can you count in 32?


7) 35
Draw the Appropriate Box
1 2 3 4 5 6 7
1 2 3 4 5
35
How many 7's can you count in 35?


6) 42
Draw the Appropriate Box
How many 6's can you count in 42?

$6\overline{)\,24}$ **Draw the Appropriate Box**

How many 6's can you count in 24?

$3\overline{)\,18}$ **Draw the Appropriate Box**

How many 3's can you count in 18?

References

ChrisOnerVidz. (2012, December 20). *Boiling water turns into snow in Siberia* [video file]. Retrieved from https://www.youtube.com/watch?v=2LFtYUUXJlE

Duncan Campbell Scott. (n.d.). In *Wikipedia*. Retrieved from https://en.wikipedia.org/wiki/Duncan_Campbell_Scott

Farber, B. (2017, January 23). *The terrible legacy of Duncan Campbell Scott*. Retrieved from http://www.huffingtonpost.ca/bernie-farber/duncan-campbell-scott-legacy_b_14289206.html

Hanson, E. (2009). *The Indian Act*. Retrieved from http://indigenousfoundations.arts.ubc.ca/the_indian_act/

Joseph, B. (2015, June 2). *21 things you may not have known about The Indian Act*. Retrieved from https://www.ictinc.ca/blog/21-things-you-may-not-have-known-about-the-indian-act-.

Wheels. (2008, July 8). *Solo to James Bay and back again*. Retrieved from https://www.wheels.ca/news/solo-to-james-bay-and-back-again/

Part III: Commentary – Fostering Relationships in the Work of Teaching Mathematics

Reidar Mosvold

A prima facie purpose of mathematics teaching is that students learn mathematics, but the work of teaching mathematics is also relational work that serves a more far-reaching function. The five chapters in this section investigate different aspects of the relational work of teaching mathematics. This commentary will address three issues. Firstly, it will provide a brief overview of some critical issues in educational research, along with some perspectives on research on mathematics teaching. Secondly, it will discuss the perspectives on mathematics teaching as relational work that are presented in the chapters. Some chapters focus on attending to students' learning and difficulties; other chapters focus more on developing relationships with the students. Thirdly, the contributions of the chapters will be discussed in terms of possibilities and constraints.

Introduction

Research on students' mathematical learning and development has long been abounding. In comparison, considerably fewer studies accentuate mathematics teaching. Among studies that seem to target mathematics teaching, many emphasize issues related to students or classroom organization more than the actual work of teaching. There is, however, a growing body of research on mathematics teaching, and efforts have been made to initiate a conceptualization of the work of teaching mathematics (e.g., Ball and Forzani 2009).[1] In their overview of research on

[1] In this chapter, the terms "mathematics teaching" and "the work of teaching mathematics" are used interchangeably to describe everything mathematics teachers do to help their students learn (cf. Ball and Forzani 2009).

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mathematics teaching and classroom practices, Franke et al. (2007) distinguish between three strands of research. A first strand focuses on teachers' facilitation of, and participation in, the mathematical discourses in classrooms. A second strand of research investigates the establishment of classroom norms for teaching and learning mathematics. Finally, a third strand of research emphasizes the building of relationships for doing and learning mathematics. The five chapters in this section relate primarily to this third strand of research.

This commentary will consist of three main sections. The first section will provide a context by elaborating on some critical issues in educational research in general along with some core aspects in research on the relational work of teaching mathematics. The second section will highlight some important perspectives in the five chapters. Finally, the last section will contain a discussion of how these chapters may contribute to research on mathematics teaching.

Research on Education and Mathematics Teaching

In the past decades, educational research has developed immensely. Political and economic interest in research has increased—in particular related to measurement in education—and international comparison studies like TIMSS (Trends in International Mathematics and Science Study) and PISA (Programme for International Student Assessment) have received more and more attention. The political acclaim of measurement and comparative studies has led to an increased focus on quality, effect and outcome in schools. Biesta (2009), who is one of many critics of this development, suggests that there is a need to revisit the question of purpose in education. In his discussion of the "learnification" of education, he makes a distinction between three core functions of education. The first describes education as *qualification*. Education has always focused on providing students with skills and knowledge that are needed to qualify them for work or further studies. The second function of education relates to how we become members in society through education, and Biesta describes this as *socialization*. Finally, a third function of education is to support the development of individuals—a function that can be labelled as *subjectification*. When discussing teaching in general and mathematics teaching in particular, it can be useful to have these three functions of education in mind—especially since there appears to be a tendency of focusing mainly on the first in discussions of mathematics teaching and learning.

In light of the developments in educational research in general, it is interesting to notice how Franke et al. (2007) emphasize the relational aspects when discussing conceptualizations of mathematics teaching. Building relationships constitutes one of three main strands in their discussion of research on mathematics teaching and classroom practices. Their discussion of teaching as relational work draws upon Lampert (2001), who is considered a main proponent of this view; Lampert argues that teaching is by and large about developing relationships (cf. Franke et al. 2007).

In concert with their managing of all the problems and demands that arise in the classroom, teachers have to develop and preserve generative relationships with students as well as content (Lampert 2010).

Fostering relationships can be described as establishing bonds between teachers and students—and among students—and understanding of other people's identities is crucial for this to happen. The fostering of relationships, then, goes beyond attending to students' mathematical thinking, and it involves getting to know their histories, the experiences they have made in and outside of school, their cultural background, and everything else that constitutes their identity. Research on the relational aspects of mathematics teaching thus typically attends to issues around identity and culture (Franke et al. 2007).

Perspectives of the Chapters

The work of teaching mathematics is complex, and no one study can attend to all its intricacies. Although all of the five chapters in the present section approach teaching as relational work, there is considerable variation among them. A main contribution of these chapters is in the richness of the relational aspects that are described. From these descriptions, two main themes can be identified. Firstly, some of the chapters appear to consider the relational work of teaching in terms of attending to students' thinking (Chorney and Hurlington). Secondly, another group of chapters concentrates more on the process of developing relational and affective perspectives (Jao, Newell, and Roth). Whereas the first group of chapters focuses more on student thinking and learning, the other group appears to focus more on developing students as whole persons. In this section, some main perspectives relating to these two themes will be highlighted.

Attending to student thinking is at the heart of teaching (Franke et al. 2007), but teachers as well as researchers might have different approaches to this core aspect of the work of teaching mathematics. Studies that are grounded in cognitivism and constructivism often tend to focus on identifying students' errors, difficulties, or misconceptions. Hurlington has a different approach to the work of attending to student thinking and learning. Hurlington contends that mathematics teachers should focus on students' strengths rather than their weaknesses. Instead of having a diagnostic approach to identifying students' weaknesses, Hurlington suggests that teachers should become "talent scouts". His theoretically based argument rests on the concept of resilience. The term is often used in material science to describe the ability of materials to recover and return to their previous state after distortion. In his chapter, Hurlington uses the term to describe students' ability to take advantage of their resources and have a good outcome despite the challenges and risks that are involved in the process. When focusing on students' strengths instead of their weaknesses, Hurlington argues, teachers might contribute to diminishing the risk of failure and support students' learning and growth.

Chorney introduces some different perspectives to the discussion of attending to student thinking when he investigates how top-down requirements regarding assessment might prohibit the work of teaching. In the empirical examples presented, we observe how two girls are involved in mathematical explorations with technological tools. Although their explorations appear promising, the teacher ends up considering the interaction with technology as a failure since the girls are not able to produce the response that is required for assessment in time. Chorney argues that some important mathematical competencies—like reasoning, justifying and communicating—are more difficult to report on than "plain" knowledge of content.

Whereas Hurlington and Chorney investigate different aspects of the relational work of teaching in terms of attending to student thinking, the other three chapters of this section focus more on how the relational work of teaching might contribute to developing students as whole persons. In her study of the teaching practices of three mathematics teachers in "at-risk contexts," Jao concentrates on the students. Unlike the three chapters in the previous category, however, Jao emphasizes the students' need to feel like they belong. She suggests that the students' level of engagement and achievement can be influenced by teachers' support, and classroom organization and features of classrooms are emphasized in the discussions of the work of teaching. In particular, building community through collaborative problem solving and student-centered approaches is highlighted.

For Newell, fostering relationships with students and their family appears to be a foundational component of the work of teaching mathematics. Where Jao concentrates on what features of classrooms and classroom organization might stimulate the building of community, Newell attends more squarely to the work of building relationships. His chapter on teaching mathematics in Cree communities indicates that this process can be extremely challenging and time-consuming. Through first-person narratives, Newell provides insights into the cultural challenges of attempting to conduct the work of teaching mathematics in the Cree culture—as an inexperienced teacher, who is considered to be an outsider by the community. The fact that the first half or so of his chapter goes by before he even starts focusing on mathematics teaching is illustrative of the intricacies that are involved in the relational work of teaching. Understanding students' culture and identity is necessary to build relationships (Franke et al. 2007), and Newell experiences that academic achievement as such is considered a threat to the culture and identity of the members in the Cree community. He concludes that it is necessary to enter such communities with humility and willingness to hear, learn and understand who other people are and how they live.

Finally, Roth presents a different set of perspectives on the relational work of teaching in his chapter on culturing affect. His primary focus is on the aspects of teaching that relate to helping students develop self-confidence and positive affects towards mathematics and themselves as mathematical learners. In doing this, Roth develops a perspective on mathematics teaching that goes beyond acquisition of knowledge. Instead of focusing on students' cognitive development, Roth accentuates the development of students' affect. He describes this aspect of the work of teaching in terms of "culturing affect," and he thereby emphasizes the communicational and relational work of helping students develop their mathematical identities—corresponding with Biesta's (2009) descriptions of subjectification as a function of education. However, Roth warns against reducing students to mathematical identities only, and instead attempts to take them into account as whole persons. Through the example of his interaction with Earl, Roth illustrates what can also be described as an act of allowing students' agency of their own learning—with the result of a change in the student's affect towards mathematics.

Possibilities and Constraints

In the beginning of his chapter, Roth observes that mathematics education research tends to adopt psychological theories of learning, and he criticizes the tendency in these theories to separate intellect and emotion. This criticism is well-founded, but another problematic aspect can be added to the discussion of this adoption of theories. Not only do the adopted psychological learning theories have a tendency to separate intellect and emotion, but most of the theories that are adopted from other fields are also theories of *learning* rather than theories of *teaching*. Using theories of learning as a starting point for studies of teaching might be problematic in and of itself. In the following discussion of possibilities and constraints of the chapters in this section, these dubious theoretical underpinnings will be a recurrent issue.

Interestingly enough, the chapter in this section that appears to have the most explicit focus on teaching is also the chapter with the least explicit theoretical foundation. Newell has a predominant focus on the work of teaching, but his account is practice-based rather than theory-based. He does not explicitly define teaching, but the implicit view of teaching appears to correspond with that of Ball and Forzani (2009), where teaching is defined as everything a teacher does in order to help students learn. From Newell's vivid account, some important insights emerge concerning challenges that might be embedded in the work of developing relationships with the students. His account also indicates that proficiency in the work of teaching needs to be developed in—or in close proximity with—practice (cf. Ball and Cohen 1999). Newell's chapter thus stands as a reminder of the importance and utility of careful investigations of records of practice in the endeavor to further develop research on mathematics teaching.

Emphasizing practice-based approaches does not, however, imply that theoretical perspectives should be disregarded in studies of teaching. Roth's chapter is illustrative of how careful and critical application of theoretical perspectives can be productively combined in analysis of practice. His chapter provides a strong reminder of considering students as whole persons, and not only to focus on the extent to which they manage to acquire a prescribed knowledge of content. This emphasis on supporting students' overall development as individuals corresponds well with Newell's perspectives. Roth also reminds us that teachers as well as students are constantly in a state of "becoming." This perspective is reminiscent of Freire's (1970) descriptions of liberating education, where teaching is described as a process where teacher and students engage in dialogue with the purpose of developing knowledge through collaborative reinvention. Engaging in such a process results in shared responsibility and agency between teachers and students, and this involves risk—a point that is shared by Roth and Hurlington.

Roth criticizes traditional cognitive theories of learning, while Hurlington accentuates student learning. Hurlington uses the metaphor of teaching as coaching, and suggests that teachers better support student learning by highlighting their strengths rather than their weaknesses. Chorney employs a theoretical perspective that differs from that of Hurlington when he describes teaching as social and sociopolitical work. This theoretical perspective seems to trigger a focus on how curriculum and policy might influence mathematics teaching in certain directions. In particular, he discusses how emphases in curriculum and policy regulations may limit the possibilities of using digital technologies in mathematics teaching. The focus in his study relates strongly to the theoretical perspectives applied, but it is, of course, important to highlight that this is a dialectical relationship.

Finally, Jao does not specify any particular theoretical foundation, but she draws upon other research that accentuates teaching in terms of supporting development of student engagement. In her investigation of the teaching practices of three mathematics teachers, she has a main focus on how teachers organize the students and the classroom environment.

Conclusion

The work of teaching mathematics is complex, and research on mathematics teaching highlights a variety of perspectives (e.g., Franke et al. 2007). The five chapters in the present section mostly attend to the relational work of teaching mathematics. Based on the highlighted perspectives and the discussions of the contributions from these five chapters, two claims can be made. One relates to constraints in the present research; the other accentuates some possibilities.

First, research on mathematics teaching might be constrained by its attempts to adopt and deploy theories from other fields. Roth argues in his chapter that many of the theories that have been adopted from psychology are limited in that they distinguish between intellect and affect, and more dynamic perspectives of the interplay between intellect and affect might be more productive. In addition, one might argue that it is problematic to adopt and use theories of learning in studies of teaching. Lortie (1975) called for the development of a professional language of teaching. Four decades later, it seems like the field of mathematics education research is still in need of both language and theories of teaching. More conceptual work needs to be done in studies of mathematics teaching, and conceptualizations of mathematics teaching should strive towards capturing the dynamic interactions between mathematical and pedagogical aspects of the work of teaching. Although the theoretical foundations used in the five chapters in this section—and in many other studies of mathematics teaching—show potential in highlighting particular aspects of either mathematical or pedagogical demands, few seem to be able to sufficiently capture the interplay between mathematics and pedagogy. Adopting theoretical frameworks from other fields might highlight certain aspects of this interplay, and studies that adopt such frameworks might end up providing useful insight into issues that are related to teaching, but these studies often fail to focus on the actual work of teaching. As a result, studies that apply adopted theoretical frameworks tend to emphasize some aspects that are relevant to teaching, but outright consideration for the complex dynamics of the actual work of teaching often seems to be missing.

A second point relates to a more general discussion about the purpose of education. Some have argued, like Biesta (2009), that there is a need to take up the question of purpose in education. Research on mathematics teaching and learning tends to have a predominant focus on qualification, which is unnecessarily narrow. Some of the chapters in this section challenge this narrow perspective on education as qualification. For instance, the chapters by Roth and Newell go beyond a discussion of qualification in their discussions of how teachers might support students' development—both as individuals and as members of a community. Further investigations along these lines might provide important contributions to the development of more integrated conceptualizations of the work of teaching mathematics that balance the different functions of education.

References

Ball, D. L., & Cohen, D. K. (1999). Developing practice, developing practitioners: Toward a practice-based theory of professional education. In G. Sykes & L. Darling-Hammond (Eds.), *Teaching as the learning profession: Handbook of policy and practice* (pp. 3–32). San Francisco: Jossey Bass.

Ball, D. L., & Forzani, F. M. (2009). The work of teaching and the challenge for teacher education. *Journal of Teacher Education, 60*(5), 497–511.

Biesta, G. (2009). Good education in an age of measurement: On the need to reconnect with the question of purpose in education. *Educational Assessment, Evaluation and Accountability, 21*(1), 33–46.

Franke, M. L., Kazemi, E., & Battey, D. (2007). Understanding teaching and classroom practice in mathematics. In F. K. Lester Jr. (Ed.), *Second handbook of research on mathematics teaching and learning* (pp. 225–256). Charlotte: Information Age.

Freire, P. (1970). *Pedagogy of the oppressed*. New York: Continuum.

Lampert, M. (2001). *Teaching problems and the problems of teaching*. New Haven: Yale University Press.

Lampert, M. (2010). Learning teaching in, from, and for practice: What do we mean? *Journal of Teacher Education, 61*(1–2), 21–34.

Lortie, D. (1975). *Schoolteacher: A sociological study*. Chicago: University of Chicago Press.

Part IV
Enhancing Problem-Based Learning

Part IV: Preface – Enhancing Problem-Based Learning

Thomas Kieren

Because the seven papers in this section (and those from other sections) are very diverse in their central interests they necessarily point to different ways in which the "enhancing" in the title is used. As I read the titles of papers Mamolo, Thomas and Frankfort; Russell; and Sterenberg and O'Connor, for example, one might say that the 'enhancement' pointed to will occur through the social focus of the problems and or the nature of teaching/learning/curriculum. In any case the titles of said papers as well as that of Savard themselves suggest that enhancement will arise through use of problems related to or appropriate to the social milieu in which the class exists. On the other hand some papers (Martin, Oliverira, & Theis; Saldanha & Thibault; Mamolo et al.) point to enhancement of problem based learning through the selection particular mathematical kinds of problems At least two (Godin; Atiya, Luca, & Kajander), at least nominally, focus on the teacher and the teaching role and the education for teaching in settings aimed at enhancing problem work. The point I am making is that such "enhancing" has many interpretations and many goals. Thus it offers the reader a diverse read but more importantly allows for the education of the reader with explicit interest in some form of enhanced problem based learning.

Perhaps even more importantly, although there are five other parts in this volume, each of these parts have papers which either add to the richness of the enhancement of problem based learning or connect to thinking that leads in that direction. In reading Part IV, it seems useful to augment this reading with selections of interest in other parts—they are certainly there. For example the chapter from Peter Taylor, Lala, Ouellet, and Knebel (Part I, this volume) is replete with historical references to the work of Whitehead and Dewey who insist that the curriculum must not errantly get stuck on a long list of skills which must be sharpened BEFORE students

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can do the rich work of problem solving—that is experience mathematics knowing as a form of living or being. Taylor et al. also cites contemporary writing of Brent Davis who makes the same point differently—he provides a set of mathematics teaching/learning actions which are life opening contrasted with related actions which are narrowing. Davis was part of a rich problem based mathematics project [see, for example, *ZDM*, Vol. 47, Apr. 2015, Preciado-Babb et al.] where the project provided evidence of mathematical growth to the school authorities using the 5 strands of mathematical activity developed and described by Kilpatrick, Swafford, and Findell in their 2001 book. In other words the problem oriented teaching learning of students could be shown to meet the variety mathematics goals [including mathematical proficiency]. Taylor himself makes the point that such proficiencies are likely to be all enhanced in a problem rich environment. And illustrates this with student approaches to sophisticated mathematical problems.

Mathematics educators have a rich base of work on problem solving and the ways it can contribute to mathematics education which even provide a mathematical base for the work discussed in Part 4 and more broadly in this book. Of course there is Pólya's famous *How to solve it* (1945 and more recently, 2010) which has been available to support mathematical problem solving in the school mathematics curriculum for decades and of course relates the enhancing problem based learning here. Polya's *Mathematics and plausible reasoning* (1954) provides a complex base for the kind of reasoning and mathematical actions which go beyond the typical school mathematics curriculum problems—such problems and actions included in the diversity of writing in Part IV. Secondly we have the work of John Mason, Burton, and Stacey (1982/2010) of *Thinking mathematically* replete with both problems and strategies which relate to both applied problems [e.g. taxation choices] and more typical mathematics problems- again supporting the diversity of enhanced problem solving advocated in part 4. Thus there is a philosophical as well as mathematical problem solving support for the diversity of enhanced problem based learning advocated here The reader will profit from pulling out the nature of enhancements of problem based learning in these papers.

Below I provide three examples of the sketches of the approaches to "Enhancing problem-based learning." The first basis for enhancement is based on the changes in matter meant, taught, and learned as affecting and being affected by changing the structuring of the environment for learning but also radically altering the structure of the instructional sequence of lessons. The second is based in enhancing the mathematical qualities of student experience. The third is the kinds of enhancing learning materials, particularly here physical materials and their uses in enhancing and extending problem based learning.

The first example arises in looking at Peter Liljedahl's on "Building a thinking mathematics classroom". In reading this paper, you will be shown what he sees as radical changes in both the environment and the content and quality of materials and inter-actions with students which he sees as inherently linked and resulting in a daily pattern of enhanced problem based learning. Each class starts with a problem, presented orally to both encourage listening but also inter-action among students, who are also working in daily changing randomly assigned groups. This focus on problem actions and student inter-actions and is enhanced by all vertical surfaces (e.g., white boards, student work on papers, models) being engaged by students to produce and inter-act about showable, shareable mathematical products. Thus students are in a 'defronted" (Liljedahl's term) classroom (the foci of attention is radically changed). Although students questions showing a "shut down" of thinking (asking, "Is this right?") are acknowledged by the teacher or others, only those questions or comments which promote further thinking are taken up and elaborated upon. Liljedahl sees this physical, interactive, mathematical classroom setting as involving a daily cycle of three different action types (of which I provide a shortened précis here): Element I: dividing up the class in random groups and situating student workspaces so as to defront the class and thus promote student interaction and autonomy; Element II: providing an orally assisted problem presentation (which may be a problem outside of the curriculum or a curricular one that is new to the class); this is followed by interaction of work in a defronted environment as well as responses to keep focus on thinking questions both aimed at autonomous student mathematical actions; Element III: the final element of the sequence involves "leveling to the bottom", providing a summation of the work of the class in such a way as to allow all students to have a sense of the ideas developed so far; then providing hints and extensions to foster extension of individual and group mathematical activities; assessment exercises and finally prompting students make mindful notes about the days work for their own use. These three segments of this cycle are repeated and renewed on a daily basis and of course situates the teacher in the middle of the interaction. I want to make 2 points with this brief description of Liljedahl's "classroom." I want to point out that the class set-up; the inter-actions; the use and meaning of tasks; and questions about work on them; the use and rhythm of assessment;and student records of and reflections on work are all different from a 'normal" secondary school mathematics class For the teacher and the students that things are different here. Second, this is but one pattern of enhancement of problem based learning, but one that shows the content, contextual, pedagogical, and instructional complexity in accomplishing such enhancement.

The second aspect of enhancing problem based learning points to a mathematical character of such change. This is vividly seen in Hoshino's paper the title of which suggests that mathematical creativity is a motor for such enhancement. You will see in this paper how Hoshino shows the difference between using "brute force in solving a mathematical problem and using mathematical information cleverly to do the same result but this result is a pattern built on patterns (e.g., problem 2). The point Hoshino seems tome to be making is that pattern perception and pattern use go hand in hand with enhanced problem-based learning—that problem based learning is enhanced by students' learning to use knowledge they already have to look at problems differently. He notes that this has a second enhancement factor which he names early in the paper: the ability to solve hard problems by converting them into (different) simpler patterns. On my reading this suggests (and is shown emphatically in Problem 4) that coming to see problems as related to one another is a mathematical basis for enhancing problem-based learning.

A third contributor to enhancing problem based learning the use of concrete and other materials (e.g., tables graphs, apps). One of the many examples of this is seen in Part VI in Ann Kajander's paper on "Learning algebra with models and reasoning". In supporting such a relationship she suggests that ' the argument that problem based learning and hands-on material based learning take too long is contradicted by the benefit of the creation of connections of ideas to other ideas, often supporting more than one set of curriculum expectations at a time. Note the connection of this argument to a parallel argument from Hoshino's paper above. Using materials and connecting these materials to mathematical expressions and ideas are mathematics eliciting acts. Being able to move with facility and between materials based mathematical actions; diagrammatic illustrations; informal but powerful symbolic schemas; and formal more standard mathematical expressions and being aware of the mathematical equivalences among them are critical aspects of enhanced problem based mathematics.

From my reading of the some of papers from Part IV and other papers in this book, suggests that the task of redeveloping mathematics curriculum and related teaching and assessment is not an easy task. But the papers I have read suggest that this volume contains solid ideas on how this might and is already happening in Canadian schools.

References

Kilpatrick, J., Swafford, J., & Findell, B. (2001). The strands of mathematical proficiency. In J. Kilpatrick, J. Swafford, & B. Findell (Eds.), *Adding it up: Helping children learn mathematics* [electronic resource] (pp. 115–155). Washington, DC: National Academy Press.

Mason, J., Burton L., & Stacey K. (1982/2010). *Thinking mathematically.* London: Pearson.

Pólya, G. (1945). *How to solve it* (1st ed.). Princeton: Princeton University Press.

Pólya, G. (1954). *Mathematics and plausible* (Vol. 1). Princeton: Princeton University Press.

Preciado-Babb, A. P., Metz, M., & Marcotte, C. (2015). Awareness as an enactivist framework for the mathematical learning of teachers, mentors and institutions. *ZDM, 47*(2), 257–268.

Building Thinking Classrooms

Peter Liljedahl

Abstract In this chapter I first introduce the notion of a thinking classroom and then present the results of over 10 years of research done on the development and maintenance of thinking classrooms. Using a narrative style I tell the story of how this research began and led first to the notion of a thinking classroom and then to a research project designed to find ways to help teachers build such a classroom. Results show that there are a number of relatively easy to implement teaching practices that can bypass the normative behaviours of almost any classroom and begin the process of developing a thinking classroom.

Keywords Problem solving · Thinking · Group work · VNPS

Motivation

My work on the research presented in this chapter began over 10 years ago when I was invited to help June implement problem solving in her grade eight classroom. June had never done problem solving with her students before, but with its prominence in the recently revised curriculum, she felt it was time. June was aware of my interest in problem solving, so she reached out to me one day late into the school year.

June, as it turned out, was neither interested in co-planning nor co-teaching. What she wanted from me was simply a collection of problems she could try with her students. I was expecting to have a greater level of involvement in the lesson, but June was firm on her conditions. We eventually arrived at a compromise whereby I would supply the appropriate problems for June to use with her grade eight students, and she would let me watch her implement them within her classroom.

The first problem I gave her to use was a problem that I had had much success with grade eight students.

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> If 6 cats can kill 6 rats in 6 minutes, how many will be needed to kill 100 rats in 50 minutes? (Lewis Carroll, cited in Wekeling 1995, p. 34)

June accepted this problem in good faith and used it the next day. It did not go well. A forest of arms immediately shot up and June began to frantically move around the room to answer questions. Many students gave up quickly so June also spent much effort trying to motivate students to keep going. In general, there was some work attempted when June was close by and encouraging the students, but as soon as she left the trying stopped. This continued for the whole 40 min period.

The following day I was back with a new problem. The results were as abysmal as they had been on the first day. The same was true of day three. Over the course of three 40 min classes we had seen little improvement in the students' efforts to solve the problem, and no improvements in their abilities to do so. So, June decided it was time to give up. Her efforts to bring problem solving to her students had been met with resistance and challenge and resulted in few, if any, rewards.

I wanted to understand why the results had been so poor, so I asked June if I could stay and observe her and her students in their normal classroom routines. She agreed to this. After three full days of observation I began to discern a pattern. That the students were lacking in effort was immediately obvious, but what took time to manifest was the realization that what was missing in this classroom was that the students were not thinking. More alarming was the realization that June's teaching was predicated on an assumption that the students either could not, or would not, think. The classroom norms (Yackel and Rasmussen 2002) that had been established in June's class had resulted in, what I now refer to as, a non-thinking classroom. Once I realized this I proceeded to visit other mathematics classes—first in the same school and then in other schools. In each class I saw the same basic behaviour—an assumption, implicit in the teaching, that the students either could not or would not think. Under such conditions it was unreasonable to expect that students were going to be able to spontaneously engage in problem solving.

What was missing for these students, and their teachers, was a central focus in mathematics on thinking. The realization that this was absent in so many classrooms that I visited motivated me to find a way to build, within these same classrooms, a culture of thinking, both for the student and the teachers. I wanted to build, what I now call, a *thinking classroom*—"a classroom that is not only conducive to thinking but also occasions thinking, a space that is inhabited by thinking individuals as well as individuals thinking collectively, learning together, and constructing knowledge and understanding through activity and discussion" (Liljedahl 2016a, p. 364).

Early Efforts

Classroom norms, once established, are difficult to change (Yackel and Rasmussen 2002). My early efforts to build thinking classrooms revealed that even when a teacher is motivated to get their students to think, their initial efforts to do so are rarely rewarded by comparable changes in student behaviour. Quite the opposite, many of the teachers I was working with were met with resistance and complaints when they tried to make changes to their practice.

From these experiences I realized that if I wanted to build thinking classrooms—to help teachers to change their classrooms into thinking classrooms—I needed a set of tools that would allow teachers to bypass any existing classroom norms. These tools needed to be easy to adopt and have the ability to provide the space for students to engage in thinking, unencumbered by their rehearsed tendencies and approaches when in their mathematics classroom.

This realization moved me to begin a program of research that would explore both the elements of thinking classrooms and the traditional elements of classroom practice that block the development and sustainability of thinking classrooms. I wanted to find a collection of teacher practices that had the ability to break students out of their classroom normative behaviour—practices that could be used by teachers that had previously entrenched the classroom norms that now needed to be broken.

In Pursuit of Thinking Classrooms

My research to find the elements and teaching practices that fostered and sustained thinking classrooms has been ongoing for over 10 years. I initially explored my own teaching, as well as the practices of more than 40 classroom mathematics teachers. From this emerged a set of 11 elements that were found to permeate mathematics classroom practice—elements that account for most of whether or not a classroom is a thinking or a non-thinking classroom. These 11 elements of mathematics teaching became the focus of my research. They are

1. the type of tasks used, and when and how they are used;
2. the way in which tasks are given to students;
3. how groups are formed;
4. student work space while they work on tasks;
5. room organization, both in general and when students work on tasks;
6. how questions are answered when students are working on tasks;
7. the ways in which hints and extensions are used while students work on tasks;
8. the autonomy students have while working on tasks;
9. when and how a teacher levels[1] their classroom during or after tasks;
10. the ways in which students record notes;
11. and assessment, both in general and when students work on tasks.

[1] Levelling is a term introduced by Schoenfeld (1985) to describe a teacher's actions to bring a whole class up to the same level of understanding. It is specifically used in the context of closing off and debriefing an activity where students have been asked to solve a task on their own. After some period of time the teacher will close of the activity by showing how the question is to best solved or by calling on a student who completed the task to do the same. Liljedahl and Allan (2013) showed that levelling produces student behavior antithetical to the goals of the teacher.

June's class, for example, was one in which

1. practice tasks were given after she had done a number of worked examples;
2. students either copied these from the textbook or from a question written on the board;
3. students had the option to self-group to work on the homework assignment when the lesson portion of the class was done;
4. students worked at their desks writing in their notebooks;
5. students sat in rows with the students' desks facing the board at the front of the classroom;
6. students who struggled were helped individually through the solution process, either part way or all the way;
7. there were no hints, only answers, and an extension was merely the next practice question on the list;
8. students had little to no autonomy in how they engaged in tasks, usually work sheets or work out of the textbook;
9. when "enough time" time had passed June would demonstrate the solution on the board, sometimes calling on "the class" to tell her how to proceed;
10. students wrote down what June wrote on the board at the front of the room;
11. and assessment was always through individual quizzes and tests.

This was not, as determined earlier, a thinking classroom.

Each of these elements were something that needed exploring and experimentation. Many were steeped in tradition and classroom norms (Yackel and Rasmussen 2002). As such, research into each of these was done using design-based methods (Cobb et al. 2003; Design-Based Research Collective 2003) within my own teaching practice as well as the practices of more than 400 teachers participating in a variety of professional development opportunities. This approach allowed me to vary the teaching around each of the elements, either independently or jointly, and to measure the effectiveness of that method for building and/or maintaining a thinking classroom. Results fed recursively back into teaching practice, each time leading either to refining or abandoning what was done in the previous iteration.

The challenge, however, was to figure out how to shift a teaching practice when it was determined that a particular teaching method needed to be abandoned. Early results indicated that small shifts in practice, in these circumstances, did little to shift the behaviours of the class as a whole. Larger, more substantial shifts were needed. These were sometimes difficult to conceptualize. In the end, a contrarian approach was adopted. That is, when a teaching method around a specific element needed to be abandoned, the new approach to be adopted was, as much as possible, the exact opposite to the practice that had shown to be ineffective for building or maintaining a thinking classroom. For example, when sitting showed to be ineffective, we tried making the students stand. When levelling to the top failed we tried levelling to the bottom. When answering questions proved to cause learned helplessness we stopped answering questions. Each of these approaches then needed further refinement through the iterative design-based research approach, but it gave good starting points for this process.

Results

Through this process a number of results eventually began, at first slowly, to emerge. In what follows I will present, in brief, the results of the research done on each of these eleven elements and discuss how they hold together as a framework to build and maintain thinking classrooms.

The Type of Tasks Used

Lessons need to begin with good problem solving tasks. In the beginning of the school year, or when first attempting to transform a classroom, these tasks are highly engaging, non-curricular, collaborative tasks that drive students to want to talk with each other as they try to solve them (Liljedahl 2008). After a period of time (usually 2–3 weeks) these should gradually be replaced with curricular problem solving tasks that permeate the entirety of the lesson and emerge rich mathematics (Schoenfeld 1985) that can be linked to the curriculum content to be 'taught' that day. These curricula tasks can simply be questions from the textbook provided they are new to the students and present something that is problematic for them.

How Tasks Are Given to Students

As much as possible, tasks need to be given orally. If there are data, diagrams, or long expressions needed these can be provided on paper or projected on the wall, but the instructions pertaining to the activity of the task need to be given orally. This very quickly drives the groups to discuss what is being asked, focuses groups on the mathematics, and reduces the urge to individually decode instructions on a page.

How Groups Are Formed

Grouping and regrouping needs to be frequent and visibly random. Ideally, at the beginning of every class a visibly random method is used to create groups of 2–3 students who will work together for the duration of the class. These groups will work together on any assigned problem solving tasks and sit together or stand together during any group or whole class discussions. Frequent randomization will fundamentally transform the social structure of the classroom within 3 weeks (Liljedahl 2014, 2016a, in press a) and build the type of community needed to autonomously maintain a thinking classroom.

Student Work Space

The work on these aforementioned tasks needs to be done with groups standing and working on vertical non-permanent surfaces such as whiteboards, blackboards, or windows. This makes visible all work being done, not just to the teacher but to the groups doing the work. To facilitate discussion, there is only one felt pen or piece of chalk per group. The use of vertical non-permanent surfaces will increase eagerness to start, increase discussion, participation, and perseverance amongst the group members, and facilitate the mobility of knowledge between groups (Liljedahl 2016a, in press a).

Room Organization

The classroom needs to be de-fronted. The teacher must let go of one wall of the classroom as being the designated teaching space that all desks are oriented towards. The teacher needs to address the class from a variety of locations within the room and, as much as possible, use all four walls of the classroom. It is best if desks are placed in a random configuration around the room, and away from the walls.

How Questions Are Answered

It turns out that students only ask three types of questions: (1) proximity questions—asked when the teacher is close; (2) stop thinking questions—most often of the form “is this right” or “will this be on the test”; and (3) keep thinking questions—questions that students ask so they can get back to work. Only the third of these types should be answered. The first two need to be acknowledged, but not answered.

How Hints and Extensions Are Used

Once a thinking classroom is established, it needs to be nurtured. Student engagement should be maintained through the teacher’s judicious and timely use of hints and extensions (Liljedahl 2016a, b, in press b). Flow (Csíkszentmihályi 1990, 1996) is a good framework for thinking about this. Hints and extensions need to be given so as to keep students in a perfect balance between the challenge of the current task and their abilities in working on it. If their ability is too high the risk is they get bored. If the challenge is too great the risk is they become frustrated.

Student Autonomy

Providing of hints and extensions in a timely fashion is difficult when there are 10–12 groups in the class. If students have autonomy to interact with other groups, however, they will manage much of this on their own as they use each other to provide help when they are stuck and to seek increased challenge when they are done (Liljedahl in press b). Simply providing this autonomy is not enough, however. Students need to be shown that this autonomy exists, and feel its value. As such, the teacher needs to build autonomy by deliberatively pushing students towards other groups when they are stuck or need an extension.

When and How a Teacher Levels Their Classroom

Rather than using levelling to bring a whole class up to the same level of understanding (levelling to the top) discussions need to happen at a level that all students can understand (levelling to the bottom). That is, when every group has passed a minimum threshold the teacher should pull the students together to debrief what they have all achieved. At this time the teacher will either go over one or more of the students' solutions or work through a new problem together with the class as a whole. This helps reify and formalize the work the students have been doing and should constitutes the 'lesson' for that particular class.

Student Notes

After the levelling has occurred students need to write some notes for themselves. These notes should be based on the work that is already existing on the boards and can come from their own work, another group's work, or a combination of work from many groups. As part of the levelling process, teachers can highlight particular parts of the work that is on the boards, but it is important that the students select themselves, and synthesize and reorganize notes on their own. Students younger than grade 8 will need guidance as to what to write down.

Assessment

Assessment in a thinking classroom needs to be mostly about the involvement of students in the learning process through efforts to communicate with them where they are and where they are going in their learning. It needs to honour the activities of a thinking classroom through a focus on the processes of learning more so than the products, and it needs to include both group work and individual work (Liljedahl 2010).

Taken Together

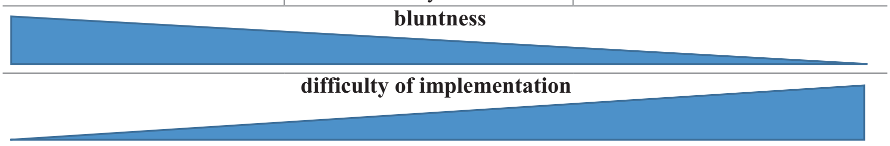
This research also showed that these are not all equally impactful or purposeful in the building and maintenance of a thinking classroom. Some of these are blunt instruments capable of leveraging significant changes while others are more refined, used for the fine-tuning and maintenance of a thinking classroom. Some are necessary precursors to others. Some are easier to implement by teachers than others while others are more nuanced, requiring great attention and more practice as a teacher. And some are better received by students than others. From the whole of these results emerged a three tier hierarchy that represent, not only the bluntness and ease of implementation, but also an ideal chronology of implementation (see Table 1).

These stages can be envisioned as a set of cycles working in sequence and together to build a thinking classroom (see Fig. 1).

Since their emergence, these eleven tools and the aforementioned stages, have been used to successfully build thinking classrooms in hundreds of mathematics classrooms from kindergarten to grade 12 (Liljedahl 2016a) with transformative effects on students' thinking, engagement, and enjoyment as well as teachers' sustained practice.

Table 1 Eleven elements as chronologically implemented

Stage one	Stage two	Stage three
Begin lessons with tasks	Use oral instructions	Level to the bottom
Form visibly random groups	Defront the classroom	Use hints and extensions to manage flow
Use vertical non-permanent surfaces	Answer only keep thinking questions	Use assessment as communication
	Build autonomy	Use mindful notes



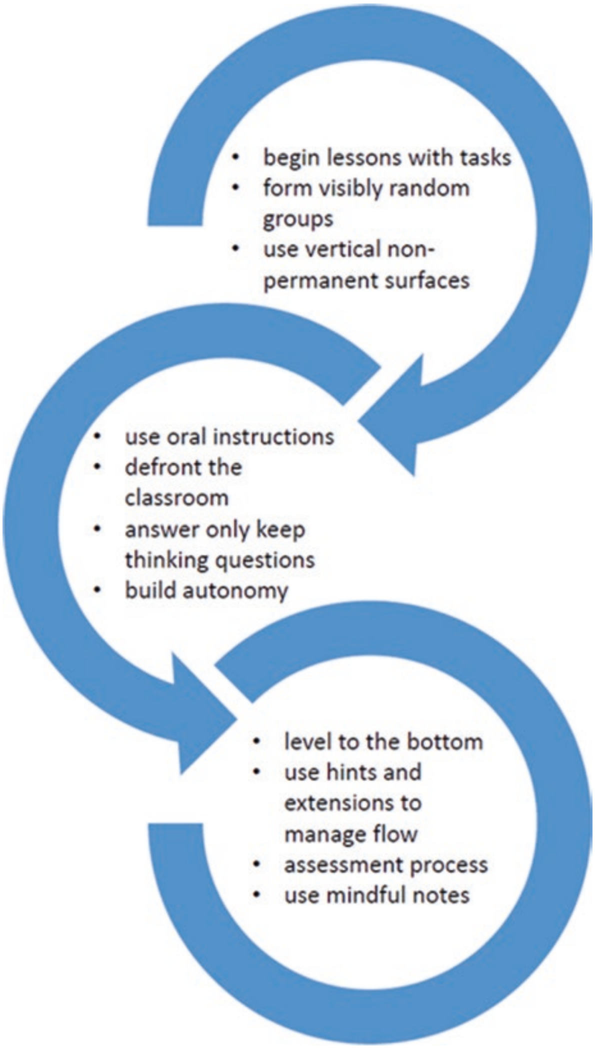


Fig. 1 The eleven tools organized into discrete cycles

References

Cobb, P., Confrey, J., di Sessa, A., Lehrer, R., & Schauble, L. (2003). Design experiments in educational research. *Educational Researcher, 32*(1), 9–13.

Csíkszentmihályi, M. (1990). *Flow: The psychology of optimal experience*. New York: Harper & Row.

Csíkszentmihályi, M. (1996). *Creativity: Flow and the psychology of discovery and invention*. New York: HarperCollins Publishers.

Design-Based Research Collective. (2003). Design-based research: An emerging paradigm for educational inquiry. *Educational Researcher, 32*(1), 5–8, 35–37.

Liljedahl, P. (2008). *The AHA! Experience: Mathematical contexts, pedagogical implications*. Saarbrücken: VDM Verlag.

Liljedahl, P. (2010). The four purposes of assessment. *Vector, 2010*(2), 4–12.

Liljedahl, P. (2014). The affordances of using visibly random groups in a mathematics classroom. In Y. Li, E. Silver, & S. Li (Eds.), *Transforming mathematics instruction: Multiple approaches and practices* (pp. 127–144). New York: Springer.

Liljedahl, P. (2016a). Building thinking classrooms: Conditions for problem solving. In P. Felmer, J. Kilpatrick, & E. Pekhonen (Eds.), *Posing and solving mathematical problems: Advances and new perspectives* (pp. 361–386). New York: Springer.

Liljedahl, P. (2016b). Flow: A framework for discussing teaching. In C. Csíkos, A. Rausch, & J. Szitányi (Eds.), *Proceedings of the 40th conference of the International Group for the Psychology of Mathematics Education* (Vol. 3; pp. 203–210). Szeged: PME.

Liljedahl, P. (in press-a). Classroom practices for supporting problem solving. In P. Liljedahl & M. Santos (Eds.), *Current themes, trends and research perspectives in mathematical problem solving*. New York: Springer.

Liljedahl, P. (in press-b). On the edges of flow: Student problem solving behavior. In S. Carreira, N. Amado, & K. Jones (Eds.), *Broadening the scope of research on mathematical problem solving: A focus on technology, creativity and affect.* New York: Springer.

Liljedahl, P., & Allan, D. (2013). Studenting: The case of "now you try one." In Lindmeier, A. M. & Heinze, A. (Eds.), *Proceedings of the 37th conference of the International Group for the Psychology of Mathematics Education* (Vol. 3; pp. 257–264). Kiel: PME.

Schoenfeld, A. (1985). *Mathematical problem solving*. Orlando: Academic Press.

Wakeling, E. (1995) *Rediscovered Lewis Carroll puzzles*. New York: Dover Publications, Inc.

Yackel, E., & Rasmussen, C. (2002). Beliefs and norms in the mathematics classroom. In G. Leder, E. Pehkonen, & G. Törner (Eds.) *Beliefs: A hidden variable in mathematics education?* (pp. 313–330). London: Kluwer Academic Publishing.

Additional Suggestions for Further Reading

Fenstermacher, G. (1986). Philosophy of research on teaching: Three aspects. In M. C. Whittrock (Ed.), *Handbook of research on teaching* (3rd ed., pp. 37–49). New York: Macmillan.

Fenstermacher, G. (1994, revised 1997). *On the distinction between being a student and being a learner*. Paper presented at the annual meeting of the American Educational Research Association, New Orleans, LA.

Liljedahl, P., & Allan, D. (2013). Studenting: The case of homework. In M. Martinez & A. Castro Superfine (Eds.), *Proceedings of the 35th annual meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education* (pp. 489–492). Chicago: University of Illinois at Chicago.

Mason, J. (2002). *Researching your own practice: The discipline of noticing*. New York: Routledge.

Mason, J., Burton, L., & Stacey, K. (1982). *Thinking mathematically*. London: Addison-Wesley Publishing Company.

Teaching Mathematics and Developing Citizenship: How to Use Contexts to Enhance Problem-Based Learning

Annie Savard

Abstract Teaching and learning mathematics through problem-based learning is highly promoted all around the world. This student-centered approach highlights the fact that students are sense makers. In this regard, the contexts presented in the problems given to students should serve the purpose of learning by providing meaning to construct. On the other hand, the contexts should also serve the purpose of teaching mathematics by designing meaningful problems for students and guiding them in the learning process of making sense of mathematics within a meaningful context. This book chapter discusses three different uses of contexts to enhance problem-based learning when teaching mathematics: for designing mathematical tasks, for interpreting or situating students' thinking, and for developing citizenship competencies. In order to illustrate the three different uses, two learning situations will be presented and analyzed using these lenses: the culture present in contexts, the mathematical knowledge targeted, and critical thinking as part of citizenship competencies.

Keywords Problem-based learning · Socio-cultural contexts · Critical thinking · Citizenship competencies

Contexts Presented in Problems

Problems presented to students might serve different purposes. For instance, they might be presented to them in order to apply new mathematical knowledge developed in different contexts, or they can be a vehicle to develop new mathematical knowledge in a contextualized situation (Savard and Polotskaia 2017). A

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contextualized situation is usually a situation where the context presented is a real-life scenario (Jackson et al. 2012). A situation usually gives more information about the context: the situation provides a problem, but also a rationale to perform a task. A task is what to do to in order to solve the problem, or, in other words, the question to investigate. The literature sometimes refers to authentic tasks, because the classroom is genuinely connected to the real world (Newmann et al. 2007). The authenticity is supposed to be a context familiar to students. For instance, it makes sense in Canada to have contexts about snow and low degrees. But it might does not make any sense for students who never experienced snow and cold winter. Authentic tasks should be real: the situation really happens in students' environment. Otherwise, this kind of task is inspired by real-life situations (Holtman et al. 2011).

A contextualized situation aims to better prepare students to face the realities and problems in society (Savard 2015). At this end, the problems to investigate should be complex, because they require multiple steps to find answers and thus lead students to make choices among the mathematical concepts and processes used and justify these choices. The complexity of these problems also refers to the fact that the immediate solution is not obvious and that many solutions or answers are possible (Stein et al. 2000). In addition to that, students will have the opportunity to explore more than one mathematical concept or process (Savard and Manuel 2016). They will have to conduct some investigations and thus present different solutions to discuss (Smith et al. 2009). They might be allowed to use different tools to perform the task: a calculator or a digital tool such as a spreadsheet or a mathematical application. In this chapter, I consider complex tasks as a learning situation, where a problem is given to students in order to explore and investigate the context using mathematical models. The task presented in the learning situation is, in fact, the problem to solve, and a learning situation might have more than one task. Thus, a learning situation should have many mathematical concepts and processes to be mobilized by students. A mobilization is the selection and the use of knowledge in situ, e.g., according to the contextualized situation. A mobilization is not only the use of knowledge: it implies a transformation of it according to the situation presented.

Use Contexts to Teach Mathematics and Develop Citizenship

In order to use contexts to enhance problem-based learning, I present how to use context in designing mathematical tasks, in interpreting or situating students' solutions, and in developing citizenship.

Using Contexts to Design Mathematical Tasks

Designing a mathematical task involves identifying the mathematical knowledge to be developed and contexts familiar to students that could be used. At this end, the situation presented to students should be accessible to them, and thus have a strong relationship between the mathematical knowledge to be developed and the context to be used. In fact, this relationship is so strong that if it is not present, it can create learning obstacles. For instance, students might not mobilize the expected mathematical knowledge or the context itself might be an obstacle. Thus, they might say that they do not understand the problem and therefore they are not able solve it.

In order to support students to learn, when designing a mathematical task, it is also important to anticipate students' thinking in regard to the whole situation: the context presented, the mathematics involved, and the potentiality to develop high thinking skills such as critical thinking and decision making. The choice of the situation is then crucial. One starting point to select the context of the situation might be to look at an event or a phenomenon coming from familiar socio-cultural contexts. The socio-cultural context is mainly about the culture present in the society. For example, socio-cultural context could be about sports (hockey, baseball, soccer), arts (music, movie, tv show), economy (shopping, saving, advertising), medias (internet, Facebook), science (scientific experiment), technology (building a bridge) or everything coming from daily life (school life, cooking, cleaning). The event or the phenomenon to be studied coming from the socio-cultural context is presented in the situation, and a problem should arise in order to create the need for students to study the event or the phenomenon. Mathematical knowledge is needed for studying the event or the phenomenon. Thus, students need to create mathematical models of the situation in order to solve the problem. The mathematical models will allow them to mobilize mathematical concepts and processes to perform the task and thus solve the problem. Using the solutions and answers when looking back at the situation might support the development of critical thinking: a citizenship competency.

In order to see the different contexts in a situation, let's look at two mathematical situations taken from the Organization for Economic Cooperation and Development (OECD) triennial Program for International Student Assessment (PISA) 2012 (2013). Those situations present two different socio-cultural contexts to be mathematically modelized by students, where critical thinking is needed to solve the task and where the situation has a strong potential to address political issues. Figures 1 and 2 present the two situations. Can you identify the elements of each context in the given situation?

Using Contexts to Interpret Students' Solutions

The three different contexts in a situation are helpful to interpret and situate students' solutions or students' thinking. For instance, if a student is not familiar with climbing in the first situation, he might not think about going down after he went up. Thus, he might choose 9 km as the distance to be used instead of 18 km. This information is helpful to know because the teacher can then make a precise intervention with the student. At this end, Tables 1 and 2 show the knowledge to be used by students for each context.

CLIMBING MOUNT FUJI

Mount Fuji is a famous dormant volcano in Japan.

Question 1: Climbing Mount Fuji
Mount Fuji is only open to the public for climbing from 1 July to 27 August each year. About 200 000 people climb Mount Fuji during this time. On average, about how many people climb Mount Fuji each day?
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Question 2: Climbing Mount Fuji
The Gotemba walking trail up to Mount Fuji is about 9 kilometres (km) long. Walkers need to return from 18 km walk by 8 pm. Toshi estimates that he can walk up the mountain at 1.5 kilometres per hour on average, and down at twice that speed. These speeds take into account meal breaks and rest times. Using Toshi's estimated speeds, what is the latest time he can begin his walk so that he can return by 8pm?

Question 3: Climbing Mount Fuji
Toshi wore a pedometer to count his steps on his walk along the Gotemba trail. His pedometer showed that he walked 22 500 steps on the way up. Estimate Toshi's average step length for his walk up the 9 km Gotemba trail. Give your answer in centimetres (cm).

Fig. 1 Climbing Mount Fuji

Using Contexts to Develop Citizenship

As we can see in the two tables, there is a possibility to capitalize on the socio-cultural and the mathematical contexts to support students in developing citizenship. In fact, two citizenship competencies are offered: gaining more information on ethical and political issues; and developing critical thinking and decision-making.

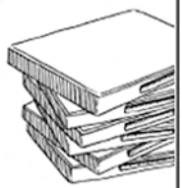
It is easy to use the sociocultural context to find connections with ethical or political issues. Those could be presented or discussed when launching the task to students in order to give a better picture of the socio-cultural context. For example,

DVD RENTAL

Jenn works at a store that rents DVDs and computer games.

At this store the annual membership fee costs 10 zeds.

The DVD rental fee for members is lower than the fee for non-members, as shown in the following table:



Non-member rental fee for one DVD	Member rental fee for one DVD
3.20 zeds	2.50 zeds

Question 1: DVD Rental
Troy was a member of the DVD rental store last year. Last year he spent 52.50 zeds in total, which included his membership fee. How much would Troy have spent if he had not been a member but had rented the same number of DVDs?

Question 2: DVD Rental
What is the minimum number of DVDs a member needs to rent so as to cover the cost of the membership fee? Show your work.
Number of DVDs: ____________________

Fig. 2 DVD rental

the socio-cultural context in the *Climbing Mount Fuji* problem (see Fig. 1) presents an out-door activity open to the public. It is relevant to address the point that the authorities have a certain control on when it is open. Discussing the reasons why it is open to the public will allow students not only to think about safety but also about the closing hours. It provides them a better access to the task by being more familiar with the context and the important information. In the case of the *DVD Rental* problem (see Fig. 2), the socio-cultural presents information about personal finances and consumerism. The mathematical context provides solutions to be discussed in the socio-cultural and in the citizenship contexts. Thus, having membership fees and different rental fees and then calculating the difference is an important aspect for making financial decision about personal finances. In this case, the solution of the problem provides important information to be discussed at the end. In fact, critical thinking might be developed after solving the task when comparing the fees and discussing when it is worth it to be a member. It is important to refer back to the other contexts and not just move on to another task when the solution is provided.

Critical thinking might be developed in many places. Critical thinking is making judgments using relevant criteria. It involves taking in to consideration the situation to be evaluated and the relevant elements to take in to consideration. It is about questioning the facts, the phenomenon and the conclusions. Making informed decisions implies using critical thinking when assessing different options or choices

Table 1 Knowledge to be used in the situation Climbing Mount Fuji

Contexts	Situation	Knowledge to be used
Socio-cultural context	1. Mount Fuji is a dormant volcano in Japan.	Determinate the days in the calendar from 1 July to 27 August;
	Mount Fuji is only open to the public for climbing from 1 July to 27 August each year.	About 200,000 people climb Mount Fuji during this time.
	2.The Gotemba walking trail up to Mount Fuji is about 9 kilometres (km) long.	Walking included going back to the starting point.
	Walkers need to return from their walk by 8 pm.	Using time until 8 pm.
	Time needed for the meal break and rest.	
	3. Toshi wore a pedometer to count his steps on his walk along the Gotemba trail.	Pedometer counts steps.
Mathematical context	1. On average, about how many people climb Mount Fuji each day?	Calculate an average using the numbers of days and the number of people;
		Performing division;
		Rounding.
	2. Using Toshi's estimated speeds, what is the latest time he can begin his walk so that he can return by 8 pm?	Walkers need to return from a18 km walk by 8 pm.
		He walks up the mountain at 1.5 kilometres per hour on average, and down at twice that speed. These speeds take into account meal breaks and rest times.
		Calculate time: 1 h = 60 min
	3. Estimate Toshi's average step length for his walk up the 9 km Gotemba trail. Give your answer in centimetres (cm).	22,500 steps on the way up for 9 km;
		Transfer m to cm;
		Proportionality.
Citizenship context	Tourism: economy and sustainability	Critical thinking:
	Climbing: sports, safety	How many people a day is possible considering 200,000 people in that time;
		Speed is faster going down;
		Climbing at night it is dangerous;
		Estimate an average step length in cm.

before selecting one. Critical thinking could be developed when discussing the socio-cultural context or the citizenship context, when modeling and solving the problem in the mathematical context, and when using the solution to refer back to the socio-cultural and the citizenship contexts. If we look at the socio-cultural and citizenship contexts in the *Climbing Mount Fuji* problem (see Fig. 1), many places could be used to develop critical thinking. As stated earlier, discussing the reasons why it is open to the public will bring students to think in terms of safety. Climbing

Table 2 Knowledge to be used in the situation DVD Rental

Contexts	Situation	Knowledge to be used
Socio-cultural context	A store rents DVD and computer games. People can be members or not and the price is different for members.	Rental fees are lower for members; There is an annual membership.
Mathematical context	1. Troy was a member of the DVD rental store last year. Last year he spent 52.50 zeds in total, which included his membership fee. How much would troy have spent if he had not been a member but had rented the same number of DVDs?	Annual membership of 10 zeds has to be deducted from the total amount in order to find the number of DVD rent. Use this number to find the amount of DVDs rented for a non-member
	2. What is the minimum number of DVDs a member needs to rent to cover the cost of the membership fee?	Find 10 zeds as a difference between minimum DVDs rents by members and non-members.
Citizenship context	Making financial decision about membership.	Critical thinking: The value of the membership, smaller rental fees for members

in the dark is not safe; therefore, the place is closed at night and thus it does make sense that the place closes at 8:00. It is also possible to develop critical thinking when making estimation about length or time when modelizing and solving problems. Thus, estimating involves a possible answer, which means that this answer should be close to the reality. This back and forth movement between the different contexts or within one context shows the complexity of solving problems. It also highlights the cognitive demand needed to make sense of the problem, modelize it and solve it, while being critical all the time, especially when making decisions about the process. In this sense, teaching mathematics using problem-based learning is more than giving a problem to solve to students: it is having them engage and participate in our world.

References

Holtman, L., Julie, C., Mbekwa, M., Mtetwa, D., & Ngcobo, M. (2011). A comparison of preferences for real life situations that could be used in school mathematics in three SADC countries. *Southern African Review of Education, 17*, 120–137.

Jackson, K., Shahan, E., Gibbons, L., & Cobb, P. (2012). Launching complex tasks. *Mathematics Teaching in the Middle School, 18*(1), 24–29.

Newmann, F. M., King, M. B., & Carmichael, D. L. (2007). *Authentic instruction and assessment: Common standards for rigor and relevance in teaching academic subjects*. Des Moines: Iowa: Department of Education.

OECDE Programme for International Student Assessment 2012. (2013). *PISA 2012 released mathematics items*. Retrieved from https://www.oecd.org/pisa/pisaproducts/pisa2012-2006-rel-items-maths-ENG.pdf

Savard, A. (2015). Making decisions about gambling: The influence of risk on children's arguments. *The Mathematics Enthusiast, 12*(1,2 & 3), 226–245.

Savard, A., & Manuel, D. (2016). Teaching statistics in middle school mathematics classrooms: Creating an intersection for intra and interdisciplinarity. *Statistics Education Research Journal, 15*(2), 239–256.

Savard, A., & Polotskaia, E. (2017). Who's wrong? Tasks fostering understanding of mathematical relationships in word problems in elementary students. *ZDM Mathematics Education*. Retrieved from https://doi.org/10.1007/s11858-017-0865-5.

Smith, M. S., Hughes, E. K., Engle, R. A., & Stein, M. K. (2009). Orchestrating discussions. *Mathematics Teaching in the Middle School, 14*(9), 548–556.

Stein, M. K., Smith, M. S., Henningsen, M. A., & Silver, E. A. (2000). *Implementing standards-based mathematics instruction: A casebook for professional development*. New York: Teachers College Press.

Additional Suggestion for Further Reading

Again, I recommend the Mathematics Education and Society website, which aims to promote discussion about the social, ethical, and political dimensions of mathematics education. The website provides access to the past conference proceedings on the latest research in mathematics education: http://www.mescommunity.info

I recommend you to take a look on the Organization for Economic Cooperation and Development (OECD) website, section Program for International Student Assessment (PISA). This international organization provides information about teaching mathematics in international environments: http://www.oecd.org/pisa/

This book chapter focuses on the implication of modelize or mathematize a situation or a problem. In order to support your students to create models before computing, I would recommend this website on virtual manipulatives. Even older students need support when mathematize: http://nlvm.usu.edu

Teaching Probability in Junior High School Through Problem Solving: Construction and Analysis of a Probabilistic Problem

Vincent Martin, Izabella Oliveira, and Laurent Theis

Abstract This text proposes a probabilistic problem aimed at teaching probability through problem solving in the first 2 years of junior high school. This problem's adaptability in terms of its level of complexity and the ease of adjusting it to students' level of mathematical development makes it highly versatile.

As we will see, the problem is rich owing to the connection it draws between the theoretical and frequentist approaches via its consideration of the characteristics of probability and probabilistic thinking. Indeed, even if connecting these two approaches is desirable in the context of teaching probability, such a connection represents a significant challenge for teachers (Martin V, Theis L. Can J Sci Math Technol Educ 16(4):1–14, 2016). This likely explains why these two approaches are rarely addressed in their multiplicity or complementarity (Caron F. Splendeurs et misères de l'enseignement des probabilités au primaire. In: Proceedings of the « Colloque annuel du Groupe de didactique des mathématiques du Québec », Trois-Rivières, Québec, 2002; Nilsson P, Eckert A. Interactive experimentation in probability—opportunities, challenges and needs of research. In Proceedings of the thirteenth international congress on mathematical education, Hamburg, Germany, 2016).

Finally, the text will conclude with a discussion of the use of this problem for teaching probability by reviewing the teacher's challenges in managing the problem, as well as its mathematical potential and the favourable context it represents for students deemed to be in difficulty in mathematics.

Keywords Problem solving · Teaching of probability · Probabilistic problem · Frequentist and theoretical approaches · Junior high school

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Teaching Mathematics Through Problem Solving

Problem solving[1] has been given a central place in mathematics learning in the context of numerous studies in mathematics education and in *didactique des mathématiques*, with the intent of bringing the student's mathematical activity closer to that of the mathematician (Mary and Theis 2007). Even so, the very notion of problem solving in mathematics is shrouded in semantic vagueness. Törner et al. (2007) have noted that problem solving has several meanings depending on the country, and that these meanings have changed over time both within and across countries.

In Québec, Lajoie and Bednarz (2012, 2014a, b) have shown that problem solving has been central to the teaching and learning of mathematics in the province's school system since the beginning of the twentieth century. In doing so, these authors have observed an evolution in the meaning of the notions of problem and problem solving over time in this context.

Among others, this echoes the reflections of Nesher et al. (2003), Francisco and Maher (2005) and Theis and Gagnon (2013), who have all noted the polysemous nature of the notion of problem.

Given the importance of (and semantic vagueness surrounding) problem solving, we believe it is important to state our definition of the term here. To do so, we take up certain elements of the definition set forth by Astolfi (1993). This definition of problem solving is largely used in the *didactique des mathématiques* community. To support our definition, we also refer to research on mathematical problem solving by Schoenfeld (1985).

Hence, we define a problem as a mathematical task that allows a student to develop new knowledge through real mathematical activity. According to Astolfi (1993), to generate such activity, the problem must be designed so as to enable the student to engage in the task without initially having all the means he/she requires to solve it. This means the problem must represent a genuine challenge for students, i.e., it must offer them sufficient resistance but without being perceived as lying beyond their grasp.

According to Schoenfeld (1985), solving a mathematical problem implies an intellectual challenge for the individual and not merely the application of algorithms. Hence, in his view, if the individual has immediate access to the solution, the task is in fact an exercise rather than a problem. This is why Schoenfeld (1985) asserts that the definition of a problem is relative. The status of a mathematical task as a problem is not inherent, but rather depends on the prior knowledge of the individual, as well as the particular relationship between the individual and the task. Astolfi (1993) likewise states that a problem allows a student to develop new

[1] In this text, rather than speaking in terms of a "problem situation" and "resolving a problem situation," we will use the expressions "problem" and "problem solving" in order to establish correspondence between the world of Francophone research in *didactique des mathématiques*, on one hand, and the world of Anglophone research on mathematics education, on the other. We are aware of the important semantic nuances between these terms, but given that this is not the topic of the text, we have chosen not to further address the issue here.

knowledge by overcoming a planned conceptual obstacle; the problem is therefore constructed specifically to allow the student to overcome a given obstacle.

Consistent with this definition, we would like to propose a probabilistic problem, which, as we will see, makes it possible to work on conceptual issues of probability and of probabilistic thinking.

Characteristics of Probability and Probabilistic Thinking

The development of probabilistic thinking in (future) citizens, whether in or outside of school, is a topic that has sparked the interest of numerous researchers for many years (Jones and Thornton 2005; Martin and Thibault 2016; Shaughnessy 1992). Indeed, "for more than half a century, numerous studies have been conducted on the development of probabilistic reasoning and on the learning and teaching of probability" (Martin and Thibault 2016, p. 80, our translation).

Several reasons may account for the importance assigned to this topic. First, probability has a significant place in our societies in a context where probabilistic thinking is useful to many professions and where random events affect individuals in their everyday lives to varying degrees (Albert 2006; Batanero et al. 2014). For example, individuals face random situations in contexts related to health, the environment, consumption, management, and recreation (diagnosis and choice of treatments, weather forecasts, gambling and lotteries, financial planning, board games, etc.).

The teaching of probability is also prescribed in the educational programs of many countries (including Canada, Italy, the United States, Australia and the United Kingdom), often starting in elementary school or at least (junior) high school (Caron 2002; Gattuso and Vermette 2013; Jones et al. 2007; Savard and DeBlois 2005).

However, probability is generally perceived by teachers as being difficult to teach and therefore raises a number of challenges (Batanero and Diaz 2012; Stohl 2005). Among other things, these challenges have to do with the conceptual complexity of probability and with the resulting impacts on developing probabilistic thinking. Examples include the non-deterministic nature of probability, which strongly sets it apart from other areas of mathematics that are all deterministic in nature (Savard 2008; Scheaffer 2006). Unlike arithmetic, for example, where the same operation always generates the same result, random experiments do not always yield the same outcome from one experiment to another owing to sample variability—even if they are conducted under the same conditions.

Furthermore, numerous studies have shown that another issue of developing probabilistic thinking is the presence, among children and adults alike, of probabilistic conceptions[2] that are generally strongly anchored and difficult to change

[2] Probabilistic conceptions, which appear as probabilistic reasoning, are associated with the notions of chance, probability and relationships to randomness. The presence of these conceptions can lead to difficulties with learning or teaching probability. However, rather than labelling them as miscon-

(Fischbein and Schnarch 1997; Kahneman et al. 1982; Pratt 1998; Savard 2014; Shaughnessy 1992).

Finally, another issue associated with developing probabilistic thinking is the existence of different probabilistic approaches depending on the situation and the context (Batanero 2014; Borovcnik and Kapadia 2016; Chernoff and Sriraman 2014; Jones et al. 2007). In the classroom context, the most frequently adopted approach is the theoretical or classical approach. In this approach, probability is calculated based on the relation between the number of favourable outcomes and the number of possible outcomes for a given event for which all outcomes are considered equiprobable. The frequentist (or frequential, experimental, experiential or empirical) approach involves measuring the relative frequency of a given event on the basis of a series of observed data. Specifically, through trials and the compilation and organization of their outcomes, a stabilization of relative frequency is observed, with a view to finally tending toward the probability of a possible event occurring. The subjective or personalist approach for its part has its roots in Bayes' theorem, and consists in an individual or group's numerical evaluation of the strength or degree of a belief through a more or less intuitive analysis of available information.

The problem we would like to propose takes into account issues related to the learning and teaching of probability and more specifically the issue of connecting the frequentist and theoretical approaches. Indeed, the problem is conducive to such a connection, which represents a significant challenge for teachers. This connection will be examined further in the text.

Description of the Probabilistic Problem

The probabilistic problem[3] we set forth here comprises three tasks respectively involving a roulette, a pair of four-sided dice, and a bottle. Depending on the curriculum, the probabilistic issues underpinning this problem are addressed at different times in students' probability instruction. For example, the problem connects with probability issues introduced at the end of elementary school under the Québec education program (Gouvernement du Québec 2006) or at the beginning of junior high school (grades 7–9) under Western Canada's Common Curriculum Framework for K-9 Mathematics (Western and Northern Canadian Protocol 2006).

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ceptions, following on Savard (2008), we choose to refer to them as probabilistic conceptions. This designation, which is neutral with respect to such reasoning, appears more accurate to us in that these conceptions are fairly common, and can develop, weaken, be modified or evolve with experience (Savard 2014).

[3] The problem we propose here was originally developed in the context of doctoral studies by Martin (2014). The author's doctoral thesis afforded the possibility of studying the didactical methods used by two 5th–6th grade elementary teachers (students 11–12 years old) to teach probability.

Fig. 1 Visual representation of the roulette

Fig. 2 Visual representation of the pair of dice

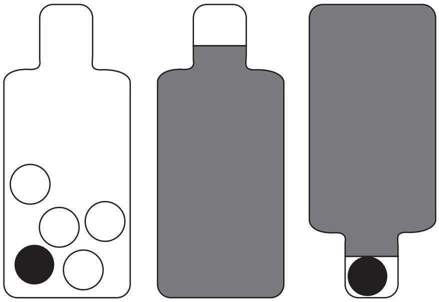

Fig. 3 Visual representation of the bottle

The aim of the problem is to bring students to determine the probability of winning associated with each task, then to compare all three in order to determine which respectively offers the highest odds of winning, the lowest odds of winning, and neither the highest nor the lowest odds of winning. The problem's structure calls for twofold action, namely determining and comparing the odds of winning. Figures 1, 2 and 3 provides a visual representation of the roulette, the pair of dice, and the bottle.

The roulette (Fig. 1) has no swivel, needle, or marble to indicate an outcome. It is simply a paper disk divided into 12 black, grey and white angular sectors (pie-shaped pieces).[4] Each colour has the same likelihood of being obtained, i.e., 1 out of 3, since the total of the angles of each angular sector for each colour is 120°.

[4] The original version of the roulette as used by Martin (2014) was a colour version with blue, red and yellow angular sectors; however, in the present text, these have been respectively replaced by grey, white and black angular sectors.

However, the number of angular sectors of each colour varies: the roulette comprises 3 black, 4 white and 5 grey angular sectors. To prevent students from directly seeing that all three colours have the same odds of being obtained, the differently coloured angular sectors have varying dimensions (non-equiprobable outcome) and are asymmetrically distributed across the roulette, with no two angular sectors of the same colour ever being adjacent. To win at the roulette, one would need to fall on one of the black angular sectors (if the roulette were capable of producing a trial outcome). Given that the black angular sectors cover a third of the roulette, the odds of winning at this task are 1 out of 3.

Next, the pair of four-sided dice (Fig. 2) is composed of a white die and a black die, which are regular tetrahedrons whose four respective faces are identical equilateral triangles.[5] To win at this task, one would need to obtain a sum of 5. There are 4 possible arrangements of outcomes in order to obtain this sum (1 and 4; 2 and 3; 3 and 2; and 4 and 1), out of a total of 16 possible outcomes. The odds of winning at this task are therefore 1 out of 4. Finally, the probabilistic bottle (Fig. 3) is inspired by Brousseau et al. (2002) and has been used in several recent studies, including those of Briand (2005, 2007), Rioux (2012), Martin (2010, 2014) and Nilsson (2014; Nilsson and Eckert 2016). The bottle contains a total of 5 marbles, i.e., 1 black marble and 4 white marbles. When turning the bottle upside down, only one marble can fit through the bottleneck, thus revealing the marble's colour. Since the bottle is opaque (with the exception of a small part of the bottleneck), trials are the only way to find out its content and the odds of winning. To win with the bottle, one must draw a black marble. The theoretical odds of winning with this task are therefore 1 out of 5.

Analysis of the Problem's Probabilistic Issues

This problem entails certain potential probabilistic issues for teachers and their students, which require analysis in order to be able to offer a rich experience with probability learning.

Didactical Examination of the Connection Between the Frequentist and Theoretical Approaches

The importance of connecting the frequentist and theoretical approaches when teaching probability has been emphasized by several authors (Batanero 2014; Jones and Thornton 2005; Prodromou 2012; Steinbring 1991; Stohl 2005). Batanero (2014) in particular notes that this connection makes it possible to address the notion

[5] Unlike other types of dice, with a four-sided die, the outcome is not given by the top face, but rather by the sum of the upper vertex of the four-sided die, with each side being numbered 1– 4. This type of die has 3 numbers on each face, i.e., one at each angle. When the die is cast on a flat surface, it is the upward facing vertex that indicates the outcome, with the same number showing on each bordering angle.

of sample variability from the non-deterministic perspective of probability, as well as to access the underlying logic of the problem to be solved. Similarly, Savard (2008) states that taking variability into account in the frequentist approach makes it possible to reason about uncertainty rather than staying restricted to deterministic reasoning. She also points out that the frequentist approach allows for working on probabilistic conceptions with students, which is impossible when exclusively using the theoretical approach.

The problem that we propose creates tension between the frequentist and theoretical approaches.[6] Indeed, given the respective characteristics of the tasks, it would be impossible to address them all using the same probabilistic approach. Whereas the roulette demands reasoning in line with the theoretical approach (given that it precludes the conducting of trials), the bottle demands reasoning in line with the frequentist approach (given that it precludes the calculation of the theoretical odds of winning based on the relation between favourable and possible outcomes).

For its part, the pair of dice[7] enables reflection on the odds of winning in line with the frequentist and theoretical approaches, given that it enables trials as well as the calculation of theoretical probability.

The respective characteristics of the tasks therefore demand parallel, or even connected, use of the two probabilistic approaches. The connection between the different approaches is therefore not just to be found in each individual task, but essentially in the effort of comparison between all three tasks.

The pair of dice and the bottle make it possible to work on the connection between the frequentist and theoretical approaches from the outset. With the pair of dice, the connection between the two probabilistic approaches can be made both ways, i.e., from the frequentist approach to the theoretical approach and vice versa. It is possible to run trials to determine a probability, then to make a hypothesis on theoretical probability; yet it is also possible to calculate the theoretical odds of winning, then to confirm them with trials.

With the bottle, the connection is necessarily made from the frequentist approach to the theoretical approach. Given the impossibility of counting the number of possible outcomes and favourable outcomes, trials in line with the frequentist approach must take place before theoretical probability can be calculated. Hence, it is the performance and systematic compilation of a sufficient number of trials that enables a hypothesis on the composition of the bottle (the number of black marbles and white marbles it contains), and thus the associated theoretical probability. This

[6] It is worth noting that the problem, like the PFEQ education program for elementary and junior high school in Quebec, entirely sets aside the subjective approach and focuses exclusively on the two probabilistic approaches.

[7] The choice to use four-sided rather than six-sided dice is based on two main arguments. First, the smaller number of possible outcomes associated with the pair of four-sided dice leads to a faster stabilization of relative frequency for the results obtained through trials. Second, the task of obtaining a certain sum of outcomes with two six-sided dice has frequently been used in textbooks and instructional research, whereas conducting the same task with a pair of four-sided dice is much less frequent. This ensures that the odds of winning associated with the different sums of possible outcomes are not known from the outset.

bridge between the stabilization of relative frequency and theoretical probability can therefore be established by the teacher through a reflection on the law of large numbers[8].

This said, hypotheses about the bottle's composition may be made at different steps of the problem solving process. It is highly likely that with the bottle, hypotheses will arise successively and be refined as the number of trials increases. Hypotheses may therefore be qualitative in nature ("there seem to be more marbles of a given colour") at first, then become quantitative when the level of certainty increases with a greater number of trials ("the bottle might have one or two black marbles," then "the bottle very likely has only one black marble").

These hypotheses on the composition of the bottle are possible because the number of marbles in the bottle is known. Knowing that the bottle contains five marbles, of which at least one is black and one white, it is possible to ascertain four possible compositions (i.e., 1b-4w, 2b-3w, 3b-2w and 4b-1w). From this point, different theoretical probabilities can be associated with these compositions:

- 1 black marble and 4 white marbles make for winning odds of 20%;
- 2 black marbles and 3 white marbles make for winning odds of 40%;
- 3 black marbles and 2 white marbles make for winning odds of 60%;
- 4 black marbles and 1 white marble make for winning odds of 80%.

If the number of marbles in the bottle were not known, this reflection on the bottle's potential composition and the associated theoretical probabilities would not be possible. This information is therefore necessary to enable the shift from the frequentist to the theoretical approach, since not knowing the number of marbles in the bottle precludes any hypothesis on the bottle's composition based on trials conducted with it.

Examination of Feedback Given to Students During Problem Solving

In Astolfi's (1993) view, it is important for a problem's structure to provide students with feedback in order to confirm different strategies for problem solving. However, the problem and its component tasks offer little feedback and few prospects of confirmation, which is directly tied to the non-deterministic nature of probability. No proof can truly confirm students' reasoning, given that each additional trial can increase their relative level of certainty, but fails to deliver an exact and final answer.

Consequently, students must essentially debate and discuss ideas in order to obtain feedback that will help confirm their hypotheses in the context of the problem. Astolfi (1993) suggests in this regard that in general, problem solving should be done in teams, and different moments of group work should be orchestrated via an instructional approach that fosters debate and information exchanges.

[8] This law states that the higher the number of trials, the closer the probability (arising from the trend observed based on the frequency of outcomes) should come to the theoretical probability (Bernouilli,1713, in Borovcnik and Peard 1996).

The teacher can also be a source of feedback and confirmation, depending on the nature of the instructional methods employed with students.

This said, a calculation of theoretical probability can be confirmed, whether by a peer or the teacher. In the case of the pair of dice, which makes it possible to approach the problem from the frequentist approach and from the theoretical approach, confirmation of the calculation of theoretical odds of winning can come from a sufficient number of trials, which shows the stabilization of frequency around the theoretical probability (expected outcome).

Moreover, to confirm the accuracy of hypotheses on the bottle's content during problem solving, a model can be used, as we have observed in a different experiment conducted with a bottle (Martin and Theis 2011). In this experiment, students were put into heterogenous groups to attempt to determine the content of bottles similar to the one used here. The students then put a certain number of black and white pieces of plastic into an open-top container in order to reproduce what they hypothesized to be the content of the bottle. The students used the model to conduct drawings in order to test their different hypotheses on the bottle's content, by comparing the outcomes obtained with the model against those obtained with the bottle. This idea of using a model to test a hypothesis on the bottle's content also featured in an experiment performed by Brousseau et al. (Brousseau et al. 2002).

Discussion on the Use of This Problem in the Classroom

To conclude this text, we would like to go beyond analysis of the problem's probabilistic issues to open a discussion on its use in the classroom, in light of the teacher's challenges in managing it in the heat of the action, as well as its mathematical potential and the favourable learning context it offers for students deemed to be in difficulty in mathematics.

The Challenges of Managing the Problem

When it comes to managing this problem in the classroom, several probability-related elements can come into play in the teacher's decision-making.

First, to be able to make an in-depth connection between the frequentist and theoretical approaches, it is fundamental for the three tasks to be addressed jointly. The bottle demands a connection from the frequentist to the theoretical approach, whereas the problem taken as a whole prompts a comparison of the probabilities stemming from different approaches—neither of which students are used to doing in school.

Establishing this connection in the classroom requires that one take into account certain aspects, such as the number of trials needed to confirm the probability associated with the bottle: What level of certainty is needed? In this case, the teacher will have to deal with the variability of the trial outcomes. If the trial outcomes do not seem to converge towards the theoretical probability, what is the teacher to do? Conclude hastily or erroneously? This would constitute an unacceptable solution. Rather, what is needed is to pursue trials and continue working on the notion of variability, for example with the help of other tools such as a simulator. These different choices lead to different actions and to different kinds of learning in students.

Another element to consider has to do with the potential strategies established by the students, for example using a possibility tree, a double-entry table, a list of possible outcomes, illustrations, bar graphs, pictograms, circular diagrams, etc. Yet the more extensively the teacher introduces (or imposes) problem solving strategies at the beginning of the sequence, the more limited the potential for students' mathematical activity. Allowing students to experiment and to compare among themselves can be an important driver of success in addressing the problem. In this vein, Astolfi (1993) maintains that the problem is designed to allow the student to refine strategies together with peers, and to overcome the conceptual obstacles in line with the development of knowledge itself. In this context, preparing to anticipate and recognize different strategies that students might use does not mean explicitly teaching these strategies from the outset.

The Problem's Mathematical Potential

The problem we have proposed in this text, in our view, has significant mathematical potential, as it constitutes a promising avenue for students (including those deemed to be in difficulty in mathematics) to experience real mathematical activity in the context of learning probability. The particular nature of probability and the place the subject is given in school are specific characteristics of this problem that make it possible to go beyond the beaten paths of traditional instruction to offer a rich learning experience, as they open the way for creative problem solving strategies and reasoning that mark a break from routine mathematics (Martin 2010, 2014). Concretely speaking, problem solving that draws a connection between the frequentist and theoretical approaches supports students' probabilistic thinking.

Certain characteristics of the problem thus give it the potential to nurture real mathematical activity in students. First, the problem calls for simultaneous work on three tasks, thus producing openness to a connection between approaches and the need to compare probabilities using various methods. Second, it leads to different odds of winning that are relatively similar, thus avoiding overly easy or even obvious comparison of the results obtained for each task. Third, the problem demands argumentation in order to express one's view on the task that offers the greatest odds of winning. This is conducive to the development of students' probabilistic thinking via the debate and sharing of ideas needed to solve the problem.

A Favourable Context for Students Deemed to Be in Difficulty in Mathematics

In our view, solving this problem constitutes a favourable learning context. Indeed, this problem, which proves complex and open from a mathematical standpoint, clearly comes under a logic of mathematical teaching-learning through problem solving. When using the bottle task (Martin 2010) or the problem at large (Martin 2014), we have observed surprising mathematical performance by students deemed to be in difficulty in mathematics through the solving of probabilistic tasks. These students have demonstrated a good level of understanding of the mathematical concepts at play. Moreover, they often were able to suggest and carry out task-related strategies that proved mathematically sound and were even recognized as valid by peers and by the teacher when concluding the activity.

For students deemed to be in difficulty in mathematics, probability thus seems to have a much less negative connotation than the other branches of mathematics studied in elementary and in junior high school. This may be because it generally has only a small place in the teaching of mathematics in the classroom. Furthermore, the little time devoted to this mathematical content within the classroom puts all the students on equal footing when faced with probabilistic tasks.

References

Albert, J. (2006). Interpreting probabilities and teaching the subjective viewpoint. In G. F. Burrill & P. C. Elliott (Eds.), *Thinking and reasoning with data and chance* (pp. 417–433). Reston: National Council of Teachers of Mathematics.

Astolfi, J.-P. (1993). Placer les élèves dans une situation-problème? *Probio-Revue, 16*(4), 311–321.

Batanero, C. (2014). Probability teaching and learning. In S. Lerman (Ed.), *Encyclopedia of mathematics education* (pp. 491–496). New York: Springer.

Batanero, C., & Diaz, C. (2012). Training school teachers to teach probability: Reflections and challenges. *Chilean Journal of Statistics, 3*(1), 3–13.

Batanero, C., Arteaga, P., Serrano, L., & Ruiz, B. (2014). Prospective primary school teachers' perception of randomness. In E. J. Chernoff & B. Sriraman (Eds.), *Probabilistic thinking: Presenting plural perspectives* (pp. 345–366). Dordrecht: Springer.

Borovcnik, M., & Peard, R. (1996). Probability. In A. J. Bishop, K. Clements, C. Keitel, & J. Laborde (Eds.), *International handbook of mathematics education* (pp. 239–287). Dordrecht: Kluwer Academic Publishing.

Borovcnik, M., & Kapadia, R. (2016). Reasoning with risk: A survival guide. In *Proceedings of the thirteenth international congress on mathematical education*, Hamburg, Germany.

Briand, J. (2005). Une expérience statistique et une première approche des lois du hasard au lycée par une confrontation avec une machine simple. *Recherches en didactique des mathématiques, 25*(2), 247–282.

Briand, J. (2007). La place de l'expérience dans la construction des mathématiques en classe. *Petit x, 75*, 7–33.

Brousseau, G., Brousseau, N., & Warfield, V. (2002). An experiment on the teaching of statistics and probability. *Journal of Mathematical Behavior, 20*(3), 363–411.

Caron, F. (2002). Splendeurs et misères de l'enseignement des probabilités au primaire. In *Proceedings of the « Colloque annuel du Groupe de didactique des mathématiques du Québec »*, Trois-Rivières, Québec.

Chernoff, E. J., & Sriraman, B. (Eds.). (2014). *Probabilistic thinking: Presenting plural perspectives*. Dordrecht: Springer Science+Business.

Fischbein, E., & Schnarch, D. (1997). The evolution with age of probabilistic intuitively based misconceptions. *Journal for Research in Mathematics Education, 28*(1), 96–105.

Francisco, J. M., & Maher, C. A. (2005). Conditions for promoting reasoning in problem solving: Insights from a longitudinal study. *Journal of Mathematical Behavior, 24*(3), 361–372.

Gattuso, L., & Vermette, S. (2013). L'enseignement de statistique et probabilités au Canada et en Italie. *Statistique et Enseignement, 4*(1), 107–129 Retrieved from http://publications-sfds.math.cnrs.fr/ojs/index.php/StatEns/article/view/141.

Gouvernement du Québec. (2006). *Programme de formation de l'école québécoise. Éducation préscolaire et enseignement primaire*. Québec: Ministère de l'Éducation, du Loisir et du Sport du Québec.

Jones, G. A., & Thornton, C. A. (2005). An overview of research into the teaching and learning of probability. In G. A. Jones (Ed.), *Exploring probability in school: Challenges for teaching and learning* (pp. 65–92). New York: Springer.

Jones, G. A., Langrall, C. W., & Mooney, E. S. (2007). Research in probability. Responding to classroom realities. In F. K. Lester (Ed.), *Second handbook of research on mathematics teaching and learning* (pp. 909–956). Reston: National Council of Teachers of Mathematics.

Kahneman, D., Slovic, P., & Tversky, A. (1982). *Judgment under uncertainty: Heuristics and biases*. Cambridge: Cambridge University Press.

Lajoie, C., & Bednarz, N. (2012). Évolution de la résolution de problèmes en enseignement des mathématiques au Québec: Un parcours sur cent ans des programmes et documents pédagogiques. *Canadian Journal of Science, Mathematics and Technology Education, 12*(2), 178–213.

Lajoie, C., & Bednarz, N. (2014a). La notion de situation-problème en mathématiques au début du XXIe siècle au Québec: rupture ou continuité? *Canadian Journal of Science, Mathematics and Technology Education, 16*(1), 1–27.

Lajoie, C., & Bednarz, N. (2014b). La résolution de problèmes en mathématiques au Québec: évolution des rôles assignés par les programmes et des conseils donnés au enseignants. *Éducation et Francophonie, XLII*(2), 7–23.

Martin, V. (2010). *Quand rien n'est sûr, tout est possible: l'apprentissage des probabilités chez des élèves à risque*. Montréal: Éditions Bande didactique.

Martin, V. (2014). *Étude des interventions didactiques dans l'enseignement des probabilités auprès d'élèves jugés ou non en difficulté en mathématiques en classes ordinaires du primaire*. Doctoral thesis. Sherbrooke: Université de Sherbrooke. Retrieved from http://savoirs.usherbrooke.ca/bitstream/handle/11143/5449/Martin_Vincent_PhD_2014.pdf?sequence=4

Martin, V., & Theis, L. (2011). La résolution d'une situation-problème probabiliste en équipe hétérogène: le cas d'une élève à risque du primaire. *Nouveaux Cahiers de la Recherche en Éducation, 14*(1), 49–70.

Martin, V., & Theis, L. (2016). L'articulation des perspectives fréquentielle et théorique dans l'enseignement des probabilités: Regard sur un changement de posture chez un enseignant du primaire. *Canadian Journal of Science, Mathematics and Technology Education, 16*(4), 1–14.

Martin, V., & Thibault, M. (2016). Regards québécois sur sept décennies de recherche liée à l'apprentissage et à l'enseignement des probabilités. *Annales de Didactique et de Sciences Cognitives, 21*, 79–115.

Mary, C., & Theis, L. (2007). Les élèves à risque dans des situations problèmes statistiques: stratégies de résolution et obstacles cognitifs. *Revue des Sciences de l'Éducation, 33*(3), 579–599.

Nesher, P., Hershkovitz, S., & Novotna, J. (2003). Situation model, text base and what else? Factors affecting problem solving. *Education Studies in Mathematics, 52*(2), 151–176.

Nilsson, P. (2014). Experimentation in probability teaching and learning. In E. J. Chernoff & B. Sriraman (Eds.), *Probabilistic thinking: Presenting plural perspectives* (pp. 509–532). Dordrecht: Springer.

Nilsson, P., & Eckert, A. (2016). Interactive experimentation in probability – Opportunities, challenges and needs of research. In *Proceedings of the thirteenth International Congress on Mathematical Education*, Hamburg, Germany.

Pratt, D. (1998). The coordination of meanings for randomness. *For the Learning of Mathematics, 18*(3), 2–11.

Prodromou, T. (2012). Connecting experimental probability and theoretical probability. *ZDM: The International Journal on Mathematics Education, 44*(7), 855–868.

Rioux, M. (2012). *Évolution des projets de formation de futurs enseignants au primaire au contact de situations probabilistes*. Doctoral thesis. Université de Montréal: Montréal. Retrieved from https://papyrus.bib.umontreal.ca/xmlui/bitstream/handle/1866/9000/Rioux_Miranda_2012_these.pdf?sequence=4&isAllowed=y

Savard, A. (2008). *Le développement d'une pensée critique envers les jeux de hasard et d'argent par l'enseignement des probabilités à l'école primaire: Vers une prise de décision*. Doctoral thesis. Université Laval: Laval, Québec. Retrieved from http://www.theses.ulaval.ca/2008/25308/25308.pdf

Savard, A. (2014), Developing probabilistic thinking: What about people's conceptions? In E. J. Chernoff & B. Sriraman (Eds.), *Probabilistic thinking: Presenting plural perspectives* (pp. 283–297). Dordrecht: Springer Science+Business.

Savard, A., & DeBlois, L. (2005). Un cadre théorique pour éclairer l'apprentissage des probabilités à l'école primaire: Vers une prise de décision à l'égard des jeux de hasard et d'argent. In *Proceedings of the « Colloque annuel du Groupe de didactique des mathématiques du Québec »*, Montréal, Québec.

Scheaffer, R. L. (2006). Statistics and mathematics: On making a happy marriage. In G. F. Burrill & P. C. Elliott (Eds.), *Thinking and reasoning with data and chance* (pp. 309–321). Reston: National Council of Teachers of Mathematics.

Schoenfeld, A. H. (1985). *Mathematical problem solving*. Orlando: Academic Press.

Shaughnessy, J. M. (1992). Research in probability and statistics: Reflections and direction. In D. A. Grouws (Ed.), *Handbook of research on mathematics teaching and learning* (pp. 465–494). New York: National Council of Teachers of Mathematics.

Steinbring, H. (1991). The theoretical nature of probability in the classroom. In R. Kapadia & M. Borovcnik (Eds.), Chance encounters: Probability in education (pp. 135–168). Amsterdam: Kluwer.

Stohl, H. (2005). Probability in teacher education and development. In G. A. Jones (Ed.), *Exploring probability in school: Challenges for teaching and learning* (pp. 345–366). New York: Springer.

Theis, L., & Gagnon, N. (2013). *L'apprentissage à travers des situations-problèmes mathématiques : Bases théoriques et réalisation pratique*. Québec City: Presses de l'Université du Québec.

Törner, G., Schoenfeld, A. H., & Reiss, K. M. (2007). Problem solving around the world: Summing up the state of the art. *ZDM, 39*(5), 353–353.

Western and Northern Canadian Protocol. (2006). *The common curriculum framework for K-9 mathematics*. Edmonton: Alberta Education.

Additional Suggestions for Further Reading

Martin, V., & Mai Huy, K. (2015). Une réflexion didactique sur des activités pour penser l'enseignement-apprentissage des probabilités et des statistiques à l'école primaire. *Bulletin de l'AMQ, 55*(3), 50–67.

Martin, V., & Theis, L. (2016). A study of the teaching of probability to students judged or not with learning difficulties in mathematics in regular elementary classes in Québec. In *Proceedings of the thirteenth International Congress on Mathematical Education*, Hamburg, Germany.

Bottles and Bridges: Sample Classroom Tasks Created by Beginning Teachers

Ibrahim Abu Atiya, Natasha Luca, and Ann Kajander

Abstract Task design forms an important aspect of the shift in a teacher's practice. In our classroom observations, we have witnessed teachers attempting problem-based tasks with students, when neither teachers nor students have had much prior experience with this type of learning. If students have not had opportunities to develop the necessary process skills to support independent problem solving, such rapid transitions may cause frustration for both students and their teachers.

As an alternative, teachers may choose to shift their practice more gradually. This allows the students to be supported as they begin to develop the capacity for more and more independent and less scaffolded mathematical work. Assessment methods need to shift along with the shift in learning environments.

This chapter provides background as well as sample classroom tasks which may provide a starting point, or middle ground, for such transitions in practice. Designed mainly by new teachers, the tasks include both context and data related to students' environment. In the first task, students actually collect the data, and in both cases students have some choice in how they present the data. Nevertheless some structure is provided—the tasks are not fully open. We have found such tasks to be effective starting points for the ongoing transition to more autonomous learning.

Keywords Secondary mathematics · Mathematics tasks · Pre-service mathematics education · Task development · Assessment

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Beginning to choose, create and use inquiry-based problem tasks in teaching may initially seem daunting for new teachers, especially those who were themselves schooled in a traditional paradigm. The transition from direct instruction to the use of problem solving tasks remains challenging for many teachers (Holm & Kajander, 2015). While other chapters in this book describe helpful strategies in making this shift, this chapter focuses on the experience by describing a specific task created by pre-service teachers. It serves as a sample of what can be achieved even by less experienced teachers.

Classroom Vignette

Consider the following scenario, drawn from previous research (Kajander et al. 2008). A young enthusiastic teacher of a Grade 9 Applied[1] (the name of the "non-academic" mathematics course stream in Ontario, for non-STEM students) class decides early on in the course to use a problem task found on some of her Ministry of Education sample lesson plans. So far this year, this class has only been taught using traditional, teacher directed lessons, which involve extended formal teacher presentations of as long as 40 min. On a given day, with no transitional experiences for students, an exploratory geometry task using Geoboards is given to the class. The task is an open-ended investigative problem involving determining areas of various shapes that students are to solve in a hands-on manner using a strategy of their choice.

What do you think happened in this scenario? Do you think the students began exploring with excitement? In fact, the lesson was not at all what the teacher had hoped for; the students claimed they had "no idea what to do," and soon the elastics from the Geoboards began flying around the room. The teacher swore she would never try this type of lesson ever again as "they don't work with these kids." The implication was that problem solving only "works" with more academically-able students.

Making the Transition

If we reflect on the above scenario, it might be obvious that the students were correct … they *did not* know what to do. This was likely not a flaw of the task at all, rather the students likely did not have experience with the kinds of skills needed to work effectively in problem-based environments, nor did the teacher provide this support. One important aspect of moving from traditional to more inquiry-based or problem-based learning environments is that students must be given the

[1] See McDougall and Ferguson (Part II this volume, para. 1) for a discussion of two of the possible Ontario pathways (Academic and Applied).

opportunity—with support—to develop the new kinds of skills required—the skills related to *problem solving*. And yes, teachers themselves may need the opportunity to develop these skills as part of supporting their students' development.

For example, students need opportunities to explicitly develop various strategies for solving new problems, such as trial and error, using a model, trying a simpler problem, working backwards, and discussing their ideas in small groups. One way to support this evolution might be to *gradually* make classroom tasks more and more open, as students develop more strategies. For example, while task instructions used early on might suggest to students that they make a chart and graph their data in order to determine what is happening, tasks used later might leave the approaches and representations much more open to students' own decisions.

Another important requirement is to find or create the *right* problems. 'Right' in this case means interesting and motivating to the students in question, and appropriate to their learning and *problem solving* capacity at the given time. Contexts should be those within the students' experiences—I remember once using a textbook problem about water levels and the locks in a canal, and being met with blank stares from my Northwestern Ontario students … locks were on door knobs, not waterways, in their world views! Just enough teacher support to keep students from getting frustrated (but no more than absolutely necessary) is helpful (see other chapters on the transition to problem solving such as Liljedahl, Part IV, this volume).

Introduction to Types of Tasks

The sample tasks provided here are examples of problems that were within the comfort level of the less experienced teachers who designed them. Thus, while the tasks strive to provide contexts that might be engaging for students, and include 'real world' elements, the actual tasks are relatively structured. This may make them less daunting entry points for both new teachers and students new to problem-based learning. The idea is that, rather than giving up on problem solving all together as did the teacher described earlier, teachers develop gradually, along with their students, as they continue to work towards fully implementing problem solving in their own classroom contexts. The shift can happen gradually, by continuing to make the tasks more and more open with expanded student choices, as students develop the necessary process skills.

As will be explained to follow, the first task may be considered an example of a *Learning Task*, and the second task as an *Assessment Task*. An important question in the transition to problem-based learning is assessment. A traditional test often does a poor job of assessing problem solving, yet assessing performance on a task, without students having opportunities to develop the kinds of problem solving abilities (such as representing or reasoning) to be assessed, may be unfair. A practical suggestion which we have found helpful in supporting such transitions with students, particularly with reluctant learners, is the idea of using a learning task/assessment task pair. Students have the opportunity to develop some of the required problem solving skills on a learning task (with teacher support and substantive feedback), and then demonstrate such development in a follow up task, which is in turn graded with a rubric.

Practical Considerations

In my (Ann's) own teaching in regions and classrooms with significant attendance issues, I have found that giving students tasks written down on problem-sheets (or available online) can be helpful. This means that if necessary, students can be working on the tasks at different times and paces. In cases with a great deal of variance, using different coloured paper to print the tasks may be a very quick way to determine which students are working on what tasks when.

The idea of learning tasks/assessment tasks is to create and use two different problem tasks which share the same curriculum expectations or topics. Both can be problem-based tasks, but they are used for slightly different purposes. The *learning task* is provided first so that students can work on a task with support from their peers and the teacher as necessary. Once completed, students can submit their work to the teacher for detailed feedback (not grades). Good descriptive feedback can help the students know what they did well, and what they might work on next to improve. Students should receive this substantive feedback *before* continuing with the next task. (A few skills-based or "textbook" questions might fit well here if a day is needed by the teacher to write this feedback—but it is critically important that the feedback be given to the students in a very timely manner and before they move to the next task.) The *assessment task* (sometimes called a performance task) looks like a different task to students in that it might involve a different problem context. However, the underlying content expectations are either the same as, or a subset of, those used on the learning task. This second task then gives students the opportunity to make use of the teacher's feedback to demonstrate improvement. The assessment task can then be graded for marks (along with providing further feedback of course). This type of lesson structure may be particularly helpful for students who have had less experience working in a problem-based environment—the learning task helps students build appropriate problem solving strategies in a collaborative environment.

If tasks are to be drawn from outside sources, it is important to revise them for your current student audience. In our experience, good tasks rely on a brainstorm combination of answers to "what excites and interests my students?" and "what are the important or overall curriculum goals I am trying to develop and assess with my students?" (see Holm, Part V, this volume, on backwards design for more on this topic). Once some ideas are at hand, the next step is to decide which will be the learning task, and which the assessment task. As a rule of thumb, since the learning task is not graded, it may be easier to use a group task, or the one with the most potential for measurement errors or different solutions, as the learning task. For example, students might collect their own data in the learning task, which means that there may be both measurement errors as well as varying solutions. For the assessment task however, grading may be streamlined if students are given data (or directed to a website from which to draw the data). If the data varies from student to student, then the teacher is faced with re-solving the problem with new data for each student task to be graded.

Sample Classroom Tasks

The following examples were designed initially as problems for a Grade 10 Academic (mathematics/science university stream, in Ontario, for students often referred to as STEM students) class. However, they might also serve as application-type tasks (meaning the students may have seen the methods or techniques before but are now applying them to new contexts) for a Grade 11 Functions course. As well, they could be modified for use in other courses, including non-academic courses.

The proposed tasks relate directly to a number of Expectations in the Ontario grade 9/10 curriculum (Ontario Ministry of Education 2005), for example,

- collect data that can be represented as a quadratic relation, from experiments using appropriate equipment and technology (e.g., concrete materials, scientific probes, graphing calculators), or from secondary sources (e.g., the Internet, Statistics Canada); graph the data and draw a curve of best fit, if appropriate, with or without the use of technology (**Sample problem**: Make a 1 m ramp that makes a 15° angle with the floor. Place a can 30 cm up the ramp. Record the time it takes for the can to roll to the bottom. Repeat by placing the can 40 cm, 50 cm, and 60 cm up the ramp, and so on. Graph the data and draw the curve of best fit.); (p. 48)
- solve problems arising from a realistic situation represented by a graph or an equation of a quadratic relation, with and without the use of technology (e.g., given the graph or the equation of a quadratic relation representing the height of a ball over elapsed time, answer questions such as the following: What is the maximum height of the ball? After what length of time will the ball hit the ground? Over what time interval is the height of the ball greater than 3 m?). (p. 49)

However, in some Western provinces, outcomes are less prescriptive as to the method in which quadratics are to be learned. For example, the Grade 11 Foundations of Mathematics curriculum in British Columbia (Western and Northern Canadian Protocol 2008) states that students are to “Solve a contextual problem that involves the characteristics of a quadratic function” (p. 64), while the Alberta outcomes refer generally to quadratics in standard form rather than the collection of primary data generating them (Alberta Education 2008). However the front matter in the grade 10–12 Program of Study does state that “problem solving is to be employed throughout all of mathematics and should be embedded throughout all the topics” (Alberta Education 2008, p. 6).

Implementing the Tasks

It is important, when designing or selecting student tasks, that teachers try the tasks themselves in advance. This allows the tasks to be tailored to students in the group, as well as alerting teachers to possible questions students may have, as well as how long the tasks are likely to take to complete. It is interesting to note just how many curriculum expectations can in fact be addressed with tasks such as these, and often teachers actually trying out the tasks themselves makes this even more evident, as teachers find themselves using more of the curriculum than expected.

The learning task provided here as an example was initially co-created by new teachers, as part of a course assignment in a teacher education program, and illustrate what is possible even for less experienced teachers in terms of task design. As you, as a teacher, progress in experience supporting problem-based learning, you can work towards designing tasks which are more and more 'open,' meaning that students are given more and more responsibility and choice in their work. It might be noted also that whether a task is more of a problem solving task (meaning the students do not have a known method or formula at hand to solve the problem and for which finding a method is what the task is all about), or an application task (meaning the students have seen appropriate methods before but now are being asked to combine or apply the methods in new ways or contexts) depends not just on the task itself, but on the students' prior knowledge (e.g., Alberta Education 2008). The tasks to follow were initially designed as 'problems' for a Grade 10 Academic course; however, if they were used in a grade 11 course where students had more experience with quadratic functions, they might be classed as 'applications'. The second provided task might be used as an assessment task, following the first task used as a learning task.

The assessment task was created for students in a particular location, using information familiar to them, and would be best if it was modified to the local area in question. Indeed the connection to the land and its geography may be particularly important to some students, particularly Indigenous students. A sample rubric is shown with the assessment task, and one should always be provided to students along with the task. It might of course be necessary to modify the rubric based on the grade level and type of course.

Take a moment to read the learning task provided at the end of this article (Appendix 1). Ideally, try it out! Predict the type of function that will result. We are carefully omitting to provide the chart of the data collected when this task was tested, as it may be tempting to (mis)use the task by actually providing the data to the students. One important point of the task is to have students collect and observe the data firsthand. However, a sample graph drawn from our test of the learning task is provided to follow, for discussion purposes; note that the x-axis represents time in seconds, and the y-axis is volume in mLs. The task itself relates to water remaining in a punctured pop bottle; it is suggested to read (and ideally try out) the task provided in Appendix 1 of this chapter before continuing on to look at the sample graph provided after the task.

In examining the graph (your own or the provided one), you may be surprised at how little variance there is of the data points from the curve of best fit. So (as teachers), we have evidence from trying the task out that the task and resultant data will be clear enough for students to observe the mathematical phenomena independently. This is an important feature of a successful classroom task.

When using a new task with students, even experienced teachers should not expect the task to be perfect. Several iterations of a particular task (or its rubric) may be needed before it really works well; this is to be expected.

To follow are the sample tasks and a suggested rubric. Readers are encouraged to try the tasks, imagining how they might use, modify or improve them in their own practice.

Appendices

Appendix 1: Learning Task

Pop Bottle Water Flow Problem

Your pop bottle was accidentally pierced by a nail when you set your grocery bag down on the ground. Oh no! There goes your pop. You wonder if you will have time to run home and empty the rest of the pop still in the bottle into a new container before it's all gone.

Your task here is to determine how fast a 2 L bottle will drain.

To simulate this situation, you will be doing an investigation of how quickly water flows from a small hole near the bottom of a 2 L pop bottle. You will work in pairs to perform the experiment.

You will need the following *materials*:

- 2, 2 L pop bottles per pair
- Graduated cylinder or measuring cup (one per pair, 100 mL minimum)
- Masking tape and several multi-coloured sharpies (permanent)
- Funnel or cone shaped paper cup with the end cut off (one per pair)
- Large container (one per pair) such as a bucket, and access to water
- Calculator and stop watch—or cell phones! (one per pair)
- Utility hook to safely pierce the bottle (one per pair) (Fig. 1)

Procedure

(A) Label the volume for every 100 mL of liquid in the pop bottle. The steps below provide one possible way to do this.

1. Start with taking a strip of masking tape that extends the entire length of the 2 L pop bottle
2. Using the graduated cylinder, measure 100 mL of water

Fig. 1 Materials

3. Pour the water into the pop bottle, then mark the water level on the masking tape
4. Label the 100 mL on the tape
5. Repeat steps 2–4 for each new 100 ml until you reach 2000 mL
6. Pour the water in the other (unmarked) pop bottle using your funnel
7. Make a hole on the (labelled) bottle using the utility hook. Your hole should be around the 100 mL mark
8. Cover your hole with a piece of masking tape
9. Pour the water back into the (labelled) bottle with the covered hole, making sure the water level is at the 2000 mL mark (Fig. 2)

(B) Next you will measure the flow of the water. Here is one possible way to do this:

1. When ready to begin, pull the masking tape from the hole, allowing the water to drain into the bucket or a sink, (this experiment could also be done outside) and begin the timing
2. After 20 s one of the team members will signal the end of that time interval, and the other member will mark the water height on the masking tape
3. Continue until the water level reaches the level of the drain hole and thus the draining stops

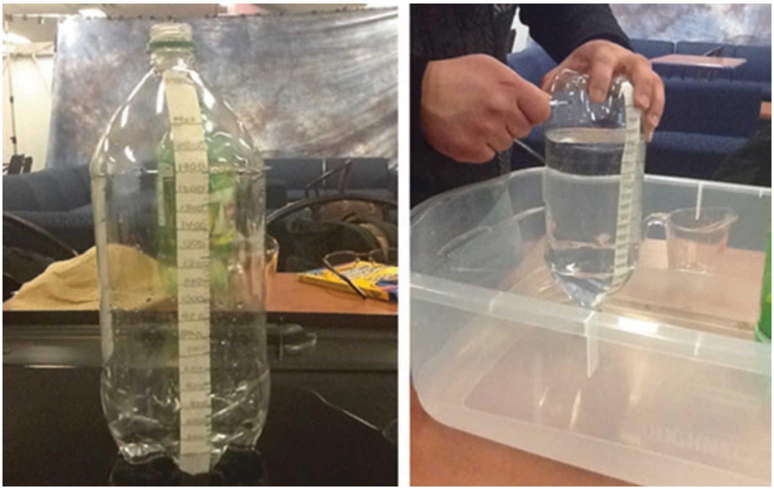

Fig. 2 Measuring the water volume

Observations and Predictions

Describe in words what you observed. For example, did the water appear to flow out at the same rate? When was it the strongest? How does this make sense?

Create a table showing the volume left in the bottle at each 20 s time interval.

Data Analysis

Graph your data. (Discuss in your group which variable to use for which axis of the graph and why). Hand sketch a curve of best fit to the data points.

Use the graph you created, make as many observations about the situation as possible. Here are a few questions you might consider (among others):

- What is the initial point you graphed? What does this represent?
- Is the graph a straight line or a curve? How does this make sense in the context?
- Is the rate of flow the same or different early on in the timing compared to later? Does this agree with your observation of the water flow? How does this relate to the shape of the graph? Why might this be the case?
- What happens when the volume remaining above the hole is very small? What is the slope of the graph as it nears this point? Why does this make sense?
- The slope of the graph represents how fast the resultant or 'dependent' (y-value) variable is changing. Discuss how the slope of your graph relates to the current context and how this makes sense.
- Use technology to find the equation of your graph. If you now graphed this new function (the one provided by the technology) would you get your exact graph? Why or why not? Does the graph of the function provided by the calculator extend beyond the graph you have? If so, what might you do to be sure the function models your graph properly? (hint: think domain and range, i.e. the regions on the graph to which the experiment applies)

Partial Solution to Learning Task

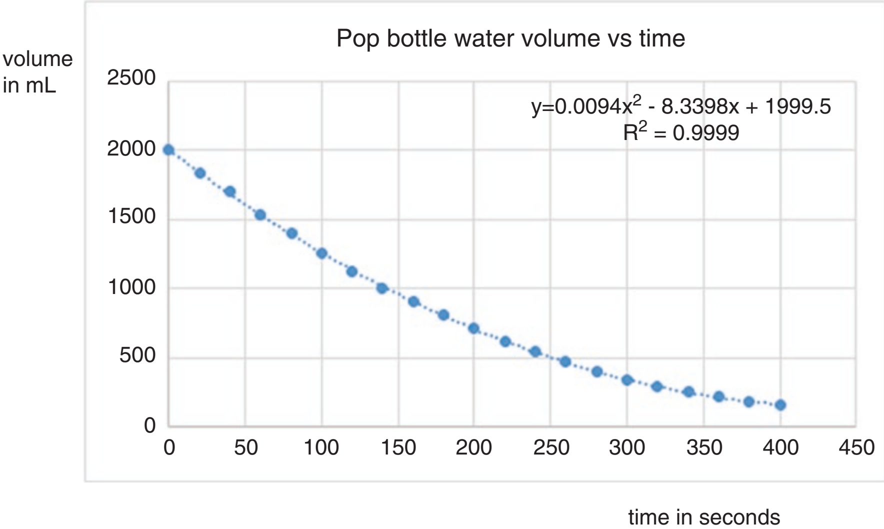

Fig. 3 Sample solution graph

- What point makes sense as the 'minimum' value of the graph?
- Use technology to identify the approximate vertex of the graph. How does the vertex relate to the minimum value?
- Use technology to change the format of the function to 'vertex form.' Explore the values given in this provided function: why might this form be called 'vertex form'? What does this algebraic form provide?
- Calculate the y-intercept using the function. How does this agree with the context of the pop bottle? Is the value what you might expect, and why or why not?
- The axis of symmetry of this type of graph is the vertical line through the vertex. Where would that be on your graph?

  And the conclusion:

- How long would you have to find another container if you wanted to be sure you had at least half of your pop left after the leak started?

Your Conclusions
Write a paragraph describing the overall results of the experiment and what you learned about modelling real world phenomena. What does the data and the mathematical model you found tell you about the rate at which the water flows from the bottle? What other contexts might have similar types of mathematical models?

Challenge Extension
If time is available, explore algebraic techniques for moving back and forth between standard form (the form likely provided by the technology) and vertex form (Fig. 3).

Appendix 2: Assessment Task

Rope Bridge Problem

Your little brother Ray has just started high school and is a bit shy. Ray would like to ask a classmate out who really wants to go to a new park, with a picturesque canyon. The canyon has a suspended rope bridge across it. The only problem is, Ray is a bit afraid of heights and after being on a rope bridge previously on a family vacation, he felt dizzy after 1 min. Ray has gone out to the park and was able to read the first three signs on the bridge, showing the horizontal distance across the gorge and the height of the bridge at that point. He couldn't see any farther without going across. Can you help Ray figure out the horizontal distance across the gorge, using your mathematics skills? Is it likely that he and his friend will make it across the bridge within 1 min at a leisurely walk before Ray feels faint? Will there be any time to stop on the bridge?

Here are the first three signs Ray is able to see, with the values showing horizontal distance, and height from the bottom of the gorge, respectively (Fig. 4).

Analyse this situation in as much detail as you can, using your knowledge of quadratic relations, since the bridge floor (the curve on which the height and width are noted) seems to be a fairly flat parabola. Note that the signs refer to the height of the floor of the bridge from the baseline as marked. Use technology as appropriate. Since you have decided to show your hard work to your math teacher, be sure to include as much mathematical detail, such as a chart, graph, equation of the bridge floor, and all of your mathematical reasoning, as possible. You may need to make some assumptions or simplifications along the way—be sure to state these. Also, find out for Ray what the lowest point of the bridge is before it starts to slope upwards again.

Make a scale model of the bridge using string, push pins, cardboard and markers. Label all relevant information.

What will you advise Ray?

Partial Solution

A quadratic regression calculator should give a = 10; b = −0.1683333…; and c = 0.0072222…. These can be simplified to fractions (Table 1).

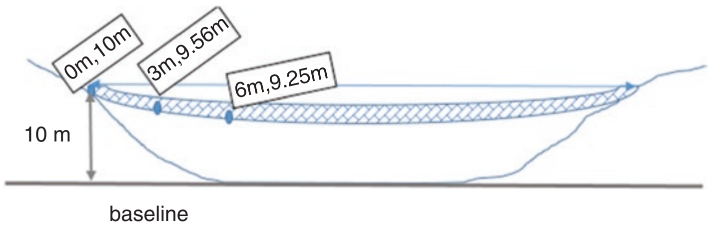

Fig. 4 Diagram of the gorge

Table 1 Sample rubric based on Ontario curriculum (Ontario Ministry of Education 2005)

Sample Rubric				
Categories	Level 4 (80–100%)	Level 3 (70–79%)	Level 2 (60–69%)	Level 1(50–59%)
Knowledge	Graphs the quadratic equation by hand or using technology with a high degree of accuracy	Graphs the quadratic equation by hand or using technology with a considerable degree of accuracy	Graphs the quadratic equation by hand or using technology with some accuracy	Graph of function is estimated or only partially accurate
	Able to calculate key features of the quadratic equation with no errors	Able to calculate key features of the quadratic equation with few errors	Able to calculate some key features of the quadratic equation with some omission or errors	A few key features of graph are provided
Thinking	Uses the function effectively to determine the key points such as vertex and width with a high degree of efficiency	Uses the function to determine the key points such as vertex and width with a considerable degree of efficiency	Uses the function or a chart to determine or estimate some key points such as vertex and width	Key points estimated from graph
	Estimates walking speed highly accurately and is able to relate it to the problem	Estimates walking speed with considerable accuracy and is able to relate it to the problem	Estimates walking speed with some accuracy and is able to relate it to the problem	Estimates walking speed less realistically and is able to relate it somewhat to the problem
Communication	Clearly explains all process and reasoning with a sophisticated explanation	Clearly explains all process and reasoning with a good explanation	Some explanation of process and reasoning provided	Explanation of process and reasoning needs improvement
	Thorough and detailed final recommendation included	Final recommendation included with considerable accuracy	Final recommendation included but has some flaws or omissions	Final recommendation omitted or inaccurate
Application	Accurately connects a realistic estimated walking speed to the context in order to solve the problem	Mostly accurately connects a realistic estimated walking speed to the context in order to solve the problem	Somewhat accurately connects a realistic estimated walking speed to the context in order to solve the problem	Estimated walking speed is inaccurate and only partly connected to the context in order to solve the problem
	Model of bridge is complete, accurate, and to scale with all important aspects clearly labelled	Model of bridge is mostly complete, accurate, and to scale with most important aspects labelled	Model of bridge is somewhat complete, accurate, and to scale with some important aspects labelled	Model of bridge is missing, incomplete, or may have flaws

References

Alberta Education. (2008). *Mathematics grades 10–12*. Retrieved from https://education.alberta.ca/media/564028/math10to12.pdf

Holm, J., & Kajander, A. (2015). Lessons learned about effective professional development: Two contrasting case studies. *International Journal of Education in Mathematics, Science and Technology, 3*(4), 262–274.

Kajander, A., Zuke, C., & Walton, G. (2008). Teaching unheard voices: Students at risk in mathematics. *Canadian Journal of Education, 31*(4), 1039–1064.

Ontario Ministry of Education. (2005). *The Ontario curriculum grades 9/10: Mathematics* (Rev. ed.). Toronto: Queen's Printer. Retrieved from http://www.edu.gov.on.ca/eng/curriculum/secondary/math910curr.pdf

Western and Northern Canadian Protocol (WNCP). (2008). *The common curriculum framework for grades 10–12 mathematics*. Edmonton: Alberta Education. Retrieved from https://www.bced.gov.bc.ca/irp/pdfs/mathematics/WNCPmath1012/2008math1012wncp_ccf.pdf.

Promoting Students' Reasoning About Statistical Inference Through Engagement with a Problem-Based Instructional Activity Involving the Use of *TinkerPlots* Software

Luis Saldanha and Mathieu Thibault

Abstract Our chapter describes the use of a problem-based sequence of instructional activities designed to provoke the emergence of high school students' reasoning about statistical inference—the web of sophisticated ideas entailed in drawing conclusions about a population with confidence based on information obtained from samples randomly drawn from that population. We illustrate the activity's potential for occasioning such reasoning with highlights from students' thinking and classroom discussions that emerged among a group of grade 9 students as they engaged with the activity, which revolved around the use of a dynamic and interactive data exploration software called *TinkerPlots*.

Keywords Statistics · Informal inference · Problem-based learning · students' reasoning · Dynamic software · Random sampling · Activity sequence · Data analysis

Introduction

Over the last three decades statistical topics and concepts have increasingly made their way into school mathematics curricula around the world (Gattuso and Vermette 2013). In North America this is evidenced by the specification of entire curricular strands and associated learning objectives devoted to statistics and probability in professional resource documents (Gouvernement du Québec 2016; National Council of Teachers of Mathematics 2000). These documents, as well as various textbook series aligned with them, evidence an increased focus on having students work with and appreciate the role of data and its analysis as part of the activity of doing statistics and engaging in statistical thinking. However typical statistics instruction in schools still relies heavily on textbook activities that present data in static forms, in contexts that focus on having students apply formulas (e.g.,

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calculating means), and that tend to convey concepts as isolated "topics" and "things to do" rather than certain ways of thinking that can help one understand and solve a problem. In our view, statistical thinking can more easily be engendered in contexts that move beyond the above-mentioned narrow aspects to present data in dynamic contexts, involving the use of simulation and software tools now widely available, to engage students in ways that promote thinking aligned with the logic of statistical inquiry.

One particular area of statistical thinking that is both accessible and important to the education of high school students is that of making data-based conclusions about a population on the basis of samples randomly drawn from it—that is the thinking entailed in making statistical inferences. Indeed, a sub-area of statistics learning and instruction generally referred to as *informal inference* (Ben-Zvi and Garfield 2004) has recently received much attention, and has benefitted from notable instructional innovations and resources emanating from the international community of statistics education researchers. Informal inferential reasoning refers to "the cognitive activities involved in informally drawing conclusions or making predictions about 'some wider universe' from data patterns, data representations, statistical measures and models, while attending to the strength and limitations of the drawn conclusions (Ben-Zvi et al. 2007)" (Garfield and Ben-Zvi 2008, p. 268). Other authors (Pfannkuch 2008; Wild and Pfannkuch 1999) have added that informal inferential reasoning involves drawing interconnections among ideas of distribution and measures of center and variability, all within a cycle of activity involving inquiry into data in context. Whereas the sophisticated machinery of formal statistical procedures and tests usually taught at the college level are thought to be inappropriate for high school students, the qualifier *informal* in the above nomenclature signals the idea that the *reasoning* and *ways of thinking* that underlie such formal procedures are both accessible to students at the high school level and worthy of learning in their own right. Beyond that, such reasoning and ways of thinking have the potential to provide a conceptual basis for understanding more formal procedures later in students' educational trajectories. Against this backdrop, and in the interest of highlighting the importance of inferential reasoning for high school students, this chapter offers an example and a vision for the teaching and learning of statistical inference that we see as a rich and viable alternative to typical approaches still evident in many school textbooks and classrooms.

Instructional Activity Sequence

An Overview

Our example consists of a sequence of instructional activities enacted within an intact grade 9 mathematics class (involving 14 and 15 year-old students) at a suburban high school in the southwestern United States. The activity sequence (adapted

A fish farmer stocked a pond with a new type of genetically engineered fish. The company that supplied the new type claims that these fish will grow to be longer than normal fish. The farmer decided to test the company's claim by stocking the pond with 625 fish of the same species, some normal and some genetically engineered. (When the fish were fully grown the farmer randomly selected a sample of fish from the pond and measured the length of each fish in his sample).

Fig. 1 The contextual situation for the instructional sequence

from Key Curriculum Press 2012)[1] revolved around the contextual situation presented in Fig. 1, involving the use of sampling to test a claim that a genetically modified version of a species of fish in a population tends to grow longer that the "normal" version of the species.

The broad aim of the instructional sequence was to provide students with occasion to investigate and develop the idea of making an inference from a random sample to a population. More specifically, the sequence emphasized the variability amongst a statistic calculated for samples of a common size chosen from a particular population, and it culminated in a comparison of the variability across distributions of that sample statistic generated from repeating the selection of such samples of different sizes. The investigation emphasized the following two big statistical ideas: (a) random sampling can be used to draw conclusions about a sampled population (the whole of which is generally inaccessible), and (b) sample statistics computed for larger samples have distributions that tend to vary less than distributions of the same statistic computed for smaller samples. These ideas provided cross-cutting themes for the investigation that was structured so as to engage students in a cycle of inquiry involving six interconnected levels of activity and thinking representing an elaboration of the fundamental stages of the investigative process of a statistical study (Franklin et al. 2005; NCTM 2000; Wild and Pfannkuch 1999): (1) posing a question; (2) generating data through sampling to inform the question; (3) representing and organizing the data graphically, and constructing a relevant sample statistic; (4) analyzing and interpreting the represented data and statistic; (5) repeating the underlying sampling process and thinking of sampling variability as a central consideration; (6) drawing conclusions about a population on the basis of the previous levels and the reasoning that emerged within them.

The investigation unfolded over an activity sequence comprised of three 60–75 min lessons.[2] The use of *TinkerPlots*—an interactive and dynamic data exploration software (Konold and Miller 2017)—played a central role in the

[1] Adapted from the activity entitled Fish-Length Distributions, downloaded from http://www.tinkerplots.com/activities/data-analysis-and-modeling-activities

[2] These lessons were part of a longer instructional sequence in which students engaged, parts of which are beyond the scope of this chapter.

investigation, both as a virtual sampling simulator and a tool for promoting the development of students' targeted imagery, as will be described shortly.

A Detailed View

Lesson 1

The opening lesson introduced students to the *Fish farmer* scenario as an anchor problem for the instructional sequence: A skeptical fish farmer wants to test the claim made by the company that supplies the fish for his pond that the genetically engineered fish tend to grow longer than normal fish. In the first part of this investigation, prior to divulging the farmer's approach to investigate the problem (as presented in the parenthetical statement in Fig. 1), students were prompted to consider how the fish farmer might go about testing this claim. Issues of data collection and selecting a representative sample were discussed, providing occasion for students to reflect on, and propose, possible ways in which the farmer might proceed. Students were then presented with the farmer's approach: when the fish were fully grown, he caught 43 fish at random from the pond and measured each of their lengths. Students then examined a *TinkerPlots* data file containing the lengths of the two types of fish in the farmer's sample of 43 fish. Working in pairs on a laptop computer, students explored this data by creating a dot plot of the sample separated into the two types of fish and using *TinkerPlots'* graphical tools to compare the lengths across the two groups (see Fig. 2).[3]

Students were subsequently asked the following sequence of questions regarding the sample they had analyzed, all of which were presented to them in structured worksheets:

- What did you notice about the sample?
- Did you conclude that the genetically modified species in the pond tends to grow longer than the normal version of the species?
- If so, how much longer do they tend to grow?

This part of the lesson concluded by having students decide, on the basis of the particular given sample, which type of species in the pond has a more variable length. Students were to support their answers to these questions by referring to the *TinkerPlots* graphs they had produced. Their responses formed the basis of whole-class discussions that syncopated their individual and paired work around the questions. A big statistical idea made explicit in this part of the lesson was that if a randomly selected sample is assumed to be representative of the sampled population, then it can be used as a basis for making claims about that larger population.

The lesson concluded with a sequence of three reflection questions designed to provoke students to anticipate the results of a thought experiment:

[3] As part of the lead-up to the instructional sequence, students had already learned to use such *TinkerPlots* tools to represent and organize univariate data sets.

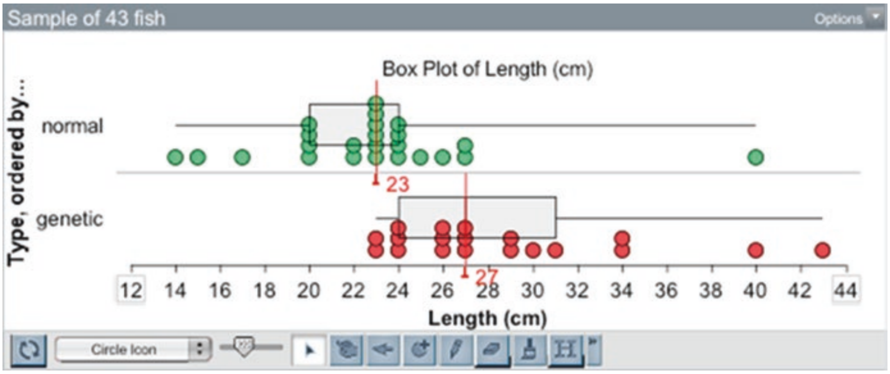


Fig. 2 A TinkerPlots dot plot of the sample of 43 fish lengths overlaid with boxplots and displaying the median length of each group within the sample

- What would you expect to see if a different sample of 43 fish was selected from the farmer's pond?
- Would you expect to draw a different conclusion about the fish in the pond based on a different sample?
- Is a sample of 43 fish large enough to draw a conclusion about all the fish in the pond, and how confident are you about this?

These concluding questions were intended to provoke the emergence of the key idea that the expected random variability amongst samples chosen from the same population problematizes the confidence one may have in an inference based on any individual sample. This idea was emphasized in instruction in an effort to motivate exploring the results of repeated sampling as a method for studying the long-term pattern of variability in the samples' statistic (i.e., its distributions) with the aim to inform one's confidence about conclusions drawn from samples to a population.

Lesson 2

The second lesson began by revisiting the issue of confidence about inferences in the context of discussing students' responses to the concluding questions from Lesson 1. Discussions recapped the idea that sampling outcomes are expected to vary from sample to sample, were the sampling process repeated under similar conditions, and that such variability therefore poses a problem for making inferences to an underlying population on the basis of any individual sample. This provided a natural segue into the main part of the lesson, which entailed explicit and systematic use of repeated sampling and attention to its resulting variability. Here students were first introduced to *TinkerPlots'* sampler as a tool to simulate the selection of random samples of 43 fish from the fish farmer's pond (the population). Then, working in pairs on a laptop computer, students used the sampler tool themselves to simulate the selection of 43 fish from the pond (a predetermined population provided to them in the *TinkerPlots* file). Students' tasks around this initial use of the sampling simulator were, first, to observe its results and explain what the sampler

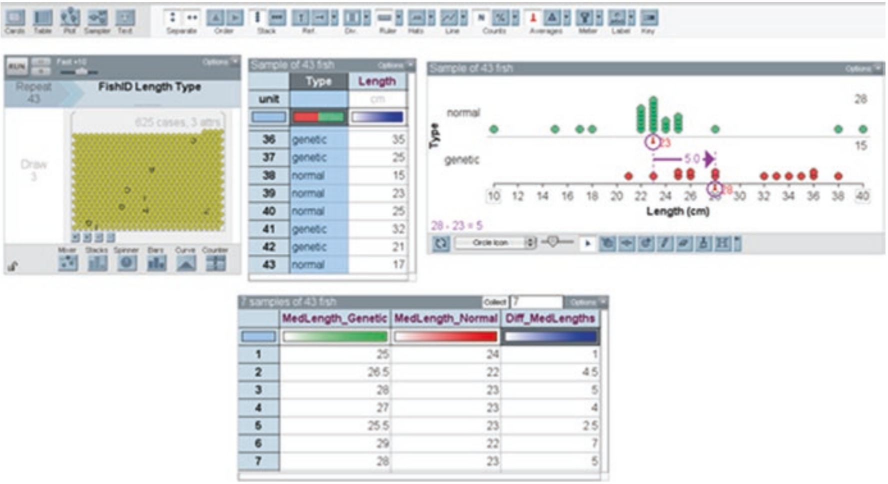


Fig. 3 A *TinkerPlots* simulation of the repeated sampling of 43 fish from a population of 625 fish. The bottom table shows the three measures recorded for seven trials of the sampling experiment

made by describing the information recorded in a data table generated in real time as the simulation unfolded, and, second, to create a graph of the resulting sample separated into two groups—lengths of the genetically modified fish and the normal fish in the sample—and displaying the median of each group (see the top part of Fig. 3). A concluding prompt asked students if, on the basis of their simulated sample, they would draw similar conclusions about the population of fish that they had in Lesson 1 (which type of fish tends to grow longer and how much longer it tends to grow).

The lesson then moved to a more systematic exploration of sampling variability by having students use the sampling simulator to first generate several samples of size 43, and then of size 15, from the simulated fish population (see Fig. 3). Students recorded the median length of each type of fish in a sample, and the difference between these medians as a measure of the group differences (i.e., difference between the median length of two types of fish in a sample). Fig. 3 displays the *TinkerPlots* set-up that the instructor and students used in this part of the investigation.

The simulator (left hand tool) used a mixer to represent the population of mixed fish. Each selected sample of 43 fish was represented in a case table displaying individual fish's type and length. The table was in turn linked to a dot plot displaying the distribution of lengths of fish in the sample, separated by type and showing the median of each type as well as the difference between medians using *TinkerPlots'* ruler tool. Each row of the table at the bottom of Fig. 3 recorded the value of the three measures for a sample (the median length of each type of fish and the difference between those medians) generated by running the simulation once. The four representations displayed in Fig. 3 were dynamically linked and automatically updated with the results of each new repetition of the simulated sampling experiment (enacted by clicking the "run" button in the mixer tool). As such, students were able to record and observe the emergence of the values of the three measures in a table and to track the variability among them as the sampling process was repeated. Students explored the patterns in these measures; they identified similarities and differences among the resulting medians and among the difference between the medians for the collection of seven samples, and they used their observations as a basis for proposing how this might help test the claim that genetically engineered fish tend to grow longer than normal fish in the larger population. Below is a sample of the specific questions posed to students around this part of the activity:

- What similarities and differences do you observe in the medians?
- Can your observations help the fish farmer make a decision about the company's claim? Please explain.

Class discussions around this part of the exploration showcased students' perceived patterns, culminating with a general consensus that the genetically engineered fish in the population were inferred to be "between 4 and 7 centimeters longer" than the normal fish. This part was then repeated for samples of 15, followed by having students compare the results for the two sample sizes and decide whether the smaller sample size was sufficient to conclude which type of fish tends to grow longer. This part unfolded around the specific questions displayed below:

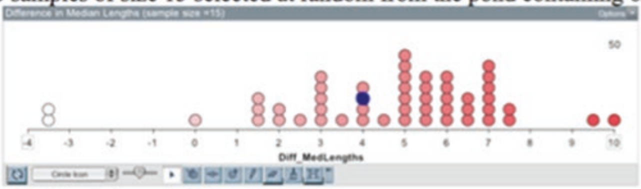
- Compare what you observed about the medians for these samples of 15 fish with what you noticed about the medians for the samples of 43 fish.
- Is 15 fish a big enough sample to decide whether the genetic fish tend to be longer than the normal fish? Please explain.
- Is 15 fish a big enough sample to decide how much longer the genetic fish tend to be than the normal fish? Please explain.

Lesson 3

The final lesson built on the activities and issues raised in Lesson 2 by having students examine the effects of sample size on the variability of the difference between median lengths of fish that they had previously explored for only seven samples. The lesson began with a demonstration and discussion of the *TinkerPlots* simulation of selecting 50 samples of size 15 from the simulated fish population (following the set up shown in Fig. 3), culminating with the presentation of a graph of the distribution of the difference in median lengths for each of the 50 simulated samples as shown in Fig. 4.

Discussions around this demonstration centered on having students track and explain the process of how the dot plot of the sampling distribution resulted from the sampler in terms of the various intermediate objects produced by the simulation and shown in the *TinkerPlots* window, as displayed in Fig. 3. This activity aimed to help students build and solidify their imagery of the repeated sampling process and their meaning for the resulting distribution of the sampling statistic displayed in Fig. 4. The accompanying activity prompts also assessed the strength and robustness of students' imagery by having them work backwards from a particular point on the dot plot and explain what it represented and the process that produced it. The

I. Here is a distribution of the difference between the median lengths of genetic and normal fish for 50 samples of size 15 selected at random from the pond containing 625 fish.



1. Describe what the darkened dot represents:

2. What information is shown by this dot?

3. How was this information obtained? Describe the sequence of steps in the sampling process that produced the information shown by the darkened dot.

Fig. 4 A sampling distribution and accompanying prompts of the opening activity of Lesson 3

activity prepared students for the subsequent part of Lesson 3, which required that they be able to decode and interpret a sequence of such dot plots coherently.

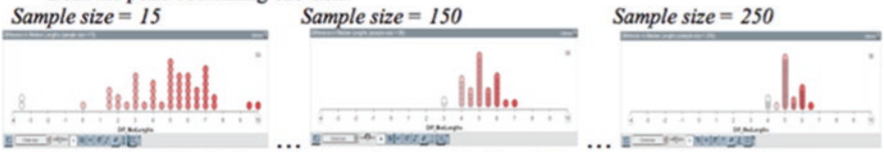
In the second part of Lesson 3, students examined and interpreted a sequence of five distributions of the difference in median lengths, each for 50 simulated samples of a different size drawn from the fish population. Students examined and interpreted these sampling distributions in relation to the increases in sample size. A subset of these distributions and the accompanying activity prompts are displayed in Fig. 5.

The first prompt (Question 4) aimed to orient students' attention to the fact that the clustering of the sample statistic becomes increasingly condensed (its variability decreases) with increasing sample size. A group discussion of this observation ensued which involved eliciting students' ideas about how to describe and measure the pattern of the observed variability. This discussion was followed by prompting students to use this pattern as a basis for choosing a sufficiently large sample in order to confidently infer whether genetically modified fish tend to grow longer than normal fish in the population (Question 5). The final prompt (Question 6) asked students to estimate *how much longer* genetically modified fish tend to grow than normal fish. These questions culminated in a group discussion about the trade-off between the competing interests of maximizing sampling accuracy and minimizing sample size.

Selected Highlights of Students' Reasoning and Engagement

As mentioned earlier, the activity sequence described above unfolded within a grade 9 mathematics class. The students had previously been exposed to some rudimentary statistical topics in their prior grades, such as computing the mean and finding the median of a set of quantitative data values, and constructing histograms and dot plots of data. However, the ideas addressed in the activity sequence, the use of

II. Here is a sequence of distributions. Each one is of the difference between the median length of genetic and median length of normal fish for 50 samples of a given size selected at random from the pond containing 625 fish.



4. Compare these distributions for the various sample sizes. What do you notice about these distributions as sample size increases?

5. What is a big enough sample of fish for the farmer to pick from the pond in order to confidently test the company's claim that genetic fish tend to grow longer than normal fish? Please explain why you think this.

6. Estimate how much longer the genetic fish in the pond tend to be than the normal fish. How confident are you about this estimate? Please explain.

Fig. 5 Task prompts and a subset of the sequence of distributions used in the final activity of Lesson 3

TinkerPlots to engender a dynamic and visual imagery of data and its analysis, and participating in activities and class discussions that encouraged them to explain and share their thinking about the ideas addressed was new to the students. It turned out that, in large part, students engaged enthusiastically with the activities and the use of *TinkerPlots*, eagerly voicing their thinking with regard to those ideas in a whole-class setting. In addition, students exhibited evidence, both individually and as a group, of having developed some habits of mind and ways of thinking that we see as coherent with statistical reasoning, albeit at an informal level. We present a few brief highlights of such evidence by way of illustrating the kinds of thinking that emerged out of the students' engagement with the activity sequence, which we believe has the potential to occasion such thinking among other groups of students. We present these highlights around the following themes that speak to two particular affordances of the instructional sequence:

Attention-Orienting Role of the Activity

The orienting prompts and reflection questions in the activities provided rich opportunities for students to notice patterns of dispersion in the distributions that resulted from the repeated sampling simulations. As a particular case in point, in Lesson 2 students' written responses to the question of what they noticed in the results of the 7 repetitions of sampling fish (see table at bottom of Fig. 3) evidenced three salient features when considered as a whole[4]:

[4] Words in bold font in the written responses indicate evidence of specific features that were salient to the students quoted.

1. A large proportion of the students noticed either a specific value or a range of values around which the median lengths of each or either type of fish was clustered around or bounded. This is illustrated by the following student response:

   S23: *The Normal fish medians stayed* ***between 22.5–25 cm****. The* ***genetic fish stayed above 25 cm*** *and the* ***normal stayed below 26****. Yes, because the median of the genetic fish was a few centimeters better than the normal. The* ***difference in size ranged from 3.5–8.5 cm****.*

2. Equally prevalent was students' explicit attention to how lengths of the two type of fish compared in terms of their medians, which enabled them to provide a "unanimous tendency" argument that the median length of the modified species exceeded that of the regular fish in *every* sample observed as a basis for inferring that the modified species is the larger one in the population. This line of reasoning is illustrated by the following student response:

   S17: ***All the medians*** *for Normal fish are around 22 and 23.* ***All the medians*** *for genetic fish are around 25–28. The genetics are* ***all longer*** *in length and range between 2 and 7 cm longer than Normal fish. Yes, they show that genetics grow larger than normal fish.* ***Out of all 7 samples, genetics were all longer****.*

3. Although less prevalent, many students also attended explicitly or implicitly to the variability in the median lengths of fish generated in the repeated sampling, as illustrated in the following student response:

   S9: *The normal medians are all around 23, while the genetics medians are all around 27.* ***There isn't much variance****. The genetic medians are always a few cm higher. The genetic fish are superior to the normal fish. The genetic fish are* ***consistently*** *an average of 4 cm longer, through multiple samples. The first sample wasn't a fluke – the normal fish are shorter, and the company was right.*

Discussion-Promoting Role of the Instructional Environment

Class discussions proved to be productive vehicles for helping students articulate and externalize their thinking about what they noticed in the activities. The excerpt below is from a discussion in Lesson 2 about the variability of the median lengths of fish observed in the samples of 15 fish versus samples of 43 fish; it focuses around one student's response to the question of whether the sample size of 15 is sufficient to draw a conclusion about the population of fish, as displayed in Fig. 6:

The associated discussion excerpt illustrates the kind of reasoning that can emerge through engagement with the activity, evidencing a sensitivity to the idea of variability in sampling data and citing too much variability in the data as a reason for not being able to make a confident claim about the underlying population (see Fig. 6):

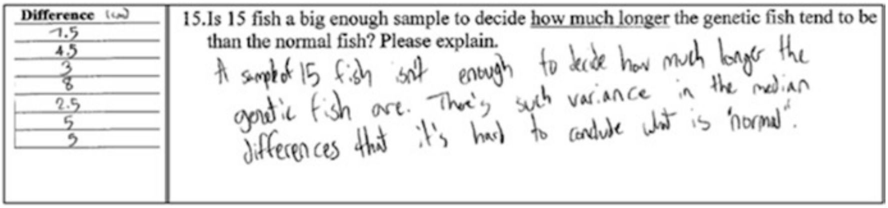


Fig. 6 One student's data values and her response to a question made on the basis of the data

Teacher: *How about the person that made these* (sampling results for both sample sizes are projected on the white board)*, S9?*

S9: *Um,* ***there was a lot more, um, variance*** *[inaudible] the fifteen fish one.*

Teacher: *Variance in what?*

S9: *Um,* ***in the differences and in the medians****.*

Teacher: *So across the samples, there's a lot more variability in the differences between medians; than is the case here* (points to the data for sample size 43). *Does anybody else see that?*

S18: *Huh huh.*

Teacher: *Really?*

S18: *Yeah, I don't see that. No, I don't.*

Teacher: *How do we help see that? Can you- can you help us out, S9? Like, what makes you see that?*

S9: ***On the last one, they go from, like, three point five to five****.*

Teacher: *Okay, so we're looking at the range of values, right?*

S9: *Yeah.*

Teacher: *Three point five to five.*

S9: ***On the right one, they go from, like, two point five to eight****.*

Teacher: *So, that's a bigger range, right? You get it?*

S9: ***Yeah, a bigger range.***

S18: *Yeah, I get it.*

Overall, the class discussions evidenced students' abilities to coordinate two things: 1) their expectation that the sample statistic's value will vary among samples, and 2) their emerging understanding that such variability is not haphazard but is instead constrained to a (possibly) predictable range of values. The ability to balance and coordinate these two ideas is an important component of statistical thinking (Rubin et al. 1990).

Concluding Remarks

As we argued at the start of our chapter, typical statistics instruction tends to still focus on having students learn procedures and calculate the value of standard measures (such as the mean of a set of data values). It is far rarer to see classrooms where data are presented in a dynamic setting, promoting a view of random sampling as a process that naturally incurs variability in its outcomes, and provoking students to explicitly attend to and take into consideration such variability when drawing conclusions about an underlying population. This focus on developing students' imagery of variability is highlighted in the statistics education literature cited earlier in our chapter as a central component of developing a meaningful and coherent understanding of statistical inference (Ben-Zvi and Garfield 2004; Garfield and Ben-Zvi 2008; Pfannkuch 2008; Rubin et al. 1990; Wild and Pfannkuch 1999). Indeed, we view the development of such imagery as a prime example of "the cognitive activities involved in informally drawing conclusions or making predictions about 'some wider universe' from data patterns, data representations, statistical measures and models, while attending to the strength and limitations of the drawn conclusions (Ben-Zvi et al. 2007)" (Garfield and Ben-Zvi 2008, p. 268). Our chapter offers one example reflecting a vision of how such imagery might be promoted in instruction. The instructional sequence we have described exploited dynamic and interactive features of the *TinkerPlots* software to support students in developing two related mental images: (1) a foundational image of sampling as a (hypothetically) repeatable process, and in turn (2) a sense of the variability they might reasonably expect if the sampling process were repeated multiple times under identical conditions.

Finally, we hasten to add that the students who participated in this sequence of activities reported having enjoyed working with *TinkerPlots*. Students also appreciated the openness of the inquiry-based context and the emergent learning milieu that encouraged them to share different points of view and ideas, as well as tools and approaches for exploring such. These aspects of the statistics learning and teaching environment described here are important not only in their own right, but also for their potential to create a milieu where students' thinking is revealed to teachers, thereby providing them with important information that they may use as a basis for understanding and evaluating such thinking and for providing feedback to students. We therefore hope that the example presented here will be of interest to teachers of statistics at the high school level, and we encourage interested readers to explore the web site http://www.tinkerplots.com/ where both the software and the activity worksheets that formed the basis of the instructional sequence described here can be obtained.

References

Ben-Zvi, D., & Garfield, J. (2004). *The challenge of developing statistical literacy, reasoning and thinking*. Dordrecht: Kluwer Academic Publishers.

Ben-Zvi, D., Gil, E., & Apel, N. (2007, August). What is hidden beyond the data? Helping young students to reason and argue about some wider universe. In D. Pratt & J. Ainely (Eds.), *Reasoning about informal statistical inference: A collection of research studies*. Proceedings of the fifth international research forum on statistical reasoning, thinking, and literacy (SRTL-5), University of Warwick, UK, August, 2007.

Franklin, C., Kader, G., Mewborn, D., Moreno, J., Peck, R., Perry, M., & Scheaffer, R. (2005). *Guidelines for assessment and instruction in statistics education* (GAISE Report). Alexandria: American Statistical Association.

Garfield, J., & Ben-Zvi, D. (2008). *Developing students' statistical reasoning: Connecting research and teaching practice*. New York: Springer.

Gattuso, L., & Vermette, S. (2013). L'enseignement de statistique et probabilités au Canada et en Italie. *Statistique et Enseignement, 4*(1), 107–129.

Gouvernement du Québec. (2016). Chapitre 6: Mathématique (original version in 2007 and updated in 2016). In *Programme de formation de l'école québécoise: Enseignement secondaire, deuxième cycle* (pp. 1–146). Québec: Ministère de l'éducation, du Loisir et du Sport.

Key Curriculum Press. (2012). *Fish-length distributions*. Amherst: Learn Troop Retrieved from http://www.tinkerplots.com/activities/data-analysis-and-modeling-activities.

Konold, C., & Miller, C. (2017) *TinkerPlots 2: Dynamic data exploration* [Computer software]. Amherst: Learn Troop.

National Council of Teachers of Mathematics. (2000). *Principles and standards for school mathematics*. Reston: Author.

Pfannkuch, M. (2008). Building sampling concepts for statistical inference: A case study. In D. De Bock, B. D. Søndergaard, B. G. Alfonso, & C. C. L. Cheng (Eds.), *Proceedings of the 11th International Congress on Mathematical Education*. Monterrey. Retrieved from http://tsg.icme11.org/document/get/476

Rubin, A., Bruce, B., & Tenney, Y. (1990). Learning about sampling: Trouble at the core of statistics. In D. Vere-Jones (Ed.), *Proceedings of the third International Conference on Teaching Statistics* (Vol. 1, pp. 314–319). Dunedin/Voorburg: International Statistical Institute Retrieved from http://iase-web.org/documents/papers/icots3/BOOK1/A9-4.pdf.

Wild, C. J., & Pfannkuch, M. (1999). Statistical thinking in empirical inquiry. *International Statistical Review, 67*, 223–265.

Additional Suggestions for Further Reading

Shaugnessy, J. M., Chance, B., & Kranendonk, H. (2009). *Focus in high school mathematics: Reasoning and sense making in statistics and probability*. Reston: National Council of Teachers of Mathematics.

TinkerPlots 2 software website: http://www.tinkerplots.com/.

TinkerPlots data sets and classroom activities: http://www.tinkerplots.com/activities.

Supporting Mathematical Creativity Through Problem Solving

Richard Hoshino

Abstract I teach at a small Canadian liberal arts and sciences university, where I offer a course called *Mathematical problem-solving*. In this course, undergraduate students develop the four key takeaways of a liberal arts education: critical thinking, creativity, oral communication, and written communication. As a teacher of mathematics, I am biased in my belief that mathematics develops these four takeaways (or skills) in a way that no other subject can. For many years, the core of my teaching practice has been developing these four skills in my students, through carefully-chosen problems ranging from logic puzzles to contest questions.

I offer several problems in this chapter, to illustrate how "applied problem solving" can develop creativity in our students. Specifically, these problems develop a key problem solving strategy or skill, the ability to solve hard problems by converting them into equivalent simpler problems.

I believe that this skill is not just essential for post-secondary students; if we can foster this mathematical problem solving ability in our secondary students, perhaps we could inspire more students with the message that mathematics is beautiful and powerful and relevant to everything in this world: challenging strong students who find classroom mathematics too easy and irrelevant while motivating weaker students who would see that mathematics is accessible, and has important applications to the issues they care about.

The problems in this chapter are based on simple ideas, but reveal surprising connections to deep mathematical ideas that are taught at the undergraduate level, including graph colourings and combinatorial enumeration.

Keywords Creativity · Problem solving · Symmetry · Enumeration · Graph colouring

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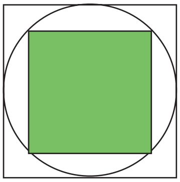
I teach at a small liberal arts and sciences university in Squamish, British Columbia, where I offer a course called *Mathematical problem-solving*. In this course, undergraduate students develop the four key takeaways of a liberal arts education: critical thinking, creativity, oral communication, and written communication. As a teacher of mathematics, I am biased in my belief that mathematics develops these four takeaways (or skills) in a way that no other subject can. For many years, the core of my teaching practice has been developing these four skills in my students, through carefully-chosen problems ranging from logic puzzles to contest questions.

I acquired this pedagogical viewpoint through a lifetime of studying mathematics, which has enabled me to make tangible impacts to my community: creating a new risk-scoring algorithm for high-risk marine cargo; reducing wait times at the Canadian border; serving as the mathematics consultant for three Canadian TV game shows; helping a billion-dollar professional baseball league design a schedule to cut down on greenhouse gas emissions; working with local organizations and companies to manage their staff scheduling; and implementing a roommate-matching program and course registration system at my university. In the process of solving these real-life problems, I have realized that a deeper skill was involved, a skill that I have worked hard to cultivate in my teaching practice: *the ability to solve hard problems by converting them into equivalent simpler problems*.

I believe that this skill is not just essential for post-secondary students; if we can foster this mathematical problem solving ability in our secondary students, perhaps we could inspire more students with the message that mathematics is beautiful and powerful and relevant to everything in this world: challenging strong students who find classroom mathematics too easy and irrelevant while motivating weaker students who would see that mathematics is accessible, and has important applications to the issues they care about.

I offer several problems in this chapter, to illustrate how "applied problem solving" can develop this key skill in our students, to recognize when a hard problem can be converted into a problem that is both equivalent and simpler.

There are numerous solutions to the problem as shown in the below figure. Readers are encouraged to try the problem before reading further!



Example Problem One: *In the diagram, a circle is inscribed in a (large) square, and a (small) square is inscribed in the circle. What is the ratio of the areas of the two squares?*

One approach is to let the small square have side length 1, and show that the side length of the large square must be $\sqrt{2}$. This can be done by noticing that the side length of the large square must equal the diameter of the circle, which must equal the diagonal of the small square, which must be $\sqrt{2}$ by the Pythagorean Theorem.

This proves that the ratio of the two areas must be $\sqrt{2} \times \sqrt{2} = 2$.

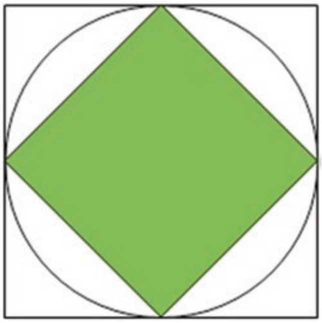


Fig. 1 By rotating the inner square 45 degrees, we develop a key insight

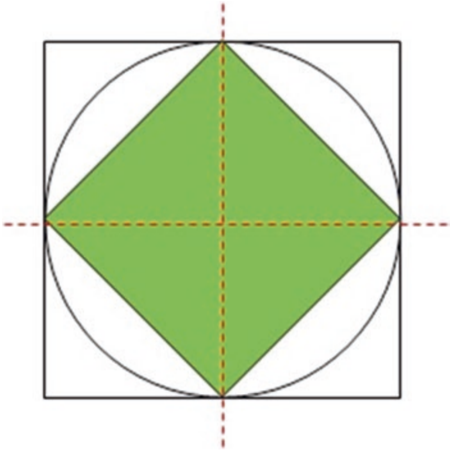


Fig. 2 A visual proof that the outer square is double the area of the inner square

But there is a cleaner solution, once we realize that we can solve this problem by converting it into an equivalent simpler problem. The key is to *recognize and exploit rotational symmetry*.

We rotate the inner square 45 degrees clockwise (Fig. 1).

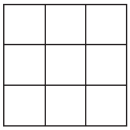
We then draw a vertical line and a horizontal line passing through the centre of the circle (Fig. 2).

Once we have done this, we can quickly see that the area of the large square has to be twice the area of the small square!

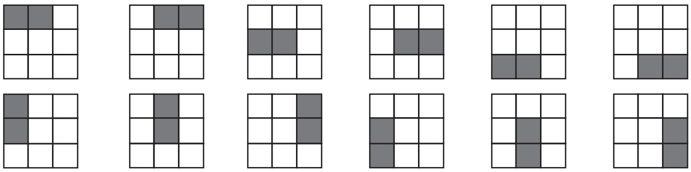
This beautiful solution illustrates that the heart of mathematics is not about memorizing formulas or rules, but rather about problem solving and detecting patterns, insight, and structure to uncover truth. Most of us assume that the inner square has to remain in a fixed position, while we can develop a better solution by breaking a self-imposed constraint, to use a simple idea (of a 45 degree rotation) to convert a difficult problem into one that is surprisingly simple and elegant as shown in the below figure

Example Problem Two:

There are many squares and rectangles, of all sizes, that appear in a 3x3 grid.



For example, there are twelve 1x2 rectangles: six horizontal and six vertical.



How many squares and rectangles, of all sizes, appear in this diagram?

Again, readers are encouraged to solve the problem themselves before reading further. As a challenge, once you have solved the problem in one way, can you think of another method?

One natural approach to solving this problem is to enumerate all possible cases, where we consider squares and rectangles of all possible dimensions, to get the correct answer of 36 (as shown in the below table).

Case	# of squares/rectangles
1 × 1 squares	9
1 × 2 rectangles	12
1 × 3 rectangles	6
2 × 2 squares	4
2 × 3 rectangles	4
3 × 3 squares	1
Total	36

But if we do this, we miss out on the underlying structure. Here is an equivalent solution, where we consider the dimensions (length and height) of each square and rectangle (as shown in the below table).

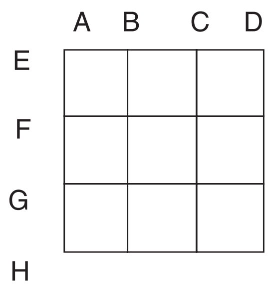
Length	Height	# of squares/ rectangles
1	1	9
1	2	6
1	3	3
2	1	6
2	2	4
2	3	2
3	1	3
3	2	2
3	3	1
Total		36

From here, we notice that the answer is $9 + 6 + 3 + 6 + 4 + 2 + 3 + 2 + 1$, which is equivalent to $(9 + 6 + 3) + (6 + 4 + 2) + (3 + 2 + 1) = 3(3 + 2 + 1) + 2(3 + 2 + 1) + 1(3 + 2 + 1) = (3 + 2 + 1)(3 + 2 + 1)$.

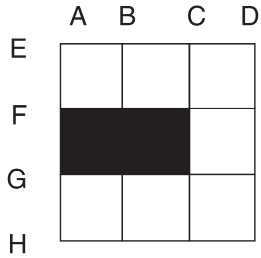
Surely there is a reason for such a beautiful answer. Take a moment to ponder this!

Indeed there is a reason. Instead of enumerating all possible cases, notice that every square or rectangle in our 3×3 grid consists of two vertical sides and two horizontal sides. Once we choose our two vertical sides and two horizontal sides, our square or rectangle is uniquely determined.

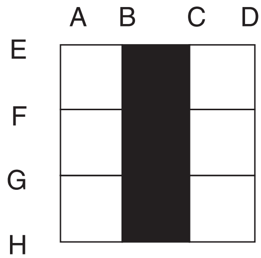
Specifically, what we do is look at each line in our 3×3 grid. We label our four vertical lines with the letters A, B, C, D, and label our four horizontal lines with the letters E, F, G, H shown in the below figure.



Notice that each selection of two vertical lines and two horizontal lines traces out a unique square or rectangle. For example, if we select the two vertical lines A and C, as well as the two horizontal lines F and G, then the four lines intersect to form this 1×2 rectangle (as shown in the below figure).



Conversely, given any square or rectangle, we can uniquely map it to the selection of two vertical lines and two horizontal lines by extending the sides until it hits the labels. For example, this 3 × 1 rectangle must map to the vertical lines B and C, and the horizontal lines E and H (as shown in the below figure).



By making this mapping, we have in fact constructed an equivalent problem which might be useful. In other words, we have shown that the problem of counting the number of squares and rectangles in a 3 × 3 grid is completely identical to the simpler problem in as shown in the below figure.

Example Problem Three:

Mrs. Smith has received four free tickets to a Justin Bieber concert, and decides to give them away: to two of her female friends, and two of her male friends.

Her female friends are Alice, Bethany, Chamique, and Diana.

Her male friends are Edwin, Fernando, George, and Harry.

Determine the number of different ways Mrs. Smith can give out the four tickets.

Do you see how this is both simpler and equivalent to the previous problem of counting squares and rectangles? Since there are six ways of choosing her female friends (AB, AC, AD, BC, BD, CD) and similarly six ways of choosing her male friends, the correct answer must be 6 × 6 = 36. In fact, this use of the Fundamental Counting Principle may be familiar to intermediate students, if they have completed a unit in elementary probability.

Thus, to solve the problem of determining the number of squares and rectangles in a 8 × 8 checkerboard grid, we do not need to enumerate all possible cases; instead, what we do is convert this problem into an equivalent simpler problem, the above "Ticket Problem" where Mrs. Smith has nine female friends and nine male friends. Can you see why the 8 × 8 checkerboard relates to the scenario with *nine* friends of each gender?

In this problem, the correct answer is 36 × 36 = 1296, since we can show that there are 36 ways that Mrs. Smith can choose two out of the nine females, and 36 ways that she can choose two out of the nine males. This is a much more elegant solution than manually enumerating all the cases.

Please explore the problem as shown in the below figure on your own or in a small group before reading on.

<u>Example Problem Four:</u>

Several students have formed various clubs, based on academic subjects that most interest them. The clubs consist of the following students:

Astronomy Club: Michael, Breanna, Joe

Biology Club: Breanna, Bonnie

Calculus Club: Joe, Caitlin

Dance Club: Joe, Gillian, Patrick

Economics Club: Caitlin, Michael, Bonnie

Food Studies Club: Bethany, Gillian, Michael

Geology Club: Bethany, Patrick

Each of the seven clubs wants to have an hour-long meeting on Friday afternoon; each person in the club must be present for the meeting. Class ends at noon, and the eight students want to get their club meetings over with as soon as possible. What is the earliest possible time at which all eight students can complete each of their hour-long meetings?

In your work, you might have noticed that the most inefficient solution is 7:00 PM, by having each of the seven clubs meeting in hour-long slots, one right after the other. We can save time by combining slots where no conflict occurs: for example, having the Calculus and Geology clubs meeting at the same time, since no student belongs to both clubs. This produces a solution whose answer is 6:00 PM.

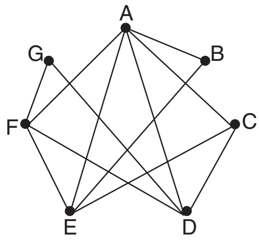


Fig. 3 Representing the problem as a conflict graph

Eventually, students find the correct answer of 3:00 PM by determining a way that the seven clubs can meet in 3 h-long slots, through trial and error. For example, here is one possible 3-h solution:

12 PM to 1 PM: Astronomy, Geology
1 PM to 2 PM: Economics, Dance
2 PM to 3 PM: Food Studies, Biology, Calculus

A common approach is to make an 8 × 7 table with students' names in the rows and clubs in the columns, to see where the possible conflicts arise. Through such a table, students determine which clubs can meet together and which ones cannot, and find a solution such as the one above.

Once students find a solution for 3 h, they are asked whether there exists a solution for 2 h. They quickly see that no 2-h solution exists, since Michael and Joe each belong to three different clubs: since each of them needs at least 3 h to complete their meetings, the entire group needs at least 3 h as well.

Despite satisfaction with solving the problem, students remark that making a 56-element table is tedious and lengthy. They realize that all is required is to determine which clubs have conflicts—e.g., Astronomy and Biology cannot meet at the same time because one individual belongs to both clubs: what matters is that there is an individual belonging to both clubs, not who that individual is.

Through this process, we motivate the key idea of solving scheduling problems using graph theory, to show that the above scheduling question can be solved by creating a "conflict graph" on seven vertices (representing the seven clubs denoted by the letters A, B, C, D, E, F, G), where two vertices are joined by an edge if and only if some individual belongs to both clubs, and would therefore have a conflict if both clubs scheduled their meetings at the same time.

For this particular problem, the conflict graph has seven vertices and twelve edges, and looks like Fig. 3.

Now we *colour* the vertices, with each colour representing a time slot. For example, if we colour vertex A red, then we see that we cannot assign red to vertices B, C, D, E, F, since each of those five clubs has a conflict with vertex A. Thus, we must use a different colour.

What is the fewest number of colours we need to ensure that no edge is joined by two vertices of the same colour? Do you see how the answer to this question must be the same as the fewest number of time slots needed to schedule all the student clubs?

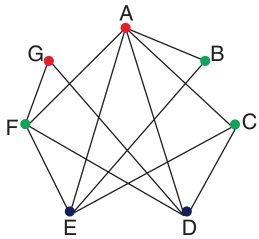


Fig. 4 Our solution, with red={A,G}, blue={D,E}, green={B,C,F}

We show that only three colours are required. To do this, we let A be red, E be blue, and F be green. Then, B and C must be green (since they are both adjacent to the red vertex A and the blue vertex E), which in turn forces D to be blue and G to be red. Therefore, we have found a valid 3-colouring (Fig. 4).

In the above picture, we see that no edge connects two vertices of the same colour. Thus, we are guaranteed a solution to the seven-club scheduling problem by simply assigning time slots to the three colours: Red = 12 PM–1 PM, Blue = 1 PM–2 PM, and Green = 2 PM–3 PM. Indeed, we can quickly verify that this is the exact same solution as what was given earlier.

A natural question is whether two colours suffice. To see why this is impossible, note that AEF is a triangle, representing the three different clubs Michael is in. And so each of these three points must be assigned different colours; thus we require at least three colours, i.e., at least three time slots.

Many mathematical problems can only be solved in routine and mundane ways. However, if secondary students see problems that can be solved in *both* a routine way and a surprising innovative way, then many unexpected benefits arise: a greater confidence in doing mathematics, a deeper appreciation for the beauty of mathematics, a development in one's creativity, as well as the opportunity to engage in applied problem solving. These skills and opportunities would help secondary students in so many ways, and serve them well for their future.

Various resources, geared towards secondary students and their teachers, are available at www.richardhoshino.com. Please use whatever you wish, free of charge.

Additional Suggestions for Further Reading

Hoshino, R. (2015). *The math Olympian*. Victoria: Friesen Press.

Hoshino, R., Polotskaia, E., & Reid, D. (2016). Problem solving: Definition, role, and pedagogy. In S. Oesterle, D. Allan, & J. Holm (Eds.), *Proceedings of the 2016 annual meeting of the Canadian Mathematics Education Study Group /Groupe Canadien d'Étude en Didactique des Mathématiques* (pp. 151–162). Kingston: CMESG/GCEDM.

Liljedahl, P. (2015). Building thinking classrooms: Conditions for problem solving. In S. Oesterle & D. Allan (Eds.), *Proceedings of the 2015 annual meeting of the Canadian Mathematics Education Study Group/Groupe Canadien d'Étude en Didactique des Mathématiques* (pp. 131–138). Moncton: CMESG/GCEDM.

Vakil, R. (1996). *A mathematical mosaic: Patterns & problem solving*. Burlington: Brendan Kelly Publishing Inc.

Exploring Math Through Social Justice Context Problems

Ami Mamolo, Kevin Thomas, and Michael Frankfort

Abstract This chapter presents tasks and task structures for incorporating socially relevant mathematical explorations in secondary school learning. We introduce and develop Social Justice Context Problems, highlighting issues in food affordability, fairness, and bullying, with connections to Canadian curricula. The structure for our tasks is discussed, and draws upon the constructs of Rich Learning Tasks and Teaching Math for Social Justice. The discussion of examples that we develop includes classroom-ready resources and techniques. We conclude the chapter with a summary of further tasks from math education literature.

Keywords Context problems · Rich tasks · Food affordability · Bullying · Infographics

Outline

- Purpose and scope – Introducing Social Justice Context Problems (SJcp)
- Structures for task design – Integrating Rich Learning Tasks and Teaching Math for Social Justice
- Examples and resources – Food equity in Canada, Issues in adolescent bullying
- Further examples of tasks – A brief selection of examples from math education literature

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Purpose and Scope

This chapter presents tasks and task structures for incorporating enriching, socially relevant mathematical explorations in secondary school learning.

1. What are Social Justice Context Problems (SJcp)?

   SJcp are in-depth, inquiry-based mathematical explorations that engage learners with authentic contexts and mathematical practices that aim to broaden and foster:

   (a) Critical awareness of social injustices in the lived experiences of Canadians;
   (b) Sophisticated mathematical thinking and decision-making while reinforcing important mathematical procedures and practices;
   (c) Awareness of the relevance of mathematics for understanding and addressing various issues affecting individuals in Canada and beyond.

2. Why use SJcp in secondary school mathematics classrooms?

   (a) Because mathematics is necessary for understanding world issues and social trends;
   (b) Because different contexts for rich mathematical activity can help teachers reach, support, and motivate a greater variety of learners;
   (c) Because students care about such issues, and these issues are important.

3. What are the goals of this chapter?

   (a) To equip teachers with structures and supports for incorporating SJcp into their practices, with different topical foci to meet the needs and interests of their particular students;
   (b) To highlight classroom examples, tools, and resources for engaging with SJcp, and that could serve as models for formulating other SJcp;
   (c) To invite critical reflection on the types of contexts and mathematical practices that we value for our students (and why).

Structures for Task Design

Structures frame how we think about SJcp, how we elicit, recognise, and value different ways of learning, and how we direct and focus student engagement.

Rich Learning Tasks (Flewelling and Higginson 2001)

Rich Learning Tasks are designed to give students opportunities to engage in inquiry, experimentation, investigation and problem solving. In doing so, students learn with understanding, use their knowledge in an integrated and authentic fashion to make sense of ideas, situations, concepts, and procedures, and develop the habits, attitudes, and skills necessary for a life of sense-making.

Rich Learning Tasks allow students to

- Explore, investigate, and challenge ideas and issues;
- Pose and answer their own questions about the situation;
- Choose and apply math and non-math skills and use them purposefully;
- Learn and practice problem solving skills in authentic ways and contexts;
- Gain independence and confidence in mathematics;
- Formulate, discuss and defend their ideas, approaches and solutions;
- Carry out, modify, and extend a plan, an idea, or an approach;
- Identify, select, and apply math tools, knowledge and procedures; and
- Justify, summarize, and communicate results and conclusions.

Rich tasks are lengthy, and are intended to last several hours or days. Sustained and persistent mathematical activity is essential for raising achievement in secondary mathematics—as Watson (2006) points out:

> It is simply impossible to learn mathematics if one is constantly changing topic, or task, or doing related but irrelevant tasks, or only doing the easy bits, or being praised for trivial performance. A problem with a fragmented, mechanistic approach to teaching mathematics is that learners who find mathematics hard are thus often taught in ways which make it hardest for them to learn it. (p.103)

Teaching Math for Social Justice (Gutstein 2006)

This framework was developed by Gutstein through his work with underprivileged middle school students. It emphasizes a needed balance between two sets of pedagogical goals for enriched student learning and engagement: Social Justice Pedagogical Goals, and Mathematics Pedagogical Goals (Table 1).

Table 1 Teaching math for social justice – pedagogical goals

Social justice pedagogical goals		Mathematics pedagogical goals	
Reading the world with mathematics	Using math to understand world issues, inequities, disparities and opportunities.	Reading the mathematical word	Developing mathematical power, understanding, and literacy.
Writing the world with mathematics	Using math to take action and initiate changes in your community and beyond.	Suceeding academically in a traditional sense	Succeeding in school, standardized tests, and post-secondary school.
Developing positive social and cultural identities	Being able to see yourself as able, confident, competent, and valuable.	Changing one's orientation to mathematics	Seeing math as relevant, connected, and powerful for understanding real issues

Examples and Resources

These examples come from our own practices; we highlight some of the prompts, contextual resources, technology, worksheets, and assessment guides used.

1. Food equity – Exploring the Nutrition North Canada program
2. Bullying in schools – Creating a safe space to play

Food Equity – Exploring the Nutrition North Canada Program

Nutrition North Canada (NNC) is a Government of Canada subsidy program to provide Northerners in isolated communities with improved access to perishable nutritious food. NNC is part of the Government of Canada's Northern Strategy (http://www.nutritionnorthcanada.gc.ca).

Context and Background

This exploration focuses on NNC's services to Nunavut, in comparison to students' local experiences. The focus on Nunavut is for two main reasons: Nunavut has the lowest (current) population density of provinces and territories, and its far north communities have a strategic purpose (arctic sovereignty).

Some facts about Nunavut:

- Nunavut has approximately 30,000 residents, with about 56% of residents under the age of 25
- Stats Canada reports that the median family income in Nunavut in 2010 was $62,680 per year, which is among the lowest in Canada
- Even with NNC program, food costs can be quite high:
  - $14 for 2 L of milk, $16.89/kg for red peppers
  - Major Canadian city approximate prices: $5 for 2 L of milk, $4.39/kg for red peppers

Overview of the Mathematical Explorations

A version of the task, including context, background reading, and some resources, is included in Appendix A. Curricular connections are presented in Appendix B, and a list of supplementary resources for exploration is included in Appendix C. The task is further discussed in Mamolo and Thomas (2014).

Table 2 Food equity – task questions

The questions (abbreviated)	Associated math activity and pedagogical goals
Q1: What is the weekly cost of groceries for a family of four in your area? Explain and compare your results with data from remote northern communities.	Explore and analyse data; make decisions about how to compare (e.g., using primary or secondary data, different groceries) and which tools to use; problem solve; formulate ideas and approaches; use math to understand possible disparity; develop math understanding, literacy, and skill.
Q2: Determine and compare costs of living and household income of a family of four in your neighbourhood. Represent the data in multiple ways; what do the different representations of the data tell you?	Investigate and analyse data; pose and answer questions; problem solve; formulate ideas and approaches; carry out and extend a plan; identify, select, and apply math tools; justify and communicate; develop math understanding; practice for academic success; see math as relevant and connected; use math to understand opportunities
Q3: Write an opinion piece about the NNC service to Nunavut. Analyse the goals of NNC; include a critique of the subsidy package, its cost effectiveness, its affordability, etc.	Justify, summarize and communicate results and conclusions; use math to understand issues; use math to take action in your community; see math as relevant, connected, and powerful for understanding world issues; develop mathematical understanding and literacy.

The *main goals* of the explorations are to critically examine the controversy around NNC and to explore mathematically whether the program is affordable, sustainable, effective, or equitable (Table 2).

Bullying in Schools – Creating a Safe Space to Play

Approximately 1 in 3 teenagers have experienced bullying (www.bullyingstatistics.org). Schools across Canada are increasingly trying to support and empower victims of physical, emotional, psychological, or cyber-bullying. Students are also expressing keen interest in fair treatment of one another, and in knowing what they can do to prevent bullying from happening.

Context and Background

This exploration focuses on understanding bullying behaviour, strategies for addressing bullying situations, and ways of supporting victims of bullying. A main focus of the exploration is on the design of a school playground which could reduce the effects of bullying in schools. The hook includes a non-verbal video called *The Bully Dance* (www.nfb.ca/film/bully_dance) (Fig. 1). It depicts some of the complexities of bullying situations and behaviours as it follows the lives of a few key characters over several days. Students can be asked to reflect on different aspects of

Fig. 1 Bully Dance (Perlman 2010)

Table 3 Bullying in schools – task questions

The questions (abbreviated)	Associated math activity and pedagogical goals
Q1: Create a proposal for a new school playground design that could reduce the effects of bullying at minimal cost.	Investigate; pose and answer questions; formulate and discuss ideas and approaches; identify and select math tools; communicate; see math as relevant and connected.
Q2: Create a blueprint of a playground that could minimize bullying, given a set of criteria and requirements, such as dimensions, types of structures, costs, and constraints.	Pose and answer questions; choose and apply skills; problem solve; formulate and discuss ideas; carry out and modify a plan; identify, select, and apply knowledge and procedures; use math to take action and initiate changes; develop math understanding; see math as relevant and connected.
Q3: Reflect individually on your playground design, bullying prevention, and bullying awareness.	Pose and answer questions; gain independence and confidence; justify, summarize, and communicate results and conclusions; see yourself as able, confident, and valuable.

the film, and to address in private communication with the teacher aspects which may have made them feel uncomfortable, surprised, or curious.

Overview of the Mathematical Explorations

A version of the task, including a preliminary assignment, a group project, and an individual reflection is included in Appendix D, along with sample rubrics. Curricular connections are presented in Appendix B.

The *main goals* of the explorations are to think critically about the impact of bullying and to explore mathematically ways of reducing bullying occurrences within pragmatic schoolyard constraints (Table 3).

Modifications and Extensions

An advantage of using Rich Learning Tasks in teaching is their receptiveness to adaptation, differentiation, and extension.

For the NNC exploration, further questions can invite investigations that touch on different curricular expectations and mathematical skills, for example:

- How is food transported to remote locations? Does time of year make a difference?
- What different routes could be taken (with what modes of transportation)? Can you find an optimum route?
- How are the subsidies being passed on to the consumer? Or distributed across communities?

For the bullying exploration:

- How common is bullying and who does it affect? Do different groups experience bullying at different rates? Or through different means?
- What if we had less money to spend on the playground? What changes would you make and why? What if we had more money to spend?
- How could your playground be designed differently, if the shape were a (circle, pentagon, etc.) with the same area as before?

Incorporating Technology

Food disparity is also a very serious world issue, and UN data can afford insight into which countries are most affected. The free, online software *GapMinder* (www.gapminder.org) provides dynamic visual representations of data that can be explored and analysed (see Fig. 2), which can provide both a scaffold for interpreting data trends, as well as a means to extend conversations, questions, and exploration. *GapMinder* allows the user to manipulate and select various parameters, and has been touted as an important tool for helping individuals read the world with mathematics (e.g., Rosling 2006).

Math literacy and communication can also be enhanced via the use of digital technologies. For example, infographics can be designed easily using free, online software such as *Piktochart* (www.piktochart.com), which offer user-friendly tools for infographic design (Fig. 3). Infographics can be used either to read the world with mathematics (presenting data to students), or to write the world with mathematics (having students create their own).

By posing and exploring their own questions about data and its visual representations, and by creating a tangible product informed by their investigations, students gain the confidence to see themselves as competent in interpreting, doing, and communicating important mathematics.

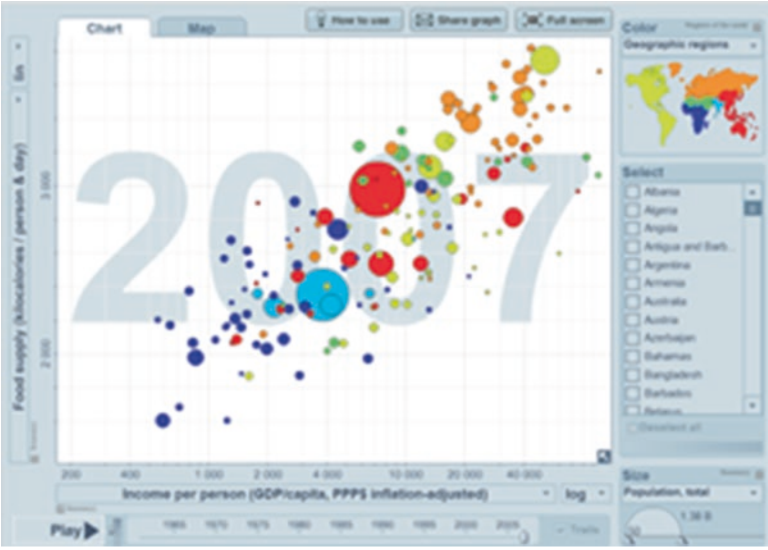

Fig. 2 Global food supply/person by income/person (gapminder.org)

Fig. 3 Food insecurity in Canada. (From google images)

Further Examples of Tasks

Researchers and teachers are actively working to incorporate social justice issues in math classrooms at all levels, and with different goals and mathematical foci.

Exploring Real Data

- Gutstein's (2005) "We are not a minority" activity uses a photo of a billboard showing Che Guevara and the caption "We are not a minority." The photo prompts students to explore the mathematics involved in this statement and to compute the percentage of the world's population made up of people of colour.
- Frankenstein (2006) developed a task to critically examine how the unemployment rate is determined in the U.S. The task provided students with different categories of labourers along with the number of people in each category. Students examine which groups should be considered unemployed, and calculate rates according to personal and governmental criteria.

Smoothing Out the Math

- Stocker's (2006) "Disabled by prejudice" lesson begins with background reading regarding challenges faced by people with physical disabilities. The lesson then includes 14 questions, some of which parallel common textbook presentations of word problems. The bite-sized problems tangentially relate the math content to the social issues addressed.
- Esmonde and Quindel (2006) created a project to help students explore globalization, labour and the environment with linear programming. The project involves an imaginary shoe company and made-up data regarding the costs for labour, materials, and shipping in both Indonesia and California. The project is supplemented with resources about globalization and impact.

Acknowledgements These tasks were developed as part of research that was funded by the Social Sciences and Humanities Research Council of Canada. We would like to extend our sincere gratitude to our research participants and partners. Early stages of this research and task development helped shaped the current work, and we would like to acknowledge those contributions from G. Gunthrope, L. Martin, K. Pain, and R. Pali, as well as T. Dai and S. Manoharan.

Appendices

Appendix A

PART A: Introduction – What is *Nutrition North Canada (NNC)*?

Nutrition North Canada is a subsidy program that seeks to improve access to perishable healthy food, as well as country or traditional foods, in isolated northern communities. It was launched on April 1, 2011 and is based on a market-driven model. The subsidy is transferred to retailers and suppliers that apply and are selected to register with the program, which then must pass on the savings to consumers. There are some conditions for communities to be eligible for the program, such as the lack of year-round surface transportation (i.e. no permanent road, rail or marine access). In some places, like Nunavut, residents are upset that non-perishable basics like flour are not subsidized under the new program. They must cope with a cost of living that can be significantly higher than many other places in Canada. You can find more facts on this program at: http://www.nutritionnorthcanada.ca/

PART B: The Project – Exploring NNC in Nunuvat

Nunavut has approximately 30,000 residents, with about 56% of residents under the age of 25. Statistics Canada reports that the median family income in Nunavut in 2010 was $ 62,680 per year, which is among the lowest in Canada. Despite this, cost of living can be quite high. Due to the challenges of vast distances, a small but growing population, the high cost of materials and labour, and extreme climate, it is difficult to maintain Canada's high standards of living in Nunavut. This is one of the reasons that the federal government has included Nunavut in the Nutrition North Canada program.

In addition to subsidizing food costs, one of the goals of NNC is to provide culturally appropriate nutrition education and health initiatives, working in conjunction with the Canadian government. However, recent media attention to the NNC has questioned the effectiveness of the program, suggesting that prices have increased since its implementation, and they question why basic necessities such as flour are not being subsidized (e.g. see: http://www.cbc.ca/news/canada/north/story/2012/06/09/north-nunavut-food-price.html). It is unclear if the needs of Nunuvat's residents are being met, and whose responsibility it is to provide for these needs.

1. What is the weekly cost of groceries for a family of four in your neighbourhood? Please explain how you determined this. Then, compare your result with data about Nunuvat from the Aboriginal Affairs and Northern Development Canada website, which has documented costs of food in remote northern communities (see: http://www.aadnc-aandc.gc.ca/eng/1100100035986/1100100035987).

2. Determine the *cost of living* of an average family of four in your neighbourhood and compare it with the *household income* of an average family of four in your neighbourhood. Represent the data in two different forms. What do the results tell you? Do the different representations give different insights into the data? Please explain.

   Note: Statistics Canada website offers helpful information for this question. Visit: http://www12.statcan.ca/census-recensement/2006/dp-pd/fs-fi/index.cfm?LANG=ENG&VIEW=D&format=flash&PRCODE=01&TOPIC_ID=7
3. A local newspaper has asked you to write an opinion piece about the Northern Nutrition Canada's service to Nunuvat. Analyse the goals of the NNC program in relation to the needs of Nunuvat's residents. Your analysis should include, but need not be limited to, a critique of the subsidy package, its cost effectiveness, and its affordability.

Appendix B: Canadian Curricular Connections

Food equity – Exploring the Nutrition North Canada program

General categories of expectations	ON	WNC
Primary data collection (surveys)	7, 8, CM11, DM12	9, FM12
Secondary data collection	7, 8, CM11, DM12	9, FM12
Data display	7, 8, CM11, DM12	9, FM12
Populations and sampling	7, 8, CM11, DM12	9, FM12
Sampling bias	7, 8, CM12, DM12	9, FM12
Use of data to make inferences and convincing arguments	7, 8, DM12	WM11, FM12
Critical analysis of data in the media	CM12, DM12	WM12, FM12
Indices (e.g. CPI)	CM12, DM12	

Bullying in schools – Creating a safe space to play

General categories of expectations	ON	WNC
Proportional relationships	7, 8, 9D, 9P, 10D, 10P	7, 8, 9, WM10, WM11, FM10, FM11
Percent, ratio, rate	7, 8, 9D, 9P, 10D, 10P	7, 8, 9, WM10, WM11, FM10, FM11
Quantity	7, 8, 9D, 9P, 10D, 10P	7, 8, 9, WM10, WM11, FM10, FM11
Area 2-D	7, 8, 9D, 9P, 10D, 10P	7, 8, 9, WM10, WM11, FM10, FM11
Volume, Capacity and Surface Area	7, 8, 9D, 9P, 10D, 10P	7, 8, 9, WM10, WM11, FM10, FM11
Whole Numbers, Decimals, Integers, Fractions	7, 8, 9D, 9P, 10D, 10P	7, 8, 9, WM10, WM11, FM10, FM11
Solving Multi-Step Problems	7, 8, 9D, 9P, 10D, 10P	7, 8, 9, WM10, WM11, FM10, FM11

Mathematics Courses Legend

Ontario (ON) 7 – Grade 7 Mathematics; 8 – Grade 8 Mathematics; 9D – Grade 9 Principles of Mathematics (Academic); 9P – Grade 9 Foundations of Mathematics (Applied); 10D – Grade 10 Principles of Mathematics (Academic); 10P – Grade 10 Foundations of Mathematics; CM11 – Foundations for College Mathematics, Grade 11; CM12 – Foundations for College Mathematics, Grade 12; DM12 – Mathematics of Data Management, Grade 12

Western and Northern Canada (WNC) 7 – Grade 7 Mathematics; 8 – Grade 8 Mathematics; 9 – Grade 9 Mathematics; WM10 - Apprenticeship and Workplace Mathematics, Grade 10; WM11 – Apprenticeship and Workplace Mathematics, Grade 11; WM12 – Apprenticeship and Workplace Mathematics, Grade 12; FM10 – Foundations of Mathematics and Pre-Calculus, Grade 10; FM11 – Foundations of Mathematics, Grade 11; FM12 – Foundations of Mathematics, Grade 12.

Appendix C: Web-Based Resources NNC Program

Consumer Price Index

http://www.statcan.gc.ca/tables-tableaux/sum-som/l01/cst01/cpis01a-eng.htm
Cost of the Revised Northern Food Basket in 2012–2013
http://www.nutritionnorthcanada.gc.ca/eng/1369313792863/1369313809684
Cost of the Nutritious Food Basket – Toronto 2013
http://www.toronto.ca/legdocs/mmis/2013/hl/bgrd/backgroundfile-61625.pdf
Eating Well with Canada's Food Guide – First Nations, Inuit and Métis
http://www.hc-sc.gc.ca/fn-an/food-guide-aliment/fnim-pnim/index-eng.php-a1
Fact Sheet: Revised Northern Food Basket
http://www.nutritionnorthcanada.gc.ca/eng/1369314079798/1369314090524
Fact Sheet: The Nutrition North Canada Program
http://www.nutritionnorthcanada.gc.ca/eng/1367932314461/1367932387670
Food Secure Canada
http://foodsecurecanada.org/community-networks/northern-remote-food
Food Mail Program
http://publications.gc.ca/collections/Collection/R2-221-2002E.pdf
A National Nutritious Food Basket
http://www.hc-sc.gc.ca/fn-an/surveill/basket-panier/index-eng.php
Nutritious Food Basket – Guidance Document
http://www.mhp.gov.on.ca/en/healthy-communities/public-health/guidance-docs/NutritiousFoodBasket.PDF
Nutrition North Canada
http://www.nutritionnorthcanada.gc.ca/eng/1351088285438/1351088295799
The Revised Northern Food Basket
http://publications.gc.ca/collections/collection_2008/inac-ainc/R3-56-2007E.pdf

Appendix D: Bully Project Worksheets and Rubrics

PLAYGROUND ANTI-BULLY PROJECT: *PROPOSAL*

School playgrounds are the time to bond with pupils around the school and to enjoy the few minutes of fresh air with friends. However, the current era lives in fear of travelling outside. May it be the weather or the legends created by students around the school, students are not willing to go outside during recess. After watching the short film "The Bully Dance", it is evident that some students are disturbed by bullies. In this assignment, you will discuss with your group members to make a school friendly playground to minimize bullying in the school.

Objective
In a group of 4–5, create a school playground that will reduce bullying and be of minimum cost.

Proposal
Brainstorm various ideas leading to the problem of the bullying in the school playground. Create a letter to the principal regarding the idea of bullying in the playground and the relation for the creation of a new playground to solve this issue. In your letter, remember to include:

– Date and Time of letter
– Principal's name and address of school
– Introduction, body, and conclusion
– Closing Statement (Example: Regards, Thanks, Sincerely, etc.)
– What bullying is Why your group is deciding to this
– What is the relationship with bullying and the playground
– How does your playground differ from the current playground
– How is your playground going to be cost effective
– How will your playground be captivating for students
– Where can students go to report bullying (Example: Teachers, Senior Students, etc.)
– Are there any danger precautions
– Will there be restrictions in your playground

Assessment Rubric

Area of achievement and expectations	Limited level 1	Some level 2	Considerable level 3	Thorough level 4
Format structure of the letter includes all the required details (such as date, school address, principal's name, paragraph structure, etc.)				
Bullying information that includes examples from the short film 'the bully dance' as well as ideas that were brought up during the class reflection.				

Area of achievement and expectations	Limited level 1	Some level 2	Considerable level 3	Thorough level 4
Draft diagram of the playground is included and contains sufficient detail in order to demonstrate the effectiveness of reducing bullying.				
Explanation and breakdown of the costs associated with building the playground.				
How will your playground reduce bullying behaviour at school and how will this be measured?				

PLAYGROUND ANTI-BULLY PROJECT: *THE DESIGN*

This assignment will explore the issue of bullying in schools. As a student, it is your responsibility to make any school environment friendly for students. In your group, discuss and brainstorm ideas that relate to the solutions of bullying in a school environment. With these ideas, create the perfect model of a playground where bullying is near negligible. Your playground must be able to accommodate all grades from kindergarten to grade 8.

Objective
Create a blueprint of a playground that minimizes all forms of bullying.

Procedure
You are given a budget $100,000 as an annual budget. Your playground will be run under a rental business and your school can only give a buffer range of $1000 if you need it. There are three land sizes:

- 4000 × 4000: $30,000
- 6000 × 6000: $40,000
- 8000 × 8000: $50,000

Your playground must have a minimum of 2 playground activities:

- double swing set: $10,800
- single slide: $10,350
- double slide: $25,000
- triple swing set: $17,500
- seesaw: $10,000
- soccer playground: $30,000 (will take 700 ft^2)
- basketball playground: $30,000 (will take 300 ft^2)
- baseball field: $30,000 (will take 850 ft^2)

Each centimetre on your layout will be measured to a scale of 25 cm in reality. There are also other restrictions:

- – the playground must be attached to the school
- – there must be 2 entrances from the playground to the school
- – there must be three teachers and teachers must be able to cover a 15 cm radius
- – there must be at least 30% of free land for school kids to roam around
- – the parking lot must be attached to the playground

Assessment Rubric

Area of achievement and expectations	Limited level 1	Some level 2	Considerable level 3	Thorough level 4
Layout/blueprint is labeled in correct format with all calculations provided (such as dimensions of land plot, total free space, estimate of activity and sizes)				
Understand the use of 3-D dimensions to portray structural placements by using correct scales to ensure safety of all structures.				
Use mathematical logistics to justify reasons behind separation of activities and areas of activity.				
Use real-life application to layout certain activities in correct areas to ensure high priority of structural stability.				

PLAYGROUND ANTI-BULLY PROJECT: *INDIVIDUAL WRITE-UP*

Description

After watching the film "The Bully Dance", you should realize that bullying is a very serious problem that happens in the real world. Reflect on what you learned after watching the short video, what are your thoughts about bullying, how your design of the playground will reduce or prevent bullying, why bullying should be stopped, what you can do to stop bullying and if bullying happens at your school. Summarize a few points on a page. You may also include your personal experiences with bullying. For example, if you've been bullied, if you witnessed someone else being bullied and what you did about it/what you should have done.

Task

Select at least 3 points to write on:

- What you learned from the video?
- What you think about bullying?
- How your design will prevent bullying?
- Experiences with bullying (if applicable)
- What are ways to prevent bullying?
- Why bullying should be stopped?
- Does bullying happen at your school?

Your write up should include at least two paragraphs covering at least three of these points.

Assessment Rubric

Area of achievement and expectations	Limited level 1	Some level 2	Considerable level 3	Thorough level 4
Summary of what bullying is from your own experience and what you learned from viewing the video.				
Explanation of why your group chose the design of your playground.				
How does your playground design promote anti-bullying.				
Explanation of how your anti-bully design will be used by all members of the school community at the present time and in the future.				

References

Canadian Broadcasting Company. (2012). *Nunavut residents protest high food prices*. Retrieved from http://www.cbc.ca/news/canada/north/nunavut-residents-protest-high-food-prices-1.1170565

Esmonde, I., & Quindel, J. (2006). Globalization, labour, and the environment. In E. Gutstein & B. Peterson (Eds.), *Rethinking mathematics: Teaching social justice by the numbers* (pp. 62–63). Milwaukee: Rethinking Schools Ltd.

Flewelling, G., & Higginson, W. (2001). *A handbook on rich learning tasks: Realizing a vision of tomorrow's mathematics classroom*. Kingston: Queen's University, Faculty of Education, Centre for Mathematics, Science and Technology Education.

Frankenstein, M. (2006). Reading the world with math. In E. Gutstein & B. Peterson (Eds.), *Rethinking mathematics: Teaching social justice by the numbers* (pp. 19–30). Milwaukee: Rethinking Schools Ltd.

Gutstein, E. (2005). We are not a minority. In E. Gutstein & B. Peterson (Eds.), *Rethinking mathematics: Teaching social justice by the numbers* (p. 118). Milwaukee: Rethinking Schools Ltd.

Gutstein, E. (2006). *Reading and writing the world with mathematics: Toward a pedagogy of social justice*. New York: Routledge.

Mamolo, A., & Thomas, K. (2014). A mathematical exploration of the nutrition North Canada Program. *Ontario Mathematics Gazette, 53*(1), 17–22.

Perlman, J. (2010). *Bully dance. National Film Board of Canada*. Retrieved from https://www.nfb.ca/film/bully_dance/

Rosling, T. (2006). *The best stats you've ever seen* [video]. Retrieved from http://www.ted.com/talks/hans_rosling_shows_the_best_stats_you_ve_ever_seen

Stocker, D. (2006). *Maththatmatters: A teacher resource linking math and social justice*. Ottawa: CCPA Education Project.

Watson, A. (2006). *Raising achievement in secondary mathematics*. UK: McGraw-Hill Education.

Social Justice and the Teaching and Learning of Mathematics

Gale Russell

Abstract Social justice seeks to identify, understand, and ultimately remove inequities from all aspects of life. This chapter discusses ways that social justice and mathematics teaching and learning can intersect within the classroom, and beyond: social justice issues as contexts for applying mathematics or as an impetus for creating of new mathematical knowledges, processes, and tools; and teaching and learning mathematics through socially just pedagogical approaches.

Keywords Social justice · Mathematics · Teaching · Learning · Worldview · Indigenous · Traditional western

Social justice has been defined in many different ways and from very different perspectives, such as political, legal, and social perspectives. In this chapter, social justice will be discussed from an educational perspective. Even within this particular context, however, social justice can be viewed very differently.

Perhaps in broadest of terms, social justice focuses on identifying, understanding, and ultimately removing inequities from social contexts. Within education, social justice is often embraced in two ways: having students explore, understand, and even take action regarding social justice issues in their school, community, or the world; and by teaching in socially just ways in the classroom.

In the first case, issues such as safe drinking water for all Canadians, global warming, or missing and murdered Indigenous women can become a topic to be studied by students through one or more of their school classes. This kind of social justice in education is often most effectively introduced by tapping into concerns and questions that the students raise rather than based upon a teacher's own interest.

The second way that social justice enters into classrooms is in response to inequities resulting from pedagogical choices, classroom structures and environments,

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and institutional policies. Teaching and learning are inherently political acts, and classroom expectations, school policies, pedagogical approaches, and other traditions determine (explicitly or implicitly) who is advantaged or disadvantaged within the classroom. The processes of privileging ultimately determine who is oppressed within the classroom, be it based upon race, gender, assumed ability or disability, and limitations placed upon learning opportunities. Classrooms grounded in social justice challenge such notions of power and sites of oppression, many of which are so engrained in the past that they have been unquestioningly accepted as inevitable in the present.

In this form of seeking to embed social justice within the practices and environments of teaching and learning, the "'ism' words—racism, classism, sexism, heterosexism, ableism, etc." (Sensoy and Diangelo 2009, p. 348) are exposed, scrutinized, and reformed. Teaching with social justice in mind recognizes that socio-economic, gender, racial, and sexual orientation (to name but a few of the isms) inequities often not only underlie, but continue to propagate gaps in student performance, poor sense of identity, and disinterest in learning. Fundamentally, teaching with social justice in mind means taking actions to counter the resulting power inequities and oppressions that enter into every classroom and demands justness for all students, regardless of who they are or where they come from.

Ultimately, in teaching and learning contexts, "Social justice is recognizing and acting upon the power that we have for making positive change" (Dell'Angelo 2014, para. 2), and as teachers, it is our responsibility to enact social justice in all of our classrooms. What such teaching and learning can look like in a mathematics classroom will now be considered.

Social Justice As a Context for Mathematics Teaching and Learning

Kumashiro (2001) tells us that:

> If science and mathematics classrooms have traditionally taught science and math in only certain contexts and attempted to answer only certain questions, then students can be invited to learn sciences and maths in different contexts (Frankenstein and Powell 1994), and use sciences and maths to answer different kinds of questions and solve different kinds of problems, especially problems relevant to their own lives (Ladson-Billings 1995). (p. 7)

In other words, using mathematics as a set of tools and ways of knowing to understand and engage with social justice issues, contexts, and questions used in the classroom plays a crucial role in learning. While an inquiry project into social justice issues related to pension plans (and the lack of pension plans) might be very relevant to a particular teacher, high school students may not see the relevance of such an undertaking either from a social justice or mathematics perspective.

A topic such as carbon taxing however, may be engaging and of great importance to students, whether they are coming from an environmental perspective, from a family farm that requires gas to operate its equipment, or as a new car owner who enjoys cruising the streets of their home community. Likewise, if the students are in a course preparing them for academic post-secondary pursuits, the topic of costs for post-secondary education and variances across Canada and the world might be one that the students are interested to explore, and possibly take action upon. Or perhaps your students have been learning that "we are all treaty people," but they are having a hard time understanding what that means and what the implications are for them and the other people of Canada. Maybe your students are already becoming politically active by taking sides on a local debate surrounding gender neutral washrooms. All of these topics of social justice can also become contexts for mathematics teaching and learning.

Using mathematics as one set of tools for exploring such issues can help students (and their teachers) come to a better understanding of the issue, how it impacts them, and what they could choose to do in response to it, as well as providing a reason for learning certain mathematical concepts and procedures. The key is to get to know your students and their interests and concerns. Listen to what they want to talk about, and then look for how you can tie the mathematics they are expected to learn to the issues they identify and discuss.

Jonathan Osler (2007) makes the following recommendations when engaging in social justice issues in the mathematics classroom:

- Make sure the mathematics connection is strong. "Fit the issues to the math,"
- Talk to your students about what issues they are interested in,
- Use broad open-ended questions about the issue for setting the context of the unit or project,
- Start the unit or project by introducing the issue, not the mathematics, and
- Introduce the mathematics as it is needed or wanted. (p. 7)

The first of these recommendations reminds us as teachers that ultimately we are responsible for teaching the students mathematics and that social justice issues are to be thought of as vehicles for facilitating that learning. Conversely, however, Osler reminds us that the social justice issue is the impetus for learning and doing the mathematics, so it is important to start with the issue and bring the mathematics in. This notion is reaffirmed in his final recommendation, that is, to hold off on introducing or developing any mathematics until the students have a need for it. In so doing, a teacher can thereby prevent the issue seeming contrived by the students.

The website http://radicalmath.org/ provides a number of examples of social justice issues that can be used at different grades and courses of mathematics, and it is organized by mathematics topic, social justice issue, and resource type. Such a resource is always helpful, but do not forget that the integration of social justice into any classroom needs to be in response to student interests and curricular content, and not the presence of a ready-made lesson.

When embarking upon this kind of integration of social justice issues and mathematics teaching and learning, assessment and evaluation are also often topics that require addressing. Assessment (without evaluation) of such instances of integration, as the collection of evidence of learning, can easily be done for both student understanding of the social justice issue and of the mathematics in whatever ways the teacher feels appropriate. However, an assessment can also become an evaluation, but at this point, the teacher needs to be very careful that what is being evaluated is what the curricular goals, outcomes, objectives, or standards are for the class they are teaching. If the curriculum is only speaking to students' learning of mathematics content, then that is what the evaluation of students should speak to—the mathematics content.

For some, this differentiation between what is assessed and what is evaluated is seen as highly problematic, questioning why one would have the students engage with the social justice issue if their mark does not reflect how well they did in that regard, and ultimately asks the question of why the social justice issue was even raised. For others, the fact that students now are more knowledgeable about the issue, and about how mathematics can be used in making sense of social justice issues is justification enough. This is a quandary that each teacher must engage with in their own terms. Many find their comfort zone in a more middle ground—where the mathematics learned is graded, but social justice issues and engagement are also reported in some separate manner (for example, a separate grade, a rubric score, or an anecdotal record) which can also be communicated to the students and parents or guardians.

Done thoughtfully and respectfully, the incorporation of social justice issues in mathematics class can bring greater meaning and value to the mathematics that students are learning. Such incorporation helps to address the question of "when am I ever going to need this stuff" (a common refrain in mathematics classrooms) as well as providing some students who may have always felt that they had no way to connect with mathematics experiences that allow them to interact with mathematics in meaningful and relational ways.

The second way that social justice and mathematics can be brought together within the mathematics classroom focuses on the ways in which mathematics is taught and learned. In this approach, the social justice of the pedagogical approaches chosen as well as the kinds of knowledge and ways of knowing being valued are under consideration.

Teaching and Learning Mathematics in Socially Just Ways

The teaching and learning of mathematics in socially just ways is directly tied to the notion of anti-racist education. Often when hearing the term "anti-racist education," people think of "multicultural education"; however, the intent of the two approaches is different, and in fact, both can occur within the same classroom. The focus of multicultural education is on helping students appreciate and understand cultures beyond their own (and often their own as well), with the underlying purpose being to prepare students to live and work successfully within multicultural settings. Anti-racist education, on the other hand, focuses on changing policies and structures within the educational system that promote the labelling of students on the basis of their ethnic or socio-economic backgrounds; physical, emotional and mental abilities; gender; and so on. So, whereas multicultural education promotes student understanding and valuing of all cultures, anti-racist education acts to ensure that no inequities based upon racial and other forms of stereotyping and oppression exist in the classroom.

For example, while multicultural education might promote students' understanding of Indigenous mathematics by providing students with experiences and knowledge related to various examples from Indigenous cultures; anti-racist education would challenge and deny the stereotyping, and resulting oppression, of Indigenous students as being mathematically deficient. This belief that Indigenous people are inherently inferior at mathematics is something which has been documented through (highly flawed) scientific research and it continues to be reinforced by some teachers and educational partners when they are interpreting statistical data showing high drop-out rates and high failure rates for Indigenous students in both mathematics and science. From an anti-racist education perspective, these beliefs would not only need to be revealed for their misinterpretation and misrepresentation of the situation, but also for the damage done by such racist beliefs and statements. Furthermore, anti-racist education would openly challenge these beliefs and theories, and deny them any power or authority within the classroom (and ultimately beyond the classroom).

Perhaps the most perplexing part of trying to engage in anti-racist education (for the purposes of social justice, or otherwise) is being able to identify when oppression is occurring and how. As Gutiérrez (2013) notes:

> The rush to move onto the next mathematical concept (or response to intervention procedure) almost ensures that we will not ask why this concept? Who benefits from students learning this concept? What is missing from the mathematics classroom because I am required to cover this concept? How are students' identities implicated in this focus? (p. 37)

These are all questions which require serious contemplation when focusing on teaching for, about, and through social justice. Blake (2015) expands on these questions by arguing that teachers and students should be expected to reflect upon questions such as:

- Who makes decisions and who is left out?
- Who benefits and who suffers?
- Why is a given practice fair or unfair?
- What is required to create change?
- What alternatives can we imagine? (para. 3)

Ultimately, all of these questions come down to an interrogation and reflection upon what kinds of knowledge and ways of knowing are being valued within the classroom, because what is being valued tells us where we are, where we need to go, and how we might get there.

One Approach to Teaching and Learning Mathematics in Socially Just Ways

Although their book does not explicitly attempt to connect to teaching and learning mathematics in socially just ways, Sullivan and Lilburn's (2002) *Good questions for math teaching: Why ask them and what to ask?* provides a possible place to start—that of creating open questions (also referred to as inquiry questions, true problems, and so on). In this resource, one approach given to creating what the author's title a "good question" out of a closed question is by working backward:

This is a three-step process.

- Identify a topic.
- Think of a closed question and write down the answer.
- Make up a question that includes (or addresses) the answer. (p. 7)

In the mathematics classroom, a closed question is one that, by the way it is written and by when it is given, there is clearly a single answer being sought and a specific procedure or algorithm to be used. Moreover, that procedure or algorithm would be one that has already been explicitly and directly taught and demonstrated to students. Let's consider a question given in the text:

> The children in the Smith family are aged 3, 8, 9, 10, and 15. What is their average age? (Sullivan and Lilburn 2002, p. 8)

In most classrooms, this would be seen as a closed question, with both the teacher and students knowing exactly what is expected to be done. Reading this question, it is easy to assume that the students have been provided an algorithm such as "add all the numbers up and divide by the number of numbers ($\frac{x_1 + x_2 + x_3 + \ldots + x_n}{n}$ as the formula)" in class, along with a few examples, and this is the typical word problem at the end of the practice exercises.

There are many changes that can be made to this question to make it more open, but let's consider just one: *The five children in a family have a mean age of 9. How old are the children?* This new question is more open in at least two ways: there are many possible answers (9, 9, 9, 9, 9; and 7, 8, 9, 10, 11 are just two of them), and with the removal of the family name, Smith, all students can better relate to who is in the family and, for some students, may even invite them to think about different kinds of families (single parent, middle-class, Muslim, Greek, same-sex parents and so on). A further editing of the question could remove the fact that there must be five children to make it even easier for the students to see themselves in the question (be they from a single child family or a family with 15 children). This further edit can in some cases add too much openness and confusion to start the students with, but it may be good for individuals that cannot relate to a family with five children.

When moving to this type of question for the purposes of instruction and learning, as a teacher you need to be prepared for students to provide challenges to your thinking and interpretation of the question and how you will respond to them. As noted earlier, five children aged nine is a possible mathematical solution, but some students (and teachers) might not say it is a realistic one. However, for some students, quintuplets (although extremely rare) may be a family reality, but so might a blended family made up of two sets of twins and one child born less than a year before or after one of the sets of twins. Thus, this question as rewritten is now allowing different kinds of knowledge (about family structures as well as mathematics) to be acknowledged and valued.

Some students may also propose children who were still-born being included, and hypothetically might argue for a solution of 0, 0, 0, 0, 0, 45. Other students might consider fractional ages 9 ½ versus 9. These proposals, if you have anticipated at least some of them, need not interfere with the intended mathematical learning. Instead, they open up the potential for greater learning and engagement by the students.

This type of learning in mathematics also considers and values the many different ways that students may be attempting to solve and answer the problem. As a teacher using this type of question, you will need to anticipate as many ways that the students might engage with the task. Will they do it concretely or visually, and if so how? Might some students build 5 towers of 9 linking cubes each and then move cubes from one tower to another to get different "ages?" Or perhaps they would put all the cubes of the towers together and "deal them out" into five piles of different sizes. What of the student who concludes "their ages must total 45." How does the student know this, and how does it relate to the other representations that are being presented? As all of the emerging approaches are being discussed and connected to each other, big ideas related to averages (or, to be mathematically correct, arithmetic means) emerge that are often missed when only the formula is taught: the idea of a total to be shared amongst a group or set and the role of balancing and compensation in the determining of an average.

Through the work on this problem, the formula stated earlier, and even this form: $\sum_{n}^{i=1} \frac{x_n}{n}$, can be drawn out through the discussion and generalized and abstracted. In this way, the students are entering into the solving of the problem through their ways of thinking and their knowledge which in turn is both valued and connected to a formula that they have co-constructed and that is now available for any student to use.

Combining the Two Approaches

Social justice experiences in the mathematics classroom can also be a combination of using social justice issues as a context for mathematics and the teaching and learning of mathematics in socially just ways. For example, consider the issue of the infectious diseases epidemics that are on the rise worldwide: Ebola, the Zika Virus, mumps, tuberculosis, antibiotic resistant superbugs, and so on. Each has substantial statistical data available to analyze mathematically through *z*-scores and probabilities, confidence intervals and margin of error, function regressions and so on. However, there is also the possibility of cultural mathematical knowledges, mathematics that has never been formally valued, even acknowledged, in Western mathematics that can enter into the classroom when exploring such topics. Consider, for example, the case of the Navajo Plague.

In 1993, the Centre for Disease Control (CDC) was called in to investigate a highly infectious and deadly disease that was ravaging the populations of the Navajo reservations in the Four Corners (the area where Arizona, Colorado, New Mexico and Utah meet). Young and healthy people in these communities were contracting a cough and then, often within a day or two, were dead. One CDC researcher, who was part Navajo, asked the communities what their healers had said about the disease, but no one had asked the healers, explaining that this was clearly a new modern infectious disease. This researcher, however, decided to go and speak to a healer. The healer showed the researcher a photo of a sand painting done many years earlier by another healer. In great detail, the sand painting told the story of at least 3 years of excessive rain, followed by the increased growth of a particular conifer, and the resulting increase of a very particular kind of mouse. This mouse was then shown going into the Navajo homes, the majority of which still had dirt floors. It then showed how the failure to properly clean up the faeces and urination of these mice resulted in the people in the homes dying from a disease with the same symptoms as what was being currently seen. The researcher was shocked, realizing that this sand painting was depicting what science knew about the transmission and prevention of transmission of the Hanta Virus. Upon reporting these findings to the head of the CDC investigation however, the researcher was told that in no way could this be the Hanta Virus because the deer mouse that carries the virus (and which was shown in the sand painting) was not endemic to the region. The CDC continued testing for other virulent and modern diseases, such as the Ebola virus, and it was only much later that they did test for the Hanta Virus, and the test was positive. By then, many people had died and communities surrounding the Four Corners reservations were banning Navajo people from their businesses for fear of their contracting of the "Navajo Plague." (For more information on this situation, see The Scalpel and the Silver Bear by Ariso Alvord and Cohen Van Peet 1999.)

So, what is the relevance of this story to mathematics understanding, teaching, and learning? There are no computations, no mathematical models, and no trend analyses present. But, there is statistical thinking and hundreds of years of evidence. It is just that the statistics and evidence were not documented, represented, and analysed in the ways that Western mathematics (and science) values. This is statistical knowledge that has been embedded in culture and oral traditions that are different from those of Western mathematics and science. It is statistical knowledge that has been shared over the centuries through stories and sand paintings. And, it is statistical knowledge that, if it had been valued, could have saved so many lives and community relations. Thus, it is mathematical knowledge worth knowing, and worth valuing.

This is not to say that Western mathematics (and science) should be replaced by Indigenous knowledges and ways of knowing. The Western mathematics absolutely serves valuable purposes, but so do other mathematical kinds of knowledge and ways of knowing. It is all about context or place, because that defines the knowledge of most value to the knower. As Kumashiro (2001) argues,

> mathematics and sciences can be taught in ways that constantly look beyond what is being learned and what is already known. … educators can approach the teaching of maths and sciences in paradoxical ways: simultaneously learn about the contributions of science and math, while exploring ways that science and math have closed off further learning, privileged only certain knowledge, and in the process, contributed to oppression. Educators can teach students to be not only mathematicians and scientists, but also math critics and science critics. (p. 7)

As future mathematics teachers, and as continuing mathematics learners, you, as pre-service teachers, have the opportunity to open up mathematics teaching and learning, shifting from the singular and authoritative view of Western mathematics and create space for many kinds of mathematical knowledge and mathematical ways of knowing. In so doing, learners of mathematics, being given the opportunity to come to mathematics in their own ways and being invited to build connections between different mathematical knowledges and ways of knowing, can develop greater mathematical competence, confidence, and fluency.

References

Arviso Alvord, L., & Cohen Van Peet, E. (1999). *The scalpel and the silver bear*. Toronto: Bantam Books.

Blake, C. (2015, May 13). *Teaching social justice in theory and practice*. Retrieved from http://education.cu-portland.edu/blog/news/teaching-social-justice/

Dell'Angelo, T. (2014, September 19). *Creating classrooms for social justice.* Retrieved from https://www.edutopia.org/blog/creating-classrooms-for-social-justice-tabitha-dellangelo

Gutiérrez, R. (2013). The socio-political turn in mathematics education. *Journal for Research in Mathematics Education, 44*(1), 37–68.

Kumashiro, K. K. (2001). "Posts" perspectives on anti-oppressive education in social students, English, mathematics, and science classrooms. *Educational Researcher, 30*(3), 3–12.

Russell, G. (2017). *Transreform radical humanism: A mathematics and teaching philosophy*. Rotterdam: Sense Publishers.

Sensoy, O., & Diangelo, R. (2009). Developing social justice literacy: An open letter to our faculty colleagues. *The Phi Delta Kappan, 90*(5), 345–352.

Sullivan, P., & Lilburn, P. (2002). *Good questions for math teaching: Why ask them and what to ask? (Grades K-6)*. Sausalito: Math Solutions.

Additional Suggestions for Further Reading

Nasir, N. S., Cabana, C., Shreve, B., Woodbury, E., & Louie, N. (2014). *Mathematics for equity: A framework for successful practice*. New York: Teachers' College Press.

Radical Math. http://radicalmath.org/main.php?id=SocialJusticeMath

Resources from the Creating Balance Conference: http://www.creatingbalanceconference.org/resources

Schoenfeld, A. H. (2004). The math wars. *Educational Policy, 18*(1), 253–286.

Stocker, D. (2008). *Maththatmatters: A teacher resource linking math and social justice* (2nd ed.). Ottawa: CCPA Education Project.

A Teacher's View – Problem Solving in the Secondary Classroom

Shawn Godin

Abstract In this chapter we examine problem solving in the secondary classroom from the point of view of a practicing teacher. Problem solving will be explored as a means of learning mathematics as well as a goal unto itself. Sample problems to engage students are discussed with references to the points in the curriculum where the problem is relevant. Connections will be made to other pedagogical practices such as the use of technology and cooperative learning. We will also discuss considerations for the classroom teacher in the assessment of student work. Finally, we will discuss some sources of nice problems.

Keywords Problem solving · Cooperative learning · Technology · Assessment

Introduction

Problem solving lies at the very heart of mathematics. All of the mathematics that we teach came from someone's need to solve a problem, their discovery of a pattern, or their generalization of some previously known facts. The most important role of mathematics in the school curriculum is to expose students to problem solving, both concrete and abstract. This chapter describes the on-going efforts of a secondary teacher of mathematics in Ontario to shift his classroom practice to have a greater emphasis on problem solving as both a means of learning mathematics and a valuable skill unto itself. "Solving problems is not only a goal of learning mathematics but also a major means of doing so. … By learning problem solving in mathematics, students should acquire way of thinking, habits of persistence and curiosity, and confidence in unfamiliar situations that will serve them well outside the mathematics classroom" (National Council of Teachers of Mathematics [NCTM], 2000, 52).

Problem solving is of such importance that it is one of the five process strands focused on by the NCTM (2000). The mathematical curricula from many regions

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use the NCTM's *Standards* as a stepping stone; as such, problem solving has a central role in these mathematics curricula. In Ontario, for example, all three Ontario mathematics curricula contain the following quote: "Problem solving forms the basis of effective mathematics programs and should be the mainstay of mathematical instruction. It is considered an essential process through which students are able to achieve the expectations in mathematics, and is an integral part of the mathematics curriculum in Ontario" (Ontario Ministry of Education 2005a, pp. 11–12, 2005b, pp. 12–13, 2007, p. 18). The Ontario mathematics curricula include seven mathematical process expectations attached to all mathematics courses. Five of these process expectations are derived directly from the five process strands of NCTM (problem solving, reasoning and proving, connecting, representing, communicating), while the other two (reflecting, selecting tools and computational strategies) have strong connections to the problem solving process. Problem solving is seen as a thread that ties all the mathematical processes together as well as a vehicle for their development. "The mathematical processes are interconnected. Problem solving and communicating have strong links to all the other processes. The problem-solving process can be thought of as the motor that drives the development of the other processes" (Ontario Ministry of Education 2007, p. 17).

In the past, problem solving may have been seen by some teachers as those hard application questions that reside in section C of the textbook exercises. Although these have their place, there are many other ways that problem solving can be used to help students learn and understand mathematics. The purpose of this chapter is to examine various ways that problem solving can be used effectively in your classroom as well as considering its coordination with other parts of your classroom activities from the perspective of a current high school mathematics teacher.

Throughout this chapter, ties to courses in the Ontario mathematics curricula will be made. In Ontario, students must earn at least three mathematics credits in order to graduate. Courses in grade 11 and 12 are divided into three categories based on the destination of the student. Students may take university preparation courses, college preparation courses or workplace preparation courses. Courses in grades 9 and 10 are broken into three categories and serve as prerequisites of courses in later grades. Academic courses[1] are prerequisites for the university preparation courses, Applied courses are prerequisites for the college preparation courses and Locally Developed courses are prerequisites for workplace preparation courses.

In Ontario, students' knowledge and skills are broken down into four broad categories: knowledge and understanding, application, communication, and thinking. The three curriculum guidelines define these categories, outline considerations for the assessment and evaluation of student work and provide a rubric, the achievement chart, to help guide teachers in the development of their own tasks and evaluation tools (Ontario Ministry of Education 2005a, pp. 18–23, 2005b, pp. 17–22, 2007, pp. 23–29).

[1] See McDougall and Ferguson (Part II this volume, para. 1) for a discussion of two of the possible Ontario pathways (Academic and Applied).

If given $\triangle ABC$ where $\angle B = 72°$, $\angle C = 53°$ and $AB = 9.5$ cm, determine, to the nearest tenth of a centimetre, the length of AC.

Fig. 1 A problem to introduce the sine law

Learning Through Investigation

In many instances students will have the background knowledge to develop fully, or partially, a concept with little scaffolding from their teacher. Encouraging students to think ideas through for themselves, or with a group, gives the concept a firmer grounding than if the students were just told it as a fact.

For example, in the Ontario Grade 10 Academic mathematics course curriculum, students need to "explore the development of the sine (cosine) law within acute triangles" (Ontario Ministry of Education 2005b, p. 51). The language implies an investigative approach, rather than just presenting students with the formula. Although student reproduction of the development of each of these laws is not required, students can benefit by exploring related problems. For example, consider the problem in Fig. 1.

Students will attack this problem in many different ways: from measurement on a scale diagram or manipulative like a geoboard; to technologically assisted solutions using dynamic geometry software; or just doing a search of the internet. Some students who have been introduced to right triangle trigonometry will explore what happens if the given triangle is decomposed into right triangles, which will lead to a solution. Students can then be prompted to generalize the situation.

The variety of methods can lead to some rich discussions about the various types of solutions. The measurement method is fairly quick but lacks a precision that may be needed in similar problems. The dynamic geometry solution gives us an answer but without further investigation, does not give us any idea where the result came from. The internet search will give us a formula but not, without further reflection, any idea of why it works.

The collection of the varied ways of solving the problem complements each other. Students who used trigonometry may not have generalized the idea into a formula, but the development of the formula will be much easier understood and retained. Similarly, students who used other methods may better appreciate the need for the formula and see its connection with prior learning. When the students are presented with the development of the law, either from a student's solution or teacher presentation, they will be intellectually invested to a higher degree than if either law was presented solely as a lecture. Teachers benefit from these types of explorations because students will often come up with novel ways of looking at things that can be incorporated into their own practice. In all cases students benefit from seeing a problem solved in different ways, because it builds stronger connections and possibly introduces methods that might be used in future problems.

An important idea, when presenting students with problems, is to let the student do the thinking. There exist many examples of problems and activities where the think-

A square is constructed on an 11 by 11 pin geoboard. What are all the possible areas for the square?

Fig. 2 An introductory problem for grade 9 students

ing is done for the students. In these problems students are presented with the steps needed to solve the problem, and they just need to follow them rather than constructing and following their own plan. You and your students are better served to strip things down as much as possible and provide support as needed (see Meyer 2010).

Engaging Prior Knowledge

I use the problem in Fig. 2 in the first few lessons of Grade 9 Academic mathematics. This problem is great for getting students to "think outside the box," since most students quickly claim to be done and provide me with the results: 1, 4, 9, 16, 25, 36, 49, 64, 81 and 100. Upon finding out that they have less than a third of the results, they get back to work. Through their exploration you will have an opportunity to remind them of areas of triangles and squares, decomposing shapes to determine area, complementary angles, the Pythagorean theorem and the fact that the process of squaring a number and taking the square root of a number are inverse operations on the positive real numbers. As well, the problem foreshadows slope and its relation to parallel and perpendicular lines from later in the course, ties into the distance between two points that they will investigate in grade 10, and ties into the problem in number theory of writing numbers as the sum of two squares.

This problem requires students to access past knowledge, and possibly see the connection between concepts that they had not seen before. Also, by being related to material students will study in the present course, it gives the chance to tie into this activity later in the year. You can see a more in-depth discussion of the problem in Godin (2011).

Turning the Student into the Teacher

In our role as teachers, we will regularly construct questions with the answer and method of solution in mind and go through a problem solving process ourselves in the act of developing a question. We can give similar tasks to our students. In the Ontario grade 12 university preparation course Calculus and Vectors, students are asked to "**4.7** solve problems relating to lines and planes in three-space … involving distances … or intersections" (Ontario Ministry of Education 2007, p. 110). A typical exercise would involve giving the equations of two lines and asking for their point of intersection or the distance between them if they do not intersect. We can turn this process on its head by asking a question like the one in Fig. 3.

> *Determine the equation of a line that is a distance of 5 units from the line with equation* $\vec{r} = (7, -2, -1) + k(8, -3, 4)$, $k \in \mathbb{R}$.

Fig. 3 Reimagining a typical exercise

> *A student has two open-topped cylindrical containers. The larger container has a height of 20 cm, a radius of 6 cm and contains water to depth of 17 cm. The smaller container has a height of 18 cm, a radius of 5 cm and is empty. The student slowly lowers the smaller container into the larger container. As the smaller container is lowered, the water first overflows out of the larger container and then eventually pours into the smaller container. Determine the depth of the water in the smaller container when the smaller container is resting on the bottom of the larger container.*

Fig. 4 An out of the ordinary volume problem

The problem in Fig. 3 is not overly difficult, but that is not the point. As it stands many students will have an entry point and be able to demonstrate some understanding of course material. What makes the problem interesting is that there are a variety of ways you can solve it, and there are many connections that can be made to material inside and outside the course. If students understand that you are interested in the thoroughness of their solutions, this one question can give insight into the students' knowledge and understanding of many of the concepts and skills from the vectors part of the course. Students could bring into their solution any, or all, of the following: algebraic and geometric representations of vectors; equations of lines and planes; parallel and skew lines; lines parallel to a plane; the dot and cross product, and other vector operations, and their properties; intersections of lines with lines or planes; length of vectors; unit vectors; spanning sets. As such, their discussion could hit on all four expectations from the Geometry and Algebra of Vectors strand.

Novel Problems

It is useful to present students with problems that are out of the ordinary. To solve the problem students need to use some very specific skills or knowledge, although their method of solution as well as other skills that are used may be varied. For example, in both the Ontario Grade 9 Academic and Grade 9 Applied mathematics curricula, students are asked to "solve problems involving … volumes of … cylinders" (Ontario Ministry of Education 2005b, pp. 37, 45). The problem in Fig. 4 is non-standard and forces students to use the idea of volume of a cylinder.

With either audience, you could accompany the problem in Fig. 4 with some experiments involving water and cylinders to have the students understand what is happening and to help formulate their method of solution. This problem originally appeared as question #25 of the 2002 Pascal contest run by the Centre for Education

You have a cylindrical vase that has diameter 10 cm and height 20 cm. Water is put in the vase to a depth of 15 cm. You also have a collection of spherical glass marbles with diameter of 2 cm. Determine the maximum number of marbles that can be added to the vase without spilling any water.

Fig. 5 A lead up problem to the problem in figure 4

in Mathematics and Computing at the University of Waterloo. You can check out the full contest and solutions online at www.cemc.uwaterloo.ca. My analysis of the problem can be found in Godin (2013).

The wonderful thing about mathematics contests is that they are full of problems that are not of the standard variety. You can find many wonderful problems that you can use in your classes and others that will inspire you to construct your own problems that may be more appropriate for your students.

The last example (Fig. 4) was the last problem of a contest, which means that it is very difficult for the target audience. As such, few students may be able to come up with a correct solution. Although there is value in incorrect and incomplete solutions, most often you want problems that will engage all of your students. With this in mind, I recently came up with the problem in Fig. 5, inspired by the cylinder problem, but which allowed a greater portion of my class to successfully complete the problem.

Much of the same ideas are used in the solution, but with fewer layers, making it easier for students to generate and execute a plan successfully. Most of the student groups solved the problem without errors, although some rounded their final result without thinking to give a close, but incorrect answer. This problem could also be used as a warm up for the original contest problem.

Real World Problems

Another great venue for problem solving is open-ended, real world problems with no exact answer. A beautiful example of this is a problem by a colleague of mine, Alex Overwijk, a master of creating real world problems.

I will summarize his problem. He had students throw cards, one at a time, into a box a fixed distance away. The students collected lots of data so that they could estimate their rate of successfully throwing cards in a box. The teacher performed the experiment as well and let the students know his rate. The problem that they were faced with is this: each student would be in a match against Alex. Alex would give each student a “head start” by starting them with some cards already in the box. The students then had to determine how long the match should last so that they beat Alex and the contest was close (to make it more exciting). Students then analyzed the problem and came up with a strategy that they could then use to determine their desired time, which they could test with an actual match. Alex used this problem with his Grade 10 Applied class with great success. The full write-up of the problem, with examples of student work, is available online (Overwijk 2015). It is worth

> *In a triangle a **median** is a line segment joining one of the vertices of the triangle to the midpoint of the opposite side. Explore medians in a triangle; determine as many properties as possible.*

Fig. 6 A dynamic geometry investigation

checking out Alex's blog, http://slamdunkmath.blogspot.ca, for other examples of his work.

Technologically Aided Explorations

One of the most powerful types of software for student mathematical discovery is dynamic geometry software like *The Geometer's Sketchpad* and *Geogebra*. These platforms allow students to search for patterns as well as discover and describe geometric theorems using their own methods and words.

Many specific, and a few overall expectations from the Ontario Mathematics curriculum, for example, mention dynamic geometry software. Some of these involve a "low end" use, for example from the Frade 11 Foundations for College Mathematics course: "**2.2** verify, through investigation using technology … the sine and cosine law (e.g., compare, using dynamic geometry software, the ratios $\frac{a}{\sin A}$, $\frac{b}{\sin B}$, and $\frac{c}{\sin C}$ in triangle *ABC* while dragging one of the vertices)" (Ontario Ministry of Education 2007, p. 73). Others are more open-ended and allow for a deeper use of the tool. In both the Ontario Grade 9 Academic and Applied mathematics courses, one of the overall expectations is "verify, through investigation facilitated by dynamic geometry software, geometric properties and relationships involving two-dimensional shapes" (Ontario Ministry of Education 2005b, pp. 36, 44), which allows for many possible tasks.

The task in Fig. 6 allows students to discover as many attributes of medians as possible, as well as possibly explaining why some work. For example, a student should be able to explain that a median cuts a triangle into two triangles of equal area. They may also be able to explain that when three medians are drawn, the six triangles that are formed all have the same area. Yet they may not be able to explain why the three medians cross at a common point, but that is fine.

Working in Groups

One of the first considerations you must make before presenting a problem to the students is how they are going to work on it. You have the two extremes: alone or in a group, as well as a range of blended options. It is important for students to have the opportunity to work on their own, as well as to collaborate with others, so you must build into your practice occasions for both to happen. You can blend the two extremes in various ways. For example, you could give a problem to the students and have them work quietly alone for 5 min to come up with a strategy, then they could be put into groups where they would share their strategies, and pick one in which they will solve the problem. If time permits, they could also resolve the problem with a different strategy so that they could better compare the two strategies. At the other end, you could have students solve the problems alone, for example as a homework assignment, then get together in groups to discuss their solutions and write up a "perfect" solution.

Assessing and Evaluating Problem Solving

You may be wondering how to assess and evaluate problem solving in the classroom. It depends on the purpose of the problem solving task. If you are using problem solving to get at deep understanding of the mathematics curriculum, or for concise and insightful selection of mathematical tools, then the evaluation tools would be ones for knowledge and application such as those given in the achievement chart (Ontario Ministry of Education 2005a, pp. 22–23, 2005b, pp. 20–21, 2007, pp. 28–29) with a written product work well. The problem in Fig. 3 would be a case where problem solving is occurring, but we, as teachers, are interested in the depth of the students' knowledge and understanding of the concepts in the course. Much of the details of the problem solving process will be lost in any written work the students may hand in. Any false starts the students made or observations by trial and error will probably not show up in their final written work. They may have also discovered an insightful concise solution, but think that the teacher is looking for something more algebraic so their write-up leads to the answer they got from the insightful connection, but it is written up in a way such that the original idea is lost. As such, in this situation it would be better to evaluate knowledge, communication and application more so than thinking (problem solving). There will be other places that we can evaluate thinking and problem solving more effectively.

If we want to assess our students' thinking, we must come up with ways to specifically get at it. One way is to have students keep a record of their problem solving process. They would write down their plan, and why they are doing it. If they run into any snags as they go through the process of solving the problem, they record it, as well as any reflections on reworking the plan. As such, you could have a student give you the solution and all their thoughts as they occurred, including errors that they made along the way, as a map of their problem solving process. You could then provide the student with some detailed feedback to help them further develop their problem solving skills. To be effective, the feedback must come quickly while the problem and process are still fresh in the student's mind. This could be accomplished either by conferencing with the student, or by collecting work from a small sample of the class to allow you to get the feedback back to the students quickly. Given the extra work being done by the students and the time constraints for the teacher to provide feedback, this is a worthwhile exercise that could be done a couple of times during the course.

A second way to get at the thinking is to interview the students after they have solved the problem. Asking the students why they did certain things, if they ran into any roadblocks and what sort of things they thought of upon completion can be quite insightful. Many times students will go through a problem correctly, but when you ask them why you find out it was based on an erroneous idea that somehow paid off. This can lead to deeper probing into the nature of the misunderstanding. On the other hand a student may hand in a very humdrum solution, but the thinking behind their decisions, and observations that they made along the way may prove to be very insightful.

Another great way to get at thinking, especially when students are working in groups, is to just listen to their conversations. Being a fly on the wall allows you to hear some great conversations between students. Insightful students are forced to communicate their ideas to their collaborators and convince them that their ideas are valid. The teacher needs to build a classroom environment such that group activities are not just about getting the answer but are about the group understanding of the problem and its solution. If it is each group member's duty to explain their ideas to others, and to question when they do not understand, rich learning takes place and the teacher can observe many things, including the thought process of the students.

Finding Good Problems

Another thought that will probably come to mind is "where can I get some nice problems?" As seen in Fig. 4, mathematics competitions are a great place to access good problems. The Centre for Education in Mathematics and Computing hosts over a dozen mathematics and computer contests each year, and their website, www.cemc.uwaterloo.ca, contains over 100 past contests as well as other resources for mathematics teachers. The Canadian Mathematical Society (CMS) also hosts its own mathematics contest, the Canadian Open Mathematics Challenge, which leads to the Canadian Mathematical Olympiad. On the CMS website, https://cms.math.ca/, are copies of past contests, plus links to other Canadian mathematics competitions as well as other resources for teachers and students. The Art of Problem Solving is an American website, set up by former members of the USA mathematical Olympiad team, to aid students in problem solving. There are hundreds of old competitions, and other free resources on their website: https://artofproblemsolving.com/. They also sell books and offer online courses that students can take to help them be better mathematical problem solvers. *Crux Mathematicorum* is an international problem solving journal published by the CMS. Although most of the current content of the journal is not accessible to most high school students, the earlier issues had more material that students could work with. All past issues are available online at https://cms.math.ca/crux/. The Mathematical Association of America is a professional organization for mathematicians, like the CMS. Although most of their material is aimed at University teachers and students, they do sell some problem solving books and have some journals on their website, www.maa.org, that could be useful to a high school teacher. Finally, the author writes a column in the *Ontario Mathematical Gazette*, available from the Ontario Association for Mathematics Education: www.oame.on.ca. There are many other international mathematics competitions, journals and websites that teachers can use as resources.

Conclusion

Doing mathematics is not about solving 20 exercises of the same type, which is more suited to building a specific skill. Doing mathematics is about solving real problems. This involves us to, as Ms. Frizzle from *The magic school bus* says, "Take chances! Make mistakes! Get messy!" (Scholastic 2017, para. 1). Mathematics is a beautiful and useful subject. Problem solving allows students to experience both of these worlds as well as better engaging them in the learning process.

References

Godin, S. (2011). What's the problem? Getting things squared away. *Ontario Mathematics Gazette, 49*(3), 8–10.
Godin, S. (2013). What's the problem? Cylinders within cylinders. *Ontario Mathematics Gazette, 51*(4), 8–9.
Meyer, D. (2010, March). *Math class needs a makeover* [Video file]. Retrieved from https://www.ted.com/talks/dan_meyer_math_curriculum_makeover.
Ontario Ministry of Education. (2005a). *The Ontario curriculum, grades 1–8, mathematics* (Rev. ed.). Toronto: Queen's Printer for Ontario. Retrieved from http://www.edu.gov.on.ca/eng/curriculum/elementary/math.html
Ontario Ministry of Education. (2005b). *The Ontario curriculum, grades 9 and 10, mathematics* (Rev. ed.). Toronto: Queen's Printer for Ontario. Retrieved from http://www.edu.gov.on.ca/eng/curriculum/secondary/math.html
Ontario Ministry of Education. (2007). *The Ontario curriculum, grades 11 and 12, mathematics* (Rev. ed.). Toronto: Queen's Printer for Ontario. Retrieved from http://www.edu.gov.on.ca/eng/curriculum/secondary/math.html.
National Council of Teachers of Mathematics. (2000). *Principles and standards for school mathematics*. Reston: Author.
Overwijk, A. (2015). *Activity: Tossing cards for student engagement*. Retrieved from http://ocdsb-flashpoints.weebly.com/flash-points/archives/12-2015
Scholastic. (2017*). The magic school bus: FAQs*. Retrieved from http://www.scholastic.com/magicschoolbus/parentteacher/faq.htm

Additional Suggestions for Further Reading

Lockhart, P. (2009). *A mathematician's lament*. New York: Bellevue Literary Press.
Overwijk, A. *Slam dunk mathematics* [blog]. Retrieved from http://slamdunkmath.blogspot.ca.
Pólya, G. (1988). *How to solve it* (2nd ed.). Princeton: Princeton University Press.

Part IV: Commentary – Characteristics of Mathematical Challenge in Problem-Based Approach to Teaching Mathematics

Roza Leikin

There is a strong consensus among mathematics educators, researchers and instructional designers that mathematical problem solving is among the central means—and ends—of school mathematics education. Different ideas, practices and studies in the field of mathematical problem solving are reflected in the volumes, chapters, and papers published over the course of the last 30-plus years (Felmer et al. 2016; Lester and Charles 2003; Schoenfeld 1985; Silver 1985). The problem-based approach to teaching mathematics assumes that students are presented with authentic problems that are meaningful for them, and that can be solved using mathematical tools available to them. The problem-based approach seeks to develop new mathematical knowledge and skills through solving such problems. Moreover, when solving these problems the students are assumed to develop appreciation for the power of mathematics to solve problems from different fields of life and science.

Cai (2010) argues that mathematics teaching is a system of interrelated dimensions that include the nature of classroom tasks, their content and context, the teacher's role, the classroom culture, mathematical tools, and concern for equity and accessibility. The collection of works in this section of the book demonstrates the richness and variability of problem-based approaches to teaching mathematics designated and advanced by the members of the Canadian mathematics education community.

In this response chapter, I address the nine manuscripts in this volume that are devoted to the problem-based approach to teaching and learning mathematics. Interestingly, the authors differ in their views on what constitutes good problems, the corresponding goals of mathematical instruction, teachers' role in the management of the problem solving process and the ways in which different participants of the problem solving session can be supported when solving the problems.

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Good Problems

Problem-based approach to learning unifies choices of "good problems" by the authors. All the tasks that the authors use are non-routine and directed at communication among students about the problems, and activation and development of critical reasoning and decision making. Overall, the authors tend to share the three preferences or their combinations: non-routine mathematical problems, context-based problems that require modeling, and curricular-related problems accompanied by specific explorative design.

Non-routine Mathematical Problems

For Liljedahl, good problem solving tasks are highly engaging, non-routine, collaborative tasks that encourage mathematical discussion among students as part of the problem solving process. These problems promote students' mathematical thinking ("developing a thinking classroom") and allow regulation of the level of mathematical challenge, such that the level of challenge can be fit to each student in the class. The importance of the non-curricular nature of mathematical tasks is also addressed by Hoshino, who suggests choosing problems from logic puzzles and contest questions, and by Godin, for whom "mathematics competitions are a great place to access good problems." Godin categorizes good problems as follows: investigation tasks, novel problems, real world problems and technologically aided explorations. The significance of technological tools is also addressed by Saldanha and Thibault, who describe activities that can be done with the use of *TinkerPlots*, an interactive and dynamic data exploration software that advances statistical reasoning.

Context-Based Meaningful Problems

In Savard's view, meaningful problems must guide students to make sense of mathematics within a meaningful context and require mathematical modelling of the situation described in the problem in order to solve it. When solving these problems, mathematical knowledge is needed to study the event or the phenomenon. Similarly, according to Martin, Oliveira and Theis, mathematical tasks should allow students to develop new knowledge through real mathematical activity.

Emphasis on the problems' context both as means and ends of the educational process is made by several authors. Interestingly, social justice and citizenship unify several chapters. Savard stresses the importance of the careful choice of the problem context to make solving problems intriguing to students, and argues that choosing a context associated with citizenship develops both students' citizenship skills and their mathematical knowledge and skills. Such problems encourage critical thinking, advance awareness of cultural context, and connect cultural context to the mathematical knowledge to be developed.

Russell maintains that social justice can be embraced by mathematics education in two ways: as a context for mathematical problem solving, and by teaching mathematics in socially just ways. In line with the first way, Mamolo, Thomas and Frankfort describe their experiences of incorporating social justice context problems related to the variety of topics that meet students' needs and interests into problem-based practices, thus making mathematics learning meaningful for students. In this context, mathematics serves as a tool for "understanding world issues and social trends." The authors introduce tasks that present authentic problems and allow students to choose which mathematical (and non-mathematical) tools and skills to use in solving. They also require students to discuss and defend their solutions. Mathematical exploration is an integral part of these good problems.

Explorative Design of Curricular Problems

While focusing on content-related areas (e.g., statistics in the chapter by Saldanha & Thibault, and probability in the chapter of by Martin, Oliveira, & Theis) the authors stress the importance of careful choice of appropriate didactical settings and problem solving approaches. For example, Martin, Oliveira and Theis stress the mathematical power of the combination of different approaches to solving probability problems while Saldanha and Thibault emphasize explorative technology-based learning involving dynamic and visual imagery of data, as well as the importance of encouraging students to share ideas and explain their thinking.

Martin, Oliveira, and Theis stress that good problems should engage students and allow them to discover for themselves some of the means needed for solving, using comparison, connection, and sharing of ideas.

Instructional Setting and Teachers' Roles in Monitoring Problem Solving Activity

One of the central roles of a teacher is devolution of good tasks to learners (Brousseau 1997; Steinbring 1998). When assigning cognitively demanding tasks to a particular classroom, teachers should "feel" their students, in order to ensure that the students are capable of solving the task. Moreover, development of students' mathematical reasoning is linked to the knowledgeable choice of challenging mathematical tasks and the integration of the tasks in appropriate settings (Choppin 2011).

Teachers ought to provide each and every student with learning opportunities that fit their abilities and motivate them to learn. Teachers should create an instructional setting that supports and advances the problem solving process. All the authors stress that problem solving should be appropriate to students' knowledge and problem solving capacity at the given time. Investigation, exploration and challenging ideas are the core elements of the problem solving activities in this section of the book. All the chapters in this section address these aspects with different levels of detail.

Liljedahl introduces 11 elements that determine the effectiveness of problem solving activities, and that encourage mathematical thought in students. These elements include starting a lesson with the task, random arrangement of small groups, use of oral instructions, defronting the classroom and answering "keep thinking" questions only. Liljedahl suggests providing students with autonomy, using hints and extensions to allow flow and "levelling to the bottom." Martin et al. suggest that one of the critical features of the problem solving setting is the problem's adaptability, that is, its level of complexity and the ease of adjusting it to students' levels of mathematical development. Another way of adjusting the instructional level is providing students with opportunities to explicitly develop various strategies for solving new problems, such as trial and error, using a model, trying a simpler problem, working backwards, and discussing their ideas in small groups (Atiya, Luca, & Kajander).

Atiya et al. focus their attention on the ways to support development of beginning teachers' proficiency and beliefs in managing problem-based instruction. They suggest gradually making classroom tasks more and more open, as students develop more strategies. An additional practice of "turning students into teachers" is suggested by Godin. Godin also acknowledges the importance of implementing a variety of settings, incorporating group work, independent work, and different combinations of the two in order to allow all students to participate actively in problem solving activities.

Mathematical Challenge As a Core Element of a Problem-Based Approach to Teaching Mathematics

Mathematical challenge, which is an interesting and motivating mathematical difficulty that a person can overcome at a particular stage (Leikin 2007, 2014), is a unifying characteristic of all the mathematical activities, tasks and problems described in this section of the volume. Here I suggest a theoretical model of a mathematical challenge embedded in a problem solving activity (Fig. 1). This model can shed light on the collection of papers observed here and suggests an additional lens for the analysis of problem solving activities suggested by the authors. The model comprises several complimentary elements, which can enhance and support each other in the creation of mathematically challenging situations. These elements include (a) intrinsic (conceptual) characteristics of mathematical problems and tasks which are in the center of mathematical activity; (b) characteristics surrounding a problem or a task such as socio-mathematical norms, instructional setting and (c) individual characteristics of the participants, such as their familiarity with the topic of the problem or their problem solving proficiency (see Fig. 1).

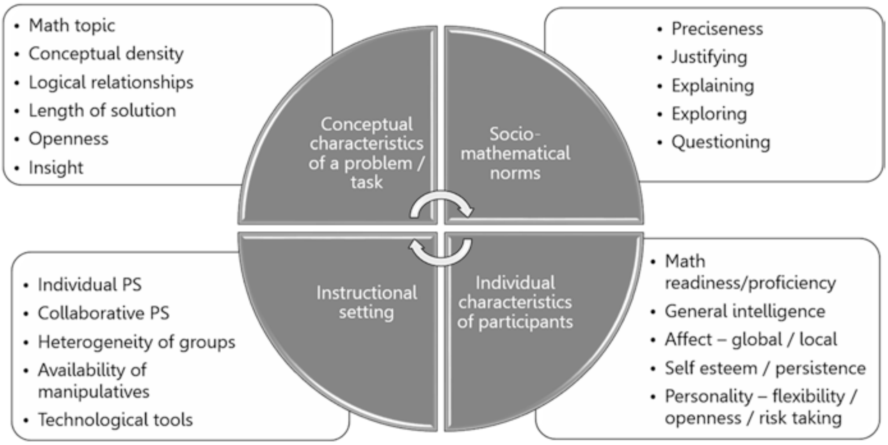

Fig. 1 Characteristics that determine mathematical challenge

The level of cognitive demand (Silver and Mesa 2011) of a particular task depends on the type and conceptual characteristics of the task, such as conceptual density, mathematical connections, and the logical relationships required for solving the problem (Leikin 2009, 2014; Silver and Zawodjewsky 1997). The openness of a problem solving task also determines its level of cognitive demand. For example, open-ended problems allow multiple answers to a problem along with critical evaluation of completeness of the set of answers (e.g., Pehkonen 1995). Open-start problems are usually multiple solution tasks (MSTs – Leikin 2007) that require solving the problem using multiple solution strategies, through activation of mental flexibility and mathematical connections. Tasks such as sorting tasks, problem posing tasks and investigation tasks are both open-start and open-end problems. Several chapters in this book section (e.g., Saldanha & Thibault and Martin, Oliveira & Theis) include excellent examples of mathematical investigation tasks. Mathematical challenge can be strengthened by the use of a non-mathematical context and the requirement to design a mathematical model that represents this context (see examples in Savard and in Martin, Oliveira & Theis). The mathematical challenge embedded in a task can be increased by socio-mathematical norms such as requirement of preciseness, explanation and justification, comparison and classification (Leikin 2014; Silver and Mesa 2011) and can be varied by instructional setting, for example, as described in the chapters by Atiya, Luca and Kajander, Liljedahl and Goding. The familiarity of the topic and associated problem solving proficiency as well as personal characteristics of the participants are additional criteria that characterize the solver rather than the problem.

The chapters in this section of the book present a variety of ideas expressed by researchers, teacher educators and mathematics teachers who share their authentic experiences and the methods that they find effective in mathematics teaching. One of the most challenging aspects of the mathematics lesson management is making the mathematical problems that students solve challenging to each student in the classroom. While different chapters refer to different components of mathematical challenge to different extents, the collective account describes a rich collection of challenging problem-based activities that can be used by teachers in their classes and by mathematics educators in teacher education settings.

Rather naturally, the practices described in these chapters can serve as an excellent playground for researchers who are interested in getting a better understanding of which approaches are more effective for the development of students' mathematical knowledge, skills and problem solving expertise along with the development of students' motivation and self-esteem in learning mathematics. The model of mathematical challenge suggested here can serve as a framework for the analysis of the quality of problem-based teaching of mathematics.

References

Brousseau, G. (1997). *Theory of didactical situations in mathematics*. Dordrecht: Kluwer.

Cai, J. (2010). Commentary on problem solving heuristics, affect, and discrete mathematics: A representational discussion. In B. Sriraman & L. English (Eds.), *Theories of mathematics education, advances in mathematics education* (pp. 251–258). Heidelberg: Springer.

Choppin, J. (2011). The role of local theories: Teachers' knowledge and its impact on engaging students with challenging tasks. *Mathematics Education Research Journal, 23*(1), 5–25.

Felmer, P., Pehkonen, E., & Kilpatrick, J. (2016). *Posing and solving mathematical problems*. Dordrecht: Springer.

Leikin, R. (2007). Habits of mind associated with advanced mathematical thinking and solution spaces of mathematical tasks. In *The fifth Conference of the European Society for Research in Mathematics Education – CERME-5* (pp. 2330–2339). Retrieved from http://ermeweb.free.fr/Cerme5.pdf

Leikin, R. (2009). Exploring mathematical creativity using multiple solution tasks. In R. Leikin, A. Berman, & B. Koichu (Eds.), *Creativity in mathematics and the education of gifted students* (pp. 129–145). Rotterdam: Sense Publisher.

Leikin, R. (2014). Challenging mathematics with multiple solution tasks and mathematical investigations in geometry. In Y. Li, E. A. Silver, & S. Li (Eds.), *Transforming mathematics instruction: Multiple approaches and practices* (pp. 59–80). Dordrecht: Springer.

Lester, F. K., & Charles, R. (2003). *Teaching mathematics through problem solving: Pre-K–grade 6*. Reston: National Council of Teachers of Mathematics.

Pehkonen, E. (1995). Introduction: Use of open-ended problems. *ZDM-International Journal of Mathematics Education, 27*(2), 55–57.

Schoenfeld, A. H. (1985). *Mathematical problem solving*. Orlando: Academic Press.

Silver, E. A. (1985). *Teaching and learning mathematical problem solving: Multiple research perspectives*. Mahwah: Erlbaum.

Silver, E. A., & Mesa, V. (2011). Coordinating characterizations of high quality mathematics teaching: Probing the intersection. In Y. Li & G. Kaiser (Eds.), *Expertise in mathematics instruction: An international perspective* (pp. 63–84). New York: Springer.

Silver, E. A., & Zawodjewsky, J. S. (1997). *Benchmarks of students' understanding (BOSUN) project* [technical guide]. Pittsburgh: LRDC.

Steinbring, H. (1998). Elements of epistemological knowledge for mathematics teachers. *Journal of Mathematics Teacher Education, 1*(2), 157–189.

Part IV: Commentary – Doing Is Not the Same As Thinking or Construing

John Mason

Introduction

It is well known and widely experienced that mathematical problem solving in general, and algebra in particular, form a watershed for many students. Something puts people off, whether in how it is presented (for example, when algebra is treated simply as "arithmetic with letters" and problems consist of burying arithmetic calculations in some context), whether in the switch from 'working forwards towards an answer' to 'working backwards starting from one or more unknowns' (Bednarz et al. 1996), whether in apparent lack of relevance or future usefulness, whether in transition to a different format of school as institution, or some combination of these and other factors. The chapters in this section delineate various ways in which teachers and teacher educators are trying to make a difference, trying to get students engaged with mathematics and involved in mathematical thinking.

As Martin, Oliveira, and Theis (this volume) point out, the very term *problem solving* is problematic, as it is used to refer to tasks ranging across a wide spectrum from the routine to the exploratory or investigative, and from the very familiar to the completely novel.

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Fostering and Sustaining Mathematical Thinking

The issue of how to engage students so as to develop their mathematical thinking and their disposition to think mathematically is not a new one: Robson (2000) suggests that 3000 years ago or more Egyptian scribes were called upon to make their mathematical exercises relevant by placing them in a practical context from outside of mathematics. The result was a collection of spurious problems involving, for example, vastly unrealistic quantities of different grades of grain. It seems that the scribes were intent on showing off computational facility, such as their ability to deal with calculations involving 1/7.

Educational researchers have taken up the baton of engagement, enquiring into it from many directions. The list of contributory or influential factors is extensive (e.g., Skilling 2014). The authors in this section continue this line of enquiry.

The mathematics curriculum up until the twentieth century was in most countries and in most instances, arrived at by stripping away any context from calculations thought to be currently important and which therefore are likely to be useful in the future. Students are then taught to carry through those calculations in the context of mathematics, or more accurately, numeracy. Put another way, teaching has been and usually still is assumed to be about training student behaviour so that learners can carry out operations on numbers. Applications to non-mathematical contexts tend to follow only after presumed mastery of procedures. Typical of this were the early mathematics texts known as Abaci, commissioned by patrons for use by the sons of merchants, as for example the book by the artist and mathematician Piero della Francesca (1412–1492) (e.g., Davis 1977). This focus belied or obscured the thrust of developments in geometry through the needs of renaissance artists, and the playfulness and inquisitiveness of mathematicians in every age.

The USMES project (Lomon et al. 1975) in the USA typified a wholescale switch to ‘relevance’ by basing instruction around work on a variety of projects or explorations, and only introducing mathematical techniques when students ran up against a need for them. At the other end of the spectrum, mathematical investigations have been promoted in the UK by the Association of Teachers of Mathematics from the 1960s to the present as the basis for teaching mathematics. Originally, these were mostly based in the context of mathematics itself, inspired by the work of John Wallis (1616–1703) and what he called ‘my method of investigation,’ which so irritated Pierre Fermat. Other projects (e.g., MES, MiS-UK and MiS-USA) have positioned themselves on this spectrum in various ways. Burkhardt (1981) parodied textbooks with his ‘alphabet’ of task types: **A**ction, **B**elievable, **C**urious, **D**ubious and **E**ducational tasks (p. 8), reinforcing disdain for the impracticality of so many problems found in mathematics textbooks through the ages.

An often overlooked dimension of engaging students is the observation that everyone takes pleasure in using their own natural powers. Small amounts of endorphins seem to be released when using your own powers; further pleasure is available from developing those powers within a mathematical context. Often described in terms of developing confidence, self-esteem, or agency, for me what lies at the heart of the matter is the use of one's own powers, rather than being subjected to the results of teachers and textbook authors displaying their powers but usurping the learners' use of their own powers. Students are then typically being called upon to 'do likewise,' when 'likewise' means carrying out routines or procedures already laid out by past experts (Gillings 1972; see also Witmer 1968). If only it were a simple matter to recognise precisely what 'likewise' implies!

There is more to 'doing' mathematics than carrying out procedures, and the complexity of this emerges through the chapters in this section. Von Glasersfeld (1988) captured this with: "To do the right thing is not enough; to be competent one must also know what one is doing and why it is right" (p. 328). Knowing in the moment to act, and knowing then how to act and why that action is appropriate, combine to make effective or competent action a complex issue, whether working with students, novice teachers, or experienced teachers (Mason and Spence 2000). There are several layers of awareness that constantly need refreshing and developing, from mathematical knowledge, maturity and sophistication, to how to work with others, be they students, teachers or teacher educators (Mason 1998).

It is well known that *mathematical knowledge*, meaning 'having mathematical actions become available when needed' (strictly, when the teacher or assessor thinks they should become available), is not a robust notion, since students appear to recall little from one institution to the next, one year to the next, one course to the next, even one week to the next. The notion of transfer was invented to try to piece together why ideas apparently mastered at one time were not available later (Detterman and Sternberg 1993), which is of course the 'likewise' issue. Transfer morphed into *situated cognition* as a description of basically the same phenomenon from a mostly sociological rather than a largely psychological perspective (Brown et al. 1989; Greeno et al. 1993; Watson and Wimbourne 2008). The abiding issue was, is, and continues to be, how gaps between contexts over which transfer occurs can be induced to widen, or how situatedness broadens, so that actions developed in one context become available in another. Here the issue is not whether learners have displayed an action in the past, but whether, in the future, an appropriate action becomes available; whether they know, in the moment, to act (Lai 2012; Mason and Spence 2000).

Engaging Learners and Teachers

Attempts to switch to an investigative, exploratory mode of problem solving often run up against an all too familiar phenomenon: students seem willing only to do what they have already been told how to do, with little enthusiasm or evidence of actually engaging, actually thinking (Atiya, Luca, & Kajander, this volume). I ran into this myself not long ago when I was called in to work with the staff of a private school who had set their students some novel and challenging problems only to have the students literally down-tools, fold-arms, and refuse to do anything because they had not been told what to do. A paradoxical feature is that these same students will spend hours trying to master a computer game, and some will do the same mastering skateboards and other sports skills.

Faced with students refusing to engage when asked to think rather than to carry out learned procedures in recognised contexts, Peter Liljedahl (this volume) advocates an extreme response. He systematically replaces institutional and teacher habits with their 'opposite' so as to force both teachers to act freshly, and students to respond freshly. The result is fresh experience and positive response from teachers and students alike. It seems to be a highly successful strategy, and highlights why so many other approaches have been at best marginally successful: habits, including expectations of oneself and of mathematics courses, are major obstacles to growth.

Jogging people out of deeply established habits is one way, perhaps even the most effective way of circumventing what has come to be known as *System 1*, the automatic reactive aspect of human psychology (of Dual Systems Theory, see Kahneman 2012), and indeed also circumventing System 1.5, the emotionally-based energy that can stimulate reactions (for an extended version of Dual Systems theory based on ancient psychology, see Mason and Metz 2017). See also Von Glasersfeld (1988) who recognised the 'reluctance to change' that we all exhibit in one way or another.

Others have approached the same problem less dramatically, by easing learners into mathematical thinking rather than dropping them in the middle of a full exploration requiring mathematical thinking. The aim is to awaken the inner witness, the monitor, the executive (Mason et al. 1982/2010; Schoenfeld 1985), so that initial impulses can be 'parked' and System 2 (cognitive consideration) can be invoked prior to acting so providing a more considered response.

Atiya and colleagues (this volume) take a gentler approach. They advocate a period of transition, gradually increasing the openness of the tasks. They also try to select tasks which they think will be interesting and motivating to the students in question, as well as appropriate to their learning and problem-solving capacity at the given time. Over a period of 25 years at the Open University we used the week-long summer schools to offer students short but challenging tasks which we knew from experience would lead to many if not most students experiencing some aspect of mathematical thinking: for example, specialising in order to get a sense of underlying structural relationships; generalising; imagining and expressing what they were imagining in pictures, diagrams and emerging notation; and modelling by isolating structural relationships, expressing them and dealing with the resulting mathematical problem (Mason 1996). This experiential, even phenomenological approach provided the basis for developing a vocabulary for mathematical actions which then became available to students to use with and for themselves (through their inner witness, monitor, or executive) in their future studies. Some indirect indications of effectiveness of this approach is evidenced by the many undergraduates at the University of Warwick over some 20 years, who took a course in their third year based on the same ideas (Mason et al. 1982/2010) and later commented on how helpful the course had been in their studies. Many expressed the wish that they could have taken the course earlier in their career. This has been echoed in universities and teacher education establishments all over the world (e.g., Gourdeau 2017; Yusof and Tall 1999).

Gauging what is relevant, or what can become relevant, is no easy matter. Young teachers may be prone to using contexts of importance to themselves (e.g., home decorating, shopping, mortgages) while older teachers may be prone to problems involving investments and pensions, neither of which cut to the heart of the interests of adolescents. On the other hand, attempts to invoke learners' particular interests can also backfire, when they are seen as an intrusion into a world the adult no longer inhabits, and indeed are likely to get 'wrong.' The *Realistic Mathematics Education* project (Treffers 1987; Treffers and Goffree 1985) makes use of Freudenthal's principle that through imagination, learners can come into contact with and 'make real for themself' contexts that may not be immediately familiar. What is 'real' is what can be imagined, not simply what one currently lives. Through the exercise of imagination learners can experience possibilities that they might never have come into contact with without the school institution, which aligns with Vygotsky's notion of *scientific concepts*: what can be encountered at school that would otherwise be unlikely to be encountered.

Problem solving can be seen as one aspect of critical thinking, creativity, and learning to communicate effectively. Richard Hoshino (this volume) focuses on these possibilities particularly. He seeks to provide learners with experience of parking the first action that becomes available to them and seeking a cleverer or more insightful approach. He also chooses his tasks so that there are unexpected connections between them which come to the surface when looking at the underlying reasoning rather than at the surface format. This could be seen as a form of the variation principle: varying the context but keeping the actual reasoning invariant (Marton 2015). Hoshino describes how he begins with relatively simple tasks, hoping that their confidence and self-esteem will grow so that they are willing to engage with challenging tasks rather than 'downing tools' at the first hurdle. He conjectures that by experiencing both routine and insightful approaches, learners will develop confidence in doing mathematics, an appreciation for the beauty of mathematics, a development of their creativity, and appreciation of mathematical thinking being used.

The Role of Context

The role of context and in particular the cultural assumptions and influences underpinning the way contexts are described and used is an ongoing question in mathematics education which is taken up by Annie Savard (this volume). This has echoes of a curious circularity in research into medieval social customs. Historians make use of mathematical word problems of the time in order to learn about social customs, assuming that textbook authors are reflecting concerns of the time. But as already mentioned in the context of ancient Egypt, teachers know perfectly well that the contexts are often spurious, based on the author's assumptions about what will interest students, and usually lying at the E-for-educational end of Burkhardt's spectrum. Nevertheless, there must be some insights into perceived cultures of a historical period by looking at the contexts provided for applying mathematical actions to non-mathematical contexts. Learning to extract relationships from contextual situations is the essence of mathematical modelling and of abstraction, and so perhaps there is no magic wand. Rather, what probably matters most is the teachers' commitment to and engagement with the contexts provided, drawing on learners to use and develop their powers.

To be sufficiently challenging, a task has to depend on some complexity of the situation so that students are called upon to draw on what they know about the context. They may have to ignore some details as irrelevant or too complex while making use of other unspoken ones. But this too has drawbacks: students may not know enough about the context, or they may know too much. They may over complicate what was intended as a dressed-up mathematical question (Cooper and Dunne 2000; Lave 1992). Savard points out that the contextual situation may act as motivation for resolving the task as set. It can of course also inhibit engagement where students believe that the context is 'not for them' or otherwise fail to recognise its potential relevance. As mentioned earlier, *relevance* is more to do with the realisable, with 'what can become real' than it is about immediate life experiences. Savard stresses the interplay between the cultural assumptions made by use of a particular context, the mathematical concepts and actions being targeted from the curriculum, and the importance of critical thinking as part of growing into full participatory citizenship.

Aiming for Social Justice

Mamolo, Thomas, and Frankfort (this volume) describe how they try to use mathematical problem solving as a vehicle for raising issues of social justice. They describe in-depth, inquiry-based mathematical explorations which engage learners with authentic contexts and mathematical practices that aim to broaden and foster a number of features: critical awareness of social injustices in the lived experiences of Canadians; sophisticated mathematical thinking and decision-making while reinforcing important mathematical procedures and practices; and awareness of the relevance of mathematics for understanding and addressing various issues affecting individuals in Canada and beyond.

Roots of such an approach go back at least to the work of Stieg Mellin-Olsen (1987) in Denmark, the "Real Problem Solving" movement typified by the work of Johnny Baker at the Open University (Open University 1980) who was in turn inspired by the USMES project. More recently the *Realistic Mathematics Education* project (Treffers 1987; Treffers and Goffree 1985) with its American offshoot *Maths in Context* (MiC-USA) and a parallel project *Realisable Mathematics* in the UK (MiC-UK) have been attracting attention, while Ole Skvovemose (1994) and colleagues in Denmark, the Shell Centre at Nottingham, and Dick Lesh and colleagues (Lesh and Fennewald 2010) are among the many who have all enriched and extended this domain. There is an international meeting with this focus biannually under the heading *Mathematics Education and Society* (MES). The issue of how to use mathematics to awaken learners to cultural-socio-political issues through using mathematics has been discussed by numerous authors in recent years.

Unfortunately there are pitfalls in using social justice themes as contexts for mathematical thinking, as Gale Russell (this volume) points out. It is for example, all too easy for the social issue to dominate to the extent that the mathematics becomes trivial. She recommends introducing topics by tapping into concerns and questions that the students raise rather than based upon a teacher's own interest. Furthermore, and equally important, it is necessary to maintain consistency between the classroom ethos and modes of interaction, and the social justice issues one might choose to highlight, because the teaching of mathematics itself, the pedagogic choices made, actions enacted and ethos developed are all part of the issue. Russell brings to the fore ways in which the teacher can deal with inequities resulting from pedagogical choices, classroom structures and environments, and institutional policies.

Russell also raises the issue of what is being assessed in courses with a strongly social equity theme. Is it learners' appreciation of the social issues, learners' appreciation and comprehension of some mathematics, or some combination of these? This raises a major question: what do students make of these courses? To what extent is the pursuit of social justice actually influencing learners, at what age are they most easily sensitised to these issues, and to what extent is the learning of mathematics developed in this way?

Mamolo et al. articulate possibilities and ways of thinking about both issues of social justice issues and of mathematics, using the slogans *reading the world with mathematics; writing the world with mathematics;* and *developing positive social and cultural identities* (through mathematics) as triggers for awareness and action. It is to be hoped that *righting the world with mathematics* is a possible outcome, at least in the long run!

Informing Task Design

As Atiya and colleagues (this volume) point out, it is not always easy to maintain a balance between what might catch learners' interest and what mathematical concepts, procedures, themes and powers form the curriculum. For example, as Russell points out, attempts to use term-long themes in primary school so as to integrate all the subjects often trivialises mathematics, displaying it as a bit of add-on arithmetic using a few numbers, but not part of the central appreciation and comprehension of the theme.

Pairing a learning task with an assessment task is but one of the interesting practices emerging from Atiya et al. It certainly makes sense from the point of view of the learners, but even more so from the point of view of a teacher, because being careful and thoughtful about the pairing could make a big difference to learners' experience. It is all too easy to select a task, then select another task for assessment purposes, without checking that the underlying reasoning and thinking are in reasonable if not full alignment (see previous remarks about transfer and situatedness).

The term *rich tasks* may have been originated by Afzal Ahmed (1987), but it has spread across the globe, at least in the English-speaking world (e.g., Flewelling and Higginson 2001). It has since become joined with the notion of *low threshold – high ceiling* which may have originated in educational computing but has been considerably developed and exploited in the NRiCH project in the UK (NRiCH). It transpires that it is not so much the task itself that is rich, but rather *how it is used*. Most tasks can be used richly, and most tasks can be used poorly, impoverishing what is potentially available. Very few tasks are 'teacher proof.' In other words, the affordances, whether pedagogic or mathematical (or indeed psychological or social) have to be within the attunements of the teacher, and accessible to the students (Gibson 1979). Put another way, the affordances of a task may not emerge unless someone, usually the teacher, is attuned to notice and exploit them, bringing them to the surface and prompting learners to articulate them for themselves. As considerable research has demonstrated, self-explanation is a powerful contribution to learning (Chi and Bassok 1989; Chi et al. 1989; Renkl 2002).

In the context of probability, Vincent Martin and colleagues (this volume) provide a detailed rationale for the construction of a specific three-fold task, based on the principle of contrasting situations concerning probability in which only a frequentist approach can be taken, situations in which only an empirical approach is possible, and situations in which both are possible. It is this kind of detail which is so desperately needed by the mathematics education community so as to inform and inspire others in their design of tasks. Martin et al. take into account both the mathematical awareness (frequentist and empirical evaluation of probability) constituting the topic, and pedagogical issues and justifications. It is to be wished that textbook authors and worksheet designers would take similar care over corresponding details.

Luis Saldana and Mathieu Thibault (this volume) make use of the notion of an anchor problem or problem setting, with variations, which reappear over a number of lessons, in related problems. This approach aligns with the *principle of variation* (Marton 2015, with roots back into the 1980s), also known as *variation theory*, which suggests that something is available to be learned only when it has been varied in proximal time and space. A Shanghai version of this principle talks about varying the problem context but keeping the problem itself the same; varying the problem but keeping the context the same; and varying both (Huang and Lee 2016). This principle has multiple levels of interpretation when addressing the mathematical notions of mean and standard deviation because mathematically, these devices are for summarising a set of data taking into account the scope of variation around some central value. In order to appreciate (the use of) and comprehend (the import and internal relationships of) a mathematical concept, students similarly need to become aware of the scope of variation or generality encompassed by definitions and theorems.

The notion of 'good tasks,' which on the surface seems so reasonable, depends on what you want to achieve. For example, Russell (this volume) uses an example of sensitivity to family size and composition to illustrate how a task with limited scope and cultural relevance (5 children with an average age of 9: what could their ages be?) might be responded to by children, and could be tinkered with, in order to open up awareness of different possible family compositions. The teacher needs to be prepared for unusual details that students might come up with, and the task can be made more or less accessible by varying the social assumptions embedded in it.

At the Open University, I found that setting a short task in which many students would naturally use mathematical thinking was most successful. They afforded an opportunity to draw attention to what many had done naturally, to label it, and so to provide a vocabulary for talking about specific actions contributing to mathematical thinking. This translated into the tasks of *Thinking mathematically* (Mason et al. 1982/2010) which was designed initially for use with teachers to introduce an effective vocabulary for talking about mathematical thinking. Recently I ran into the same issue again in which experienced mathematicians, mathematics educators and teachers struggled with some problems I had set them, and struggled even more to articulate what mathematical actions they tried and what they could, in retrospect have tried.

Pedagogic Choices

Guy Brousseau (1997) formed the notion of a *didactic tension* which is always present in classrooms:

> The more clearly and precisely the teacher describes the behaviour being sought, the easier it is for the learners to display that behaviour without generating it from themselves (Mason and Davis 1989). (p. 284)

In other words, *training of behaviour* appears to be easier and more direct than provoking learners to *educate their awareness* (Mason 1994). Vincent Martin and colleagues (this volume) formulate a variation of the tension:

> the more extensively the teacher introduces (or imposes) problem solving strategies at the beginning of the sequence, the more limited the potential for students' mathematical activity. Allowing students to experiment and to compare among themselves can be an important driver of success in addressing the problem.

I read this as a plea for an experiential or phenomenological stance to 'teaching problem solving,' which might more usefully be called 'teaching through problem solving' or even 'teaching investigatively' (Jaworski 1994; Williams 1989). Shawn Godin (this volume) describes his attempts in this direction, seeing problem solving not as an add-on but as the heart and core of mathematics. In this he is in complete alignment with Paul Halmos (1980) among many others.

The effect of direct instruction in which labels are presented prior to experiences for which those labels might be useful, is to dull the students potential for enriching their own experience, turning what is to be learned into 'things to be memorised' rather than emerging through and based on experience. Returning to the issue of transfer and situatedness mentioned earlier, having an action become available is much more likely if that action is linked to vibrant experience.

As with any learning environment, affordances are not necessarily recognised, either by the teacher or by the students. That is why pedagogy remains a vital contribution mediating between learner and task. Examples include promoting learners in considering and discussing modelling assumptions, and the influence on the task and or its resolution coming from social forces and individual psychology.

Learner Experience

The *didactic transposition* put forward by Chevellard (1985) suggests that expert awareness is transformed into instructions in behaviour for students. When an expert comes across a mathematical relationship, or a way of approaching a concept, and experiences a desire or hope that learners could experience a similar awareness, they are tempted to construct a task for learners that will in some way, they hope, reproduce that awareness. They embark on a transformation of their own experience into a sequence of instructions for learners to follow. Unfortunately the transformation is usually too great for learners to get even a taste of what the expert experienced. Instead, learners carry out the instructions but gain little or nothing from the experience. As William James (1890) said in a different context, "A succession of feelings does not add up to a feeling of that succession" (p. 628), which can be recast as "A succession of experiences does not add up to an experience of that succession" (Mason and Davis 1989, p. 275). Put another way, "one thing we do not seem to learn from experience, is that we do not often learn from experience alone: something more is required" (Mason 2002, pp. 8, 68). That 'something more' is *reflection on action*, as Schön (1983) put it. But even more powerful is when there is awareness in the moment so as to inform a fresh choice of action, whether mathematical or pedagogical. Schön called this *reflection in action*, and it is the invocation of System 2 of dual systems theory (Kahneman 2012). This is the aim and work of the *discipline of noticing* (Mason 2002, 2012).

Luis Saldanha and Mathieu Thibault (this volume) bring to the surface again the issue of whether students become aware of their thinking when they are immersed in responding to the questions and prompts from teachers. When students are immersed in a task, experience suggests that they are unlikely to be aware of the nature of the questions, probes and prompts provided by the author or the teacher. A major role for teachers is to act as 'consciousness for two" (Bruner 1986, pp. 75–76), prompting learners to withdraw briefly from the action so as to reflect upon, to become explicitly aware of, actions that are proving to be effective, and actions that are not. Labelling these can then provide reference points so that in the future teachers and learners both can refer back to previous experience in order to enrich current experience. This is the essence of *scaffolding and fading* (Seeley Brown et al. 1989; see also Love and Mason 1992). It seems to be the most effective way of informing action in the future.

Exploring for Oneself

It is tempting to describe details of a pedagogic approach using the word 'should', and to berate oneself when things don't go according to hope and expectation, again by using 'I should have …'. I have found it useful to replace 'should' with 'could.' It stops me from feeling guilty about what I have done, and emphasises what I could do, and would prefer to do, in the future. This applies equally to students and to teachers. I have great admiration for Peter Liljedahl's 'contrarian approach' which is proving to be both practical and effective in generating novel pedagogic actions. The combination of novelty with being able to imagine oneself carrying out such an action in one's own situation is indeed powerful. I am also sure that when he is working with teachers the contrariness comes across creatively rather than negatively, as more of an invitation than a confrontation.

What is certain, and what these chapters articulate, is that no matter how confining or restricting the institutional and peer pressures, there is always room for experimentation, for trying to enact fresh pedagogic actions, and for trying to find alternative ways to bring learners into contact with core mathematical themes and actions, as well as social and psychological forces. One valuable contribution to such an enquiring stance is to work with colleagues on mathematical tasks, as Atiya and colleagues (this volume) point out. At first these can be working through tasks that are going to be offered to learners, so as to become as deeply aware of possibilities, affordances, and necessary attunements as possible. Discussing which actions proved to be effective, and which not, and describing salient moments to each other serves to enrich and refine a vocabulary for pedagogic and mathematical actions.

Seeking Evidence

On the face of it, it seems perfectly reasonable to ask for evidence that this or that approach to engaging learners is effective. However the situation is much more complicated than it appears on the surface (Biesta 2007). To start with, the notion of 'an approach' is far too vague, too indefinite to be reliably tested for, either by observation or by randomised control trials. It is very rarely possible to reproduce the classroom ethos, the pedagogical choices and awarenesses, the mathematical experience and maturity which underpin someone's description of 'what they do.' As a metaphor for transformations that take place in and beyond classrooms, cause-and-effect is far too limited, too imperfect to be used as the basis for decisions

(Mason 2016). A much more fruitful metaphor is of a chemical 'soup' in which transformations take place and equilibria are established around which conditions vary, aided or impeded by various catalysts. Other peoples' articulation of their explorations provides insight and inspiration for others, but not recipes.

Although medicine is often held up as a model to be followed in education, there is a huge difference between administering drugs or other such treatments and trying to provoke people to develop their powers, modify their dispositions, and internalise new concepts and new procedures, in short, to *educate their awareness* (Gattegno 1970). Even doctors find that there are rarely trials that take into account all of the details and factors that they know about their particular patients. Furthermore, just as you would not consult an actuary about your personal life-prospects, it would not make sense to consult a report of some large randomised controlled trial for what is likely 'to work' in your particular situation, unless all of the known influences are explicitly taken into account in the trial. Statistics give indications about an overall picture, not about the specifics of individuals. Without some claim for, indeed evidence of influence, if not cause-and-effect, correlational studies can be at best indications for consideration.

Thus it is that I am unable to provide either statistical studies or even systematic observational studies for the effectiveness of courses based on *Thinking mathematically* (Mason et al. 1982/2010), and this is mirrored in chapters in this section. No attempt is given to prove that such and such an approach 'works.' Rather, what these chapters provide is detailed description of what the authors do and have actually done. They are laying out the details of their 'approach'. What we cannot know is whether there are other factors and influences at play than those articulated by the authors, which contribute to making their approach effective. What learners appear to be doing, whether in class or on an examination, tells us little about what they are actually thinking and construing.

References

Ahmed, A. (1987). *Better mathematics*. London: HMSO.

Bednarz, N., Kieran, C., & Lee, L. (Eds.). (1996). *Approaches to algebra: Perspectives for research and teaching*. Dordrecht: Kluwer.

Biesta, G. (2007). Why 'what works' won't work: Evidence based practice and the democratic deficit in educational research. *Educational Theory, 57*(1), 1–22.

Brousseau, G. (1997). *Theory of didactical situations in mathematics: Didactiques des mathématiques, 1970–1990* (N. Balacheff, M. Cooper, R. Sutherland, & V. Warfield, Trans.). Dordrecht: Kluwer.

Bruner, J. (1986). *Actual minds, possible worlds*. Cambridge, MA: Harvard University Press.

Burkhardt, H. (1981). *The real world of mathematics*. Glasgow: Blackie.

Chevallard, Y. (1985). *La transposition didactique*. Grenoble: La Pensée Sauvage.

Chi, M., & Bassok, M. (1989). Learning from examples via self-explanation. In L. Resnick (Ed.), *Knowing, learning and instruction: Essays in honour of Robert Glaser*. Hillsdale: Erlbaum.

Chi, M., Bassok, M., Lewis, P., Reiman, P., & Glasser, R. (1989). Self-explanations: How students study and use examples in learning to solve problems. *Cognitive Science, 13*, 145–182.

Cooper, B., & Dunne, M. (2000). *Assessing children's mathematical knowledge: Social class, sex and problem-solving*. Buckingham: Open University Press.
Davis, M. (1977). *Piero della Francesca's mathematical treatises: The Trattato d'abaco and Libellus de quinque corporibus regularibus*. Ravenna: Longo.
Detterman, D., & Sternberg, R. (Eds.). (1993). *Transfer on trial: Intelligence, cognition, and instruction*. Norwood: Ablex Publishing.
Flewelling, G., & Higginson, W. (2001). *A handbook on rich learning tasks.* Centre for Mathematics, Science, and Technology Education. Kingston: Queen's University.
Gattegno, C. (1970). *What we owe children: The subordination of teaching to learning*. London: Routledge & Kegan Paul.
Gibson, J. (1979). *The ecological approach to visual perception*. London: Houghton Mifflin.
Gillings, R. (1972, reprinted 1982). *Mathematics in the time of the pharaohs*. New York: Dover.
Gourdeau, F. (2017). Doing mathematics in teacher preparation: Giving space and time to think, reflect, share and feel. In S. Cho (Ed.), *Selected regular lectures from the 12th International Congress on Mathematics Education* (pp. 223–234). London: Springer.
Greeno, J., Smith, D., & Moore, J. (1993). Transfer of situated learning. In D. Detterman & R. Sternberg (Eds.), *Transfer on trial: Intelligence, cognition, and instruction* (pp. 99–167). Norwood: Abbex.
Halmos, P. (1980). The heart of mathematics. *American Mathematical Monthly, 87*(7), 519–524.
Huang, R., & Lee, Y. (2016). *Teaching and learning mathematics through variation: Confucian heritage meets western theories*. Rotterdam: Sense.
James, W. (1890, Reprinted 1950). *Principles of psychology* (Vol. 1). New York: Dover.
Jaworski, B. (1994). *Investigating mathematics teaching: A constructivist enquiry*. London: Falmer Press.
Kahneman, D. (2012). *Thinking fast, thinking slow*. London: Penguin.
Lai, K. (2012). Knowing to act in the moment: Examples from Confucius' analects. *Asian Philosophy: An International Journal of the Philosophical Traditions of the East, 22*(4), 347–364. https://doi.org/10.1080/09552367.2012.729324.
Lave, J. (1992). Word problems: A microcosm of theories of learning. In P. Light & G. Butterworth (Eds.), *Context and cognition: Ways of learning and knowing* (pp. 74–92). New York: Harvester.
Lesh, R., & Fennewald, T. (2010). Modeling: What is it? Why do it? In R. Lesh, C. Haines, P. Galbraith, & A. Hurford (Eds.), *Modeling students mathematical modeling competencies* (pp. 5–15). New York: Springer.
Lomon, E., Beck, B., & Arnbetter, C. (1975). Real problem solving in USMES: Interdisciplinary education and much more. *School Science and Mathematics, 75*(1), 63–74.
Love, E., & Mason, J. (1992). *Teaching mathematics: Action and awareness*. Milton Keynes: Open University.
Marton, F. (2015). *Necessary conditions for learning*. Abingdon: Routledge.
Mason, J. (1994). Only awareness is educable. In A. Bloomfield & T. Harries (Eds.), *Teaching, learning and mathematics* (pp. 28–29). Derby: Association of Teachers of Mathematics.
Mason, J. (1996). *Twenty-six years of problem posing: Mathematical investigations used in Open University mathematics summer schools 1971–1996*. Milton Keynes: Open University Centre for Mathematics Education.
Mason, J. (1998). Enabling teachers to be real teachers: Necessary levels of awareness and structure of attention. *Journal of Mathematics Teacher Education, 1*(3), 243–267.
Mason, J. (2002). *Researching your own practice: The discipline of noticing*. London: Routledge Falmer.
Mason, J. (2012). Noticing: Roots and branches. In M. Sherin, V. Jacobs, & R. Phillip (Eds.), *Mathematics teacher noticing: Seeing through teachers' eyes* (pp. 35–50). Mahwah: Erlbaum.
Mason, J. (2016). Rising above a cause-and-effect stance in mathematics education research. *JMTE, 19*(4), 297–300. Retrieved from http://rdcu.be/npTf.
Mason, J., & Davis, J. (1989). The inner teacher, the didactic tension, and shifts of attention. In G. Vergnaud, J. Rogalski, & M. Artigue (Eds.), *Proceedings of PME XIII* (Vol. 2, pp. 274–281). Paris.

Mason, J., & Metz, M. (2017). Digging beneath dual systems theory and the bicameral brain: Abductions about the human psyche from experience in mathematical problem solving. In U. Eligio (Ed.), *Understanding emotions in mathematical thinking and learning* (pp. 379–407). Cambridge: Elsevier/Academic.

Mason, J., & Spence, M. (2000). Beyond mere knowledge of mathematics: The importance of knowing –to act in the moment. In D. Tirosh (Ed.), *Forms of mathematical knowledge: Learning and teaching with understanding* (pp. 135–161). Dordrecht: Kluwer.

Mason, J., Burton L., & Stacey K. (1982/2010). *Thinking mathematically.* London: Pearson.

Mellin-Olsen, S. (1987). *The politics of mathematics education*. Dordrecht: Reidel.

MES. www.mescommunity.info. Accessed Nov 2017.

MiS (UK). educationendowmentfoundation.org.uk/projects-and-evaluation/projects/maths-in-context. Accessed Nov 2017.

MiS (USA). qualifications.pearson.com/en/qualifications/edexcel-mathematics-in-context/mathematics-in-context.html. Accessed Nov 2017.

NRiCH. nrich.maths.org/7701/index. Accessed Nov 2017.

Open University. (1980). *PME233: Mathematics across the curriculum* (a distance learning course). Milton Keynes: Open University.

Renkl, A. (2002). Worked-out examples: Instructional explanations support learning by self-explanations. *Learning and Instruction, 12*, 529–556.

Robson, E. (2000). Mesopotamian Mathematics: some historical background. In V. Katz (Ed.), *Using history to teach mathematics: An international perspective* (pp. 149–158). Washington, DC: Mathematical Association of America.

Schoenfeld, A. (1985). *Mathematical problem solving*. New York: Academic.

Schön, D. (1983). *The reflective practitioner: How professionals think in action*. London: Temple Smith.

Seeley Brown, J., Collins, A., & Duguid, P. (1989). Situated cognition and the culture of learning. *Educational Researcher, 18*(1), 32–42.

Skilling, K. (2014). Teacher practices: How they promote or hinder student engagement. In J. Anderson, M. Cavanagh, & A. Prescott (Eds.), *Curriculum in focus: Research guided practice*. Proceedings of the 37th annual conference of the Mathematics Education Research group of Australasia (pp. 589–596), MERGA.

Skovsmose, O. (1994). *Towards a philosophy of critical mathematics education*. Dordrecht: Kluwer.

Treffers, A. (1987). *Three dimensions: A model of goal and theory description in mathematics education*. Dordrecht: Reidel.

Treffers, A., & Goffree, F. (1985). Rational analysis of realistic mathematics education – The Wiskobas program. In L. Streefland (Ed.), *Proceedings of PME 9* (Vol. 2, pp. 97–121). Utrecht: PME.

Von Glasersfeld, E. (1988). The reluctance to change a way of thinking. *Irish Journal of Psychology, 9*(1), 83–90.

Watson, A., & Winbourne, P. (2008). *New directions for situated cognition in mathematics education*. New York: Springer.

Williams, H. (1989). *Tuning-in to young children: An exploration of contexts for learning mathematics*. Unpublished M. Phil. thesis, Open University, Milton Keynes, England.

Witmer, T. (Trans.). (1968). *Ars magna or the rules of algebra, Girolamo Cardano*. New York: Dove.

Yusof, Y., & Tall, D. (1999). Changing attitudes to university mathematics through problem-solving. *Educational Studies in Mathematics, 37*, 67–82.

Part V
Planning and Assessment

Part V: Preface – Planning and Assessment: Teachers and Students As Central Actors

Carolyn Kieran

Part V of this volume offers the reader six chapters dedicated to the theme of "Planning and assessment." While five of the six chapter titles suggest a greater emphasis on the assessment aspect, consideration is also given to planning in all chapters. Current perspectives on assessment view the planning for effective learning and for the kind of assessment that will be used to provide feedback related to that learning as being closely intertwined (Black and Wiliam 1998). One way of thinking about planning and assessing is that of two points on a continuum, points that can be overlapping or even far apart, depending on one's particular focus at the moment. Yet another way of interpreting the planning-assessing continuum is from the actor's point of view, that is, according to whether the center of interest is the teacher or the learner or both. The recently coined phrase, *assessment as learning* (OME 2010)—wherein students monitor and assess their own learning—highlights the attention being given to students within the planning-assessing process, an attention that is reflected in the chapters herein.

In her chapter on "Enhancing mathematics teaching and learning through sound assessment practices," Christine Suurtamm provides a useful overview of the current state of the assessment field:

> Currently, with a different understanding of how students learn that recognizes that students need to work with mathematical ideas in order to develop an understanding of those ideas, assessment is seen as on-going, constantly looking at students' understanding, and making teaching and learning decisions based on what that understanding looks like. … Thus there is a move away from assessing merely through paper-and-pencil tests to the use of a range of assessment strategies that recognize the multi-faceted mathematical actions that are part of doing mathematics and provide multiple opportunities for students to show what they know and can do.

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While assessment can have many different purposes, Suurtamm argues that its main purpose is to support and enhance student learning. In line with this argument, she then elaborates on what she sees as the four central components of sound classroom assessment:

- ongoing and embedded in instruction;
- using a variety of assessment strategies;
- reflecting meaningful mathematics;
- including students in the assessment process.

In developing the idea that sound assessment requires planning, Suurtamm suggests the use of a long-term-planning template that includes recording of the daily lessons or activities, the curricular expectations addressed by each, the corresponding mathematical processes, and the assessment opportunities offered by each lesson. The assessment could be diagnostic, formative, or summative; the assessment strategy could involve, for example, an observation, a quiz, or student conferencing; and the focus of the assessment could be some component of the curricular expectation, informal assessment of prior learning, a particular mathematical process, or perhaps a learning skill.

Adopting a similar perspective on classroom assessment as an ongoing process, the chapter by Jimmy Pai, titled "Assessment: Broadening our conceptions to improve our practice," emphasizes as well that (i) assessment is a humanistic activity wherein the ways in which teachers elicit, attend, interpret and act depend on many factors, (ii) a positive classroom culture is necessary for student learning and for assessing that learning, and (iii) assessment can serve formative, summative, and/or interpersonal functions, depending on the actual circumstances. He encapsulates his stance as follows: "Assessment processes are always in play for a mathematics teacher, and include preparations for, acting in, and reflections upon, moments that support learning in the classroom."

Pai argues that, in order to access the learning that is occurring, student thinking must be elicited during the assessment process. He plans for this by employing—within a variety of problems, some of them student-generated—such strategies as, for example, having students write on vertical whiteboards their emerging ideas, which are easily sharable and revisable in the light of classroom discussion. He also has students maintain a mathematical processes portfolio that documents their reflections on the improvements they have been making with respect to seven specific mathematical processes. However, eliciting student thinking is only part of the assessment process; equally important are attending and responding to student thinking.

One of Pai's strategies in this latter regard is to observe different groups in the midst of their problem-solving, sometimes to listen carefully to what they are saying and at other times to ask for clarifications—depending on whether the students are still trying to grapple with understanding an idea or are a little further along in their thinking. In his descriptions of these strategies, Pai's attention to the *interpersonal* function of assessment is palpable. As noted earlier in his chapter, "the dualism [of the summative and formative functions of assessment] does not capture the emotional dynamics of the assessment processes, which may be helpful to consider if assessment is to be envisioned as a process of being *with* students, as its root word *assidere* suggests." Such sensitivity of the teacher toward the learner throughout the entire assessment process, and the accompanying strategies for operationalizing this sensitivity—along with the chapter's insightful approaches to formative and summative assessment—do indeed contribute to a broadening of conceptions of the assessment process.

A useful tool for analyzing student thinking within assessment is offered in the chapter by Priscila Dias Corrêa, titled "Observing for mathematical proficiency in secondary mathematics education." Corrêa suggests that the model of mathematical proficiency described in Kilpatrick et al. (2001) can serve as a foundation for interweaving task design and assessment within a framework involving five vital components of mathematical proficiency: conceptual understanding, strategic competence, procedural fluency, adaptive reasoning, and productive disposition.

Corrêa presents two fairly complex modeling tasks that were engaged in by a class of senior high school students (11th grade), who worked in small groups and, as they worked, progressively recorded individually in journals their assumptions about the tasks and how they were attempting to find appropriate solutions. By means of fragments collected from students' journals, audio and video recording transcripts, and post-class interview transcripts, Corrêa assembles detailed portraits of two students' modeling activity—portraits that are finely annotated and then discussed within the context of the five mathematical proficiencies.

In her chapter, Corrêa argues that analyzing students' mathematical activity in terms of Kilpatrick et al.'s model of mathematical proficiency not only allows for assessing students' work in a more holistic way rather than simply assessing for the correct answer, but also attests to the various abilities that can be promoted by mathematical tasks designed to invoke the various strands of mathematical proficiency. She concludes that by "recognizing the assorted nuances of students' mathematical thinking and skills, teachers are potentially prepared to identify students' needs and plan their classes accordingly."

Planning is the main theme of the chapter by Jennifer Holm, titled "Planning a unit by starting with the end in mind: Unit and lesson planning." Her focus begins with an examination of curricular expectations with their specific outcomes. These outcomes are then turned into learning goals. Holm argues that each learning goal should be created with six characteristics in mind: (1) a learning goal identifies knowledge and skills from the curriculum expectations, (2) a learning goal is incremental and scaffolded, (3) a learning goal is expressed in language meaningful to students, (4) a learning goal uses clear, concise language, (5) a learning goal is specific and observable, and (6) a learning goal is stated from the student's perspective. Once the learning goals for the unit have been determined, Holm emphasizes the importance of anticipating the difficulties that students might encounter, as well as considering the issue of essential prior knowledge for the unit in question. Holm follows up on her suggestions for unit planning with the planning of a lesson, which she exemplifies with one of her favourite tasks, the "Popcorn picker" task.

While, at first glance, this chapter may appear to be focusing on planning in general, it does in fact carry a strong message on a central aspect of planning that is related to assessment. It concerns the clarity of the learning goals. In order for learning goals to be effective, students must know the milestones they need to hit in order to meet the unit goal. Students need to be aware of the goals of a lesson, as well as understand what they mean, so that they can monitor their own learning progression. Black and Wiliam (1998) define *assessment* as “all those activities undertaken by teachers—and by their students in assessing themselves—that provide information to be used as feedback to modify teaching and learning activities” (p. 140). Holm’s chapter, with its dual focus on the formulation of explicit learning goals by the teacher, and on students’ monitoring their progress with respect to these goals, addresses the double-actor aspect of assessment involving both teachers and students that has been emphasized by Black and Wiliam.

In the chapter by P. Janelle McFeetors, titled “Improving students’ approaches to learning high school mathematics,” we obtain insights into students’ assessing and improving their own processes of learning. McFeetors argues that typical study-resources tend to offer rather general strategies for managing time, completing homework, and taking tests, but stop short of addressing how students might personalize these strategies, how they might further develop them from their current approaches to learning mathematics. Wanting to explore this issue more deeply, she worked with a group of grade 12 students who were enrolled in a course called *Mathematical learning skills* that was oriented toward assisting them in succeeding in their pure mathematics course. At the beginning of the research project, the students listed in their journal some of the prescribed ways of learning mathematics such as study, review, copy notes, work with others, and do homework; but these descriptors did not elaborate on the steps that students would need to take in order to operationalize these learning strategies.

By means of three vignettes, McFeetors illustrates how the students gradually developed “processes for learning; examples of these processes of learning (juxtaposed with [the more general] strategies in parentheses) included creating summary sheets (study), making and using cue cards (review), creating various forms of notes (copy notes), collaborating with peers (work with others), and learning from homework (do homework).” These learning processes were both personal and dynamic in that they evolved as the students learned to perfect them.

McFeetors concludes by offering three key ideas that supported students in developing learning processes and which provided them with the means to assess and improve their own mathematical understanding: (i) the students benefitted from having opportunities to discuss and work on their approaches to learning mathematics; (ii) the students willingly engaged in developing learning processes when they noticed a teacher listened to what they were already doing and where they wanted to improve in order to succeed in mathematics class; and (iii) the students developed learning processes from suggestions that were offered rather than prescriptions told.

The student is also central to the chapter by Tina Rapke, Jennifer Hall, and Richelle Marynowski, titled “Re-framing testing to better fit within problem-solving classrooms: Ways to create and review tests.” The chapter focuses on the use of two assessment strategies developed by the researchers: strategies that permit instructors to design assessments based on student thinking related to assessments and strategies that assist students in assessing themselves.

The first assessment strategy involved students in developing exams to help them prepare for exam writing. During this process, some students created questions that others did not understand, leading to discussion of these questions and further learning. The instructor then created the "actual" exam by choosing some of the questions, with a few adaptations, from the students' practice exams, thereby using student thinking about exam questions as a means for developing the actual assessment.

The second strategy combined exam writing, exam reviewing, and response revision in a process that involved the use of summative exams in formative ways. This strategy began with students writing an actual closed-book exam. This was followed by the reviewing of instructor-selected student responses to exam questions and the subsequent revising by students of their own exam responses. Rapke et al. point out that the exam-reviewing process, which focused on actual student responses to exam questions, gave students the opportunity to see examples of their peers' thinking and consequently to analyze and revise their own thinking: "Students indicated that they learned by comparing, analyzing, critiquing, and providing feedback on student responses."

Rapke et al. argue that their approach, despite the fact that it comprises the use of "traditional" exams, illustrates how such exams can be utilized in meaningful ways, when students are actively involved in, and their thinking is at the heart of, the exam-preparation and exam-reviewing process. This chapter thus provides an additional tool for learners to assess their own understanding and for teachers to gather feedback on that understanding.

To conclude, Part V of this volume offers many novel insights on the multifaceted complexity of planning and assessment in mathematics education, insights that will be of interest not only to the seasoned reader of the assessment literature but also to the beginner in this field. Along with its contributions to the various courses of action by which planning for learning and for assessment might be approached, as well as different ways of conceptualizing and engaging in assessment, it also highlights the manner in which teachers and students co-participate as central actors within the planning-assessing process.

References

Black, P., & Wiliam, D. (1998). Inside the black box: Raising standards through classroom assessment. *Phi Delta Kappan, 80*(2), 139–148.

Kilpatrick, J., Swafford, J., & Findell, B. (Eds.). (2001). *Adding it up: Helping children learn mathematics*. Washington, DC: National Academy Press.

Ontario Ministry of Education [OME]. (2010). *Growing success: Assessment, evaluation, and reporting in Ontario schools, grades 1 to 12* (1st ed.). Retrieved from http://www.edu.gov.on.ca/eng/policyfunding/growSuccess.pdf.

Planning a Unit by Starting with the End in Mind: Unit and Lesson Planning

Jennifer Holm

Abstract This chapter starts with using a grade 10 content strand from the Alberta Program of Studies to illustrate how to plan a unit by starting with the end curricular goals of the unit. A framework is presented and filled out within the chapter to provide concrete examples of one possible way to plan a secondary mathematics unit. From the completion of the unit planning guide, a discussion is built on how to decide where start the unit by focusing on where students currently are in their understandings. Next, the chapter explores planning a lesson using a modification of the three-phase lesson plan. An example of a lesson is created and described in the chapter. Strategies on how to both plan and implement an inquiry-based lesson are discussed throughout the example.

Keywords Unit planning · Three-phase lesson plan · Backward planning · Secondary mathematics planning

Lesson planning is one activity that teachers engage in every day of their career. "Effective lessons establish a clear purpose and objectives for both teacher and students. They connect with students' prior knowledge, capture students' interest, and provide opportunities for meaningful practice inside and outside the classroom" (Expert Panel on Student Success in Ontario 2004, p. 46). Having a clear goal for teaching makes planning a lesson much simpler by focusing on the essential understandings to be covered each day.

The chapter begins by looking at a framework to use to plan a unit in mathematics. The framework begins with the curricular expectations set out for the grade level. From here, the planning looks at writing learning goals, anticipating student struggles and identifying prior knowledge, and then considering planned support for

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the unit. Using a completed template of the unit plan, a discussion about how to plan a single lesson within the unit is discussed in order to provide an illustrative example of planning within mathematics.

Unit Planning

Alberta Assessment Consortium (2015) and the Ontario Literacy and Numeracy Secretariat (n.d.) both note that planning should begin by looking at the end goals first. What is meant by this is that all planning needs to begin with the curriculum document that details what all students should learn during the school year. Across Canada, provincial Ministries of Education have created guidelines for determining what needs to be taught in schools. Alberta calls theirs a Program of Study and Ontario uses the term Curriculum. For the basis of this chapter, I have chosen to use Alberta's Program of Study for grade 10C in the strand of Measurement to show as an illustration.

Since planning should begin by examining the expectations set out for the grade level, I will begin there as well. Figure 1 gives a blank template that I will fill in as I move through the chapter to illustrate how a teacher might plan a mathematics unit. This template was adapted from templates found in the Alberta Assessment Consortium (2014) and Pilot and Walter-Rowan (2014). Beginning with the Measurement strand in grade 10C, there is only one general outcome specified: Develop spatial sense and proportional reasoning (Alberta Education 2008). Within this outcome, there are four specific outcomes to serve as the guideline for planning the unit (see Fig. 2): solve problems that involve linear measurement, using: SI and imperial units of measure, estimation strategies, and measurement strategies; apply proportional reasoning to problems that involve conversions between SI and imperial units of measure; solve problems, using SI and imperial units, that involve the surface area and volume of 3-D objects, including: right cones, right cylinders, right prisms, right pyramids, and spheres; and develop and apply the primary trigonometric ratios (sine, cosine, tangent) to solve problems that involve right triangles (Alberta Education 2008, p. 13). By starting with the expectations or outcomes from the Ministry, a clear picture of what should be accomplished in any given unit by the end of the school year is established. To ensure a strong curricular link in a mathematics program, teachers can set out each of the planning sheets for the different units within the curriculum first and then find areas of overlap or where some multi-strand units may be supported. This will also provide some ideas of how to revisit concepts throughout the year to ensure deep understanding is achieved and maintained. For the purposes of this chapter, only the single unit will be addressed to serve as an illustration. These specific outcomes are then turned into the learning goals that will drive the learning within the classroom.

Grade: __________ Unit of Study: ______________________________

Curriculum Outcomes	Learning Goals	Anticipated Struggles / Prior Knowledge	Planned Support / Resources

Fig. 1 Unit planning template

Grade: 10C Unit of Study: Develop spatial sense and proportional reasoning.

Curriculum Outcomes	Learning Goals	Anticipated Struggles/ Prior Knowledge	Planned Support / Resources
1. Solve problems that involve linear measurement, using: • SI and imperial units of measure • estimation strategies • measurement strategies.	I will solve problems using SI and imperial units of measure. I will estimate measurements for linear problems. I will use different measurement strategies.	List from 1-9 curriculum where these come in Cannot remember conversion between SI units. Struggles with imperial units.	NCTM *Illuminations* www.mathclips.ca (Geometry)
2. Apply proportional reasoning to problems that involve conversions between SI and imperial units of measure.	I will convert between SI and imperial units of measure. I will solve problems that involve conversions between units of measure.	Difficulty with multiplication/division operations. Remember conversions.	
3. Solve problems, using SI and imperial units, that involve the surface area and volume of 3-D objects, including: • right cones • right cylinders • right prisms • right pyramids • spheres.	I will determine and apply the surface area formula of (each shape individually). I will determine and apply the volume formulas of (each shape individually). I will solve problems using surface area and volume of right cones. I will solve problems using surface area and volume of right cylinders. I will solve problems using surface area and volume of right prisms. I will solve problems using surface area and volume of right pyramids. I will solve problems using surface area and volume of spheres.	Unable to identify the different shapes. Needs to know what surface area and volume are. Has difficulty with area of 2-D shapes (or cannot identify the 2-D shapes)	www.edugains.ca/newsite/math/tips.html: Unit 7: Surface area and volume
4. Develop and apply the primary trigonometric ratios (sine, cosine, tangent) to solve problems that involve right triangles.	I will develop and apply sine, cosine, and tangent to solve problems with right triangles	Can identify a right triangle. Struggles with remembering where the hypotenuse, opposite and adjacent sides are. Difficulty with using a protractor to measure sides. Struggles with multiplication/division operations.	Trigonometric Functions: http://mathies.ca/activities.html Trigonometry: www.mathclips.ca www.edugains.ca/newsite/math/tips.html: Unit 2:Trigonometry

Fig. 2 Sample grade 10 unit on spatial sense and proportional reasoning

Learning Goals

"Learning is easier when learners understand what goal they are trying to achieve, the purpose of achieving the goal, and the specific attributes of success" (Chappuis and Stiggins 2002). Each learning goal should be created with six characteristics in mind (Ontario Ministry of Education 2011) in order to make them useful for teachers and students. First, a learning goal "identifies knowledge and skills from the curriculum expectations" (Ontario Ministry of Education 2011, p. 11). Learning goals in this unit would focus around the specific skills students would need in order to meet the unit goal of developing spatial sense and proportional reasoning. Each of the specific outcomes details specific skills for students to master to meet the overall unit general outcome.

Second, a learning goal is "incremental and scaffolded" (Ontario Ministry of Education 2011, p. 12). In order for a learning goal to be effective, students would know the milestones they need to hit in order to meet the unit goal. By focusing on the specific steps to follow, teachers would be able to plan lessons no matter where students currently are at in their learning. In the example of this unit, before students can "solve problems, using SI and imperial units, that involve the surface area and volume of … right cylinders" (Alberta Education 2008, p. 13), students would need to "calculate surface area and volume of right cylinders." To do this they would also need to have a working definition and understanding of right cylinders, surface area, and volume. Previous outcomes related to an understanding of circles would also be important in order to fully understand how surface area and volume of a right circular cylinder can be derived and understood.

Third, a learning goal is "expressed in language meaningful to students" (Ontario Ministry of Education 2011, p. 13). If a student does not understand the learning goal, then it is not useful for him or her to support learning. In *Growing success* (Ontario Ministry of Education 2010), assessment as learning is described as students monitoring their own learning as they progress toward the learning goals. Students would need to be aware of the goals of the lesson, as well as understand what they mean so that they can help to monitor their own learning progression.

Fourth, a learning goal uses "clear, concise language" (Ontario Ministry of Education 2011, p. 45). Learning goals should be written in such a way that students can easily understand their meaning. Making the statements brief will allow teachers to focus on the important concepts of the lesson, but also convey to students what their target is for the lesson. It is also important to keep the mathematical terminology in the learning goal so that students are being exposed to the correct and precise mathematical vocabulary. It would then be important for a teacher to ensure students understand what the mathematical terms mean through the lesson progression.

Fifth, a learning goal is "specific and observable" (Ontario Ministry of Education 2011, p. 14). It is important to choose verbs that are associated with what a teacher will see students doing in order to evaluate if they are meeting the expectations. A verb like "understand" is not measurable since it could mean different things to different people. How would one decide if a student really "understands" a concept and to what level they understand it? Using the example unit planning guide, for the second curriculum outcome, a learning goal like "I will understand SI and imperial units of measure" does not give information about what students are expected to do. Verbs like identify, solve, and convert would detail specific actions that can be observed and measured as a result of the lesson. In this example, a learning goal of "I will convert between SI and imperial units of measure" details exactly what students would be doing (or not doing) if they are working toward that curriculum outcome.

Finally, a learning goal is "stated from the student's perspective" (Ontario Ministry of Education 2011, p. 14). If students are to be monitoring their own learning, then they need to see that the goals are set for them. I like to use "I will…" statements for my learning goals in order to convey to students that this is something they are striving for and that I do not expect them to already have the knowledge.

In most cases, the curriculum outcomes in the example could just be restated as learning goals by separating them and putting them into the student's perspective. In some cases, however, the outcomes may need to be adapted in order to create learning goals that are appropriate for classroom use. For example, "derive, through the investigation and examination of patterns, the exponent rules for multiplying and dividing monomials, and apply these rules in expressions involving one and two variables with positive exponents" from Ontario would need to be adapted (Ontario Ministry of Education 2005, p. 5). Since the outcome is not scaffolded, clear or concise, it would need to be altered to make it an effective learning goal. This expectation would most likely need to be split into several smaller learning goals in order for it to be useful within the classroom and to conform to the suggestions for writing learning goals. An example learning goal for this expectation could be "I will derive exponent rules for multiplying and dividing monomials." See Fig. 2 for sample learning goals created for this unit. Once the learning goals for the unit have been determined, the next task would be to determine the anticipated struggles of students as well as what prior knowledge is essential for the different areas within the unit, as is described next.

Anticipated Struggles/Prior Knowledge

An important area that I have found is often overlooked in planning is anticipating struggles. As a new teacher, this could be difficult to consider in advance if a teacher has not taught the specific grade level prior to this. Keep this section available so that it can be expanded on as the unit progresses and to give ideas for the next time the unit is taught. Considering anticipated struggles prior to a lesson can help with deciding what resources and supports may be needed before the lesson hits a rough patch. Prior knowledge is another area to consider at this point in the planning. By identifying what topics a student should already have learned in the previous year(s) (or unit as the case may be), a teacher may have a potential source for areas when students are struggling in the current unit. For example in looking at the learning goal "I will solve problems, using SI and imperial units, that involve the surface area and volume of right cylinders," students should have mastered the outcomes related to circumference and area of circles from grade 7 in order to support development of right circular cylinders. Figure 2 shows some other examples of previous knowledge and some anticipated struggles for the unit. This example is not meant to be an exhaustive list; instead it is an illustration for the purposes of this discussion.

The Literacy and Numeracy Secretariat (n.d.) discusses anticipating student responses and stresses that teachers need to try their activities prior to using them in a classroom. Once deciding where students may struggle and what they should have already learned, then having some supports already planned out can be a time saver when a student starts to struggle in the unit.

Planned Support/Resources

For this section, consider different resources or programs available to help students if they are struggling with the unit. These resources could be other textbooks or activities that are known to support planning. I use this section to also list some provoking questions for the unit so that I can use them once I have decided which learning goals to address first with my students.

Once a higher-level plan for the unit is created, the next step is to look at individual tasks or lessons to use with students. Figuring out where the class is beginning from, a teacher can begin to create lessons and activities that are designed to move students forward from where they are towards the unit goals. The next section of the chapter illustrates how to use the unit planning guide to create a specific lesson.

Lesson Planning

There are different formats that are suggested for creating a lesson in mathematics. Especially early in a teaching career, a good lesson plan can be a life saver. There are many templates online, for example, eworkshop.on.ca has a three-part lesson plan template with a description of each of the parts to help with planning (http://www.eworkshop.on.ca/edu/pdf/Mod18_lesson_template.pdf). One nice feature of this template is that each section discusses both what the teacher would be doing, as well as what students should be doing. "Student voice is recognized and valued during the three-part lesson, as students articulate and reflect on their understanding" (Literacy and Numeracy Secretariat n.d., p. 4). In Ontario the three-part lesson plan uses before, during, and after as the names of the phases. My favourite lesson plan to use is a modification of the three-part lesson with a little bit of a different bend on the three parts. Dan Meyer calls this modification the Three-Act Task on his blog (http://blog.mrmeyer.com/). In this type of lesson plan, students would not be given a problem and all the information up front to explore, but would instead be given some sort of prompt or video to discuss and ask questions about. The mathematics lesson would come from one of the questions created about the prompt, and then students would determine the types of information that they need in order to answer that question. In this lesson there is also a fourth part to the lesson. Figure 3 shows a summary all of the parts of the lesson plan.

In order to find good problems or tasks to use there are many different resources. A good resource would be to refer to Part IV within this book. I like to use the Three-Act Tasks from Dan Meyer's blog, so I will be using one as an example throughout the remaining portion of this chapter.

Lesson Plan Title

Grade: ________________ Topic: ______________________________

Learning goals	**Success criteria**
From unit plan	Specific behaviours that would show students have met the learning goals.
Part 1: Setting up the Lesson Show a prompt/video that engages students in the lesson. Allow for student discussion and questioning of the topic. Establish the question for students to explore based on the prompt. Review vocabulary needed to solve the problem (if necessary) but do not discuss how to solve the problem or task.	
Part 2: Exploration Allow for student discussion and exploration. Teacher is there to facilitate and ask questions as students are working to keep them working. May bring in other information that is needed to solve the task if all the information is not given up front (this is done as students determine a need for the information).	
Part 3: Discuss task/Summary Discussion focussed on student solution methods. Allow students to share ideas in different formats. *While observing the lesson decide which solutions are important to share and in what order (Meikle, 2016, provides a good summary of how to sequence ideas). Keep discussion focussed around the big idea of the lesson (what is it students should have learned through the activity).	
Part 4: Extending learning Create any questions that have come out of the task. This could be used to set up future activities that can be explored based on the knowledge gained during the lesson. *This may happen in a future class period.	

Fig. 3 Lesson plan template and description of the sections

In looking back at my unit planning guide, I would first determine where students are in order to plan a lesson that would push their understandings forward. For this example, I am going to concentrate on Specific Outcome #3 (see Fig. 2), specifically addressing right circular cylinders. Since I would first assess where my students are in their understanding, I would now know that my students are able to find the area of circles and rectangles. They are also familiar with the shape of a right circular cylinder, so I know I could build my next lesson from there to push their understandings further. An example of a task that students could now do would be *Popcorn*

Popcorn Picker

Grade: 10C Topic: Exploring right cylinders

Learning goals	**Success criteria**
I will solve problems using surface area and volume of right cylinders. I will apply the volume formula of right cylinders.	I can determine the volume of a right cylinder. I can determine the relationship between the two pieces of paper and the effect it has on the volume of the cylinder.

Part 1: Setting up the Lesson
Show Act 1 video from http://threeacts.mrmeyer.com/popcornpicker/. Give time for student questions (what do they notice and wonder about the video). Question "Which container will hold more popcorn?" Have students write down their guess and reasoning for the answer.
Part 2: Exploration
Allow for student discussion and exploration. If students ask about the paper size, show Act 2 video of the two cylinders unrolled. Monitor for students struggling with the volume formula or what to do with the paper dimensions. *Possible error-using the base dimension as the radius instead of calculating the radius from the circumference.
Part 3: Discuss task/Summary
Big idea: The size of the radius is going to have the greatest impact on the volume. After discussing student solutions, show Act 3 as the final summary.
Part 4: Extending learning
Sample questions from the Popcorn Picker task to explore further: Can a rectangular piece of paper give you the same amount of popcorn no matter which way you make the cylinder? Prove your answer. How many different ways could you design a new cylinder to double your popcorn? Which would require the least extra paper? Is there a way to get more popcorn using the exact same amount of paper? How can you get the most popcorn using the same amount of paper?

Fig. 4 Sample lesson plan for exploring right cylinders

Picker from Dan Meyer's collection of math tasks (http://threeacts.mrmeyer.com/popcornpicker/). The next portion of the chapter will detail each of the parts using this task; however, see Fig. 4 for the completed lesson plan.

Part 1 of the lesson would be to activate the learning by setting up an interesting context. In the lesson plan illustrated here it would be showing the first video (listed as Act 1). This is a chance for students to examine a video related to the context or something to spark their interest. After putting out the idea, ask students to generate any questions that they thought of as they were listening or watching. Stress that the questions do not necessarily have to be mathematical, but what did they wonder about as they were observing. In this example, the video shows a gentleman (Dan Meyer) making a right circular cylinder out of a sheet of paper and then filling it with popcorn. He uses a blank piece of paper and then makes a cylinder length-wise and width-wise. At this point, I would ask students what do they wonder about the video. Hopefully one of my students would bring up the question “Which cylinder holds the most popcorn?” I would then ask students to use what they know about right cylinders to make an estimation about which cylinder they think holds more, or if they think they hold the same, and why. We could debate the ideas for a little while in order to set up the inquiry for the rest of the class period. At this point, I would move into the second part of the lesson. A teacher knows it is a good task when the question becomes obvious to students.

Part 2: Let them explore the task. Do not give all the information up front. Part of the fun of this portion of the lesson is to let students decide what they need to know and how to figure out that information. What is important? What do they notice? Let them engage with the mathematics, but keep a close eye and be there to deal with questions and ask questions when groups or individuals get stuck. In the example lesson plan, students would need to know the size of the paper and how it was orientated. In this example, there is an image in Act 2 to specify to students that both cylinders are made out of 8 ½ by 11 paper, with one being 11 inches tall and the second being 8 ½ inches tall. Students would then be able to work with the ideas to create calculations. As the teacher, I would be moving between groups and asking questions about what they are doing. I would not be telling them where to go and would be avoiding validating their answers. This is a time for the students to explore and for me to observe and facilitate their learning. During this portion of the lesson it would also be important to have materials for the students to physically act out the problem to gain an understanding of the problem.

Part 3: Discuss the task and summarize findings. The Literacy and Numeracy Secretariat (2010) discusses some different ways to have students share their work such as a Gallery Walk or Math Congress. The idea would be to have students present their solutions and solution methods in order to start the conversation about the goals of the lesson. During the consolidation portion, it is imperative to help students see the connections between the mathematical concepts (Expert Panel on Student Success in Ontario 2004). “Without consolidation, mathematics remains a set of isolated facts and algorithms” (Expert Panel on Student Success in Ontario 2004, p. 47). In my experience, this is where inquiry-based mathematics lessons are most likely to fall apart. I have seen fabulous lessons where this part is either skipped or the conversations get away from the teachers. Van de Walle et al. (2015) extend this idea that this portion of the lesson “is often neglected in the planning process and short-changed when class time runs out” (p. 55). A great task can just be a task if there is not a chance to bring students together on the mathematical ideas. Van de Walle et al. (2015) note that in this third part of the lesson, “the critical piece is helping students make connections between strategies and to highlight the mathematics” (p. 55). In this case, it is important to return to the learning goal of the lesson “I will

The Goodsmell perfume producing company has a new line of perfume and is designing a fancy new bottle for it. Because of the expense of the glass required to make the bottle, the surface area must be less that 150 cm^2. The company also wants the bottle to hold at least 100 mL of perfume. The design under consideration is the shape of a cylinder. Determine the maximum volume possible for a clyindrical bottle that has a total surface of less than 150 cm^2. Determine the volume to the nearest 10 mL. Report the dimensions of the bottle and corresponding surface area and volume.

Fig. 5 Grade 9 exemplar problem (Ontario Ministry of Education 2000)

solve problems, using SI and imperial units, that involve the surface area and volume of right cylinders" and decide what are the important mathematical understandings within the task. In this task, students should come to the understanding that the surface area of the curved surfaces in both cylinders is the same; however, the volume is different. A discussion about why the volume is different is important at this point and how students could predict which cylinder has the larger volume based on what they know about right circular cylinders, and specifically circles.

Part 4: Providing extensions or new questions that arise from the task. Dan Meyer calls this part of the problem the "sequel". His suggestions of the possible sequels to this task are "Can a rectangular piece of paper give you the same amount of popcorn no matter which way you make the cylinder? Prove your answer." "How many different ways could you design a new cylinder to double your popcorn? Which would require the least extra paper?" "Is there a way to get more popcorn using the exact same amount of paper? How can you get the most popcorn using the same amount of paper?" Another possible extension for the task would be the Perfume problem since it applies extra knowledge about the surface area and volume (Fig. 5).

Conclusion

Starting planning at the beginning of the year by looking first at the curriculum goals can lead to ensuring that learning is always focused on the outcomes of the class. Creating learning goals up front can allow for the topics to be revisited throughout the year in order to create a program of learning that is goal oriented and allows for connections between topic areas. Once creating the unit plans for the entire grade, it can be easier to look for cross overs between the topics to allow for multiple units to be discussed during different lessons. This also helps to move away from this idea of a "laundry list" and toward thinking about the topics of study in a broader context. As Peter Taylor (2017) noted in his presentation, "The high school mathematics laundry list. Our students do not need it. Neither does the world. Those few who need it will master it—because they need it and they love it" (slide).

References

Alberta Assessment Consortium. (2014). *AAC performance assessment development template*. Retrieved from https://aac.ab.ca/

Alberta Assessment Consortium. (2015). *AAC key visual*. Retrieved from http://www.aac.ab.ca/updated-aac-key-visual/

Alberta Education. (2008). Mathematics grades 10-12. Retrieved from https://education.alberta.ca/media/564028/math10to12.pdf

Chappuis, S., & Stiggins, R. (2002). Classroom assessment for learning. *Educational Leadership, 60*(1), 40–43.

Expert Panel on Student Success in Ontario. (2004). *Leading math success: Mathematical literacy grades 7–12*. Toronto: Queen's Printer for Ontario.

Literacy and Numeracy Secretariat. (2010). *Communication in the mathematics classroom* (Secretariat Special Edition #13). Retrieved from http://www.edu.gov.on.ca/eng/literacynumeracy/inspire/research/CBS_Communication_Mathematics.pdf

Literacy and Numeracy Secretariat. (n.d.). *The three-part lesson in mathematics: Co-planning, co-teaching and supporting student learning: Viewers guide*. Toronto: Queen's Printer for Ontario Retrieved from http://www.curriculum.org/secretariat/coplanning/files/CoPlanningGuide.pdf.

Meikle, E. M. (2016). Selecting and sequencing students' solution strategies. *Teaching Children Mathematics, 23*(4), 226–234.

Meyer, D. (n.d.). *Popcorn picker*. Retrieved from http://threeacts.mrmeyer.com/popcornpicker/

Ontario Ministry of Education. (2000). *The Ontario curriculum exemplars: Grade 9 Mathematics*. Toronto: Queen's Printer for Ontario. Retrieved from http://sw1math.wikispaces.com/file/view/Grade%209%20Math%20Exemplar.pdf

Ontario Ministry of Education. (2005). *The Ontario curriculum, grades 9 and 10: Mathematics* (Rev. ed.). Ontario: Queen's Printer. Retrieved from http://www.edu.gov.on.ca/eng/curriculum/secondary/math910curr.pdf

Ontario Ministry of Education. (2010). *Growing success: Assessment, evaluation and reporting in Ontario schools: First edition, covering grades 1–12*. Retrieved from Ontario Ministry of Education website http://www.edu.gov.on.ca/eng/policyfunding/growSuccess.pdf

Ontario Ministry of Education. (2011). *Learning goals and success criteria: Assessment for learning video series: Viewing guide.* Retrieved from http://www.edugains.ca/resourcesDI/ProfLearningModules2011/DiffInst_AssessmentandEvaluation/LearningGoalsSuccessCriteriaViewingGuide2011.pdf

Pilot, J., & Walter-Rowan, N. (2014, August). *Assessment and evaluation…more than learning goals and success criteria: Effective planning for learning*. Presentation at Lakehead University's welcome retreat: Faculty of Education, Thunder Bay, ON.

Taylor, P. (2017, October). The aesthetics of mathematical modelling for the classroom. Presentation at MathEd Forum, The Fields Institute for Research in Mathematical Sciences, Toronto.

Van de Walle, J. A., Karp, K. S., Bay-Williams, J. M., McGarvey, L. M., & Folk, S. (2015). *Elementary and middle school mathematics: Teaching developmentally* (4th Canadian ed.). Toronto: Pearson Allyn & Bacon.

Additional Suggestions for Further Reading

Dan Meyer's blog (including additional Three-act tasks): http://blog.mrmeyer.com/.

Learning goals and success criteria video library: http://www.edugains.ca/newsite/aer/aervideo/learninggoals.html.

Observing for Mathematical Proficiency in Secondary Mathematics Education

Priscila Dias Corrêa

Abstract One of the critical aspects of a mathematics teacher's work is analysing learners' mathematical work. Mathematics tasks can prompt the development of different students' mathematical abilities, and it is of relevance to address the various promoted skills when assessing students' mathematical work. However, this is a challenging task, given that students' work is commonly based on procedural knowledge, which is only one of the skills students can work on. Attentive to teachers' needs to assess students' learning (summatively and formatively), this chapter uses Kilpatrick J, Swafford J, Findell B, The strands of mathematical proficiency. In: Kilpatrick J, Swafford J, Findell B (eds) Adding it up: helping children learn mathematics [electronic resource]. National Academy Press, Washington, DC, pp 115–155, 2001) model of mathematical proficiency to study examples of students' work when solving mathematical tasks. From those examples we observe how the five strands of mathematical proficiency are demonstrated by learners as they bring forward various mathematics through their mathematical work. By observing students' work on mathematical proficiency, teachers are able to acknowledge students' mathematical abilities, address students' difficulties, assist students' knowledge development and, as a consequence, plan and structure their classes and activities focused on students' needs.

Keywords Mathematical proficiency · Mathematical abilities · Secondary mathematics · Classroom practice

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Introduction

Several research studies in mathematics education currently address the teaching and learning of mathematics at the secondary level. Studies may suggest pedagogical approaches to be used in the daily teaching of mathematics, or may suggest formative and summative assessment approaches to evaluate students' mathematical work. Recent research about the design of mathematical tasks and assessments (Burkhardt and Swan 2013; Swan and Burkhardt 2012) highlights important aspects that should be considered when assessing students' mathematical work.

Preferably, mathematical tasks and mathematical assessments should be interwoven. That is to say that, the assessment of students' mathematical work when solving mathematical tasks should consider all the skills that are supposed to be prompted and fostered by the task. In other words, tasks and assessment should be developed and thought of together so that students' mathematical learning can take maximum advantage of this relation.

This chapter intends to shed some light in the analysis of students' mathematical work, by means of connecting Kilpatrick et al.'s (2001) theory of mathematical proficiency to the work students present when doing mathematical tasks. The main advantage of looking into mathematical proficiency is that it analyses students' work based on a holistic theory that takes into consideration the range of students' abilities that should be sought when planning and creating a mathematical task. In addition, this mathematical proficiency theory can take into account various expressions of how students solve their mathematics tasks. Finally, a strength of using mathematical proficiency as the basis of this analysis is that mathematical proficiency does not overvalue procedures in students' mathematical work. Procedures are important aspects of students' mathematical work, but so too are other features of their work, such as concepts, strategies, reasoning and attitudes.

This chapter starts off by briefly explaining the theoretical background that supports the current analysis in terms of assessment needs. Then, it describes Kilpatrick et al.'s (2001) mathematical proficiency model and uses it to analyse the work done by two different students in two different tasks. Finally, the chapter concludes with some conjectures and comments for the teaching of mathematics.

Theoretical Background

Swan and Burkhardt (2012) assert that for high-quality assessment eight principles should be taken into consideration. According to the authors an assessment tool should (1) address curriculum requirements in a balanced way; (2) encompass tasks that are acknowledged as worthwhile; (3) fit its purpose; (4) be challenging but still accessible; (5) focus on reasoning instead of on results; (6) present tasks about genuine contexts; (7) encourage decision making; and (8) be clear in terms of its demands. Swan and Burkhardt's principles speak to the consistency that should exist between the assessment tool and the assessment itself. By following these principles, an assessment tool can potentially promote mathematical thinking and mathematical abilities that should be acknowledged and evaluated. However, Swan and Burkhardt argue that many current assessment practices do not address these assessment principles. Assessment practices are usually focused on individual and disconnected elements, based mainly in procedural knowledge (Swan and Burkhardt 2012).

In a similar perspective, Henningsen and Stein (1997) discussed and recognized that building conceptual connections with previous knowledge; emphasizing meaning; and requiring explanations, thinking processes and strategies are essential for maintaining engagement in high-level thinking mathematical tasks. These features highlight that engaging in and solving mathematical tasks is not only about procedural knowledge. Other skills are necessary and desired. Therefore, these skills should also be taken in consideration when assessing students' mathematical work. Consistent with that, Burkhardt and Swan (2013) speak to the necessity of a reliable tool for students' assessment; they assert that "research is needed to show how student performances on conceptual and problem solving tasks might be reliably measured and reported. Otherwise examiners and teachers will continue to assess fragments rather than complete performances" (p. 438).

In sum, if students are to face tasks that (1) address genuine situations, (2) demand mathematical curriculum knowledge, (3) are challenging, (4) require students' autonomy and decision making, (5) focus on reasoning, (6) acknowledge mathematics worthiness, and (7) are engaging, then students' work will definitely encompass and work on different and various mathematical skills that should be assessed. The present chapter offers Kilpatrick et al.'s (2001) mathematical proficiency model as an assessment option that looks into comprehensive factors. Kilpatrick et al. assert that "initial learning with understanding can make learning more efficient" (p. 123). As such, the authors' model of mathematical proficiency draws attention not only for procedural knowledge but also to other factors that contribute to students' mathematical understanding. For Kilpatrick et al., the five essential strands that compose students' mathematical proficiency are *conceptual understanding, strategic competence, procedural fluency, adaptive reasoning* and *productive disposition* (Table 1).

Procedural fluency is commonly the only strand focused on in assessment. This might be due to the fact that in mathematics students are commonly encouraged to write down the procedures they use to solve tasks, instead of explaining their understandings, strategies and/or reasoning. In some situations, understandings, strategies and reasoning can be inferred from procedures, but these inferences are not always a reliable picture of students' mathematical thinking and proficiency. In other words, students' written materials are predominantly about *procedural fluency*, which reflects only part of students' mathematical proficiency. Bearing in mind that students' mathematical learning should be ideally assessed formatively and as a whole, Kilpatrick et al.'s (2001) mathematical proficiency model is a good option to holistically analyse students' work when solving mathematical tasks. On the other hand, it is relevant to note that students write down more than procedures,

Table 1 Description of Kilpatrick et al.'s (2001) strands of mathematical proficiency

Conceptual understanding	Speaks to students' retrieval, connection and comprehension of mathematical ideas, content and representations. Allows students to better retain and (re) construct knowledge, gets students ready to detect conceptual errors, requires students to "learn less" (once content is interrelated), and provides students with confidence.
Strategic competence	Involves formulating, representing and solving problems. Allows students to realize similarities in tasks' structure, to identify relations between mathematical elements, and to figure out different solving models or representations when faced with non-routine situations, which stimulates flexibility and productive thinking.
Procedural fluency	Refers to the appropriate and flexible use of procedures, as well as to the comprehension of them. Without enough *procedural fluency*, procedures may be compartmentalized, students' mathematical understanding may be superficial, and problem solving skills may be impaired.
Adaptive reasoning	Alludes to the ability of building logical connections between mathematical ideas, contents and circumstances. It is not only about formal proofs, but also about argumentation, justification and reasoning.
Productive disposition	It is necessary if students are to develop the other four strands; and it is determinant on students' academic achievement. It is about acknowledging and believing in the benefits of mathematics, and about trusting in one's own ability to do and learn mathematics.

only if they are prompted to. That is to say that, mathematics tasks should encourage or directly ask for students' understanding, strategies and reasoning. This chapter next describes two examples that show how the five strands of mathematical proficiency can be expressed in students' mathematical work.

Assessing Students' Work Through Mathematical Proficiency

Task 1 (Fig. 1) and Task 2 (Fig. 3) were implemented in a senior high school in Alberta, Canada, in a grade 11 mathematics course. Modeling tasks were used in order to prompt and allow for the emergence and use of relevant mathematical skills. Students worked in groups of three or four, discussing and sharing their ideas. Students had two consecutive 80 min classes to work on each task and they had individual journals to record their work.

Following each task, there are two interpretive diagrams (Figs. 2 and 4) illustrating the work done for each task by two different students. These diagrams present fragments of students' work and discourse while doing the task. Fragments were collected from students' journals, audio and video recordings transcripts, and post-class interview transcripts (for one of the students only). Italics are used to represent students' verbatim quotations. A grey ellipse-shaped arrow shows the chronological order in which task fragments were gathered. Fragments that are not overlapped by the grey arrow were collected during interviews. Fragments are categorized according to the five strands of Kilpatrick et al.'s (2001) mathematical proficiency model,

<u>Task 1</u>:

The distance medley relay is an athletic event in which four athletes compete as part of a relay. Unlike most track relays, each member of the team runs a different distance. A distance medley relay is made up of a 1200 meter leg, or three laps on a standard 400 meter track; a 400 meter leg, or one lap; an 800 meter leg, or two laps; and a 1600 meter leg, or four laps - in that order. The total distance run is 4000 meters or nearly 2.5 miles. Aside from the 400 meter segment, which is a sprint, all legs are a middle distance run. Prior to going metric, the distance medley relay consisted of a 440 yard leg, an 880 yard leg, a 1320 yard leg and a mile leg. The total distance for the old distance medley relay was 4400 yards and the total distance for the current metric distance medley relay is 4374.45 yards - a little over 25 yards shorter than the old race. (Adapted from Distance medley relay, n.d.)

You are the coach of a women's distance medley relay team. In a few months, your team will be participating in a competition, and your goal is to have them complete the task in 11 minutes and 15 seconds or less. The difference between the speed of your fastest runner and your slowest runner is 1.6 m/s at maximum. Other factors can influence athletes' performance as well. Decide which athlete will run each relay leg. What is the desired minimum speed you should be looking for from each athlete during the training period?

Fig. 1 Task 1 – Distance medley relay

revealing different nuances of students' work that are present in an everyday mathematics class. These nuances can and should be acknowledged and assessed.

Students were supposed to analyse Task 1 (Fig. 1) and make their own assumptions to solve it. There were many possible correct answers, and different solutions and conjectures were welcome. This chapter presents the interpretive diagram designed out of Rick's work for Task 1 (Fig. 2), in which initial investigation is based on the fact that each runner is supposed to run a different fraction of the total medley distance. As seen in fragment one of Rick's interpretive diagram, Rick retrieves his mathematical knowledge about ratios and connects with the given data in the problem, demonstrating *conceptual understanding*. In the post-class interview (fragment ten), Rick explains his notes from fragment one, confirming his *conceptual understanding* about the whole process. When asked about why he used fractions (fragment eight), he clarifies that by using fractions he gets exact values instead of approximate values, which would be the case if he was working with decimals. This explanation shows Rick's *adaptive reasoning*, given his ability to logically relate fractions and decimals.

Going back to Rick's classroom notes, in fragment two, he demonstrates *adaptive reasoning* by presenting an argument based on runners' stamina to explain why a certain runner should be running a certain distance. Still based on this reasoning, in fragment three, Rick shows *strategic competence* by supposing that the runner of the 400 meter leg (shortest leg) should be at maximum 1.6 m/s faster than the runner of the 1600 meter leg (longest leg). That is the strategy he understands will make the problem reasonable. In the interview (fragment nine), Rick explains his strategy and logically relates the different speeds and the different distances. He connects his

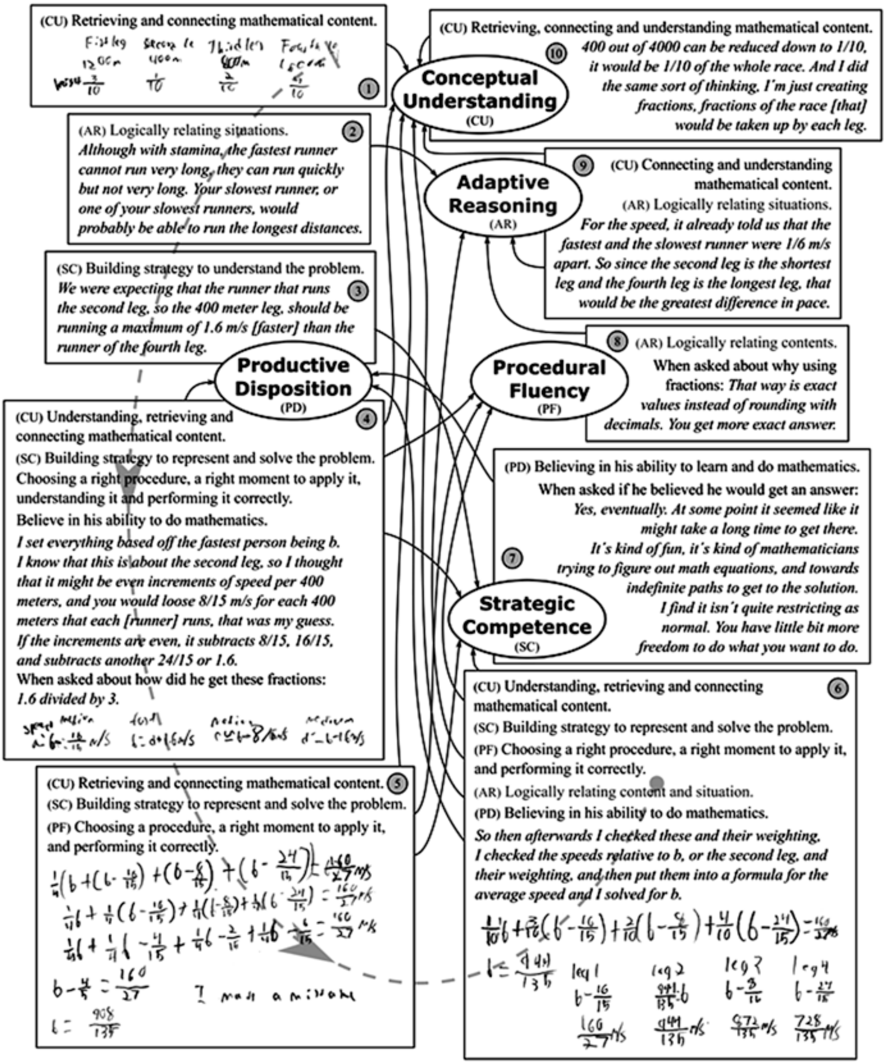

Fig. 2 Rick's mathematical work for Task 1

analysis with mathematical content by stating that 1.6 m/s would represent the maximum difference between speeds. His explanation on this matter speaks both to *conceptual understanding* and *adaptive reasoning*.

In fragment four, Rick shows *conceptual understanding* when retrieving his knowledge about fractions and connecting them to the task. He then chooses a correct procedure based on fractions and correctly applies it to find out runners' speeds, demonstrating *procedural fluency*. Rick explains he has one runner as his main reference and assumes even speed increments for every 400 meters increase in distance, which speaks to his *strategic competence*. He obtains these speed increments

(8/15) by dividing the maximum difference (1.6 m/s) by 3. His explanation about the procedure confirms his *conceptual understanding* and his *procedural fluency*. By assuming one runner as a reference, by affirming he knows what the problem is about, and by making informed guesses to solve the task, Rick is in fact presenting evidence of his belief in his ability to do mathematics, which in turn demonstrates his *productive disposition*.

In fragment five, Rick demonstrates *conceptual understanding* by retrieving his previous knowledge about average, connecting it to the task and coming up with an equation to balance out runners' speeds. This equation serves the purpose of representing and solving the problem, which is about *strategic competence*. Fragment five also speaks to Rick's *procedural fluency* as it illustrates the procedure he chooses to correctly solve the equation. However, as Rick acknowledges, he makes a mistake and needs to come up with a new equation. This new equation is presented in fragment six, alongside an explanation that makes clear his *conceptual understanding* about what was wrong in the first equation. The mistake is that Rick does not consider a weighted average in his first equation. Rick shows *conceptual understanding* by retrieving mathematical content again and connecting to the task, but this time mathematical content about weighted average. *Strategic competence* is demonstrated again when Rick balances runners' speeds out with a weighted average in his final step to represent and solve the task. To make this move, Rick's notes demonstrate *adaptive reasoning* because he logically relates weighted average with the fact that different runners run different legs in size. His explanation in fragment six speaks to his *conceptual understanding* about weighted average as well. Rick shows *procedural fluency* when he chooses a correct procedure to solve the equation, at a right moment, and performs it correctly, getting to his final answer this time. By admitting he made a mistake, by redoing the equation and solving it again, Rick shows *productive disposition* by means of his persistence and confidence in his ability to do mathematics.

As fragment seven shows (an interview fragment), Rick believes he would eventually get to an answer, although sometimes it seemed it would take longer. He also says it was fun to engage in a task similar to what mathematicians do. Finally, he affirms that this kind of task is not as restricting as usual mathematics tasks, which gives him more autonomy to make his own choices. All of these assertions reinforce Rick's belief in his capacity to do and learn mathematics, which speaks to his *productive disposition* towards mathematics.

In Task 2 (Fig. 3), students were welcome to justify their conjectures based on examples, given that they did not formally study logarithms before. Clara's work is analysed for Task 2. In fragment one of Clara's interpretive diagram (Fig. 4), she starts off by correctly performing a binary search procedure in order to understand the task. Her approach shows an example of how *procedural fluency* can be used to understand a task. Clara also asks questions to better understand the mathematical content behind the binary search, which speaks to her *conceptual understanding*. When asking if the searched items in the binary search should be in order or not, she logically relates an ordered binary search to an unordered one, demonstrating her *adaptive reasoning*. In fragment two, Clara again shows *procedural fluency* by

Task 2:

In computer science, a binary search or half-interval search algorithm finds the position of a *specified input value (the search "key") within an array sorted by key value. For binary search, the array should be arranged in ascending or descending order. In each step, the* algorithm *compares the search key value with the key value of the middle element of the array. If the keys match, then a matching element has been found and its index, or position, is returned. Otherwise, if the search key is less than the middle element's key, then the algorithm repeats its action on the sub-array to the left of the middle element or, if the search key is greater, on the sub-array to the right. If the remaining array to be searched is empty, then the key cannot be found in the array and a special "not found" indication is returned.*

(http://en.wikipedia.org/wiki/Binary_search_algorithm)

Binary search is a common tool to search databases such as: dictionaries, library catalogues, phone books etc. Linear search, which examines a disordered list by looking at each item at a time, is not as common. Binary search seems to be preferred in relation to linear search, based on the claim that binary search is faster than linear search. To better illustrate this idea, consider a system looking for the name *Miller* in a database with 10 clients last names. The binary search can be done as follows. The system will retrieve all information in position 6, which corresponds to client Miller.

10	9	8	7	6	5	4	3	2	1
Taylor	**Smith**	**Parker**	**Moore**	**Miller**	**Jones**	Johnson	Hill	Harris	Davis

10	9	8	7	6	5	4	3	2	1
Taylor	Smith	**Parker**	**Moore**	**Miller**	Jones	Johnson	Hill	Harris	Davis

10	9	8	7	6	5	4	3	2	1
Taylor	Smith	Parker	**Moore**	**Miller**	Jones	Johnson	Hill	Harris	Davis

10	9	8	7	**6**	5	4	3	2	1
Taylor	Smith	Parker	Moore	**Miller**	Jones	Johnson	Hill	Harris	Davis

Consider you have a phone book list with ***n*** entries and is searching for one specific number. How would you justify the aforementioned claim based on the worst-case scenario?

Fig. 3 Task 2 – Binary search

correctly performing another binary search and also a linear search to better understand the task. Then she works on her *adaptive reasoning* by comparing the two results to make sense of both kinds of search.

In fragment three, Clara demonstrates *strategic competence* when she comes up with a way to calculate the number of searches in the linear case. As such, she also shows *conceptual understanding* about the linear search, given that this understanding is necessary for her to figure out the number of necessary searches. Then, in fragment

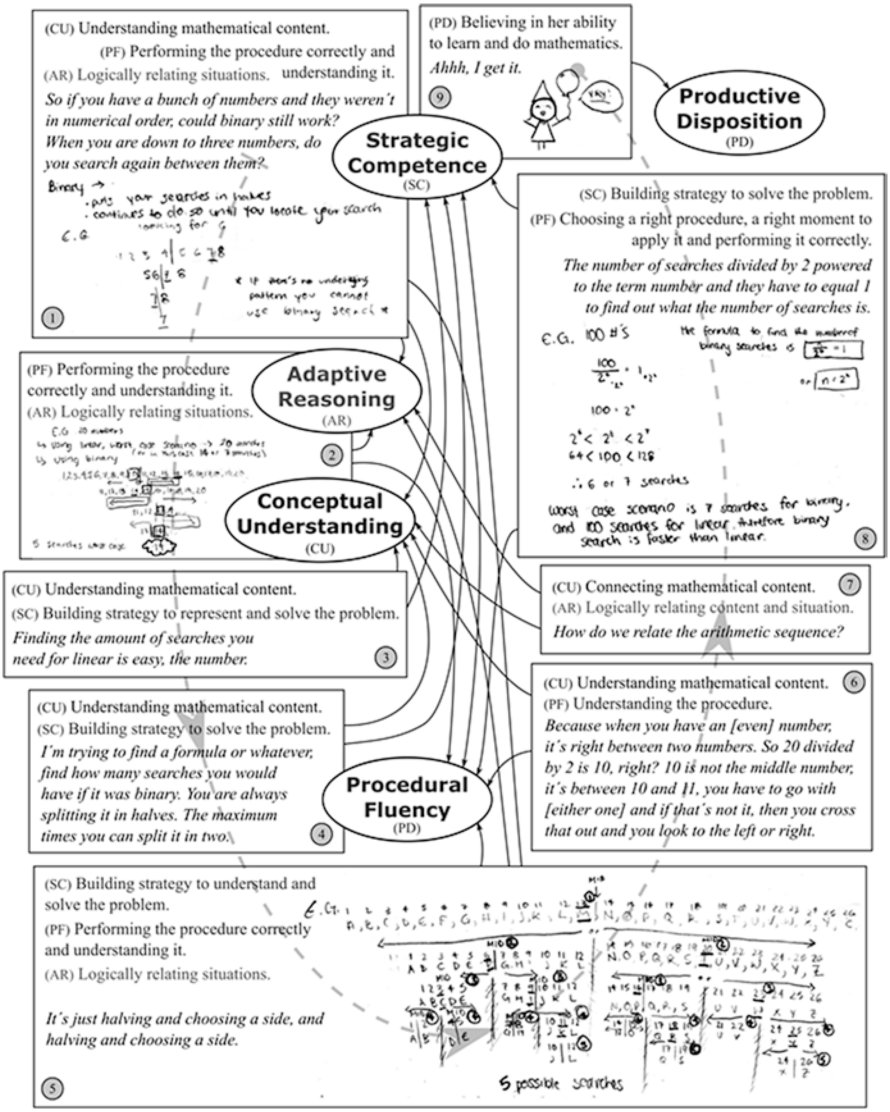

Fig. 4 Clara's mathematical work for Task 2

four, she shows *conceptual understanding* about the binary search procedure, because she connects the procedure to the fact that it is necessary to keep halving the number of searched items. Based on this understanding, she demonstrates *strategic competence* by figuring out what she needs to do to find out the number of searches in the binary case. In fragment five, Clara chooses to reproduce the binary search procedure again, however, at this time, she presents a different sort of *strategic competence*. She decides to look into the discarded options as well. In this way, she works on her *adaptive reasoning* by logically comparing both situations under analysis in order to try to understand the problem, find differences and similarities, and figure out a solution. When she states that it is just about "halving and choosing a side" she demonstrates understanding about the procedure she is implementing, which speaks to her *procedural fluency*. In fragment six, her discourse demonstrates both *conceptual understanding* and *procedural fluency* because she not only correctly explains the procedure, but she also identifies, analyses and explains the particular situation in which there is an even number of items to be searched.

Finally, based on the teacher's input about arithmetic and geometric sequences, Clara works on her *conceptual understanding* by trying to connect her work so far with arithmetic sequences (fragment seven). She also demonstrates *adaptive reasoning* when wondering about the logical relation between the arithmetic sequences content and the linear and binary searches. Nevertheless, she does not present a conclusion about this possible connection. Clara needs some prompting to formalize her ideas mathematically. After visualizing how she could do that, in fragment eight, she uses an example to validate what she called "the formula to find the number of binary searches" ($n = 2^t$). She shows *procedural fluency* by choosing a correct procedure and a right moment to apply it, and she also describes and performs the procedure correctly. Solving this example speaks to her *strategic competence*, given that the example presents the strategy she uses to answer the task's initial question, that is, the strategy she uses to justify why the binary search is faster than the linear search in the worst case scenario. As for *productive disposition*, researcher's field notes attest to Clara's interest in the task. Although she has moments of discouragement, she asks for help and tries to understand the prompts she is given. At the end of the task, she expresses *productive disposition* by drawing and verbalizing her contentment for being able to learn and do mathematics, as can be seen in fragment nine. Clara's work assessment is entirely based on her in-class work since she did not participate in an interview.

Final Considerations

The above examples were not provided to exhaust the analysis of students' work in terms of Kilpatrick et al.'s (2001) model of mathematical proficiency. Rather, they were intended to have a two-fold purpose: (1) explore how Kilpatrick et al.'s mathematical proficiency model assesses more than solely mathematical procedures, addressing students' work in a more holistic way; and (2) attest for the various abilities that can be promoted by mathematical tasks, which in turn are of relevance for the development of mathematical proficiency.

As the interpretive diagrams illustrate, students' journals, students' comments during tasks or during interviews, and researcher's field notes provide evidence that the five strands of mathematical proficiency were in place during students' work when solving the tasks. Indeed, when students analyse the task, and develop a strategy to tackle and solve it, different and varied abilities and thinking processes can be observed. These skills should definitely be acknowledged when assessing students' work, because they address the different nuances of students' mathematical thinking that compose the learning processes that tasks are to promote. These skills can be categorized under Kilpatrick et al.'s (2001) five strands of mathematical proficiency.

To illustrate the importance of the assessment of all five strands of mathematical proficiency in students' work, Rick's situation is further discussed in this closing section. Rick's interpretive diagram (Fig. 2) shows that before he balances runners' speeds out into an equation to solve the task—which might be considered by some as the most relevant part of the solution—many other important aspects of his mathematical thinking and abilities were in place. Rick demonstrates *strategic competence* when structuring his solution, for instance, when defining which runner was running each leg and when defining the increment in runners' speed. Then, Rick works on *conceptual understanding*, *adaptive reasoning*, and *productive disposition*, given that he respectively expresses understanding about the concepts involved in the task, logical reasoning that supports his thinking, and confidence while developing his solution. What this means is that even if Rick had not formalized an equation to finish the task, or had finished the task with the first wrong equation he got, he would still have worked on at least four of the five strands of mathematical proficiency, which indicates a lot in terms of students' mathematical learning.

It is up to us—mathematics teachers—to look at students' work and acknowledge when there is more than formal procedures or final right answers to assess. When a student does a lot of relevant work before getting to the final answer and—based on teacher's assessment—mistakenly concludes it has no value, this student may believe that mathematics is too hard and that (s)he has no capacity to do mathematics. This may be the moment when the teacher "loses" her/his student. Therefore, it is extremely relevant to highlight the conceptual, the strategic, the logical, the procedural and the attitudinal value of students' work. Kilpatrick et al.'s (2001) mathematical proficiency model serves the purpose of helping teachers in this challenging but essential work. By recognizing the assorted nuances of students' mathematical thinking and skills, teachers are potentially prepared to identify students' needs and plan their classes accordingly.

References

Burkhardt, H., & Swan, M. (2013). Task design for systemic improvement: Principles and frameworks. In C. Margolinas (Ed.), *Task design in mathematics education – proceedings of ICMI study 22* (pp. 431–439). Oxford: Springer International Publishing.

Henningsen, M., & Stein, M. K. (1997). Mathematical tasks and students cognition: Classroom-based factors that support and inhibit high-level mathematical thinking and reasoning. *Journal for Research in Mathematics Education, 28*(5), 524–549.

Kilpatrick, J., Swafford, J., & Findell, B. (2001). The strands of mathematical proficiency. In J. Kilpatrick, J. Swafford, & B. Findell (Eds.), *Adding it up: Helping children learn mathematics* [electronic resource] (pp. 115–155). Washington, DC: National Academy Press.

Swan, M., & Burkhardt, H. (2012). A designer speaks: Designing assessment of performance in mathematics. *Educational Designer – Journal of the International Society for Design and Development in Education*, *2*(5). Retrieved from http://www.educationaldesigner.org/ed/volume2/issue5/article19/

Additional Suggestions for Further Reading

Groth, R. E. (2017). Classroom data analysis with the five strands of mathematical proficiency. *The Clearing House: A Journal of Educational Strategies, Issues, and Ideas, 90*(3), 103–109.

Suh, J. M. (2007). Tying it all together: Classroom practices that promote mathematical proficiency for all students. *Teaching Children Mathematics, 14*(3), 163–169.

Re-Framing Testing to Better Fit Within Problem-solving Classrooms: Ways to Create and Review Tests

Tina Rapke, Jennifer Hall, and Richelle Marynowski

Abstract We offer two alternative strategies to simply giving paper-and-pencil mathematics tests that use student thinking as a basis, which we identify as a key underpinning of teaching in problem-solving classrooms. Using student thinking as a basis refers to the idea that teaching is inseparable from, grounded in, and formed by students' ideas. Specifically, we discuss (1) involving students in developing tests to help them prepare for writing tests and (2) reviewing test material by having students compare, analyze, and critique their classmates' test responses and subsequently revise their own work. These two strategies are re-castings of the traditional paper-and-pencil test. Teachers can use the strategies to promote deep approaches to learning and, as a result, help students to perform better on tests.

Keywords Novel assessment methods · Testing · Student thinking

We offer two alternative strategies to simply giving paper-and-pencil tests that use student thinking as a basis, which we identify as a key underpinning of teaching in problem-solving classrooms. Using student thinking as a basis refers to the idea that teaching is focused on and has student contributions at its heart. In this chapter, we describe two assessment strategies that use student thinking as a basis and that have been implemented and researched in post-secondary mathematics classes that were aimed at helping students transition from secondary school. Specifically, we discuss (1) involving students in developing tests to help them prepare for writing tests and (2) reviewing test material by having students compare, analyze, critique, and

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provide feedback to their classmates' test responses and subsequently revise their own work. These two strategies are re-castings of the traditional paper-and-pencil test. Teachers can use the strategies to promote deep approaches to learning and as a result help students to perform better on tests.

Student Thinking As a Basis for Teaching in Problem-solving Classrooms

The idea of using student thinking as a basis in problem-solving classrooms can be seen in various publications. For example, the National Council of Teachers of Mathematics (2014) asserts that effective mathematics teaching "engages students in solving and discussing tasks that promote mathematical reasoning and problem-solving" and "uses evidence of student thinking to assess progress toward mathematical understanding and to adjust instruction continually in ways that support and extend learning" (p. 56). In other words, effective teaching in problem-solving classrooms focuses on students' ideas, as their ideas are essential and required to adjust instruction.

Similarly, the classrooms that inspired Yackel and Cobb (1996) to develop their theory of sociomathematical norms were problem-solving-based and emphasized student thinking. Specifically, lessons in these classrooms began with a problem and ended in "whole-class discussions where children explain and justify the interpretations and solutions they develop during small-group work" (Yackel and Cobb 1996, p. 460), thus making student thinking part of the classroom discourse. Schoenfeld (2014) also advocates for teaching that uses student thinking as a basis. He created a rubric that teachers can use to reflect on their teaching, on which the highest rating is achieved when "the teacher solicits student thinking and subsequent instruction responds to those ideas, by building on productive beginnings or addressing emerging misunderstandings" (p. 408). Without student thinking, it is not possible for teaching to achieve a high rating because there would be no productive beginnings or misunderstandings shared. Accordingly, in Schoenfeld's conception, teaching emphasizes and draws on students' ideas.

The point being that if students' ideas are foundational for mathematics teaching then they should also be for assessment. Galbraith (1993) warns us that teaching and assessment should share in core tenets, as "it is inappropriate to look at alternative modes of assessment without asking about associated teaching approaches and assumptions underlying both" (p. 79).

Deep Approaches to Learning

The importance of students' perspectives on their learning context (including assessment) and the influence of students' perspectives on their learning can be traced back to Marton and Säljö's (1976) foundational work on how students conceptualize learning. In their work, they looked at deep and surface approaches to learning. Surface approaches to learning involve ideas about "hoop jumping" and memorizing, whereas deep approaches to learning are associated with learning to understand and seeing things from different points of view. Specifically, Marton et al. (1993) came up with six ways that people conceptualize learning: "(a) increasing one's knowledge, (b) memorizing and reproducing, (c) applying, (d) understanding, (e) seeing something in a different way and (f) changing as a person" (p. 277). These researchers explain that the first three categories are linked to surface approaches to learning while the last three categories are descriptions of deep approaches to learning. In other words, deep approaches to learning can be evidenced by students describing their learning in terms of understanding or seeing things from different perspectives/angles.

Assessment Strategies That Use Student Thinking As a Basis

In the following sections, we discuss two practical strategies that focus on students' ideas in problem-solving classrooms. While these strategies were implemented and researched in post-secondary mathematics classrooms, they can easily be translated into practice at the secondary level, as the post-secondary classrooms either involved secondary mathematics content or focused on helping students transition from secondary to post-secondary mathematics courses.

Strategy 1: Involving Students in Developing Exams to Help Them Prepare for Exam Writing

The first strategy was researched (Rapke 2016, 2017) in a Canadian technical institute's upgrading mathematics course, which included topics from the province's Grade 11 curriculum. The course was taken by students who lacked experience with secondary mathematics topics. They took the course as a prerequisite to prepare them to complete technical programs such as lab technician, electrician, and legal assistant. In this course, the students worked with the instructor to co-develop a final exam, as per the following steps, which took two three-hour sessions to complete:

- In groups of three or four, students developed practice exams that consisted of six open-ended questions and accompanying solution keys, including point allocation. Students crafted original questions, as well as modified the values within questions from class, assignments, and the course textbook. At least one of the six questions had to be original.
- In these groups, students practised for the "actual" exam by writing each other's practice exams (e.g., students in Group 1 wrote Group 2's practice exam and vice-versa). They completed each other's practice exams individually, and were

instructed not to talk with classmates or use resources other than a pen or pencil in order to mimic the "actual" exam conditions.
- The same paired groups assessed their classmates' written responses to the practice exams and assigned grades to the practice exams.
- The instructor posted all practice exams and student-generated solution keys on the course website for study purposes.
- The instructor created the "actual" exam by choosing and modifying the values of five questions from students' practice exams, as well as crafting one additional question.

This strategy uses student thinking as a basis because creating practice exams would not be possible without students' ideas on what questions to include on the exam. Furthermore, results from the study, which involved post-course interviews with students, establish that developing exams with students promotes the use of deep approaches to learning when preparing to write an exam (Rapke 2016). The deep approaches to learning reported in the interviews were evident in this process through reports of seeing things from different viewpoints and learning for understanding (Marton et al. 1993).

In terms of learning for understanding, one participant commented that the process of creating the exam "was really good 'cause [pause] we get to review and if there was something that I missed and I still don't understand it, hopefully, I can understand it now that we are going to put that on the final." This quotation illustrates deep approaches to learning as the participant specifically mentioned understanding. In terms of changing the values within existing questions to craft practice exam questions, another participant said that:

> You're looking at math from a different point of view. If you can actually [pause] now if I look at a question, I kind of see, like, [pause] you can kind of think of how it was created too, to solve it. Like, how did someone put this together to make this equation?

Many comments that the students made about seeing things from different perspectives referred to changing the numbers within questions about solving polynomial equations for real roots. For example, one student said that he experienced difficulty in "mak[ing] the questions and having them work out. I was a little anxious in not knowing what to put in there." The student clarified that "what ended up happening was we changed our numbers so there was [sic] no zeros." That is, he saw a new relationship between the coefficients of a polynomial and its zeros. These students' comments provide strong evidence of deep approaches to learning, as per Marton et al.'s (1993) conception, as these students described seeing things from different points of view. Moreover, the quotations evidence how this process supported students as they prepared to write an exam. Thus, having students create practice exams uses student thinking as a basis, provokes deep approaches to learning, and helps students prepare for writing exams.

Strategy 2: Reviewing Tests by Having Students Compare, Analyze, Critique, and Provide Feedback to Their Classmates' Test Responses

Another strategy successfully implemented in a first-year university mathematics course involved students writing a traditional closed-book test. This course was focused on supporting students who were mathematics majors to transition from the secondary to post-secondary level. However, rather than the instructor simply grading and returning the tests, the students engaged in a process, termed the Active Exam Review Process (AERP), that involved reviewing instructor-selected student responses to test questions and students revising their own exam responses (for more information, see Rapke and Hall 2016). Thus, the AERP involves the use of summative assessments (assessments that summarize learning) in formative ways (assessments that are used to move learning forward).

Specifically, the AERP was comprised of the following steps:

- Individually, students wrote a closed-book mathematics test, using only a pen or pencil.
- The instructor reviewed all students' responses to each test question and selected both acceptable responses and responses that were representative of misunderstandings. For instance, misunderstandings could include text that was logically flawed, did not flow, or could have been more concise.
- The selected responses were anonymized (i.e., names removed) and photocopied for use in the in-class portion of the AERP.
- Working in small groups, students reviewed the provided responses to each test question and identified acceptable responses. Students compared, analyzed, critiqued, and provided feedback to their classmates' test responses.
- Then, the original tests (without any feedback; the instructor provided feedback and marks on photocopies of the tests) were returned to the students, who had the chance to revise their own responses based on their discussions with peers about the provided responses. Each student could earn an additional few points on her/his initial test grade based on these revisions.

To illustrate the process, we provide an example from the first-year university mathematics course where the AERP was employed. The process was completed for the test question: Prove that for every non-negative integer n, $2^n > n$. Figure 1 shows an example of feedback that students provided to the instructor-selected responses. Specifically, the student is indicating that the response contains a logical flaw (Notice that $2k \ngeq k + 1$ for all non-negative integers). The student identified the misconception in their feedback (purple writing) by underlining the "flawed" text and asking the question, "What if $k = 0$?"

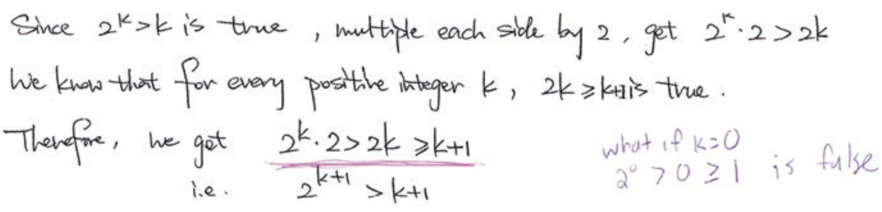

Fig. 1 Sample student work for test question

As with the first strategy discussed, the AERP is focused on students' mathematical ideas as it relies on students' test responses. Namely the use of student responses to test questions gave students the chance to see examples of their peers' thinking about the mathematical topics on the test and provided the instructor with opportunities to address misunderstandings that arose on the test.

As discussed in detail elsewhere (Rapke and Hall 2016), having students engage in the AERP and consider their classmates' ideas is conducive to deep approaches to learning. Markedly, in student surveys about the AERP, nearly all the students (17 of 18, 94.4%) discussed the learning opportunities inherent in analyzing others' mistakes and correcting their own mistakes. One student indicated that she was using deep approaches to learning in the AERP when she said that the process "lets you understand what you did wrong and improve so for next time you know the proper way of doing it." This comment evidences deep approaches to learning, as the student used the word "understand" to describe her experiences. There is also evidence that the AERP promotes deep approaches to learning as the students were able to see things in different ways through being exposed to new approaches to respond to test questions. For example, a student asserted that "I learned what mistakes I made and got a new approach to address the questions better." The AERP promotes deep approaches to learning by supporting students to understand the mistakes that they made on tests and having students engage with different/better approaches to solve a mathematics problem.

Conclusion

Assessment in problem-solving classrooms should not be something that is done *to* students but rather *with* students by focusing on and having students' ideas at the heart of assessment. It is in this way that students employ deep approaches to learning and thus experience more success on tests.

The first strategy is clearly focused on student thinking. The teacher made the "actual" test from a pool of student-developed questions. During the process, some students identified questions that they did not understand. Thus, developing the practice exam supported them to understand the questions. In the second strategy, student thinking was a basis as students' test responses were used to review the test. Students indicated that they learned by comparing, analyzing, critiquing, and providing feedback on student responses. Specifically, students said that they were offered different approaches/perspectives to solve problems, which would help them avoid making the same mistakes on future tests.

While many scholars have critiqued the continued focus on "traditional" tests as they can lead students to memorize and reproduce information, as well as spend time on test-taking strategies rather than meaningful learning (Miller and Parlett 1974; Struyven et al. 2005; Willingham 2002), in this chapter, we have provided ways to utilize such tests in meaningful ways, resulting in deep approaches to learning. Hence, some of the deficiencies of "traditional" tests can be overcome in problem-solving classrooms through the use of novel strategies for teaching and learning. Furthermore, the re-casting of the traditional paper-and-pencil test described here aligns teaching and assessment in problem-solving classrooms because both share the core tenet of using student thinking as a basis.

References

Galbraith, P. (1993). Paradigms, problems and assessment: Some ideological implications. In M. Niss (Ed.), *Investigations into assessment in mathematics education* (pp. 73–86). Dordrecht: Springer.

Marton, F., & Säljö, R. (1976). On qualitative differences in learning I: Outcome and process. *British Journal of Educational Psychology, 46*(1), 4–11. https://doi.org/10.1111/j.2044-8279.1976.tb02980.x.

Marton, F., Dall'Alba, G., & Beaty, E. (1993). Conceptions of learning. *International Journal of Educational Research, 19*(3), 277–300. https://doi.org/10.1016/0883-0355(93)90015-C.

Miller, C., & Parlett, M. (1974). *Up to the mark: A study of the examination game*. Guildford: Society for Research into Higher Education.

National Council of Teachers of Mathematics. (2014). *Principles to actions: Ensuring mathematics success for all*. Reston: Author.

Rapke, T. (2016). A process of students and their instructor developing a final closed-book mathematics exam. *Research in Mathematics Education, 18*(1), 27–42. https://doi.org/10.1080/14794802.2015.1134342.

Rapke, T. (2017). Involving students in developing math tests. *Mathematics Teacher, 110*(8), 612–616. Retrieved from http://www.nctm.org/publications/mathematics-teacher/.

Rapke, T., & Hall, J. (2016). Summative ~ formative: Exploring novel uses of written exams in an undergraduate mathematics course. In C. A. Shoniregun & G. A. Akmayeva (Eds.), *Canada International Conference on Education (CICE-2016) proceedings* (pp. 48–51). Mississauga: Infonomics Society.

Schoenfeld, A. H. (2014). What makes for powerful classrooms, and how can we support teachers in creating them? A story of research and practice, productively intertwined. *Educational Researcher, 43*(8), 404–412. https://doi.org/10.3102/0013189X14554450.

Struyven, K., Dochy, F., & Janssens, S. (2005). Students' perceptions about evaluation and assessment in higher education: A review. *Assessment & Evaluation in Higher Education, 30*(4), 325–341. https://doi.org/10.1080/02602930500099102.

Willingham, D. T. (2002, Summer). Allocating student study time: "Massed" versus "distributed" practice. *American Educator, 26*, 37–39.

Yackel, E., & Cobb, P. (1996). Sociomathematical norms, argumentation, and autonomy in mathematics. *Journal for Research in Mathematics Education, 27*(2), 458–477. https://doi.org/10.2307/749877.

Enhancing Mathematics Teaching and Learning Through Sound Assessment Practices

Christine Suurtamm

Abstract This chapter reflects current thinking in assessment and provides concrete examples of what this might look like in a secondary mathematics classroom. The chapter begins with an overview of principles of sound classroom assessment, particularly as they relate to mathematics teaching and learning.

The chapter then discusses the components of sound classroom assessment that include assessment as ongoing and embedded in instruction, using a variety of assessment strategies, reflecting meaningful mathematics, and including students in the assessment process. Discussion of these components includes practical examples, strategies, or templates that teachers can use at the secondary level in mathematics. A variety of assessment strategies are explored and include observations, conferencing, questioning, performance assessment, and portfolios. Suggestions for building an overall assessment plan conclude the chapter.

Keywords Assessment · Assessment strategies · Formative assessment · Planning for assessment · Secondary mathematics

The word "assessment" is regularly used to mean many different things. It might bring to mind large-scale assessment, or end of unit tests or quizzes, or it might mean the ongoing questioning, listening, and responding to student thinking that teachers do in every aspect of their classroom activity. During the mid-1900s, when models of learning were more closely aligned with an acquisition model that saw students as taking in facts and procedures, assessment often was viewed as a means to see how much of the information students actually ingested through performance on an end-of-unit test. Currently, with a different understanding of how students learn that recognizes that students need to work with mathematical ideas in order to develop an understanding of those ideas, assessment is seen as on-going, constantly looking at students' understanding and making teaching and learning decisions

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based on what that understanding looks like. Current perspectives of mathematics teaching and learning value mathematical understanding through student engagement in problem solving and argumentation. Furthermore, these perspectives recognize the importance of aligning assessment and instruction, including mathematical actions in assessment tasks, and designing and using assessments in equitable ways. Thus there is a move away from assessing merely through paper-and-pencil tests to the use of a range of assessment strategies that recognize the multi-faceted mathematical actions that are part of doing mathematics and provide multiple opportunities for students to show what they know and can do.

Purposes of Assessment

Assessment has many different purposes from reporting student progress to parents, determining what program a student might enter, providing feedback to students about next steps, or providing information to teachers to guide their instructional moves. Although assessment may be conducted for many reasons, the central purpose of assessment should be to support and enhance student learning (Joint Committee on Standards for Educational Evaluation 2003; Wiliam 2007). This message is reflected in many provincial curriculum and assessment policy documents. For instance, the mathematics curriculum documents of the Ontario Ministry of Education state "The primary purpose of assessment and evaluation is to improve student learning" (Ontario Ministry of Education 2005, p. 18).

This focus on improving student learning puts students at the heart of the assessment process. It is interesting to note that assessment derives from the Latin word *assidere*, meaning "to sit beside or with" (Wiggins 1993). Thus the origin of the word assessment further suggests that it is a process done with students, not to students (Klein 1966).

Assessment discussions often focus on the distinctions between formative and summative assessments. If evidence is used to inform teaching and learning with a view to improve learning, then the assessment would be considered to have a formative purpose (Black and Wiliam 2009). If, instead, the evidence gathered from an assessment is used to report on student learning at a particular point in time, then the assessment could be considered to be serving a summative purpose (Black and Wiliam 2009; Wiliam 2015).

However, assessments themselves are neither formative nor summative (Wiliam 2015). Rather, it is how the evidence generated by the assessment is used and the types of inferences that are made that make an assessment formative or summative. For instance, although a teacher might design an end of unit test as a summative task, the feedback provided on that test might also inform teaching or provide information to students to improve future learning (Brookhart 2001; Earl 2013). Thus, the assessment is serving both formative and summative purposes.

Increasingly there has been an emphasis on including more formative assessment in teaching mathematics as the use of formative assessment has been shown to enhance student learning for all students, but particularly for those who might be deemed struggling learners (Black and Wiliam 1998).

Components of Sound Classroom Assessment

Sound classroom assessment is ongoing and embedded in instruction, uses a variety of assessment strategies, reflects meaningful mathematics, and includes students in the process. The following discusses each of these components and includes examples of what this might look like in a mathematics classroom.

Ongoing and Embedded in Instruction

Research suggests that assessment should be integrated into all aspects of teaching and learning in a seamless manner to inform instructional decisions in ways that promote student learning (Carless 2007). Students' learning is supported when the moment-by-moment actions and decisions that teachers make during teaching are informed by evidence of students' understanding (Leahy et al. 2005). However, these actions and decisions require focused attention in order to make students' mathematical thinking and understanding evident.

Terry, a Grade 10 teacher, integrates formative assessment within her lessons, particularly in the way that she questions, listens, and responds to student thinking. For instance, Terry began one lesson by modeling a linear relationship with stacked linking cubes. She asked the students, "What kind of relation is this?" She allowed many students to answer as she was looking for the different ways that students could describe relations. After they focused their attention on the relation being linear she asked, "Why is it linear?", "How do you know?" and "Is there anything else that you notice?" to promote a discussion that prompted students to listen to other students' responses to encourage their reasoning and to expand their understanding of particular concepts.

Terry then presented color-coded linking cube models of quadratic relationships such as in Fig. 1. Terry worked with the class to create a table of values for the model, and together they determined and discussed the characteristics of the relationship.

Groups were then asked to create a quadratic relationship with specific criteria using different colored blocks. After students had created their relationships, Terry led a discussion about the models and asked "What does that mean that those two models are equal?" She paraphrased a student response and then asked, "… how could I verify, how could I prove that they were equal?" Terry encouraged all of her students to respond and worked with their responses, whether they were correct or incorrect.

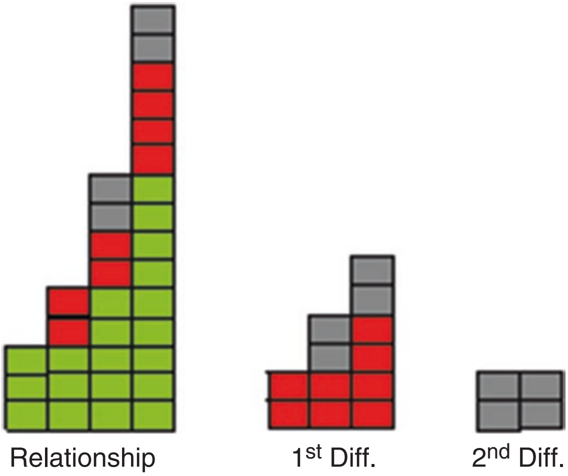

Fig. 1 Model of quadratic relationship and its first and second differences

Throughout this lesson, Terry encouraged her students to consider not only what they were doing and how they were doing it, but to question why, by thinking about what makes a relation linear or quadratic. Terry claimed that her questioning not only provided feedback to the students but that she learned a great deal about her students' prior knowledge, misconceptions, and current understanding of mathematical ideas through creating a dialogue with thought-provoking questions, attentive listening, and responses that develop the conversation. Terry also used the responses of students to adjust her pedagogical moves (Suurtamm 2012).

There are many ways that teachers provide opportunities to elicit and listen to student thinking such as observations during problem solving, informal interviews during class, or using focused questions during mathematical discussions, as Terry did. These methods allow teachers to be responsive to students' understandings and adjust instruction as well as deal with particular understandings with individual students. These opportunities to elicit student thinking can be incorporated into lessons, even in the planning stages. It might be useful to think ahead of time of the kinds of questions that could be asked to make student thinking visible. These questions could occur in a whole class discussion, in individual interviewing, or in conferencing with small groups as they work on problem solving.

Using a Variety of Assessment Strategies

There are several reasons why using a variety of assessment strategies is important. One is that using a variety of strategies takes into account that students show their understanding in different ways. Some students may perform well on a test whereas others may be able to verbally explain their thinking or demonstrate their thinking using mathematical thinking tools such as graphing software or concrete materials. Another reason to use a variety of assessment tools is so that students have multiple opportunities to show what they know and can do. In other words, we are not just

Table 1 Assessment tools and strategies

Strategies	Tools
Tests	Marking scheme
Quizzes	Rubric
Interviews	Check-bric
Rich tasks	Checklist
Conferencing	Comments
Projects	

Criteria	Levels			
	1	2	3	4
Clear descriptor of criteria 1				
Clear descriptor of criteria 2				
Clear descriptor of criteria 3				

Fig. 2 Template for a Check-bric

giving them one chance to demonstrate their achievement of a particular curriculum expectation or standard. A third reason to use a variety of assessment strategies is that mathematics is a complex process with multiple actions. Assessment strategies need to assess the full range of mathematical actions (Suurtamm et al. 2010). Thus, assessment practices that include observations, interviews, performance tasks, reflective journals, projects, portfolios, presentations and self-assessments are an essential part of implementing current approaches to teaching mathematics (NCTM 1995; Wiliam 2007). The following table (Table 1) provides some of the assessment strategies (things students do) and tools (used by teachers to assess the things students do) that might be useful to mathematics teachers.

Tests, quizzes, and marking schemes might be some of the more familiar assessment methods in mathematics, however, other assessment strategies are also emerging in mathematics classes (Suurtamm et al. 2010). A teacher might, for instance, give the students a rich task, or problem, to work on that could take an entire instructional period. The teacher might use this task as a way to assess how students engage in mathematical processes, or to assess an expectation that requires students to make conjectures or engage in investigation. She may use portable technology to video record groups of students as they work and thus have a record of her observations. Rather than using a marking scheme to assess the student work, the teacher may want to use a combination of an observation rubric where she records how students engage in the problem or the types of models they build or use. She might also use another rubric to assess the student's thinking in the written work that the student submits that explains his/her solution and thinking. Because the solutions will be qualitatively different, the use of a rubric with descriptors would seem more appropriate than a marking scheme. In creating a rubric, a teacher needs to consider the criteria that will be assessed. Some teachers find it useful to create a check-bric, a combination of a rubric and a checklist, (see Fig. 2) which clearly lists the criteria that the teacher is looking for on the left hand side. It is assumed that the teacher would have established a clear understanding with the students as to what each of the levels corresponding to each criteria might look like.

Reflecting Meaningful Mathematics

Since classrooms focus on developing student reasoning and sense making, and on the mathematical processes that students engage in when doing mathematics, then assessment must take these into account (NCTM 2009). Assessments should reflect the mathematics that is important to know and do and should present a comprehensive picture of what mathematics is (NCTM 1995, 2000, 2014). Assessing complex processes is not an easy task and generally cannot be done through an easy-to-score paper and pencil test. All too often what is assessed is what is easiest to assess, such as manipulation of symbols or an application of a formula, rather than what is more complex but which more closely resembles the important process of doing mathematics, such as problem solving or reasoning and proving.

In considering the types of assessments that might be included in the planning process for a unit or term, consideration needs to be given to the curriculum expectations that are being assessed. For instance, consider how to assess the following examples of curriculum expectations:

1. "Students will make connections, through investigation with technology, between changes in a real-world situation that can be modelled using a periodic function and transformations of the corresponding graph (e.g., investigate the connection between variables for a swimmer swimming lengths of a pool and transformations of the graph of distance from the starting point versus time)" (grade 11; Ontario Ministry of Education 2007, p. 66).
2. "Students will analyze puzzles and games that involve spatial reasoning, using problem-solving strategies" (grade 10; Alberta Education 2008, p. 32).

For the first example, a paper and pencil test might not be able to determine how students make connections between real-world situations and graphical models through using technology in investigations. It might be necessary to have an observation component while students are investigating as well as a written component explaining their thinking. In the second example, students might do a project where they focus on two or three games and discuss how spatial reasoning is involved. This might be through a presentation to the class and the teacher might use an observation rubric during the presentation or conference with the student after the presentation to be sure that the student explains his or her thinking.

Thus, a teacher needs to consider the wide array of curriculum expectations being assessed and align the assessment strategies to the curriculum expectations. It is helpful to pay attention to the verbs in the curriculum expectations as the verbs tend to tell us the student actions that we should be observing and assessing. The nouns help to tell us the mathematical concepts that we should see developing.

Assessing mathematics includes assessing the mathematical processes to better understand not just what students have learned but how they learn, as this helps teachers determine next steps (Hunsader et al. 2014).

Including Students in the Assessment Process

Current approaches to classroom assessment emphasize the role of the student in the assessment process (c.f. Earl 2013). Wiliam (2007) suggests that as teachers engage students in formative assessment, students develop ownership for their own learning and act as resources for one another. Students might take part in developing and applying assessment criteria or improving their peer- and self-assessment skills (Bleiler et al. 2015; Moss and Brookhart 2009; Shepard 2001). Engaging students in the co-creation of assessment criteria and using samples of student work to discuss criteria helps students recognize high quality work and helps them to improve their own work. When students are clear about criteria they are able to provide or select evidence of their own learning. For instance, Bleiler et al. (2015) provide evidence of the value of students co-constructing rubrics in college mathematics classes.

Many teachers use portfolios that require students to select evidence of their achievement of content areas or their use of mathematical processes. Within these portfolios, students also explain why they have selected the particular pieces of student work and how those pieces demonstrate their learning. Thus, students develop the metacognitive skills to self assess.

Many teachers regularly conference with students to review with each student the evidence of student achievement that the teacher has recorded. In this way the teacher and student might determine whether the student needs other assessment opportunities to show what they know and can do (Suurtamm and Arden 2017). In this way, assessment is transparent and involves the student in the assessment process.

Making an Assessment Plan

When doing a unit plan, it is helpful to also consider the assessment opportunities within that plan. In determining what those assessment opportunities might look like, focus on the mathematical actions suggested by the verbs in the curriculum expectations, as well as the mathematical concepts that are to be assessed. Consider what types of assessments might address both the nouns and verbs in the curriculum expectations. Also, consider the ways that you will know what students know at the beginning of a unit, during the unit and at the end of the unit. At the beginning of a unit, you might consider having students work in pairs solving a problem that connects to their prior knowledge on the first day of the unit. In this way, you can

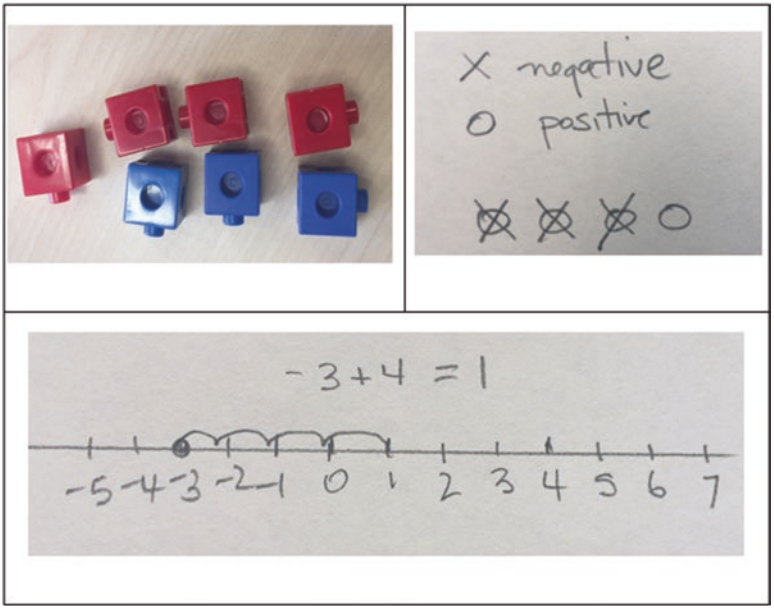

Fig. 3 Possible student representations of −3 + 4

circulate throughout the class and listen to students working together to hear the ways students already understand the concepts that underlie the new unit. For instance, in starting a unit on operations with integers, a teacher might have students "teach" one another some simple operations with integers so that the teacher can hear what students remember but also how they remember it. For instance, in adding integers, are students drawing a number line, using the model of a thermometer, using colour coding, or talking about "good guys" and "bad guys"? (See Fig. 3 for possible student representations).

By listening to the conversations, the teacher can then build on the models and the language that the students already know. For instance, the teacher will know whether or not most students are familiar with a number line and therefore might choose to start the unit with that model. Midway through the unit, the teacher might consider a quiz and provide descriptive feedback to the students on areas they seem to know and areas where they need some extra support. The unit may also address expectations that call for students to investigate, build models, describe, or explain. This might call for a performance task that engages students in an activity so that the teacher can observe or conference with students as they work on the task and record the ways in which they engage in mathematical actions.

Sound assessment requires planning. A template similar to the one in Fig. 4 might help a teacher to think ahead about assessment opportunities in their unit or long range planning. In this template, the teacher would record the daily lessons or activities in the first column, the curriculum expectations that are being addressed in the second column, and the mathematical processes in the third column. Then

			Assessment Opportunities		
Activity/Lesson	Curriculum Expectation(s)	Mathematical Processes	Purpose	Strategy	What is being assessed

Fig. 4 Template for including assessment in unit plan

assessment opportunities for each lesson would be considered. The activity or lesson might provide the opportunity for either a diagnostic, formative, or summative assessment. The strategy to be used (e.g., observation, quiz, student conferencing) would be specified as well as an indication of the focus of the assessment. It might be on some component of the curriculum expectation, informal assessment of prior learning, a particular mathematical process, or perhaps a learning skill (such as team work or perseverance).

This planning allows the teacher to see the range of assessment opportunities and to choose which will be used and when. Consideration should be given to what is being assessed and how it is being assessed so that a full range of mathematical actions are assessed and a variety of strategies are used, thus taking into account many of the components of sound classroom assessment discussed above.

References

Alberta Education. (2008). *The Alberta 10–12 mathematics program of studies with achievement indicators*. Edmonton: Alberta Education.

Black, P., & Wiliam, D. (1998). Assessment and classroom learning. *Assessment in Education: Principles, Policy, and Practice, 5*(1), 7–74.

Black, P., & Wiliam, D. (2009). Developing the theory of formative assessment. *Educational Assessment Evaluation and Accountability, 21*, 5–31.

Bleiler, S. K., Ko, Y.-Y., Yee, S. P., & Boyle, J. D. (2015). Communal development and evaluation of a course rubric for proof writing. In C. Suurtamm & A. Roth McDuffie (Eds.), *Assessment to enhance teaching and learning* (pp. 97–108). Reston: NCTM.

Brookhart, S. M. (2001). Successful students' formative and summative uses of assessment information. Assessment in education: Principles, policy, and practice*,* 8(2), 153–169.

Carless, D. (2007). Learning-oriented assessment: Conceptual bases and practical implications. *Innovations in Education and Teaching International, 44*(1), 57–66.

Earl, L. (2013). *Assessment as learning: Using classroom assessment to maximize student learning*. Thousand Oaks: Corwin.

Hunsader, P. D., Thompson, D. R., & Zorin, B. (2014). Mathematical practices: Small changes in assessments = big benefits. In K. Karp & A. Roth McDuffie (Eds.), *Using research to improve instruction* (pp. 205–214). Reston: NCTM.

Joint Committee on Standards for Educational Evaluation (JCSEE). (2003). *The student evaluation standards: How to improve evaluation of students*. Thousand Oaks: Corwin Press and ETS.
Klein, E. (1966). *A comprehensive etymological dictionary of the English language: Dealing with the origin of words and their sense development thus illustrating the history of civilization and culture*. New York: Elsevier.
Leahy, S., Lyon, C., Thompson, M., & Wiliam, D. (2005). Classroom assessment: Minute by minute, day by day. *Educational Leadership, 63*(3), 18–24.
Moss, C. M., & Brookhart, S. M. (2009). *Advancing formative assessment in every classroom*. Alexandria: ASCD.
National Council of Teachers of Mathematics (NCTM). (1995). *Assessment standards for school mathematics*. Reston: Author.
National Council of Teachers of Mathematics (NCTM). (2000). *Principles and standards for school mathematics*. Reston: Author.
National Council of Teachers of Mathematics (NCTM). (2009). *Focus on high school mathematics: Reasoning and sense making*. Reston: Author.
National Council of Teachers of Mathematics (NCTM). (2014). *Principles to action: Ensuring mathematical success for all*. Reston: Author.
Ontario Ministry of Education. (2005). *The Ontario curriculum: Grades 9 and 10 mathematics*. Toronto: Queen's Printer for Ontario. Retrieved from http://www.edu.gov.on.ca/eng/curriculum/secondary/math910curr.pdf
Ontario Ministry of Education. (2007). *The Ontario curriculum: Grades 11 and 12 mathematics*. Toronto: Queen's Printer for Ontario.
Shepard, L. A. (2001). The role of classroom assessment in teaching and learning. In V. Richardson (Ed.), *Handbook of research on teaching* (4th ed., pp. 1066–1101). Washington, DC: American Educational Research Association.
Suurtamm, C. (2012). Assessment can support reasoning and sense making. *Mathematics Teacher, 106*(1), 29–33.
Suurtamm, C., & Arden, A. (2017). Using assessment to enhance mathematics mathematics teaching and learning. In D. Spangler & J. Wanko (Eds.), *Enhancing classroom practice with research behind principles to actions* (pp. 141–152). Reston: NCTM.
Suurtamm, C., Koch, M. J., & Arden, A. (2010). Teachers' emerging assessment practices in mathematics: Classrooms in the context of reform. *Assessment in Education: Principles, Policy, and Practice, 17*(4), 399–417.
Wiggins, G. P. (1993). *Assessing student performance: Exploring the purpose and limits of testing*. San Francisco: Jossey-Bass.
Wiliam, D. (2007). Keeping learning on track: Classroom assessment and the regulation of learning. In F. K. Lester Jr. (Ed.), *Second handbook of research on mathematics teaching and learning* (pp. 1053–1098). Charlotte: Information Age Publishing.
Wiliam, D. (2015). Assessment: A powerful focus of the improvement of mathematics education. In C. Suurtamm & A. Roth McDuffie (Eds.), *Assessment to enhance teaching and learning* (pp. 247–254). Reston: NCTM.

Improving Students' Approaches to Learning High School Mathematics

P. Janelle McFeetors

Abstract Within the routines of high school mathematics classrooms, learning strategies like copying notes, doing homework, and studying for tests play a prominent role for students. These strategies can become meaningless labels for students when implemented with a goal to complete a product in relation to systemic expectations. Students can bring into view how they learn mathematics through the process of learning to learn mathematics, thereby improving both how they learn and their mathematical understanding. In this chapter, I report on a study with grade 12 students where they developed meaningful approaches to learning high school mathematics. Vignettes of three of the students' experiences illustrate how our conversations and working together led to their development of learning processes. Viewed as dynamic and authentic, learning processes are approaches used to make sense of mathematical content which students develop and refine for themselves. They fulfill a primary intention to learn and are responsive to who the students are as learners. Three ideas with specific suggestions to support learning to learn mathematics in high school are given: provide opportunities to create and improve learning processes, listen to students' current approaches and intentions, and invite improvements in learning through suggestions.

Keywords Learning processes · Learning to learn · Conversation

High school mathematics students often do homework and study for tests without support to consider how these actions could aid their mathematical learning. However, students can attend to how they learn mathematics through the process of learning to learn mathematics. Learning to learn is making sense of approaches to learning, improving the approaches, and using them for greater success. Teachers can integrate elements into mathematics class to scaffold students' learning to learn. In this chapter, I explore the limitations of learning strategies and suggest the

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development of customized learning processes. These alternative approaches are illustrated through the use of three student vignettes and are followed by suggestions on how routines can be incorporated into mathematics class for developing learning processes.

What Are Learning Strategies?

It is relatively easy to find how-to books on increasing achievement in school by using described strategies. Sometimes they focus on mathematics class (e.g., Bass 2013), and in other instances they are about school subjects broadly (e.g., Muchnick and Muchnick 2013). Generally, the books present prescribed learning strategies of simple steps to manage time, do homework, and take tests. As a starting point these tips could be helpful, but often stop short of addressing how students could develop their own personalized approaches to fit who they are as learners.

While learning strategies exist in many high school mathematics classrooms, I studied them as I worked with grade 12 students over 4 months. They attended a high school located in a Western Canadian city and known for its superior academics. The students took *Mathematics Learning Skills*, a course to assist them in succeeding in their grade 12 pure mathematics course. Within the *Learning Skills* course, students were self-directed as they chose what homework (mathematics or other courses) to work on and often requested help from the teacher. The teacher answered mathematics content questions, provided extra practice, and led the class in setting study goals. As a former mathematics teacher and as a researcher, I assisted students with mathematics questions and coached their improvement of approaches to learning mathematics.

At the beginning of the research project, the students provided a list of learning strategies prescribed to them to learn mathematics: study, review, copy notes, work with others, and do homework. Learning strategies were ways students were told by their teachers to work on mathematics. Learning strategies became labels that did not reveal steps students would need to use to enact the strategies, emphasized work and product over learning and process, and Landers (2013) notes they do not necessarily hold the intention of meaning-making of mathematics content. These procedures are systematized by the normative structure of school without consideration of particular students and were externally imposed. The students tried to use strategies without making sense of or personalizing the approaches, simultaneously perceiving continued struggle in their mathematics learning. Against this backdrop, the students and I explored how to improve their ways of learning mathematics.

What Does the Development of Learning Processes Look Like?

Through the project, students began to reform their goals for learning mathematics. They shifted from a memorization-based approach to an interest in understanding why the procedures given worked on the questions they were completing. However, without any ideas of how to use homework, studying, copying notes and other strategies to learn in personally meaningful ways, students were frustrated with their struggles to make sense of mathematical ideas. It is within the search to use strategies to any effect where I captured the development of learning processes.

To begin, I offer three vignettes of students' experiences in developing learning processes to illustrate the range of approaches in supporting learning to learn mathematics. Stories of particular individual's growth are powerful moments from which we can learn (e.g., Bruner 1986; Clandinin and Connelly 2000). These vignettes are based on data collected in the study, including interactive journal writing, transcripts of group sessions where processes were developed, two transcribed interviews with each student, students' working papers, and field notes. Over time, working individually as well as in small groups each of the students showed progress in their approaches to learning mathematics. The three student examples were selected to show several learning processes out of a broader range exhibited by the students.

Vignette #1: Kylee's Refining of Cue Cards

Kylee identified that she was "trying to improve my study habits." Discussing approaches to studying often implied getting ready for a unit test in mathematics class. Tests were the primary form of assessment. Rather than following strategy advice like practice more questions to study, Kylee independently chose a different approach to make cue cards. After success with cue cards in biology, Kylee considered, "Okay, well maybe this will be useful in math because there's a hundred and ten examples here but I only really need to know two of them. Right?" She began using the same process that helped her succeed in biology for mathematics.

She realized, "How much of my time I waste making Q-cards [sic] before my test when I could instead be studying them." The limitation arose from directly transporting a study approach from one subject to another. In response to her journal I suggested, "For math class, would it help to make them after each new idea is presented in class so that you have a complete set to use the day before the test?" This led to a conversation, where Kylee noted the last five minutes of mathematics class would be better used making the cue cards than starting on homework. She left *Learning Skills* class excited to try the adaptation.

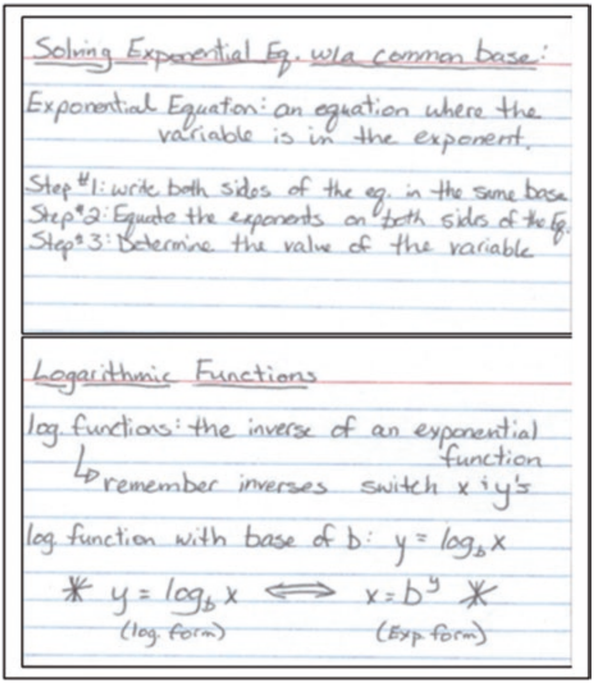

Fig. 1 Two of Kylee's cue cards

Kylee began creating cue cards at the end of each mathematics class. Deciding what to put on cue cards required more than copying out what occurred in the lesson. During class, the teacher wrote out symbolic steps for examples from a workbook on the board with no record of word-based explanations for steps. Kylee, in order to create a cue card, needed to generalize from the specific examples toward word-based explanations to work on similar types of problems. She explained the process as, "I just wrote down…the main important points and maybe two examples. Writing it down is definitely learning it in my own ways instead of just how it is in the workbook." Identifying ideas imbedded within the teacher's worked examples allowed Kylee to learn from her homework beyond following symbolic steps. Figure 1 contains examples of two cue cards from an Exponential and Logarithmic Functions unit.

Kylee increased the complexity of intentions with the cue cards. Kylee also self-assessed when she used the cue cards as an informal check. She recounted her assessment focus: "To make sure I understand and then—it's all definitely quite an active process." At the end of the unit, Kylee used her cue cards to make connections across lessons which had been presented as discrete mathematics facts. Laying cue cards with an exponential graph and a logarithmic graph side-by-side, for example, helped her understanding the meaning of switching *x* and *y* variables symbolically. Her peer, Ashley, helped explain, "All the ideas come one after the other and then you understand…this is why this happens in this section because of what I learned last lesson." While early use focused on instrumental understanding, her review with cue cards was a nascent approach to relational understanding (Skemp 1976/2006). Kylee was making the mathematical content her own, rather than being given content through worked examples.

Kylee found a way to refine her existing approach of cue cards, an exemplar of students' capability to generate their own learning processes. She demonstrates that possibilities to refine approaches emerge in conversation. In contrast to mandating a standard approach as with strategies, I viewed my offers of ideas as suggestions. Kylee confirmed this perspective when she stated, "You weren't telling me to do something or getting mad because I did that on a math test. You were just encouraging." She shaped a learning process of cue cards that was dynamic based on her intentions for use from generalizing steps to connecting ideas, and emphasized the process of learning over completing steps to produce a product.

Vignette #2: Grace's Multiple Ways of Explaining Mathematics Ideas

About 2 months into working with Grace, I presented ways of learning I had observed or heard her talk about using: take good notes, complete assignments, complete homework, list ideas after homework, review questions, break down and write steps, notice types of questions, and work with classmates. Grace responded, "Wow, that's a lot!…Oh, I thought I only had two or three ways to learn math, kind of thing. Just never really think about it." Drawing into Grace's view the ways she was learning mathematics enabled her to improve her approaches. Over time, Grace developed processes to explain mathematical ideas both in oral and written forms.

Students are sometimes told by their mathematics teachers to work with peers. To implement her teacher's instruction, Grace met regularly with peers to work on homework. Grace's work with peers evolved from comparing answers to

> But now, we actually talk about what's actually happening with the question. 'Cause when you know the answer, it's like, "Oh, you know the answer. Whatever."…So now we pick a question and then we all try to do it. And then we stop and then we explain, "You're supposed to start with this." Or "Oh, you have the wrong idea."…And we discuss why you're doing it.

The initial learning strategy pointed to a superficial aim of getting an answer. Grace came to see that in mathematics "the process is the most important. It's not in the answer." Developing a learning process as a collaborative approach to explain her thinking aloud fit who she was as a mathematics learner and illustrates early attempts to describe how a learning process evolved.

Grace also wrote down explanations on paper. She heeded a directive to "take all the notes," which her peer Shane added was "like a drone—copy down all the notes." Grace described that beyond scribing her teacher's symbolic steps for examples, she would "write down little notes for myself. Side notes…It's writing the numbers and then beside it why you did it. It's in my own words!" Grace's generative approach is an example of "to comprehend a text means to transform it in such a way as to produce understanding, that is, to duplicate the author's creative role and not simply the author's message" (Borasi and Siegel 1990, p. 5). Creating *why* steps were sensible, not just how to perform them, illustrates how Grace's side notes transformed her teacher's text for understanding. Grace adapted writing side notes for mathematics ideas with an authorial stance to watching supplementary Khan Academy videos. She reported, "I've been going by the videos and just writing it down, making my own notes." She applied the approach of explanations in a new context with the added benefit of making sense of specific ideas.

Grace shifted from submitting to external expectations toward intending explaining in multiple ways as a meaningful approach to learning mathematics. Her growth in a process of creating various kinds of explanations—both oral and written—throughout a unit allowed her to put mathematical ideas in her own words and impacted all of her learning processes. Grace articulated that using a variety of ways to explain over a unit allowed her to arrive at a "big theme, but then there's actually so many—it's almost like a tree branch. So there's a lot of things branching out." Furthermore, Grace was aware of her learning to learn mathematics, connecting her learning process of explanations to confidently say, "You can tell how I'm improving. …I actually understand what I'm doing." Grace's learning process demonstrated her authority in how to learn mathematics meaningfully with an emerging identity as a mathematics learner.

Vignette #3: Laurel's Meaningful Approach to Homework

Laurel's shift came in how she approached homework. Perhaps because of the relentlessness of assigned practice, homework as a strategy figured prominently in the students' talk about mathematics class. Most students viewed homework as a task to be completed, typified in Shane's question, "Is there a way to do homework well?" Laurel concurred:

> Everyone's always said, "Do your homework. The home study is the biggest thing." So I noticed that I never did any homework, and it kind of caught up to me, 'cause there were questions in your homework right on the [unit] exam. And you're like, "Well, you could have known that before the exam, and that's an easy question."

At first, the reward of marks for easy questions on tests caused Laurel to do homework. Her account continues, "And then I started noticing that I understood it all, and I didn't need to know the question. I just needed to know how to do the question." Laurel signals a different way of looking at homework questions: from remembering specific questions for the test to recognizing that questions could be

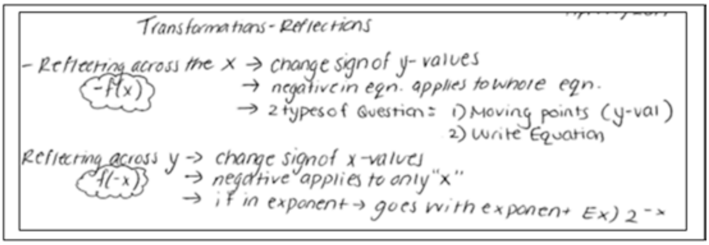

Fig. 2 Laurel's summary for reflecting functions

completed to make sense of a procedure. I see this as a critical moment of leaving behind a prescribed learning strategy and developing a learning process.

Laurel identified how she learned from homework: "So then once I start working through it and there's examples, it makes sense to me, and I see the pattern and it—then it starts making sense." Laurel saw two types of patterns within her homework. The first pattern was "not just doing the same questions over and over again. They all look the same, but there's different steps to them and different ways you have to solve them." The mathematical concept was similar across questions, but the concept was applied differently. The second pattern was "if you notice the steps are all the same, it's a different question, but it's the exact same steps." The procedure was the same regardless of the question, which often differed by numerical coefficients. Laurel's pattern-noticing in her homework illustrates that she was learning by looking for similarities across questions and solutions allowing her to state the main ideas of a lesson. Sfard (2003) emphasizes such pattern-noticing as attending to structure: "Learning mathematics implies seeing structures on many different levels" (p. 360). Figure 2 shows part of Laurel's summary of main ideas from reflecting functions on a plane created after attending to patterns in homework.

Laurel's shaping homework as a learning process demonstrates that homework can be a site for learning given a student's intentions toward learning. In this case, Laurel began with a superficial intention for a learning strategy of completing homework because teachers valued it. Seeing positive results on test marks, she was then able to attend to what was occurring. In addition to the mathematical aspects of homework in noticing patterns, Laurel was able to express an approach that improved her disposition toward learning mathematics: "I come into every class just thinking, 'I'm going to understand it.'…It's about learning how you learn, which is kind of the basis of high school." Her success indicated to Laurel that she was capable of both learning mathematics and improving how she learned. Laurel's learning process positioned her dynamically as a learner of mathematics and empowered to continue shaping her learning.

What Are Learning Processes?

As students inquired into the learning strategies they were told to use to learn mathematics, they were developing processes for learning. Examples of these processes for learning (juxtaposed with strategies in parentheses) included creating summary sheets (study), making and using cue cards (review), creating various forms of notes (copy notes), collaborating with peers (work with others), and learning from homework (do homework). Learning processes for mathematics are approaches students were agentic in developing to make sense of mathematical content and were aimed toward learning rather than completing. Learning processes have particular techniques that are responses both to who the students are as learners and to where they are used within a unit of instruction. In this study, learning processes grew out of existing strategies or from ideas students had generated themselves, and the growth was demonstrated in customizing an approach that supported learning. In contrast to learning strategies, learning processes were perceived by the students as being dynamic and authentic.

Learning processes were dynamic because the students continued to shape them and noticed their peers doing the same. Students' sense-making of mathematics content was supported by learning processes, suggesting the discipline of mathematics was seen as malleable and the content was of their own making. Learning processes positioned students as dynamic persons who developed a sense of authority in their approaches to learning mathematics, growing as mathematical learners. Learning processes were also authentic because students were aware of the effectiveness of a process as they developed it and could describe how it was developed, akin to Bigg's (1988) deep learning where students "become actively involved and can reflect upon what they are doing so that they may improve their approach" (p. 135). Moving beyond a strategy as a label, learning processes were meaningful because students chose how to use them in accordance with their own intentions for learning. They were succeeding in their learning.

How Can Students Develop Learning Processes for Mathematics?

The vignettes of students' movement from learning strategies to learning processes offer insight into the possibilities for learning processes, but hint implicitly as to how these processes were developed by students. A focus on learning to learn in high school mathematics occurred through conversations among the students and me, especially as I shifted attention from solely a content-based focus to exploring with the students how they were making sense of the content.

Just as learning processes need to be developed by students to fit who they are and their intentions for learning mathematics, the processes of developing approaches to learning mathematics need to be responsive to particular teachers and students within their classroom context. While a generalized step-by-step procedure for incorporating learning to learn in classrooms is problematic, in looking across the three students' vignettes and considering the experiences of all of the students in the study three key ideas arose in helping students develop learning processes. Given the perspective of how students benefitted from these three key ideas, implications for high school mathematics teaching practice can be gained.

First, students benefitted from being given opportunities to discuss and work on approaches to learning mathematics. With an absence of these opportunities previously, Kylee provides an example of trying to do this independently in her initial use of cue cards. The discussions had to be purposeful on my part, especially on a day-to-day basis. For example, after assisting a student with a difficult homework question I would add a prompt at the end like, "What helped you get unstuck? How might you use that approach on your own?" The shift in attention supported students in beginning to think about how they were learning mathematics and showed that I valued not only the mathematics but also how they could succeed in learning. The opportunities often arose spontaneously through requests for help.

The moments also came in time allocated to work on learning processes. One small group of students met for about 20 min every three weeks to create a study approach of summary sheets, showing connected mathematics ideas and procedures across a whole unit. Ashley wished her teacher would "do one entire mind map of the chapter on big poster board with the class" in imagining regular development of a learning process in class. This kind of approach could be seamlessly incorporated into a review class at the end of a unit. Rather than another task to add into an already packed class time, opportunities to discuss and work on learning processes could happen as existing classroom routines are modified to focus on learning.

Second, students willingly developed learning processes when they noticed a teacher listened to what they were already doing and where they wanted to improve to increase their success in mathematics class. I interacted with the students by first listening to their experiences. Grace's surprised reaction in viewing her list of ways of learning mathematics was not just about the length of the list, but also that someone had listened closely enough to make a list. A listening-based stance provided opportunities for me to hear the uniqueness of how each student approached learning. Van Manen (1986) states, "pedagogic thoughtfulness is sustained by a certain kind of seeing, of listening, of responding" (p. 12). From within a pedagogic relationship, a teacher is positioned to listen intently to students as individuals and learners.

Finding time to listen to each student individually within class time is difficult. An approach I have found effective as a teacher and researcher is interactive journal writing with students on a bi-weekly basis (Mason and McFeetors 2002). As I prompted students to describe their approaches to learning mathematics, I used interpretive listening in my replies to them by affirming what was working, by highlighting common themes that arose, and by making use of the students' words as they described specific approaches. When I asked the students why they had volunteered for the research project, Laurel explained, "I just feel that it would be interesting if we could actually work something out and figure out how everything worked for me." She was willing to engage in developing learning processes which fit who she was as a learner in the context of having a conversation partner who listened. Interacting with students begins with listening to the students with a responsiveness that meets students where they are and envisioning together possibilities for improved learning.

Third, students developed learning processes from suggestions offered rather than prescriptions told. Many of the students had often been told to do more practice or to spend more time using existing strategies to little avail. Suggestions, rather, are ideas building from the students' current capabilities and approaches. The perspective is not to fix what might appear to be deficient, but to personalize an approach with an intention to learn mathematics meaningfully. Often, I only happened to notice moments to offer suggestions by listening intently to what students were saying. My suggestions were framed with a tentative statement like "I wonder if …" or a prompt of "How does this element help you …?" and were invitations to a student to respond, to try, to imagine. Suggestions open up space to consider and then incorporate.

Peers were also sources of suggestions. More assertive than my tentative wondering, their description of a personal experience was a powerful yet non-prescriptive suggestion. Ashley explained the importance of "sitting down with other people and talking to them and asking them, 'How are you doing this?'" Being able to ask peers for suggestions generated novel approaches to learning for many of the students. When going over a previous day's homework in mathematics class, students could work in pairs or small groups to respond to questions like "Pick a more difficult question you succeeded with from your homework. How did you do the question? What part of yesterday's notes does the question illustrate?" This approach would allow students to both confirm answers to homework questions and notice how they learned, freeing up the teacher to work with small groups of students on developing a learning process simultaneously. I found students willingly shared their learning processes because it helped them recognize they had ideas for learning worth incorporating.

What Are the Possibilities from Here?

In my experience, high school mathematics students are willing and eager to engage in conversations about how to improve their approaches to learning. Kylee, Grace, and Laurel each provide specific examples of what it looks like when they find support at critical moments when they are trying to improve how they learn. No longer were learning strategies seen as tasks to complete because they were expected to; learning processes were imbued with meaning by the students and were taken up because the students had shaped the approach and the intentions for its use. The students exhibited a sense of authority as they expressed in what ways and for what effect they were creating particular learning processes.

While students are interested in developing learning processes, they do need the support of their high school mathematics teachers. It begins in the forging of a pedagogical relationship where a teacher begins to see students as capable learners. Given small moments of time, conversations which begin with listening and contain suggestions for adaptation support students' exploration of novel processes for learning. It is demanding work to bring into view learning as it is enacted. In order to succeed in learning high school mathematics, our students deserve opportunities to not only make sense of mathematical ideas but also opportunities to make sense of their own learning processes.

Through the developing of learning processes, students can begin to see themselves differently in relation to mathematics class. Many students came to see themselves as mathematical thinkers: persons able to understand mathematical ideas. The growth can also be profound, in that the possibility for empowerment exists when students see themselves as mathematical learners. The voices of two students close this chapter, illustrating the positive possibility for shaping who the students are through the development of learning processes:

Elise: But this class has really helped me figure out that there are ways that I can be creative, how I am, in order to learning something I find so boring. … I can learn it in a way that I know works for me.

Shane: All I'd have to do is just take note of how I learned this, and it will be useful for future reference. Like in university, when I'm trying to learn something, I could just say, "This is what kind of learner I am, and I could do it this way, and this is how I learn."

References

Bass, A. (2013). *Math study skills* (2nd ed.). Boston: Pearson Education.

Biggs, J. (1988). The role of metacognition in enhancing learning. *Australian Journal of Education, 32*(2), 127–138.

Borasi, R., & Siegel, M. (1990). Reading to learn mathematics: New connections, new questions, new challenges. *For the Learning of Mathematics, 10*(3), 9–16.

Bruner, J. (1986). *Actual minds, possible worlds*. Cambridge: Harvard University Press.

Clandinin, D. J., & Connelly, F. M. (2000). *Narrative inquiry: Experience and story in qualitative research*. San Francisco: Jossey-Bass.

Landers, M. G. (2013). Towards a theory of mathematics homework as a social practice. *Educational Studies in Mathematics, 84*(3), 371–391.

Mason, R. T., & McFeetors, P. J. (2002). Interactive writing in mathematics class: Getting started. *Mathematics Teacher, 95*(5), 532–536.

Muchnick, C. C., & Muchnick, J. R. (2013). *Straight-a study skills: More than 200 essential strategies to ace your exams, boost your grades, and achieve lasting academic success*. Avon: Adams Media.

Sfard, A. (2003). Balancing the unbalanceable: The NCTM standards in light of theories of learning mathematics. In J. Kilpatrick, W. G. Martin, & D. Schifter (Eds.), *A research companion to principles and standards for school mathematics* (pp. 353–392). Reston: National Council of Teachers of Mathematics.

Skemp, R. R. (1976/2006). Relational understanding and instrumental understanding. *Mathematics Teaching in the Middle School, 12*(2), 88–95.
Van Manen, M. (1986). *The tone of teaching*. Richmond Hill: Scholastic.

Additional Suggestions for Further Reading

Jackson, D. B. (2014). Algebra homework: A sandwich! *The Mathematics Teacher, 107*(7), 528–533.
Manninen, J. L. (2014). Teaching perseverance through differentiated homework. *Mathematics Teaching in the Middle School, 19*(8), 502–504.
Marzano, R. J., & Pickering, D. J. (2007). The case for and against homework. *Educational Leadership, 64*(6), 74–79.
McFeetors, P. J. (2015). Opportunities for learning-based conversations in high school mathematics. *Alberta Journal of Educational Research, 61*(2), 147–165.
Otten, S., Cirillo, M., & Herbel-Eisenmann, B. A. (2015). Making the most of going over homework. *Mathematics Teaching in the Middle School, 21*(2), 98–105.
Smith, C. M. (1999). Meta-learning in mathematics: How can teachers help students learn how to learn? In F. Hitt & M. Santos (Eds.), *Proceedings of the twenty-first annual meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education* (pp. 792–797). Columbus: ERIC Clearinghouse for Science, Mathematics, and Environmental Education.

A Teacher's View – Broadening Our Conceptions of Assessment to Improve Our Practice

Jimmy Pai

Abstract In this chapter, I delve into the idea of assessment as being about much more than measuring student achievement. Instead, I argue that assessment is a process that involves eliciting, attending to, interpreting, and responding to student thinking. This process can serve formative, summative, and interpersonal functions, depending on the circumstances of the classroom. First, I establish a practical definition of assessment, drawing from the literature on assessment and noticing. I then elaborate on this definition by considering how student thinking can be elicited, attended to, interpreted, and responded to in secondary mathematics. Classroom examples from my own practice are used to illustrate statements and connections. I then discuss the summative functions of assessment, since these are often a source of particular concern for teachers. I conclude by reiterating that assessment processes are embedded in all aspects of teacher practice that involve interaction with student thinking, and that an expansion of our definition of assessment may subsequently support student learning.

Keywords Classroom assessment · Formative assessment · Summative assessment

Introduction

Assessment is an important consideration in mathematics education, garnering much attention from educators, researchers (e.g., Suurtamm et al. 2016), and policy makers (e.g., National Council of Teachers of Mathematics [NCTM] 2000; Ontario Ministry of Education [OME] 2010). In my own experience as a secondary mathematics teacher, appreciation for the importance of assessment often depends on one's perception of the form and function of assessment. For example, some of my teacher colleagues have often used the word 'assessment' interchangeably with 'quizzes' or 'tests.' However, the literature on assessment indicates that assessment

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is far more powerful than a process for arriving at numbers that supposedly represent student learning (e.g., Harlen 2012; Suurtamm et al. 2016; Wiliam and Leahy 2007; Yorke 2011). In this chapter, I describe my own conceptualization of assessment as the processes of eliciting, attending, interpreting, and responding to student thinking. I begin by establishing a definition of assessment that has informed my practice.

A Practical Definition of Assessment

The word *assessment* originated (Klein 1966) from the Latin word *assidere*, meaning "to sit beside or with." This suggests that assessment involves supporting students on their learning journey, and not simply a process of arriving at a number that describes student achievement. In light of this, we must ask ourselves: What does assessment mean?

Assessment can be understood through its functions.[1] Assessment functions *summatively* when teachers elicit, interpret, and act on available information as part of their efforts to *sum* up evidence of student learning, which is often represented as a test score or grade. On the other hand, assessment functions *formatively* when teachers elicit, interpret, and act on available information in order to support students to *form* understandings. It is important to note that many researchers (e.g., Harlen 2012; Wiliam and Leahy 2007) have noted that an assessment may serve either or both summative or formative functions, depending on how the information has been used. This dualism, however, does not capture the emotional dynamics (Stiggins 2007) of the assessment processes, which may be helpful to consider if assessment is to be envisioned as a process of being *with* students, as its root word *assidere* suggests. In this vein, Pai (2016) suggested that assessment may also function *interpersonally*, which can improve, or make more difficult, the possibilities for ongoing or future assessment processes to serve formative or summative functions.

Purposefully paying attention to moments in the classroom, reflecting on them, and using them to inform future practices are among the critical processes of assessment. I found Mason's (2002) work on noticing as "experiencing and exploiting moments of complete and full attention" (p. 27) to be helpful for better understanding assessment. As a teacher collects and reflects upon accounts of his or her interactions with students, he or she might "[develop] sensitivities by seeking threads among those accounts" (Mason 2002, p. 87). Paying attention to his or her own experiences in this way also encourages the teacher to break out of habitual responses.

[1] I note that 'assessment for/of/as learning' can also be used to describe the functions of assessment (Daugherty and Ecclestone 2006; Earl 2003), and that there are many intersecting and interconnected ideas between assessment for/of/as learning and formative/summative assessment. However, for the sake of brevity and clarity, in this chapter, I primarily utilize the terms *formative* and *summative* in subsequent discussions.

In the effort to focus on what teachers think and do, I draw from the above literature to establish a practical definition of assessment, which will serve as a foundation for the discussion in the rest of this chapter. The definition below is grounded in the literature on formative and summative assessment (e.g., Harlen 2012; Wiliam and Leahy 2007), as well as on noticing (e.g., Mason 2002):

> Classroom assessment is an ongoing process of eliciting, attending, interpreting, and responding to student thinking, which may be influenced by teacher knowledge, experiences, and goals, as well as considerations for student experiences and classroom culture. Assessment may function formatively, summatively, and/or interpersonally, and particular functions that the process has served can only be determined retrospectively.

I deem this definition practical because I have found it helpful in thinking about and informing my practice. I elaborate below on the aspects of the definition that I have found to be particularly meaningful:

- Assessment is a process (e.g., NCTM 2000; Wiliam and Leahy 2007) that involves eliciting, attending, interpreting, and responding to student thinking (Pai 2016). These processes are always in play for a classroom teacher. Snapshots of learning, such as written tests, are products that cannot represent the rich tapestry of learning that is woven over time. Instead, assessment processes naturally involve moments in the classroom: observations, conversations, and interactions. This aspect of the definition helps me focus more on the dynamic action in the classroom rather than on isolated events, such as written tests, as I reflect on student achievement.
- How teachers elicit, attend to, interpret, and act on student thinking depend on many factors (Son and Sinclair 2010; Watson 2006) that might be categorized under teacher, student, relationships, and contexts (Pai 2016). This implies that assessment is a human (rather than mechanistic) process, and that there is no one-size-fits-all assessment strategy. This aspect of the definition helps me to break free from the illusion that assessments are or can be designed to be objective.
- Positive classroom culture is an important consideration in the classroom (e.g., Heitink et al. 2016). Without students' active participation in classroom activities, it becomes difficult for students to learn, and, for the purposes of assessment processes, difficult for the teacher to support learning. This aspect of the definition reminds me of the importance of paying attention to how students feel about mathematics and about themselves in relation to mathematics, and of fostering positive attitudes about mathematics and one's abilities in mathematics.
- The descriptors 'formative,' 'summative,' or 'interpersonal' can only be determined retrospectively—that is, after an assessment process has occurred (e.g. Harlen 2012; Wiliam and Leahy 2007). Put another way, activities and teacher actions are not effective in and of themselves—their effectiveness in achieving a particular aim can only be evaluated in hindsight. This aspect of the definition reminds me that simply believing that certain activities have been helpful in giving rise to student learning does not automatically make it true. Instead, I need to listen carefully to students in order to make appropriate pedagogical decisions.
- Offering a mark is only one aspect of assessment (a summative function, in particular), and assessment encapsulates far more than grading (Harlen 2012;

Wiliam and Leahy 2007). This aspect of the definition reminds me that I must focus on facilitating an effective learning environment.

In summary, assessment, as I have framed it, is involved in every aspect of my teaching practice that involves eliciting, listening, and responding to students' thinking. Assessment processes are always in play for a mathematics teacher, and include preparations for, acting in, and reflections upon, moments that support learning in the classroom.

Eliciting Student Thinking During the Assessment Process

Teachers cannot read minds. In order to assess, therefore, teachers first need to access student thinking and learning. Thus, the eliciting aspect of the assessment process is about getting students to talk, write, and do mathematics. It is as simple and as complex as that. The following examples of eliciting are not meant to be presented as fail-safe strategies. Instead, they are ways of eliciting information about student learning that have worked for me, or considerations that have continued to improve my teaching.

For many reasons, vertical non-permanent surfaces (Liljedahl 2016) in my classroom help to elicit information about student learning by encouraging student actions and conversations. The use of vertical non-permanent surfaces (VNPS) involves students working together to tackle problems while standing and recording their thinking with non-permanent writing tools such as chalk or erasable markers. The non-permanence of the recordings allows students treat the writings as helpful, yet temporary representations of their thinking. The fact that these representations are displayed on vertical surfaces allows students to easily share and discuss their ideas. When students are intrigued by strategies from other groups, the vertical boards help to facilitate conversations, and give students the opportunity to consider how others have tackled a problem when they feel lost while problem solving. For these reasons, I have found VNPS to be a helpful tool in eliciting information about student thinking.

Of course, in order to share their thinking, students also need problems to think about and to discuss. Godin (Part IV, this volume) discussed several types of problems and their roles in engaging students in problem solving, and provided some examples of how he presents tasks to students. It should be noted, however, that teacher decisions about the kinds of problems to use, as well as how he or she will present them, often depends on his or her goals, students, and classroom dynamic. In addition to the strategies that Godin (Part IV, this volume) has illustrated, I have also had success with engaging students in posing their own problems. One way I do so in my own classroom is through a series of related activities, spanning several days, where students examine a scenario and then develop their own related questions. To give an example, at the beginning of one such series, I showed students an image of a Lego Star Wars play set (Fig. 1) and a Lego Friends play set (Fig. 2) (Pai 2015), and asked students to share what they noticed and wondered.

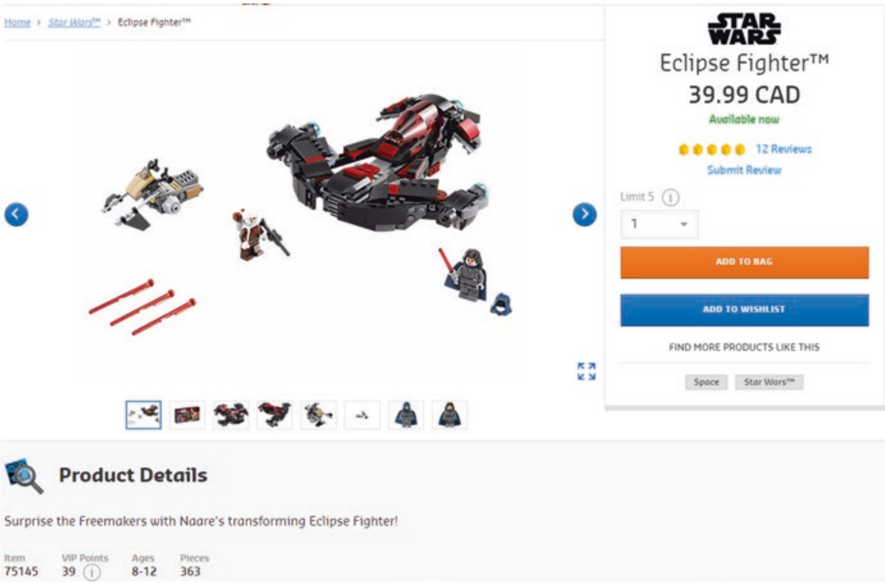


Fig. 1 Lego Star Wars play set, retrieved from https://shop.lego.com/en-CA/

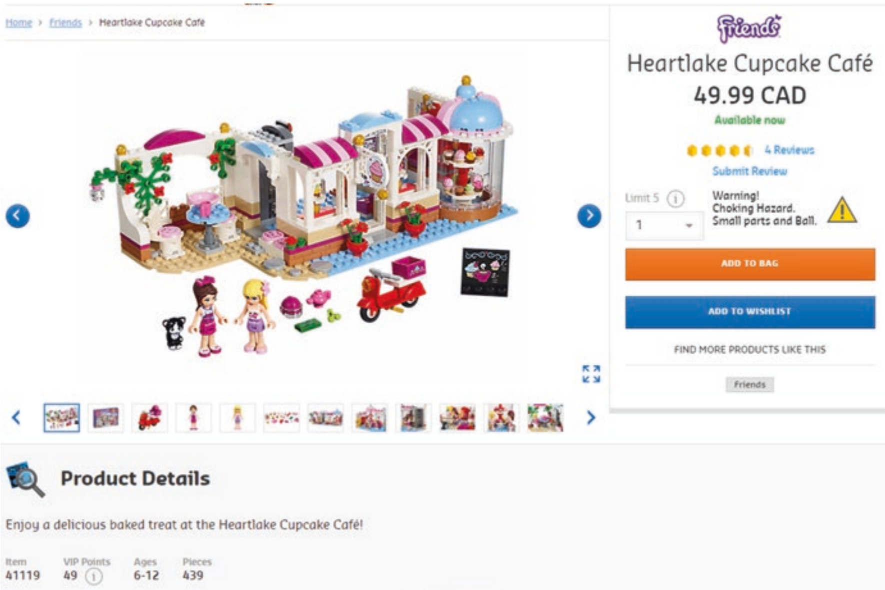


Fig. 2 Lego Friends play set, retrieved from https://shop.lego.com/en-CA/

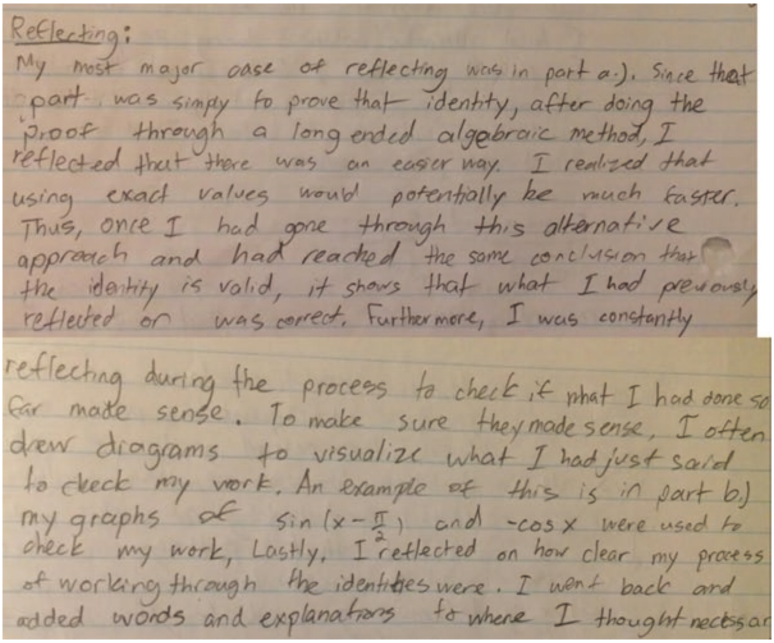

Reflecting:
My most major case of reflecting was in part a.). Since that part was simply to prove that identity, after doing the proof through a long ended algebraic method, I reflected that there was an easier way. I realized that using exact values would potentially be much faster. Thus, once I had gone through this alternative approach and had reached the same conclusion that the identity is valid, it shows that what I had previously reflected on was correct. Furthermore, I was constantly reflecting during the process to check if what I had done so far made sense. To make sure they made sense, I often drew diagrams to visualize what I had just said to check my work. An example of this is in part b.) my graphs of $\sin(x - \frac{\pi}{2})$ and $-\cos x$ were used to check my work. Lastly, I reflected on how clear my process of working through the identities were. I went back and added words and explanations to where I thought necessar

Fig. 3 Sample Grade 12 student entry for portfolio

Students engaged in conversations about stereotypes (e.g., why one set had more pink pieces than the other) and about mathematics (e.g., how much they cost per unit). Often discussions about social justice raised questions or made claims that mathematics was subsequently helpful for supporting. This then led to discussions about mathematics that were rooted in the contexts of their lived experiences. Students were thus willingly entering conversations involving mathematics, which, in the process of assessment, helps to generate information that I, as the teacher, can attend to, interpret, and act upon.

Besides mathematical 'content' such as factoring polynomials, I also believe it is important to elicit how students think mathematically as they engage in mathematical processes.[2] I begin with explicitly discussing mathematical processes by co-constructing what it means to, for instance, problem solve. My students maintain a 'mathematical processes portfolio' throughout the semester documenting their reflections on their improvements with each mathematical process (see Fig. 3 for an example from a student in my Grade 12 Advanced Functions class). I have found that explicit acknowledgements and discussions about mathematical processes are helpful. Students are able to see that I value their thinking processes over 'perfect

[2] For example, the Ontario Ministry of Education (e.g., OME 2007) identified the following 7 aspects of mathematical processes: problem solving, reasoning and proving, reflecting, selecting tools and computational strategies, connecting, representing, and communicating.

answers.' For example, 'representing' mathematics is one of the seven mathematical processes described in OME (2007). Throughout the semester, students see that developing multiple representations helps to better visualize strategies, to illustrate their thoughts, or to arrive at solutions. Unpacking students' mathematical processes is important for both the students and the teacher. For the students, a focus on the processes helps to alleviate anxiety around solutions because it provides value to making mistakes. For me, the focus helps me to purposefully integrate mathematical processes in my interactions with students. Furthermore, by eliciting student thinking about mathematical processes, I can then listen and respond in ways that may improve, for example, their ability to represent mathematical ideas in different ways when solving a problem.

Finally, it is important to note that students may not always verbalize their thinking. It is important, then, to structure opportunities that allow students to represent their thinking through concrete materials, such as manipulatives. There are often mathematics inherent in the structure of manipulatives that afford thinking. For instance, as my students were using linking cubes to explore the painted cube task (e.g., Youcubed 2016), one group became stuck. They then decided to use colours to count and subsequently account for the cubes. This colourful three-dimensional representation then led them to develop several conjectures about possible patterns. In this case, the existence of the linking cubes helped to elicit their mathematical thinking and supported further conversations. In addition, student actions with the linking cubes also help me, often a fleeting observer, quickly attend to the mathematical thinking that has surfaced.

Attending and Responding to Student Thinking During the Assessment Process

Hunger is not alleviated simply by cooking—it also requires eating. In other words, it is not enough for a teacher to simply elicit mathematical thinking—he or she needs to also attend to, interpret, and respond to it. As indicated in my definition, it is important to recognize that assessments may serve formative, summative, and/or interpersonal functions, and that these functions are interrelated. In this section, I will briefly (constrained by the length of this chapter) discuss some ways of attending, interpreting and responding to opportunities for assessment, focusing primarily on teacher interpretations and teacher actions as part of the assessment process.

In the classroom, I often join different groups of students as they work on their VNPS. It is important that I *listen to*[3] students when I am there. This means that I attend carefully to what students are saying and, as necessary, seek clarification on what they are thinking about as they work on a task. As I interpret what students are

[3] Davis' (1994, 1997) work on listening has been influential in my thinking and in my practice.

saying, I need to remember that they may still be negotiating meaning and developing their understanding, and as such, their explanations may not immediately represent their thinking. In some cases, I may say nothing at all, as students sometimes simply need to verbalize their thoughts in order to continue thinking. In other cases, I may wonder together with students—for instance, when the strategies that students use are unexpected but are supported by their reasoning. When this happens, I find it is powerful for me to follow students' reasoning and to continue thinking with them.

Attending to student thinking, or listening, is an important action in and of itself. Listening to students is helpful in several ways. First, it helps me to establish a better understanding of what the students might have understood and what they are working toward understanding. In other words, listening may elicit more information about student thinking. Second, the presence of a listener may help students get 'unstuck' while problem solving. This is because, in order to explain, students need to reiterate aspects of what they have done. Reiterations may lead to reflections, which may in turn lead to realizations about alternative strategies, representations, or solutions. This means that listening is a teacher action that can also further student thinking. Third, it is important for me to model listening so that my students learn to value the input from their classmates. This supports a positive classroom culture and may encourage more conversations in the classroom, which may facilitate future eliciting of student thinking.

Attending and responding to mistakes is another powerful teacher action: not only are mistakes valuable opportunities for learning, they also contribute to how students see themselves in relation to mathematics, which may subsequently impact how they participate (or not) in classroom activities. I find it important, then, to pay attention to instances when students attempt unfruitful strategies or reason inappropriately, and to be tactful when responding to their thinking. Depending on the student, I may ask him or her to explain the strategy or reasoning; in addition, I may ask for more examples or alternative representations that illustrate the points. Besides the teacher being able to better interpret the perceived mistake, in clarifying and thinking further, the student might identify inconsistencies on his or her own, and subsequently resolve the mistake. Another possible teacher response might also be to direct the student to think about how the strategies of other groups cohere with his or her thinking. The presence of VNPS is helpful in this situation, because the student is able to simply look over at other whiteboards without stepping away from his or her workspace.

As we attend to and interpret students' mathematical thinking, we also cannot ignore existing power dynamics in the room. The most obvious one is the perception of a teacher in the position of power. In particular, I need to be cognisant of the fact that when I speak, at least in the beginning of a semester, my words carry weight. One implication of this, for me, is that I cannot only wonder and question when mistakes are made—I also need to offer wonderings and questions when students are successful with their strategies, lest students think that I only offer input when they are wrong. Besides perceptions about the teacher, students also hold perceptions about each other, and I believe it to be helpful to recognize these and respond accordingly. For example, if a student is extremely hesitant about speaking to other students, I might give him the erasable marker and ask him to note down the thoughts of others on the group whiteboard. In addition, I might task this student with the role of asking clarifying questions to the others in the group, and to reiterate these ideas to me when I drop by. These considerations of power dynamics are important for being able to attend and respond to student thinking in a way that supports their learning.

When conversing with students, I also pay attention to mathematical processes. After introducing and co-constructing the meaning of terms related to mathematical processes (e.g., problem solving, connecting, reasoning), I sometimes explicitly refer to these terms when discussing strategies with students or posing questions about their thinking. Since my students keep a process portfolio throughout the semester, I also dedicate class time to discuss and encourage student reflection about mathematical processes. I believe it is important to refer to these metacognitive processes because it helps students see that I value these thinking processes as much as the products of their thinking. This in turn may help to improve how students think mathematically and view themselves in relation to mathematics, and may subsequently encourage mathematical activity in the classroom.

Functions of the Assessment Process

My descriptions of the assessment processes in the previous sections may serve formative, summative, and/or interpersonal functions. It should be noted, however, that none of the teacher actions in my examples are meant to be presented as ideal. This is because acting appropriately is not algorithmic (e.g., Davis and Sumara 2006), meaning that it is *not* true that certain actions will always yield the best outcomes, even in similar situations. Similarly, whether or not the assessment processes described above serve formative, summative, and/or interpersonal functions depends on a wide variety of factors, including context and the individuals involved. In the many instances I described, the actions may serve formative functions when the assessment process leads to an improvement in students' mathematical thinking. Interpersonal functions may be served when the assessment process improves the rapport between teacher-student and student-student such that students are more likely to continue mathematical discussions and therefore allow for other assessment processes to take place. Summative functions can also be served, but sufficient information must be accumulated and considered before I am able to 'sum' up what a student has learned about a particular topic, strategy, or way of thinking mathematically. In my experience, the summative functions of assessment often give my colleagues (and myself) the most headaches; for this reason, the next section focuses on some aspects of the summative function.

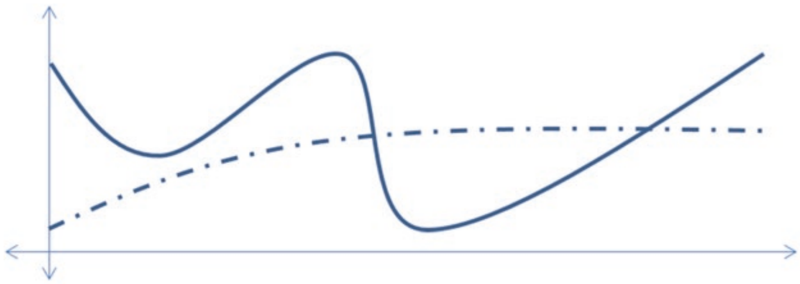
The Summative Conundrum

What does it mean to 'sum' up learning if learning is an ongoing process? To sidestep the question somewhat, I believe that it is particularly difficult for a single assessment process to function summatively. In order to glimpse student understanding, it is necessary to create an image out of *incomplete* and *interpreted* puzzle pieces. These puzzle pieces are collected from several interactions with students and their thinking about mathematics, and can take permanent (e.g., written) or ephemeral (e.g., verbal) forms (e.g., Harlen 2012; Pai 2016). As such, the interactions mentioned in previous sections all have the potential to be an account of and/or an account for[4] students' mathematical thinking, depending on how they are used. Here, I elaborate on some strategies that have been helpful in serving the summative functions of assessment.

My students also add reflections (with respect to the mathematical processes) about their mathematical work into their portfolios. These portfolios, for example, can serve as mosaics that represent the ongoing process of learning (summative), but that also encourage conversations (interpersonal) and invite reflections and feedback about how to move learning forward (formative). The continued use, and reference to, portfolios is also helpful because it means that my students and I are following up on the co-constructions that we had worked on. This helps to further illustrate to students that I value how they think more than whether the products of their thoughts are 'correct.' During the semester, I often conference with individual students while they reflect on the contents of their portfolios. These instances provide me with opportunities to listen and offer feedback and wonderings. For example, I ask about their decisions to include certain pieces of work and not others, or how they elaborated on particular instances of their thinking that demonstrated mathematical processes.

Besides ongoing projects such as portfolios, there are also tests and quizzes in my classroom that provide students with opportunities to demonstrate learning. My tests and quizzes often include open questions that invite students to demonstrate their understanding through a variety of strategies, representations, or reflections (an example is provided in Fig. 4). Moreover, as tests and quizzes are returned with only feedback and no marks, I often act in ways that illustrate to students that these tests and quizzes are learning opportunities, and not isolated events. Students are then often given class time to respond to and reflect on the feedback and participate in one-on-one interviews. During these interviews, I ask specific questions that give students another opportunity to demonstrate what they were unable to show in writing on the test or quiz. Thus, in framing tests and quizzes as part of an ongoing process that helps both myself and the students better understand their learning, students are better able to focus on improving their mathematical thinking.

[4] Mason (2002) distinguished between giving an account-of an event and accounting-for it. He identified an 'account-of' as an attempt to draw attention to something, and an 'account-for' as explaining the something that was accounted.



(Example from a grade 12 advanced functions test)
Laura and Abdi weren't happy with the results of the cup-stacking race from class. After 30 long years of feuding, they decided to settle things by making robots and having the robots compete through cup-stacking. The graph above tracks the velocity vs time of their robots.
Who won? What more information might you need? How would that help? Justify your reasoning and showcase what you know.

Fig. 4 Robot cup stacking question from Grade 12 Advanced Functions test

The way that I design my course is also significant in helping students see learning as an ongoing process. During both teacher- and student-generated activities, several curriculum expectations are typically involved in students' explorations. The series of activities involving Lego mentioned previously, for example, naturally involves many aspects of the grade 9 mathematics curriculum in Ontario (including linear relations, measurement and geometry, number sense, and algebra). Throughout the semester, then, students build familiarity with the concepts through repeated exposure and connections within different contexts. Since topics are revisited, each time in greater depth, students have many opportunities to strengthen their understanding, and to demonstrate their understanding to me. In other words, since 'topic units' no longer exist, the doors are never closed on students who are continuing their learning throughout the semester.

Grading

The end of the semester is a different story. While some ways of summing up learning (e.g., conferencing with students or clinical interviews) can often also serve formative and interpersonal functions, in Ontario, as in many provinces and states, secondary teachers are required to provide a final grade in the form of a numerical value. Grades, often in the form of percentages, give the illusion of precision. Yorke (2011) pointed out that "finely graded scales […tend] to seduce assessors into believing that assessment can be conducted with a precision which it manifestly does not possess […], [calling for] the eradication of the false consciousness regarding precision" (p. 265). Nonetheless, most teachers are faced with the immediate requirement of providing a grade.

				R	1			2			3			4		
Strands	Overall Expectation	Task	Achievement	R	1-	1	1+	2-	2	2+	3-	3	3+	4-	4	4+
Number Sense and Algebra	A2	Q2	1-		Q2											
	A2	OA	R	OA												
	A2	OA	1-		OA											
	A2	T1	1			T1										
	A2	OA	3									OA				
	A2	Q4	2						Q4							
	A2	OA	4												OA	
	A2	Q3	3-								Q3					
	A2	OA														
	A2	T2	3									T2				

Fig. 5 Evidence record

When forced to provide an aggregate number for a student, I lean on a variety of evidence as much as possible. Throughout the different conversations that I have with students, as well as through portfolios, interviews, and projects, I keep note of my interpretations and assumptions about particular students' understanding. Using a Google spreadsheet, I share an evidence record with each student, which is organized in such a way that one can visually identify growth with respect to a particular topic. The example shown in Fig. 5 refers to the same topic (manipulating polynomials) evaluated over time through different tasks, such as observations and conversations (OA), major tasks or tests (T), as well as quizzes and formal interviews (Q). This document is shared with students so that my perceptions regarding their achievement is communicated to them. The evidence record also serves as an invitation for students to discuss their progress in the course. As I conference with students, we often begin with this document and move on to specific strategies that might help to improve their grades. I must note that the current structure of my evidence records is a work in progress, much like the individual evidence records of my students. I do not claim that it is a perfect system for deducing students' level of achievement,[5] but merely that it has been helpful for my navigation toward an aggregate numerical representation of student learning.

Arriving at a mark is not a perfect process. A realization of this imperfection, then, implies both freedom and significant responsibility on the part of the teacher. Teachers are no longer restricted to formulas, averages, and medians that spit out high stakes numbers; at the same time, they are unable to hide behind algorithms that feign a sense of objectivity. Personally, I feel a need to stand on rationales built on a multitude of varied experiences, as well as to invite students to discuss their perception of their progress.

[5] In Ontario, levels R, 1, 2, 3, 4 are qualitative descriptors that integrate considerations of knowledge, understanding, thinking, communication, and application (OME 2010) for different topics in the course. The mathematical processes are also woven into the same considerations.

Conclusion

Assessment is far more than numbers, and it is far more than particular events, such as quizzes or tests. Rather, the assessment process embodies all teacher practices that involve eliciting, attending to, interpreting, and responding to student thinking. Throughout this chapter, I have elaborated on what I have established as a practical way of thinking about assessment in the mathematics classroom. For me, continuing to reflect on both the definition of assessment, as well as on assessment strategies, has helped improve my teaching practice. I hope the ideas in this chapter provide possibilities for readers to broaden their definitions of assessment in ways that honour its etymological roots—to sit beside or with their students on their journey of learning mathematics.

References

Daugherty, R., & Eccletsone, K. (2006). Constructing assessment for learning in the UK policy environment. In J. Gardner (Ed.), *Assessment and learning* (pp. 149–168). London: Sage.

Davis, B. (1994). *Listening to reason: An inquiry into mathematics teaching*. Unpublished doctoral dissertation, University of Alberta, Edmonton, Canada.

Davis, B. (1997). Listening for differences: An evolving conception of mathematics teaching. *Journal for Research in Mathematics Education, 28*(3), 355–376.

Davis, B., & Sumara, D. (2006). *Complexity and education: Inquiries into learning, teaching, and research*. New York: Routledge.

Earl, L. M. (2003). *Assessment as learning: Using classroom assessment to maximize student learning*. London: Sage.

Harlen, W. (2012). On the relationship between assessment for formative and summative purposes. In J. Gardner (Ed.), *Assessment and learning* (pp. 87–102). Thousand Oaks: Sage.

Heitink, M. C., Van der Kleij, F. M., Veldkamp, B. P., Schildkamp, K., & Kippers, W. B. (2016). A systematic review of prerequisites for implementing assessment for learning in classroom practice. *Educational Research Review, 17*, 60–62.

Klein, E. (1966). *A comprehensive etymological dictionary of the English language: Dealing with the origin of words and their sense development thus illustrating the history of civilization and culture*. New York: Elsevier.

Liljedahl, P. (2016). Building thinking classrooms: Conditions for problem-solving. In P. Felmer, E. Pehkonen, & J. Kilpatrick (Eds.), *Posing and solving mathematical problems: Advances and new perspectives* (pp. 361–386). Cham: Springer.

Mason, J. (2002). *Researching your own practice: The discipline of noticing*. New York: Routledge.

National Council of Teachers of Mathematics. (2000). *Principles and standards for school mathematics*. Reston: Author.

Ontario Ministry of Education. (2007). *The Ontario curriculum: Grades 11 and 12 mathematics*. Toronto: Queen's Printer for Ontario.

Ontario Ministry of Education. (2010). *Growing success: Assessment, evaluation, and reporting in Ontario schools, covering grades 1 to 12* (1st ed.). Toronto: Queen's Printer for Ontario.

Pai, J. (2015). *Lego: Star wars with friends* [Web log post]. Retrieved from http://intersectpai.blogspot.ca/2015/12/lego-star-wars-with-friends.html

Pai, J. (2016). *In-the-moment decisions: A preliminary investigation on observations and conversations as assessment in secondary mathematics classrooms*. Extended paper presented at the 13th International Congress on Mathematics Education in Hamburg, Germany.

Son, J.-W., & Sinclair, N. (2010). How preservice teachers interpret and respond to student geometric errors. *School Science and Mathematics, 110*(1), 31–46.

Stiggins, R. J. (2007). Conquering the formative assessment frontier. In J. A. McMillan (Ed.), *Formative classroom assessment: Theory into practice* (pp. 8–28). New York: Teachers College Press.

Suurtamm, C., Thompson, D. R., Kim, R. Y., Moreno, L. D., Sayac, N., Schukajlow, S., Silver, E., Ufer, S., & Vos, P. (2016). Assessment in mathematics education: Large-scale assessment and classroom assessment. In G. Kaiser (Ed.), *ICME-13 topical surveys* (pp. 1–30). Hamburg: University of Hamburg.

Watson, A. (2006). Some difficulties in informal assessment in mathematics. *Assessment in Education, 13*(3), 289–303.

Wiliam, D., & Leahy, S. (2007). A theoretical foundation for formative assessment. In J. A. McMillan (Ed.), *Formative classroom assessment: Theory into practice* (pp. 29–42). New York: Teachers College Press.

Yorke, M. (2011). Summative assessment: Dealing with the ‘measurement fallacy. *Studies in Higher Education, 36*(3), 251–273.

Youcubed. (2016). *Painting youcubed*. Retrieved from https://www.youcubed.org/task/painting-youcubed/

Additional Suggestions for Further Reading

Baird, J. A., Hopfenbeck, T. N., Newton, P., Stobart, G., & Steen-Utheim, A. T. S. (2014). *State of the field review: Assessment and learning (OUCEA Report 13/4697)*. Retrieved from University of Oxford, Oxford University Centre for Educational Assessment website http://bit.ly/1qZ52Mq

Zager, T. J. (2017). *Becoming the math teacher you wish you'd had: Ideas and strategies from vibrant classrooms*. Portland: Stenhouse Publishers.

Part V: Commentary – Planning and Assessment from a South African Perspective

Vijay Reddy

Abstract This commentary focusses on the six papers related to Planning and Assessment through the lens of a South African researcher. Both countries acknowledge the role of assessments to improve student learning outcomes. This set of papers focusses on different forms of assessment and how the teacher uses assessments inside classrooms to improve the learning experience. There is a silence in the papers around the learning context and the history associated with the context. Drawing on the South African experience, the commentary explains how the socio-economic context influences the learning processes.

I feel privileged to provide one of the international commentaries for the "Planning and assessment" section of the book *Teaching and learning secondary school mathematics: Canadian perspectives in an international context*. South Africa, a country in the Southern Hemisphere, is located 15,000 km from Canada.

Reading the six papers that constitute the Planning and Assessment section, I am struck by the commonalities of assessment issues in our countries, by the silences around context and history, and a yearning that one day my country will be able to focus much more on school and classroom practices to improve the teaching and learning of mathematics.

The six papers in the section emphasise the importance of assessments (in all its forms) and acknowledge that the key purpose of assessment is to improve student learning. The papers focus on pedagogical processes inside mathematics classrooms and make arguments for assessments as an integral part of good teaching practices, and providing students with skills on how to learn. The assessment process shifts from simply testing the acquisition of information to a continuous process of

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finding out what students know and think using activities other than formal tests. In all of these activities which aim to improve the learning experiences, the role of the teacher is very important: the teacher must be engaged and incorporate the new ideas and strategies into lessons and the curriculum. The unspoken assumption in the papers is that all students would start from the same place.

My commentary perspective is as a researcher, embedded in a highly diverse country in terms of people, ethnic and racial backgrounds, languages, politics, religions, social stratifications and histories. South Africa is a young democracy (since 1994) and only 12% of the 25–64 year old age group has a post-secondary education (Statistics South Africa (Stats SA) 2016) compared to Canada where this figure is 64% (Statistics Canada 2011). South Africa as a lower middle income country, and has an economy one fifth the size of the Canadian economy (Central Intelligence Agency (CIA) n.d.). South Africa is characterised by high levels of income inequality with the Gini co-efficient at 0.63 in 2011 compared to Canada where the index was 0.34 in 2014 (World Bank n.d.). South Africa is grappling with overcoming the historical legacies of apartheid[1] and aspires to be a country with lower levels of social and economic inequality.

Like other countries, South Africa has prioritised education and believes that higher educational levels would lead to better social and economic outcomes for individuals and the country. Since 1994, there have been massive investments and interventions to improve educational infrastructure; physical, pedagogical, human resources; and financial resources for children from low income households. We have observed some improvement in our educational outcomes: in the last 20 years our mathematical performance has moved from being classified as *very low* to *low* achievement (Reddy et al. 2016). South Africa and Canada participate in the Trends in International Mathematics and Science Study (TIMSS) and in TIMSS 2015, South Africa scored 372 and Canada 527 TIMSS points for mathematics.

South Africa participates in a number of international, regional and national assessment studies to measure the health of our educational system. These studies provide a deeper understanding of the health of our educational system. There is also school based assessment, which is made up of a continuous assessment component and a year-end examination assessing knowledge and the application of knowledge. The school based assessment has the dual purpose of assessing what students know and can do, leading to a judgement of pass or fail, and provides feedback to students and teachers to help improve practice and hence learning. A major assessment milestone in the schooling system is the exit examination in the last year of schooling (grade 12). This ‘matriculation’ examination is a high stakes national examination, as these results determine the subsequent educational trajectory of each student. The assessment outcome from international studies and the matriculation examination stimulates robust debate among the South African public. The low annual achievement results, and the position as the lowest in the rank order in the TIMSS, leads to newspaper headlines along the lines of ‘South African pupils are

[1] A discrimination policy used by the previous government regime of separate development based on racial classifications.

the dunces of Africa,' 'Bottom of the class in maths', and 'Grade 3 flunkers sound a warning about our schools.'

Since 1994, South Africa has introduced a number of policies to inform the education and training system. In 1995, the National Qualifications Framework (NQF) outlined a ladder of qualifications linking the training and education systems. This link between education and training shifted the focus away from what teachers were required to teach to what the student is required to understand and be able to do, i.e., the learning outcomes. In 1998, the country's first outcome based education (OBE) school curriculum, Curriculum 2005, was implemented. Curriculum 2005 represented a radical change from the apartheid curriculum and the goal of the transformation of society was included in its objectives. While the intentions of C2005 were noble, there was a serious mismatch with the skills set of teachers and the conditions at schools required to implement this curriculum. Critique from many sectors of society led to the review of C2005 which confirmed that it concentrated too much on skills and the processes of learning, without sufficient specification of content and knowledge. One of the consequences of this radical and un-implementable curriculum was that the average mathematics achievement score in TIMSS 1995, 1999 and 2003 did not change—a major disappointment for the new government who had implemented many structural reforms in education.

In 2003, the government introduced the Revised National Curriculum Statements (RNCS) which were an improvement, but still difficult for schools and teachers to implement. This has led to a proliferation of policy documents from national, provincial and district departments trying to make it more understandable for the average, poorly trained South African teacher with limited subject knowledge—a legacy of apartheid and the uneven quality of teacher education (Hofmeyr 2010). In 2010, the government introduced the concise Curriculum and Assessment Policy Statement (CAPS) to provide more guidance and structure for teachers. This curriculum specified the content and provided details on what teachers ought to teach and assess. Schools were provided with structured textbooks, workbooks and teaching support materials.

The present CAPS documents gives expression to the knowledge, skills and values worth learning in South African schools. Like in Canada, the central purpose of assessment is to support and enhance student learning. However, given our history, the principles of social transformation, human rights, inclusivity, environmental and social justice and valuing the Indigenous knowledge systems remain important. I would like to expand further on "inclusivity" as one of the tenets for the organisation, planning, teaching and assessment of mathematics in school.

South Africa is a highly unequal country with a national curriculum: students start school at different points, but all students are expected to achieve the same outcomes. Given the high levels of income poverty and unemployment, government has subsidised the school fees for two thirds of school going students—thus our public schools are categorised as fee paying and no-fee paying schools. The general description for students in non-fee-paying schools is that they come from low income households, live in impoverished communities, attend schools with fewer resources and are taught by less knowledgeable teachers. One third of students attend fee paying school, generally living in middle class neighbourhoods and attending better resourced schools. Given the vastly different starting points of students entering school, the South African response to improve educational performance is to focus on both what happens inside classrooms (pedagogical inputs and classroom climate) and the context in which students live and learn.

To illustrate the differences between students who attend fee paying and no-fee paying schools, I will use information generated from the TIMSS 2015 data[2] (Mullis et al. 2016). The general achievement level of students in fee paying schools is 3.5–4 grades higher than those attending no-fee paying schools (452 vs. 343 TIMSS points). Parental education is a powerful indicator of the social capital of the household and is a predictor of future educational outcomes. 37% of students in no-fee schools and 65% of students in fee paying schools reported that their mothers' have a post grade 12 educational qualification. This home educational difference translates to 11% students in no-fee schools and 34% students in fee-paying schools have at least one parent with a job in a professional occupation.

The physical resources in the home are different: 59% students in no-fee schools versus 75% students in fee paying schools reported having running tap water in their homes. South Africa has 11 official languages, but the language of teaching and learning is generally in one of two languages (English or Afrikaans). Test language proficiency is low, with 14% of students in no-fee schools and 59% of students in fee paying schools reporting that they 'always' speak the language of the test at home. The internet is a source of information for any student: 28% of students in no-fee schools and 53% of students in fee paying schools have internet connection at home. All of these household, educational and social assets matter in the kinds of learning experiences which students have and subsequently the educational outcomes they achieve.

Home educational activities are different for students in these two environments: one third of parents in no-fee schools and a corresponding 41% in fee paying schools reported reading books to their children. 30% of parents with children who attend no-fee schools and 49% in fee paying schools reported that their children played games with shapes prior to entering Grade 1.

Given the difference in home environments, parents from the two school types rated their children's readiness for school differently: 46% of children attending no-fee schools could recognise the alphabet and 22% could count up to 100 prior to entering Grade 1. The corresponding result for children attending fee-paying schools was 55% and 36% respectively.

Given these unequal starting points for children entering Grade 1, the expectation is that the schools will provide high quality inputs to reduce the inequality gap. While there have been inputs to schools and households to reduce inequality, the same has not yet been achieved inside schools and classrooms. There have been some changes in achievement outcomes, but this has been at a very slow pace. To improve achievement outcomes also requires having a cadre of high calibre teachers, with strong disciplinary knowledge, and good teaching, learning and assessment practices, especially in no-fee schools. It is only when this is in place that we can claim that we are on track to achieve the goals of inclusivity.

[2] This is for the study conducted at the grade 5 level.

As South African society works towards becoming more equal, we aspire for the discussions for improving mathematics learning shift towards examining the curriculum and pedagogical practices in the classroom. At the moment, the South African challenge is both about improving what happens inside and outside schools and classroom. As Canada becomes more diverse, I would imagine it would be challenged to take into consideration, more significantly, outside classroom contexts to improve the learning for students.

References

Central Intelligence Agency (CIA). (n.d.). *The world factbook*. Retrieved from https://www.cia.gov/library/publications/the-world-factbook/

Hofmeyr, J. (2010). *Is this a new curriculum I see before me?* Retrieved from https://www.ieducation.co.za/is-this-a-new-curriculum-i-see-before-me/

Mullis, I. V. S., Martin, M. O., Foy, P., & Hooper, M. (2016). *TIMSS 2015 international results in mathematics*. Boston: TIMSS & PIRLS International Study Center.

Reddy, V., Visser, M., Winnaar, L., Arends, F., Juan, A., Prinsloo, C., & Isdale, K. (2016). *Highlights of mathematics and science achievement of grade 9 South African learners*. Pretoria: Human Sciences Research Council.

Statistics Canada. (2011). *National household survey profile, Canada, 2011*. Retrieved from http://www12.statcan.gc.ca/nhs-enm/2011/dp-pd/prof/details/page.cfm?Lang=E&Geo1=PR&Code1=01&Data=Count&SearchText=canada&SearchType=Begins&SearchPR=01&A1=Education&B1=All&Custom=&TABID=1

Statistics South Africa. (2016). *Community survey 2016: Statistical release*. Retrieved from http://cs2016.statssa.gov.za/wp-content/uploads/2016/07/NT-30-06-2016-RELEASE-for-CS-2016-_Statistical-releas_1-July-2016.pdf

World Bank. (n.d.). *GINI index (World Bank estimate)*. Retrieved from https://data.worldbank.org/indicator/SI.POV.GINI

Part V: Commentary – On Measures of Measurement and Mismeasurement: A Commentary on Planning and Assessment

Bharath Sriraman

Introduction

There is an old Indian fable which goes along the following lines:

> A student who has memorized all night long for an examination the next day is seen walking very carefully along a cobble stoned path. On encountering a friend along the way, the student is asked why he is stepping so carefully. The student replies, "I am afraid if I fall down, I will not remember all the information I have memorized all night long, it could get easily jumbled up."

This story has many morals and interpretations—and it is up to the reader, in this case the mathematics educator holding this book to decide what it really is. Before doing so, the six chapters in this section could very well serve as a compass or a guide to our interpretation because they delve deeply into issues surrounding planning and assessment in a mathematics classroom keeping the triage of the student, the lesson and the teacher in mind.

Several themes emerge from the six chapters which can be subsumed under the categories of homework and testing, alternative assessment, student understanding, and problem solving. Then there is also the issue of "scale"—which relates to the title of this commentary, i.e., what are we measuring through planning and assessment: an individual student, an entire classroom, entire schooling systems, or an entire country? The question of scale is important since it relates to recommendations made in each of the six chapters on what is possible at a particular micro-level. More often than not, recommendations for planning and assessment that work at a classroom level are often not implementable at a school level or even a district level. For now, we leave these larger considerations aside and focus on the contents of the six chapters.

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The Six Chapters

Holm's chapter presents a necessary provocation—namely can we not plan a lesson with an end in mind? Is she suggesting we teach to the test? No! The end in this case is to further student understanding of concepts in mensuration. After a teacher implements an inquiry based lesson that relates to provincial standards in Ontario, Canada, Holm argues based on experience that great tasks simply remain "tasks" if no effort is made to consolidate and discuss what students have learned (and understood) from engaging with the task. Allusions are made to what this consolidation and discussion might look like—in this case to specifically target understanding of the relationship between volumes and surface areas of right cylinders. And even more specifically basic understanding of how the radii of circles form a building block for 3-dimensional circular objects, and using the proper units of measurement. The chapter closes with a specific extension task that conceivably "measures" all these aforementioned understandings. As an example of a lesson (task) that had a specific end in mind, it achieves the goals of measuring understanding by having students express their reasoning in words, and reveal their technical knowledge about surface areas and volumes of 3-D circular objects. However, what is not clear is ways in which a teacher might assign a score or a grade to this extension task, and whether or not is can be reformulated in a way that it can even be used as question that generated several related multiple choice item?

This question can be understood when seen through the lens of Corrêa's chapter which follows Holm's because it analyzes the relationship between ability and demonstrated work. In other words, can the numerous formative tasks that led to the summative extension task be used to evaluate the entire spectrum of ability in the demonstrated mathematical work? Arguably a multiple choice assessment offends the reader's sensibilities in some ways as it is tacitly assumed that open-ended items are better than multiple choice items. And this tacit assumption is further rooted in associating the former with "conceptual understanding" and the latter, viz., multiple-choice items with procedural understanding. We will come back to this at a later stage in the commentary in the context of situation the notion of "memory."

Corrêa presents two very interesting tasks and uses Kilpatrick et al. (2001) mathematical proficiency model as an assessment tool. The first task, namely the distance medley relay task is used as an example where a coding scheme is revealed to the reader to gauge student understanding of proportional reasoning. I find this of interest as proportional reasoning is one area of mathematics education where many diagnostic instruments are available to pinpoint student errors (and deficiencies in understanding). Several well validated theories suggest that with age and experience students can tackle more complex proportional problems. In a nutshell, students move beyond the use of constant differences strategies to a building up strategy, and then to a multiplicative approach such as the unit-rate method. The development of these proportional reasoning concepts is not immediate, but rather a gradual process based on continual growth and progress coming from more progressive problems and strategies (Lamon and Lesh 1992). Taking a Piagetian stance, this development happens at adolescent stages of maturity where additive approaches turn into multiplicative strategies that do not necessarily allow for the generalization to all cases. However, a formulation of a law concerning proportions becomes established that fits the scheme for solving proportions in its various situations (Lesh et al. 1988). This may explain why a student will utilize different approaches for individual tasks (Inhelder and Piaget 1958). The coding of the mathematical work of Rick on the distance medley task using Kilpatrick et al. (2001) model reveals the interplay between procedural skill, conceptual understanding and elements of the problem solving cycle. While this is useful for the reader to appreciate both the student's thought processes and the teacher's ability to understand the student's thinking, the question of whether such an approach to assessment can be scaled up or not is left unaddressed. While it does suggest that Kilpatrick's mathematical proficiency model is a useful assessment tool, the constraints of time make it impractical for a teacher to be able to use this on an entire class homework set. This first caveat is important for the reader to note. Single cases are relevant as examples of good assessment practices, but become complicated when implementation is asked for on a larger measurement scale. The second caveat is that of inter-rater reliability—would teachers from the same school teaching a similar classroom, and using similar tasks, code the student work in the same way?

These caveats offer the perfect segue to delve into Rapke, Hall and Marynowski's chapter which scales things up for summative assessments that involve elements of mathematical problem solving. The question posed by these authors, is whether we can reframe testing to work within a classroom that emphasized problem solving. The mismatch between classroom assessment and classroom practice is one of the holy grails of the discipline of mathematics assessment. Without having technology as a confounding variable in this debate, it is well known that while activities in the classroom can emphasize student thinking, it is difficult to capture and thereby assess this on both formative and summative measurements (unless portfolios with fair rubrics are used). These authors make the bold suggestion of involving students in the development of tests—which can sound like heresy to the measurement orthodoxy. The second suggestion is to involve students as peer reviewers in assessing other responses on tests. In a sense what is being proposed harks back to the days of oral testing, where students were posed problems by the teacher in front of the entire classroom, and provided feedback on their solution in the spirit of constructive criticism. While the claim of these authors is that these strategies are "re-castings of the traditional paper-and-pencil test [w]hich, teachers can use …[t]o promote deep approaches to learning and as a result help students to perform better on assessments" (Rapke et al., this volume) in a sense it is an appeal to a very old tradition of assessment going back to the Socratic method of elenchus-proof-refutation, which is found both in the Moore Method, as well as the Lakatosian heuristic (Sriraman and Dickman 2017).

Suurtamm calls for us to move beyond the century old notion of aligning models of learning that emphasized facts and procedures easily measurable through end-of-unit tests, into acknowledging that students learn differently in this century, and are able to develop and convey their mathematical understandings in multi-modal ways.

She writes that "current perspectives of mathematics teaching and learning value mathematical understanding through student engagement in problem solving and argumentation." The upshot is a need to focus on alternatives to traditional paper-pencil testing which can "provide multiple opportunities for students to show what they know and can do."

In a similar vein, the following chapter from McFeetors calls for customizing student learning and alternative approaches to a one-size-fits-all approach to homework and classroom routines. It is obvious that this chapter calls for differentiating instruction when possible and for being cognizant to multi-modal ways of representation available for lesson delivery, when students are able to work in customized classroom settings. Some limitations are also provided to such an approach.

Pai argues for expanding our current notions of the very idea of assessment, by calling on us to go beyond the idea of "measuring." An argument is put forth for a more holistic process that provides "formative, summative, and interpersonal functions, depending on the circumstances of the classroom." In other words, it calls for a boutique like attention to the needs of specific classroom milieus in the context of the material that has been learned. This chapter contains a nostalgic undertone of the teacher being able to mathematically journey through the different stages of sophistication of the student with mathematics. It also presumes a Lakatosian like ideal to a classroom where tangential questions and topics can become objects of curiosity and mathematical attention. Alas, as we all know—the day to day reality in a public school classroom is quite different from the lofty ideals on this chapter.

What Is Assessment, Really?

So, we now return back to the fable and examine what it means in light of the six chapters that have addressed assessment and planning of mathematics lessons. In essence assessment is really about aligning instructional outcomes with (correct) responses on tests. It is also a reflection of the teacher, the teaching methods, and the learner and their learning methods. A class which performs poorly on a test can pose the following questions:

> Is the test designed and written to assess (and thereby measure) what I have learned?
> Or
> Is the test designed and written to assess (and thereby measure) what I have not learned?

This is a fundamental question that is addressed to the writer of a test (or assessment). Are learners being assessed based on outcomes that can be fairly measured on a test. i.e., are they posed problems at various levels of "sophistication" (e.g., based on Bloom's taxonomy) on the mathematical material that has been covered? Is there a balance between items that call for recalling information (factual memory), and those that ask for applying and synthesizing information (deep memory)?

An adaptation of a game theoretic diagram illustrates what the basic constituents of an assessment look like. Assuming an assessment is construed as a "test" or "game" or "battle" between the teacher and student, then four basic win-loss combinations can be generated from the constituents. In our case it is important to define what the terms "win" and "loss" mean in this situation to establish our own norm of "Nash equilibrium," and the reader is urged to do so, based on the recommendations of the six chapters.

Teacher

Win Loss

Aligned to learning (A)	Factual Memory (fM)	Student
Non-Alignment to learning (Na)	Deep Memory (dM)	

Loss Win

For instance an assessment that is aligned to learning and also calls for deep memory results in a win-win situation for the student, and the teacher (reflected in the measured score), as opposed to complete non-alignment to learning and reliance on factual memory which is NOT a win-win combination for neither student nor teacher (which will also reflect in the measured score). The former results in the equilibrium we desire and the latter in disequilibrium.

Scenario 1:	A-fM	Win-Loss
Scenario 2:	A-dM	Win-Win
Scenario 3:	Na-fM	Loss-Loss
Scenario 4:	Na-dM	Loss-Win

Scenarios 2 and 3 are in need of no explanation to the reader and can serve as tautological cases of the diagram (see explanation above). Scenarios 1 and 4 are Win-Loss and Loss-Win situations which are interesting to unpack as the reader would undoubtedly want to know who the winners and losers are in these situations.

Scenario 1 is a win for the teacher and a loss for the student because it results in a situation where students are taught behaviorally (think back to the push for mathematical reform in the 1970s in the U.S) and assessed exactly for each subskill or specific content covered. Think of a timed test with 20 problems that all call for the "skill" of addition. The scores might reflect well on the teacher as this form of learning does result in a "winning" score on a test, but is a loss for a student who has compartmentalized addition as the repeated invoking of factual memory—e.g., align the numbers from right to left in columns, carry over when the sum of the digits is over 9, etc.

Scenario 4 is a loss for the teacher and a win for the student because a situation has arisen where the teacher has failed to design an assessment that can provide a student an opportunity to demonstrate deep memory but the latter is sufficient to do fairly well on any kind of assessments. Having introduced a rather old-fashioned term "memory" into this commentary on chapters that extol the virtues of problem-solving based classrooms and alternative modes of assessment, an apology (defense) is needed. So this commentary concludes with this defense.

An Apology on Memory

> The power of memory is great, very great, (my God). It is a vast and infinite profundity. Who has plumbed its bottom? This power is that of my mind and is a natural endowment, but I myself cannot grasp the totality of what I am. (St. Augustine of Hippo, Book X.8.15)

It is fashionable these days to confuse memory with memorization or rote learning. Mathematical memory is not this at all. To paraphrase St. Augustine of Hippo and much later Vadim Krutetskii, in the language of modern cognitive psychology, mathematical memory is that associated with an explicit memory system, thereby representing information which can be consciously recalled and explained. It is that which has resulted in deep structural insights into the nature of mathematics, at having experienced its fluidity in abstraction and generalization processes in the context of the mathematics being learned. It is a deeper kind of memory that allows for persons to retrieve relevant information when needed and be adept at using it or adapting it. For instance not knowing the derivative of a function can be overcome with the mathematical memory that contains its definition as the limit of a difference function evaluated at zero—and being able to apply this to derive canonical derivatives. It is not the knee-jerk recall of information that is simply memorized for a particular day or test, but one that is retrievable due to the cultivation of all previous mathematical structures within the memory.

The moral of the story based on the six chapters and this interpretation of what mathematical memory means, is that the student who memorized all night long was about to take a test for which his deeper mathematical memory would not be accessed, and his performance on the test would be based on the ability to recall different pieces of information (un-related or even dystopic from one another). We are of course assuming the student is taking a math test and is not about to give a recitation or performance requiring memorized information. In other words, the test would not measure his true knowledge, one that has accumulated over time, but that which is superficial and easily cast from the mind. The six chapters provide ideas, suggestions, tasks and reflections about the different assessments that have worked, and ways in which these target deeper learning objectives, as opposed to simple recall of information or process. Mass testing for deeper student understanding and problem solving is not even at its infancy yet, as we have learned from tests like

PISA, item development and item coding is riddled with problems of cultural incongruence, relevance, and mismatch with local or even national curricular goals. In mathematics, it makes more sense to adopt alternative modes of assessment starting at the grass roots level as many of the chapters in this section advocate. In time, we can hope to achieve this on a much larger scale provided this is one of our goals as researchers.

References

Inhelder, B., & Piaget, J. (1958). *The growth of logical thinking from childhood to adolescence*. New York: Basic Books.

Kilpatrick, J., Swafford, J., & Findell, B. (2001). The strands of mathematical proficiency. In J. Kilpatrick, J. Swafford, & B. Findell (Eds.), *Adding it up: Helping children learn mathematics* [electronic resource] (pp. 115–155). Washington, DC: National Academy Press.

Lamon, S. J., & Lesh, R. (1992). Interpreting responses to problems with several levels and types of correct answers. In R. Lesh & S. J. Lamon (Eds.), *Assessment of authentic performance in school mathematics* (pp. 320–335). Washington, DC: AAAS Press.

Lesh, R., Post, T., & Behr, M. (1988). Proportional reasoning. In J. Hiebert & M. Behr (Eds.), *Number concepts and operations in the middle grades* (pp. 93–118). Reston: National Council of Teachers of Mathematics, Inc.

Sriraman, B., & Dickman, B. (2017). Mathematical pathologies as pathways into creativity. *ZDM Mathematics Education, 49*, 137–145.

Part VI
Broadening Mathematical Understanding Through Content

Part VI: Preface – Broadening Mathematical Understanding Through Content

Gila Hanna

To help students improve their grasp of mathematics and make a successful transition from secondary to tertiary mathematics, the chapters in this part share the belief that it is necessary to broaden the students' mathematical understanding by addressing, at the secondary level, topics such as number theory, probability and statistics, mathematical modelling, and classroom projects making use of innovative technology.

Rina Zazkis discusses the correct order of arithmetic operations by examining two mnemonics in current use, namely the Canadian BEDMAS (Brackets, Exponents, Division, Multiplication, Addition, and Subtraction) and the USA PEMDAS (Parentheses, Exponents, Multiplication, Division, Addition, and Subtraction), where in the latter the order of division and multiplication (D and M) is reversed. She then argues that the order of operations is not and should not be an arbitrary convention, and that there are sound mathematical reasons for it to be "B (brackets); E (exponents); DM (division and multiplication, whichever is first from left-to-right); and AS (addition and subtraction, whichever is first from left-to-right)."

The importance in the curriculum of mathematical models, as simplified representations of real life situations, is the topic of the next two chapters. France Caron argues that the mathematics curriculum could benefit from an enhanced use of mathematical modeling. She presents examples of mathematical models, and points out to their potential contribution to the teaching of specific topics such as, linear, polynomial, and exponential functions. Ann Kajander draws attention to the value of modelling by citing a successful classroom example in which using a model to discover the rules for factoring a quadratic and completing the square helped students solve a problem and promoted their understanding of high-level mathematical

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processes. She also argues that modeling may well have the secondary benefit of giving students an effective tool to help them become more independent learners.

Statistics and probability are discussed in two of the chapters. Jeff Gardner argues that teaching statistics need not be just about plugging numbers into formulas such as those for mean, median, and standard deviation, as unfortunately commonly perceived by students. Rather it is about looking at a body of data and figuring out what it has to say. He shows how the teaching of statistics and the interpretation of data require and foster the development of analytical skills. At the same time he concedes that it is not an easy task, because one has to help students grasp many concepts, such as bias, inference, variability, and the reliability of a conclusion. Nat Banting, Ilona Vashchyshyn, and Egan Chernoff note that the topic of conditional probability gets short shrift in the mathematics and statistics curriculum. Through the analysis of a number of examples and the use of reverse engineering, they suggest a better way of teaching Bayes' theorem.

Edward Barbeau presents an example of composition of linear polynomials designed to encourage students to explore the idea of functions and combination of functions, to investigate the role of variables, to make conjectures, and to try to prove these conjectures. He argues that such a rich teaching situation has the advantage of fostering students' general competence and fluency while leading them to greater attention to detail, to structure and meaning, to reasonableness, and to consistency.

Andrijana Burazin and Miroslav Lovric reflect on the transition from secondary to tertiary mathematics, which in their opinion is nothing short of a "culture shock." The chapter gives examples of topics to show just how poorly prepared secondary-school students often are for that transition. Burazin and Lovric point out that students can be unfamiliar with mathematical symbols to the extent that they get confused when presented with a simple expression such as $f(x, y)$, where *both* x and y represent independent variables, and that students have failed to develop logical reasoning skills due to the deplorable fact that proving theorems is not part of the Ontario mathematics curriculum.

“Canada Is Better”: An Unexpected Reaction to the Order of Operations in Arithmetic

Rina Zazkis

Abstract In Canadian schools the acronym BEDMAS is used as a mnemonic, which is supposed to assist students in remembering the order of operations: Brackets, Exponents, Division, Multiplication, Addition, and Subtraction. In the USA schools the prevailing mnemonic is PEMDAS, where P denotes parentheses, and it further assists memory with the phrase “Please Excuse My Dear Aunt Sally”. Note that while ‘parentheses’ and ‘brackets’ are synonyms, the order of division and multiplication (D and M) is reversed in PEMDAS vs. BEDMAS.

I present my extended reaction to the following claim:

> According to the established order of operation in arithmetic, division should be performed before multiplication.

I was deeply surprised by this assertion, which was voiced by an experienced secondary school teacher of Mathematics. However, as a way of addressing my surprise, I presented the claim for discussion in two classes: a class of secondary mathematics teachers and in a class of prospective elementary school teachers. I share with the reader what happened in each class: surprising realizations, respectful disagreements, reliance on mnemonics, search for counterexamples, attempts to deal with disconfirming evidence, robustness of prior knowledge, and … a declaration of national pride.

Keywords Order of operations · BEDMAS · Scripting task · Conventions

Having worked in mathematics teacher education in Canada for over 25 years, I am still surprised with some widely shared conventional approaches and deeply rooted teaching practices, uprooting which appears extremely hard, if at all possible. I attend to one such teaching convention in this chapter.

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"*According to the established order of operations in arithmetic, division should be performed before multiplication*." I was deeply surprised by this assertion, which was voiced by an experienced secondary school teacher of Mathematics. However, as a way of addressing my surprise, I presented the claim for discussion in two classes: in a class of secondary mathematics teachers, and in a class of prospective elementary school teachers. I share with the reader what happened in each class, presenting the above assertion as a student's suggestion. Following the two stories, I introduce several theoretical constructs that serve as a lens for analyzing the stories.

Order of Operations: Mathematical Convention and Acronyms

When several arithmetic operations appear in the same expression, confusion may occur with respect to the order in which operations are performed. To eliminate this confusion, an agreed upon convention is that multiplication and division precede addition and subtraction, exponentiation precedes multiplication and division, and brackets are introduced to suggest deviations of this order.

In Canadian schools[1] the acronym BEDMAS is used as a mnemonic, which is supposed to assist students in remembering the order of operations: Brackets, Exponents, Division, Multiplication, Addition, and Subtraction. In the USA schools the prevailing mnemonic is PEMDAS, where P denotes parentheses, and it further assists memory with the phrase "Please Excuse My Dear Aunt Sally". Note that while 'parentheses' and 'brackets' are synonyms, the order of division and multiplication (D and M) is reversed in PEMDAS vs. BEDMAS.[2]

While there are researchers and educators who argue against the use of mnemonics, as it does not support conceptual understanding and may lead to mistakes (e.g., Ameis 2011; Hewitt 2012), it is still a shared practice among many teachers, which is occasionally reinforced in textbooks.

For example, a variation on the following text from Musser et al. (2006), appears in a variety of textbooks used in mathematics courses for elementary school teachers.

> To eliminate any ambiguity, mathematicians have agreed that the *proper order of operations* [italics in original] shall be Parentheses, Exponents, Multiplication and Division, Addition and Subtraction (PEMDAS). Although multiplication is listed before division, these operations are done left-to-right in order of appearance. Similarly, addition and subtraction are done left-to-right in order of appearance. The pneumonic devise *Please Excuse My Dear Aunt Sally* is often used to remember this order. (p. 141)

[1] With apology to Francophone colleagues, I refer here to Anglophone Canada. However, I learned recently that in some Francophone schools PEMDAS is used.

[2] In the UK the analogous mnemonic is BIDMAS referring to Brackets, Indices, Division, Multiplication, Addition and Subtraction (Hewitt, 2012). Google search also reveals occasional use of BOMDAS, POMDAS or PODMAS.

Order of Operations: Unintended Ambiguities, Learners’ Difficulties and Pedagogical Ideas

Several research studies focus on the inconsistencies that surface when arithmetic operations are interpreted by various calculation devices. For example, when trying to calculate 2 + 3 × 5 with the help of a hand-held calculator and entering the symbols 2, +, 3, ×, 5, = in that order, the obtained result could be either 25 or 17, depending on whether or not the device is programed to acknowledge the order of operations. Additional difficulty is presented by so called ‘implied multiplication’, to which some calculating devices are programed to give priority (e.g., Joseph 2014). Consider for example the expression 6 ÷ 2(1 + 2) (see Fig. 1), which has been a source of considerable confusion and ongoing web-based discussions.[3]

Calculators aside, order of operation errors are common among secondary school students (e.g., Blando et al. 1989; Linchevski and Livneh 1999), among college students (e.g., Pappanastos et al. 2002), as well as among prospective teachers (e.g., Glidden 2008). In particular, Glidden (2008) reported a significant deficiency in prospective elementary school teachers’ knowledge of order of operations. While the majority of the 381 participants in his study acknowledged that multiplication has priority over addition (in 3 + 4 × 2), over a third erred in performing multiplication before division (in 24 ÷ 2 × 3) and addition before subtraction (in 9 − 4 + 3), which indicated an over-reliance on PEMDAS. In fact, no participant answered correctly all five simple arithmetic calculations in the administered questionnaire.

Acknowledging the problematic notion of order of operations, different ideas on teaching the topic are introduced. For example, Hewitt (2012) described an

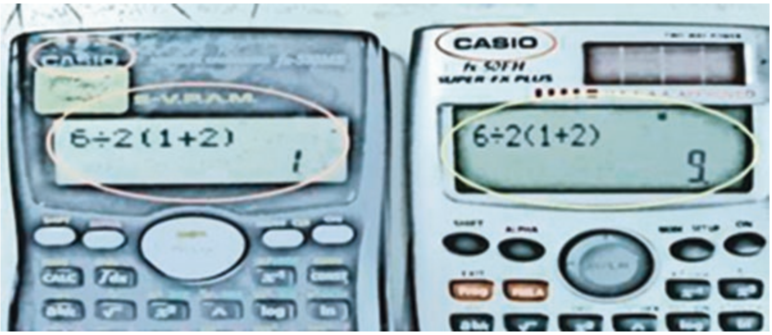
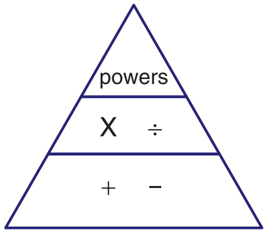


Fig. 1 Example of a calculator-enforced confusion

[3] e.g., http://www.askamathematician.com/2011/04/q-how-do-you-calculate-6212-or-48293-whats-the-deal-with-this-orders-of-operation-business/; https://productforums.google.com/forum/#!topic/websearch/kZkTv_WTSxA; https://answers.yahoo.com/question/index?qid=20110427155042AACb7d8

Fig. 2 Order of operations triangle. (Adapted from Ameis 2011)



instructional software package 'Grid Algebra' that enables students to learn the order of operations without explicit instruction and without the use of mnemonics.

In order to emphasize the conventional order of operations in instruction, Gunnarsson et al. (2016) added 'superfluous brackets.' That is, they recorded expressions of the form $a \pm b \times c$ as $a \pm (b \times c)$ in order to emphasize the priority of multiplication. In doing so, they followed a suggestion of Linchevsky and Livneh (1999) that brackets emphasize the intended structure of the expression and make the conventional rules more explicit. Their study did not find benefits in using superfluous brackets, and, in fact, students who did not use brackets for unnecessary emphasis performed better on order of operations tests.

Ameis (2011) recommended representing the order of operations as a triangle, in which the operation on the 'top' has the top priority (see Fig. 2). He further recommended, working with prospective teachers, to rewrite division as multiplication, turning for example, $30 \div 2 \times 15$ to $30 \times \frac{1}{2} \times 15$, in order to emphasize that division and multiplication have the same priority.

Similar suggestions include presenting hierarchy of operations as a chart or a list, where multiplication and addition appear on the same 'level,' above the level of addition and subtraction (e.g., Karp et al. 2015).

Writing stories that involve multiplication/addition or subtraction with multiplication or division situations is suggested as an activity that reinforces the understanding or order of operations for middle school students (Golembo 2000) as well as for prospective teachers (Ameis 2011). Following the idea of story writing, Blackwell (2003) wrote a play for students to act out, in which students assume roles of different doctors and clarify how a complex operation has to be performed.

Order of Operations Convention: Arbitrary or Necessary?

Is the conventional order of operations an arbitrary choice of mathematicians, or is there a mathematical reason behind this choice?

Hewitt (1999), considering school mathematics, distinguished between arbitrary and necessary features of mathematics. He described 'arbitrary' knowledge as knowledge one can get only by external means, such as read in a book or being told by a teacher, and "needs to be memorized to be known" (p. 3). On the other hand, he referred to 'necessary,' as aspects of mathematics of which students do not need to be informed. That is, "these are things which students can work out for themselves and know to be correct," or "properties which can be worked out from what someone already knows" (p. 4). For example, the convention of writing x before y in the Cartesian coordinates is arbitrary, as alternative decisions could have been made. The choice to use a 'decimal point' as a symbol that separates integer and fractional parts of a number is arbitrary, as in fact, in several European countries the conventional symbol is a 'decimal comma' rather than a point. However, that the sum of the interior angles in a triangle is 180 degrees is a necessary outcome of the arbitrary decision to measure angles in degrees.

While Hewitt described most mathematical conventions as arbitrary, Kontorovich and Zazkis (2017) noted that some seemingly arbitrary (for students) mathematical conventions can be explained as choices for a reason. This position was exemplified pointing to the choice of superscript (−1), which is used to denote a reciprocal of a fraction and an inverse for a function. Considering a reciprocal as a multiplicative inverse explains the choice of the same symbol in both contexts (see also Zazkis and Kontorovich 2016). But what about the conventional order of operations?

While the conventional order of operations may appear as an arbitrary decision of mathematicians, it is actually a necessary result of interpreting/rewriting multiplication as repeated addition,[4] and of interpreting/rewriting exponentiation as repeated multiplication. Consider for example $2 + 5 + 5 + 5 + 5$ vs. $2 + 4 \times 5$. Obviously, $5 + 5 + 5 + 5$ can be rewritten as 4×5. In order to assure that both expressions lead to the same result, multiplication should be performed before addition. Similarly, exponentiation should precede multiplication to assure that, for example, $5 + 2^3$ and $5 + 2 \times 2 \times 2$ lead to the same result.

As such, the prescribed order is necessary, it is not an arbitrary decision of mathematicians imposed to confuse students. However, regardless how the order of operations is introduced in school, what is often memorized and applied is a mnemonic, which, if not interpreted as intended, may lead to mistakes.

Order of Operations: Attending to M-D and A-S Order

Researchers and teachers, mostly focusing on PEMDAS, which is most popular in the USA, noted students' incorrect answers when the mnemonic is interpreted as the appropriate rigid order (e.g., Glidden 2008; Jeon 2012; Watson 2010). That is, students' mistakes resulted from giving priority to multiplication over division, or from giving priority to addition over subtraction.

[4] Note that I do not claim that multiplication *is* a repeated addition, but that it *can be* interpreted/rewritten as such.

However, in Canada, the prevailing mnemonic is BEDMAS, which may lead to an alternative confusion. This is because carrying out division before multiplication out of order actually works. Consider for example $10 \times 6 \div 2$. Performing division before multiplication leads to the same result—$10 \times 6 \div 2 = 10 \times 3 = 30$—as carrying out the operations in order left-to-right.

Is this a coincidence? In other words, is it a "general case" that

$$a \times b \div c = a \times (b \div c)?$$

The situation is easily resolved attending to (a) division is an inverse operation of multiplication and (b) multiplication is associative. Therefore, division can be performed before multiplication and "out of order," as

$$a \times b \div c = a \times b \times \frac{1}{c} = a \times \left(b \times \frac{1}{c}\right) = a \times (b \div c).$$

The fact that 'division first' (out of order) and 'left-to-right' (in order) execution lead to the same result presents a surprise for students who recall the 'proper conventional' order, as well as to those that argue for 'division first' based on BEDMAS.[5]

But what about A-S? While addition before subtraction, when out of order, results in an error, subtraction before addition, when out of order, leads to a correct result. The reason is similar to the one described above: subtraction can be rewritten as addition of the opposite, and addition is associative. To elaborate,

$$a + b - c = (a + b) - c = a + (b - c)$$

because

$$a + b - c = a + b + (-c) = a + [b + (-c)].$$

The fact that D-M order works, but A-S does not, may present an additional layer of confusion to students exposed to BEDMAS.

BEDMAS and Teachers

I follow the narrative inquiry methodology, where "narrative inquiry is aimed at understanding and making meaning of experience" (Clandinin and Connelly 2000, p. 80). I rely on Mason (2002) in distinguishing between account-of and

[5] In fact, Bay-Williams and Martinie (2015) noted that in Kenya students are taught to carry out division before multiplication, which seemingly contradicts that 'left-to-right' order as related to division and multiplication.

accounting-for. The term ‘account-of’ provides a brief description of the key elements of the story, suspending as much as possible emotion, evaluation, judgment or explanations. This serves as data for ‘accounting-for’, which provides explanation, interpretation, value judgment or theory-based analysis. In what follows the account-of is presented in two stories describing my experiences in two classes of teachers. The accounting-for is presented as my reflection, in which several theoretical constructs are used to analyze the stories.

Story 1: Secondary Teachers

My first story is situated in the course “Foundations of Mathematics” for secondary mathematics teachers (n = 16), which is a part of a Master’s program in mathematics education. Building and strengthening connections between advanced mathematics and school mathematics was my explicit goal as I taught this course.

Background: Conventions Task

One of the assignments for secondary mathematics teachers (to whom I also refer here as ‘students,’ as they were students in the course I taught) was to consider mathematical conventions. This assignment followed discussion on the choice of a particular mathematical convention, the use of superscript (−1) in different contexts. In prior research, prospective secondary teachers’ explanations of the “curious appearance” of superscript (−1) in the two contexts—inverse of a function and reciprocal of a fraction—were studied by Zazkis and Kontorovich (2016). It was found that the majority of participants do not attend to the notion of ‘inverse’ with respect to different operations, that is, do not view “reciprocal” as multiplicative inverse. Rather, the differences between the contexts were emphasized and analogies were made to other words and symbols, whose meaning is context dependent.

In the conversation with students about the superscript (−1) similar ideas were initially voiced, but later an agreement converged towards a group-theoretic perception of inverse, as exemplified in two different contexts. This provoked interest in the choice of other mathematical conventions, conventions that are often introduced and perceived as arbitrary, rather than necessary (Hewitt 1999), without any particular explanation. The “Mathematical conventions task” was designed to address this interest. (For extended discussion on mathematical conventions see Kontorovich and Zazkis 2017.)

The idea behind this task was to extend a conversation on the choice of conventions, and acknowledge either the arbitrary nature or the reasoning underlying some of these choices. The students were asked to write a script for a dialogue between a teacher and students, or between students, where interlocutors explore a particular mathematical convention and a reason behind it. The particular conventions were left for the students’ choice. The detail of the scripting task is found in Fig. 3.

Choose a mathematical convention and consider possible explanations for the particular choice.

IN YOUR SUBMISSION:

1. Reflect on the process of choosing the particular mathematical convention for this task. Share alternative conventions that you considered for this task and explain why they were not chosen. (1-2 paragraphs)

2. Write a script for a dialogue in which interlocutors consider possible explanations for the convention you explored. The dialogue should reflect possible doubts, uncertainties and arguments regarding the suggested explanations. The dialogue should end either with an explanation that interlocutors accept or a summary of the disagreement between the characters. (3-5 pages). The dialogue can begin in the following way:

Sam: Hey Dina, have you ever noticed that (the chosen convention)?
Dina: Well, everybody knows that.
Sam: Yes, but did you ever think about why it is so?
Dina: Why should I think about it? It's a convention.
Sam: But, still... Can you propose an explanation?
Dina: Maybe, this is because...

Feel free to modify the proposed beginning of the dialogue.

3. What have you learned, if anything, from completing this task? (1-2 paragraphs)

Fig. 3 Scripting task on mathematical conventions

The task of writing a dialogue belongs to the family of scripting tasks developed and explored in prior research (e.g., Zazkis and Zazkis 2014; Kontorovich and Zazkis 2016). In my earlier research with colleagues (e.g., Zazkis et al. 2009, 2013, scripting tasks were referred to as "lesson plays". It was acknowledged that scripted dialogues provide a lens for observing how script-writers imagine interaction with or among learners, and expose script-writers' understanding of mathematics and their pedagogical moves.

Attending to Order of Operations Convention

One of the repeated examples for a convention was order of operations when performing arithmetic calculations. Below is an excerpt from the script written by Andy, who describes a conversation occurring in a Grade 8 class.

Sam: Hey Mr. X, a couple of us can't decide on answer to the following question:

$$25 + 5 \times 7 - 2 \times 10 \div 5$$

Mr. X: What do you mean?

Mary: I bet them a dollar that they couldn't get the correct answer to a question I made:

$$25 + 5 \times 7 - 2 \times 10 \div 5$$

Sam: Well I got 40. Jane says it’s 56. Tom believes it’s 436, and no one can agree on a solution.
Tom: Mine is correct! I know it.
Mr. X: Tom why do you say that?
Tom: I had a process of how I did mine.
Mr. X: How so?
Tom: I just did one operation after another: 26 plus 5 times 7 and so on. See:

$$25 + 5 \times 7 - 2 \times 10 \div 5$$

$$30 \times 7 - 2 \times 10 \div 5$$

$$210 - 2 \times 10 \div 5$$

$$218 \times 10 \div 5$$

$$2180 \div 5$$

$$436$$

Sam: I did something similar, but I started on the right side of the problem:

$$25 + 5 \times 7 - 2 \times 10 \div 5$$

$$25 + 5 \times 7 - 2 \times 2$$

$$25 + 5 \times 7 - 4$$

$$25 + 5 \times 3$$

$$25 + 15$$

$$40$$

Mary: You guys did the operations in the wrong order.
Jane: I agree with you Mary.
Mr. X: What order would you suggest?
Jane: Well I did the division first followed by multiplication, addition and subtraction.

$$25 + 5 \times 7 - 2 \times 10 \div 5$$

$$25 + 5 \times 7 - 2 \times 2$$

$$25 + 35 - 4$$

$$60 - 4$$

$$56$$

Tom: I don't understand why you started with division. Why would you start there?

Jane: Everybody knows that's the proper order to do operations.

Sam: Mr. X is that correct?

Mr. X: Jane is correct. That is the correct order to do those operations.

Sam: But why?

Mr. X: A long time ago a group of people had a very similar situation that we have now. They were confused and couldn't figure out who had the correct solution to a problem that involved the very operations you are having problems with. It happened around the early fifteenth century in a small European kingdom, it was called the Kingdom of Math. The King of Math, as it were, was a very intelligent leader and believed that his people should always come together to solve their problems.

Jane: Really, Mr. X, a kingdom of math?

Mr. X: Oh yeah, they were a very progressive country. Several of the King's subjects had come to him to settle a problem that they were having. They couldn't decide on an order of the operations that needed to be used. Sound familiar?

Mary: Very funny, Mr. X.

Tom: So what did the King do?

Mr. X: The king commanded his most trusted advisors, members of the Order of Knowledge, to look into the problem. It took several months before the Order had a response for the King. They proposed that the only way to solve this problem was for the King to proclaim an order to the operations so that everyone would know the correct way to solve the mathematical problem.

Sam: That makes sense. Then everyone would follow the same order and no one would be confused about what steps to do first.

Andy offered the following comment at the end of the assignment:

> I felt that there was only one reason that I could [tell] students: "We need to have an order that everyone follows so we can be consistent". "This is the way we all do it". "We" being us in the math community. Whether you're in France, New Zealand, or Canada it's the same. This is because we've all agreed to use the same order so as to have the same understanding of the operations. I tried to find an actual history of the order of operations, but couldn't find anything concrete. So I decided to make up a story that would hopefully give them some connection to the problem and some entertainment along the way.

It is clear from Andy's commentary that accompanied the script that he perceives the convention of order of operations as an arbitrary decision. The reasoning behind this choice, other than the need for consistency, was unclear to Andy and was not found when sought.

The teacher-character's agreement with the student statement, "division first followed by multiplication, addition and subtraction" could have been overlooked, as the result was correct. It is further unclear from the script whether the listed order refers to the general convention, or to the particular case. Nevertheless, both the claim of "division first", and the order in which the operations were performed in Jane's example, attracted my attention. (Note also that addition appears to the left of subtraction in the explored example.) For simplicity, let us consider only the last short computation that involves multiplication and division only, $2 \times 10 \div 5$.

Performing "division first" means interpreting this calculation "as if" there are parenthesis around the operation of division $2 \times (10 \div 5)$. But actually, "division first" and "in order of appearance" yields the same result. I wondered whether "division first" is a shared belief among teachers, Andy's classmates.

Addressing "Division First" Assertion in Class

As a consequence of the "division first" suggestion in Andy's script, the following assertion was presented to a class discussion:

> Assertion: "*According to the established order of operations, division should be performed before multiplication*."

It was presented as a student's claim, for which a teacher's response was sought. The teachers immediately started to explain the conventional order of operations, initially ignoring that the assertion related to multiplication and division only. When the focus was established, four students (out of a class of 16) agreed with the claim, while others insisted on the "left to right" order, when only division and multiplication appear in a computation. BEDMAS was the presented argument that supported the assertion.

However, the majority of students claimed that "division first" was wrong and attempted to find a counterexample, where giving priority of division over multiplication vs. performing these operations in order they appear will lead to different results. (An analogous idea of "multiplication first" or "order does not matter" was suggested, but immediately rejected by a counterexample.)

When "simple" computations did not lead to a counterexample, students turned to more complicated examples. These examples included a longer chain of computations, fractions, and negative numbers. An additional conjecture was voiced that "division first" works only in case there is divisibility between the chosen numbers for division. This resulted in more complicated examples, but the conjecture was refuted after several tests.

In a class session, a search for a counterexample lasted for about 25 min. Counterexamples were sought for both sides of the argument, with a hope to find inconsistency between "division first" (out of order) and "left-to-right in order" order of computation. There were occasional exclamations of "Eureka!", which eventually resulted in double checking that uncovered computational errors. A failure to come up with a counterexample resulted in a conjecture that prioritizing division over multiplication will always work.

Considerable scaffolding was needed to prove this conjecture. When someone suggested that "it works" because "division is just an inverse of multiplication", I countered the claim with "multiplication is just an inverse of division" and "it doesn't work". The suggestion of associativity was voiced only after the students were asked explicitly to consider in what ways division and multiplication are different. As a result of focusing on this difference, in the Lakatosian tradition (Lakatos 1976) of refining claims in light of new evidence, the assertion was rephrased: *Division can be (rather than should be) performed before multiplication*.

Robustness of Belief

The assertion presented for discussion for secondary school teachers caused surprise to both supporters and objectors. Those who supported the conjecture based on BEDMAS were surprised to find out that performing addition before subtraction (A before S) does not lead to an expected result. Those who believed that conjecture was false, and claimed that division and multiplication have the same priority and should be performed in left-to-right order, were surprised to find out that giving priority to division indeed "works."

Each group exhibited a robust "strength of belief" (Ginsburg 1997), based on knowledge that was entrenched and never questioned, as evident in a lengthy search for a counterexample. Extending the example space in search for counterexamples indicates, in accord with Zazkis and Chernoff (2008), that different counterexamples have different convincing power.

The justification and reformulation of the assertion, based on associativity of multiplication, was readily accepted, and even came with an "AHA!" experience for some teachers. As such, it is curious why such an argument was hard for teachers to find on their own.

Story 2: Prospective Elementary School Teachers

My next story is situated in a methods course on teaching mathematics for prospective elementary school teachers (PESTs, n = 21). This can be looked at as a "sample of convenience": I was both concerned and excited with what happened in the class of secondary teachers, I wondered about the extent of the phenomenon and investigated it in the next course I taught.

Background: Written Questionnaire

As a preparation for a discussion on the order of operations, prospective elementary school teachers (in what follows, I also refer to them as students, as they were students in the course I taught) were asked to respond to the questionnaire, see Fig. 4 (the actual questionnaire had space for response between the items). The questionnaire consisted of five calculation tasks involving multiplication and division with "easy" numbers, so that the calculation did not require a calculator (Task 1). It

Your name or pseudonym ___________________________________

Task 1: Solve the following without the use of brackets or fractions:

60÷5×2 = ____ 10×4÷2 = ____ 100÷10÷2 = ____ 100÷5×4 = ____ 200÷2×50 = ____

Task 2: Mary, David, Larry and Ned considered numerical expressions that have only multiplication and division, and disagreed on how the calculations should be carried out.

Mary says that you must always do the multiplication operation first.
Is she right? Please circle: YES / NO
If yes, please explain your reasoning. If no, please explain your reasoning and provide a counterexample.

David says that you must always do the division operation first.
Is he right? Please circle: YES / NO
If yes, please explain your reasoning. If no, please explain your reasoning and provide a counterexample.

Larry says that you must always do the operations from left-to-right in the order they appear.
Is he right? Please Circle: YES / NO
If yes, please explain your reasoning. If no, please explain your reasoning and provide a counterexample.

Ned says that the order does not matter when there is only multiplication and division in the calculation.
Is he right? Please Circle: YES / NO
If yes, please explain your reasoning. If no, please explain your reasoning and provide a counterexample.

Fig. 4 Questionnaire (adapted) on multiplication and division order

further presented different claims of four characters on how the calculation should be carried out (Task 2). The names of the characters were chosen for ease of reference in a follow up discussion: Mary for "multiplication before division" claim, David for "division before multiplication," Larry for "left-to-right order," and Ned for no difference, that is, for "order does not matter" claim. The students had to express agreement or disagreement with each character.

Class Discussion

Following completion of the questionnaires, a class discussion opened with asking the students with which character they agreed. A show of hands revealed that no one agreed with Ned or Mary, while five students agreed with Larry (performing operations left-to-right in order) and 16 students agreed with David (giving priority to division over multiplication), when only multiplication and division appear in a calculation. Arguments for either position were limited to "that's just the rule" or "because of BEDMAS". (Note that this vote is slightly different from the responses to the Task 2 of the questionnaire, and from the order implied in the results of calculations in Task 1, summarised in Zazkis and Rouleau (2018)).

When offered an example of the form a × b ÷ c, the students appeared surprised that performing the operations left-to-right gave the same answer as giving priority to division. They recalculated the items in Task 1 of the questionnaire and became determined to find an example where a different order of operations resulted in a different answer. Examples of the form a ÷ b × c, where the order did matter, served as confirmation of the general disagreement with Mary, and rejection of the 'multiplication first' idea. (Note that this result is different from the studies with students exposed to PEMDAS, e.g., Glidden 2008).

When no example of the form a × b ÷ c was found with whole numbers, PESTs, similarly to secondary school teachers, attempted to involve calculations with negative numbers or fractions. However, unlike the secondary school teachers, PESTs attempted examples with 'large' numbers and used calculators to obtain the results. After unsuccessful attempts to find a disconfirming example, that is, an example where "division first" (out of order) and "left-to-right in order" result in a different number, a conjecture was raised, that either way 'works.' However, no one was able to explain why both approaches (in a × b ÷ c) led to the same answer.

Even after I led the class through a lesson on associativity, some students were still convinced that a counterexample could be found. The mismatch between the robust prior knowledge and the presented 'evidence' caused a significant discomfort to the participants. As one student explained later, "There were some people that definitely, almost still after the class seemed like "I don't know if I believe it" because they'd always been taught to do it in a certain way."

I then focused the class on the of A-S (addition/subtraction) order implied by BEDMAS. While no one gave priority to addition in calculations of the form a − b + c, the fact that 'subtraction first' in a calculation of the form a + b − c leads to a correct result was overlooked. The PESTs appeared to have two ways of thinking about calculations containing addition and subtraction. While a few viewed subtraction as addition of the opposite number, the majority claimed that BEDMAS was applicable only when there was also multiplication or "higher order operations" involved in a calculation. The latter claim was explained referring to a common instructional sequence in elementary school, "You do not need BEDMAS if there are no other operations, like multiplying." This claim can be seen as a reflection on the common exercises that students are exposed to when learning order of operations. Such exercises usually involve a longer sequence with different 'levels' of operations and do not invite learners to reconsider the order of the operations "on the same level."

It appeared that the fact that "left-to-right" order of computation yields the same result as "division before multiplication" used to reinstate the belief in BEDMAS, contrary to my goal. The fact that S-A order "works" rather than A-S was insufficient evidence to abandon, or at least to reconsider and reinterpret the mnemonic.

Beyond Canada

I shared with the PESTs that in some countries the order of operations is taught without any mnemonic, relying on personal experience. The students questioned how learners in those countries would know how to proceed.

In an attempt to question the mnemonic conventional in Canada, I then introduced PEMDAS, which is prevalent in the USA. The class reacted with initial disbelief. Several students attempted to ‘google[6]’ PEMDAS in order to check the information provided by their teacher. When google searches confirmed the information I provided, some students still expressed surprize. This appeared to be based on their conviction that BEDMAS was a standardized international convention for working with the order of operations.

The observation that performing multiplication prior to division, which may result from PEMDAS and cause errors, did not lead to the expected re-interpretation of the mnemonic. Rather, it resulted in a claim that “our acronym is better,” or “Canada is better!” While I would not argue with the claim itself, I find the reason for this declaration of the national pride rather intriguing.

Reflecting on the Two Stories

There are a lot of similarities in the reactions of the two classes to the “division first” assertion. These include the perception that the order of operations is an arbitrary invention, initial surprise when “division first” and “left-to-right in order” give the same result, search for counterexamples, and the repeated mention of BEDMAS. The main difference, however, is the reaction to the presented explanation as to “why” both methods result in the same answer. Secondary teachers accepted the explanation with the feeling of “I should have thought of this myself.” Elementary teachers expressed no interest in the explanation, repeating that “BEDMAS works” and ignoring the A-S part. In what follows I use several theoretical constructs to analyze the stories.

On Cognitive Conflict

> Cognitive conflict is a psychological state involving a discrepancy between cognitive structures and experience, or between various cognitive structures (i.e., mental representations that organize knowledge, beliefs, values, motives, and needs). This discrepancy occurs when simultaneously active, mutually incompatible representations compete for a single response. (Waxter and Morton 2012, p. 585)

In education in general (e.g., Piaget 1977), and in mathematics education in particular, cognitive conflict is considered as an impetus for learning. Several studies explicitly used cognitive conflict as a strategy to explore students’ ideas related to various mathematical concepts, such as division (Tirosh and Graeber 1990), average and measurement (Watson 2007), or fractions (Shahbari and Peled 2015).

Researchers suggested that overcoming or resolving a cognitive conflict is beneficial in building understanding of a related mathematical content. In presenting the “division first” assertion a cognitive conflict was invoked in both groups, as the

[6] Obviously Google is a greater authority than the teacher, especially when the teacher questions conventions.

calculations pointed to discrepancy between the expectations and the results. However, the two groups resolved the conflict in different ways. Secondary teachers accepted the conflict resolution based on their understanding of associativity, while prospective elementary school teachers retrieved to the familiar BEDMAS insisting that “it works.”

On Mathematical Landscape

Wasserman (2016) introduced the topological metaphor of mathematical landscape. He considered the *local mathematical landscape* to be the mathematics being taught and the nonlocal mathematical landscape, as consisting of “ideas that are farther away” (p. 380). He suggested that this division “tackles the notion of mathematical knowledge beyond what one teaches” (p. 380). These metaphors are linked to the notion of “knowledge at the mathematical horizon”, focusing on teachers’ (rather than students’ or curricular) horizons, which Zazkis and Mamolo (2011) re-conceptualized as advanced mathematical knowledge used in teaching.

Applying Wasserman’s distinction to the conventional order of operations, it appears that “how to” belongs to teachers’ local mathematical landscape, but “why” belongs to the nonlocal mathematical landscape. Wasserman asserted that “teachers’ development of and understandings about nonlocal mathematics must not only relate to the content of school mathematics, but to the teaching of school mathematics content” (p. 386). This is because exposure to advanced mathematics helps teachers in developing Key Developmental Understandings (KDUs) (Simon 2006), which change perceptions about content and influence mathematical connections, so in turn, have an impact on teaching.

I suggest that the notion of associativity, even if not so “advanced,” does not belong to teachers’ local mathematical landscape. That is to say, neither secondary, nor elementary school teachers explicitly teach associativity, and even when this property is acknowledged together with other properties of arithmetic operations, it is mentioned together with commutativity. To elaborate, operations discussed in school mathematics are either both commutative and associative, or neither commutative nor associative, which results in frequent confusions between the two (Hadar and Hadass 1981; Zaslavsky and Peled 1996). Associativity appears as a property “on its own” when considering groups and their structure. As such, while the notion itself does not require advanced background, knowledge of advanced mathematics reshapes how associativity is perceived.

For both groups of teachers, associativity appeared to be found in the nonlocal environment, and the connection between local and nonlocal mathematics was not immediately articulated. Furthermore, I suggest that the discussion of the assertion helped secondary teachers in connecting nonlocal mathematics (associativity) to local mathematics (order of operations) in a potential situation of contingency in their teaching. The notion of associativity, however, did not resonate with prospective elementary school teachers, possibly because of the larger gap between their local and nonlocal mathematics.

In relation to teachers' mathematical knowledge Wasserman (2016) uses the terms 'nonlocal' and 'advanced' as almost synonymous, referring to knowledge of mathematics beyond what is taught in school (nonlocal) and knowledge that is acquired in university studies (advanced). While for a large variety of concepts nonlocal and advanced coincide, the example of order of operations demonstrates that 'nonlocal' is not necessarily 'advanced,' but situated beyond teachers' "active repertoire" of knowledge used in teaching.

On "Met-Before"

Tall (2013) used the term 'met-before' to refer to "a structure we have in our brains now as a result of experiences we have met before" (p. 23). He noted that a met-before can be supportive in some contexts, but problematic in others. For example, the met-before 'take-away leaves less' is supportive when considering whole numbers or measuring lengths of objects. However, the idea becomes problematic in the context of integers or infinite cardinal numbers. Tall maintains that curriculum designers focus mainly on supportive met-befores as a basis for future development. He suggests, however, that "a sensible approach to learning requires not only the building towards powerful ideas that will be encountered in the future but also addressing problematic issues in the present that may have long-term consequences" (ibid, p. 116).

My two stories demonstrate that BEDMAS met-before is indeed a "problematic issue" with "long-term consequences". The robustness of BEDMAS presents difficulty to abandon it altogether, or at least to interpret it consistently with the convention. This is because the mnemonic is located not only on teachers' local mathematical landscape, but also—extending Wasserman's constructs—on teachers' *shared local mathematical landscape*, that is, it refers to a broadly shared teaching approach. A cognitive conflict experienced in a class session in teacher education course cannot uproot prior robust knowledge, but it may plant a seed for revisiting the issue when the subject is taught or when students ask questions.

Conclusion with a Note on Teaching

With respect to the conventional order of operations in arithmetic, should division have priority over multiplication? If yes, why so? If not, does giving priority to division lead to an incorrect result? These questions, and unexpected answers, were explored with two groups of teachers, secondary and prospective elementary. In the two stories above I described surprising realizations, respectful disagreements, reliance on mnemonics, search for counterexamples, attempts to deal with disconfirming evidence, the robustness of prior knowledge, and … a declaration of national pride.

Acknowledging the problematics presented by the popular mnemonic, particular ideas on how to avoid the unintentional order imposed by the mnemonic are introduced by several authors. For example, Van de Walle et al. (2011) suggest "just use the acronym "BEDMAS" with the letters listed in rows to indicate order:

B = brackets
E = exponents
DM = division and multiplication (whichever is first from left-to-right)
AS = Addition and subtraction (whichever is first from left-to-right) (p. 492).

A similar presentation is reiterated by other authors (e.g., Golembo 2000).

Rather than writing the acronym's letters in rows, teachers suggested recording it as $BE\frac{D}{M}\frac{A}{S}$ or BE(DM)(AS) in order to highlight that operations of the same priority should be considered together (personal communication).

Surprisingly, these ideas center around how to interpret the mnemonic properly, rather than argue against it. However, I join several researchers and educators (e.g., Dupree 2016; Kalder 2012; Watson 2010) who call to abandon the use of mnemonics in general, focusing on stronger connections among the arithmetic operations. In particular, Kalder (2012) pointed out that the particular mnemonics associated with the order of operations are confusing in introducing both correct (multiplication before addition) and incorrect (addition before subtraction) ideas.

It is unavoidable that some ideas of elementary mathematics have to be relearned as their domain of applicability was limited to early experiences (Zazkis 2011). Those are unavoidable met-befores. In other cases, such as BEDMAS, relearning would not be necessary if there were no misleading met-before. However, the mnemonic passed over from a teacher to a student creates a vicious cycle of mis-information.

I strongly believe that the acronym met-before can be avoided, rather than created and then followed by suggestions on how it can be surmounted. After all, there are millions of students in non-English speaking countries that learn to apply the order of operations without any mnemonics. Tall (2013) emphasized that the term met-before applies to experiences that have both supportive and problematic aspects. The supportive aspect of BEDMAS is of course in assisting memory. But the problematic aspect is not only in incorrect applications of order in some situations. The problematic aspect is in reinforcing the view of mathematics as a collection of arbitrary rules.

References

Ameis, J. A. (2011). The truth about pemdas. *Mathematics Teaching in the Middle School, 16*(7), 414–420.

Bay-Williams, J. M., & Martinie, S. L. (2015). Order of operations: The myth and the math. *Teaching Children Mathematics, 22*(1), 20–27.

Blackwell, S. B. (2003). Operation central: An original play teaching mathematical order of operations. *Mathematics Teaching in the Middle School, 9*(1), 52.

Blando, J. A., Kelly, A. E., Schneider, B. R., & Sleeman, D. (1989). Analyzing and modeling arithmetic errors. *Journal of Research in Mathematics Education, 20*(3), 301–308.

Clandinin, D. J., & Connelly, F. M. (2000). *Narrative inquiry: Experience and story in qualitative research*. San Francisco: Jossey-Bass.

Dupree, K. M. (2016). Questioning the order of operations. *Mathematics Teaching in the Middle School, 22*(3), 152–159.

Ginsburg, H. P. (1997). *Entering the child's mind: The clinical interview in psychological research and practice*. Cambridge: Cambridge University Press.

Glidden, P. L. (2008). Prospective elementary teachers' understanding of order of operations. *School Science and Mathematics, 108*(4), 130–136.

Golembo, V. (2000). Writing a pemDAs story. *Mathematics Teaching in the Middle School, 5*(9), 574.

Gunnarsson, R., Sönnerhed, W. W., & Hernell, B. (2016). Does it help to use mathematically superfluous brackets when teaching the rules for the order of operations? *Educational Studies in Mathematics, 92*, 91–105.

Hadar, N., & Hadass, R. (1981). Between associativity and commutativity. *International Journal of Mathematics Education in Science and Technology, 12*, 535–539.

Hewitt, D. (1999). Arbitrary and necessary: Part 1. A way of viewing the mathematics curriculum. *For the Learning of Mathematics, 19*(3), 2–9.

Hewitt, D. (2012). Young students learning formal algebraic notation and solving linear equations: Are commonly experienced difficulties avoidable. *Educational Studies in Mathematics, 81*(2), 139–159.

Jeon, K. (2012). Reflecting on PEMDAS. *Teaching Children's Mathematics, 18*(6), 370–377.

Joseph, K. N. (2014). *College students' misconceptions of the order of operations*. Unpublished doctoral dissertation, State University of New York at Fredonia, Fredonia, NY.

Kalder, R. S. (2012). Are we contributing to our students' mistakes? *Mathematics Teacher, 106*(2), 90–91.

Karp, K. S., Bush, S. B., & Dougherty, B. J. (2015). 12 Math rules that expire in the middle grades. *Mathematics Teaching in the Middle School, 21*(4), 208–215.

Kontorovich, I., & Zazkis, R. (2016). Turn vs. shape: Teachers cope with incompatible perspectives on angle. *Educational Studies in Mathematics, 93*(2), 223–243.

Kontorovich, I., & Zazkis, R. (2017). Mathematical conventions: Revisiting arbitrary and necessary. *For the Learning of Mathematics, 37*(1), 29–34.

Lakatos, I. (1976). *Proofs and refutations*. Cambridge: Cambridge University Press.

Linchevski, L., & Livneh, D. (1999). Structure sense: The relationship between algebraic and numerical contexts. *Educational Studies in Mathematics, 40*, 173–196.

Mason, J. (2002). *Researching your own practice: The discipline of noticing*. London: Routledge Falmer.

Musser, G. L., Burger, W. F., & Paterson, B. E. (2006). *Mathematics for elementary teachers: A contemporary approach*. New York: Wiley & Sons.

Pappanastos, E., Hall, M. A., & Honan, A. S. (2002). Order of operations: Do business students understand the correct order? *Journal of Education for Business, 78*(2), 81–84.

Piaget, J. (1977). *The development of thought: Equilibration of cognitive structures*. New York: Viking.

Shahbari, J. A., & Peled, I. (2015). Resolving cognitive conflict in a realistic situation with modeling characteristics: Coping with a changing reference in fraction. *International Journal of Science and Mathematics Education, 13*(4), 891–907.

Simon, M. (2006). Key developmental understandings in mathematics: A direction for investigating and establishing learning goals. *Mathematical Thinking and Learning, 8*(4), 359–371.

Tall, D. (2013). *How humans learn to think mathematically: Exploring the three worlds of mathematics*. New York: Cambridge University Press.

Tirosh, D., & Graeber, A. O. (1990). Evoking cognitive conflict to explore preservice teachers' thinking about division. *Journal for Research in Mathematics Education, 21*(2), 98–108.

Van de Walle, J. A., Folk, S., Karp, K. S., & Bay-Williams, J. M. (2011). *Elementary and middle school mathematics: Teaching developmentally* (3rd Canadian ed.). Toronto: Pearson Canada.

Wasserman, N. (2016). Nonlocal mathematical knowledge for teaching. In C. Csíkos, A. Rausch, & J. Szitányi (Eds.), *Proceedings of the 40th conference of the international group for the psychology of mathematics education* (Vol. 4, pp. 379–386). Szeged: PME.

Watson, J. (2007). The role of cognitive conflict in developing students' understanding of average. *Educational Studies in Mathematics, 65*(1), 21–47.

Watson, J. (2010). BODMAS, BOMDAS and DAMNUS. *Cross Section, 20*(4), 6–9.

Waxter, M., & Morton, J. B. (2012). Cognitive conflict and learning. In N. M. Seel (Ed.), *Encyclopedia of the sciences of learning* (pp. 585–587). Boston: Springer.

Zaslavsky, O., & Peled, I. (1996). Inhibiting factors in generating examples by mathematics teachers and student-teachers: The case of binary operation. *Journal for Research in Mathematics Education, 27*(1), 67–78.

Zazkis, R. (2011). *Relearning mathematics: A challenge for prospective elementary school teachers*. Charlotte: Information Age Publishing.

Zazkis, R., & Chernoff, E. (2008). What makes a counterexample exemplary? *Educational Studies in Mathematics, 68*(3), 195–208.

Zazkis, R., & Kontorovich, I. (2016). A curious case of superscript (−1): Prospective secondary mathematics teachers explain. *Journal of Mathematical Behavior, 43*, 98–110.

Zazkis, R., & Mamolo, A. (2011). Reconceptualising knowledge at the mathematical horizon. *For the Learning of Mathematics, 31*(2), 8–13.

Zazkis, R., & Rouleau, A. (2018). Order of operations: On convention and met-before acronyms. *Educational Studies in Mathematics, 97*(2), 143–162.

Zazkis, R., & Zazkis, D. (2014). Script writing in the mathematics classroom: Imaginary conversations on the structure of numbers. *Research in Mathematics Education, 16*(1), 54–70.

Zazkis, R., Liljedahl, P., & Sinclair, N. (2009). Lesson plays: Planning teaching vs. teaching planning. *For the Learning of Mathematics, 29*(1), 40–47.

Zazkis, R., Sinclair, N., & Liljedahl, P. (2013). *Lesson play in mathematics education: A tool for research and professional development*. Dordrecht: Springer.

Modelling in Secondary Mathematics Education: Moving Beyond Curve Fitting Exercises

France Caron

Abstract Mathematical modelling is gaining greater attention as a goal of mathematics education, but its transposition to school often passes by the main characteristics and benefits of its integration. This chapter aims at providing a broader view on the integration of modelling in secondary mathematics, one that would better reflect the richness of the modelling process, for understanding both mathematics and the world around us.

Keywords Mathematical modelling · Modelling process · Mathematics education · Competencies · Technology · Interdisciplinarity

Introduction

Mathematical modelling has been a topic of growing interest in mathematics education. Conceived as a real life problem solving activity that has contributed to the development of mathematics, it can be promoted as a means to help learn mathematics and motivate for new content, as connections with real world problems can enrich the meaning of the mathematics learned and allow for deeper understanding. Mathematical modelling can also be associated with a set of competencies that are key to tackling the increasingly complex problems that we as individuals or as society may face, and as such, their development can be viewed as an important goal of mathematics education.

It is therefore no surprise that a number of secondary school curricula now refer to modelling as part of their objectives or suggested approaches for mathematics education. Yet, finding the right balance between the use of modelling and applications for enriching the content taught and the development of true modelling skills is not a straightforward task (Blum et al. 2002). A lack of tradition and the fear

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of taking time away from the teaching of mathematics have often led to a reduced version of what modelling entails and of how it can be integrated in secondary school mathematics.

One such reduction has been to restrict to functions the set of models considered in the last years of secondary and to limit the modelling activity to finding the curve of best fit for a set of data points, typically with the help of technology. While curve fitting may be a valuable skill in approaching an unknown relationship between two variables, it can only bring a limited contribution to the understanding of a situation and of the mathematics that is used.

This chapter thus aims at providing a broader view on the integration of modelling in secondary mathematics, one that would better reflect the richness of the modelling process, the variety of models that can be used, and the many ways in which the learning of mathematics, science and other disciplines can benefit from a greater attention to modelling.

Understanding Modelling

The word "modelling" has been used extensively in education, with a meaning much different than what is meant by mathematical modelling. Promoting mathematical modelling in school by no means suggests multiplying the number of occurrences where a teacher of mathematics acts as a "model" and exhibits the mathematical skills and attributes to be reproduced or developed by the students. Although there may be moments when a teacher of mathematics may elect to think aloud or show explicitly what should or could be done, this is not what mathematical modelling is about.

Mathematical modelling has to be understood as an aspect of the mathematical practice, one that is done by a practitioner of mathematics, be it a mathematician, an economist, a biologist, any citizen, a teacher or a student. Mathematical modelling is the process for building a mathematical model out of a real-life situation, and using that model to analyse the situation, understand it, address an issue, solve a problem, make a decision, etc. Mathematical models can help us do all this, because they are simpler than the situation from which they were built, they have a level of abstraction that makes them valid for a large class of problems and they can be turned into computation and simulation tools.

A mathematical model is indeed a simplified representation of a situation, one that captures the essential aspects of that situation with respect to a given goal the modeller has or has been given. Such representation can be done using different types of register or languages: graphical (with figures, diagrams, graphs, sketches), symbolic (typically a set of equations), numerical (table of values) and even verbal. A computer program can also be seen as a model, as it organizes in a programming language, entities (objects, variables, parameters), data, relationships (structures, equations) in a way so as to solve a problem or simulate and explore a situation. In a related way, a model of a situation can also be made out of concrete material. With

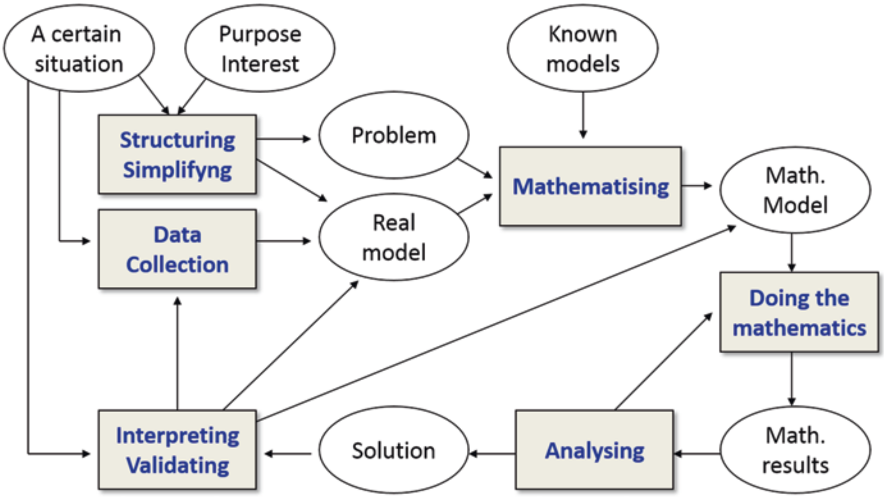

Fig. 1 A representation of the modelling process

the variety of possible models, it is important to recognize that not all models have the same explanatory capacity (this also depends on who will use the model), the same degree of generality (the size of the class of problems or situations that a model can address) or the same power for generating solutions or predictions.

Mathematical modelling can be seen as a rather demanding and creative form of problem solving, one that acknowledges and addresses the complexity and messiness of real-life situations. It has been described as a multi-stage and cyclic process (Blum et al. 2002). Many diagrams have been proposed to represent this process, some with as few as three phases, others with many more. I present mine (Fig. 1), which is strongly connected to the description found in Blum et al. (2002), with the addition of some elements which I feel are important to include.

Modelling starts with a given real-world situation and a goal for which that situation must be examined. The goal might be to predict the evolution of the situation, to understand how a particular phenomenon has emerged, to make a decision, to develop or optimize a system that would help manage the situation, etc. The situation is then simplified by making some assumptions that restrict the scope of the possibilities to consider and by extracting from the initial situation some key entities (objects, attributes, variables, etc.) that characterize the situation with respect to the goal that is being pursued. Reformulating the goal in terms of these elements leads to the problem to be solved. Between the chosen elements of the situation, some relations can be anticipated, sketched or sometimes, even at that stage, clearly defined. These connections are typically derived from principles known by the modeller, but they can also be partly inferred from existing data, and together with the objects and attributes to which they refer, they form a structure with which the situation can be examined; a representation of this simplified version of the situation, which has not yet been fully mathematized, is what has been referred to as the *real model*.

Turning this real model into a mathematical model is the object of the following phase; although the resulting mathematical model can be original, at least for the modeller, one often builds on known mathematical models, by selecting them, combining them, and adapting them to the situation and for the problem to be solved. Reuse of known models was not included explicitly in the description of Blum et al. (2002), but we believe it is worth mentioning, as a way of acknowledging both the collective nature of model development and the efficiency of the process. Simply said, the wheel is not reinvented every time a new situation is encountered.

Once a mathematical model has been defined, one can call upon mathematical techniques to try to solve the problem and produce a solution or explore various scenarios. In order for a solution to be considered valid, it must not only be consistent with the model of the situation (e.g. by verifying the equations), but it must also be compatible with the initial situation; it must make sense within that context. This may lead the modeller to take a step back and revisit some of the initial assumptions, consider more variables, collect more (or better) data if deemed appropriate, attempt to generalize, and resume the whole process again.

In a recent document (Garfunkel and Montgomery 2016) produced for teachers by the Consortium for Mathematics and Its Applications (COMAP) and the Society for Industrial and Applied Mathematics (SIAM), a good example is provided to show how a simple question (*Is it worth the drive across town for less expensive gas?*) can get students to engage in the modelling cycle and iterate as they refine their analysis. They may start with a very modest *data collection* by looking for the current price for gas in two stations between which the distance is known. In structuring the situation into a real model, they will identify key variables (cost, distance, fuel use) and try to establish connections between them (e.g., *the more you drive the more fuel you use*). In mathematising the model, they will produce equations. They will "do the math" in using these equations for a particular set of data. They will get to more sophisticated models as they as they attempt to generalize (*At what point does the difference between the two prices makes it worth the drive?*). They may redefine the *real model* in envisioning different road configurations, in revisiting assumptions (*Is the extra-distance to drive really the double of the distance between the two gas stations? Is the fuel consumption really constant? Or the same on both roads?*). They may eventually put things in a larger perspective, where new variables, parameters and a potentially revised purpose would be considered (*What is the value of my time? Shouldn't we consider also the cost on the environment?*).

This is a very good illustration of how rich modelling can be in developing and refining critical analysis, and of the way this kind of activity may represent a change from typical learning activities in mathematics. First, the activity is very much iterative, and works best if it is conducted within a team where there is an exchange of ideas and a will to revisit initial assumptions and decisions. Specific cases and numerical values can be used at first to get a better feel for what is at play, and used again to validate a more general model. But it is really with a general model that one can answer the initial question by providing, in terms of different variables and parameters, the tipping point at which the decision should change. For this general model to appear, students must move away from their well-learned skill of substituting values in expressions, and instead accept playing with the variables and the various relations that they will generate to structure and mathematize the situation. They do that by formalising some knowledge they might have of the situation (e.g., fuel consumption is measured in litres/100 km), making assumptions (e.g. fuel consumption is constant) and/or using proportional reasoning (e.g. the longer the drive, the more fuel we use).

But what may appear fundamentally different in such a modelling activity when compared to a typical school mathematics task is the fact that there is not a single "good" answer; there is a trade-off to be made between the level of sophistication of the model and its usability for producing results in solving the problem or exploring the situation in the amount of time allotted for the task. The final model produced depends as much on the goals pursued by the modellers, the aspects of the situation that they want to address, the degree of simplification and generalisation that they are aiming for, as it does on their knowledge or experience of both the situation and the various mathematical tools that could help them address it. Consequently, a same situation can be presented to modellers of different levels of expertise or backgrounds, and lead to completely different models that may address well different aspects of the situation, or with different degrees of accuracy or generalisation. In a classroom, it is often at the moment of comparing the different models produced that a deeper understanding can be gained, of both the real-life situation and the mathematical tools that were (or could be) used to address it.

Yet, despite all the potential benefits, the time restrictions that may come with the curriculum often act as obstacles to greater presence of modelling activities in the class of mathematics. The iterative nature of the modelling process, the time required for a modelling activity to reap most of its anticipated benefits, the lack of guarantee that even a carefully chosen open-ended modelling problem will lead to some of the mathematical concepts and skills aimed for by the curriculum, may lead teachers to reduce the scope of modelling activities so as to realign them with the content to be covered. As the learning of functions and the solving of associated equations typically characterize the end of secondary mathematics, this has often led to consider curve fitting exercises as an appropriate compromise for opening to real-life applications of the functions taught at that level while developing the skills to extract a functional model from data. The next section will show the presence, benefits and limitations of this approach.

Modelling Through Curve Fitting

Across the different school curricula in Canada, the capability of extracting a function of one variable from a set of data, with or without technology, is a skill often associated with learning objectives (or outcomes) in mathematics for the end of secondary. As example, *The Common curriculum framework for grades 10–12 mathematics*, produced in 2008 by the Northern and Western and Northern Canadian Protocol (WNCP 2008) for Alberta, British Columbia, Manitoba, Northwest

Territories, Saskatchewan and Yukon Territory, and adapted in 2015 for the Nova Scotia grade 12 curriculum (NSDEECD 2015), has several expectations that use the following format:

- Graph data and determine the [specific] function that best approximates the data.
- Interpret the graph of a [specific] function that models a situation, and explain the reasoning.
- Solve, using technology, a contextual problem that involves data that is best represented by graphs of [specific] functions, and explain the reasoning.

Depending on the section of the curriculum where it appears, the expression [specific] takes the value of either “polynomial,” “exponential or logarithmic” or “sinusoidal.”

Looking for the function that best approximates a set of data, interpreting it and using it for solving a problem are thus presented as skills associated with modelling that must be developed. Not only are tasks that call upon these skills seen as applications of the functions taught, which might motivate for their study, but such tasks are generally perceived as opportunities to develop a greater working knowledge of these functions, their parameters and properties.

For instance, a teacher may ask her students to approximate authentic monthly average temperature data for their city collected over a 4 year interval with a sinusoidal function f such that $f(x) = a \sin (b * (x - h)) + k$, where x is the month. Such an exercise, with data that has so much meaning to Canadian students, may enrich their understanding of the different parameters of the sinusoidal function, especially if the reflection for finding the optimal values is not completely bypassed with a one-step sine regression procedure handled by a technological tool. Using a spreadsheet or any other graphing tool where the original data is compared with the value produced from a sinusoidal function of which the parameters are settable can open to trial and error investigation: although students may start with random trials, the complexity of having to find all four parameters will have them move to a more reflective approach; they will associate amplitude a with about half the difference between maximum and minimum values; they will seek how to stretch with the help of parameter b the 2π period over a 12 month period, they will see how the value of h can provide a phase shift to Spring; and they might get pleasant confirmation that the median temperature for a year (k) is above 0. Such investigation from authentic empirical data can also make students appreciate the inherent variability of nature, by seeing the variations in the monthly averages from one year to the next and studying the distribution of the errors (sinusoidal? uniform? normal?) made by “the best” functional model (Fig. 2).

Students can even explore climate change by wondering if variability alone could account for the difference in functions identified for two four-year intervals that would be thirty or fifty years apart. This may raise some new questions: can we still consider it a periodic function or must we account for the addition of an increasing function? Could we make reliable long term extrapolations from this data?

Encouraging such questions not only gives authentic data and the phenomenon of which they might testify the respect they deserve, but it also favours the

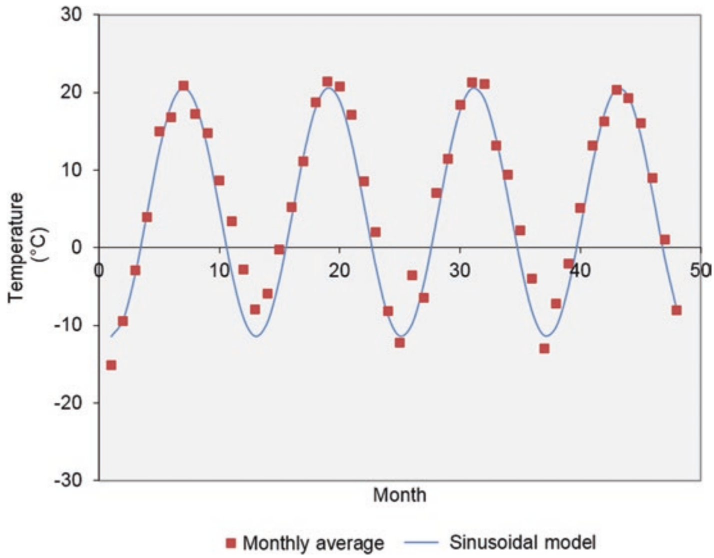

Fig. 2 Looking for a sinusoidal model for monthly averages of Montreal temperature over 4 years

development of critical judgement at validation stage. This can go further with questions such as the following: even if monthly average temperatures follow a yearly cycle, is a sinusoidal function the best periodic function to consider, and if so, why? What could be the meaning of a continuous function between monthly averages? Should we remain with a discrete model and simply associate with each month a sequence of its average temperatures over the years? What is the goal of the model? What are we trying to achieve here?

Again, lack of time may prevent such discussion on models to take place in the mathematics class; curriculum orientation may even lead to manipulation of the data so that it almost perfectly fits the functions under study. Tides are an example of that, as they are often proposed as context where pure sinusoidal functions can be used to model their height at a given location; this can be seen in both textbooks in Quebec and in the Ontario curriculum sample problems. Yet, associating their height to a pure sinusoidal function ignores a fundamental characteristic of tides, in that their amplitude is not constant. The flux that makes tides results from a combination of forces exerted by the moon, the sun and the rotation of the earth, and these forces vary with distinct periodicities. As a result, there are days where the tides are stronger, a phenomenon well known by residents of coastal areas. Overlooking this aspect is an oversimplification of reality which is neither useful for practical purposes nor beneficial for the learning of science. And it does not do justice to mathematical modelling either.

A sum of sinusoidal functions is typically used to capture the complex behaviour of the height of tides, and the finding of the many parameters involved is well above the capacities of typical regression tools used by students. The computer models

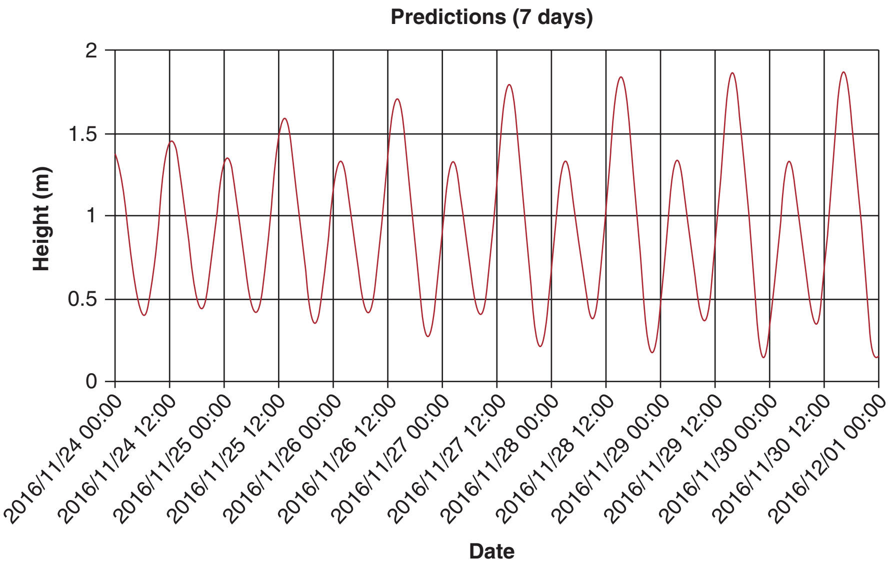

Fig. 3 Prediction of height of tides for a week in Bathurst (http://www.tides.gc.ca/)

which produce predictions such as the one below are generally based on a deeper understanding and structuring of the physical phenomena at play (Fig. 3).

It would be very tedious for a student to come up through trial and error with an approximate model that could generate such a curve, even with only two sinusoidal functions (and a total of eight parameters) to account for the dominating forces. However, the exploration alone could provide interesting insight into both superposition of waves and addition of trigonometric functions.

While curve fitting and regression may help approach unknown relationship between two variables and develop initial intuitions, they can only bring limited contribution to the understanding of the situation at play. In fact, these techniques can be associated with the *empirical paradigm* of modelling (Maull and Berry 2001) for which the main goal is to predict some behaviour. If the task of finding these relations is mainly left to the calculator or some other computing tool, then the experience of the modelling process, from the student perspective, is reduced to going directly from *data* to the *mathematical model*, with no control over the *structuring* and *mathematising* phases. Consequently, the epistemic contribution of such activities is rather low; it is mainly in the *interpretation* phase, when looking for the reasons behind the different parameters, that the student might gain back some of the benefits associated with modelling. In addition, the systematic use of preprogrammed regression procedures of which the theory is not taught at secondary level also runs the risk of implicitly promoting both the use of black boxes in mathematics and "subversion of reality by choices available on the menus of calculators" (Galbraith 2007, p. 81). Not only should the quality of the predictions that are made under these conditions be questioned, but one should also wonder about the impact on students' conception of science (pure or applied) of producing models in such a way.

In order to understand the inner working of a situation and maintain control over the mathematical model used, one needs to move from the *empirical paradigm* (simply aimed at predicting from data) to the *theoretical paradigm* (aimed at understanding) in modelling the situation (Maull and Berry 2001). The next section will show how that can be achieved with secondary mathematics.

Modelling Through Structuring: An Example

Adopting the theoretical paradigm entails "that the processes underlying the phenomenon be studied, that appropriate laws suggested, and the laws compared with the data obtained" (Maull and Berry 2001, p. 80). The next activity provides an illustration of the passage from the empirical to the theoretical paradigm.

The activity originated from a team of students in a graduate course on the integration of modelling in mathematics education. As part of the assignments for the course was an open-ended project that required students to find an authentic situation where they had a goal to meet or a problem to solve, for which they could get some data, and that would benefit from mathematical modelling. This particular team, of which a member was a teacher of mathematics at a secondary school, wanted to develop a better appreciation of the impact of waste disposal. After a few phone calls, they managed to get some city data on the filling over the last 4 years of an urban landfill site that was nearing saturation. They then set for themselves the objective of studying the filling of the site and determining the year by which the landfill would no longer be usable.

The data consisted, for each of the 4 years, of the volume of waste that had been routed to the site as well as the remaining capacity, as assessed from aerial photos. Knowing the total capacity of the site and the remaining capacity at any given year, they could easily deduct the volume of accumulated waste in the site for each for the years. But because these values could not be linked by simple additive relationships with the amount of waste sent to the site, the team members fell back on curve fitting for the evolution of the volume of waste over time (Fig. 4).

From the appearance of the curve for the volume of waste accumulated in the site and with the trendline options supported in Excel, they opted for a logarithmic function for which the optimal parameters provided by the tool gave them a correlation coefficient of 0.99.[1] However, despite its almost perfect fit to the data points they had, the resulting curve did not seem to go in the right direction for extrapolation purposes. Moreover, in trying to interpret the model, the students could not make sense of this logarithmic function with respect to the situation (Fig. 5).

[1] The years were renumbered from 1 to 4, as the Excel trendline tool only supports curve fitting with logarithmic functions of the form $y = a \ln x + b$. Leaving the years to be greater than 2000 could not allow for such rapid growth, and the "curve of best fit" was much further from the points.

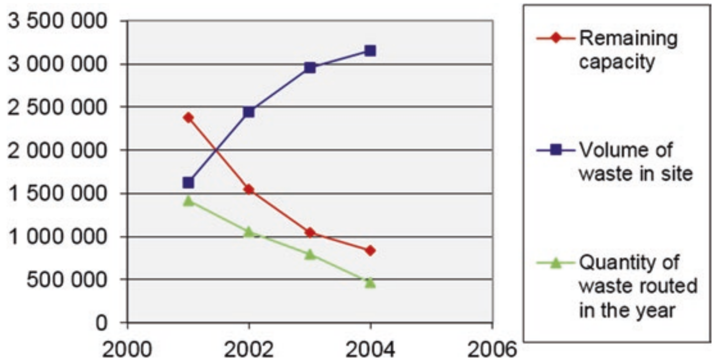

Fig. 4 Data for the landfill problem

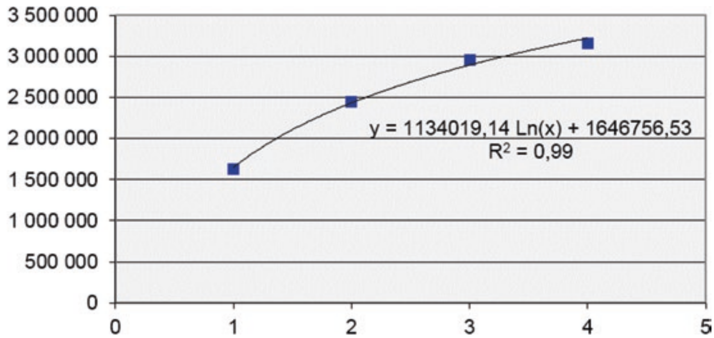

Fig. 5 Looking for a curve of best fit for the landfill problem

A paradigm shift was in order, which had the students move from the empirical to the theoretical, from a sceptical use of a regression black box to an iterative and more comprehensive experience of the modelling cycle.

The first step was coming back to the most intuitive relationships that had first been envisioned for approaching the situation. This led to defining the following recurrence relation for volume V of waste accumulated at the end of year n:

$$V(n) = V(n-1) + Q(n)$$

where $Q(n)$ is the value of waste brought to the site in year n. As shown in the left part of Fig. 6, the values generated by this very simple model with an initial value for V were systematically greater than the actual waste volume in the site, as deducted from the remaining capacity data. In looking for an explanation, one can figure out that some compaction must have taken place. A new model was thus created which introduced a mean compaction factor (c), applicable to all the waste:

$$V(n) = c\left(V(n-1) + Q(n)\right)$$

A value of 0.92 for c provided rather close approximation to data. The model was further improved by considering two different compaction rates, one (a) for the old waste, already present in the site, and the other one (b) for the new waste routed to the site.

$$V(n) = aV(n-1) + bQ(n)$$

Iterative manual adjustment of both parameters led to an excellent fit with the data for the volume used by the waste (using $a = 0.94$ and $b = 0.85$) (Fig. 6).

This iterative quest for the optimal parameters opened to an interesting interdisciplinary reflection: of the old and the new, what waste will compress more? What is the meaning of a greater compaction factor? Would the waste at the bottom of the accumulation not be subject to greater pressure? Should the compression factor for the old waste vary with the amount of new waste introduced? Should more layers be considered?

What makes this work different from a typical curve fitting exercise is the fact that the fine-tuning of the model is done after some basic principles (of accumulation and compaction of volume) have been formulated to structure the situation and define a generic model. The capacity to identify appropriate principles may depend on the experience with similar situations and/or knowledge of other disciplines. If crossing into another discipline can be an enriching opportunity for collaboration with a teacher of this other discipline, the difficulty of making it happen in a school, with both curricular and logistics constraints, can also act as an obstacle to introducing real-life situation modelling in the learning of mathematics. Yet, as we have shown here, with both this landfill site and the drive for cheaper gas, there are situations where the expression of applicable principles or reasonable substitutes can be

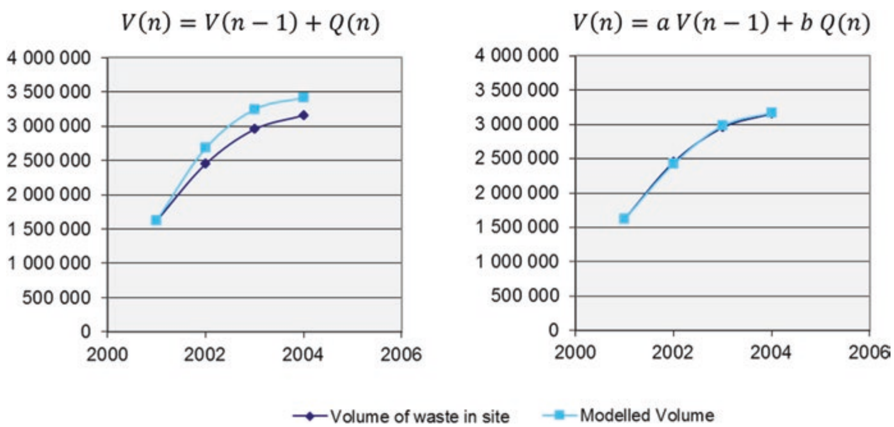

Fig. 6 Modelling the landfill problem with recurrence relations

quite intuitive, based on day-to-day experience or common sense. Generally one looks for what could change and how, what could cause growth or reduction, where proportionality might apply, or what would remain constant in a situation of change. The idea here is not so much to find "the right model" or to identify "the right principles" up front, but to develop this cyclic approach of formulating principles and assumptions, expressing them mathematically, reviewing them with the data, and refining them if deemed necessary.

It is also worth noting that the model that was kept by the students as the most useful for their landfill problem was not expressed as a one variable real function, but stayed in the form of a recurrence relation. In fact, it is often much easier to characterize change than it is to come up with an adequate closed-form expression for the evolution of some variable over time, especially if that variable depends on the evolution of many other variables. This is the reason for the ubiquity of differential equations in modelling nature; we may know locally how entities influence each other or interact with one another, but it is much more difficult to predict the global outcome over time of their multiple interactions. There may not even be an analytical function to describe such evolution. But when recurrence relations are used to characterize change over time, the solution can be generated as a sequence, and this can be accomplished very efficiently with controlled use of technology (e.g., with spreadsheets).

As another element of interest in the selected model, the evolution of the waste volume depends on another variable (the amount of waste routed to the site) for which very little is known, except that it seems to decrease linearly over time. As this variable is controlled by human decision and may reflect conditions external to the system considered here (the availability of other landfill sites, the cost of diverting waste to these alternate sites, etc.), leaving this variable free and testing different scenarios was thought to be a better option than extrapolating it from the line of best fit. Recurrence relations also allow for such flexibility, and they also support the introduction of random variables.

One can also make use of system dynamics modelling software (such as Stella, VenSim or Insight Maker) to model the evolution of a situation where many variables interact. With these tools, models are built as diagrams through an iconographic interface: cumulative variables are modelled as *stocks*, and their interaction with the rest of the system as incoming or outgoing *flows*, which can be modulated with other variables, functions and equations. The tools are responsible for integrating over time the cumulative effects of these interactions. As the size of the time step is set by the user, one can move from discrete to "almost continuous" models, from difference equations to differential equations. As a possible way of modelling the landfill problem, the waste volume could be made a stock for which the routed and freshly compacted stock would act as inflow, while compaction of the old waste could be modelled as a "loss" (or an outflow) of $(1 - a) \times$ (old waste volume) (Fig. 7).

As models like this can easily be grown to include more variables and interactions, the system dynamics approach has been used by biologists, geologists, economists, to simulate a wide array of phenomena ranging from alcohol absorption or

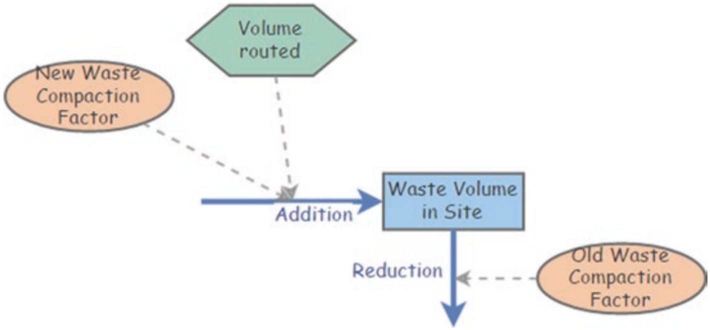

Fig. 7 Modelling the landfill problem with system dynamics software (Insight Maker)

spread of Lyme disease to the dynamics of societal inequality or the impact of fossil fuel burning on carbon dioxide concentration in air. Introducing students to this modelling approach could help them envision their possible contribution in tackling some of todays' major challenges and the role mathematics could play in that respect.

Enriching Modelling in Secondary Mathematics

Valuing *structuring* and *mathematisating* in school mathematical modelling activities is not out of reach, as it can build on what is already done.

For one thing, recursion is already called upon, explicitly or implicitly, when students learn about the properties of some of the functions (linear, polynomial, exponential), or when they determine the probability of multiple events in traversing a tree diagram. In fact, when they are introduced to algebra from pattern sequences, students naturally approach such sequences in terms of recurrence relations (e.g., "it's always 3 more than the one before"). But this way of modelling tends to disappear once the "real" function has set in, as students are often guided into replacing such expression of the relationship with a closed form for the general term, a form that will bring them closer to the function. In a way, students learn to move away from expressing change, as if there were always a direct analytical expression that could help predict the future. There may be a need to reaffirm the relevance of recurrence relations and difference equations as legitimate and powerful tools for modelling a situation, even within a curriculum that tends to focus on continuous functions. In the same way that we value multiple representations to describe the same mathematical object, students should also learn to value the multiple ways of establishing a relationship between variables, as one can enlighten the other. For instance, they could make use of systems dynamics software to investigate what happens when the value of the inflow is made proportional to the value of the stock, and have exponential functions emerge from such exploration.

Then they could learn to add the effect of a limiting capacity on the growth rate and generate the logistics function.

Evidently, this is not to say that functions cannot or should not be used for structuring a situation. Simple functions can often describe different aspects of a complex situation. For instance, power functions and trigonometric functions are particularly useful in structuring space through geometric relationships. As shown in the activity proposed by Van Maanen (1991), an elaborate model that falls outside the regression capacities of a typical technological tool can be built through combination of simple functions, linking them with operations, variable substitution or function composition. Exploring geometry problems with dynamic geometry software can also serve as context for learning to model change through structuring, a context that may feel more manageable to address in a mathematics class.

But one should not overlook the value of using real life situations with their intrinsic complexity and authentic data, as this is where the need for simplification and structure is most solicited; this is where modelling can be quite creative and empowering. Generalized Fermi problems (*How many dental hygienists should there be in a city?*) or generic fairness questions (*What should be a fair price for an airline ticket?*) can be interesting entry points for learning to identify variables and express assumptions and principles. Although there may be a tradition of calculating without units in mathematics, maintaining their use throughout a modelling activity, as do our colleagues from other disciplines, can only help validate the relationships that are built. And one cannot overestimate the role of validation in maintaining control over a problem for which there is no longer a unique and straightforward solution.

References

Blum, W., et al. (2002). ICMI Study 14: Applications and modelling in mathematics education – Discussion document. *Educational Studies in Mathematics, 51*, 149–171.

Galbraith, P. (2007). Beyond the low hanging fruit. In W. Blum, P. Galbraith, M. Niss, & H.-W. Henn (Eds.), *Modelling and applications in mathematics education: The 14th ICMI Study* (pp. 79–88). New York: Springer.

Garfunkel, S., & Montgomery, M. (2016). *Guidelines for assessment and instruction in mathematical modeling education*. Bedford/Philadelphia: COMAP/SIAM.

Maull, W., & Berry, J. (2001). An investigation of student working styles in a mathematical modelling activity. *Teaching Mathematics and its Applications, 20*(2), 78–88.

Nova Scotia Department of Education and Early Childhood Development (NSDEECD). (2015). *Mathematics 12* (Implementation Draft, June 2015). Retrieved from http://www.ednet.ns.ca/files/curriculum/Math12-ImplDraft-June2015-web.pdf

Van Maanen, J. (1991). L'Hôpital's weight problem. *For the Learning of Mathematics, 11*(2), 44–47.

Western and Northern Canadian Protocol (WNCP). (2008). *The common curriculum framework for grades 10–12 mathematics*. Edmonton: Alberta Education. Retrieved from https://www.bced.gov.bc.ca/irp/pdfs/mathematics/WNCPmath1012/2008math1012wncp_ccf.pdf

Learning Algebra with Models and Reasoning

Ann Kajander

Abstract In traditional mathematics classrooms, the use of representations is often restricted to the "show your work" instruction, perhaps involving a diagram or graph. As such, models and representations may have a more limited use. Mathematicians however, often use rough diagrams, gestures, etc. to think about and discuss mathematical ideas as they evolve. Hence in a problem-based classroom, encouraging students to make use of such models and representations as tools to think with can have a powerful effect. Such a use is very different from the assumption that models, such as in the form of physical classroom manipulatives, are tools to support struggling students. Rather, this chapter takes the stance that a vigorous use of models and reasoning can be highly mathematical, and effective for all students.

Many current Canadian curricula, as well as the *Principles and standards for school mathematics* (NCTM, Principles and standards for school mathematics. Author, Reston, 2000), have both Representation as well as Reasoning listed as learning processes. The use of models and modelling can support both of these processes.

As well as providing some background, this chapter provides a specific classroom example, which may be useful in grade 9 or 10 classrooms. The lesson has been field tested in both grade levels, as well as in relatively more and less high level courses. Without exception, students have been able to discover the rules for factoring a simple quadratics for themselves. As well as generating a useful factoring method, students are left with a conceptual understanding of how and why the method works, which may be helpful in terms of not having to memorize a mysterious procedure.

A secondary benefit of using models in learning may be that they may be a structure that help teachers move more and more towards encouraging increased student autonomy in learning. Once students have some effective tools with which to think, they may be able to become increasingly independent as learners.

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Keywords Mathematics teaching · Mathematics learning · Models and modelling in learning · Models and reasoning · Teacher development · Learning algebra

The purpose of this chapter is to examine a *models and modelling approach* (Lesh and Doerr 2003) in classroom learning by exploring a sample mathematics topic. Such an approach may also support the transition of teacher practice from a more traditional paradigm to a conceptual one. A sample lesson using a models and modelling approach (sometimes also called a models and reasoning approach, the term used here to emphasis the active nature of it) is provided at the end of the chapter to illustrate the concept.

Previous research (e.g., Holm and Kajander 2015; Kajander et al. 2008) suggests that it may be risky for teachers to attempt a problem-based lesson without building, for themselves and subsequently their students, the necessary mathematical understandings particularly of the mathematical processes. A models and reasoning approach may support such development, for both teachers and students. Further, models can be used in relatively more teacher-directed learning, as well as being helpful in supporting more and more student autonomy. Hence a models and reasoning approach may also be a helpful learning paradigm to use as teachers begin to shift their practice because the approach can be used with relatively more, as well as less, teacher direction and control.

Learning with Understanding

A cornerstone assumption of problem-based learning is that, through the construction of meaning and understanding in a rich environment, students will develop conceptual understanding rather than isolated and decontextualized rule-based skills. An interesting topic of professional discussion is to brainstorm reasons why some experienced teachers remain hesitant to change their practice in the face of this mounting evidence; concerns may include nervousness about the right skills being learned, worries about curriculum coverage, beliefs about what students are and are not capable of, and perceived lack of classroom control. It may initially feel daunting to try to create lessons that involve students wandering around the classroom, using materials, and making noise, when one is used to having strict control with students sitting quietly at individual desks working on assigned textbook questions. Yet, even more traditionally oriented teachers tend to agree that it is important that students develop conceptual understanding, gaining a sense of *why* the rules and methods make sense. The tension between these two sets of goals and beliefs may contribute to the slowness with which mathematics reform is being adopted at the secondary classroom level.

One way which may be helpful in supporting the transition of classroom environments from traditional structures to more conceptually oriented ones is the use of a *models and modelling approach* in learning (Lesh and Doerr 2003), referred to here as a models and reasoning approach. In such an approach, different types of carefully selected concrete or visual models allow and encourage students to explore problems with gradually increasing autonomy. It is common in even traditional practices to ask students to finish or "explain" a given answer by including a model in the form of a graph or even a diagram, where the model is to be used to simply *explain* an answer in the sense of verifying it. However in the context of this chapter, and in line with what mathematicians often do when working on a new problem, a model can also be used as a *tool with which to think and reason*, and it is this latter use that is of interest here. This latter interpretation involves learning processes such as those mentioned in many Canadian curricula (e.g., Ontario Ministry of Education 2005), as well as by the National Council of Teachers of Mathematics (NCTM 2000), namely Representation, and also Reasoning and Proving. Using these processes, we are able to help students understand the processes that mathematicians often use in their work, such as developing a *representation* or model of a given problem, and using it as a tool to help *reason* about a given problem—as opposed to simply "explaining" an answer after the thinking has taken place.

It should be noted that a mathematical model cannot only be a diagram or physical model made with manipulatives, but it can also be a mental image, and later, even an algebraic model. Gesture may also play a role (De Freitas and Sinclair 2014). The modelling process then involves not just the construction of a suitable model, but its use, modification and manipulation, which are all part of the thinking process—and thus the modelling process draws in the mathematical learning and thinking processes of reasoning and proving along the way.

As mentioned, the use of models and modelling, or models and reasoning, as an approach to learning can be used even to some extent in a more teacher-directed classroom, and in this sense their use may also support a more gradual and comfortable transition to increased levels of problem-based learning. For example, a teacher might propose a problem to students, and then ask them to create a model to help think about the problem. Teachers can support the process by offering suitable manipulatives, or reminding students of other types of models used previously. These representations can then be shared in a whole group discussion. Ideally, students would continue to work on the problem themselves using a model of their choice, but this process may also be handled in a slightly more Socratic manner with the teacher asking suitable questions to prompt constructions. Teachers and students can work collectively to develop more and more student voice in the explorations.

A great benefit of the use of models is that they may make mathematical processes both evident, as well as, visibly interconnected. The argument that problem-based learning and hands-on learning takes too long is contradicted by the benefit of the creation of connections of the ideas to other ideas, often supporting more than one set of curriculum expectations at a time. One of many possible examples of such a classroom use of models in learning is summarized in the sample lesson to follow, aimed at the early secondary level.

Sample Lesson: Discovering the Rules for Factoring a Quadratic

The example to follow is simple for the teacher to set up, and directly builds on earlier conceptions of a particular mathematical model called the *area* model. The game-like quality of the lesson appeals to many intermediate students, and no algebra whatsoever is needed in the first part of the activity, giving it a low floor and thus making it accessible to many learners.

While it is hoped that it would now be rare that students had neither seen an area model to represent multiplication, nor the use of algebra tiles by grade 9, these structures may still be used less frequently than might be ideal (Holm and Kajander 2015). Thus, before embarking on the lesson to be described, a brief review of these ideas from the elementary curriculum may be needed for the students, and a summary of such a recap will be included to follow, to which the reader may refer as needed. The following outline is presented to review the salient development of content around this model in the curriculum from the elementary level up to early secondary. Depending on what students recall (or ever experienced), some aspects of these concepts may need reviewing with your students. In my own past classroom practice, I found that at least, a model of an example such as 23 × 14 was helpful in reviewing how two factors (say 23 and 14) generate an area and hence the product. Algebra tiles can then be introduced (or reviewed) and the idea extended to binomial products such as (x + 3) (x + 4).

The Area Model as the Product of Two Numbers

From early on in elementary mathematics, children understand 2 × 3 as *two groups of three* or *three groups of two*. It is not immediately obvious that these are the same. However, when the operation is viewed using an *area model* (see Fig. 1) we can identify the ideas of the two factors (here, 2 and 3), the product (here, 6), and the commutativity of multiplication, all embodied in this powerful representation. The factors are the side lengths, and the product is the area—and this is the fundamental premise of the model. More examples allow students to come to the conclusion that the construction of this model can always be used to model the process of multiplication of two quantities.

Moving along in grade level from primary to junior (middle elementary) grades, the area model is a representation that allows the development of double digit

Fig. 1 Area model of 2 × 3

Fig. 2 A ten and a hundreds piece

Fig. 3 Sample numeric methods

24	24
× 35	× 35
600	20
120	100
100	120
20	600
840	840

multiplication methods. For example, using base ten blocks, we recall that the ten or "long," as well as the 100's piece or "flat" (a 10 by 10 square) can also be used to illustrate 10 and 100 respectively (Fig. 2).

For example, creating an area model to represent 24 × 35 yields a rectangle with four distinct regions, namely 600, 100, 120 and 20. Take a moment yourself to sketch (or build) the rectangle using base ten blocks (or sketches of them) and locate these four sub-products, as well as how they are related to the initial two numbers. Explore a bit further to convince yourself that these sub-products also exactly combine in a prescribed way to generate the traditional multiplication procedure. (By the way, after learning about multiplying using an area model, you may find yourself preferring to multiply by a method more similar to a numeric version of the model, i.e., by adding the four sub-products in any order—I know I do!) A few of the possible numeric methods are shown in Fig. 3—and others are possible. Note that the second example shown in Fig. 3 aligns most directly with the "traditional" method, although the first one makes more sense in some ways—if we consider the largest sub-product as the most important quantity in terms of its effect on the answer, then this order of working is logical.

The beauty of the area model is that its use extends to the development of algebraic ideas. Indeed the area model with two digit factors is almost directly generalised into the area model for binomial products such as $(x + 4)(x + 5)$ or even $(2x + 4)(3x + 5)$, making rules about how to expand these expressions obsolete. The key here is to have students imagine the *variable* length piece as a length *that can change*. When introducing these ideas to students, take a moment to have your students visualise the "x" piece growing and shrinking in their minds, or demonstrate it using virtual algebra tiles. Figure 4 shows various "lengths" of such a variable tile.

Next, we need to think about how to represent x^2. Given that 100 (or 10^2) is a 10 by 10 square, it makes sense that x^2 is also a square, with side length equal to the

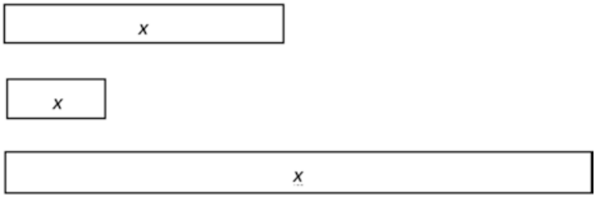
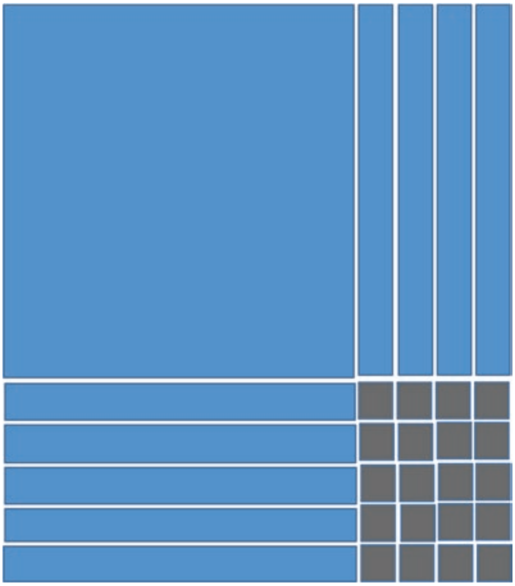


Fig. 4 Different representations of the variable x

Fig. 5 The area model for $x^2 + 9x + 20$



length of x. So now we are ready to model $(x + 4)\ (x + 5)$. Before reading further, consider drawing a sketch yourself, or building a model with algebra tiles, of this expression.

Figure 5 shows one representation of such a model. The side lengths of the rectangular model are $(x + 4)$ and $(x + 5)$. Adding up the pieces here, we see that the model of $(x + 4)\ (x + 5)$ contains one x^2 piece, nine x pieces, and 20 units. The expanded expression is thus $x^2 + 9x + 20$. We see that the expression $x^2 + 9x + 20$ is 'factorable' because the region can *be made into an exact rectangle*. Hence there are two terms (the side lengths) which can be multiplied together to yield the area. In other words, there are two (here, binomial) expressions which form the side lengths of a rectangle, whose product is the expanded form, shown in the area.

The concept of factoring is really the same idea (in reverse) as expanding, it's just we are moving in the opposite direction. So the question "does $x^2 + 9x + 20$ factor, and if so what are the factors", relates to the exploration "if I have an x^2 piece, nine x pieces, and 20 units, can they be arranged into a rectangle?" Before even mentioning any of this—or even the word "factor," students might be offered the following activity to follow. We are now ready for the actual classroom lesson set up!

Sample Lesson—Make a Rectangle Activity

Set Up (Teacher Preparation)

A classroom set of algebra tiles, together with some ziplock bags (one filled bag per pair of students) is all that is required. In each bag, place sufficient algebra tiles to form a factorable expression, such as (using the previous example) an x^2 piece, nine x pieces, and 20 units. Fill enough bags with different similar polynomials so that there are enough bags for each pair of students. A simple way to do this is to take a list of textbook factoring practice problems and use those as a guide. However, I highly recommend including at least a few that are not factorable—it is fairly easy to take a factorable expression and simply add or remove something from the final constant to create an unfactorable expression. For example, an expression such as $x^2 + 5x + 6$ can be easily made unfactorable by adding one more unit tile. You may wish to number the bags so students know if they have used a given set yet.

Introduction

Before beginning, review algebra tiles (and other content as above as necessary). Students should be able to recognise the x and x^2 pieces in the bags, as well as of course the units. Students are to work in pairs, each being provided with a ziplock bag of materials as described. First, ask the students to explore the contents of the bag they received, and identify the pieces.

Task

Students can be challenged to see if the tiles in their bag can be arranged into an exact rectangle and record their data about each example. If a rectangle can be made, they are also to record the side lengths of the rectangle. They are then to trade bags with another student pair and try another set. The goal of the lesson is for students to be able to devise a way to predict *if the contents of a given bag can be arranged into a rectangle (eventually without actually doing so), and if so what the side-lengths of the rectangle are (again, predicting without actually needing to build it)*. Once students have articulated a conjecture, they are to test it on several more cases.

At some point during the discussion it may become clear to students that they are in fact devising the rules of factoring simple quadratics, as some students will recall that the side lengths of the rectangle are the *factors*. It will likely be helpful to actually have this whole group discussion at some suitable point during or after the exploration. In a lesson structure called a three part lesson, such a sharing or consolidation phase is the third part of the lesson. It can be the hardest to plan because it must emerge from the students' observations. However, teachers must know what to expect, and how to probe students to share their new understandings.

Sharing

Student pairs can be asked to share their methods for predicting the factors with the whole group. The sum-product factoring method easily emerges as the shared outcome; however, its understanding is drawn from a visual model which may endure in students' minds and help with future recall. It is also important that the ways the idea is stated in the classroom are drawn from the students' descriptions of their methods. Terminology and efficiency can be added later. Student conceptual understanding is the first goal.

Summary

The one sample lesson provided in this chapter was meant to be illustrative of the use of models and reasoning as the basis of a lesson. While it is still the case that some teachers consider the use of manipulatives and models as a crutch for weaker students (Holm and Kajander 2015), it is argued here that the creation of models and their use in problem solving is an important and high level mathematical process. Many similar such lessons based on models are possible (for examples see Kajander and Boland 2014). Technology may also be helpful in model creation but it is also possible to deeply understand many ideas using physical manipulative constructions. As further examples of the many possibilities, completing the square literally can mean just that, and the definitions of many of the conics lend themselves readily to physical models, involving perhaps thread, Styrofoam and pins. And as your lessons become more and more about (literally) *constructing* knowledge, so too will everyone's deep understanding grow—including likely that of the teacher!

References

De Freitas, E., & Sinclair, N. (2014). *Mathematics and the body: Material entanglements in the classroom*. New York: Cambridge University Press.

Holm, J., & Kajander, A. (2015). Lessons learned about effective professional development: Two contrasting case studies. *International Journal of Education in Mathematics, Science and Technology, 3*(4), 262–274.

Kajander, A., & Boland, T. (2014). *Mathematical models for teaching: Reasoning without memorization*. Toronto: Canadian Scholars' Press.

Kajander, A., Zuke, C., & Walton, G. (2008). Teaching unheard voices: Students at risk in mathematics. *Canadian Journal of Education, 31*(4), 1039–1064.

Lesh, R., & Doerr, H. (2003). *Beyond constructivism: Models and modeling perspectives on mathematics problem solving, learning, and teaching*. Mahwah: Lawrence Erlbaum.

National Council of Teachers of Mathematics. (2000). *Principles and standards for school mathematics*. Reston: Author.

Ontario Ministry of Education. (2005). *The Ontario curriculum: Grades 9 and 10 mathematics* (Rev. ed.). Toronto: Queen's Printer for Ontario Retrieved from http://www.edu.gov.on.ca/eng/curriculum/secondary/math910curr.pdf.

Additional Suggestions for Further Reading

Kajander, A. (2013–2016). MB4T: Math by and for teachers. Monthly column in the *Ontario Mathematics Gazette*.

In No Uncertain Terms: Encouraging a Critical Stance Toward Probability in School

Nat Banting, Ilona Vashchyshyn, and Egan J Chernoff

Abstract In this chapter, we question the tendency in secondary schools to present the notion of probability exclusively through scenarios where all necessary information is readily available and appears to be wholly reliable. Such scenarios create the impression that probability is a fixed attribute of the objects that generate an event, obscuring the larger idea that the probability of an event is, rather, an enumeration of available information about the event in question, which may be subject to change. We highlight a family of problems suitable for use in the secondary school classroom where determining a solution requires not only a consideration of possible outcomes, but also the uncovering of assumptions regarding the process through which information about the event was obtained. The ambiguity provided through the presence of a middleman illustrates how differing sets of information may affect probability assessments, and encourages students to take a critical stance toward probability calculations.

Keywords Probability · Probability literacy · Secondary school mathematics

The widespread adoption of the study of probability into Canadian secondary school programs of study is about to enter its fourth decade. The topic became widely recognized as an important aspect of a robust education in mathematics through a sequence of standards documents from the National Council of Teachers of Mathematics—first, the *Curriculum and evaluation standards* (1989), and later, the *Principles and standards for school mathematics* (2000)—that cemented

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expectations for data analysis and probability in the North American school classroom. Similar reform documents were issued at this time around the globe (see Jones et al. 2007), including the Western and Northern Canadian Protocol for Collaboration in Education (WNCP 2006, 2008).

However, although the topic has now become an established feature of mathematics curricula across Canada, we suggest that the nature of probability is often left unexplored, and potentially misrepresented, through problems that are composed entirely of events arising in overly-simplified contexts, whose underlying processes are either fully known or assumed to be fully known. Students are then left with the impression that probability is an unchanging mathematical feature of an object, where every possible event has a single, true probability of occurring that can be calculated from the given set of information. Through such sanitized contexts, the teaching and learning of probability tends to misrepresent the nature of probability, which is not a characteristic or attribute of the object or process that generates an event, but rather a quantification of information about the event in question. Providing problems where students have the opportunity to interrogate the assumptions underlying given information may help students appreciate the nature of probability as a statement of certainty, susceptible to change based on new information and on assumptions about what, how, and when information was obtained (Falk 1992; Nickerson 1996). The opportunity to engage with such problems may additionally help to foster a critical stance towards probabilistic claims and encourage the development of probability literacy. In this chapter, we highlight a family of problems that may provide students with opportunities to inquire into the nature of probability and to foster the development of a critical stance regarding probability claims.

The Nature of Things

> A standard, fair, six-sided die is rolled. What is the probability that the die lands 6-up?
> Two standard, fair, six-sided dice are rolled. If one of the dice lands 6-up, what is the probability that the sum of the two dice is 7?

Whether they feature dice, cards, coins, or balls in urns, probability problems in schools all tend to be presented in a similar way. First, a random generator is offered (in the above examples, a die), which is either declared to be fair, standard, or, in the rare case, explicitly given to be biased in a predictable and known way. If a reporter of information is named, they are assumed to have presented the information in an accurate, unbiased way. Second, although not always, a conditioning event is provided (in the second example above, it is revealed that one of the two dice lands 6-up). Finally, *the* probability of an event involving the random generator(s) is requested.

Sanitized, seemingly free of ambiguity, and asking for the learner's input only at the final step, probability problems in schools tend to eliminate any possibility of interrogation. Such problems are carefully worded so as to present clear and straightforward parameters in which the given probabilistic experiments take place.

We thus become habituated to think of every deck of cards as a standard deck of cards, every die as a fair die, and every reporter of information as unbiased. Even when there appears to be a gap in the given information (as we will present in this chapter), it is often left unexplored, and any space for interrogation is filled with unarticulated assumptions. Consequently, such problems tend to, somewhat paradoxically, eliminate any and all uncertainty about the processes underlying events, the information known about the events, and how that information came to be known by the solver. As a result, the only uncertainty that appears to remain is the outcome of the experiment.

There is an additional, unintended consequence of encountering the notion of probability through such sanitized situations: the implication that every event has one true, unique probability of occurrence. For instance, in the first example above, "the" probability that a standard, fair die lands 6-up when it is rolled is typically determined as follows: A standard die has six sides, numbered from 1 to 6, and each side has an equal chance of being rolled, given that the die is fair. The desired outcome is one among six possible outcomes, and therefore "the" probability is 1/6. While the second example presents a slightly greater computational challenge, the solution may be determined in a similar way. In both cases, and in the absence of further analysis, the computation of probability appears to uncover an attribute of the die, in the same way that we might measure its height, mass, or density. However, unlike height, mass, or density, probability is not a static feature of an event, but a dynamic one that changes depending on the status of our knowledge about the event in question.

As an example, consider the following simple classroom experiment proposed by Devlin (2014). A class of students is divided into two groups, and one group is given a die. The die is rolled and all of the members of the first group see the outcome, whatever it happens to be. The instructor then asks the second group, who did not see the outcome of the roll, to assign a probability to the event that the die landed 6-up. The theoretical calculation, based on the symmetrical attributes of the die, results in a response of 1/6. However, if the instructor assigns the same task to the first group, who saw the outcome of roll, they are forced to respond either with a probability of zero or one, because for them, there is no uncertainty about the outcome. It is easier to see that the assigned probability, from the perspective of the first group, is based on available information regarding the scenario, rather than a preassigned theoretical value. However, *both* evaluations are in fact based on an enumeration of information, with the second group being privy to less information than the first. In other words, no probability exists within the die; the computation is a measure of what a group knows about the properties of the die with regards to the toss in question. It is in this sense that probabilities quantify our information about the world at any given moment in time.

However, in the absence of opportunities to gather information and question the assumptions underlying the calculations that generate probabilities, students are liable to develop the belief that events, both inside and outside the classroom context, have a unique probability (Devlin 2014). Even Canadian curriculum documents have advocated this stance, stating, for example, that "many important properties in mathematics do not change when conditions change. Examples of constancy include […] the theoretical probability of an event" (Western and Northern Canadian Protocol 2008, p. 11). However, in our world, events either happen or do not happen, and consequently, probability cannot be a tangible characteristic of events, but rather represents "the *status of our knowledge*" based on the current set of given information, and allows us to build a stochastic model of what we perceive to be the reality (Borovcnik and Kapadia 2014, p. 42). In other words, probability is a quantified perception, an enumeration of the information available about a future event that one is privy to in advance (Devlin 2014; Gal 2005).

This fact is obscured by problems that present a world to learners that is assumed to be fully-known and unquestionable. In sum, problems of probability in schools tend to be, paradoxically, presented in no uncertain terms. It is this typical presentation of probability through contexts where all information is readily available and appears wholly reliable that denies students the opportunity to pose critical questions about the underlying processes or assumptions through which the information was obtained.

Probability Literacy

While the distinction between probability as a characteristic of an event and probability as the quantification of information about an event may seem overly exacting or too abstract to be relevant at the school level, experiencing probability in this vein is critical. According to Konold (1989), viewing events as having a true, unique probability results in the pursuit of certainty that a probabilistic hypothesis is unequivocally correct—that is, not liable to change. However, if students focus on labeling events with specific, static probabilities but ignore the conditions under which these probabilities arise, Konold contends, somewhat hyperbolically, that "all of probability theory will evade them" (p. 92). Given this warning, the provision of tasks with the potential to inquire into the nature of probability as a quantification of available information is of importance to the teaching and learning of probability. The additional value of such tasks, where assumptions must be unearthed and the effects of differing sets of information may be observed, has the potential to help students foster a critical stance towards probabilistic claims, a key aspect of probability literacy.

A relatively recent addition to the gamut of literacies, a working model of probability literacy was proposed by Gal (2005), which includes both knowledge elements and dispositional elements (see Table 1). Notably, figuring probabilities (i.e., technical or computational skills) and language (i.e., terms related to the quantification of uncertainty, such as *possible*, *probable*, and *certain*) are acknowledged to be key elements of probability literacy. However, these aspects tend to constitute the overwhelming focus of the study of probability in schools (Jones et al. 2007), to the exclusion of other elements of probability literacy. And, as Gal (2005) argues, "an

Table 1 Probability literacy – building blocks (Gal 2005, p. 51)

Knowledge elements
1. *Big ideas*: Variation, randomness, independence, predictability/uncertainty.
2. *Figuring probabilities*: Ways to find or estimate the probability of events.
3. *Language*: The terms and methods used to communicate about chance.
4. *Context*: Understanding the role and implications of probabilistic issues and messages in various contexts and in personal and public discourse.
5. *Critical questions*: Issues to reflect upon when dealing with probabilities.
Dispositional elements
1. *Critical stance.*
2. *Beliefs and attitudes.*
3. *Personal sentiments regarding uncertainty and risk (e.g., risk aversion).*

instructional focus only on one or two of the elements will not be sufficient to develop 'probability literate' behavior" (p. 50).

We highlight, in particular, the limited opportunities that students have to ask critical questions and to develop a critical stance when engaging with typical problem situations involving uncertainty. As Gal (2005) notes, most students will go on to become adults who are consumers, rather than producers, of probabilistic and statistical information, so in a world where stochastic information may be, and has been, used to distort or deceive, it is essential that students not only gain proficiency in computing probabilities, but also develop a "critical perspective on information one receives from presumably 'official' sources or from experts" (p. 49). This requires both the knowledge of and the disposition to ask a number of critical questions about the context, source, and process used to arrive at the claim being made.

Importantly, teachers cannot assume that students who have been exposed to a variety of computational exercises will consequently be able to ask such questions and, in general, think critically about diverse probabilistic situations (Gal 2005; Jones et al. 2007). We are guided by the understanding that "if children are to develop beliefs such as 'it is legitimate to be critical about probabilistic messages,' these kinds of messages need to be built into the content of tasks and experiences that the students face" (Jones et al. 2007, p. 942). In this chapter, we propose one way in which the teaching of probability may provide students with opportunities to ask critical questions and to foster the development of a critical stance regarding probability claims. Our focus is on revealing and evaluating the consequences of various assumptions embedded in problem situations involving uncertainty. More precisely, the problems discussed in this chapter focus on assumptions surrounding the process by which information came to be known. As Nickerson (1996) argues, both scientists and laypersons need to understand that probabilistic claims are often based on unstated assumptions, and that an essential part of determining whether to accept the conclusion of a probabilistic claim is to make those assumptions explicit. Equally valuable is the important insight into the nature of probability that such problems offer: If different sets of reasonable assumptions can lead to different probabilities, then probability cannot be an inherent, empirical quality of an outcome, but rather a measure of the information about the uncertain outcome that we have at our disposal.

Hidden Assumptions in Middleman Problems

We now turn our attention to a family of problems that demonstrate how different assumptions and additional information can result in a revision of the probability assigned to an event. We explore the possibilities in several of these problems below. Despite differences in the cover story, the basic mathematical structure of such problems is the same: an uncertain target event is presented; additional information, involving another event in the sample space, is then provided; and finally, the reader is tasked with finding the revised probability of the target event (Falk 1992). These problems also include the presence of a key character whom we will call the *middleman*, an agent who relays information and can exercise some selectivity in what information they report.

The presence of a sentient, human agent who makes decisions and provides information to the solver plants a seed of further unpredictability, and can serve to demonstrate how seemingly small situational differences can prove consequential in determining probabilities. To this point in the literature, such non-routine probability problems and the reasoning they sponsor have largely been studied from a psychological perspective. We suggest, however, that they also have the potential to offer key educative opportunities to be critical of information, to discuss ambiguities, and to explicate assumptions with regard to the information provided to the solver, all the while illustrating how probability is dependent on the assumptions made about the objects in question and the process by which information came to be known. In the next section, we demonstrate how a middleman can be introduced into a classic problem situation, the Three Cards problem, which allows us to explore how different assumptions about the middleman can lead to different probabilities of the desired event.

The Three Cards Problem

The Three Cards problem is a classic problem in probability involving one card that is red on both sides, one card that is blue on both sides, and one card that is blue on one side and red on the other.[1] One of the cards is selected at random (say, out of a

[1] The original formulation of the Three Cards problem was proposed by Joseph Bertrand in his 1889 work *Calcul des probabilités*. The problem involved three boxes: one box containing two gold coins, a second box containing two silver coins, and a third box containing one gold and one silver coin.

hat), and only one side of the card is observed. The problem that readers are tasked with is the following: If you draw the card and observe one of the sides to be blue, what is the probability that the other side is blue?[2] As stated, the problem is virtually unambiguous. A typical solution is as follows: Since the card cannot be the red-red card, it must either be the blue-blue or blue-red card. Among these possibilities, there are four sides, three of which are blue and only one of which is red. Given that one of the sides has been observed to be blue, the probability that the other side is blue is 2/3.

However, what happens if we introduce a middleman to relay information about one of the sides? Let us suppose now that a friend, Shania, pulls a card out of the hat without showing it to you, and states that at least one of the sides is blue. What is the probability that the other side of the card is blue? With the subtle change of the information now being provided by a third party, the problem becomes ambiguous from a computational point of view. To start, the problem as stated suggests that "the" unique probability of the other side being blue is somewhere out there to be found. However, we know that this cannot be so, because from the perspective of Shania, who is holding the card and can see both sides, the probability is either 0 or 1—either the other side is also blue, or it is not—while the probability is most likely somewhere between these two values for the observer, since they have not seen both sides of the card. (This situation is analogous to Devlin's dice experiment, described earlier.) Let us assume, then, that we are interested in the probability of the other side being blue from the perspective of the person who has not drawn the card. Let us also assume that this friend is honest, and will not, for example, state that one of the sides is blue if she is holding the red-red card. A third ambiguity, to be explored below, is how Shania decides to choose which colour to report.

Keeping in mind a variety of possible ways in which Shania may choose to report information, consider the following four cases.

(i) *Shania is colour-unbiased*. In this situation, we will assume that Shania picks a card randomly, looks at both sides, and does not show it to you. We will also assume that if she is holding the red-red card, she will say "at least one of the sides is red"; if she is holding the blue-blue card, she will say "at least one of the sides is blue"; and if she is holding the blue-red card, she will secretly flip a coin and say "at least one of the sides is blue" if she gets heads (P = 1/2) and "at least one of the sides is red" if she gets tails (P = 1/2).

The possibilities and probabilities are enumerated in Table 2. Consider, for example, the case that Shania draws the blue-blue card and reports "at least one of the sides is blue." Drawing randomly, she will draw the blue-blue card with probability 1/3, and upon drawing this card, will always say "at least one of the sides is blue"; knowing that she will always do so, we assign this event a probability of 1. Hence, the probability of this scenario occurring is (1/3)(1) = 1/3, which appears in the corresponding cell in Table 2. As another example, consider the case that Shania

[2] Our brief treatment of this problem is not meant to suggest that the above solution is intuitive (see, e.g., Bar-Hillel 1989; Bar-Hillel and Falk 1982; Rubel 2002).

Table 2 Probabilities for the Three Cards problem given that the reporter is colour-unbiased

	Shania reports		
Card drawn	Blue	Red	Total
Blue-Blue	$\frac{1}{3} \times 1 = \frac{1}{3}$	$\frac{1}{3} \times 0 = 0$	$\frac{1}{3}$
Red-Red	$\frac{1}{3} \times 0 = 0$	$\frac{1}{3} \times 1 = \frac{1}{3}$	$\frac{1}{3}$
Blue-Red	$\frac{1}{3} \times \frac{1}{2} = \frac{1}{6}$	$\frac{1}{3} \times \frac{1}{2} = \frac{1}{6}$	$\frac{1}{3}$
Total	$\frac{1}{2}$	$\frac{1}{2}$	1

draws the blue-red card and reports "at least one of the sides is red." Drawing randomly, she will draw the blue-red card with probability 1/3, and upon drawing this card, she will secretly flip a coin and say "at least one of the sides is red" if she gets tails, which occurs with probability 1/2. Hence, the probability of this scenario occurring is (1/3)(1/2) = 1/6, which appears in the corresponding cell in Table 2.

Assuming you are privy to the information that Shania is colour-unbiased, the probability that she is holding the blue-blue card given that she reports "at least one of the sides is blue" is found by determining the probability that she draws the blue-blue card and reports that at least one of the sides is blue (P = 1/3) and dividing this by the probability that she reports "at least one of the sides is blue" no matter what card she is holding (P = 1/2). Hence, the probability is (1/3)/(1/2) = 2/3. Notably, this is the same probability given for the original problem, computed previously; however, it rests on a particular set of assumptions, without which the modified problem would be ambiguous.

(ii) *Shania is blue-biased*. In this situation, we will assume again that Shania picks a card randomly, looks at both sides, and does not show it to you. We will also assume that if she is holding the red-red card, she will say "at least one of the sides is red," and if she is holding the blue-blue or the blue-red card, she will say "at least one of the sides is blue." In other words, in this situation, Shania will always say "at least one of the sides is blue" when possible, but will never lie. The possibilities and probabilities are enumerated in Table 3.

In keeping with the same process of analysis, the probability that Shania is holding the blue-blue card, given that she is blue-biased and reports that "at least one of the sides is blue," is (1/3)/(2/3) = 1/2. This is, of course, assuming you are privy to the information that Shania is blue-biased. Notably, this probability aligns with the intuitive probability often provided as a solution to the original problem (e.g., Bar-

Table 3 Probabilities for the Three Cards problem given that the reporter is blue-biased

	Shania reports		
Card drawn	Blue	Red	Total
Blue-Blue	$\frac{1}{3} \times 1 = \frac{1}{3}$	$\frac{1}{3} \times 0 = 0$	$\frac{1}{3}$
Red-Red	$\frac{1}{3} \times 0 = 0$	$\frac{1}{3} \times 1 = \frac{1}{3}$	$\frac{1}{3}$
Blue-Red	$\frac{1}{3} \times 1 = \frac{1}{3}$	$\frac{1}{3} \times 0 = 0$	$\frac{1}{3}$
Total	$\frac{2}{3}$	$\frac{1}{3}$	1

Hillel 1989; Bar-Hillel and Falk 1982; Rubel 2002), but depends crucially on the assumption of Shania being biased towards saying blue.

(iii) *Shania is red-biased*. In this situation, we will assume again that Shania picks a card randomly, looks at both sides, and does not show it to you. We will assume that if she is holding the red-red or the blue-red card, she will say "at least one of the sides is red," and if she is holding the blue-blue card, she will say "at least one of the sides is blue." In other words, in this situation, Shania will always say "at least one of the sides is red" when possible, but will never lie. The possibilities and probabilities are enumerated in Table 4.

Assuming you are privy to the information that Shania is red-biased, then the probability that she is holding the blue-blue card given that she reports "at least one of the sides is blue" is (1/3)/(1/3) = 1. Thus, in this situation, you can be certain that Shania is holding the blue-blue card if she states that at least one of the sides is blue.

(iv) *Shania observes only one side*. In this last situation, we will assume that Shania picks a card randomly and only looks at one side, reporting the colour that she sees to you. Shania is not given a decision to make and only gains access to one side of information. This, coupled with the fact that she always reports truthfully, creates a problem that is isomorphic to the original. The probability table is therefore identical to Table 2, even though the assumptions about Shania's process of selecting a colour to report differ. In this situation, the probability that Shania is holding the blue-blue card given that she looks at one of the sides and reports that "at least one of the sides is blue" is (1/3)/(1/2) = 2/3.

Here, we should be clear in noting that our treatment of this problem is not new: The Three Cards problem and other analogous probability puzzles, including the Monty Hall problem, the Three Prisoners problem, and the Two-Child problem,

Table 4 Probabilities for the Three Cards problem given that the reporter is red-biased

	Shania reports		
Card drawn	Blue	Red	Total
Blue-Blue	$\frac{1}{3}\times 1=\frac{1}{3}$	$\frac{1}{3}\times 0=0$	$\frac{1}{3}$
Red-Red	$\frac{1}{3}\times 0=0$	$\frac{1}{3}\times 1=\frac{1}{3}$	$\frac{1}{3}$
Blue-Red	$\frac{1}{3}\times 0=0$	$\frac{1}{3}\times 1=\frac{1}{3}$	$\frac{1}{3}$
Total	$\frac{1}{3}$	$\frac{2}{3}$	1

have been analyzed extensively both in the research literature (e.g., Bar-Hillel 1989; Bar-Hillel and Falk 1982; Falk 1992; Khovanova 2011; Nickerson 1996; Rubel 2006) and in popular culture (e.g., Haddon 2003; Rees and Williams 2011). With the Three Card problem, the sets of assumptions for Shania's reporting represent only a few of those that she may have followed before revealing information about one side of the card; others that result in more exotic probabilities may be envisioned, and present a worthwhile extension task.

The crucial takeaway is that the assumptions that are made about a problem situation *matter*, and that the seemingly intuitive assumptions (e.g., Shania is colour-unbiased) are not the only ones that could be made (nor are they universally intuitive; see Nickerson 1996). As the first and last cases above demonstrate, different sets of assumptions do not always lead to different probability assignments (and yet, they can and do, as the second and third cases reveal). When the problem involves a middleman who provides a piece of information, we should be concerned not only about updating the relevant sample space after new information is provided, but also about the procedure by which the middleman chose what information to give (e.g., Bar-Hillel 1989; Khovanova 2011; Nickerson 1996). As Bar-Hillel and Falk (1982) write, and as we have seen, "information cannot, as a rule, be divorced from its sources" (p. 120), and "what matters for reaching a correct solution to many probability problems is often not only the given information, but also the manner by which it has been obtained" (Falk 1992, p. 217).

Introducing a middleman into the original problem context renders the situation ambiguous. In particular, without receiving more information or introducing assumptions about the middleman's behavior, the solution is indeterminate. As a consequence, no solution to the problem can be said to be *the* correct solution, given that it is sensitive to the assumptions made. Whether any particular set of assumptions is more reasonable than another is a matter of judgment (Nickerson 1996), and our goal is not to champion one set of assumptions over another. On the contrary, we contend that the educative possibility of such problems lies not in choosing the best set of assumptions or even in computing the resulting probabilities, but in *recognizing* the ambiguity and the role assumptions play in their solution. Offering students the opportunity to consider the role of assumptions and to explore their consequences may help to foster a critical stance towards probability statements that is too often neglected in the classroom context.

Unfortunately, in textbooks and curriculum documents, the phrasing "the probability of A given B" and the notation P(A|B) typically sidesteps the issue of how the event B came to be known, since the term 'given' supplies the conditioning event (Bar-Hillel and Falk 1982). As Bar-Hillel (1989) suggests, "the standard textbook problem is of this sanitized type, and when it isn't, this is seldom by design" (p. 352). This is also the case in the original version of Three Cards problem. Phrased as "What is the probability that you are holding the blue-blue card given that one of the sides is blue?", it is difficult to argue that the probability is anything but 2/3, assuming that the cards were well-mixed and the drawing was fair. However, as Bar-Hillel and Falk (1982) argue, such situations are unrealistic at best, and dishonest at worst:

> Outside the never-never land of textbooks […] real-life problems […] need to be modeled before they can be solved formally. And for the selection of an appropriate model (i.e., probability space), the way in which information is obtained (i.e., the statistical experiment) is crucial. (p. 121)

It is valuable, then, to consider how ambiguity might be recognized or reintroduced into typical curricular problem situations so as to allow students to experience how various pieces of information, and the process by which they were obtained, may affect the probability of outcomes.

Unearthing Ambiguity

The examples presented below were drawn or adapted from Canadian resources, illustrating that opportunities to engage students in discussions about the nature of probability and in asking critical questions about probability computations can arise quite naturally during classroom action, provided that these opportunities are recognized and harnessed.

The Loonie Problem

Consider the following problem, a slight variation on an exercise from the *Foundations of Mathematics 12* textbook (Canavan-McGrath et al. 2012), a WNCP-aligned resource used in Grade 12 mathematics courses across Western Canada:

> Alanis has one hand in her pocket. In it, she has 10 coins, 3 of which are loonies. She reaches into her pocket and pulls out two coins at random. She observes that the first coin is a loonie, but does not look at the second coin. Determine the probability of both coins being loonies. (p. 310)

Posed as such, the problem is virtually unambiguous, a classic example of drawing without replacement. (For curious readers, *loonie* is a term commonly used for the Canadian one dollar coin, which bears the image of a common loon.) After observing that one of the coins is a loonie, nine coins remain, two of which are loonies. The probability of both coins being loonies in this case is therefore $2/9 \approx 0.222$. Consider, now, the following modification of the problem:

> Alanis has one hand in her pocket. In it, she has 10 coins, and 3 of these coins are loonies. She reaches into her pocket and pulls out a coin at random, not showing it to you. She does the same with another coin. After observing both coins, Alanis then tells you, "At least one of the coins is a loonie." Determine the probability of both coins being loonies.

By reassigning the role of middleman to Alanis, and the role of observer to the reader, ambiguity has been introduced into the problem. Therefore, the question of how Alanis decided to provide the given information emerges as salient. Here, we consider three possibilities:

(i) *Alanis is coin-unbiased*. In this case, Alanis looks at both coins and reports that "at least one coin is a loonie" if both coins are loonies, and she reports that "at least one coin is not a loonie" if both coins are not loonies. She chooses to report the type of coin randomly (e.g., by secretly flipping one of the coins) if the coins are of different types. The possibilities and probabilities are enumerated in Table 5.

Table 5 Probabilities for the loonie problem given that the reporter is unbiased

	Alanis reports		
Coins drawn	"At least one coin is a loonie"	"At least one coin is not a loonie"	Total
LL	$\frac{3}{10}\times\frac{2}{9}\times 1=\frac{1}{15}$	$\frac{3}{10}\times\frac{2}{9}\times 0=0$	$\frac{1}{15}$
LN	$\frac{3}{10}\times\frac{7}{9}\times\frac{1}{2}=\frac{7}{60}$	$\frac{3}{10}\times\frac{7}{9}\times\frac{1}{2}=\frac{7}{60}$	$\frac{7}{30}$
NL	$\frac{7}{10}\times\frac{3}{9}\times\frac{1}{2}=\frac{7}{60}$	$\frac{7}{10}\times\frac{3}{9}\times\frac{1}{2}=\frac{7}{60}$	$\frac{7}{30}$
NN	$\frac{7}{10}\times\frac{6}{9}\times 0=0$	$\frac{7}{10}\times\frac{6}{9}\times 1=\frac{7}{15}$	$\frac{7}{15}$
Total	$\frac{3}{10}$	$\frac{7}{10}$	1

L loonie, *N* not-loonie

In this situation, the probability that both coins are loonies, provided that Alanis has revealed that at least one is a loonie and that you are aware of Alanis' process for choosing which coin to report, is (1/15)/(3/10) = 2/9 ≈ 0.222. This probability is identical to that of the original problem, although it was determined under different assumptions.

(ii) *Alanis is loonie-biased*. In this situation, Alanis looks at both coins and always reports that "at least one coin is a loonie" if she can, but she never lies. By modifying Table 5, we find that the probability that both coins are loonies, given that Alanis has revealed that at least one is a loonie and assuming that we are aware of her loonie-bias, is (1/15)/(8/15) = 1/8 = 0.125.

(iii) *Alanis is anti-loonie*. In this situation, Alanis looks at both coins and does not report that one of the coins she is holding is a loonie if she can avoid it, but she never lies. By modifying Table 5, we find that the probability that both coins are loonies, given that Alanis has revealed that at least one is a loonie and assuming that we are aware of Alanis' process of choosing which coin to report, is (1/15)/(1/15) = 1. In other words, in this situation we can be certain that Alanis is holding both loonies.

(iv) *Alanis always reports the type of the first coin*. In this last situation, we assume that Alanis randomly chooses two coins, but always reports the type of the first coin. By slightly modifying Table 5, we find that the probability that both coins are loonies, given that Alanis has revealed that at least one is a loonie (which was the first coin), is again (1/15)/(3/10) = 2/9 ≈ 0.222. The fact that Alanis is not given a decision to make, coupled with the fact that she always reports truthfully, creates a problem that is isomorphic to the original. Case (i) and Case (iv) further demonstrate that different information does not always lead to different probability assignments.

As with the Three Cards problem, modifying the Loonie problem to introduce a middleman offers students the opportunity to investigate the effect of assumptions in a way that the original problem does not, thus offering valuable insight into the nature of probability. We encourage readers to consider how this strategy may be applied to problems in their local resource. However, modifying problems in this way is not always necessary, as the following example will show.

Cohen and Celine

The following problem also appears (in identical format, save for the students' names) in the *Foundations of Mathematics 12* textbook (Canavan-McGrath et al. 2012):

> Cohen asks Celine to choose a number between 1 and 40 and then say one fact about the number. Celine says that the number she chose is a multiple of 4. Determine the probability that the number is also a multiple of 6. (p. 346)

By now, the reader will undoubtedly recognize at least one of the ambiguities in the problem situation as described: namely, how does Celine, the middleman who is relaying information, choose what kind of information she discloses about the number she draws? This problem represents a rare non-sanitized textbook problem (to adopt Bar-Hillel's terminology), although this was unlikely an intentional design. Only one solution is presented and no discussion about assumptions follows the problem statement, suggesting to readers that other solutions are not possible. As such, the educative possibilities of the problem may be easily overlooked.

Various assumptions about Celine's process may be made, each of which may affect the sought-after probability. To give a few examples (the details of which are omitted for brevity):

(v) *Celine is four-biased.* In this situation, Celine always says that the chosen number is a multiple of 4 if she can (i.e., when the number is 4, 8, 12, 16, 20, 24, 28, 32, 36, or 40), but never lies. Assuming that Cohen is aware of Celine's four-bias and that Celine has revealed that it is a multiple of 4, the probability that the number is also a multiple of 6 is 3/10 = 0.3, given that three of the multiples of 4 listed above are also multiples of 6 (12, 24, and 36). This is the answer offered in the text, suggesting that these were the implicit assumptions drawn on in determining the solution.

(vi) *Celine is factor-unbiased.* In this situation, if Celine picks, for example, the number 16, she will randomly choose one of its factors using a suitable random generator. In this case, assuming that Cohen is aware of Celine's lack of bias and Celine reveals that the number is a multiple of 4, there is a $145/652 \approx 0.222$ probability that the number is also a multiple of 6, which is smaller than the probability offered in the text.

(vii) *Celine is factor-unbiased, but will only say 1 if the number is 1.* In this situation, if Celine picks, for example, the number 16, she will randomly choose (using a suitable random generator) one factor among 2, 4, 8, and 16. She will not say that the number is a multiple of 1 unless the number in question is 1, knowing that this information is not very useful. In this case, assuming that Cohen is aware of this process and Celine reveals that the number is a multiple of 4, there is a $393/1927 \approx 0.204$ probability that the number is also a multiple of 6, which is also smaller than the probability offered in the text.

Other cases may be envisioned, such as the case of Celine being biased towards revealing a particular factor (e.g., 8) or never revealing information that will allow Cohen to guess the number immediately (e.g., stating that the number is a multiple of 35). Students might be encouraged to conceive of others. As the above analysis reveals, different assumptions may lead to different probability assignments, none of which are unquestionably correct, given the incompleteness of the problem statements. For instance, is it more reasonable to assume that Celine will always declare that the number is a multiple of 4 if this is true, or that she will reveal any of its factors with equal probability? Arguably, unless one is led to believe that Celine has a particular affinity for the number 4, the latter is a more natural assumption; and yet, the former assumption appears to be the one to have been implicitly adopted by the textbook authors.

It is worth highlighting again that, while the calculations of the probabilities resulting from the various assumptions adopted are interesting and worthwhile, it is not the multiple opportunities for computation but rather the ambiguity of the problem that has the potential to offer students deeper insight into the nature of probability and encourage a critical stance towards probability statements. In failing to recognize this potential, it may be tempting to try to "improve" the Cohen and Celine problem, or a similar middleman problem, by making assumptions explicit before the problem reaches students so as to avoid confusion. Imagine, for example, that the problem is revised as follows:

> Cohen asks Celine to choose a number between 1 and 40 and then say one fact about the number. *Celine will always say that the number is a multiple of 4 whenever this is true*. Celine draws a number and says that the number she chose is a multiple of 4. Determine the probability that the number is also a multiple of 6.

Or, alternatively:

> Cohen asks Celine to choose a number between 1 and 40 and then say one fact about the number. Celine says that the number she chose is a multiple of 4. Determine the probability that the number is also a multiple of 6 *if*:
>
> (a) *Celine always says that the number is a multiple of 4 whenever this is true.*
> (b) *Celine chooses factors with equal probability.*
> (c) *Celine always says that the number is a multiple of 8 whenever this is true.*

Although these revisions address the ambiguity surrounding the process by which Celine chooses to report information, they do so through a dilution of the experience. Making assumptions explicit in this way, before they can be recognized and questioned in context, removes the need for students to be critical in their analysis of the information obtained about the events. In this sense, the problem loses the very feature that made it valuable for gaining insight into the nature of probability as a quantification of available information. The middleman, a provider of information who has some choice in the information they provide, is replaced with a robotic alternative no different from a fair die or standard deck of cards. This serves to disarm students from posing questions about the assumptions made, and, implicitly, advocates a passive approach to reasoning in the face of uncertainty.

We thus arrive at a point of synergy between the two goals of introducing middleman problems in secondary school: revealing the notion that probability is a quantification of information about a scenario, rather than an attribute of the process or event in question, and fostering among students a critical stance toward probabilistic situations. In fact, we believe that these two goals mutually sustain each other. Ultimately, the understanding that probability is a quantification of available information gives students permission to question the assumptions made and the information provided in a probabilistic scenario. This active, critical stance unlocks the understanding that probability is dependent on available information and the assumptions made about the event in question. In fact, the original Cohen and Celine problem, with its inherent ambiguity, offers more educative possibilities—provided that these possibilities are harnessed. The teacher is then tasked with designing an ecology that can provide opportunities for students to ask critical questions in situations of uncertainty, and to determine the impact of differing sets of assumptions and information on the subsequent probability calculations.

Conclusion

This chapter has advocated for the use of middlemen problems in the teaching of probability, where the underlying model for revealing information needs to be explicated in order for the problem to be well-posed. We believe that the use of these problems in the classroom holds a twofold benefit. First, by revealing how various, correct solutions result from different, but equally reasonable assumptions, they illustrate a key aspect of the nature of probability: namely, that probability is not a characteristic of an event, but rather an enumeration of the information about the event in question. They accomplish this by introducing ambiguity through an agent who provides information—a middleman. Under these conditions, the probability of an event changes based on the process, or biases, that generated the reported information. In this case, allowing ambiguity to creep into the problem through the actions of the middleman provides a clearer illustration that probability is dependent not only on *what* you know, but also on *how* you came to know it.

It is not our intent, however, to suggest that ambiguities in computing probabilities only arise when a human information-passing agent is involved (Nickerson 1996). Likewise, we do not aim to suggest that middleman problems are the only way to illustrate to students the notion that probability is a quantification of the information known about an event; other problem situations that illustrate this notion, such as the case of different people in the same situation receiving different information, may also offer students the opportunity to engage with this key aspect of the nature of probability. Most important, then, is that such problems indicate the need for assumptions in the quantification of probability, and illustrate how disagreements about the probability of an event can arise when differences in the assumptions that are made do not come to light.

However, on their own, ambiguous problem situations will not guarantee that the role of assumptions in the quantification of uncertainty will be recognized. If students are to develop the capacity to ask critical questions about probabilistic situations, they must be given the opportunity to question and investigate the consequences of various assumptions in classroom tasks. Herein lies the second benefit of middlemen problems. This critical outlook on probability is the dispositional groundwork for probability literacy, where a critical stance and skills in calculation are mutually supportive. Operating within this understanding reorients the interrogation of information about the probabilistic scenario out from the realm of the assumed or unstated into the realm of the critically necessary.

References

Bar-Hillel, M. (1989). How to solve probability teasers. *Philosophy of Science, 56*(2), 348–358.

Bar-Hillel, M., & Falk, R. (1982). Some teasers concerning conditional probabilities. *Cognition, 11*, 109–122.

Bertrand, J. (1889). *Calcul des probabilités*. Paris: Gauthiers Villars.

Borovcnik, M., & Kapadia, R. (2014). From puzzles and paradoxes to concepts in probability. In E. J. Chernoff & B. Sriraman (Eds.), *Probabilistic thinking: Presenting plural perspectives* (pp. ix–xiii). Dordrecht: Springer.

Canavan-McGrath, C., Desrochers, S., MacDonald, H., Pruner, M., Reinbold, H., Samra-Gynane, R., Shaw, C., Teshima, R., & Trufyn, D. (2012). *Foundations of mathematics* (Vol. 12). Toronto: Nelson Education.

Devlin, K. (2014). The most common misconception about probability? In E. J. Chernoff & B. Sriraman (Eds.), *Probabilistic thinking: Presenting plural perspectives* (pp. ix–xiii). Dordrecht: Springer.

Falk, R. (1992). A closer look at the probabilities of the notorious three prisoners. *Cognition, 43*, 197–223.

Gal, I. (2005). Towards "probability literacy" for all citizens: Building blocks and instructional dilemmas. In G. A. Jones (Ed.), *Exploring probability in school: Challenges for teaching and learning* (pp. 43–70). Dordrecht: Kluwer Academic Publishers.

Haddon, M. (2003). *The curious incident of the dog in the night-time*. Toronto: Random House of Canada.

Jones, G. A., Langrall, C. W., & Mooney, E. S. (2007). Research in probability: Responding to classroom realities. In F. Lester (Ed.), *Second handbook of research on mathematics teaching and learning* (pp. 909–955). Charlotte: Information Age.

Khovanova, T. (2011). Martin Gardner's mistake. *The College Mathematics Journal, 43*(1), 20–24 Retrieved from the arXiv database at: https://arxiv.org/abs/1102.0173.

Konold, C. (1989). Conceptions of probability. *Cognition and Instruction, 6*(1), 59–98.

National Council of Teachers of Mathematics. (1989). *Curriculum and evaluation standards for school mathematics*. Reston: Author.

National Council of Teachers of Mathematics. (2000). *Principles and standards for school mathematics*. Reston: Author.

Nickerson, R. S. (1996). Ambiguities and unstated assumptions in probabilistic reasoning. *Psychological Bulletin, 20*(3), 410–433.

Rees, P., & Williams, C. (Writers). (2011, November 23). Wheel of mythfortune [Television series episode]. In A. Dallow (Producer). *MythBusters*. Silverspring: Discovery Communications.

Rubel, L. H. (2002). *Probabilistic misconceptions: Middle and high school students' mechanisms for judgments under uncertainty*. Unpublished doctoral dissertation, Teachers College, Columbia University, New York, NY.

Rubel, L. H. (2006). Good things always come in threes: Three cards, three prisoners, and three doors. *Mathematics Teacher, 99*(6), 401–405.

Western and Northern Canadian Protocol. (2006). *The common curriculum framework for K–9 mathematics*. Edmonton: Alberta Education Retrieved from http://www.wncp.ca/media/38765/ccfkto9.pdf.

Western and Northern Canadian Protocol. (2008). *The common curriculum framework for grades 10–12 mathematics*. Edmonton: Alberta Education. Retrieved from http://www.wncp.ca/media/38771/math10to12.pdf.

Additional Suggestions for Further Reading

Gal, I. (2002). Adults' statistical literacy: Meanings, components, responsibilities. *International Statistics Review, 70*(1), 1–51.

Rossman, A. J., & Short, T. H. (1995). Conditional probability and education reform: Are they compatible? *Journal of Statistics Education, 3*(2). Retrieved from https://ww2.amstat.org/publications/JSE/v3n2/rossman.html.

Encouraging Able Students: An Example of Composition of Linear Polynomials

Edward J. Barbeau

Abstract In order for students of science, technology, engineering and mathematics to have a proper algebraic foundation, the secondary curriculum must go beyond memorizing results and technicalities. Algebra is a powerful tool that can be wielded only if its user has a sophisticated appreciation of its value as a language, its structure and its role in proving results. In this chapter, we look at an example that is accessible to material in the early high school syllabus in which students are encouraged to deepen their grasp of the subject.

Keywords Secondary mathematics · Algebra · STEM education

Algebra is the core of the secondary syllabus, both for every educated citizen and for those who need a solid preparation in mathematics for study or employment. Students, particularly those headed for a STEM program in university or college, need to appreciate the role of algebraic notation and process as a language for expressing relationships, as a tool for setting up and solving equations and as a powerful means of disposing of a broad range of problems.

STEM students require not only a broader and more technical syllabus, but tasks that shed varied light on the concepts and practices of algebra. There are many subtle distinctions that can only be understood by exposing them to a range of examples that bring out salient features. A helpful viewpoint is to regard an algebraic expression as a bearer of information, of which some mathematical facts can be easily read off and some are latent. Technical manipulation of an algebraic expression changes its form so that the particular information needed is readily visible.

In order for this side of algebra to be real for students, it should be presented in a context that means something to them. As argued in earlier chapters, this context might relate to their life and experience. But it can be purely mathematical, if it has

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its own coherence and is attractive on its own terms. At the secondary level, it is not easy to find applications that are both accessible and deal with aspects of mathematical usage and analysis that students should be exposed to. One might hope that courses like physics, chemistry and geography are sufficiently mathematical as to fulfil some of this necessity.

However, it is in mathematics courses themselves that one should develop skills of investigation, analysis and reasoning. STEM students in particular need to be attuned to the subtleties of notation, usage and reasoning that will make them confident and competent users of the discipline. Accordingly, in following the syllabus, one should avoid the sense of just doing one topic after another, and build in questions to be answered and goals to be reached. The example in this paper is purely mathematical, and builds on one of the very first topics in the secondary syllabus, namely the study of linear functions and equations.

A linear function has the form $f(x) = ax + b$, where a and b are constants. Students can be asked to create and then evaluate their own linear functions at different values of x. It is helpful to think of a linear function as a transformation, $x \to ax + b$, that takes the real number line to itself. It then becomes natural to ask whether any number x is taken to itself under this transformation. This involves finding the values of x to satisfy the linear equation

$$ax + b = x.$$

Such a value is said to be a fixed point for f. This can be investigated using numerical values of a and b as well as by sketching the graph of the equation $y = f(x)$ and seeing if and where it intersects the graph of the equation $y = x$.

The important intellectual step here is to now think of a and b, as parameters that stand in for the numerical coefficients, and to express the fixed point in terms of these parameters. It turns out that there are essentially three situations. If $a = 1$, $b = 0$, then every point is fixed. If $a = 1$, $b \neq 0$, then there are no fixed points. This can be seen geometrically since the mapping $x \to x + b$ represents a translation of the line by a nonzero quantity. Finally, whenever $a = 1$, there is exactly one fixed point.

We now bring in a second strand that will be related to what we have just discussed. If we introduce a second linear function, $g(x) = cx + d$, then we can evaluate one of the two functions at a particular point and then evaluate the second at the result of the first evaluation. In mathematical terms, we consider $f(g(x))$ and $g(f(x))$, the compositions of the two functions.

A natural question is whether the order of composing the functions makes a difference to the final result. Students can once again investigate particular cases and perhaps be led to some tentative hypotheses. At the end, we can treat a, b, c, d as parameters and set up and solve the linear equation $f(g(x)) = g(f(x))$.

The idea of composition should not be completely new to the student, especially if we regard a linear polynomial as implementing a transformation of the real line. Some students will have studied geometrical transformations such as reflections, rotations, translations and dilatations in elementary school and already experienced situations in which the transformation do not commute under composition. Indeed, we can describe $x \to ax + b$ as a dilatation with factor a followed by a translation by distance |b| to the right or left depending on sign.

Despite the attractiveness of this topic, it does not appear to have received much attention in the educational literature. The references include two articles in the National Council of Teachers journal, *The Mathematics Teacher*, and one a quarter century ago in an expository mathematics journal.

Before discussing pedagogical issues any further, I will devote a section just to the presentation of the mathematics. The reader is invited to work through the material and consider how it might be framed for a secondary class. In any situation where an extra topic is introduced, there is a trade-off to be made. The first reaction might be that it is quite enough to get through the prescribed curriculum without introducing any side issues. On the other hand, one should always ask whether there is any value added by having students go through a mathematical experience that may improve their skills and insights to the extent that later topics can be handled more expeditiously.

Commuting Linear Functions: The Mathematical Issues

Suppose that we have two linear polynomials, say $f(x) = 3x - 2$ and $g(x) = 2x + 5$. We can compose them in two ways to form two new functions: $f \circ g(x) = f(g(x)) = 3(2x + 5) - 2 = 6x + 13$ and $g \circ f(x) = g(f(x)) = 2(3x - 2) + 5 = 6x + 1$. As you can see, the order of composing makes a difference to the result. Is it possible to find linear functions for which the result of the two options is the same? In other words, under what conditions do two linear polynomials commute under composition?

Investigation of this question leads us to distinguish two separate roles for letters in algebra, that of *parameters* that play the role of constants and represent for example numerical coefficients, and that of *variables* which represent values in the domain of some function or expression and allow us to link numbers in the domain and range.

To deal with linear polynomials in general, we can denote them by $f(x) = ax + b$ and $g(x) = cx + d$, with a, b, c, d being the parameters. Composing them in both orders leads to

$$f\left(g\left(x\right)\right) = a\left(cx + d\right) + b = acx + ad + b;$$

$$g\left(f\left(x\right)\right) = c\left(ax + b\right) + d = acx + bc + d.$$

At this point, we should make clear what sort of equality is at stake. One perspective is to see $f \circ g$ and $g \circ f$ as new functions created by performing an operation on the pair f and g. Then we may ask under what conditions are the functions $f \circ g$ and $g \circ f$ the same? For this to occur, they must have the same domain and take the same value at each point of the domain: $f(g(x)) = g(f(x))$ for every real number x. (This is usually expressed by saying that $f(g(x)) = g(f(x))$ is an *identity* in x.) Here we are looking for conditions on the parameters a, b, c, d.

A second perspective is to consider the functions f and g to be given (*i.e.*, a, b, c, d represent a particular choice of numerical coefficients), and ask for which values of x we have $f(g(x)) = g(f(x))$. (In other words, $f(g(x)) = g(f(x))$ is a conditional equation for x.)

The equation $f(g(x)) = g(f(x))$ is equivalent to $ad + b = bc + d$. This is notable in that there is no dependence on x. What is the significance of this? It means that $f(g(x)) = g(f(x))$ for *some* value of x implies that $f(g(x)) = g(f(x))$ for *all* values of x. For a geometrical take on this, observe that the graphs of $y = f(g(x))$ and $y = g(f(x))$ have the same slope, so that they are either distinct and parallel, or they coincide. Thus they have either no point in common or every point in common. A consequence is that if we want to check commutativity of two linear polynomials under composition, we just have to check it for one value in the domain.

The condition $ad + b = bc + d$ that f and g commute is not very informative as it stands, so we transform it so that we can dig out other information more readily. For example, it can be rewritten as

$$(a-1)d = (c-1)b.$$

This is helpful, because we are in a position to collect to one side of the equation terms that pertain only to a and b (*i.e.* to the function f) and to the other side the terms that pertain only to c and d. However, we can do this only if we are sure we are not dividing by 0, so we must take care of that possibility first.

If $a = c = 1$, then the condition is satisfied and we can check that the functions $f(x) = x + b$ and $g(x) = x + d$ commute and that the composite in either order is $x + b + d$. Geometrically, each function represents a translation of the real line. Their composition represents a translation through the sum of the distances of its components.

If $a = 1$ and $b = 0$, then $f(x) = x$, and this commutes with any function. In fact $f \circ g = g \circ f = g$ so that f is an *identity* function. The case $c = 1$ and $d = 0$ is similarly handled.

Finally, if $b = d = 0$, then $f(x) = ax$ and $g(x) = cx$. Geometrically, each function represents a dilatation of the real line, and the factor of the composite dilatation is the product of the factors of the component dilatations.

Excluding these cases, we can now carry out the division and get the condition for commuting in the form

$$\frac{b}{1-a} = \frac{d}{1-c} \tag{1}$$

For example, if $f(x) = 5x + 2$, then $g(x)$ must have the form $(2d + 1)x + d$ for some real number d. It is straightforward to check that this works.

However, this is not the end of the matter, because the condition turns out to signify a striking relationship between the two functions. Solving the equation $x = f(x)$ for the fixed point of f leads to

$$x = \frac{b}{1-a}$$

Similarly, we find that the fixed point of g is given by

$$x = \frac{d}{1-c}$$

Thus Eq. (1) tells us that the two functions in question commute if and only if they have a common fixed point.

With some reflection, we realize that this is not surprising. If f and g have a common fixed point p, then $f(g(p)) = f(p) = p = g(p) = g(f(p))$, so that $f \circ g$ and $g \circ f$ take the same value at p. But we already noted that this implies they take the same value everywhere, and so are equal as functions.

On the other hand, suppose that $f(p) = p$ and $g(q) = q$. Excluding the possibility that either is the identity function x, we note that we obtain the fixed point by solving a linear equation which has exactly one solution. Thus, p and q are unique. What happens if we take on board the hypothesis that $f \circ g = g \circ f$? Then

$$g\left(f\left(q\right)\right) = f\left(g\left(q\right)\right) = f\left(q\right)$$

so that $f(q)$ is a fixed point of g. Therefore $f(q) = q$ by the uniqueness of the fixed point of g. But the fixed point of f is also unique, so that $p = q$.

Commuting Linear Polynomials: Pedagogical Issues

As students advance in mathematics, they absorb greater levels of abstraction, beginning with the notion of number itself. The symbols of algebra which first play the role of placeholders of numbers become entities in their own right, as do functions and polynomials in particular with their own structure. This process will continue as students take on board in their later education such things as vector spaces and groups. A similar evolution takes place in other areas, such as geometry and combinatorics, and the syllabus should be taught keeping the need for such maturation in mind.

The basic technical requirements for this example, involving as it does linear equations, is within the range of a grade 9 course. But the level of sophistication is high, and teachers who take it on must first take ownership of it by working through the details on their own terms. Then follows a number of strategic decisions as to the goals to be achieved and how to get there. A careful and sensitive approach is needed that works out from what is familiar to the student and allows a steady intellectual progression towards a textured approach that will help them enjoy future success in mathematics. For the teacher has the task of not just securing technical proficiency, but also of fostering judgment as well as precision and critical thinking.

The teacher can be likened to a conductor of an orchestra. The syllabus and the topics in it are the score. It is the conductor who has to bring it to life, a task that requires scholarship, judgment, technical skill and empathy. The conductor has to know the context and bring a unifying vision to the whole, make sure that the level of skill of the orchestra is adequate and communicate his vision to the players and the audience. While the music performed should be true to its composer and its authenticity respected, each conductor has his own particular take that distinguishes the performance.

So it is with the teacher. The mathematics she teaches should be reliably and honestly presented, but she brings into the task her own experiences and analysis, a sense of context and a treatment appropriate to the students in front of her. Before the conductor meets with the orchestra, it is necessary to make a deep study of the score and decide what its essence is. The teacher is in a similar position. She has to live with the mathematics herself so that she gets her own feeling for what is significant and pleasing. Good teaching is done from a foundation of experience—the teacher is a witness to her subject.

From the get-go this example presents a challenge for a typical class. In my experience, the flow of discussion can vary considerably from one group of students to another. The following discussion is not intended to be prescriptive but only to indicate issues that might arise or directions that teachers may find productive. The composition of functions will likely be a new idea, as indeed is the idea of a function itself. Is there something in the student's previous experience that can be formulated in terms of functionality? It is hard to convey composition using words alone, and efficiency of communication demands that some notation be invented. But this is part of what algebra is all about. Even at its most basic level, we need to introduce variables to describe relationships, such as $A = \pi r^2$, and solve algorithmically word problems that would be difficult if left in the realm of arithmetic.

Thus, the first task is to introduce the notation $f(x)$ and provide examples. Then one can discuss $f(g(x))$ and $g(f(x))$ for particular pairs, such as $(f(x), g(x)) = (3x - 4, 7x + 6)$. One should be sure to deal with the special case $f(x) = x$ and make the point that it is an identity, playing the role that 0 plays for addition of numbers and 1 for multiplication. While working out examples of composition can be a way of having the students do manipulative practice, the lesson can be given a bit of direction by noting that $f \circ g$ and $g \circ f$ seem to be generally different and asking students to seek examples for which they are the same. (The idea of a noncommutative operation will not be completely new, as students at this stage should have been exposed to exponentiation and be aware that, for example, 2^3 and 3^2 are different; however, the two directions of exponentiation can agree in special cases, as $2^4 = 4^2$.)

Having students search for examples is a wonderful teaching tool, as it forces then to pay attention to the details of the artefacts they are working with. If would be interesting, at this stage, to see whether pairs of the form $(f(x), g(x)) = (ax, cx)$ and $(f(x), g(x)) = (x + b, x + d)$ emerge. If so, this provides the opportunity to look at the geometric significance of the commutativity.

In general, it will probably be difficult for students to conjure up pairs, so a strategy will be needed to generate them. Why not take a particular example, such as $f(x) = 3x - 4$, and ask what it takes for c and d for g for commute with f? First, this is where the parameter-variable distinction weighs in, and it needs to be understood that we are looking for particular values of c and d to make the commutativity work.

Let us look at this particular example in more detail. The condition $f(g(x)) = g(f(x))$ leads to $3(cx + d) - 4 = c(3x - 4) + d$. This equation bears information, but it is not clear what we are supposed to do with it. To begin with, we can remove brackets just to see what happens. A small miracle occurs; the terms in x are the same on both sides of the equation, and we are left with $3d - 4 = -4c + d$, which in turn simplifies to $d + 2c = 2$, or $d = 2(1 - c)$. This is the only condition that has to be satisfied, so that there are infinitely many possibilities.

This illustrates a concept that is useful in physics, that of *degrees of freedom*. The two variables c and d represent potentially two degrees of freedom in that, without restriction, we can make independent choices of values for them. However, the equation $d = 2(1 - c)$ is a restriction that ties one to the other. So there is a net of one degree of freedom, and we can choose only one arbitrarily.

It is not a bad idea to check the answer. In particular, will any student realize that any function has to commute with itself, so that taking $(c, d) = (3, -4)$ should work? We know that the identity function x commutes with everything, so that $(c, d) = (1, 0)$ should also satisfy the restriction. Students should be encouraged to find other numerical values for c and d and verify that they work, and then check $g(x) = cx + 2(1 - c)$ generally. This is a cheap way of getting them to practice their manipulative skills.

The equation $d = 2(1 - c)$ is linear in c and d, so students should be encouraged to plot this line in the c-d plane and identify various points on it.

We are not done with this example. Consider $f(x) = 3x - 4$ and a function that commutes with it, say $g(x) = -x + 4$, where the common composition is $-3x + 8$. (Each student can select his very own example other than $-x + 4$, $3x - 4$ and x, perform the following and then display the result before the whole class.) In the standard x-y plane, plot the graphs of the two equations $y = 3x - 4$ and $y = -x + 4$, or whatever the student picks. If will be noted that the two lines always intersect; ask them to name the point of intersection. In every case, the lines will pass through the point (2, 2). What is the meaning of this? Since the coordinates are the same, the point lies on the line $y = x$. In fact $f(2) = 2$ and $g(2) = 2$, so this provides the incentive to introduce the notion of a fixed point. It is not hard to check that 2 is a common fixed point of $f(x) = 3x - 4$ and $g(x) = cx + 2(1 - c)$.

Further examples can be studied that lead towards the formulation of the conjecture that commuting of the functions $f(x) = ax + b$ and $g(x) = cx + d$ (when a and b are distinct from 1) is equivalent to their having a common fixed point. Now we are in a position to tackle the general situation: under what conditions do $f(x) = ax + b$ and $g(x) = cx + d$ commute? Follow the analysis in the previous section to analyze the equation $ad + b = bc + d$. This is a place where one has to proceed cautiously to make sure that students maintain control over the connotations of the variables. We might think of $f(x)$ as a given example and see the problem as finding a corresponding restriction of c and d that ensure that f and g commute. Or we can see the matter more symmetrically as a mutual relationship between the functions f and g. It is vitally important first that the teacher think through the situation first on her own in order to, first, decide on her own mathematical perspective, and, secondly, to determine what preparation is needed for the members of her class to handle the situation. She also needs to be prepared for whatever ideas might come from the students themselves, whether they arise from misconceptions or from a competing viewpoint.

A benefit of this situation is that, because it involves a theorem, it takes us into the realm of proof (an area that is often seen only in the context of Euclidean geometry). Ideally, it would be nice if the students were led towards a conjecture about the common fixed point characterizing commuting functions, perhaps through checking out many examples of commuting and noncommuting pairs and making observations. Important things to emphasize include the need for a restrictive hypothesis (that a and c are not to be equal to 1) and the fact that the result is the equivalence of two properties.

Basically, we would begin by solving for the fixed point of each function. Then the structure of the reasoning would be as follows: f and g commute if and only if (1) is true and if and only the expressions for the fixed points of f and g are equal.

There is an alternative way of proceeding after one has discussed a particular case such as $f(x) = 3x - 4$ above. After looking at the intersections of the pairs of lines, one can move to the general case and look at where the graphs of two commuting functions intersect. As an exercise, students might be required to deal with the general situation: Suppose that $f(x) = ax + b$ and $g(x) = cx + d$ commute under composition. Determine the intersection of the graphs of $y = f(x)$ and $y = g(x)$, and show that this point lies on the line with equation $y = x$. This is not an easy question. The abscissa (first coordinate) of the intersection point is found by solving the equation $ax + b = cx + d$ to get

$$x = \frac{d - b}{a - c}$$

Plugging this into $y = ax + b$ to get the ordinate (second coordinate) of the intersection point leads to

$$y = \frac{ad - bc}{a - c}$$

which does not look at all like the abscissa. But we have not yet fed in the hypothesis that the functions commute. The condition for this is that $ad + b = bc + d$, so $ad - bc = d - b$ and we find that the two coordinates are indeed equal. (As a check, the student might deal with $g(x)$ rather than $f(x)$.)

Again, this gets into the construction of a proof and the invoking of a hypothesis to lead to a desired conclusion.

Conclusion

Since the foregoing represents a considerable investment of time, there are a number of questions that the teacher needs to settle. What group of students are likely to be receptive to it? For whom will it have value? The topic might not be presented to the whole class, but could be pursued by a group of students as a project or provided as enrichment in a mathematics club.

It will be argued that this topic is not on the list of expectations. But this depends on what sort of expectations we are considering. There are expectations of topic and expectations of practice and affect. The Ontario curriculum, for example, mentions several desiderata that are relevant: Problem Solving, Reflecting, Connecting, Critical Thinking, Reasoning and Proving. Such expectations cannot be inculcated in isolation, but are meant to inform the different mathematical topics to be covered and are best realized in a situation where there is particular program of investigation, discovery and proving. The task in this article allows students to "reason, connect ideas, make connections, apply knowledge and skills." It should provide the teacher with the opportunity to assess student understanding of concepts, and possibly provide the students with some enjoyment.

Even if it is not on the syllabus, in prosecuting the example, will it support the syllabus by requiring techniques and concepts that are part of the curriculum? What is the cost-benefit analysis? The cost is not only one of the time of setting up, but of taking students into fairly deep waters that they may not reap the value of until later in their algebraic life. The benefit might be a better connection with the mathematics that makes later learning easier. I argue that, with very simple materials, one raises matters of algebraic thinking and practice that will foster competence and fluency to a degree not normally seen among high school students. Here we touch on the idea of function and their combinations, the role of variables as unknowns, parameters and domain descriptors for functions, investigation and conjecturing, and at the end the proof of a rather interesting result. Above all, it teaches the important lesson that students must pay attention to details and meaning, and not approach algebra as an automaton.

A rich situation like this prefigures aspects of algebra that might not be important at the moment but will emerge as students mature in the subject. In the appendix, I look at how quadratic functions, a later topic in the syllabus, might impinge.

Modern education is often criticized because many students do not know the "basics" and are maladept at any sort of technical task. To be sure, there is truth to this, but it is not the whole story. A more fundamental reason for student difficulty is their misconception of what mathematics is. They seem to feel that it is a body of fixed automatic processes that will lead inevitably to an answer, and so is something that can be learned solely by rote. Rather it is a way of thinking that requires one to pay attention to structure, be careful about details and check for accuracy, reasonableness and consistency. There is strength in this consistency; while one may look at a mathematical situation in many different ways, each of them supports and enriches the others. Anything that fosters flexible critical thinking in the classroom will strengthen the student; anything that encourages a mindless formulaic approach will work against the student.

Appendix

Students should not be deceived into thinking that, for functions f and g in general, the truth of $f(g(x)) = g(f(x))$ for one value of x implies its truth for all x. A simple counterexample that might be possible for Grade 9 students and certainly possible when students learn about quadratic functions is to see what commutes with the square function. If we let $f(x) = ax + b$ and $h(x) = x^2$, and ask for the circumstance under which $f(h(x)) = h(f(x))$, we are led to the condition

$$\left(a^2 - a\right)x^2 + 2abx + \left(b^2 - b\right) = 0. \tag{2}$$

In this case, the condition that $f \circ h = h \circ f$ requires that the quadratic Eq. (2) in x is satisfied for all x.

This highlights an important point about polynomials that does not arise in the normal course of events when quadratic equations are taught, and that is what sort of polynomial equation will be satisfied for all values of x. Write the quadratic equation in the form

$$px^2 + qx + r = 0.$$

There are different ways of looking at the situation. If this quadratic is to vanish for *all* x, then it must vanish for *each particular* x. If we make three substitutions for x, then we obtain three homogeneous linear equations for the three variables p, q and r. It turns out that the only solution for this system is $(p, q, r) = (0, 0, 0)$. For example, if $x = 0$, $x = 1$ and $x = -1$, we get

$$r = 0;$$

$$p + q + r = 0;$$

and

$$p - q + r = 0.$$

It is easily found that the three equations are satisfied simultaneously only by $(p, q, r) = (0, 0, 0)$.

Another way of looking at the situation is to note that if not all the coefficients *p, q, r* vanish, then we have a nontrivial polynomial equation of degree not exceeding 2. We can always solve such an equation and find that there are at most two solutions.

Returning to the requirement that the Eq. (2) should be satisfied for all *x* leads to

$$0 = a(a-1) = 2ab = b(b-1),$$

which is satisfied only by $(a, b) = (1, 0)$ and $(a, b) = (0, 1)$. The first possibility leads to $f(x) = x$, the identity function which commutes with every function. The second possibility leads to $f(x) = 1$, a constant function that takes the value 1 everywhere. Indeed, $f(h(x)) = 1 = h(f(x)) = 1^2$.

It is however possible that, for particular values of *a* and *b* that (2) has two solutions. For example, let $f(x) = 2x + 1$ and $h(x) = x^2$. Then $f(h(x)) = 2x^2 + 1$ and $h(f(x)) = (2x + 1)^2$, two different functions. However, $f(h(0)) = 1 = h(f(0))$ and $f(h(-2)) = 9 = h(f(-2))$.

Additional Suggestions for Further Reading

Avital, S., & Barbeau, E. (1992). Commutativity of polynomials. *College Mathematics Journal, 23*(5), 386–395.

Ehrenpreis, W., & Iannone, M. (1972). Commutativity of linear functions. *Mathematics Teacher, 65*(3), 217–220.

Seguin, C. P. (1965). Commuting linear functions and fixed points. *Mathematics Teacher, 58*(8), 702–704.

Transition from Secondary to Tertiary Mathematics: Culture Shock – Mathematical Symbols, Language, and Reasoning

Andrijana Burazin and Miroslav Lovric

Abstract Our education is marked by large-magnitude discontinuities called *transitions*, during which significant changes—which require more than just academic reconstruction—occur over a relatively short period of time. The passage from high school to university, i.e., the *secondary-to-tertiary transition*, is the subject of this chapter. A key ingredient in helping students transition successfully lies in the two-way communication between high school teachers and university instructors. The case studies we discuss illustrate what the topics of these conversations could be. For instance, Ontario high school mathematics curriculum expectations do not adequately address mathematics language and logical reasoning. However, university mathematics instructors assume that their students have experience in working with definitions, universal and existential quantifiers, in constructing simple implications, or providing counterexamples. Surprisingly, standard university textbooks that review high school material do not even have hints or guidelines about understanding mathematics language and "mathematics culture," nor do they provide examples illustrating rules of logical deduction. In another case study we investigate difficulties that students face as they navigate trough a myriad of mathematical symbols, and work with their changing, content-dependent meanings. Case studies presented in this chapter could be included into high school teachers' horizon knowledge. An ability to see and understand how mathematical ideas and reasoning develop over a longer time scale can inform teaching, and thus better prepare students for their transition to tertiary mathematics. For exactly the same reasons, these case studies should find their way into university teaching.

Keywords Secondary to tertiary transition in mathematics · Mathematics symbols · Number bias · Language of mathematics · Logical reasoning · Horizon knowledge

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Formal education, from elementary school to university, is by no means a straightforward, smooth and continuous process. Perhaps the easiest way to define a *transition* is to qualify it as referring to a large-magnitude, singular discontinuity, such as changing from having one teacher for all subjects in elementary school to teacher-specialists in high school, or finishing high school and starting university. The latter discontinuity, usually called the *secondary-tertiary transition*, is the subject of this chapter.

There is no doubt that a key ingredient in helping students transition successfully lies in the two-way communication between high school teachers and university instructors. As triggers for topics of such dialogues, we discuss four case studies—mathematical symbols, "number bias," language and culture of mathematics, and logical reasoning—which cover areas where we identified significant gaps (discontinuities) between high school and university treatments.

University mathematics courses require proficiency in navigating through a large number of *mathematical symbols*, as well as their changing, content-dependent meanings, especially when discussing applications. We use the term "*number bias*" to discuss students' expectations that numbers involved in calculations, as well as in answers to mathematics questions, are certain special types of numbers, such as integers or simple fractions. In the section on *language and culture of mathematics* we identify situations which, while routinely (and correctly) understood by mathematicians and mathematically mature students, are often a source of confusion and misconceptions for novices. Proper use of *mathematics language* and *logical reasoning* (i.e., principles of mathematical logic) are usually not covered in high school.[1] However, university mathematics instructors assume that their students are familiar with them and have experience in working with definitions, quantifiers ("for every," "there exists"), in constructing simple implications, providing counterexamples, and so on. Surprisingly, standard university textbooks (calculus and linear algebra, for instance) that review high school material do not even have hints or guidelines on understanding mathematics language, nor provide examples illustrating rules of logical deduction.

Besides outlining these themes and illustrating with specific examples, we suggest ways in which they could inform teaching practice, both in high school and in university.

Mathematical Symbols

Even something as "straightforward" and "simple" as familiarity with mathematics symbols demands time and adjustment in transition. Using our own province as an example, the Ontario grades 9–10 and grades 11–12 curriculum documents (Ontario

[1] The word "definition" does not appear even once in Ontario grades 9–10 and 11–12 curriculum documents; the word "define" appears several times, but not as a suggestion to actually write down a formal, precise mathematical statement. For instance, there is no suggestion to define the term "asymptote."

Ministry of Education, 2005, 2007) use x exclusively to denote an independent variable, and y or $f(x)$, or sometimes $g(x)$ or $h(x)$, to denote a dependent variable. Although students might be exposed to a larger variety of notation for variables in their high school classes, the deep bias toward using "standard" x and $f(x)$ notation can cause problems and difficulties in university.

For instance, some students prefer to use x and $f(x)$ instead of a more suitable notation, such as t and $P(t)$, when studying population change. Faced with a body mass index formula (covered in a life sciences mathematics course) $BMI = \frac{m}{h^2}$ (mass divided by height squared, in SI units) students do not find it obvious that the graph of BMI as a function of m is a line through the origin with a slope of $1/h^2$. They have even more difficulty graphing BMI as a function of h. Likewise, many have problems recognizing that the function in the exponent in a cell-survival formula $S(D) = e^{-\alpha D - \beta D^2}$ is a parabola with $D = 0$ as one intercept.

As yet another example, while students do not have a problem to integrate $5x^3 + 12$, they typically find the integral $\int (At^m + B)\, dt$ which involves parameters and a "non-standard" symbol t for the independent variable, much more challenging. Further confusion is caused when implicit functions are studied, i.e., when a cognitive model of a function developed in high school needs to accommodate for the fact that the equation $5x^3 + y^2 = 10$ can be interpreted as a "usual" function $y = f(x)$, but also as a function $x = g(y)$, i.e., as a function of the independent variable y. Even further accommodation is needed in a study of functions of several variables, such as $f(x, y)$, where *both* x and y represent independent variables.

We have noticed that providing extensive opportunities to use a wide variety of symbols and notations facilitates students' learning and increases their comfort levels in our calculus classes. Based on our practice and experience, we suggest that high school teachers:

- Use the "standard" x and $f(x)$ notation in defining new terms and developing theory (thus enabling students to focus on the concepts), but then suggest a wide variety of symbols for variables and parameters in exercises, routine algebraic manipulations, as well as in problem solving activities;
- Discuss families of curves, i.e., functions whose formulas involve parameter(s), such as $y = mx$ (what happens as m changes?), $y = ax^2 + b$ (what feature of the graph is controlled by a, and what happens when we change b from positive to negative values?), or $y = \sin(ax)$ (how does the period depend on a?);
- Ask students to graph functions such as $s(D) = -\alpha D - \beta D^2$, label coordinate axes appropriately, and use terms such as "D-axis" and "D-intercept." When working on exercises related to applications, ask students to select appropriate symbols for variables and parameters (and remind them that this is common practice—for instance, illustrate with formulas from physics!).

As mentioned earlier, insisting that x represents an independent variable whereas y is used exclusively for a dependent variable could be a cause of misunderstandings and conceptual problems. When computing an inverse function, students recall that they have to "switch *x* and *y* and then solve for *y*." For instance, to find the inverse of $f(x) = \frac{2x-3}{x-7}$ this routine suggests that they write

$$y = \frac{2x-3}{x-7}$$

then switch *x* and *y*

$$x = \frac{2y-3}{y-7}$$

and then solve for *y*. Although it yields a correct answer in the end (assuming no algebraic errors are made), this method is conceptually unsound as it may not be clear to students why this works. As well, what is lost on most students is that *y* in $y = \frac{2x-3}{x-7}$ represents the function $f(x)$, whereas the same symbol *y* in the next line, $x = \frac{2y-3}{y-7}$, represents the inverse function $f^{-1}(x)$.

This routine becomes problematic when variables no longer represent abstract quantities; having to invert the degrees Fahrenheit to the degrees Celsius conversion formula

$$C = \frac{5}{9}(F-32)$$

using the process of switching the variables produces

$$F = \frac{5}{9}(C-32)$$

which is an incorrect formula (of course, one can proceed to compute *C* from it, and then at the end switch *C* and *F* again to obtain a correct formula; needless to say, conceptual understanding of the inverse is completely lost).

To avoid these problems, finding the inverse function routine should be rephrased as "solve for the independent variable," and followed by several worked examples[2] which use both standard and non-standard notations for the variables.

[2] Cognitive models are highly robust; even after we discuss these issues and illustrate with examples, our students will inevitably ask if they can still use what they learned in high school—i.e., "switch *x* and *y*".

"Number Bias"

We use the term *number bias* to refer to students' expectations that all numbers involved in calculations, as well as in the results (answers) are "nice".[3] A brief look at Ontario grades 9–10 and 11–12 curriculum documents (Ontario Ministry of Education 2005, 2007) reveals that there are very few places where it is suggested explicitly that students work with "non-nice" numbers, even in application problems. In the sample problem,[4] we read "The distance, d metres, travelled by a falling object in t seconds is represented by $d = 5t^2$" (instead of $d = 4.9t^2$; moreover, there is no indication that, for the given formula to hold, the vertical axis needs to point downward).

Similarly,[5] students are invited to investigate the graph of

$$f\left(x\right)=\frac{1}{x+n}$$

where n is an integer. Instead, n should have been a real number, with values such as −0.16 and 11.29 (it is somewhat unusual to insist on integers in the context of a calculus course, which is about *real numbers* and *real-valued* functions). Almost all examples of polynomials and rational functions (*ibid.*) involve integer coefficients. This is a root of the problem we witness when students in our university calculus classes have difficulties factoring expressions such as $x^2 - 0.01$, or completing the square in $a^2 - 0.52a$.

Applications are a good opportunity to work with "non-nice" numbers, and to demonstrate to our students that real-life problems demand that we use such numbers. For instance, when working with exponential functions, instead of discussing the function $y = 3x^5$, one can discuss the formula $Sk = 0.49Sp^{0.84}$ that relates the skull length to the spine length of a larger dinosaur. Or, in modeling the population of Canada, we could abandon using rounded numbers (31.6 and 33.5 million), and use thousands as units, thus working with 31,613 and 33,477 instead. As well, it is beneficial to study (and graph!) human daily temperature oscillation.

$T\left(t\right)=36.8+0.34\cos\left(\frac{2\pi\left(t-14\right)}{24}\right)$ after studying an abstract function such as $f(x) = 2 + 3\cos(4t - \pi)$. In our experience, working with (many!) models and

[3] Numbers 12, 3/4 and 0.5 are viewed (declared) as "nice," however, 23/18 and 0.00102 are not considered "nice." Students sometimes refer to the latter as "unexpected."

[4] Ontario grades 11–12 Curriculum (Advanced Functions MHF4U, Understanding Rates of Change, item 1.6), page 96.

[5] Ontario grades 11–12 Curriculum (Advanced Functions MHF4U, Understanding Rates of Change, item 2.1), page 92.

applications significantly lowers the number bias levels, and modifies students' expectations of kinds of numbers their answers are supposed to contain.[6] As well, such applications give meaning and purpose to the underlying algebra.

Language and Culture of Mathematics

It is important to emphasize[7] to our students that mathematics consists of, and deals with concepts, objects, and algorithms which are precise, unambiguous, and well-defined. When we encounter something that we are not clear about, it is always a good idea to ask—what exactly is this about? What is the meaning of this word/phrase?

In other words, we need to know definitions and employ theorems, algorithms and other procedures appropriately, and with great care. We illustrate this claim in a few examples.

Consider the infinite sum

$$S = 1 - 1 + 1 - 1 + 1 - 1 + 1 - 1 + \dots$$

Using our "finite sum" cognitive model, we "cancel" (here, subtract out) terms starting from the first term, and obtain

$$S = (1-1) + (1-1) + (1-1) + (1-1) + \dots = 0$$

However, if we keep the first term, and cancel the remaining terms, we obtain

$$S = 1 + (-1+1) + (-1+1) + (-1+1) + (-1+1) + \dots = 1$$

Clearly, we have a problem—what is the correct answer? It cannot be 0 and 1 at the same time. Our finite experiences and notions (adding numbers, "cancelling" numbers) do not generalize to infinite sums, and we need to know (i.e., we have to *define*) what is meant by the sum of infinitely many numbers. Once this is done, we get a clear answer—the above sum is divergent, i.e., it does not have a numeric value.

When working with prime numbers, we recall the definition:

> A prime number is a natural number that has exactly two distinct divisors: number 1 and itself.

To make sense of this definition, we need to know what natural numbers are, and what a divisor is. But on top of that, we must pay attention to the part "two *distinct*

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[6] In our experience, providing ample opportunities for (carefully designed, and motivated) practice goes a long way.

[7] And keep repeating!

divisors," since it rules out the number 1 as being prime.[8] Thus, to start our list of prime numbers, we write 2, 3, 5, 7, and so on.

In our view, it does not make sense to discuss whether or not 1 is a prime number. It is not, and the definition is clear about it. What we can say to our students is that definitions are made for a reason—in this case (horizon knowledge!) the reason is to make the unique factorization theorem[9] work.

The fact that 0.99999… = 1 becomes clear once the infinite decimal representation of a number is given its precise[10] meaning as an infinite sum of numbers

$$0.99999\ldots = \frac{9}{10} + \frac{9}{100} + \frac{9}{1000} + \frac{9}{10{,}000} + \frac{9}{100{,}000} + \ldots$$

and when the definition of the sum of a series[11] is employed. In our view, discussions about the "last digit" (or the absence of one) in the expression 0.99999… are not worth much.

With time and through exposure, we learn that, although mathematics language is mostly clear and unambiguous, there are exceptions.

The most common exception is related to the use of the indefinite article "a." In some cases, such as in the statement "a differentiable function is continuous," the article "a" means "any," or "all," and thus represents a universal quantifier. Likewise, in the statement "for a real number x, the graph of the function $y = e^x$ lies above the x-axis," the article "a" means "for all."

In some cases, however, "a" refers to an existential quantifier. For instance, in the sentence "find a prime number between 10 and 1000," we interpret the article "a" as meaning that we need to find *one* (i.e., *any*) prime number between 10 and 1000, but not all of them. That this is a real issue, can be seen from students' replies to the true/false question.[12]

If a function has a critical point at c, then it has an extreme value at c.

A student, interpreting "a" as "some" (existential quantifier), will say that the statement is true (and indeed, for *some* functions, it is true). However, an instructor, interpreting "a" as a universal quantifier (as is common practice) will mark student's answer as incorrect. The roots of students' beliefs that an example can constitute a proof could easily be traced back to this misinterpretation.

Together with learning mathematics and its language, we also need to become familiar with its *culture*. The indefinite article case illustrates one aspect of it. There are many others, and as illustrations, we mention a few.

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[8] Of course, there are different ways to phrase the definition, but the meaning is always the same.

[9] Every natural number greater than one is a prime number, or can be written in a *unique* way as a product of prime factors. If 1 were a prime number, then uniqueness would be lost; for instance, $24 = 2^3 \cdot 3$, but also $24 = 1 \cdot 2^3 \cdot 3$. $24 = 1^5 \cdot 2^3 \cdot 3$, and so on.

[10] And only possible.

[11] In this case, the sum of a geometric series.

[12] Identify the statement as true or false.

Although the domain is part of the definition of a function, we do not write it explicitly in all situations. For instance, a common question

Find the derivative of the function $y = \sqrt{x^2 - 3x + 2}$.

usually comes without "for all x for which it is defined." Likewise, we ask students to find vertical asymptotes of the function

$$f(x) = \frac{2x+4}{x^2-4}$$

without adding "defined for all $x \neq -2, 2$." We need to clearly communicate to our students that the assumptions are always there, even when we do not write them out explicitly. A root cause of students' erroneous work with theorems (using a conclusion of a theorem without checking assumptions) might be related to this issue.

Ignoring assumptions leads to all kinds of errors. For instance, when solving the equation $x^2 = 7x$ students routinely divide both sides by x, and, forgetting that at that step the assumption $x \neq 0$ has been made, obtain a single solution $x = 7$. Or, asked to compute the composition $f \circ g$ where $f(x) = \ln x$ and $g(x) = \ln(\cos x)$, they routinely calculate $(f \circ g)(x) = f(g(x)) = f(\ln(\cos x)) = \ln(\ln(\cos x))$ without realizing that the composition makes no sense: the range of $g(x) = \ln(\cos x)$ consists of zero and negative numbers; the assumption for the composition[13] $f \circ g$ to be defined does not hold, and thus the composition does not exist![14]

An example of an imprecise mathematics statement is a common question such as

Where (for which x values) is the function $f(x) = x^2 + 3$ increasing?

Of course, we mark the answer (0, 1) as incorrect and $(0, \infty)$ as correct, because we[15] expect the question to be understood as:

Identify the largest interval of real numbers on which the function $f(x) = x^2 + 3$ is increasing.

A common calculus question

Find all x where the function $f(x) = x^{-1/3}$ is not continuous

is a cause of confusion: what numbers x do we consider—those which are in the domain of $f(x)$, or all real numbers? By checking the answer ($x = 0$) we realize that it is the latter.[16] Often, we ask students to "find the limit …" even when the answer to the question is that the limit does not exist.

[13] The range of g is contained in the domain of f.

[14] Not all is lost—if we look at the formula for the composition and ask what the domain is, we'll figure it out.

[15] Teachers, instructors, and all others familiar with mathematics culture (i.e., "math nature").

[16] This particular situation is left vague in many calculus textbooks.

Although these (and many other) situations present no problems for teachers, instructors, or experts, who may be aware of the inherent embedded assumptions, they could be (and are!) quite confusing to novices. In some cases, it is easy to avoid confusion (for instance, by rephrasing the limit question as "find the limit or else say that it does not exist"); however, in general, as they are encountered, such situations have to be clearly identified and their precise meaning revealed.

Logical Reasoning

Since the word "theorem" does not appear in the grades 9–10 and 11–12 Ontario curriculum documents, it is safe to assume that the logical structure of a theorem (namely, the implication[17]) is not at all discussed in high school, at least in Ontario.[18] A theorem consists of one or more statements which constitute assumption(s), and of one or more statements which form its conclusion(s); its logical structure is expressed in the English language as "if assumption(s) then conclusion(s)."

Once we verify that all assumptions are true for a given problem, we draw the conclusions. The most common belief that students hold is that if an assumption in a theorem is not true, then the conclusion is not true either. It is easy to show that this is not so: consider the theorem

If the last digit of a number N is 4, then N is even.

Clearly, if the last digit of N is not 4 (assumption not satisfied), N could still be even (say, $N = 18$), i.e., the conclusion of the theorem still holds.[19]

It is clear that the reverse of the above theorem, i.e., the statement

If N is even, then the last digit of N is 4.

is not true. However, when this same construction is cloaked in abstract context, things are no longer as obvious. The best evidence is the theorem

If the series $\sum_{i=1}^{\infty} a_i$ *is convergent, then* $\lim_{i\to\infty} a_i = 0$.

which is often reversed to, and used in the incorrect form

If $\lim_{i\to\infty} a_i = 0$, *then the series* $\sum_{i=1}^{\infty} a_i$ *is convergent.*

We have noticed that students understand logic better if we discuss an "obvious" logical structure first ("obvious" meaning in an easy-to-understand, familiar context), and then apply it in the abstract situation, as we have done with the if-then statement above.

---

[17] Or equivalence ("if an only if").

[18] However, university mathematics instructors routinely assume that students are familiar with it.

[19] But the theorem does not apply.

Consider another example. Students are asked to determine whether the following statement is true or false, and to justify their answer:

For every natural number n, the number $n^2 + n + 41$ is prime

A common difficulty is the strategy, as students are not sure how to prove that their answer (true or false) is correct. It helps to consider a simple statement, such as "Every dog in Ontario is black" and ask students to articulate what would it take to prove that this statement is true (we have to check that every single dog in Ontario is black) or false (we need to find one dog in Ontario which is not black). Armed with this understanding, students are now more confident that by saying that when $n = 41$ the number $n^2 + n + 41$ has a factor of 41 and thus they have proved that the given statement is false.

An appropriate suggestion to help students overcome the difficulties illustrated here and to improve their mathematical reasoning skills is to use writing assignments, in particular expository[20] and excogitative[21] types of writing. DeDieu and Lovric (2018) are exploring ways in which students benefit from having to write in the context of a differential equations course. Burazin and Lovric (2015) suggest that working on, and keeping an archive of one's mathematical work (including narratives, of course) in the form of a learning portfolio can further enhance student learning.

Conclusion

We presented a few cases of the *culture shock* situations that students experience in transition to university mathematics. With time and through teachers' and students' active involvement, these transitional issues can be minimized.

Case studies presented here could be included into high school teachers' horizon knowledge. An ability to see and understand how mathematical ideas and reasoning develop over a longer time scale can inform teaching (and lesson planning), and thus better prepare students for the transition to tertiary mathematics. For exactly the same reasons, such case studies should find their way into university teaching.

References

Burazin, A., & Lovric, M. (2015). *Learning portfolio in mathematics at McMaster University*. Hamilton, ON: McMaster Institute for Innovation and Excellence in Teaching and Learning (62 pages).

DeDieu, L., & Lovric, M. (2018). Student perceptions of the use of writing in a differential equations course. *Primus, 28*(2), 166–185.

[20] Use narratives to describe and explain a mathematical idea, theorem, or definition.

[21] Carefully, and in detail, explain the reasoning in a mathematical argument or in an algorithm.

Ontario Ministry of Education. (2005). *The Ontario curriculum grades 9 and 10: Mathematics* (Rev ed.). Toronto: Queen's Printer for Ontario. Retrieved from http://www.edu.gov.on.ca/eng/curriculum/secondary/math910curr.pdf

Ontario Ministry of Education. (2007). *The Ontario curriculum grades 11 and 12: Mathematics* (Rev ed.). Toronto: Queen's Printer for Ontario. Retrieved from http://www.edu.gov.on.ca

Suggestions for Further Reading

Clark, M., & Lovric, M. (2008). Suggestion for a theoretical model for the secondary-tertiary transition in mathematics. *Mathematics Education Research Journal, 20*(2), 25–37.

Clark, M., & Lovric, M. (2009). Understanding secondary-tertiary transition in mathematics. *International Journal of Mathematical Education in Science and Technology, 40*(6), 755–776.

De Guzman, M., Hodgson B., Robert, A., & Villani, V. (1998). Difficulties in passage from secondary to tertiary education. In *Proceedings of the International Congress of Mathematicians*, Berlin 1998 (Vol. III: Invited Lectures, pp. 747–762).

Kajander, A., & Lovric, M. (2005). Transition from secondary to tertiary mathematics: McMaster University experience. *International Journal of Mathematics Education in Science and Technology, 36*(2–3), 149–160.

Kajander, A., & Lovric, M. (2009). Mathematics textbooks and their potential role in supporting misconceptions. *International Journal of Mathematics Education in Science and Technology, 40*(2), 173–181.

Tall, D. O. (1995). Mathematical growth in elementary and advanced mathematical thinking. In L. Meira & D. Carraher (Eds.), *Proceedings of PME-19* (Vol. 1, pp. 61–75). Recife, Brazil.

Tall, D. (1997). From school to university: The effects of learning styles in the transition from elementary to advanced mathematical thinking. In M. O. J. Thomas (Ed.), *Proceedings of the seventh annual Australasian Bridging Network mathematics conference* (pp. 9–26). Auckland: University of Auckland.

Ward, C., Bochner, S., & Furnham, A. (2001). *The psychology of culture shock*. Hove: Routhledge.

A Teacher's View – Teaching a University Bound Statistics Course

Jeff Gardner

Abstract During university, most of the mathematics students I knew took the minimum number of required statistics credits. The collective complaint of statistics was that plugging numbers into formulas (find the mean, median and mode) and looking values up in tables (the way it was done in the early 1980s) was boring. In comparison, calculus was flashy and it demanded the user to think to find the solution. Through my teaching career, I have noted that it is a majority of mathematics teachers who shy away from teaching statistics. I have come to realise that it was not because of the mundane crunching of numbers. Instead, the teacher would be facing students who have problems with typical (high school) mathematical concepts such as factoring, completing the square, translations of functions and the order of operations of exponent rules and in a university statistics course those same students would need to be taught how to analyze data, which does not default to a set of algorithmic steps. What the statistical processes tell us about the data is the big deal. Statistics is about interpreting the data so that anyone interested in the same data will understand its story when they read the work. So how might a teacher get their students to ask those critical questions?

Keywords Analyse (analysis) · Checklist · Correlation · Rubric · Statistics

In preparing to teach a university bound statistics course there are two basic areas the students must achieve: the calculations and the analysis. The "Recalling statistical methods" section has some websites I found worthwhile in reminding me of the formulas used as well as ideas for teaching them. High school students are always interested in knowing where they would use "this" in real life, so I have provided some sources of "Real world data" to get you started. Since "Analysing data" is a much more organic task I have included some tidbits that helped get my students pushed in the right direction. The skill level of your class is also a

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determining factor of how you will teach this material, so I have provided an example of "Simplifying complex calculations" as well as some of my philosophy behind "Scaffolding large projects." In the end teachers are responsible for providing a mark and so I have included some ideas on "Evaluation."

Recalling Statistical Methods

Online courses, such as Coursera (https://www.coursera.org), offer plenty of free statistics courses (first year statistics, advanced statistics, statistics for business and courses in the R programming language used by many statisticians) if you feel your own knowledge needs a touch up. These courses usually start when university/college terms start (i.e., September and January). There are also MOOCs—Massive Open Online Courses for Educators—which have video lectures and online small group work laboratories with coaching for classroom practice. There is a MOOC newsletter (https://www.mooc-list.com) which includes notifications of course offerings.

The Statistics Canada site also contains well written mini-lessons, such as how to plan a survey (http://www.statcan.gc.ca/edu/power-pouvoir/toc-tdm/5214718-eng.htm#tphp) and using the data (http://www.statcan.gc.ca/pub/11-533-x/2007001/4072249-eng.htm). These are especially useful if you do not have access to textbooks.

Real World Data

The Statistics Canada website (http://www.statcan.gc.ca) has a mass of Canadian data, including but not limited to population, health, housing, work, education, and Anishinaabe to create various one (mean, median, mode and standard deviation) and two variable (correlation and scatter plot) exercises with real world data for your students. Unfortunately, there is a gap in the data due to the 2006–2015 Conservative government's anti-scientific philosophy.

For more specific data such as health, the World Health Organization (http://www.who.int/whosis/en) and the Centers for Disease Control and Prevention (http://www.cdc.gov/nchs/fastats) have websites the link to good data for student assignments and projects.

Organizations such as the Statistical Society of Canada (https://ssc.ca/en) have experts that can answer questions and direct you to other sources of data.

Analysing Data

Teaching how to analyse statistical data is harder than teaching computations because there are not a set of algorithmic steps you can simply lay out for your students. In teaching analysis of bias, inference, variability and statistical accuracy students must have multiple scenarios to practice. Diversifying the scenarios can help deepen student understanding of the process of analysis and the increase chance of course success. As a gift, I received a daily desk calendar with an outrageous but statistically based statement on each page (e.g., "Eight percent of the world's population regularly eats insects"). So for the first month of the semester, I started each class by discussing one day's statement. I started the first week with statements specifically designed to elicit an emotional response (like the statement above) and progressed towards statements more statistical in nature (e.g., "Think about how stupid the average person is, and then realize that half of them are stupider than that"—George Carlin). The purpose of this class opener is to teach students how to analyse a statement by questioning the accuracy, author and context. At the end of the month most students were able to question a statement's qualities competently. By the end of the term students were commenting on bias, the conclusion's accuracy, inference, variability, et cetera.

Sources for discussion topics are abundant. Trish Hennessy writes a piece called *A number is never just a number*, which is published monthly. *Harper's Magazine* has something similar. You can get these from game shows such as *Family Feud*. The *USA Today* has an info-graphic pretty much every day. Analyzing graphs is also a learned skill you can help your students achieve. Billboards, especially in the United States, are also good sources for provocative discussion seeds, such as Sellner (2016).

My Twitter assignment sprouted from such a trip. I set up a Twitter identity that all my class followed, although now school and board networks can handle this concept (which would be a much better idea due to professionalism issues). Each week students were responsible for tweeting one statistic, which the entire class would receive. The intention was to have the students looking for statistics out in the real world. No statistics from other courses were accepted.

Simplifying Complex Calculations

Keep in mind that these statistics students are rarely from your crop of the university mathematics or STEM pathway. Some students may have just passed their grade 11 mathematics course. For example, consider a formula like the Pearson Correlation Coefficient, $r = \frac{\sum_{i=1}^{n}(x_i - \bar{x})(y_i - \bar{y})}{\sqrt{\sum_{i=1}^{n}(x_i - \bar{x})^2 \sum_{i=1}^{n}(y_i - \bar{y})^2}}$. That will be a big, scary formula to

	x_i	y_i	$x_i - \bar{x}$	$y_i - \bar{y}$	$(x_i - \bar{x})(y_i - \bar{y})$	$(x_i - \bar{x})^2$	$(y_i - \bar{y})^2$
	A	B	C	D	E	F	G
1	26.1	$535	-4.0	-$131	519.6	15.7344	17161
2	30.4	$568	0.3	-$98	-32.7	0.1111	9604
3	27.7	$544	-2.4	-$122	288.7	5.6011	14884
4	33.5	$737	3.4	$71	243.8	11.7878	5041
5	31.8	$645	1.7	-$21	-36.4	3.0044	441
6	30.9	$967	0.8	$301	250.8	0.6944	90601
7	**30.1**	**$666**		**sums:**	**1233.9**	**36.9**	**137732**

$$\sum_{i=1}^{n}(x_i - \bar{x})^2 \sum_{i=1}^{n}(y_i - \bar{y})^2 = 5086901.87$$

$$\sqrt{\sum_{i=1}^{n}(x_i - \bar{x})^2 \sum_{i=1}^{n}(y_i - \bar{y})^2} = 2255.416$$

$$\frac{\sum_{i=1}^{n}(x_i - \bar{x})(y_i - \bar{y})}{\sqrt{\sum_{i=1}^{n}(x_i - \bar{x})^2 \sum_{i=1}^{n}(y_i - \bar{y})^2}} = 0.5471$$

Fig. 1 Calculating the Pearson product-moment correlation coefficient using a table method

many of those students—what are those "E" things? To make the formula easier to calculate, the work can be laid out as a chart, as follows:

Find the Pearson Correlation of the data of Average Daily Temperature versus Ice Cream Sales recorded over 6 days at Burlington Beach:

(26.1, $535), (30.4, $568), (27.7, $544), (33.5, $737), (31.8, $645), (30.9, $967). The given data was entered into an Excel® table as shown in Fig. 1.

Notice the headings. The chart headings relate to the formula. It helps students to understand how to use the formula, especially if the students have to include the headings in their work. There is an outlier in the data to allow for a wider variety of interpretations. Many statistics formulae can be simplified like this.

Sometimes, the chart does not provide enough simplicity. In that case the formula can be broken into algorithmic steps for the students having problems, as follows:

1. Write both variables' data in a column (A & B resp)
2. Find the mean of each column (A7 & B7 resp)
3. Subtract each datum from its associated mean (C & D resp)
4. Multiply the differences (E = C*D)
5. Find the sum (E7)
6. Square each of the differences columns (F & G resp)
7. Find each column's sum (F7, G7)
8. Multiply the sums of the squared differences (F7*G7 = E9)
9. Square root step 8 (sqrt(E9) = E13)
10. Divide step 5 by step 8 (E7 ÷ E13 = E17)

Although having a student understand what Pearson's Linear Regression Correlation Coefficient formula is actually doing may not be possible, it is interesting to note with students the similar constructs in it and the Standard Deviation formula. By showing them the comparable algebraic elements it makes the Pearson's formula less menacing.

Scaffolding Large Projects

The Ontario Curriculum (Ontario Ministry of Education 2007), for example, requires a "Culminating data management investigation." My predecessor's activity was a probability game, for which students invented, calculated all the probabilities (or odds) and then played all the games in the final class in a casino format—Casino Day. I have seen another activity where students wrote a book examining one of the concepts of the course. I decided my students would do statistical research. Whatever the nature of this investigation, the students need a framework because of the sheer girth of this project and the high cost of failure if it is done badly or it is incomplete. My expectation is for my students to submit parts of this project during the semester as a safety net. These stages are worth partial marks towards this project (because some students need that motivation) but more importantly to provide deadlines which students must meet to help avoid a catastrophe at the end of the semester. The idea is to try to stop things from being rushed at the last minute and instead have the projects well planned and thought out so that elements are not forgotten. This framework includes keeping logs of searches for source material and peer-reviews. Some handouts have been included (which evolved from other teachers' work) to assist students in planning and conducting their investigation and then reporting it.

Some of those handouts are distributed on the first day of class (such as the following two page handout in Fig. 2). Other documents (e.g., Hypothesis Proposal in Fig. 3) are handed out at specific points in the semester, with submissions that are marked and count a certain percentage towards the overall mark of the project.

The marking criteria of the research project rubric (see Fig. 4) are also used for correcting exercises and grading assignments. One of the things students tell me that they like about mathematics courses in general is that they know what is expected from them (the questions on tests are like the exercises they practice in their homework) and how to get marks—though they cannot always "do" those things. I have tried to replicate familiarity for them by using similar looking marking criteria on assignments and the project. By using the same gauges students get plenty of practice writing up reports for assignments and receiving feedback before submitting their major research project. That means there should be fewer surprises.

Using peer-reviews, where one student reviews another student's project before the actual submission, help both students clarify their understanding of expectations. These handouts can take the form of checklists that further delineate elements of the course which should be in the final report. These check lists serve to assist the project's student author by indicating the necessary elements still in need of clarification or that are missing in their project. The checklists also assist the student reviewer as a concrete exercise of locating the elements. It means the student reviewer must understand the elements. There have been valuable discussions that have taken place (sometimes close to arguments) as to whether an element has been adequately fulfilled—I try never to intervene and simply point the contenders towards their notes. The student reviewer can also benefit after the peer-review exercise by reflecting

DATA MANAGEMENT CULMINATING RESEARCH PROJECT

MDM4U ________________________ name:

OVERALL EXPECTATIONS

1. design and carry out a culminating investigation* that requires the integration and application of the knowledge and skills related to the expectations of this course;
2. communicate the findings of a culminating investigation and provide constructive critiques of the investigations of others.

All topics **MUST be approved by Mr. Gardner before** you begin researching.

Stage	Brief Description*	Evaluation	Due Date
Identification of the research topic	Identify an area of interest to be investigated. ref: pp 482 to 484		
Initial data collection	Hand-in LOG Sheet (it will be copied and returned) with preliminary data search in the suggested bibliographical form. Variables of your research defined.	Unit Test weight	
Research Project Proposal	Oral discussion of Hypotheses	Quiz weight	
	Develop a hypothesis statement. ref: pp 486 to 487 & p594	Unit Test weight	
Complete data collection	Accumulate sufficient data specific to the research question **with citations and LOG sheet**. Peer-observation of data collected	Appendix*	
Data Analysis	Apply concepts and techniques taught in class to analyse data.	Appendix*	
Written Report	Research Project Report with conclusions, based on your analysis, regarding your hypothesis. ref: pp491 to 493	10% Final Evaluation	
Oral Presentation	Research Project Presentation ref: pp 493 to 495	Unit Test weight	

More detail regarding several of these stages will be provided at the appropriate times.

Fig. 2 Data management culminating research project overview

ALL submissions (your assignments and the culminating project) must be typed and conform to the following:

1. Use 12 point Times Roman font;
2. Double spaced;
3. Cite all references (below and see http://owl.english.purdue.edu/owl/resource/949/01/)
4. Use grammar and spell checking, but also use proper statistical terminology.

Marks will be assigned for the written of your assignments (and the culminating project) using the Composition Rubric (below).

	Level 1	**Level 2**	**Level 3**	**Level 4**
	Explanations and justifications are partially understandable by MrG	Explanations and justifications are understandable by MrG, but would likely be unclear to others	Explanations and justifications are clear for an MDM4U audience	Explanations and justifications are clear for a wide range of audiences
Composition Degree of clarity in explanations	Unfocused and vague expression of ideas *and/or* Only rudimentary responses *and/or* Grammar and spelling mistakes	Repetitive or wandering expression of ideas *and/or* Partial undertaking of responses *and/or* Some grammar and spelling mistakes	Clear expression of ideas *and* Adequate undertaking of responses *and* Few grammar and spelling mistakes	Concise expression of ideas *or* Thorough undertaking of responses *and/or* No grammar and spelling issues

Assignments NOT handed in by the due date are late (and will be graded as a zero, until submitted) with penalties applied as referenced on the mathematics information sheet.

Be sure to review bibliographical standards either in the text (`Canton 597`), <http://owl.english.purdue.edu/owl/resource/949/01/> or below:

You must cite the source within your text any time you use any other authors' work, facts, ideas, statistics, diagrams, charts, drawings, music, or words in your paper. We will use the following standard for any webpage `(Columbia College of Missouri)` which is defined on the log sheet (that you will receive later). A tweet is a little different: Using "the text" of that tweet," you can incorporate the facts" or details into your research `(3. Tweet: URL)`.

And then (preferably) at the bottom of the page or on a separate bibliography page:

`1. Canton, Mathematics for Data Management, McGraw Hill, 2002: 597`

`2. Columbia College of Missouri. Date of access.` <https://web.ccis.edu/en/Offices/ /AcademicResources/WritingCenter/EssayWritingAssistance/~/media/Files/Academic%20Resources/Writing%20Center/mla_examples.pdf>

`3. Last Name, First Name. (@User Name),` Twitter Post. `Date, Time.`

Fig. 2 (continued)

back on her own work. The final peer review checklist for the research project is shown in Fig. 5.

Most of these reports involve secondary data. Each student in the class receives samples of statistical reports to model. For example, did you know Kansas is in fact flatter than a pancake (Fonstad et al. 2003)? Students are expected to find data on a

CULMINATING DATA MANAGEMENT RESEARCH PROJECT: Thesis Proposal

MDM4U This proposal is the equivalent of a unit test mark name____________________

OVERALL EXPECTATIONS

1. design and carry out a culminating investigation* that requires the integration and application of the knowledge and skills related to the expectations of this course;

2. communicate the findings of a culminating investigation and provide constructive critiques of the investigations of others.

See pp 486 to 487 for more direction.

In concise and non-repetitive sentences explain/ describe the following using proper terminology at every opportunity:

State your hypothesis

- make your hypothesis a **statement** (a sentence not a question)
- your hypothesis MUST take a position (If A happens then B will increase)
- be specific

State what variables will be used to support your hypothesis

- what variables are you comparing (you must be making a comparison of two or more variables)
- are the variables used in your thesis; if not how do the variables relate to your thesis
- there must be numbers to work with
- will the variables you have chosen directly support your thesis
- what do the data need to show to support your thesis
- how will you measure those variables to support your thesis
- indicate if you are using your own survey

Background

- Have you checked to see if your hypothesis has been studied already?
- Are there similar hypotheses?
- Do the preliminary data support your hypothesis?

Why this topic?

- "Because I thought it would be interesting" is **<u>not</u>** enough -- explain why you think "it is/ would be interesting".
- how do your belief(s) relate to your hypothesis?
- what do you believe you will find?

Proposals not graded must be corrected and re-submitted until it is graded at a satisfactory level. All previously attempted proposals must be submitted (in reverse order). The late penalty is only applied to the *first* proposal.

Culminating tasks will not be accepted (and may be considered late) until completing a satisfactory proposal.

Fig. 3 Culminating research project hypothesis proposal

CULMINATING DATA MANAGEMENT INVESTIGATION: **Proposal Rubric**

	Level 1	Level 2	Level 3	Level 4
Communication	• lacks proper use of terminology	• some proper use of terminology	• good and proper use of terminology	• a lot and proficient use of terminology
Hypothesis	Hypothesis statement unclear or has no point of view	Hypothesis still needs modification	Hypothesis summarizes a point of view	Hypothesis is a specific, clear statement, summarizing a point of view
Variables	Unclear how the numbers support hypothesis or variables unclear	Hypothesis uses 1 Variable data study, **which is below grade level for MDM4U**	Hypothesis uses 2 (or more) Variable data, but measurement of variables in question	2 (or more) Variable data supports hypothesis
Background	Unclear if background was explored	Some discussion of background	Adequate discussion of background	Thorough discussion of background
Why this topic?	Unclear of decision for choice of hypothesis	Ambiguous discussion for how choice of hypothesis was made	Choice of hypothesis clear	Thorough discussion of choice of hypothesis

Fig. 3 (continued)

topic of their own choosing. Topics have ranged from a comparison of women's versus men's Olympic sprint times, gun violence in the US schools, the increase in concussions in the NHL, to the increase of female obesity in North America. Sometimes while gathering their data my students have questions about this secondary data (e.g., What were the outliers footnoted in the report?). I have encouraged those students to reach out, usually through email, to the authors who have created this data for answers. I have observed that most authors take time to answer questions for someone who has taken an interest in their work.

The students who decide to collect their own data are given extra readings on how to do a survey (e.g., https://www.mathsisfun.com/data/survey-questionnaire.html) and tutored through the process of survey design. Two projects that stand out were testing the relationship between weather and headaches; and rating someone's level of "extrovertedness" by the number and type of selfies on their Facebook page.

CULMINATING DATA MANAGEMENT INVESTIGATION: **Report**

MDM4U name ______________________

OVERALL EXPECTATIONS

1. design and carry out a culminating investigation* that requires the integration and application of the knowledge and skills related to the expectations of this course;

2. communicate the findings of a culminating investigation and provide constructive critiques of the investigations of others.

A **one** variable study (including a one variable longitudinal study) ***counts less than*** a **two** variable study

Studies involving primary data **earn more** than those using secondary data

Criteria	**Level 1**	**Level 2**	**Level 3**	**Level 4**
Communication	· lacks a statistical point of view · scarce use of statistical terms	· partial statistical point of view · some use of statistical terms	· written with a statistical point of view · adequate use of statistical terms	· explanations are explicit and concise with a solid statistical point of view · fluent use of statistical terms
Bibliography and Citations (**w/** URLs)	Sources not included but previously submitted	Some sources cited *with* errors	All sources cited *with* few minor errors	All sources cited correctly
Finalize … numerical summaries of one and two variable data D1.1, 2.1	Data collected DOES NOT support Hypothesis *and/or* Few statistics correct *and/or* Needs assistance to use statistical tools	Data collected *sort of* supports Hypothesis *and/or* Some statistics correct	Many statistics correct	All statistics correct
Generate, using technology, the relevant graphical summaries … **D1.3, 2.3**	Selection of uncited graphical displays or *Many* errors or omissions in (user generated) graphs	Selection of cited graphical displays or *Some* errors or omissions in (user generated) graphs	F*ew* errors or omissions in (user generated) graphs	No errors or omissions in (user generated) graphs
Make inferences & justify conclusions from statistical summaries of one and two variable data D1.5, 2.5	Minimal interpretation of trends, similarities and differences, correlation, outliers and lurking variables in the data	Some interpretation of trends, similarities and differences, correlation, outliers and lurking variables in the data	Sufficient interpretation of trends, similarities and differences, correlation, outliers and lurking variables in the data	Effective interpretation of trends, similarities and differences, correlation, outliers and lurking variables in the data
Determine the validity of the data… are reliable, unbiased, and current **C1.2, D3.2**	Superficial discussion any of: bias, stat error, variability, authority or ambiguity in data, or limitations of the study	Discussion of *some* of: bias, stat error, variability, authority or ambiguity in data, and limitations of the study	Discussion of *most* of: bias, stat error, variability, authority and ambiguity in data, and limitations of the study	Discussion of *all* of: bias, stat error, variability, authority and ambiguity in data, and limitations of the study

*The *letter-number codes* at the end of each criterion (i.e. C1.2, D3.2 above) reference the 2007 Ontario curriculum.

Fig. 4 Marking criteria for culminating research project

A reminder:

Remember you must cite the source within your project any time you use *any other author's* work, facts, ideas, statistics, diagrams, charts, drawings, music, or words in your paper. Citations are covered in our text[1]. We will use the following standard for any webpage[2] which is defined on the log sheet (that you will receive later). A tweet is a little different: Using "the text" of that tweet," you can incorporate the facts" or details into your research[3].

For citing sources please see the following:
1. Canton, Mathematics for Data Management, McGraw Hill, 2002: 597
2. Columbia College of Missouri. Date of access. <https://web.ccis.edu/en/Offices//AcademicResources/WritingCenter/EssayWritingAssistance/~/media/Files/Academic%20Resources/Writing%20Center/mla_examples.pdf>
3. Last Name, First Name. (@User Name), Twitter Post. Date, Time.

Fig. 4 (continued)

Evaluation

My students are assessed and evaluated in a number of different ways. I test most of the mathematical concepts such as capacity with the Pearson Correlation Coefficient in a hand written paper quiz format with the assistance of a calculator. The students then practice regression on a computer using Excel® and check their calculations using the built-in macro for correlation. Next the class is given an assignment where regression is a necessary part of the analysis of the data. Most students will include regression in their projects. Finally, the students write a traditional final exam, since this course is a university track course, which again will test the Pearson Correlation Coefficient but in a formalized setting.

As a final thought, since I am finishing this chapter during the examination period of the semester, the question may rise: cheat sheet or not? I have talked with many (and I mean MANY) university and college professors about allowing high school statistics students to use a cheat sheet. It got a resounding 'why not?' I was not overly surprised; after all, I was allowed one for my university statistics courses way back in early 1980s. Interestingly, only two thirds of my students of this current class took advantage to make one. For my class, one-sided cheat sheets may include any information the student wishes from formulae to theory to specific examples but must be hand written, not photocopied and they may not bring a microscope or magnifying glass for assistance. The cheat sheets are also submitted with the examination but are not marked.

CULMINATING DATA MANAGEMENT INVESTIGATION:

MDM4U Research Project Peer Checklist E2.4 ____________________ name

Communication of Analysis	**How many statistical words did you find?** ☐ Uses 12 point Times Roman font ☐ Double spaced ☐ Clean from grammar and spelling errors ☐ Written in third person ☐ Fluent use of statistical terms (see word list from 1 variable statistics unit)
Hypothesis ***and*** **Background Information** **(i.e. definitions)** E1.1, 1.2	☐ Population of project is identified ☐ Sample(s) from secondary data used for project are identified ☐ Sample(s) is representative of the population ☐ The **hypothesis** *clearly* stated; data collected supports thesis ☐ The **variables** and terms used in the research have been defined (explained) ☐ Background is **1-2 sentences** (enough to understand the author's motivation) ☐ Included all *relevant* data ☐ There are enough **data (≥ 30)** to support the hypothesis
Bibliography **and** **Citations(w/** URLs) E1.3	**Are URLs provided for every bibliographical entry?** ☐ Proper citations are used for charts, graphs, stats and other authors work ☐ Original data sources been referenced ☐ Original data sources been verified
Finalize … relevant numerical summaries of one and two variable data D1.1, 2.1	**Put calculations in an appendix, if your project does NOT use them.** ☐ Mean ☐ standard deviation ☐ Median ☐ correlation ☐ Mode ☐ equation of line of best fit ☐ Calculations were checked for math errors
Generate, using technology, the relevant graphical summaries … D1.3, 2.3	☐ Data interpreted correctly to support the hypothesis ☐ Data presented in context to support the hypothesis ☐ Data (i.e discrete or continuous)is illustrated on appropriate graphs ☐ Embedded charts and graphs simplify discussion ☐ Every graph (borrowed or constructed) references the data it is analyzing ☐ Every borrowed graph is cited ☐ Every graph is properly labeled and titled with coincident axes
Make inferences and justify conclusions from statistical summaries of one and two variable data D1.5, 2.5	**Does explanation of the numerical summaries actually explain what the data shows?** ☐ **NO** insinuation of the writer of the thesis being ***right/ wrong*** ☐ or ***correct/ incorrect*** ☐ Discussion is understandable to all readers ☐ Discussion makes references to the data without regurgitating the numbers ☐ Statistical tools have been used appropriately within the discussion ☐ Mathematics is included for every statistical tool used (see above) ☐ Every statistical tool references the data it is analyzing ☐ Outlier, causality and lurking variables do not take up the majority of the discussion
Determine the validity of the data… are reliable, unbiased, and current **C1.2, D3.2**	☐ There is discussion of possible sources of statistical error or bias in the data ☐ There is discussion of other inconsistencies in the data ☐ There is discussion of the limitations of the study ☐ There is discussion of why the data may vary within different surveys ☐ There is discussion of the possible extensions of this thesis

Fig. 5 Peer-review checklist for culminating research project

Finally

We use Excel® because my statistics classroom is a computer lab. I try to have my class on the computers every day crunching and analysing data—but it is probably more like twice or three times a week. I assume the students have no knowledge of Excel® at the start of the semester and teach them how to do both calculations and graphing.

This course is fun to teach. I can teach it outside the (mathematics discipline) box. I can try things that non-mathematics teachers do with their classes (such as using rubrics and the mastery learning component of solidifying the hypothesis proposal) but still keep the mathematical edge (students' clear understanding of what is expected). I have had students come back (after graduation) and thank me for specifically making them rewrite (repeatedly) their proposal until it was specific and detailed and without filler. I saw strengths (such as their artistic talent, writing skills, geographical knowledge, computer programming ability and even personal sufferance) in my students I would not normally have seen from them in a calculus bound mathematics course. And the students can see a direct link between what being taught and its significance to the real world.

References

Fonstad, M., Pugatch, W., & Vogt, B. (2003). Kansas is flatter than a pancake. *Annals of Improbable Research (AIR), 9*(3). Retrieved from http://www.usu.edu/geo/geomorph/kansas.html.

Ontario Ministry of Education. (2007). *The Ontario curriculum: Grades 11 and 12 mathematics*. Toronto: Queen's Printer for Ontario.

Sellner, S. (2016, March). *85 break-ins today in ABQ*. Retrieved from http://www.securitysales.com/news/albuquerque_alarm_company_faces_heat_for_misrepresenting_burglary_data_on_b

Part VI: Commentary – Broadening Mathematical Understanding Through Content

Plinio Cavalcanti Moreira

Introductory Remarks

First of all, I should say that my reading of the texts in Part VI was both pleasant and interesting. I have found them thoughtful as a source for this Commentary Chapter, being challenging and appealing, both as mathematics teacher education and as school mathematics material. The chapters, taken as a whole, constitute a nice set of reports that address issues related to the processes of teaching and learning mathematics in the secondary grades. Furthermore, all of them are welcome to the specialized literature as they certainly contribute to broaden mathematics understanding, aligned with the title of this part of the book. As to the potential need for adapting the activities suggested in each chapter to real school classrooms, I think it will vary from country to country and, possibly, from school to school, within a country. I will be returning to this point later in this Commentary Chapter.

Let me now briefly describe the point of view from which I have read the chapters and comment on them in this one. In the first place, from my perspective, it is relevant to distinguish between the professional practice of mathematicians (primarily valued as a practice of producing new mathematical results, at the boundary of contemporary mathematical knowledge) and the professional practice of mathematics schoolteachers, which is part of a long-lasting and complex social process that aims to educate young people. Viewing these two practices as distinctive, I shall understand they demand different kinds of knowledge. Thus, mathematical knowledge demanded by the practice of teaching at school may be reckoned as fundamentally different from academic mathematics (i.e., the kind of mathematical knowledge demanded by the professional practice of mathematicians).

It is also important, in this line of reasoning, to keep in mind the distinctive values entrenched by each one of these two professional practices. For example, a

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schoolteacher may have to use different arguments to convince students of different grades that an assertion is correct (or incorrect), perhaps being some of these arguments unacceptable as a proof, within the range of academic mathematics values. The fact is that schoolteachers work with intellectually (as well as physically) developing young people and therefore face different pedagogical challenges along different stages of a basic education process—which, by the way, usually takes 12 years to be completed. On the other hand, mathematicians are adults who have chosen, though under socially conditioning influences, to work professionally with mathematics, often having, backing her/his choice, a history of success in dealing with this discipline at school. In accomplishing the tasks of obtaining new results and convincing the peers' community that these results are correct, mathematicians must, ultimately, value the rigorous axiomatic approach, formal proofs, the use of precise language, the search for generalization and abstraction taken as further as possible, and so on. As mentioned above, these values might not prevail in the schoolteachers' professional practice.

It should be observed, nonetheless, that accepting these two kinds of knowledge as distinctive does not imply that they are totally antagonistic to each other. One can even deem them as having a part in common—that part named school mathematics, i.e., the mathematics dealt with in the school education process, according to the prevailing curriculum. This kind of mathematical knowledge is usually called elementary (in contrast to advanced) mathematics. However, naming it "elementary" already marks a distinction, since it is not usually suitable for a teacher (if at all possible) to isolate its so-called elementary part from the amalgamated non-elementary aspects, when the context is that of teaching and learning mathematics at different stages of school education. Thus, while mathematics schoolteachers' professional knowledge may be regarded, from a particular point of view, as contained in academic mathematics, I argue that once situated in the school teaching context, a professional filter applies and "cleans" school mathematics from values that are not relevant to this context, aggregating other values that are inherent to school teaching practice. This process leads, as I see it, to an essentially different way of knowing mathematics, adjusted to this particular professional practice. Accordingly, it has been claimed in the literature that academic mathematics and mathematical knowledge demanded in school teaching practice bear important conflicting elements (Moreira and David 2008). Summing up all this, I may say that academic mathematics, as knowledge impregnated with deep-rooted values inherent to the mathematicians' professional practice, though maybe potentially useful to teachers in some specific instances, is neither necessary nor sufficient in schoolteachers' professional practice.

As a final word on this matter, I should remark that the perspective described above is not purely idiosyncratic. It has been built with support on research studies, and, to an extent, the ideas conforming this particular point of view align with some more general ones, developed in the field of mathematics education, especially since the 1980s decade, which suggest that professional practice of teaching mathematics at school demands a particular and specific kind of mathematical knowledge (not restricted to what is usually called "content knowledge"). Researchers have been identifying the particulars and the specifics of this mathematical knowledge along the last decades, though in directions not yet in a level of recognizable consensus (e.g., Ball et al. 2008; Bednarz and Proulx 2009; Davis and Renert 2014; Moreira 2004; Lins 2006; Moreira and David 2005).

As to the comments on the chapters of this Part VI, I will be making some notes on ideas expressed by the respective authors. If not properly situated on my perspective, described above, those notes may sound as if I were criticizing the authors' ideas in an absolute manner. Quite to the contrary, I hope my comments will illustrate the fact that the chapters are rich enough to foster interesting reflections, even when analyzed from an external (possibly divergent) point of view. Therefore, in authoring this Commentary Chapter, my purpose is that of producing a hopefully consistent counterpoint, *in the musical sense*, to the ideas presented by the authors in their chapters.

The Chapters

In her chapter, Zazkis presents an interesting discussion on the rules that establish the order for performing the operations in an arithmetic expression. In developing the chapter, issues related to difficulties in using the conventional order by students at different levels of instruction are discussed, and strategies for teaching these rules at school are suggested. As to the suggestions, I would like to comment briefly on one by Ameis, mentioned by Zazkis this way: "*He further recommended, working with prospective teachers, to rewrite division as multiplication, turning for example, 30 ÷ 2 × 15 to 30 × 1/2 × 15, in order to emphasize that division and multiplication have the same priority*." Some textbooks in Brazil also suggest this strategy, even though we do not have a confusing mnemonic (in fact, as far as I know, we do not have any general mnemonic for the rules on the order of operations). However, I observe that this rewriting of division as multiplication (by the inverse of 2, not of 2 × 15) presupposes the understanding that these operations have the same priority and therefore should be performed in the order they appear from left to right, in which case the suggested emphasis seems no longer necessary.

In discussing the nature of the rules in terms of whether or not they are arbitrary, Zazkis' chapter conveys the idea that multiplication must be performed prior to addition because this is "*a necessary result of interpreting/rewriting multiplication as repeated addition*." To me, this discussion is important in teacher education because accepting arbitrariness in mathematics seems problematic for both teaching and learning at school, even though we have to deal with many conventions that could have been established otherwise (in some cases, actually were) and, in this sense, may be considered arbitrary (e.g., notation for fractions, for square root, for decimal representation of real numbers, positioning the dividend, divisor and quotient in the division algorithm, the signs for the usual operations of addition (+), subtraction (−), multiplication (× or .), division (÷) and so on). Nevertheless, even after reading the chapter, I kept asking myself what would be the problem with establishing a convention/agreement which prioritizes addition over multiplication, so that one would have to write the sum $3 + 5 + 5 + 5 + 5$ as $3 + (4 \times 5)$. Would it happen to cause intrinsically "bad" consequences? It does not seem so to me. One may argue in favor of considering "economy," since there would be no need for parenthesis to write the same expression under the "old" rules, that is, $3 + 4 \times 5$. But, on the other hand, if we wish to express $3 + 4 + 3 + 4 + 3 + 4 + 3 + 4 + 3 + 4$ we could dispense with the parenthesis and write just $5 \times 3 + 4$ instead of $5 \times (3 + 4)$, since, according to the (new) rules, addition would have priority over multiplication. Similarly, if multiplication had priority over exponents, we would have to write $5 \times (3^4)$ for $5 \times 3 \times 3 \times 3 \times 3$,but 5×3^4 for $5 \times 3 \times 5 \times 3 \times 5 \times 3 \times 5 \times 3$. Economy would be lost in the case of stating the distributive law, since with the current rules we would use one less parenthesis: $a \times (b + c) = a \times b + a \times c$, while with prioritizing addition over multiplication we would have: $a \times b + c = (a \times b) + (a \times c)$. Would this be enough to say that prioritizing multiplication over addition is necessary?

Of course, even in the case such changes prove to be "good" ones (that is, no "bad" consequences), the above examples, though suggesting that the current rules are not a necessary assumption in mathematics, should not lead us to the conclusion that these rules result from a completely arbitrary choice of the mathematicians. In fact, there is reference in the literature to relatively recent historical periods, during which a consensus had not yet been established internationally on this matter, and, then, different authors recommended different rules (see Cajori 1928, vol. I, p. 274). Why these rules we use today became generally accepted and prevalent is, as far as I am concerned, still waiting for a sound explanation.

Now, I turn to the last issue I would like to comment on in this chapter. The author, using Wasserman's idea of local and non-local mathematical knowledge, attributes to the associative property of multiplication of numbers a non-local character, in the sense that it may not be part of an "active repertoire" of knowledge used by teachers in their professional practice. Zazkis, then, goes on to say:

> To elaborate, operations discussed in school mathematics are either both commutative and associative, or neither commutative nor associative, which results in frequent confusions between the two [...]. Associativity appears as a property 'on its own' when considering groups and their structure. As such, while the notion itself does not require advanced background, knowledge of advanced mathematics reshapes how associativity is perceived.

While in general agreement with most of what is said in the citation just above, I shall put forward a few remarks. Though for the most basic operations on numbers discussed in school mathematics it is true that they are either both associative and commutative or neither one of these, multiplication of $n \times n$ matrices and composition of functions from R to R, are examples of operations dealt with in secondary mathematics, which are associative, but non-commutative. What I find like gold dust in school mathematics are examples of non-associative operations. This fact may give rise to a way of looking at this property as naturally valid, so apparently dispensing with the necessity of discussing its meaning, use and role in school mathematics situations.

As to bringing about a view of associativity "on its own" and eventually reshaping how secondary teachers perceive this property, it seems also possible to work within the scope of the school mathematics curriculum, considering sets and structures that are actually objects of teaching at the school level. For example, teachers or prospective teachers may be asked to justify the validity of the associative and commutative properties of addition and multiplication of natural numbers, adding the requirement that the arguments used must be understandable to students in elementary as well as in secondary grades. In accomplishing this task, student teachers may perceive and explore distinctive attributes of each one of the properties under consideration. Such a task proposal may be followed by a debate (or another type of classroom activity) on whether or not these properties extend (and the respective justifications adapt) to division and subtraction of natural numbers. This could be done with both elementary and secondary student teachers. In the latter case however, one could also wonder about the validity of these two properties (and justify her/his conclusion), now applied to addition and multiplication of polynomials, addition and multiplication of $n \times n$ matrices and composition of functions from R to R (the specification R to R here is just to guarantee the composition is always defined). This kind of approach to associative and commutative properties may challenge student teachers within a range of situations directly associated with school mathematics curriculum, while deepening the comprehension of a piece of knowledge easily perceived by them as aligned with the demands of their (future) professional practice at school.

In her chapter, France Caron argues for a more fruitful integration of modeling into school mathematics education, emphasizing the possibilities this process may open to the development of mathematical (and other types of) competencies, through construction of models for understanding, and possibly predicting, the behavior of variables in a real world phenomena. She makes a case for the idea that school curriculum should surpass curve fit tasks and go further to develop mathematical models for real world situations, as this process motivates learning, unfolds meanings for learned mathematical concepts, allows for deeper understanding and fosters the development of competencies in using mathematics. She then discusses the process of constructing and analyzing mathematical models put forward by students in prospective teachers' courses. She further presents examples of issues that arise when constructing and validating a model, discusses difficulties (including some of non-mathematical nature) involved in the modeling process, possible limitations of the model and so on.

From the "technical" point of view on teaching and learning mathematics at school, her arguments seem sound to me. Problems may arise (not in her arguments though), when one considers certain elements that influence the conditions under which teaching practice is usually developed at school. These elements relate to factors such as teacher education program's curriculum (including questions relative to time versus deepness versus volume of issues to be covered in teachers' preparation for school practice), school students' beliefs on the role of schooling as an instrument for socio-economic mobility, the prescribed school curriculum's scope, the objectives of school education, etc. I see no problem in that the author has not focused in detail on these external (so to speak) elements that may affect teachers' decisions of "going beyond curve fit" in secondary grades, even when this "going beyond" is suggested by the prescribed curriculum. Firstly, I consider the possibility that these elements are not so relevant in Canada, as they seem to be in Brazil. Secondly, as I see it, each researcher approaches each problem from her/his own point of view, and when selects certain aspects to focus, naturally has to leave others behind. Thus, it is due to the research community, as a whole, to eventually produce a multiple-sided view of the phenomenon under consideration. Nevertheless, I would like to present here some ideas that came to my mind while reading and appreciating Caron's arguments in favor of "going beyond curve fit."

First, I shall note that she herself mentions important elements that may undermine going beyond what is usually done in school, in terms of modeling. She says, for example: "*A lack of tradition and the fear of taking time away from the teaching of mathematics have often led to a reduced version of what modeling entails and of how it can be integrated in secondary school mathematics*." No doubt that these are two real obstacles to fully integrate modeling in secondary school education. However, I think they do not come into play isolated from other equally important aspects to be considered, though I am not sure about how relevant a role these other aspects play in the Canadian school teaching system. For example, mathematics education literature has pointed out that promoting students' development of mathematical models for real phenomena demands teachers' disposition to abandon the control they usually have over what happens in school classrooms, in terms of students' interactions and behavior, as well as over what kind of knowledge is going to be mobilized and dealt with along the process. Altering this control may bring about highly resistant attitudes from teachers and even, in some cases, from students.

Another point that deserves attention in this context of fully integrating modeling in secondary school is the question of a finite (usually short) curriculum time versus the amount of interesting topics suggested to be included. Every time I see a proposal of including something in the (school or teacher education) curriculum, it immediately comes to my mind the following question: supposing school curriculum time is already properly determined, what might be taken out in order to put in whatever is being suggested? As to this point, the author says, in a sensible way:

> Yet, despite all the potential benefits, the time restrictions that may come with the curriculum often act as obstacles to greater presence of modeling activities in the class of mathematics. The iterative nature of the modeling process, the time required for a modeling activity to reap most of its anticipated benefits, the lack of guarantee that even a carefully chosen open-ended modeling problem will lead to some of the mathematical concepts and skills aimed for by the curriculum, may lead teachers to reduce the scope of modeling activities so as to realign them with the content to be covered.

Lastly, another question this chapter offered me the opportunity to raise, though not approached directly in the text, is the following: is there a (sufficient) target for the process of developing mathematical reasoning through curricular activities along K-12 grades? I bring this up because when we globally examine the school curriculum, we observe that mathematics is just one of various disciplines dealt with in this stage of education. Furthermore, differently from Canada and USA, for example, in some countries (including Brazil), students are not offered the possibility of choosing between courses within K-12 grades. Besides that, in Brazil, for example, according to official documents, this time period of schooling (K-12) is not just a propaedeutic stage for college or further education; it is called Basic Education and has its own end and purposes (though one might say, reasonably, that this is just the official discourse). However, I have often seen people forgetting that school students are (usually) children and adolescents, being educated in basic terms (i.e., general, non-specialized, non-professional terms), in various different disciplines. In such a context, even considering that many school students (perhaps the majority of them, in some countries) might proceed to tertiary education, and a part of these eventually enter a mathematics based university course, the school curriculum must leave out some topics, as choices have inevitably to be made in composing any curriculum. One is then led to this kind of thinking: should secondary students, no matter their future education and/or work choices, undergo a development of refined mathematical thinking, under the justification that it helps to "better" (mathematically) understand real world phenomena (that is, it helps examining those phenomena through a relatively sophisticated lens)? Should this experience be part of basic education? Of course there are no simple answers to these questions but, in any case, I would say that whether or not the school courses that include a deeper approach to modeling (as proposed in the chapter) are mandatory should be considered.

In her chapter, Kajander suggests the models and reasoning approach to school mathematics, with an illustration that applies to early secondary grades. She believes this approach may "*support the transition of teacher practice from a more traditional paradigm to a conceptual one*" since it is known to be difficult to take this move all of a sudden. That is an important observation, since experience tells us that proposals of radical changes in school teaching might face resistance. The author briefly elaborates on this point, listing a few sources for this resistance as "*nervousness about the right skills being learned, worries about curriculum coverage, beliefs about what students are and are not capable of, and perceived lack of classroom control.*" By the way, this may turn us back to Caron's chapter, where some of these sources of teachers' resistance (to go beyond curve fit, in that case) may apply.

As Kajander puts it, the use of a models and reasoning approach may make visible some otherwise invisible (for the learner) mathematical processes. In the case of factoring a quadratic polynomial, as shown in the chapter, the area model for multiplication and the geometrical manipulation of squares and rectangles to compose a single rectangle can effectively turn visible and acceptable what may be perceived, by a beginner in algebra, as mysterious or "magic," when obtained by pure algebraic procedures. In general, this experience with the models may point to the schoolteacher a cognitively significant relation between two kinds of mathematical reasoning, exemplified by two different solutions for a problem: one based in analytical reasoning and the other in the so-called synthetic approach. While this distinction in the form of reasoning may sound irrelevant for the development of contemporary mathematics, it is interesting, from the school teaching point of view, to note that even around the sixteenth century, a mathematician as respectable as François Viète used to add a confirming phase, based on geometric construction, to other essentially algebraic phases of his method for solving geometrical problems. That indicates the power of geometric construction in establishing the dependability of a solution, especially for those still not sufficiently familiar with algebraic methods, as is the case of school students at early secondary.

As a last comment on Kajander's chapter, I shall briefly mention a point that may be relevant to schoolteachers. The author refers to leading students eventually to conjecture rules—possibly related to sum-product observation, deduced from the experiments with the manipulative pieces in the bags—to be applied, in order to decide whether or not the quadratic expression in a given ziplock bag is factorable. She also refers to a third (Sharing) phase of the sample lesson, where the students are supposed to share and discuss their strategies, convictions and conclusions. Though understanding (and agreeing) that "*terminology and efficiency can be added later. Student conceptual understanding is the first goal*," I have missed a reference to a pedagogical strategy teachers might use (perhaps at the Sharing Phase) so as to prompt students to check if their conjectures were correct or, otherwise, needed some repair, so to speak. For example, if they arrive at the conclusion that a quadratic expression (in the ziplock bag) is factorable (in two first grade polynomials with real coefficients) if and only if it is "possible" to compose a rectangle using the given pieces, the teacher, as I see it, should be prepared to question, in appropriate ways for the moment, the correctness of this conclusion (and the meaning of the word possible, in this context), since otherwise the students might internalize a piece of knowledge that could function, in the near future, as an obstacle to learning, when terminology and efficiency are in time to be added. I think it is important to mention the need for this didactical strategy because the author (wisely) suggested that some of the ziplock bags should be filled with pieces that lead to non-factorable quadratic expressions. However, as the example $x^2 + 3x + 1$ shows—and the pieces for this example could have been put in one of the bags—the process of actually constructing a rectangle using the pieces on the bags may not be possible (in the manipulative sense described in the sample lesson) in some R[x]-factorable cases.

In his chapter, E. Barbeau points to the discussion of a mathematical result (a necessary and sufficient condition for the commuting of the composition of two linear polynomials) as a way to involve secondary students in "*pure mathematical skills of reasoning in algebraic problems*." In the last two sections, the author comments on pedagogical issues related to the eventual school classroom use of the material presented.

As to the activity proposed, I comment on a few ideas advanced by the author. I agree with his point in that to make mathematics meaningful for school students, though a common strategy is connecting it to student's life and experience, it need not to be always this way. A purely mathematical context may as well appeal to students. I would just add that difficulties might arise in this "pure" context. For example, in case the course is obligatory to all secondary students, it may be difficult for a teacher to opt for this kind of activity without being convinced that its potential benefits (some of them referred to by the author in the "Pedagogical issues" and the "Conclusion" sections) could not as well be achieved through usual curriculum activities, which may be attractive or interesting to a larger spectrum of secondary students.

I also observe, considering another aspect of working at school in a purely mathematical context, that it might induce the idea of arbitrariness, especially to beginners. Separated from any reference to a "real" context, purely abstract mathematics may seem (to many secondary students) to operate in a restriction-free atmosphere, that is, no conditions under which definitions, objectives, lines of reasoning, procedures, etc. are to be created, accepted, chosen, etc., except for the strong requirement of avoiding internal contradictions. Hence, the idea of school mathematics as an arbitrary system of non-contradictory rules may arise (I remember Morris Kline (1974) in *Why Johnny can't add*, writing something like this: a child must understand that a + b equals b + a not because the commutative property of addition is valid, but quite to the contrary, addition has this property because a + b always equals b + a). In this sense, schoolteachers who decide to work with the author's suggested activity may be prepared to deal with potential questions like why is composition of function defined in a way that it happens to be non-commutative? In other words, what really "makes" an "interesting" definition of composition of functions non commutative? Yet another way to put it: could composition not have been defined so as to be commutative, and then in such a case this kind of activity (the search for a necessary and sufficient condition under which the composition of two linear polynomials commute) would not make sense? Or perhaps issues like: accepting the established definition of composition of functions, what kind of conditions on higher degree polynomials would make the composition of two of them commute? What about conditions (for composition to commute) in the case of other functions as logarithmic, exponential, trigonometric, and so on? If these cases are not to be dealt with, why study the case of first-degree polynomials? Is there an explanation on why this fixed-point condition only applies to linear polynomials?

Finally I would like to emphasize that I raise these questions not in disagreement with the author, but only as a tentative enlargement of his list of pedagogical issues presented in the chapter.

In their chapter, Burazin and Lovric discuss three aspects that usually impact the transition from secondary to university mathematics, namely mathematical symbols, language, and reasoning. The examples they list are amazingly coincident with those I have been encountering in my mathematics classes at the university, which indicates that the issues discussed in the chapter are rather of international concern than strictly Canadian-only issues. I would add that these issues should be of special concern to mathematics teacher education programs all over the world.

A distinguishing point in the chapter is the authors' assertion that facilitating transition is a duty that should be taken care of by the school *and* the university. Going a little further, I think it should be kept in mind that school has to provide basic general education to every member of the society (as to whether or not this is really to be accomplished makes room for a particularly interesting debate, but not exactly appropriate to this Commentary Chapter), while university education usually constitutes a more specific and individual choice. This choice, of course, is strongly influenced by social factors in general (e.g., social attractiveness of each profession), and, in particular, the greater or lesser attraction of the student towards the specialized knowledge required in the corresponding professional practice. Therefore, it seems reasonable to argue that university specific courses should take a greater share on the duty of facilitating transition from secondary to tertiary mathematics.

In what follows, I express a somewhat different view over some of the authors' contention in the chapter. As to the first two cases discussed (i.e., mathematical symbols and number bias), I see the examples and suggestions presented as highly pertinent to mathematics schoolteachers' practice. Furthermore, it seems to me that promoting, in school education, a flexible way of giving meaning to school mathematics symbols as well as stimulating the development of a "non-biased relationship with numbers" (as explained in the chapter), should be an important part of basic education, independent of considerations on difficulties that may arise in transition from secondary to university mathematics. The same arguments advanced by the authors in their text (e.g., eventual expansion of the competence for using school mathematics in different situations and contexts) will support the above assertion of mine. I would like to add a reference to the case of mathematical symbols: two articles that provide an interesting view on interpreting symbols in the learning of mathematics (Gray and Tall 1993, 1994).

The third case discussed—language and mathematical culture—starts with a controversial statement that may incite different reactions, according to the perspective each reader may hold, relative to what mathematics consists of. The authors say, "*It is important to emphasize to our students that mathematics consists of, and deals with concepts, objects, and algorithms which are precise, unambiguous, and well-defined*." My comments on this statement come from my particular point of view on school mathematics education. First of all, the assertion leads me to wonder what kind of mathematics we are talking about when we affirm it consists of, and deals with precise, unambiguous, and well-defined concepts, objects, and algorithms. In school mathematics, one may not be able to define, in such a well-defined way, even the set of natural numbers. While in many cases it might be necessary to produce a school-validated definition for a mathematical object, it seems to me pedagogically questionable to emphasize to (school) students that mathematics consists of, and deals with well-defined concepts and objects, because, as far as my experience goes, it is not pedagogically viable (neither necessary) to deal with many fundamental concepts in school mathematics in such a precise, unambiguous and well-defined way (e.g., the set of real numbers, addition of natural numbers, addition and multiplication of real numbers, among others). In this sense, it may be regarded as educationally inadequate to land school students with a view of mathematics that does not fit to the reality actually experienced in the learning (and teaching) process of this discipline at school.

Going a little further in this issue, I would like to invite the reader (and the authors) to another front of reflection: looking back at some historical period, as Isaac Newton's times for example, could we say that mathematics deals with precise, unambiguous and well-defined objects? Newton himself could not tell what a real number or a function is, in a (today considered) precise, unambiguous and well-defined way. Nevertheless, this didn't impair him of doing what he did in favor of the development of mathematical knowledge.

Still on the themes of mathematical culture related to precise, unambiguous, and well-defined objects mathematics deals with, I shall comment on the discontinuity of the function $f(x) = \frac{1}{\sqrt[3]{x}}$ in $x = 0$. Can we say that it makes sense, nowadays, from a strictly mathematical point of view, to teach Calculus students that a function is discontinuous in a point for which it is not defined (that is, where it does not even exist)? Other questions the chapter may induce are: how does mathematical culture help students to accept this fact (a function being discontinuous in a point where it doesn't exist), considering that it is just provisional, until a course in Real Analysis comes about with a more precise (for the present time) definition of continuity at a given point, according to which a function can only be continuous or discontinuous in a point of its domain? How does this quarrel between Calculus and Real Analysis textbook definitions of continuity of a function at a point suit the assertion that mathematics deals with precise, unambiguous, and well-defined objects?

To finish my commentary on Burazin and Lovric's chapter, I would like to say a word on how one might react to the assertion "*In our view, it does not make sense to discuss whether or not 1 is a prime number. It is not, and the definition is clear about it*." As a teacher educator, I would say that, in my view, it does make sense to discuss whether or not 1 is a prime number whenever a student or a schoolteacher happens to have any doubt about this matter, independent of the clearness of the formal definition. In my experience, I see this issue as one both prospective and in practice teachers often bring up to discussion in teacher education courses. I would even suggest, if I may, two references on this matter. From the point of view of uncovering a network of teacher's professional knowledge, potentially associated with a discussion around the question of whether or not 1 is a prime, it may be interesting to see chapter 3 of Davis and Renert (2014). From a more general point of view, but still relevant towards a better understanding of the role of definitions in the learning and teaching of mathematics, I would suggest the article by Vinner (1991).

In the last chapter of this Part VI, Gardner describes a university bound course on Statistics he has been teaching at secondary school. I must say, at this point, that I do not have any experience in teaching statistics, neither at school nor at university. Therefore, my comments on this chapter may be viewed as coming from an outsider, so to speak.

I start by saying that, in Brazil, we do not have this kind of course (university bound) at school, not even as a possibility for a student's choice. All students in each grade, in a given school, have the same (annual) courses, though this is in a process of changing, possibly from 2019 on. Statistical basic concepts like measures of central tendency and measures of dispersion are dealt with as a topic in standard school mathematics courses. Thus the course Gardner describes seems a long way ahead of what we usually do in the teaching of statistics within Brazilian Basic Education (1–12 grades). It may be, as far as I am concerned, somewhat equivalent to the (only) one semester course in statistics that prospective mathematics teachers usually have to take in their undergraduate program in Brazil.

I should also say that I appreciate the way the course is organized, confirming what is said at the beginning of the chapter, related to the matter of students knowing as precisely as possible what is to be done in the course and the criteria for assessment. The activities along the course look rich and productive, as, for example, the peer review of the project students must develop. In general, I see the course as a demanding one (though not excessively), but at the same time offering a corresponding level of support to the students. As to the results, the text indicates that they are good enough to allow the author report students coming back to the teacher, thanking him for the way he handles the course. Therefore, I could say nothing else but congratulate the author for the planning and execution of the course he describes in the chapter.

A Concluding Note

Mathematics school teaching demands, at least theoretically, an ingenious professional, capable of creating and mobilizing a blend of pedagogical, curricular, psychological, didactical, and mathematical knowledge, both when planning and when actually teaching in the classrooms. Furthermore, as participating in a mandatory social process of general education, the mathematics teacher has also to act wisely in balancing his professional duty of teaching what is to be taught, with the personal, though highly socially conditioned, interests of each student in learning what it is to be learnt. A third factor that gets into this balancing challenge may be referred to as the "operation mode" of the school, viewed as the running of an ultimately social institution. These considerations, in conjunction with the context of the comments I have just presented, lead me to a reflection that can be synthetized this way: whilst all chapters in this Part VI are, together with many other studies that can be found in the Mathematics Education research literature, potentially contributive to international school mathematics teaching practice (of course with the eventually necessary adaptations), one might anticipate a historical (and likely intensity-varying across countries) difficulty to convert the potentiality of these contributions into day-to-day reality of school teaching practice. Though some plausible hypothesis have been raised in the literature, I still hunger for studies that could help to clarify why that happens.

References

Ball, D. L., Thames, M. H., & Phelps, G. (2008). Content knowledge for teaching: What makes it special? *Journal of Teacher Education, 59*(5), 389–407.

Bednarz, N., & Proulx, J. (2009). Knowing and using mathematics in teaching: Conceptual and epistemological clarifications. *For the Learning of Mathematics, 29*(3), 11–17.

Cajori, F. (1928). *A history of mathematical notation* (Vol. I). London: Open Court Company.

Davis, B., & Renert, M. (2014). *The math teachers know: Profound understanding of emergent mathematics*. New York: Routledge.

Gray, E. M., & Tall, D. O. (1993). Success and failure in mathematics: The flexible meaning of symbols as process and concept. *Mathematics Teaching, 142*, 6–10.

Gray, E. M., & Tall, D. O. (1994). Duality, ambiguity and flexibility: A proceptual view of simple arithmetic. *Journal for Research in Mathematics Education, 26*, 115–141.

Kline, M. (1974). *Why Johnny can't add: The failure of the new math*. New York: Vintage Books.

Lins, R. C. (2006). Characterizing the mathematics of the mathematics teacher from the point of view of meaning production. In M. Niss (Ed.), *10th International Congress on Mathematical Education* (pp. 1–16). Copenhagen.

Moreira, P. C. (2004). *O conhecimento matemático do professor: Formação na licenciatura e prática docente na escola básica*. Unpublished doctoral dissertation, Faculdade de Educação UFMG, Belo Horizonte, MG, Brazil.

Moreira, P. C., & David, M. M. (2005). *A formação matemática do professor: Licenciatura e prática docente escolar*. Belo Horizonte: Autêntica.

Moreira, P. C., & David, M. M. (2008). Academic mathematics and mathematical knowledge needed in school teaching practice: Some conflicting elements. *Journal of Mathematics Teacher Education, 11*, 23–40.

Vinner, S. (1991). The role of definitions in the teaching and learning of mathematics. In D. Tall (Ed.), *Advanced mathematical thinking* (pp. 65–81). Dordrecht: Kluwer.

Final Commentary: The Challenge Continues

This edited volume started off as a practical project of the first editor to fill a self-identified void of not having a singular Canadian text that could be used as a source of relevant literature in teaching future secondary school mathematics teachers. Enter the Canadian Mathematics Education Study Group (CMESG), specifically and perhaps appropriately, a conversation during a meal at the 40th anniversary meeting of CMESG, in Kingston, Ontario, and a much bigger vision was born. By the end of a scenic boat ride, we began collaborating as editors, and the project was greatly expanded, eventually bringing together a collection of Canadian authors who work and research within Canadian secondary classrooms in some capacity. Broader vision and context would be provided by having a contributor who was foundational within the Canadian mathematics education community, begin each section with a preface. The 'landscape' reference was purposeful; we wanted to represent our varying culture, geography, and context as broadly as possible. As well as including Indigenous and Francophone perspectives, each section of chapters also included the voice of a current classroom teacher ("A teacher's view"), in order to provide a practical, grassroots examination of some aspect of the section theme from a practicing teacher's perspective. In the end, the addition of the international authors who graciously offered to read the pieces and use their own contexts to comment on the works greatly added to the depth of what could be accomplished within this single volume and allowed for us to see the possibilities for connecting our Canadian contexts to those beyond our own borders. Below we detail some final thoughts on how we put the sections together and take a look at how the commentaries in each of the six parts have challenged or supported the writings within each section.

The collected volume began with a highly meaningful preface by Edward Doolittle that encompassed a fundamental vision for putting together this collection in the Canadian context. As he noted, "Indigenous culture and issues are foundational to Canada; and …continue to be a necessary part of anything Canadian" (Preface, this volume). We were deeply honored by the connections made between

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the work of the Canadian authors that we had compiled for this volume to this fundamental part of being Canadian and working in a Canadian school context. Doolittle's moving preface highlighted how all of the authors did in fact support this context to some degree. In this vein, he also brought the volume to an international audience in his final comments on how these are issues that all peoples face and added, "the question is what we [as Indigenous peoples] have to offer to Canada, and the world" (Doolittle, this volume). It is with this initial overview of connections to Indigeneity that situates all of the sections and chapters under one broader theme and extends the link to secondary school mathematics.

We chose to begin the collection of authors' work with a look at "The changing landscape of teaching and learning mathematics" because we have all been participants and observers in how teaching in secondary schools has changed and continues to change. We were honoured that two Canada Research Chairs (Rina Zazkis and Nathalie Sinclair), as well as mathematician and long-standing mathematics education advocate Walter Whiteley, contributed prefaces to this section. In this first section, we chose to situate the volume in the historical as well as cultural landscape of our context. Beginning the contained chapters with Peter Taylor was an obvious choice because he has been pivotal in how Canadian mathematics education has been shaped over the last four decades, which he describes in his chapter. As a 'founding father' of CMESG, an organisation with which all of the various section preface authors have been involved, Peter has forever left a mark on teaching and learning mathematics in Canada. We also wanted to include a strong focus on the Indigenous knowledges that both inform and shape our Canadian classrooms. As a more recent aspect of our context, Canadian schools are beginning to focus on emotional well-being and mental health so this was also deemed a necessary aspect of this first section. Kaino's commentary on this section looked at how the chapters were illustrative of the changing portions of school from his own context and perspective. As he noted, this section provides a chance to "re-think innovative and better ways of teaching and learning mathematics" (Kaino, Part I commentary, this volume). This need for innovative changes answers the call of many mainstream media headlines that question the effectiveness and status of mathematics education in Canada and many other areas of the world. In the end, Kaino supports the driving vision of today's classrooms (and this volume): to "provide ways for long-term retention of mathematical knowledge" (Part I commentary, this volume). Part I provided the broad strokes of the challenges facing today's secondary classrooms in preparing the learners to be mathematics users (and not just learners), and the remaining sections each tackle specific areas relating to classroom teaching.

"Shifting to a culture of inclusion," as introduced by David Pimm, a well-known mathematics educator in Canada, focussed attention on helping all learners to succeed in mathematics classrooms, not just those who would go on to become mathematics educators and mathematicians later in life. Our choices for the chapters examined those who are most at-risk in our classrooms including students who are coming from other parts of the world. Although most of the chapters around at-risk classrooms were Ontario based, we felt the stories were not unique to this part of the country or even the world, and the stories could highlight learners who need the most support in our classrooms. Through these chapters, we reiterate the fundamental idea that the information included is supportive of the quote by the Expert Panel on Student Success in Ontario (2004) as well as other learning initiatives (e.g., Ontario Ministry of Education 2013): "good for all, necessary for some" (p. 42). Beswick, in her commentary, responds to the chapters by beginning with an overview of some of the difficulties in enacting the ideas in the chapters in this section. The first challenge she brings forward is the complex issue of how a teacher's beliefs interact (or interfere) with how pedagogy is taken up and implemented in the classroom. This idea is a theme throughout the rest of the book as many of the chapters challenge traditional ideas of what teaching mathematics at the secondary school level is and should be.

Part III in the volume, introduced by Elaine Simmt, another well-known Canadian mathematics education researcher, discusses how relationships can be fostered in classroom environments. As stated by Boland and Tranter in Part I, the basis of all teaching experiences is the relationship that is formed with students and the school community. The chapters in this section critically look at different ways relationships can help foster greater student success in academics by focussing on behaviours and other characteristics. Mosvold notes, "The fostering of relationships, then, goes beyond attending to students' mathematical thinking, and it involves getting to know their histories, the experiences they have made in and outside of school, their cultural background, and everything else that constitutes their identity" (Part III commentary, this volume). His commentary on the section notes that the connections among chapters attend to the different aspects that constitute the identities of secondary students in the classroom. As he notes, two of the chapters specifically focus on relationships within the classroom and student thinking; whereas, the other three focus on the development of the whole person through the relationships. Mosvold ends his commentary with a concern of the tendency to take theories of learning from other fields and then use them to apply to teaching by assuming them as theories of teaching. He turns his focus to how the chapter with the strongest link to teaching is also the one without an explicit theoretical foundation (Newell). The noted chapter is an interesting treatise into looking at teaching as all that a teacher does in an effort to support the learning of students. Mosvold concludes that "more conceptual work needs to be done in studies of mathematics teaching, and conceptualizations of mathematics teaching should strive towards capturing the dynamic interactions between mathematical and pedagogical aspects of the work of teaching" (Part III commentary, this volume). The commentary ends with a discussion about the complex nature of teaching and how this section has shown the need to conceptualize teaching as more than just certifications. Following the more theoretical nature of this section, the edited volume moves into specific pedagogy for teaching mathematics.

The part entitled "Enhancing problem-based learning" was meant to serve as a collection of chapters that focus on specific examples of using this type of learning environment in the classroom. All of the work in the section was meant to answer the call of mandates by the Canadian *Manifesto* (Whiteley and Davis 2003/2016), the National Council of Teachers of Mathematics (2000), and others, to include more tasks and explorations in the classroom. The focus in the section is on using problems in a way that allows students to explore the mathematics, and not just use routine problems to practice previously known formulas. Tom Kieren, long standing, influential mathematics education scholar, provided the preface. The chapters in this section are illustrative of how the mathematics comes from the "doing," rather than for students to repetitively use something that a teacher has shown them to do. Leikin and Mason provide two very different commentaries on the chapters from different international perspectives. Leikin takes the viewpoint of looking at the chapters as unified through the concepts presented in a model of characteristics that determine mathematical challenge (Part IV commentary, this volume). Her presented model focusses on the conceptual characteristics of the problem, socio-mathematical norms, instructional setting, and individual characteristics of participants as the four foci, and uses the model as a framework to assess the mathematical challenge of tasks. For her, the chapters in this section present ideas that can be summarized through this model and notes that each of the chapters shows the authors sharing their "authentic experiences" in using problem solving methods in the classroom (Leikin, Part IV commentary, this volume). Mason frames the chapters in this section through the lens of "doing" mathematics in a way that is more than just an execution of mathematical procedures. As he notes, the unifying theme in the section is that the authors "are trying to make a difference, trying to get students engaged with mathematics and involved in mathematical thinking" (Mason, Part IV commentary, this volume). Through his discussion of the history of mathematical problem solving, Mason notes that these chapters differ from the idea that teaching mathematics "is assumed to be about training student behaviour so that learners can carry out operations on numbers" (Part IV commentary, this volume). Both Mason and Leikin key into the role of context within many of the chapters in this section of the volume, albeit from different lenses. Leikin focusses on the social justice and citizenship aspects in the chapters, and Mason approaches the ideas from a context standpoint with a goal of reaching for social justice. Mason explicitly links this idea to the concerns raised by Russell (Part IV, this volume) in making sure that the social justice themes do not eclipse the mathematics. Mason concludes by reiterating that the chapters in this section are not meant to "prove" their approaches but rather to serve as a description of how the authors in the chapters have used the techniques in their own experiences.

Part V, prefaced by Carolyn Kieran, another very prominent mathematics education researcher, particularly well known for her work in algebra, provided concrete examples of planning and assessment to bring directly into secondary classrooms. These chapters were intentionally chosen to challenge the ideas that teaching is all about direct transmission of knowledge from teacher to student and that assessment is all about exams and quizzes that "test" how well students have retained what has been shown to them. Reddy and Sriraman both provide commentaries on the chapters in this section through their differing experiences and contexts. Reddy (this volume) provides an interesting commentary on both the commonalities within the six chapters in this section, but also how the ideas relate to the struggles of secondary teaching in South Africa. He begins with a look at the vast differences between Canada and South Africa in terms of the economic and political struggles of his home country; however, strengthens his position on the importance of education to move the country forward in an effort to alter the glaringly negative headlines that Reddy mentions in his commentary, as well as to improve the future conditions of the students. He points to the difficulties with the Curriculum 2005 (Reddy, Part V commentary, this volume) that highlighted a challenge with the skill set of the teachers who would be implementing the curriculum, a difficulty that is taken for granted as not an issue within the chapters in this section of the book. This commentary reminds us of the relative position Canada maintains in providing secondary teachers with rigorous schooling to prepare them to enact the content in this section on planning and assessment. The “learning outcome” discussion of Reddy in relation to the South African curriculum notes the curriculum revision included “skills and the processes of learning, without sufficient specification of content and knowledge” (Reddy, Part V commentary, this volume) serving as an interesting parallel with Canadian curriculum as a blend of the two: based on what students should be able to do and know by the end of the school year. Reddy concludes with a discussion of the social inequities facing the South African school system and mounts a challenge: “As Canada becomes more diverse, I would imagine it would be challenged to take into consideration, more significantly, outside classroom contexts to improve the learning for students” (Part V commentary, this volume). This statement echoes the underpinning themes within Part II of this volume, so although not specifically addressed in this section, shows a need for Canadian teachers to focus on more than just content when working with students. Sriraman approaches the chapters in this section from a different vantage point: an overarching question of what is actually being measured and notes that the chapters in the section leave concerns over how these ideas can be implemented on a larger scale beyond the single classroom that they seem to address (and in some cases beyond the single cases discussed in the chapter to an entire classroom). His initial commentary pushes back on the chapters as not being supportive of the use of multiple choice questions in testing student understanding. He attests that the chapters imply there is a link between open tasks and conceptual understanding, and multiple-choice questions and procedural understanding. He ends with a commentary on what assessment really is and asks for the reader to question the possible win-loss scenarios inherent in testing. He calls for assessment to be aligned to learning as well as deep memory in order to create a “win-win” situation for both the student and the teacher. He further extends this to state that memory in mathematics is not simply rote memorization or recitation but something deeper, flexible, and stronger for students to retrieve, adapt, and use the information in situational experiences. His final comments seem to suggest agreement with the chapters' push to consider assessment (and planning) as not just to prepare for or use multiple choice tests.

The final part of this book, "Broadening mathematical understanding through content," focusses on specific instances of content within the secondary curriculum. The goal in placing these chapters together was to look at example spaces within the secondary classroom that are closely tied to the content students are expected to learn. Gila Hanna, known internationally for her work in many areas particularly around notions of proof in mathematics education research, provides the preface, while Moreira provides the international context to the section through his view on the chapters and their ability to "broaden" mathematical understanding as the section claims. He begins with a separation between the "professional practice of mathematicians" and the "professional practice of mathematics schoolteachers" in order to frame the chapters (Moreira, Part VI commentary, this volume). He does caution that although these are distinct practices and sets of knowledge, they are not opposing, though the knowledge of the mathematicians is not "necessary nor sufficient" for the knowledge needed by schoolteachers (Moreira, Part VI commentary, this volume). He provides an additional challenge to each of the chapters in his description meant to push these understandings of mathematics towards the knowledge of mathematicians and where this adds to or separates from the knowledge of mathematics schoolteachers. He notes that the issues raised by Burazin and Lovric, although based in a Canadian context, parallel international issues connected to his own work. In the end, Moreira comments on the complexities of what teachers must know and do to teach mathematics effectively, which provides an overall statement that reinforces the entirety of the collection of authors in this volume.

While not all invitees were able (or willing) to contribute, the breadth and range of contributing voices (totalling 85 individual contributors in all), including many of the founders of CMESG, editors of various mathematics education journals in Canada and from around the world, authors of current curriculum documents, new and experienced classroom teachers, and a breadth of international scholars from five continents, left us deeply humbled.

This volume was not meant to serve as an exhaustive collection of all the current issues and challenges facing Canadian secondary school teaching and learning. Rather, it was meant to showcase the variety and range of research and resources to both expand and deepen the conversations around the issues and challenges facing both the research community as well as today's secondary school teachers of mathematics. The speed with which technology is changing makes it impossible to know precisely how to prepare students for the future, hence developing the ability to think and reason mathematically may be the best possible preparation for students to be ready to face the currently unknown mathematical challenges that are awaiting us all in the coming years.

References

Expert Panel on Student Success in Ontario. (2004). *Leading math success: Mathematical literacy grades 7–12*. Toronto: Queen's Printer for Ontario.

National Council of Teachers of Mathematics. (2000). *Principles and standards for school mathematics*. Reston: Author.

Ontario Ministry of Education. (2013). *Learning for all: A guide to effective assessment and instruction for all students, Kindergarten to grade 12*. Retrieved from http://www.edu.gov.on.ca/eng/general/elemsec/speced/LearningforAll2013.pdf

Whiteley, W., & Davis, B. (2016). A mathematics curriculum manifesto. In P. Liljedahl, D. Allan, O. Chapman, F. Gourdeau, C. Lajoie, S. Oesterle, E. Simmt, & P. Taylor (Eds.), *Canadian Mathematics Education Study Group: 40 years of CMESG (Les 40 ans du GCEDM)* (pp. 193–197). Burnaby, BC: CMESG/GCEDM. (Reprinted from *Proceedings of the 2003 annual meeting of the Canadian Mathematics Education Study Group/Groupe Canadien d'Etude en Didactique des Mathématiques*, pp. 79–83, by E. Simmt & B. Davis, Eds., 2003, Edmonton, AB: CMESG/GCEDM)

References

Adams, B., & Vallance, J. (1984). *Summer of '69* [Recorded by B. Adams]. On *Reckless.* Santa Monica: A&M Records.

Ahmed, A. (1987). *Better mathematics*. London: HMSO.

Aikenhead, G. S. (1996). Science education: Border crossing into the subculture of science. *Studies in Science Education, 27*(1), 1–52. https://doi.org/10.1080/03057269608560077.

Aikenhead, G. S. (2006). *Science education for everyday life: Evidence-based practice*. New York: Teachers College Press.

Aikenhead, G. S. (2008). Objectivity: The opiate of the academic? *Cultural Studies of Science Education, 3*, 581–585.

Aikenhead, G. (2017a). Enhancing school mathematics culturally: A path of reconciliation. *Canadian Journal of Science, Mathematics and Technology Education, 17*(2), 73–140. https://doi.org/10.1080/14926756.2017.1308043.

Aikenhead, G. (2017b). *School mathematics for reconciliation: From a 19th to a 21st century curriculum*. Retrieved from https://www.usask.ca/education/documents/profiles/aikenhead/index.htm

Aikenhead, G. S., & Elliott, D. (2010). An emerging decolonizing science education in Canada. *Canadian Journal of Science, Mathematics and Technology Education, 10*, 321–338.

Aikenhead, G., & Huntley, B. (1999). Students learning Western and Aboriginal science. *Canadian Journal of Native Education, 23*(2), 159–175.

Aikenhead, G., & Michell, H. (2011). *Bridging cultures: Indigenous and scientific ways of knowing nature*. Toronto: Pearson Education Canada.

Aikenhead, G., Brokofsky, J., Bodnar, T., Clark, C., … Strange, G. (2014). *Enhancing school science with Indigenous knowledge: What we know from teachers and research*. Saskatoon: Saskatoon Public School Division. Retrieved from http://www.amazon.ca/Enhancing-School-Science-Indigenous-Knowledge/dp/149957343X

Albert, J. (2006). Interpreting probabilities and teaching the subjective viewpoint. In G. F. Burrill & P. C. Elliott (Eds.), *Thinking and reasoning with data and chance* (pp. 417–433). Reston: National Council of Teachers of Mathematics.

Alberta Assessment Consortium. (2014). *AAC performance assessment development template*. Retrieved from https://aac.ab.ca/

Alberta Assessment Consortium. (2015). *AAC key visual*. Retrieved from http://www.aac.ab.ca/updated-aac-key-visual/

Alberta Education. (2006). *Common curriculum framework for K-9 mathematics: Western and Northern Canadian protocol*. Edmonton: Author.

A. Kajander et al. (eds.), *Teaching and Learning Secondary School Mathematics*, Advances in Mathematics Education,
https://doi.org/10.1007/978-3-319-92390-1

Alberta Education. (2008a). *Mathematics grades 10–12*. Retrieved from https://education.alberta.ca/media/564028/math10to12.pdf

Alberta Education. (2008b). *The Alberta 10–12 mathematics program of studies with achievement indicators*. Edmonton: Alberta Education.

Alberta Learning. (2002). *First Nations, Métis and Inuit education policy framework.* Edmonton: Government of Alberta. Retrieved from http://education.alberta.ca/media/164126/framework.pdf

Ameis, J. A. (2011). The truth about pemdas. *Mathematics Teaching in the Middle School, 16*(7), 414–420.

Antone, E. M. (2000). Empowering Aboriginal voice in Aboriginal education. *The Canadian Journal of Native Education, 24*(2), 92–101.

Arbaugh, F. (2003). Study groups as a form of professional development for secondary mathematics teachers. *Journal of Mathematics Teacher Education, 6*, 139–163.

Archambault, I., Janosz, M., Morizot, J., & Pagani, L. (2009). Adolescent behavioural, affective, and cognitive engagement in school: Relationship to dropout. *Journal of School Health, 79*(9), 408–415.

Arviso Alvord, L., & Cohen Van Peet, E. (1999). *The scalpel and the silver bear*. Toronto: Bantam Books.

Assembly of First Nations (AFN). (2010). *First Nations control of First Nations education*. Ottawa: Author. Retrieved from http://www.afn.ca/uploads/files/education/3._2010_july_afn_first_nations_control_of_first_nations_education_final_eng.pdf

Astolfi, J.-P. (1993). Placer les élèves dans une situation-problème? *Probio-Revue, 16*(4), 311–321.

Atlantic Provinces Education Foundation. (1996). *Foundation for the Atlantic Canada mathematics curriculum*. Halifax: Atlantic Provinces Education Foundation.

Australian Curriculum, Assessment and Reporting Authority. (2017). *Australian curriculum: Mathematics F-10*. Retrieved from http://v7-5.australiancurriculum.edu.au/mathematics/curriculum/f-10?layout=2#page=1

Avalos, B. (2011). Teacher professional development in teaching and teacher education over ten years. *Teaching and Teacher Education, 27*(1), 10–20.

Balacheff, N. (1988). Aspects of proof in pupils' practice of school mathematics. (Pimm, D., Trans.). In D. Pimm (Ed.), *Mathematics, teachers and children* (pp. 216–235). London: Hodder and Stoughton.

Balfanz, R., & Byrnes, V. (2006). Closing the mathematics achievement gap in high-poverty middle schools: Enablers and constraints. *Journal of Education for Students Placed at Risk, 11*(2), 143–159.

Balfanz, R., Mac Iver, D. J., & Byrnes, V. (2006). The implementation and impact of evidence-based mathematics reforms in high-poverty middle schools: A multi-site, multi-year study. *Journal for Research in Mathematics Education, 37*(1), 33–64.

Ball, P. (2013, December 16). Polynesian people used binary numbers 600 years ago. *Nature*. Retrieved from https://www.nature.com/news/polynesian-people-used-binary-numbers-600-years-ago-1.14380

Ball, D. L., & Bass, H. (2000). Interweaving content and pedagogy in teaching and learning to teach: Knowing and using mathematics. In J. Boaler (Ed.), *Multiple perspectives on the teaching and learning of mathematics* (pp. 83–104). Westport: Ablex.

Ball, D. L., & Cohen, D. K. (1999). Developing practice, developing practitioners: Toward a practice-based theory of professional education. In G. Sykes & L. Darling-Hammond (Eds.), *Teaching as the learning profession: Handbook of policy and practice* (pp. 3–32). San Francisco: Jossey Bass.

Ball, D. L., & Forzani, F. M. (2009). The work of teaching and the challenge for teacher education. *Journal of Teacher Education, 60*(5), 497–511.

Ball, D. L., Thames, M. H., & Phelps, G. (2008). Content knowledge for teaching: What makes it special. *Journal of Teacher Education, 59*(5), 389–407.

Bang, M., & Medin, D. (2010). Cultural processes in science education: Supporting the navigation of multiple epistemologies. *Science Education, 94*, 1009–1026.

Bar-Hillel, M. (1989). How to solve probability teasers. *Philosophy of Science, 56*(2), 348–358.
Bar-Hillel, M., & Falk, R. (1982). Some teasers concerning conditional probabilities. *Cognition, 11*, 109–122.
Bartlett, S., & Burton, D. (2007). *Introduction to education studies* (2nd ed.). London: Sage.
Barton, B. (2008). *The language of mathematics: Telling mathematical tales*. New York: Springer.
Barton, B., Fairhall, U., & Trinick, T. (1998). Tikanga Reo Tātai: Issues in the development of a Māori mathematics register. *For the Learning of Mathematics, 18*(1), 3–9.
Barwell, R. (2005). Working on arithmetic word problems when English is an additional language. *British Educational Research Journal, 31*(3), 329–348.
Barwell, R. (2009). *Mathematics in multilingual classrooms: Global perspectives*. Bristol: Multilingual Matters.
Barwell, R. (2012). Heteroglossia in multilingual mathematics classrooms. In H. Forgasz & F. Rivera (Eds.), *Towards equity in mathematics education: Gender, culture and diversity* (pp. 315–332). Heidelberg: Springer.
Barwell, R., Clarkson, P., Halai, A., Kazima, M., Moschkovich, J., Planas, N., Setati Phakeng, M., Valero, P., & Villavicencio Ubillús, M. (Eds.). (2016). *Mathematics education and language diversity: The 21st ICMI study*. Cham: Springer.
Bass, A. (2013). *Math study skills* (2nd ed.). Boston: Pearson Education.
Basso, K. H. (1996). *Wisdom sits in places: Landscape and language among the Western Apache*. Albuquerque: University of New Mexico Press.
Bastien, B. (2004). *Blackfoot ways of knowing*. Calgary: University of Calgary Press.
Batanero, C. (2014). Probability teaching and learning. In S. Lerman (Ed.), *Encyclopedia of mathematics education* (pp. 491–496). New York: Springer.
Batanero, C., & Diaz, C. (2012). Training school teachers to teach probability: Reflections and challenges. *Chilean Journal of Statistics, 3*(1), 3–13.
Batanero, C., Arteaga, P., Serrano, L., & Ruiz, B. (2014). Prospective primary school teachers' perception of randomness. In E. J. Chernoff & B. Sriraman (Eds.), *Probabilistic thinking: Presenting plural perspectives* (pp. 345–366). Dordrecht: Springer.
Bateson, G. (1972). *Steps to an ecology of mind*. New York: Ballantine.
Battiste, M. (1987). Mi'kmaq linguistic integrity: A case study of a Mi'kmawey school. In J. Barman, Y. Hébert, & D. McCaskill (Eds.), *Indian education in Canada: The challenge* (Vol. 2, pp. 102–125). Vancouver: UBC Press.
Battiste, M. (2002). *Indigenous knowledge and pedagogy in First Nations education: A literature review with recommendations*. Ottawa: National Working Group on Education and Indian and Northern Affairs Canada.
Baumert, J., Kunter, M., Blum, W., Brunner, M., Voss, T., Jordan, A., et al. (2010). Teachers' mathematical knowledge, cognitive activation in the classroom, and student progress. *American Educational Research Journal, 47*(1), 133–180.
Bay-Williams, J. M., & Martinie, S. L. (2015). Order of operations: The myth and the math. *Teaching Children Mathematics, 22*(1), 20–27.
Bednarz, N., & Proulx, J. (2009). Knowing and using mathematics in teaching: Conceptual and epistemological clarifications. *For the Learning of Mathematics, 29*(3), 11–17.
Bednarz, N., Kieran, C., & Lee, L. (Eds.). (1996). *Approaches to algebra: Perspectives for research and teaching*. Dordrecht: Kluwer.
Bednarz, N., Maheux, J., & Barry, S. (2007). Context-based professional development in mathematics education: Analysis of a collaborative research process aimed at supporting it. In T. Lamberg & L. R. Wiest (Eds.), *Proceedings of the 29th annual meeting of the North American chapter of the international group for the psychology of mathematics education* (pp. 788–795). Stateline: University of Nevada, Reno.
Beltzner, K. P., Coleman, A. J., & Edwards, G. D. (1976). *Mathematical sciences in Canada background study No. 37*. Ottawa: Science Council of Canada.
Benard, B. (2004). *Resiliency: What we have learned*. San Francisco: WestEd.
Benard, B. (2005). *What is it about Tribes?* (1st ed.). Windsor: CenterSource Systems.

Benson, P. L. (2007). Developmental assets: An overview of theory, research, and practice. In R. K. Silbereisen & R. M. Lerner (Eds.), *Approaches to positive youth development* (pp. 33–58). Thousand Oaks: Sage.

Ben-Zvi, D., & Garfield, J. (2004). *The challenge of developing statistical literacy, reasoning and thinking*. Dordrecht: Kluwer Academic Publishers.

Ben-Zvi, D., Gil, E., & Apel, N. (2007, August). What is hidden beyond the data? Helping young students to reason and argue about some wider universe. In D. Pratt & J. Ainely (Eds.), *Reasoning about informal statistical inference: A collection of research studies*. Proceedings of the fifth international research forum on statistical reasoning, thinking, and literacy (SRTL-5), University of Warwick, UK, August, 2007.

Berliner, D. C. (2005). The near impossibility of testing for teacher quality. *Journal of Teacher Education, 56*(3), 205–213.

Berry, M., White, H., & Foster, P. (2002). Streaming reviewed: Some reflections from an inner-city, multi-ethnic primary school. In O. McNamara (Ed.), *Becoming an evidence-based practitioner: A framework for teacher-researchers* (pp. 141–151). New York: Routledge Falmer.

Bertrand, J. (1889). *Calcul des probabilités*. Paris: Gauthiers Villars.

Beswick, K. (2012). Teachers' beliefs about school mathematics and mathematicians' mathematics and their relationship to practice. *Educational Studies in Mathematics, 79*(1), 127–147.

Beswick, K. (2017). Raising attainment: What might we learn from teachers' beliefs about their best and worst mathematics students? In C. Andrà, D. Brunetto, E. Levenson, & P. Liljedahl (Eds.), *Teaching and learning in math classrooms. Emerging themes in affect-related research: Teachers' beliefs, students' engagement and social interaction* (pp. 95–106). New York: Springer.

Beswick, K., Fraser, S., & Crowley, S. (2017). "The best ever": Mathematics teachers' perceptions of quality professional learning. In B. Kaur, W. K. Ho, T. L. Toh, & B. H. Choy (Eds.), *Proceedings of the 41st conference of the International Group for the Psychology of Mathematics Education* (Vol. 2, pp. 169–176). Singapore: PME.

Biesta, G. (2007). Why 'what works' won't work: Evidence based practice and the democratic deficit in educational research. *Educational Theory, 57*(1), 1–22.

Biesta, G. (2009). Good education in an age of measurement: On the need to reconnect with the question of purpose in education. *Educational Assessment, Evaluation and Accountability, 21*(1), 33–46.

Biggs, J. (1988). The role of metacognition in enhancing learning. *Australian Journal of Education, 32*(2), 127–138.

Bishop, A. J. (1988). The interactions of mathematics education with culture. *Cultural Dynamics, 1*(2), 145–157.

Bishop, A. J. (1990). *Western mathematics: The secret weapon of cultural imperialism*. Thousand Oaks: Sage.

Black, P., & Wiliam, D. (1998a). Inside the black box: Raising standards through classroom assessment. *Phi Delta Kappan, 80*(2), 139–148.

Black, P., & Wiliam, D. (1998b). Assessment and classroom learning. *Assessment in Education: Principles, Policy, and Practice, 5*(1), 7–74.

Black, P., & Wiliam, D. (2009). Developing the theory of formative assessment. *Educational Assessment Evaluation and Accountability, 21*, 5–31.

Blackwell, S. B. (2003). Operation central: An original play teaching mathematical order of operations. *Mathematics Teaching in the Middle School, 9*(1), 52.

Blake, C. (2015, May 13). *Teaching social justice in theory and practice*. Retrieved from http://education.cu-portland.edu/blog/news/teaching-social-justice/

Blando, J. A., Kelly, A. E., Schneider, B. R., & Sleeman, D. (1989). Analyzing and modeling arithmetic errors. *Journal of Research in Mathematics Education, 20*(3), 301–308.

Bleiler, S. K., Ko, Y.-Y., Yee, S. P., & Boyle, J. D. (2015). Communal development and evaluation of a course rubric for proof writing. In C. Suurtamm & A. Roth McDuffie (Eds.), *Assessment to enhance teaching and learning* (pp. 97–108). Reston: NCTM.

Blum, W., et al. (2002). ICMI Study 14: Applications and modelling in mathematics education – Discussion document. *Educational Studies in Mathematics, 51*, 149–171.
Boaler, J. (2015). *Mathematical mindsets: Unleashing students' potential through creative math, inspiring messages and innovative teaching*. San Francisco: Wiley.
Boaler, J., & Dweck, C. (2016). *Mathematical mindsets: Unleashing students' potential through creative math* (1st ed.). Mississauga: Wiley.
Boaler, J., & Sengupta-Irving, T. (2016). The many colors of algebra: The impact of equity focused teaching upon student learning and engagement. *The Journal of Mathematical Behavior, 41*, 179–190.
Boaler, J., Wiliam, D., & Brown, M. (2000). Students' experiences of ability grouping—Disaffection, polarisation, and the construction of failure. *British Educational Research Journal, 26*(5), 631–648.
Bobis, J., Anderson, J., Martin, A., & Way, J. (2011). A model for mathematics instruction to enhance student motivation and engagement. In D. Brahier (Ed.), *Motivation and disposition: Pathways to learning mathematics, National Council of Teachers of Mathematics Seventy-third Yearbook* (pp. 31–42). Reston: NCTM.
Borasi, R., & Siegel, M. (1990). Reading to learn mathematics: New connections, new questions, new challenges. *For the Learning of Mathematics, 10*(3), 9–16.
Borovcnik, M., & Kapadia, R. (2014). From puzzles and paradoxes to concepts in probability. In E. J. Chernoff & B. Sriraman (Eds.), *Probabilistic thinking: Presenting plural perspectives* (pp. ix–xiii). Dordrecht: Springer.
Borovcnik, M., & Kapadia, R. (2016). Reasoning with risk: A survival guide. In *Proceedings of the thirteenth international congress on mathematical education*. Germany: Hamburg.
Borovcnik, M., & Peard, R. (1996). Probability. In A. J. Bishop, K. Clements, C. Keitel, & J. Laborde (Eds.), *International handbook of mathematics education* (pp. 239–287). Dordrecht: Kluwer Academic Publishing.
Boylan, M. (2007). Teacher questioning in communities of political practice. *Philosophy of Mathematics Education Journal,* 20. Retrieved from http://socialsciences.exeter.ac.uk/education/research/centres/stem/publications/pmej/pome20/index.htm
Brahier, D. J., & Schäffner, M. (2004). The effects of a study-group process on the implementation of reform in mathematics education. *School Science and Mathematics, 104*(4), 170–178.
Bransford, J., Darling-Hammond, L., & LePage, P. (2005). Introduction. In L. Darling-Hammond & J. Bransford (Eds.), *Preparing teachers for a changing world: What teachers should learn and be able to do* (pp. 201–231). San Francisco: Jossey-Bass.
Briand, J. (2005). Une expérience statistique et une première approche des lois du hasard au lycée par une confrontation avec une machine simple. *Recherches en didactique des mathématiques, 25*(2), 247–282.
Briand, J. (2007). La place de l'expérience dans la construction des mathématiques en classe. *Petit x, 75*, 7–33.
British Columbia Ministry of Education. (2013). *Transforming curriculum and assessment – Mathematics*. Retrieved from https://curriculum.gov.bc.ca/curriculum/Mathematics
British Columbia Ministry of Education. (2015). *Building student success – BC's new curriculum.* Victoria: Queen's Printer. Retrieved from https://curriculum.gov.bc.ca/curriculum/mathematics/9
British Columbia Ministry of Education. (2016). *Mathematics: Introduction*. Retrieved from https://curriculum.gov.bc.ca/curriculum/mathematics/introduction
Broderick, A. A., Hawkins, G., Henze, S., Mirasol-Spath, C., Pollack-Berkovits, R., Clune, H. P., & Steel, C. (2012). Teacher counternarratives: Transgressing and 'restorying' disability in education. *International Journal of Inclusive Education, 16*(8), 825–842.
Brofenbrenner, U. (1979). *The ecology of human development: Experiments by nature and design*. Cambridge, MA: Harvard University Press.
Brookhart, S. M. (2001). Successful students' formative and summative uses of assessment information. *Assessment in education: Principles, policy, and practice, 8*(2), 153–169.

Brousseau, G. (1997a). *Theory of didactical situations in mathematics*. Dordrecht: Kluwer.

Brousseau, G. (1997b). *Theory of didactical situations in mathematics: Didactiques des mathématiques, 1970–1990* (N. Balacheff, M. Cooper, R. Sutherland, & V. Warfield, Trans.). Dordrecht: Kluwer.

Brousseau, G., Brousseau, N., & Warfield, V. (2002). An experiment on the teaching of statistics and probability. *Journal of Mathematical Behavior, 20*(3), 363–411.

Brown, R. S. (2008). *The grade 9 cohort of fall 2002: A five-year cohort study, 2002–2007*. Technical report to Toronto District School Board. Toronto: Toronto District School Board.

Brown, B. (2012). *Daring greatly: How the courage to be vulnerable transforms the way we live, love, parent, and lead*. New York: Gotham Books.

Brown, J. H., D'Emidio-Caston, M., & Benard, B. (2001). *Resilience education*. Thousand Oaks: Corwin Press.

Bruce, C. D. (2007). Student interaction in the math classroom: Stealing ideas or building understanding. *What Works? Research into Practice. Research Monograph # 1*. Retrieved from http://www.edu.gov.on.ca/eng/literacynumeracy/inspire/research/Bruce.pdf

Bruce, C., & Ross, J. A. (2008). A model for increasing reform implementation and teacher efficacy: Teacher peer coaching in grades 3 and 6 mathematics. *Canadian Journal of Education, 31*(2), 346–370.

Bruner, J. (1986). *Actual minds, possible worlds*. Cambridge, MA: Harvard University Press.

Burazin, A., & Lovric, M. (2015). *Learning portfolio in mathematics at McMaster University*. Hamilton, ON: McMaster Institute for Innovation and Excellence in Teaching and Learning (62 pages).

Burkhardt, H. (1981). *The real world of mathematics*. Glasgow: Blackie.

Burkhardt, H., & Swan, M. (2013). Task design for systemic improvement: Principles and frameworks. In C. Margolinas (Ed.), *Task design in mathematics education – proceedings of ICMI study 22* (pp. 431–439). Oxford: Springer International Publishing.

Burton, L. (2004). *Mathematicians as enquirers: Learning about learning mathematics*. Dordrecht: Springer.

Cai, J. (2010). Commentary on problem solving heuristics, affect, and discrete mathematics: A representational discussion. In B. Sriraman & L. English (Eds.), *Theories of mathematics education, advances in mathematics education* (pp. 251–258). Heidelberg: Springer.

Cajete, G. (1994). *Look to the mountain: An ecology of Indigenous education*. Durango: Kivaki Press.

Cajete, G. A. (1999). *Igniting the sparkle: An Indigenous education science education model*. Skyland: Kivaki Press.

Cajete, G. (2000). *Native science: Natural laws of interdependence*. Santa Fe: Clearlight Publishers.

Cajori, F. (1928). *A history of mathematical notation* (Vol. I). London: Open Court Company.

Callahan, R. (2005). Tracking and high school English learners: Limiting opportunity to learn. *American Educational Research Journal, 42*, 305–328.

Canadian Broadcasting Company. (2012). *Nunavut residents protest high food prices*. Retrieved from http://www.cbc.ca/news/canada/north/nunavut-residents-protest-high-food-prices-1.1170565

Canavan-McGrath, C., Desrochers, S., MacDonald, H., Pruner, M., Reinbold, H., Samra-Gynane, R., Shaw, C., Teshima, R., & Trufyn, D. (2012). *Foundations of mathematics* (Vol. 12). Toronto: Nelson Education.

Carless, D. (2007). Learning-oriented assessment: Conceptual bases and practical implications. *Innovations in Education and Teaching International, 44*(1), 57–66.

Carnegie Council on Adolescent Development. (1989). *Turning points: Preparing American youth for the 21st century*. Washington, DC: Carnegie Council on Adolescent Development.

Caron, F. (2002). Splendeurs et misères de l'enseignement des probabilités au primaire. In *Proceedings of the « Colloque annuel du Groupe de didactique des mathématiques du Québec»*, Trois-Rivières, Québec.

Central Intelligence Agency (CIA). (n.d.). *The world factbook*. Retrieved from https://www.cia.gov/library/publications/the-world-factbook/

Cerbin, W. (2011). *Lesson study: Using classroom inquiry to improve teaching and learning in higher education*. Sterling: Stylus Publishing.
Chappuis, S., & Stiggins, R. (2002). Classroom assessment for learning. *Educational Leadership, 60*(1), 40–43.
Chernoff, E. J., & Sriraman, B. (Eds.). (2014). *Probabilistic thinking: Presenting plural perspectives*. Dordrecht: Springer Science+Business.
Chevallard, Y. (1985). *La transposition didactique*. Grenoble: La Pensée Sauvage.
Chi, M., & Bassok, M. (1989). Learning from examples via self-explanation. In L. Resnick (Ed.), *Knowing, learning and instruction: Essays in honour of Robert Glaser*. Hillsdale: Erlbaum.
Chi, M., Bassok, M., Lewis, P., Reiman, P., & Glasser, R. (1989). Self-explanations: How students study and use examples in learning to solve problems. *Cognitive Science, 13*, 145–182.
Choppin, J. (2011). The role of local theories: Teachers' knowledge and its impact on engaging students with challenging tasks. *Mathematics Education Research Journal, 23*(1), 5–25.
Chorney, S., Ng, O., & Pimm, D. (2016). A tale of two more metaphors: Storylines about mathematics education in Canadian national media. *Canadian Journal of Science, Mathematics and Technology Education, 16*(4), 402–418.
ChrisOnerVidz. (2012, December 20). *Boiling water turns into snow in Siberia* [video file]. Retrieved from https://www.youtube.com/watch?v=2LFtYUUXJlE
Christopherson, R. W. (2002). *Geosystems: An introduction to physical geography* (4th ed.). Upper Saddle River: Prentice-Hall.
Clandinin, D. J., & Connelly, F. M. (2000). *Narrative inquiry: Experience and story in qualitative research*. San Francisco: Jossey-Bass.
Clarkson, P. C. (2007). Australian Vietnamese students learning mathematics: High ability bilinguals and their use of their languages. *Educational Studies in Mathematics, 64*(2), 191–215.
Clarkson, P. C. (2009). Mathematics teaching in Australian multilingual classrooms: Developing an approach to the use of classroom languages. In R. Barwell (Ed.), *Multilingualism in mathematics classrooms: Global perspectives* (pp. 145–160). Bristol: Multilingual Matters.
Clements, D., & Sarama, J. (2014). *Learning and teaching early math: The learning trajectories approach* (2nd ed.). New York: Routledge.
Cobb, P., & Yackel, E. (1996). Constructivist, emergent, and sociocultural perspectives in the context of developmental research. *Educational Psychologist, 31*(3/4), 175–190.
Cobb, P., Confrey, J., di Sessa, A., Lehrer, R., & Schauble, L. (2003). Design experiments in educational research. *Educational Researcher, 32*(1), 9–13.
Cole, P. (1998). An academic take on 'Indigenous traditions and ecology. *Canadian Journal of Environmental Education Research, 3*(1), 100–155.
Cole, A., & Knowles, G. (2000). *Researching teaching: Exploring teacher development through reflexive inquiry*. Toronto: Allyn and Bacon.
Collins, R. (2004). *Interaction ritual chains*. Princeton: Princeton University Press.
Collins English dictionary. (1994). Glasgow: HarperCollins Publishers.
Condly, S. J. (2006). Resilience in children: A review of literature with implications for education. *Urban Education, 41*(3), 211–236. https://doi.org/10.1177/0042085906287902.
Connelly, F. M., & Clandinin, J. (1985). Personal practical knowledge and the modes of knowing: Relevance for teaching and learning. In E. Eisner (Ed.), *Learning and teaching the ways of knowing* (pp. 174–198). Chicago: National Society for the Study of Education.
Cooper, B., & Dunne, M. (2000). *Assessing children's mathematical knowledge: Social class, sex and problem-solving*. Buckingham: Open University Press.
Copsey-Haydey, D., Zakaluk, B. L., & Shaw, S. (2010). The changing face of content area teaching. *Journal of Applied Research on Learning, 3*(3), 1–29.
Corbiere, A. I. (2000). Reconciling epistemological orientations: Toward a wholistic Nishaabe (Ojibwe/Odawa/Potowatomi) education. *Canadian Journal of Native Education, 24*(2), 113–119.
Corrigan, D., Gunstone, R., Bishop, A., & Clarke, B. (2004, July). *Values in science and mathematics education: Similarities, differences and teacher views*. A paper presented at the 35th

annual meeting of the Austalasian Science Education Research Association, Armidale, NSW, Australia.

Corsiglia, J., & Snively, G. (1995). Global lessons from traditional science of long-resident peoples. In G. Snively & A. MacKinnon (Eds.), *Thinking globally about mathematics and science education* (pp. 5–46). Vancouver: Centre for the Study of Curriculum and Instruction, University of British Columbia.

Costa, A. L., & Kallick, B. (Eds.). (2008). *Learning and leading with habits of mind: 16 essential characteristics for success*. Alexandria: ASCD.

Csíkszentmihályi, M. (1990). *Flow: The psychology of optimal experience*. New York: Harper & Row.

Csíkszentmihályi, M. (1996). *Creativity: Flow and the psychology of discovery and invention*. New York: HarperCollins Publishers.

Cummins, J. (2000). *Language, power and pedagogy: Bilingual children in the crossfire*. Clevedon: Multilingual Matters.

Cuoco, A., Goldenberg, E. P., & Mark, J. (1996a). Habits of mind: An organizing principle for a mathematics curriculum. *Journal of Mathematical Behavior, 14*(4), 375–402.

Cuoco, A., Goldenberg, E. P., & Mark, J. (1996b). Habits of mind: An organizing principle for mathematics curricula. *Journal of Mathematical Behaviour, 15*, 375–402.

D'Ambrosio, U. (2016). Ethnomathematics: A response to the changing role of mathematics in society. In P. Ernest, B. Sriraman, & N. Ernest (Eds.), *Critical mathematics education: Theory, praxis and reality* (pp. 23–34). Charlotte: Information Age Publishing.

Darling-Hammond, L., & Sykes, G. (1999). *Teaching as the learning profession*. San Francisco: Jossey-Bass.

Darling-Hammond, L., Wei, R. C., Andree, A., Richardson, N., & Orphanos, S. (2009). State of the profession: Study measures status of professional development. *Journal of Staff Development, 30*(2), 42–50.

Daschuk, J. (2013). *Clearing the plains: Disease, politics of starvation, and the loss of aboriginal life*. Regina: University of Regina Press.

Daugherty, R., & Eccletsone, K. (2006). Constructing assessment for learning in the UK policy environment. In J. Gardner (Ed.), *Assessment and learning* (pp. 149–168). London: Sage.

Davidson, D. M. (2002). Teaching mathematics to American Indian students: A cultural approach. In J. E. Hankes & G. R. Fast (Eds.), *Changing the faces of mathematics: Perspectives on Indigenous people of North America* (pp. 19–24). Reston: National Council of Teachers of Mathematics.

Davis, M. (1977). *Piero della Francesca's mathematical treatises: The Trattato d'abaco and Libellus de quinque corporibus regularibus*. Ravenna: Longo.

Davis, B. (1994). *Listening to reason: An inquiry into mathematics teaching*. Unpublished doctoral dissertation, University of Alberta, Edmonton, Canada.

Davis, B. (1996). *Toward a sound alternative*. New York: Routledge.

Davis, B. (1997). Listening for differences: An evolving conception of mathematics teaching. *Journal for Research in Mathematics Education, 28*(3), 355–376.

Davis, B. (2001). Why teach mathematics to all students? *For the Learning of Mathematics, 21*, 17–24.

Davis, B., & Renert, M. (2014). *The math teachers know: Profound understanding of emergent mathematics*. New York: Routledge.

Davis, B., & Simmt, E. (2003). Understanding learning systems: Mathematics education and complexity science. *Journal for Research in Mathematics Education, 34*, 137–167.

Davis, B., & Sumara, D. (2006). *Complexity and education: Inquiries into learning, teaching, and research*. New York: Routledge.

Davis, B., & The Spatial Reasoning Study Group. (2015). *Spatial reasoning in the early years: Principles, assertions, and speculations*. New York: Taylor and Francis, Routledge.

Davis, G., Ellis, D., Hope, J., Rolheiser, D., Sufrin, D., Van Bergeyk, C., Van Bergeyk, D., & Zimmer, D. (2010). *Foundations of pre-calculus mathematics 10*. Toronto: Pearson Canada.

Davis, B., Sumara, D., & Luce-Kapler, R. (2015). *Engaging minds: Cultures of education and practices of teaching* (3rd ed.). New York: Routledge.
Dawe, L. (1983). Bilingualism and mathematical reasoning in English as a second language. *Educational Studies in Mathematics, 14*(4), 325–353.
De Freitas, E., & Sinclair, N. (2014). *Mathematics and the body: Material entanglements in the classroom*. New York: Cambridge University Press.
DeDieu, L., & Lovric, M. (2018). Student perceptions of the use of writing in a differential equations course. *Primus, 28*(2), 166–185.
Dell'Angelo, T. (2014, September 19). *Creating classrooms for social justice.* Retrieved from https://www.edutopia.org/blog/creating-classrooms-for-social-justice-tabitha-dellangelo
Design-Based Research Collective. (2003). Design-based research: An emerging paradigm for educational inquiry. *Educational Researcher, 32*(1), 5–8 35–37.
Detterman, D., & Sternberg, R. (Eds.). (1993). *Transfer on trial: Intelligence, cognition, and instruction*. Norwood: Ablex Publishing.
Devlin, K. (2014). The most common misconception about probability? In E. J. Chernoff & B. Sriraman (Eds.), *Probabilistic thinking: Presenting plural perspectives* (pp. ix–xiii). Dordrecht: Springer.
Dewey, J. (1938). *Experience and education*. New York: Kappa Delta Pi.
Donald, D., Glanfield, F., & Sterenberg, G. (2012). Living ethically within conflicts of colonial authority and relationality. *Journal of the Canadian Association for Curriculum Studies, 10*(1), 53–76.
Donnon, T., & Hammond, W. (2007). A psychometric assessment of the self-reported youth resiliency: Assessing developmental strengths questionnaire. *Psychological Reports, 100*, 963–978.
Doolittle, E. (2006). Mathematics as medicine. In P. Liljedahl (Ed.), *Proceedings of the 2006 annual meeting of the Canadian Mathematics Education Study Group* (pp. 17–25). Burnaby: CMESG/GCEDM.
DuFour, R. (2004). The best staff development is in the workplace, not in a workshop. *National Staff Development Council, 25*(2), 63–64.
DuFour, R., & Eaker, R. (1998). *Professional learning communities at work: Best practices for enhancing student achievement*. Bloomington: National Educational Service.
Duncan Campbell Scott. (n.d.). In *Wikipedia.* Retrieved from https://en.wikipedia.org/wiki/Duncan_Campbell_Scott
Dupree, K. M. (2016). Questioning the order of operations. *Mathematics Teaching in the Middle School, 22*(3), 152–159.
Dweck, C. (2006). *Mindset: The new psychology of success*. New York: Ballantine Books.
Eaker, R. (2002). Cultural shifts: Transforming schools into professional learning communities. In R. Eaker, R. DuFour, & R. Burnette (Eds.), *Getting started: Reculturing schools to become professional learning communities* (pp. 9–29). Bloomington: National Educational Service.
Eaker, R., DuFour, R., & Burnette, R. (2002). Getting started: A conceptual framework for creating a professional learning community. In R. Eaker, R. DuFour, & R. Burnette (Eds.), *Getting started: Reculturing schools to become professional learning communities* (pp. 1–8). Bloomington: National Educational Service.
Earl, L. M. (2003). *Assessment as learning: Using classroom assessment to maximize student learning*. London: Sage.
Earl, L. (2013). *Assessment as learning: Using classroom assessment to maximize student learning*. Thousand Oaks: Corwin.
Eccles, J., Lord, S., & Midgley, C. (1991). What are we doing to early adolescents? The impact of educational contexts on early adolescents. *American Journal of Education, 99*(4), 521–542.
Education Quality and Accountability Office. (n.d.). *Highlights of the provincial results: Assessments of reading, writing and mathematics, primary division (Grades 1–3) and junior division (Grades 4–6) English-Language students, 2016–2017.* Retrieved from http://www.

eqao.com/en/assessments/results/communication-docs/provincial-report-highlights-elementary-2017.pdf

Education Quality and Accountability Office. (2012). *An analysis of questionnaire and contextual data for grade 9 students in the Academic and Applied mathematics courses*. Toronto: EQAO Retrieved from http://www.eqao.com/Research/pdf/E/Analysis_Questionnaire_ContextualDataG9_en.pdf

Education Quality and Accountability Office (EQAO). (2014). *Grade 9 assessment of mathematics 2014: Released assessment questions*. Toronto: Queen's Printer for Ontario.

Edwards, L., Radford, L., & Arzarello, F. (2009). Gestures and multimodality in the teaching and learning of mathematics. *Educational Studies in Mathematics, 70*(2), 91–215.

Egadawatte, G., McDougall, D. E., & Stoilescu, D. (2011). The effects of teacher collaboration in Grade 9 Applied mathematics. *Educational Research for Policy and Practice, 10*(3), 189–209.

ENGAGE. (n.d.). *Spatial visual skills* [links]. Retrieved from https://www.engageengineering.org/spatial/whyitworks, https://www.nature.com/scitable/content/chapter-5-spatial-skills-18255038

Ernest, P. (1988). *The impact of beliefs on the teaching of mathematics*. Retrieved from http://webdoc.sub.gwdg.de/edoc/e/pome/impact.htm

Ernest, P. (1989). The impact of beliefs on the teaching of mathematics. In P. Ernest (Ed.), *Mathematics teaching: The state of the art* (pp. 249–254). London: Falmer Press.

Ernest, P. (1991). *The philosophy of mathematics education.* London: Routledge-Falmer. Retrieved from https://p4mriunpat.files.wordpress.com/2011/10/the-philosophy-of-mathematics-education-studies-in-mathematicseducation.pdf

Ernest, P. (2016a). Mathematics education ideologies and globalization. In P. Ernest, B. Sriraman, & N. Ernest (Eds.), *Critical mathematics education: Theory, praxis and reality* (pp. 35–79). Charlotte: Information Age Publishing.

Ernest, P. (2016b). The problem of certainty in mathematics. *Educational Studies in Mathematics, 92*, 379–393.

Esmonde, I., & Quindel, J. (2006). Globalization, labour, and the environment. In E. Gutstein & B. Peterson (Eds.), *Rethinking mathematics: Teaching social justice by the numbers* (pp. 62–63). Milwaukee: Rethinking Schools Ltd.

Expert Panel on Student Success in Ontario. (2004). *Leading math success: Mathematical literacy grades 7–12*. Toronto: Queen's Printer for Ontario.

Ezeife, A. N. (2002). Mathematics and culture nexus: The interactions of culture and mathematics in an Aboriginal classroom. *International Education Journal, 3*(3), 176–187.

Falk, R. (1992). A closer look at the probabilities of the notorious three prisoners. *Cognition, 43*, 197–223.

Farber, B. (2017, January 23). *The terrible legacy of Duncan Campbell Scott*. Retrieved from http://www.huffingtonpost.ca/bernie-farber/duncan-campbell-scott-legacy_b_14289206.html

Felmer, P., Pehkonen, E., & Kilpatrick, J. (2016). *Posing and solving mathematical problems*. Dordrecht: Springer.

Festinger, L. (1957). *A theory of cognitive dissonance*. Evanston: Row, Peterson.

Firmender, J. M., Gavin, M. K., & McCoach, D. B. (2014). Examining the relationship between teachers' instructional practices and students' mathematics achievement. *Journal of Advanced Academics, 25*(3), 214–236.

First Nations Education Steering Committee (FNESC). (2011). *Teaching mathematics in a First Peoples context: Grades 8 and 9*. Vancouver: Author. Retrieved from http://www.fnesc.ca/wordpress/wp-content/uploads/2015/05/PUB-LFP-Math-First-Peoples-8-9-for-Web.pdf

Fischbein, E., & Schnarch, D. (1997). The evolution with age of probabilistic intuitively based misconceptions. *Journal for Research in Mathematics Education, 28*(1), 96–105.

Flewelling, G., & Higginson, W. (2001a). *A handbook on rich learning tasks: Realizing a vision of tomorrow's mathematics classroom*. Kingston: Queen's University, Faculty of Education, Centre for Mathematics, Science and Technology Education.

Flewelling, G., & Higginson, W. (2001b). *A handbook on rich learning tasks.* Centre for Mathematics, Science, and Technology Education. Kingston: Queen's University.

Fonstad, M., Pugatch, W., & Vogt, B. (2003). Kansas is flatter than a pancake. *Annals of Improbable Research (AIR), 9*(3). Retrieved from http://www.usu.edu/geo/geomorph/kansas.html

Foster, C. (2013). Mathematical études: Embedding opportunities for developing procedural fluency within rich mathematical contexts. *International Journal of Mathematical Education in Science and Technology, 44*, 765–774.

Fouconnier, G., & Turner, M. (2002). *The way we think: Conceptual blending and the mind's hidden complexities*. New York: Basic Books.

Francisco, J. M., & Maher, C. A. (2005). Conditions for promoting reasoning in problem solving: Insights from a longitudinal study. *Journal of Mathematical Behavior, 24*(3), 361–372.

Franke, M. L., Kazemi, E., & Battey, D. (2007). Understanding teaching and classroom practice in mathematics. In F. K. Lester Jr. (Ed.), *Second handbook of research on mathematics teaching and learning* (pp. 225–256). Charlotte: Information Age.

Frankenstein, M. (2006). Reading the world with math. In E. Gutstein & B. Peterson (Eds.), *Rethinking mathematics: Teaching social justice by the numbers* (pp. 19–30). Milwaukee: Rethinking Schools Ltd.

Franklin, C., Kader, G., Mewborn, D., Moreno, J., Peck, R., Perry, M., & Scheaffer, R. (2005). *Guidelines for assessment and instruction in statistics education* (GAISE Report). Alexandria: American Statistical Association.

Fraser, M. W., & Galinsky, M. J. (1997). Toward a resilience-based model of practice. In M. W. Fraser (Ed.), *Risk and resilience in childhood: An ecological perspective* (pp. 265–276). Washington, DC: NASW Press.

Fredricks, J. A., Blumenfeld, P. C., & Paris, A. H. (2004). School engagement: Potential of the concept, state of the evidence. *Review of Educational Research, 74*, 59–109.

Freire, P. (1970). *Pedagogy of the oppressed*. New York: Continuum.

Fullan, M. (2001). *The new meaning of educational change* (3rd ed.). New York: Teachers College Press.

Fullan, M. (2005). *Leadership and Sustainability*. Thousand Oaks, CA: Corwin Press.

Fullan, M., Hill, P., & Crevola, C. (2006). *Breakthrough*. Newbury Park: Corwin Press.

Furuto, H. L. (2014). Pacific ethnomathematics: Pedagogy and practices in mathematics education. *Teaching Mathematics and Its Applications, 33*, 110–121.

Gadamer, H.-G. (1990). *Truth and method*. New York: Continuum.

Gadanidis, G. (2012). Trigonometry in grade 3? In *What works? Research into practice* (Research monograph 42). Ottawa: The Literacy and Numeracy Secretariat.

Gadanidis, G., Borba, M., Hughes, J., & Lacerda, H. D. (2016). Designing aesthetic experiences for young mathematicians: A model for mathematics education reform. *International Journal for Research in Mathematics Education, 6*(2), 225–244.

Gal, I. (2005). Towards "probability literacy" for all citizens: Building blocks and instructional dilemmas. In G. A. Jones (Ed.), *Exploring probability in school: Challenges for teaching and learning* (pp. 43–70). Dordrecht: Kluwer Academic Publishers.

Galbraith, P. (1993). Paradigms, problems and assessment: Some ideological implications. In M. Niss (Ed.), *Investigations into assessment in mathematics education* (pp. 73–86). Dordrecht: Springer.

Galbraith, P. (2007). Beyond the low hanging fruit. In W. Blum, P. Galbraith, M. Niss, & H.-W. Henn (Eds.), *Modelling and applications in mathematics education: The 14th ICMI Study* (pp. 79–88). New York: Springer.

Gamoran, A. (2002). *Standards, inequality, and ability grouping in schools* (CES Briefing No. 25). Edinburgh: Centre for Educational Sociology, University of Edinburgh.

Gamoran, A., & Berends, M. (1987). The effects of stratification in secondary schools: Synthesis of survey and ethnographic research. *Review of Educational Research, 57*(4), 415–435.

Gamoran, A., & Mare, R. D. (1989). Secondary school tracking and educational inequality: Compensation, reinforcement or neutrality? *American Journal of Sociology, 94*(3), 1146–1183.

Gamoran, A., Nystrand, M., Berends, M., & LePore, P. C. (1995). An organizational analysis of the effects of ability grouping. *American Educational Research Journal, 32*(4), 687–715.

Garfield, J., & Ben-Zvi, D. (2008). *Developing students'statistical reasoning: Connecting research and teaching practice*. New York: Springer.

Garfunkel, S., & Montgomery, M. (2016). *Guidelines for assessment and instruction in mathematical modeling education*. Bedford/Philadelphia: COMAP/SIAM.

Garmezy, N. (1991). Resiliency and vulnerability to adverse developmental outcomes associated with poverty. *American Behavioral Scientist, 34*(4), 416–430. https://doi.org/10.1177/0002764291034004003.

Garrison, E. (1995). Navajo scientists of the next century–Laanaa Hason. *Journal of Navajo Education, 12*(3), 3–7 Retrieved from http://mfnerc.org/wp-content/uploads/2012/11/4_Ezeife.pdf.

Gattegno, C. (1970). *What we owe children: The subordination of teaching to learning*. London: Routledge & Kegan Paul.

Gattuso, L., & Vermette, S. (2013). L'enseignement de statistique et probabilités au Canada et en Italie. *Statistique et Enseignement, 4*(1), 107–129 Retrieved from http://publications-sfds.math.cnrs.fr/ojs/index.php/StatEns/article/view/141.

Gee, J. P. (2003). *What video games have to teach us about learning and literacy*. New York: Palgrave Macmillan.

Gibson, J. (1979). *The ecological approach to visual perception*. London: Houghton Mifflin.

Gillings, R. (1972., reprinted 1982). *Mathematics in the time of the pharaohs*. New York: Dover.

Ginsburg, H. P. (1997). *Entering the child's mind: The clinical interview in psychological research and practice*. Cambridge: Cambridge University Press.

Ginsburg, H. P. (2002). Little children, big mathematics: Learning and teaching in the pre-school. In A. D. Cockburn & E. Nardi (Eds.), *Proceedings of the 26th conference of the International Group for the Psychology of Mathematics Education* (Vol. 1, pp. 3–14). University of East Anglia, UK: Psychology of Mathematics Education.

Glazerman, S., McKie, A., & Carey, N. (2009). *An evaluation of the teacher advancement program (TAP) in Chicago: Year one impact report*. Washington, DC: Mathematical Policy Research.

Glidden, P. L. (2008). Prospective elementary teachers' understanding of order of operations. *School Science and Mathematics, 108*(4), 130–136.

Goddard, R. D. (2001). Collective efficacy: A neglected construct in the study of schools and student achievement. *Journal of Educational Psychology, 93*(3), 467–476.

Godin, S. (2011). What's the problem? Getting things squared away. *Ontario Mathematics Gazette, 49*(3), 8–10.

Godin, S. (2013). What's the problem? Cylinders within cylinders. *Ontario Mathematics Gazette, 51*(4), 8–9.

Gojmerac, A., & Cherubini, L. (2012). Connecting for greater purpose: An examination of a district professional learning community. *Journal of Teaching and Learning, 8*(1), 43–58.

Golembo, V. (2000). Writing a pemDAs story. *Mathematics Teaching in the Middle School, 5*(9), 574.

Goodenow, C. (1993). *The psychological sense of school membership among adolescents: Scale development and educational correlates*. *Psychology in the Schools, 30*(1), 79–90 Retrieved from https://doi.org/10.1002/1520-6807(199301)30:1<79::aid-pits2310300113>3.0.co;2-x.

Gottfried, A. E., Fleming, J. S., & Gottfried, A. W. (2001). Continuity of academic intrinsic motivation from childhood through late adolescence: A longitudinal study. *Journal of Educational Psychology, 93*(1), 3–13.

Gottfried, A. E., Marcoulides, G. A., Gottfried, A. W., Oliver, P. H., & Wright Guerin, D. (2007). Multivariate latent change modeling of developmental decline in academic intrinsic math motivation and achievement: Childhood through adolescence. *International Journal of Behavioral Development, 31*(4), 317–327. https://doi.org/10.1177/0165025407077752.

Gourdeau, F. (2017). Doing mathematics in teacher preparation: Giving space and time to think, reflect, share and feel. In S. Cho (Ed.), *Selected regular lectures from the 12th International Congress on Mathematics Education* (pp. 223–234). London: Springer.
Gouvernement du Québec. (2001). *Québec education program: Preschool education and elementary education*. Québec: Ministère de l'Éducation.
Gouvernement du Québec. (2006). *Programme de formation de l'école québécoise. Éducation préscolaire et enseignement primaire*. Québec: Ministère de l'Éducation, du Loisir et du Sport du Québec.
Gouvernement du Québec. (2016). Chapitre 6: Mathématique (original version in 2007 and updated in 2016). In *Programme de formation de l'école québécoise: Enseignement secondaire, deuxième cycle* (pp. 1–146). Québec: Ministère de l'éducation, du Loisir et du Sport.
Graham, P. (2007). Improving teacher effectiveness through structured collaboration: A case study of a professional learning community. *RMLE Online, 31*(1), 1–17.
Grant, T. J., Hiebert, J., & Wearne, D. (1994). *Teachers' beliefs and their responses to reform-minded instruction in elementary mathematics*. In Paper presented at the annual meeting of the American Educational Research Association, New Orleans. Retrieved from ERIC database online. (No. ED376170)
Gray, E. M., & Tall, D. O. (1993). Success and failure in mathematics: The flexible meaning of symbols as process and concept. *Mathematics Teaching, 142*, 6–10.
Gray, E. M., & Tall, D. O. (1994). Duality, ambiguity and flexibility: A proceptual view of simple arithmetic. *Journal for Research in Mathematics Education, 26*, 115–141.
Greene, R. R. (Ed.). (2002). *Resiliency: An integrated approach to practice, policy and research*. Washington, DC: NASW Press.
Greeno, J., Smith, D., & Moore, J. (1993). Transfer of situated learning. In D. Detterman & R. Sternberg (Eds.), *Transfer on trial: Intelligence, cognition, and instruction* (pp. 99–167). Norwood: Abbex.
Gregoire, M. (2003). Is it a challenge or a threat? A dual-process model of teachers' cognition and appraisal processed during conceptual change. *Educational Psychology Review, 15*(2), 147–179.
Grønmo, L. S., Lindquist, M., Arora, A., & Mullis, I. V. S. (2013). TIMSS 2015 mathematics framework. In I. V. S. Mullis & M. O. Martin (Eds.), *TIMSS 2015 assessment frameworks* (pp. 11–27). Chestnut Hill: TIMSS & PIRLS International Study Center, Boston College.
Gunnarsson, R., Sönnerhed, W. W., & Hernell, B. (2016). Does it help to use mathematically superfluous brackets when teaching the rules for the order of operations. *Educational Studies in Mathematics, 92*, 91–105.
Guskey, T. (1995). Professional development in education: In search of the optimal mix. In T. Guskey & M. Huberman (Eds.), *Professional development in education: New paradigms and practices* (pp. 114–132). New York: Teachers College Press.
Guskey, T. (2000). Does it make a difference? Evaluating professional development. *Educational Leadership, 59*(6), 45–51.
Guskey, T. (2003). What makes professional development effective? *Phi Delta Kappan, 84*(10), 748–750.
Gutiérrez, R. (2013). The socio-political turn in mathematics education. *Journal for Research in Mathematics Education, 44*(1), 37–68.
Gutstein, E. (2005). We are not a minority. In E. Gutstein & B. Peterson (Eds.), *Rethinking mathematics: Teaching social justice by the numbers* (p. 118). Milwaukee: Rethinking Schools Ltd.
Gutstein, E. (2006). *Reading and writing the world with mathematics: Toward a pedagogy of social justice*. New York: Routledge.
Hadar, N., & Hadass, R. (1981). Between associativity and commutativity. *International Journal of Mathematics Education in Science and Technology, 12*, 535–539.
Haddon, M. (2003). *The curious incident of the dog in the night-time*. Toronto: Random House of Canada.
Hall, E. T. (1976). *Beyond culture*. Toronto: Doubleday.

Halmos, P. (1980). The heart of mathematics. *American Mathematical Monthly, 87*(7), 519–524.
Hamlin, D., & Cameron, D. (2015). *Applied or academic: High impact decisions for Ontario students*. Toronto: People for Education.
Hammerness, K., Darling-Hammond, L., Bransford, J., Berliner, D., Cochran-Smith, M., McDonald, M., & Zeichner, K. (2005). How teachers learn and develop. In L. Darling-Hammond & J. Bransford (Eds.), *Preparing teachers for a changing world: What teachers should learn and be able to do* (pp. 258–289). San Francisco: Jossey-Bass.
Hand, V. M. (2010). The co-construction of opposition in a low-track mathematics classroom. *American Educational Research Journal, 47*(1), 97–132.
Handal, B., & Herrington, A. (2003). Mathematics teachers' beliefs and curriculum reform. *Mathematics Education Research Journal, 15*(1), 59–69.
Hannula, M. (2006). Motivation in mathematics: Goals reflected in emotions. *Educational Studies in Mathematics, 63*(2), 165–178.
Hanson, E. (2009). *The Indian Act*. Retrieved from http://indigenousfoundations.arts.ubc.ca/the_indian_act/
Hargreaves, A., Earl, I., & Ryan, J. (1996). *Schooling for change: Reinventing education for early adolescents*. Bristol: Falmer.
Harlen, W. (2012). On the relationship between assessment for formative and summative purposes. In J. Gardner (Ed.), *Assessment and learning* (pp. 87–102). Thousand Oaks: Sage.
Harvard. (1995). *Harvard balanced assessment project*. Retrieved from http://hgse.balancedassessment.org/amuse.html
Hattie, J. A. C. (2002). Classroom composition and peer effects. *International Journal of Educational Research, 37*, 449–481.
Hattie, J. A. C. (2009). *Visible learning: A synthesis of 800+ meta-analyses on achievement*. Oxford: Routledge.
Hawley, W., & Valli, L. (1999). The essentials of effective professional development: A new consensus. In L. Darling-Hammond & G. Sykes (Eds.), *Teaching as the learning profession: Handbook of policy and practice* (pp. 127–150). San Francisco: Jossey-Bass.
Heider, F. (1946). Attitudes and cognitive organization. *The Journal of Psychology, 21*(1), 107–112.
Heitink, M. C., Van der Kleij, F. M., Veldkamp, B. P., Schildkamp, K., & Kippers, W. B. (2016). A systematic review of prerequisites for implementing assessment for learning in classroom practice. *Educational Research Review, 17*, 60–62.
Henderson, J. Y. (2000). Ayukpachi: Empowering aboriginal thought. In M. Battiste (Ed.), *Reclaiming Indigenous voice and vision* (pp. 248–278). Vancouver: UBC Press.
Henderson, N., & Milstein, M. (2003). *Resiliency in schools: Making it happen for students and educators: Updated edition* (1st ed.). Thousand Oaks: Corwin Press, Inc.
Henningsen, M., & Stein, M. K. (1997). Mathematical tasks and students cognition: Classroom-based factors that support and inhibit high-level mathematical thinking and reasoning. *Journal for Research in Mathematics Education, 28*(5), 524–549.
Hersh, R. (1986). Some proposals for reviving the philosophy of mathematics. In T. Tymoczko (Ed.), *New directions in the philosophy of mathematics* (pp. 9–28). Boston: Birkhäuser.
Hewitt, D. (1999). Arbitrary and necessary: Part 1. A way of viewing the mathematics curriculum. *For the Learning of Mathematics, 19*(3), 2–9.
Hewitt, D. (2012). Young students learning formal algebraic notation and solving linear equations: Are commonly experienced difficulties avoidable. *Educational Studies in Mathematics, 81*(2), 139–159.
Hofmeyr, J. (2010). *Is this a new curriculum I see before me?* Retrieved from https://www.ieducation.co.za/is-this-a-new-curriculum-i-see-before-me/
Holm, J. (2014). *Improving mathematics teaching through professional learning groups*. Unpublished doctoral dissertation, Lakehead University, Thunder Bay, ON.
Holm, J., & Kajander, A. (2015). Lessons learned about effective professional development: Two contrasting case studies. *International Journal of Education in Mathematics, Science and Technology, 3*(4), 262–274.

Holtman, L., Julie, C., Mbekwa, M., Mtetwa, D., & Ngcobo, M. (2011). A comparison of preferences for real life situations that could be used in school mathematics in three SADC countries. *Southern African Review of Education, 17*, 120–137.
Hord, S. M. (2009). Professional learning communities: Educators work together toward a shared purpose—Improved student learning. *Journal of Staff Development, 30*(1), 40–43.
Hord, S. M., & Sommers, W. A. (2008). *Leading professional learning communities: Voices from research and practice*. Thousand Oaks: Corwin Press.
Horn, I. S. (2006). Lessons learned from detracked mathematics departments. *Theory Into Practice, 45*(1), 72–81.
Huang, R., & Lee, Y. (2016). *Teaching and learning mathematics through variation: Confucian heritage meets western theories*. Rotterdam: Sense.
Huberman, M. (1995). Professional careers and professional development: Some intersections. In T. R. Guskey & M. Huberman (Eds.), *Professional development in education: New paradigms and practices* (pp. 193–224). New York: Teachers College Press.
Huffman, D., Thomas, K., & Lawrenz, F. (2003). Relationship between professional development, teachers' instructional practices, and the achievement of students in science and mathematics. *School Science and Mathematics, 103*(8), 378–387.
Hunsader, P. D., Thompson, D. R., & Zorin, B. (2014). Mathematical practices: Small changes in assessments = big benefits. In K. Karp & A. Roth McDuffie (Eds.), *Using research to improve instruction* (pp. 205–214). Reston: NCTM.
Hurlington, K. (2010, February). Bolstering resilience in students: Teachers as protective factors [research monograph #25]. *What Works? Research into Practice*. Retrieved from http://www.edu.gov.on.ca/eng/literacynumeracy/inspire/research/WW_bolstering_students.pdf
Hutchinson, N. L. (2016). *Inclusion of exceptional learners in Canadian schools: A practical handbook for teachers* (5th ed.). Toronto: Pearson Education Canada.
Il'enkov, E. V. (1977). *Dialectical logic: Essays on its history and theory*. Moscow: Progress.
Inglis, S. (2004). 400 years of linguistic contact between the Mi'kmaq and the English and the interchange of two world views. *The Canadian Journal of Native Studies, 24*(2), 389–402.
Inhelder, B., & Piaget, J. (1958). *The growth of logical thinking from childhood to adolescence*. New York: Basic Books.
Ireson, J., Hallam, S., Hack, S., Clark, H., & Plewis, I. (2002). Ability grouping in English secondary schools: Effects on attainment in English, mathematics and science. *Educational Research and Evaluation: An International Journal on Theory and Practice, 8*(3), 299–318.
Jackiw, N. (2001). *The Geometer's Sketchpad*. Emeryville: Key Curriculum Press.
Jackson, K., Shahan, E., Gibbons, L., & Cobb, P. (2012). Launching complex tasks. *Mathematics Teaching in the Middle School, 18*(1), 24–29.
Jacobbe, T., & Millman, R. S. (2009). Mathematical habits of the mind for preservice teachers. *School Science and Mathematics, 109*(5), 298–302.
James, W. (1890, Reprinted 1950). *Principles of psychology* (Vol. 1). New York: Dover.
Jao, L. (2013). Peer coaching as a model for professional development in the elementary mathematics context: Challenges, needs and rewards. *Policy Futures in Education, 11*(3), 290–297.
Jao, L., & McDougall, D. (2015). The collaborative teacher inquiry project: A purposeful professional development initiative. *Canadian Journal of Education, 38*(1). Retrieved from http://journals.sfu.ca/cje/index.php/cje-rce/article/view/1600.
Jao, L., & McDougall, D. (2016). Moving beyond the barriers: Supporting meaningful teacher collaboration to improve secondary school mathematics. *Teacher Development, 20*(4), 557–573.
Jäppinen, A.-K., Leclerc, M., & Tubin, D. (2016). Collaborativeness as the core of professional learning communities beyond culture and context: Evidence from Canada, Finland, and Israel. *School Effectiveness and School Improvement, 27*(3), 315–332.
Jardine, D. W. (2003). *On the nature of inquiry: Choosing a topic*. Retrieved from http://galileo.org/teachers/designing-learning/articles/choosing-a-topic/
Jaworski, B. (1994). *Investigating mathematics teaching: A constructivist enquiry*. London: Falmer Press.
Jeon, K. (2012). Reflecting on PEMDAS. *Teaching Children's Mathematics, 18*(6), 370–377.

Joint Committee on Standards for Educational Evaluation (JCSEE). (2003). *The student evaluation standards: How to improve evaluation of students*. Thousand Oaks: Corwin Press and ETS.

Jones, G. A., & Thornton, C. A. (2005). In G. A. Jones (Ed.)., *Exploring probability in school: Challenges for teaching and learning An overview of research into the teaching and learning of probability* (pp. 65–92). New York: Springer.

Jones, G. A., Langrall, C. W., & Mooney, E. S. (2007a). Research in probability. Responding to classroom realities. In F. K. Lester (Ed.), *Second handbook of research on mathematics teaching and learning* (pp. 909–956). Reston: National Council of Teachers of Mathematics.

Jones, G. A., Langrall, C. W., & Mooney, E. S. (2007b). Research in probability: Responding to classroom realities. In F. Lester (Ed.), *Second handbook of research on mathematics teaching and learning* (pp. 909–955). Charlotte: Information Age.

Jorgensen, R. (2016). The elephant in the room: Equity, social class, and mathematics. In P. Ernest, Sriraman, & N. Ernest (Eds.), *Critical mathematics education: Theory, praxis and reality* (pp. 127–145). Charlotte: Information Age Publishing.

Joseph, K. N. (2014). *College students' misconceptions of the order of operations*. Unpublished doctoral dissertation, State University of New York at Fredonia, Fredonia, NY.

Joseph, B. (2015, June 2). *21 things you may not have known about The Indian Act*. Retrieved from https://www.ictinc.ca/blog/21-things-you-may-not-have-known-about-the-indian-act-

Kahneman, D. (2012). *Thinking fast, thinking slow*. London: Penguin.

Kahneman, D., Slovic, P., & Tversky, A. (1982). *Judgment under uncertainty: Heuristics and biases*. Cambridge: Cambridge University Press.

Kajander, A., & Boland, T. (2014). *Mathematical models for teaching: Reasoning without memorization*. Toronto: Canadian Scholars' Press.

Kajander, A., & Mason, R. (2007). Examining teacher growth in professional learning groups for in-service teachers of mathematics. *Canadian Journal of Mathematics, Science and Technology Education, 7*(4), 417–438.

Kajander, A., & Zuke, C. (2007). *Factors contributing to success for intermediate students of mathematics: The needs of the teachers and the characteristics and needs of students at-risk*. Thunder Bay: Northern Ontario Educational Leaders.

Kajander, A., Zuke, C., & Walton, G. (2008). Teaching unheard voices: Students at risk in mathematics. *Canadian Journal of Education, 31*(4), 1039–1064.

Kalder, R. S. (2012). Are we contributing to our students' mistakes? *Mathematics Teacher, 106*(2), 90–91.

Karp, K. S., & Voltz, D. L. (2000). Weaving mathematical instructional strategies into inclusive settings. *Intervention in School and Clinic, 35*(4), 206–215.

Karp, K. S., Bush, S. B., & Dougherty, B. J. (2015). 12 Math rules that expire in the middle grades. *Mathematics Teaching in the Middle School, 21*(4), 208–215.

Katz, M. (1997). *On playing a poor hand well: Insights from the lives of those who have overcome childhood risks and adversities*. New York: Norton.

Katz, C. (2001). Vagabond capitalism and the necessity of social reproduction. *Antipode, 33*(4), 709–728.

Katz, M. (2016). *Children who fail at school but succeed at life: Lessons from lives well-lived* (1st ed.). New York: W. W. Norton & Company.

Kawasaki, K. (2002). A cross-cultural comparison of English and Japanese linguistic assumptions influencing pupils' learning of science. *Canadian and International Education, 31*(1), 19–51.

Kazemi, E., & Franke, M. L. (2004). Teacher learning in mathematics: Using student work to promote collective inquiry. *Journal of Mathematics Teacher Education, 7*, 203–235.

Kelly, K. (2010). *What technology wants*. New York: Penguin.

Key Curriculum Press. (2012). *Fish-length distributions*. Amherst: Learn Troop Retrieved from http://www.tinkerplots.com/activities/data-analysis-and-modeling-activities

Khisty, L. (1995). Making inequality: Issues of language and meanings in mathematics teaching with Hispanic students. In W. G. Secada, E. Fennema, & L. B. Adajian (Eds.), *New directions for equity in mathematics education* (pp. 279–297). New York: Cambridge University Press.

Khovanova, T. (2011). Martin Gardner's mistake. *The College Mathematics Journal, 43*(1), 20–24 Retrieved from the arXiv database at: https://arxiv.org/abs/1102.0173.

Kieren, T., & Simmt, E. (2002). Fractal filaments: A simile for observing collective mathematical understanding. In D. Mewborn, P. Sztajn, D. White, H. Wiegel, R. Bryant, & K. Nooney (Eds.), *Proceedings of the 24th annual meeting of the North American chapter of the International Group for the Psychology of Mathematics Education* (Vol. 2, pp. 865–874). Columbus: Eric Clearinghouse for Science, Mathematics, and Environmental Education.

Kilpatrick, J., Swafford, J., & Findell, B. (2001a). The strands of mathematical proficiency. In J. Kilpatrick, J. Swafford, & B. Findell (Eds.), *Adding it up: Helping children learn mathematics [electronic resource]* (pp. 115–155). Washington, DC: National Academy Press.

Kilpatrick, J., Swafford, J., & Findell, B. (Eds.). (2001b). *Adding it up: Helping children learn mathematics*. Washington, DC: National Academy Press.

King, A. J. C., Warren, W. K., Boyer, J. C., & Chin, P. (2005). *Double cohort study phase 4 report for the Ontario Ministry of Education*. Kingston: Queen's University.

Klein, E. (1966). *A comprehensive etymological dictionary of the English language: Dealing with the origin of words and their sense development thus illustrating the history of civilization and culture*. New York: Elsevier.

Kline, M. (1974). *Why Johnny can't add: The failure of the new math*. New York: Vintage Books.

Knorr-Cetina, K. (1999). *Epistemic cultures: How the sciences make knowledge*. Cambridge, MA: Harvard University Press.

Knuth, E. J., Stephens, A. C., McNeil, N. M., & Alibali, M. W. (2006). Does understanding the equal sign matter? Evidence from solving equations. *Journal for Research in Mathematics Education, 36*, 297–312.

Konold, C. (1989). Conceptions of probability. *Cognition and Instruction, 6*(1), 59–98.

Konold, C., & Miller, C. (2017). *TinkerPlots 2: Dynamic data exploration [Computer software]*. Amherst: Learn Troop.

Kontorovich, I., & Zazkis, R. (2016). Turn vs. shape: Teachers cope with incompatible perspectives on angle. *Educational Studies in Mathematics, 93*(2), 223–243.

Kontorovich, I., & Zazkis, R. (2017). Mathematical conventions: Revisiting arbitrary and necessary. *For the Learning of Mathematics, 37*(1), 29–34.

Kumashiro, K. K. (2001). "Posts" perspectives on anti-oppressive education in social students, English, mathematics, and science classrooms. *Educational Researcher, 30*(3), 3–12.

Lai, K. (2012). Knowing to act in the moment: Examples from Confucius' analects. *Asian Philosophy: An International Journal of the Philosophical Traditions of the East, 22*(4), 347–364. https://doi.org/10.1080/09552367.2012.729324.

Lajoie, C., & Bednarz, N. (2012). Évolution de la résolution de problèmes en enseignement des mathématiques au Québec: Un parcours sur cent ans des programmes et documents pédagogiques. *Canadian Journal of Science, Mathematics and Technology Education, 12*(2), 178–213.

Lajoie, C., & Bednarz, N. (2014a). La notion de situation-problème en mathématiques au début du XXIe siècle au Québec: rupture ou continuité? *Canadian Journal of Science, Mathematics and Technology Education, 16*(1), 1–27.

Lajoie, C., & Bednarz, N. (2014b). La résolution de problèmes en mathématiques au Québec: évolution des rôles assignés par les programmes et des conseils donnés au enseignants. *Éducation et Francophonie, XLII*(2), 7–23.

Lakatos, I. (1976). *Proofs and refutations*. Cambridge: Cambridge University Press.

Lamon, S. J., & Lesh, R. (1992). Interpreting responses to problems with several levels and types of correct answers. In R. Lesh & S. J. Lamon (Eds.), *Assessment of authentic performance in school mathematics* (pp. 320–335). Washington, DC: AAAS Press.

Lampert, M. (2001). *Teaching problems and the problems of teaching*. New Haven: Yale University Press.

Lampert, M. (2010). Learning teaching in, from, and for practice: What do we mean. *Journal of Teacher Education, 61*(1–2), 21–34.

Landers, M. G. (2013). Towards a theory of mathematics homework as a social practice. *Educational Studies in Mathematics, 84*(3), 371–391.
Landscape. (2017). In *English Oxford living dictionaries*. Retrieved from https://en.oxforddictionaries.com/definition/landscape
Lapkin, S., Hart, D., & Turnbull, M. (2003). Grade 6 French immersion students' performance on large-scale reading, writing, and mathematics tests: Building explanations. *Alberta Journal of Educational Research, 49*(1), 6–23.
Lave, J. (1992). Word problems: A microcosm of theories of learning. In P. Light & G. Butterworth (Eds.), *Context and cognition: Ways of learning and knowing* (pp. 74–92). New York: Harvester.
Layton, D. (1981). The schooling of science in England, 1854–1939. In R. MacLeod & P. Collins (Eds.), *The parliament of science* (pp. 188–210). Northwood: Science Reviews.
Leahy, S., Lyon, C., Thompson, M., & Wiliam, D. (2005). Classroom assessment: Minute by minute, day by day. *Educational Leadership, 63*(3), 18–24.
Leap, W. L. (1988). Assumptions and strategies guiding mathematics problem solving by Ute Indian students. In R. Cocking & J. Mestre (Eds.), *Linguistic and cultural influences on learning mathematics* (pp. 165–186). New York: Routledge.
Leikin, R. (2007). Habits of mind associated with advanced mathematical thinking and solution spaces of mathematical tasks. In *The fifth Conference of the European Society for Research in Mathematics Education – CERME-5* (pp. 2330–2339). Retrieved from http://ermeweb.free.fr/Cerme5.pdf
Leikin, R. (2009). Exploring mathematical creativity using multiple solution tasks. In R. Leikin, Berman, & B. Koichu (Eds.), *Creativity in mathematics and the education of gifted students* (pp. 129–145). Rotterdam: Sense Publisher.
Leikin, R. (2014). Challenging mathematics with multiple solution tasks and mathematical investigations in geometry. In Y. Li, E. A. Silver, & S. Li (Eds.), *Transforming mathematics instruction: Multiple approaches and practices* (pp. 59–80). Dordrecht: Springer.
Lesh, R., & Doerr, H. (2003). *Beyond constructivism: Models and modeling perspectives on mathematics problem solving, learning, and teaching*. Mahwah: Lawrence Erlbaum.
Lesh, R., & Fennewald, T. (2010). Modeling: What is it? Why do it? In R. Lesh, C. Haines, P. Galbraith, & A. Hurford (Eds.), *Modeling students mathematical modeling competencies* (pp. 5–15). New York: Springer.
Lesh, R., Post, T., & Behr, M. (1988). Proportional reasoning. In J. Hiebert & M. Behr (Eds.), *Number concepts and operations in the middle grades* (pp. 93–118). Reston: Lawrence Erlbaum & National Council of Teachers of Mathematics.
Lester, F. K., & Charles, R. (2003). *Teaching mathematics through problem solving: Pre-K–grade 6*. Reston: National Council of Teachers of Mathematics.
Lewis, B. F., & Aikenhead, G. S. (2001). Introduction: Shifting perspectives from universalism to cross-culturalism. *Education, 85*(1), 3–5.
Lewis, C. C., Perry, R. R., & Hurd, J. (2009). Improving mathematics instruction through lesson study: A theoretical model and North American case. *Journal of Mathematics Teacher Education, 12*(4), 285–304. https://doi.org/10.1007/s10857-009-9102-7.
Li, Y., & Lerner, R. M. (2011). Trajectories of school engagement during adolescence: Implications for grades, depression, delinquency, and substance use. *Developmental Psychology, 47*, 233–247.
Libbey, H. P. (2004). Measuring student relationships to school: Attachment, bonding, connectedness, and engagement. *Journal of School Health, 74*(7), 274–283.
Lieberman, J., & Wilkins, E. (2006). The professional development pathways model: From policy to practice. *Kappa Delta Pi Record, 42*(3), 124.
Liljedahl, P. (2008). *The AHA! Experience: Mathematical contexts, pedagogical implications*. Saarbrücken: VDM Verlag.
Liljedahl, P. (2010). The four purposes of assessment. *Vector, 2010*(2), 4–12.

Liljedahl, P. (2012). Building thinking classrooms: Conditions for problem solving. In P. Felmer, E. Pekhonen, & J. Kilpatrick (Eds.), *Posing and solving mathematical problems: Advances and new perspectives* (pp. 361–386). New York: Springer.

Liljedahl, P. (2014). The affordances of using visibly random groups in a mathematics classroom. In Y. Li, E. Silver, & S. Li (Eds.), *Transforming mathematics instruction: Multiple approaches and practices* (pp. 127–144). New York: Springer.

Liljedahl, P. (2016a). Building thinking classrooms: Conditions for problem-solving. In P. Felmer, E. Pehkonen, & J. Kilpatrick (Eds.), *Posing and solving mathematical problems: Advances and new perspectives* (pp. 361–386). Cham: Springer.

Liljedahl, P. (2016b). Flow: A framework for discussing teaching. In C. Csíkos, A. Rausch, & J. Szitányi (Eds.), *Proceedings of the 40th conference of the International Group for the Psychology of Mathematics Education* (Vol. 3, pp. 203–210). Szeged: PME.

Liljedahl, P. (in press-a). Classroom practices for supporting problem solving. In P. Liljedahl & Santos (Eds.), *Current themes, trends and research perspectives in mathematical problem solving*. New York: Springer.

Liljedahl, P. (in press-b). On the edges of flow: Student problem solving behavior. In S. Carreira, Amado, & K. Jones (Eds.), *Broadening the scope of research on mathematical problem solving: A focus on technology, creativity and affect.* New York: Springer.

Liljedahl, P., & Allan, D. (2013). Studenting: The case of "now you try one". In A. M. Lindmeier & A. Heinze (Eds.), *Proceedings of the 37th conference of the International Group for the Psychology of Mathematics Education* (Vol. 3, pp. 257–264). Kiel: PME.

Lim, K., & Selden, A. (2009). Mathematical habits of mind. In S. L. Swars, D. W. Stinson, & S. Lemons-Smith (Eds.), *Proceedings of the 31st annual meeting of the North American chapter of the International Group for the Psychology of Mathematics Education* (Vol. 5, pp. 1576–1583). Atlanta: Georgia State University.

Linchevski, L., & Livneh, D. (1999). Structure sense: The relationship between algebraic and numerical contexts. *Educational Studies in Mathematics, 40*, 173–196.

Lins, R. C. (2006). Characterizing the mathematics of the mathematics teacher from the point of view of meaning production. In M. Niss (Ed.), *10th International Congress on Mathematical Education* (pp. 1–16). Copenhagen.

Lipka, J. (1994). Culturally negotiated schooling: Toward a Yup'ik mathematics. *Journal of American Indian Education, 33*(3), 14–30.

Lipka, J., & Adams, B. (2004). *Culturally based math education as a way to improve Alaska Native students' math performance* (Working paper no. 20). Athens: Appalachian Center for Learning, Assessment, and Instruction in Mathematics.

Lipka, J., Hogan, M. P., Webster, J. P., Yanez, E., Adams, B., Clark, S., & Lacy, D. (2005). Math in a cultural context: Two case studies of a successful culturally based math project. *Anthropology and Education Quarterly, 36*(4), 367–385.

Literacy and Numeracy Secretariat. (n.d.). *The three-part lesson in mathematics: Co-planning, co-teaching and supporting student learning: Viewers guide*. Toronto: Queen's Printer for Ontario Retrieved from http://www.curriculum.org/secretariat/coplanning/files/CoPlanningGuide.pdf

Literacy and Numeracy Secretariat. (2010). *Communication in the mathematics classroom* (Secretariat Special Edition #13). Retrieved from http://www.edu.gov.on.ca/eng/literacynumeracy/inspire/research/CBS_Communication_Mathematics.pdf

Little, J. W. (2003). Inside teacher community: Representations of classroom practice. *Teachers College Record, 105*(6), 913–945.

Little Bear, L. (2000a). Jagged worldviews colliding. In M. Battiste (Ed.), *Reclaiming Indigenous voice and vision* (pp. 77–85). Vancouver: UBC Press.

Little Bear, L. (2000b). Foreword to Native science: Natural laws of interdependence. In G. Cajete (Ed.), *Native science: Natural laws of interdependence* (pp. ix–xii). Santa Fe: Clearlight Publishers.

Lomon, E., Beck, B., & Arnbetter, C. (1975). Real problem solving in USMES: Interdisciplinary education and much more. *School Science and Mathematics, 75*(1), 63–74.

Long, J. (2008). *Caring for students and caring for mathematical ideas in an elementary classroom.* Unpublished doctoral dissertation. Edmonton: University of Alberta.

Long, J. (2011). Labeling angles: Care, indifference and mathematical symbols. *For the Learning of Mathematics, 31*(3), 2–7.

Lortie, D. (1975). *Schoolteacher: A sociological study*. Chicago: University of Chicago Press.

Love, E., & Mason, J. (1992). *Teaching mathematics: Action and awareness*. Milton Keynes: Open University.

Lowe, K., Lee, L., Schibeci, R., Cummings, R., Phillips, R., & Lake, D. (2010). Learning objects and engagement of students in Australian and New Zealand schools. *British Journal of Educational Technology, 41*(2), 227–241.

Luke, A., Elkins, J., Weir, K., Land, R., Carrington, V., Dole, S., Pendergast, D., Kapitzke, C., van Kraayenoord, C., Moni, K., McIntosh, A., Mayer, D., Bahr, M., Hunter, L., Chadbourne, R., Bean, T., Alverman, D., & Steven, L. (2003). *Beyond the middle: A report about literacy and numeracy development of target group students in the middle years of schools*. Canberra: Commonwealth Department of Education, Science and Training.

Lunney Borden, L. (2010). *Transforming mathematics education for Mi'kmaw students through mawikinutimatimk.* Unpublished doctoral dissertation. Fredericton: University of New Brunswick.

Lunney Borden, L. (2011). The "Verbification" of mathematics: Using the grammatical structures of Mi'kmaq to support student learning. *For the Learning of Mathematics, 31*(3), 8–13.

Lunney Borden, L. (2012). What's the word for…? Is there a word for…? How understanding Mi'kmaw language can help support Mi'kmaw learners in mathematics. *Mathematics Education Research Journal., 25*, 5. https://doi.org/10.1007/s13394-012-0042-7.

Lunney Borden, L., & Wagner, D. (2011). Qualities of respectful positioning and their connections to quality mathematics. In B. Atweh, M. Graven, W. Secada, & P. Valero (Eds.), *Mapping equity and quality in mathematics education* (pp. 379–391). New York: Springer.

Lunney Borden, L., & Wagner, D. (2013). Naming method: "This is it, maybe, but you should talk to…". In R. Jorgensen, P. Sullivan, & P. Grootenboer (Eds.), *Pedagogies to enhance learning for Indigenous students* (pp. 105–122). New York: Springer.

Lunney Borden, L., Wagner, D., & Johnson, N. (in press). Show me your math: Mi'Kmaw community members explore mathematics. In C. Nicol, S. Dawson, J. Archibald, & F. Glanfield (Eds.), *Living culturally responsive mathematics curriculum and pedagogy: Making a difference with/in Indigenous communities*. Rotterdam: Sense Publications.

Luthar, S. (Ed.). (2003). *Resilience and vulnerability: Adaptation in the context of childhood adversities*. Cambridge: Cambridge University Press.

Luthar, S. S., Cicchetti, D., & Becker, B. (2000). The construct of resilience: A critical evaluation and guidelines for future work. *Child Development, 71*(3), 543–562. https://doi.org/10.1111/1467-8624.00164.

Ma, L. (1999). *Knowing and teaching elementary mathematics: Teachers' understanding of fundamental mathematics in China and the United States*. Mahwah: Lawrence Erlbaum Associates.

Macaulay, A. V. M. (2015). *Effective practices in Grade 9 Applied mathematics*. Unpublished doctoral dissertation, Ontario Institute for Studies in Education, University of Toronto, Toronto, ON.

MacIvor, M. (1995). Redefining science education for Aboriginal students. In M. Battiste & J. Barman (Eds.), *First Nations education in Canada: The circle unfolds* (pp. 72–98). Vancouver: UBC Press.

MacNeil, A. J., Prater, D. L., & Busch, S. (2009). The effects of school culture and climate on student achievement. *International Journal of Leadership in Education, 12*(1), 73–84.

Mamolo, A., & Thomas, K. (2014). A mathematical exploration of the nutrition North Canada Program. *Ontario Mathematics Gazette, 53*(1), 17–22.

Marks, H. M. (2000). Student engagement in instructional activity: Patterns in the elementary, middle, and high school years. *American Educational Research Journal, 37*(1), 153–184.

Martin, V. (2010). *Quand rien n'est sûr, tout est possible: l'apprentissage des probabilités chez des élèves à risque*. Montréal: Éditions Bande didactique.
Martin, V. (2014). *Étude des interventions didactiques dans l'enseignement des probabilités auprès d'élèves jugés ou non en difficulté en mathématiques en classes ordinaires du primaire*. Doctoral thesis. Sherbrooke: Université de Sherbrooke. Retrieved from http://savoirs.usherbrooke.ca/bitstream/handle/11143/5449/Martin_Vincent_PhD_2014.pdf?sequence=4
Martin, V., & Theis, L. (2011). La résolution d'une situation-problème probabiliste en équipe hétérogène: le cas d'une élève à risque du primaire. *Nouveaux Cahiers de la Recherche en Éducation, 14*(1), 49–70.
Martin, V., & Theis, L. (2016). L'articulation des perspectives fréquentielle et théorique dans l'enseignement des probabilités: Regard sur un changement de posture chez un enseignant du primaire. *Canadian Journal of Science, Mathematics and Technology Education, 16*(4), 1–14.
Martin, V., & Thibault, M. (2016). Regards québécois sur sept décennies de recherche liée à l'apprentissage et à l'enseignement des probabilités. *Annales de Didactique et de Sciences Cognitives, 21*, 79–115.
Martin, L. C., & Towers, J. (2003). Collective mathematical understanding as an improvisational process. *International Group for the Psychology of Mathematics Education, 3*, 245–252.
Martin, L. C., & Towers, J. (2015). Growing mathematical understanding through collective image making, collective image having, and collective property noticing. *Educational Studies in Mathematics, 88*(1), 3–18.
Marton, F. (2015). *Necessary conditions for learning*. Abingdon: Routledge.
Marton, F., & Booth, S. A. (1997). *Learning and awareness*. New York: Psychology Press.
Marton, F., & Säljö, R. (1976). On qualitative differences in learning I: Outcome and process. *British Journal of Educational Psychology, 46*(1), 4–11. https://doi.org/10.1111/j.2044-8279.1976.tb02980.x.
Marton, F., Dall'Alba, G., & Beaty, E. (1993). Conceptions of learning. *International Journal of Educational Research, 19*(3), 277–300. https://doi.org/10.1016/0883-0355(93)90015-C.
Marx, K., & Engels, F. (1978). *Werke band 3* [Works vol. 3]. Berlin: Dietz.
Mary, C., & Theis, L. (2007). Les élèves à risque dans des situations problèmes statistiques: stratégies de résolution et obstacles cognitifs. *Revue des Sciences de l'Éducation, 33*(3), 579–599.
Mason, J. (1994). Only awareness is educable. In A. Bloomfield & T. Harries (Eds.), *Teaching, learning and mathematics* (pp. 28–29). Derby: Association of Teachers of Mathematics.
Mason, J. (1996). *Twenty-six years of problem posing: Mathematical investigations used in Open University mathematics summer schools 1971–1996*. Milton Keynes: Open University Centre for Mathematics Education.
Mason, J. (1998). Enabling teachers to be real teachers: Necessary levels of awareness and structure of attention. *Journal of Mathematics Teacher Education, 1*(3), 243–267.
Mason, J. (2002a). *Researching your own practice: The discipline of noticing*. New York: Routledge.
Mason, J. (2002b). *Researching your own practice: The discipline of noticing*. London: Routledge Falmer.
Mason, J. (2012). Noticing: Roots and branches. In M. Sherin, V. Jacobs, & R. Phillip (Eds.), *Mathematics teacher noticing: Seeing through teachers' eyes* (pp. 35–50). Mahwah: Erlbaum.
Mason, J. (2016). Rising above a cause-and-effect stance in mathematics education research. *JMTE, 19*(4), 297–300. Retrieved from http://rdcu.be/npTf
Mason, J., & Davis, J. (1989). The inner teacher, the didactic tension, and shifts of attention. In G. Vergnaud, J. Rogalski, & M. Artigue (Eds.), *Proceedings of PME XIII* (Vol. 2, pp. 274–281). Paris.
Mason, R. T., & McFeetors, P. J. (2002). Interactive writing in mathematics class: Getting started. *Mathematics Teacher, 95*(5), 532–536.
Mason, J., & Metz, M. (2017). Digging beneath dual systems theory and the bicameral brain: Abductions about the human psyche from experience in mathematical problem solving. In U.

Eligio (Ed.), *Understanding emotions in mathematical thinking and learning* (pp. 379–407). Cambridge: Elsevier/Academic.

Mason, J., & Spence, M. (1999). Beyond mere knowledge of mathematics: The importance of knowing-to act in the moment. *Educational Studies in Mathematics, 38*, 135–161.

Mason, J., & Spence, M. (2000). Beyond mere knowledge of mathematics: The importance of knowing –to act in the moment. In D. Tirosh (Ed.), *Forms of mathematical knowledge: Learning and teaching with understanding* (pp. 135–161). Dordrecht: Kluwer.

Mason, J., Burton L., & Stacey K. (1982/2010). *Thinking mathematically.* London: Pearson.

Masten, A. S. (2001). Ordinary magic: Resilience processes in development. *American Psychologist, 56*(3), 227–238. https://doi.org/10.1037/0003-066x.56.3.227.

Masten, A. S., & Coatsworth, J. D. (1998). The development of competence in favorable and unfavorable environments: Lessons from research on successful children. *American Psychologist, 53*(2), 205–220. https://doi.org/10.1037//0003-066x.53.2.205.

Masten, A. S., & Obradovic, J. (2006). Competence and resilience in development. *Annals of the New York Academy of Sciences, 1094*(1), 13–27. https://doi.org/10.1196/annals.1376.003.

Maull, W., & Berry, J. (2001). An investigation of student working styles in a mathematical modelling activity. *Teaching Mathematics and its Applications, 20*(2), 78–88.

McDougall, D. (2004). *PRIME school mathematics improvement: Leadership handbook*. Toronto: Nelson.

McDougall, D. E., & Jao, L. (2009). *Report on the Grade 9 Applied math project for the learning consortium*. Report for Toronto District School Board, Toronto Catholic District School Board, Peel District School Board and York Region District School Board. Ontario Institute for Studies in Education, Toronto, ON.

McDougall, D., Ross, J. A., & Ben Jaafar, S. (2006). *PRIME ten dimensions of mathematics education: Research study*. Toronto: Thompson Nelson.

McDougall, D. E., Jao, L., Maguire, K., Stoilescu, D., & Egodawatte, G. (2010). *Collaborative teacher inquiry project: Grade 9 Applied mathematics*. Report for Toronto District School Board and Toronto Catholic District School Board. Ontario Institute for Studies in Education, Toronto, ON.

McDougall, D. E., Yan, X. H., Kam, M., Wang, Z. Y., Kwan, K., & Ferguson, S. (2013). *Collaborative teacher inquiry project: Grade 8 mathematics*. Report for Toronto District School Board. Ontario Institute for Studies in Education, Toronto, ON.

McGarvey, L. M. (2014). Intentional play-spaces for teaching/learning mathematics with young children. *For the Learning of Mathematics, 34*(2), 9–11.

McGarvey, L., Davis, B., Glanfield, F., Martin, L., Mgombelo, J., Proulx, J., Simmt, E., Thom, J. S., & Towers, J. (2015, November). *Collective learning: Conceptualizing the possibility in the mathematics classroom*. Discussion group at the 37th annual meeting of the North American chapter of the International Group for the Psychology of Mathematics Education, East Lansing, MI.

McGarvey, L., Glanfield, F., Mgombelo, J., & Thom, J. S. (2017, April). *Monitoring the vital signs of classroom life*. Research symposium at the 2017 National Council of Teachers of Mathematics research conference, San Antonio, TX.

McNeal, B., & Simon, M. A. (2000). Mathematics culture clash: Negotiating new classroom norms with prospective teachers. *The Journal of Mathematical Behavior, 18*(4), 475–509.

Mead, G. H. (1972). *Mind, self, and society: From the standpoint of a social behaviorist*. Chicago: University of Chicago Press.

Mehta, R., Mishra, P., & Henriksen, D. (2016). Creativity in mathematics and beyond—Learning from Fields medal winners. *TechTrends, 60*(1), 14–18. https://doi.org/10.1007/s11528-015-0011-6.

Meikle, E. M. (2016). Selecting and sequencing students' solution strategies. *Teaching Children Mathematics, 23*(4), 226–234.

Mellin-Olsen, S. (1987). *The politics of mathematics education*. Dordrecht: Reidel.

Mendrick, H. (2008). What's so great about doing mathematics like a mathematician? *For the Learning of Mathematics, 28*(3), 15–16.
Mergendoller, J. R., Marchman, V. A., Mitman, A. L., & Packer, M. J. (1988). Task demands and accountability in middle-grade science classes. *Elementary School Journal, 88*(3), 251–265.
MES. www.mescommunity.info. Accessed Nov 2017.
Meyer, D. (n.d.). *Popcorn picker*. Retrieved from http://threeacts.mrmeyer.com/popcornpicker/
Meyer, D. (2010, March). *Math class needs a makeover* [Video file]. Retrieved from https://www.ted.com/talks/dan_meyer_math_curriculum_makeover
Michell, H, Vizina, Y, Augustus, C., & Sawyer, J. (2008). *Learning Indigenous science from place*. Retrieved from http://iportal.usask.ca/docs/Learningindigenousscience.pdf
Middleton, J. A., & Spanial, P. A. (1999). Motivation for achievement in mathematics: Findings, generalizations, and criticisms of the research. *Journal for Research in Mathematics Education, 30*(1), 65–88.
Mikhailov, F. T. (2001). The "other within" for the psychologist. *Journal of Russian and East European Psychology, 39*(1), 6–31.
Miller, C., & Parlett, M. (1974). *Up to the mark: A study of the examination game*. Guildford: Society for Research into Higher Education.
Ministry of Education. (2014). *New Zealand curriculum online: Mathematics and statistics*. Retrieved from http://nzcurriculum.tki.org.nz/The-New-Zealand-Curriculum/Mathematicsand-statistics. Supporting software INZight: https://www.stat.auckland.ac.nz/~wild/iNZight/index.php
MiS (UK). educationendowmentfoundation.org.uk/projects-and-evaluation/projects/maths-in-context. Accessed Nov 2017.
MiS (USA). qualifications.pearson.com/en/qualifications/edexcel-mathematics-in-context/mathematics-in-context.html. Accessed Nov 2017.
Mitchell, R., Charalambous, C., & Hill, H. (2014). Examining the task and knowledge demands needed to teach with representations. *Journal of Mathematics Teacher Education, 17*, 37–60.
Moore, K. A., & Lippman, L. (Eds.). (2005). *What do children need to flourish?: Conceptualizing and measuring indicators of positive development*. New York: Springer.
Moreira, P. C. (2004). *O conhecimento matemático do professor: Formação na licenciatura e prática docente na escola básica*. Unpublished doctoral dissertation, Faculdade de Educação UFMG, Belo Horizonte, MG, Brazil.
Moreira, P. C., & David, M. M. (2005). *A formação matemática do professor: Licenciatura e prática docente escolar*. Belo Horizonte: Autêntica.
Moreira, P. C., & David, M. M. (2008). Academic mathematics and mathematical knowledge needed in school teaching practice: Some conflicting elements. *Journal of Mathematics Teacher Education, 11*, 23–40.
Moschkovich, J. N. (1999). Supporting the participation of English language learners in mathematical discussions. *For the Learning of Mathematics, 19*(1), 11–19.
Moschkovich, J. N. (2009). How language and graphs support conversation in a bilingual mathematics classroom. In R. Barwell (Ed.), *Multilingualism in mathematics classrooms: Global perspectives* (pp. 78–96). Bristol: Multilingual Matters.
Moschkovich, J. N. (Ed.). (2010). *Language and mathematics education: Multiple perspectives and directions for research*. Charlotte: Information Age Publishing.
Moschkovich, J. N. (2013). Principles and guidelines for equitable mathematics teaching practices and materials for English language learners. *Journal of Urban Mathematics Education, 6*(1), 45–57.
Moss, C. M., & Brookhart, S. M. (2009). *Advancing formative assessment in every classroom*. Alexandria: ASCD.
Moss, J., Beatty, R., Barkin, S., & Shillolo, G. (2008). "What is your theory? What is your rule?" Fourth graders build an understanding of functions through patterns and generalizing problems. In C. E. Greenes & R. Rubenstein (Eds.), *Algebra and algebraic thinking in school mathematics* (pp. 155–168). Reston: National Council of Teachers of Mathematics.

Moss, J., Hawes, Z., Naqvi, S., & Caswell, B. (2015). Adapting Japanese Lesson Study to enhance the teaching and learning of geometry and spatial reasoning in early years classrooms: A case study. *ZDM Mathematics Education, 47*, 377–390.

Muchnick, C. C., & Muchnick, J. R. (2013). *Straight-a study skills: More than 200 essential strategies to ace your exams, boost your grades, and achieve lasting academic success*. Avon: Adams Media.

Muir, T. (2008). Principles of practice and teacher actions: Influences on effective teaching of numeracy. *Mathematics Education Research Journal, 20*(3), 78–101.

Mukhopadhyay, S., & Greer, G. (2012). Ethnomathematics. In J. A. Banks (Ed.), *Encyclopedia of diversity in education* (pp. 857–861). Thousand Oaks: SAGE Publication.

Mullis, I. V. S., Martin, M. O., Foy, P., & Hooper, M. (2016). *TIMSS 2015 international results in mathematics*. Boston: TIMSS & PIRLS International Study Center.

Murray, C. (2009). Parent and teacher relationships as predictors of school engagement and functioning among low-income urban youth. *Journal of Early Adolescence, 29*(3), 376–404.

Musser, G. L., Burger, W. F., & Paterson, B. E. (2006). *Mathematics for elementary teachers: A contemporary approach*. New York: Wiley & Sons.

Nasir, N. S., Hand, V., & Taylor, E. V. (2008). Culture and mathematics in school: Boundaries between "cultural" and "domain" knowledge in the mathematics classroom and beyond. *Review of Research in Education, 32*, 187–240.

National Council of Teachers of Mathematics. (1989). *Curriculum and evaluation standards for school mathematics*. Reston: Author.

National Council of Teachers of Mathematics (NCTM). (1995). *Assessment standards for school mathematics*. Reston: Author.

National Council of Teachers of Mathematics (NCTM). (2000). *Principles and standards for school mathematics*. Reston: Author.

National Council of Teachers of Mathematics (NCTM). (2009). *Focus on high school mathematics: Reasoning and sense making*. Reston: Author.

National Council of Teachers of Mathematics (NCTM). (2014a). *Principles to action professional learning toolkit: Bike and truck*. Retrieved from http://www.nctm.org/PtAToolkit/

National Council of Teachers of Mathematics (NCTM). (2014b). *Principles to action: Ensuring mathematical success for all*. Reston: Author.

Nelson, T., & Slavit, D. (2008). Supported teacher collaboration inquiry. *Teacher Education Quarterly, 35*(1), 99–116.

Nelson-Barber, S., & Lipka, J. (2008). Rethinking the case for culture-based curriculum: Conditions that support improved mathematics performance in diverse classrooms. In M. Brisk (Ed.), *Language, curriculum & community in teacher preparation* (pp. 99–123). Mahwah: Lawrence Erlbaum Associates.

Nemirovsky, R. (2003). Perceptuo-motor activity and imagination in mathematics learning. In N. Pateman, B. Dougherty, & J. Zilliox (Eds.), *Proceedings of the 27th conference of the International Group for the Psychology of Mathematics Education* (Vol. 1, pp. 103–135). Honolulu: PME.

Nesher, P., Hershkovitz, S., & Novotna, J. (2003). Situation model, text base and what else? Factors affecting problem solving. *Education Studies in Mathematics, 52*(2), 151–176.

Nespor, J. (1987). The role of beliefs in the practice of teaching. *Journal of Curriculum Studies, 19*(4), 317–328.

Newman, F., & Holzman, L. (1996). *Unscientific psychology*. Westport: Praeger.

Newmann, F. M., King, M. B., & Carmichael, D. L. (2007). *Authentic instruction and assessment: Common standards for rigor and relevance in teaching academic subjects*. Des Moines: Department of Education.

Ng, O., & Sinclair, N. (2015). Young children reasoning about symmetry in a dynamic geometry environment. *ZDM – The International Journal on Mathematics Education, 51*(3), 421–434.

Nickerson, R. S. (1996). Ambiguities and unstated assumptions in probabilistic reasoning. *Psychological Bulletin, 20*(3), 410–433.

Nicol, C., & Archibald, J. (2009, April). *Living culturally responsive education and research: Teaching and learning with/in Indigenous communities*. Paper presented at the annual American Education Research Association conference, San Diego, CA.

Nilsson, P. (2014). Experimentation in probability teaching and learning. In E. J. Chernoff & B. Sriraman (Eds.), *Probabilistic thinking: Presenting plural perspectives* (pp. 509–532). Dordrecht: Springer.

Nilsson, P., & Eckert, A. (2016). Interactive experimentation in probability – Opportunities, challenges and needs of research. In *Proceedings of the thirteenth International Congress on Mathematical Education*, Hamburg, Germany.

Nova Scotia Department of Education and Early Childhood Development (NSDEECD). (2015). *Mathematics 12* (Implementation Draft, June 2015). Retrieved from http://www.ednet.ns.ca/files/curriculum/Math12-ImplDraft-June2015-web.pdf

NRiCH. nrich.maths.org/7701/index. Accessed Nov 2017.

Nye, B., Konstantopoulos, S., & Hedges, L. V. (2004). How large are teacher effects? *Educational Evaluation and Policy Analysis, 26*(3), 237–257.

O'Connor, K. B. (2006). *Investigations into Indigenous research and education through an experiential and place-based lens*. Unpublished master's thesis, McGill University, Montreal, QC.

O'Connor, K. (2009). *Puzzles rather than answers: Co-constructing a pedagogy of experiential, place-based and critical learning in Indigenous education*. Unpublished doctoral thesis, McGill University, Montreal, QC

Oakes, J. (1982). The reproduction of inequity: The content of secondary school tracking. *Urban Review, 14*(2), 107–120.

Oakes, J. (1985). *Keeping track: How schools structure inequality*. New Haven: Yale University Press.

Oakes, J. (1986). Keeping track. Part 1: The policy and practice of curriculum inequality. *The Phi Delta Kappan, 68*(1), 12–18.

OECD. (2016). *PISA 2015 results: Excellence and equity in education* (Vol. I). Paris: OECD Publishing. Retrieved from https://doi.org/10.1787/9789264266490-en

OECDE Programme for International Student Assessment 2012. (2013). *PISA 2012 released mathematics items*. Retrieved from https://www.oecd.org/pisa/pisaproducts/pisa2012-2006rel-items-maths-ENG.pdf

Oesterle, S., Olson, M., Hwang, S., Slovin, H., Olson, J., Murawska, J., et al. (2016). Mathematical habits of mind. In L. C. Hart, S. Oesterle, S. Swars Auslander, & A. Kajander (Eds.), *The mathematics education of elementary teachers: Issues and strategies for content courses* (pp. 51–85). Charlotte: Information Age Publishing.

Ogawa, M. (1995). Science education in a multiscience perspective. *Science Education, 79*(5), 583–593.

Ontario Ministry of Education. (2000). *The Ontario curriculum exemplars: Grade 9 Mathematics*. Toronto: Queen's Printer for Ontario Retrieved from http://sw1math.wikispaces.com/file/view/Grade%209%20Math%20Exemplar.pdf.

Ontario Ministry of Education. (2005a). *TIPS4RM for grades 7, 8, 9 applied and 10 applied math*. Toronto: Queen's Printer for Ontario.

Ontario Ministry of Education. (2005b). *The Ontario curriculum grades 9/10: Mathematics* (Rev. ed.). Toronto: Queen's Printer.

Ontario Ministry of Education. (2005c). *The Ontario curriculum, Grades 1–8: Mathematics*. Toronto: Queen's Printer for Ontario. Retrieved from http://www.edu.gov.on.ca/eng/curriculum/elementary/math18curr.pdf

Ontario Ministry of Education. (2005d). *The Ontario curriculum, grades 1–8, mathematics* (Rev. ed.). Retrieved from http://www.edu.gov.on.ca/eng/curriculum/elementary/math.html

Ontario Ministry of Education. (2005e). *The Ontario curriculum: Mathematics, grades 9 and 10*. Toronto: Queen's Printer for Ontario. Retrieved from http://www.edu.gov.on.ca/eng/curriculum/secondary/math910curr.pdf

Ontario Ministry of Education. (2007a). *Ontario First Nation, Métis and Inuit education framework: Delivering quality education of Aboriginal students in Ontario's provincially funded schools*. Toronto, ON: Government of Ontario. Retrieved from http://www.edu.gov.on.ca/eng/aboriginal/fnmiFramework.pdf

Ontario Ministry of Education. (2007b). *The Ontario curriculum, grades 11 and 12, mathematics* (Rev. ed.). Toronto: Queen's Printer for Ontario. Retrieved from http://www.edu.gov.on.ca/eng/curriculum/secondary/math.html

Ontario Ministry of Education. (2007c). *Professional learning communities: A model for Ontario schools* (Secretariat: Special Edition #3). Toronto: Ontario Ministry of Education. Retrieved from http://www.edu.gov.on.ca/eng/literacynumeracy/inspire/research/PLC.pdf

Ontario Ministry of Education. (2010). *Growing success: Assessment, evaluation and reporting in Ontario schools: First edition, covering grades 1 to 12*. Toronto: Queen's Printer for Ontario. Retrieved from Ontario Ministry of Education website http://www.edu.gov.on.ca/eng/policyfunding/growSuccess.pdf

Ontario Ministry of Education. (2011a). *Bansho (board-writing)*. Toronto: Queen's Printer for Ontario.

Ontario Ministry of Education. (2011b). *Learning goals and success criteria: Assessment for learning video series: Viewing guide.* Retrieved from http://www.edugains.ca/resourcesDI/ProfLearningModules2011/DiffInst_AssessmentandEvaluation/LearningGoalsSuccessCriteriaViewingGuide2011.pdf

Ontario Ministry of Education. (2012). *Paying attention to proportional reasoning K-12*. Toronto: Queen's Printer for Ontario.

Ontario Ministry of Education. (2013a). Appendix C: Map, globe and graphing skills – A continuum. In The Ontario curriculum: Social studies grades 1 to 6 and history and geography grades 7 and 8 (pp. 191–198). Toronto: Queens Printer for Ontario. Retrieved from http://www.edu.gov.on.ca/eng/curriculum/elementary/sshg18curr2013.pdf

Ontario Ministry of Education. (2013b). *Learning for all: A guide to effective assessment and instruction for all students, Kindergarten to grade 12.* Retrieved from http://www.edu.gov.on.ca/eng/general/elemsec/speced/LearningforAll2013.pdf

Ontario Ministry of Education [OME]. (2010). *Growing success: Assessment, evaluation, and reporting in Ontario schools, grades 1 to 12* (1st ed.). Retrieved from http://www.edu.gov.on.ca/eng/policyfunding/growSuccess.pdf

Open University. (1980). *PME233: Mathematics across the curriculum (a distance learning course)*. Milton Keynes: Open University.

Overwijk, A. (2015). *Activity: Tossing cards for student engagement*. Retrieved from http://ocdsb-flashpoints.weebly.com/flash-points/archives/12-2015

Pai, J. (2015). *Lego: Star wars with friends* [Web log post]. Retrieved from http://intersectpai.blogspot.ca/2015/12/lego-star-wars-with-friends.html

Pai, J. (2016). *In-the-moment decisions: A preliminary investigation on observations and conversations as assessment in secondary mathematics classrooms*. Extended paper presented at the 13th International Congress on Mathematics Education in Hamburg, Germany.

Pan-Canadian Joint Consortium for School Health. (2015). *Strategic plan of the Pan Canadian Joint Consortium for School Health 2015–2020.* Retrieved from http://www.jcsh-cces.ca/images/JCSH_strategic_Plan_2015-2020_Final.pdf

Papert, S. (1980a). *Mindstorms*. New York: Harvester Press.

Papert, S. (1980b). *Mindstorms: Children, computers, and powerful ideas*. New York: Basic Books.

Pappanastos, E., Hall, M. A., & Honan, A. S. (2002). Order of operations: Do business students understand the correct order. *Journal of Education for Business, 78*(2), 81–84.

Peat, F. D. (2002). *Blackfoot physics*. York Beach: Red Wheel/Weiser.

Pehkonen, E. (1995). Introduction: Use of open-ended problems. *ZDM-International Journal of Mathematics Education, 27*(2), 55–57.

People for Education. (2013). The trouble with course choices in Ontario high schools. *Should low income = high Applied?* Retrieved from http://www.peopleforeducation.ca/wp-content/uploads/2013/04/trouble-with-course-choices-in-high-school-2013.pdf

Perlman, J. (2010). *Bully dance. National Film Board of Canada*. Retrieved from https://www.nfb.ca/film/bully_dance/

Pfannkuch, M. (2008). Building sampling concepts for statistical inference: A case study. In D. De Bock, B. D. Søndergaard, B. G. Alfonso, & C. C. L. Cheng (Eds.), *Proceedings of the 11th International Congress on Mathematical Education*. Monterrey. Retrieved from http://tsg.icme11.org/document/get/476

Philipp, R. A. (2007). Mathematics teachers' beliefs and affect. In F. Lester Jr. (Ed.), *Second handbook of research on mathematics teaching and learning* (pp. 257–315). Charlotte: Information Age Publishing.

Piaget, J. (1977). *The development of thought: Equilibration of cognitive structures*. New York: Viking.

Pilot, J., & Walter-Rowan, N. (2014, August). *Assessment and evaluation…more than learning goals and success criteria: Effective planning for learning*. Presentation at Lakehead University's welcome retreat: Faculty of Education, Thunder Bay, ON.

Pinar, W. F. (1975). *The method of "currere". Paper presented at the annual meeting of the*. Washington, DC: American Educational Research Association.

Pirie, S., & Kieren, T. (1989). A recursive theory of mathematical understanding. *For The Learning of Mathematics, 9*(3), 7–11.

Pirie, S., & Kieren, T. (1992). Creating constructivist environments and constructing creative mathematics. *Educational Studies in Mathematics, 23*, 505–528.

Pirie, S., & Kieren, T. (1994). Growth in mathematical understanding: How can we characterise it and how can we represent it? *Educational Studies in Mathematics, 26*(2–3), 165–190.

Planas, N. (2014). One speaker, two languages: Learning opportunities in the mathematics classroom. *Educational Studies in Mathematics, 87*(1), 51–66.

Pólya, G. (1945). *How to solve it* (1st ed.). Princeton: Princeton University Press.

Pólya, G. (1954). *Mathematics and plausible* (Vol. 1). Princeton: Princeton University Press.

Possin, K. L. (2010). Visual spatial cognition in neurodegenerative disease. *Neurocase, 16*, 466–487 Retrieved from NIH Public Access.

Postman, N. (1995). *The end of education: Redefining the value of school*. New York: Knopf.

Postman, N., & Weingartner, C. (1969). *Teaching as a subversive activity*. New York: Dell Publishing Co., Inc.

Pratt, D. (1998). The coordination of meanings for randomness. *For the Learning of Mathematics, 18*(3), 2–11.

Preciado-Babb, A. P., Metz, M., & Marcotte, C. (2015). Awareness as an enactivist framework for the mathematical learning of teachers, mentors and institutions. *ZDM, 47*(2), 257–268.

Pressley, M., Dolezal, S. E., Raphael, L. M., Mohan, L., Roehrig, A. D., & Bogner, K. (2003). *Motivating primary-grade students*. New York: Guilford.

Prodromou, T. (2012). Connecting experimental probability and theoretical probability. *ZDM: The International Journal on Mathematics Education, 44*(7), 855–868.

Pui-wah, D. C. (2008). Meta-learning ability: A crucial component for the professional development of teachers in a changing context. *Teacher Development, 12*(1), 85–95.

Punch, K. F. (2009). *Introduction to research methods in education*. Thousand Oaks: SAGE Publications, Inc.

Radford, L. (2009). Signifying relative motion: Time, space and the semiotics of Cartesian graphs. In W.-M. Roth (Ed.), *Mathematical representations at the interface of the body and culture* (pp. 45–69). Charlotte: Information Age Publishers.

Radford, L. (2012). Education and the illusions of emancipation. *Educational Studies in Mathematics, 80*(1), 101–118.

Radford, L. (2014). On teachers and students: An ethical cultural-historical perspective. In P. Liljedahl, C. Nicol, S. Oesterle, & D. Allan (Eds.), *Proceedings of the joint 38th meeting of the* *conference of the International Group for the Psychology of Mathematics Education (PME) and the 36th North American chapter (PME-NA)* (Vol. 1, pp. 1–20). Vancouver: PME.

Radford, L., & Roth, W.-M. (2011). Intercorporeality and ethical commitment: An activity perspective: An activity perspective on classroom interaction. *Educational Studies in Mathematics, 77*, 227–245.

Radford, L., Miranda, I., & Guzmán, J. (2008). Relative motion, graphs and the heteroglossic transformation of meanings: A semiotic analysis. In O. Figueras, J. L. Cortina, S. Alatorre, T. Rojano, & A. Sepúlveda (Eds.), *Proceedings of the joint 32nd conference of the International Group for the Psychology of Mathematics Education (PME) and the 30th North American chapter (PME-NA)* (Vol. 4, pp. 161–168). Morelia: Cinvestav-UMSNH.

Raphael, L. M., Pressley, M., & Mohan, L. (2008). Engaging instruction in middle school classrooms: An observational study of nine teachers. *The Elementary School Journal, 109*(1), 61–81.

Rapke, T. (2016). A process of students and their instructor developing a final closed-book mathematics exam. *Research in Mathematics Education, 18*(1), 27–42. https://doi.org/10.1080/14794802.2015.1134342.

Rapke, T. (2017). Involving students in developing math tests. *Mathematics Teacher, 110*(8), 612–616. Retrieved from http://www.nctm.org/publications/mathematics-teacher/

Rapke, T., & Hall, J. (2016). Summative ~ formative: Exploring novel uses of written exams in an undergraduate mathematics course. In C. A. Shoniregun & G. A. Akmayeva (Eds.), *Canada International Conference on Education (CICE-2016) proceedings* (pp. 48–51). Mississauga: Infonomics Society.

Reddy, V., Visser, M., Winnaar, L., Arends, F., Juan, A., Prinsloo, C., & Isdale, K. (2016). *Highlights of mathematics and science achievement of grade 9 South African learners*. Pretoria: Human Sciences Research Council.

Rees, P., & Williams, C. (Writers). (2011, November 23). Wheel of mythfortune [Television series episode]. In A. Dallow (Producer). *MythBusters*. Silverspring: Discovery Communications.

Reid, D., & Zack, V. (2010). Observing the process of mathematics teacher change – Part 1. *Journal of Mathematics Teacher Education, 13*, 371–374.

Renkl, A. (2002). Worked-out examples: Instructional explanations support learning by selfexplanations. *Learning and Instruction, 12*, 529–556.

Richards, J., Hove, J., & Afolabi, K. (2008). *Understanding the Aboriginal/non-Aboriginal gap in student performance: Lessons from British Columbia* (Commentary No. 276). Toronto: C.D. Howe Institute.

Rickard, A. (2005). Constant perimeter, varying area: A case study of teaching and learning mathematics to design a fish rack. *Journal of American Indian Education, 44*(3), 80–100.

Rioux, M. (2012). *Évolution des projets de formation de futurs enseignants au primaire au contact de situations probabilistes*. Doctoral thesis. Université de Montréal: Montréal. Retrieved from https://papyrus.bib.umontreal.ca/xmlui/bitstream/handle/1866/9000/Rioux_Miranda_2012_these.pdf?sequence=4&isAllowed=y

Robson, E. (2000). Mesopotamian Mathematics: some historical background. In V. Katz (Ed.), *Using history to teach mathematics: An international perspective* (pp. 149–158). Washington, DC: Mathematical Association of America.

Rodney, S., Rouleau, A., & Sinclair, N. (2016). A tale of two metaphors: Storylines about mathematics education in Canadian national media. *Canadian Journal of Science, Mathematics and Technology Education, 16*(4), 389–401.

Rosenfeld, L. B., Richman, J. M., & Bowen, G. L. (2000). Social support networks and school outcomes: The centrality of the teacher. *Child and Adolescent Social Work Journal, 17*(3), 205–226.

Rosling, T. (2006). *The best stats you've ever seen* [video]. Retrieved from http://www.ted.com/talks/hans_rosling_shows_the_best_stats_you_ve_ever_seen

Ross, J. A., & Gray, P. (2005). School leadership and student achievement: The mediating effects of teacher beliefs. *Canadian Journal of Education, 29*(3), 798–822.

Roth, W.-M. (2005). *Auto/biography and auto/ethnography: Praxis of research method*. Rotterdam: Sense Publishers.

Roth, W.-M. (2011). *Geometry as objective science in elementary classrooms: Mathematics in the flesh*. New York: Routledge.

Roth, W.-M. (2017). *The mathematics of mathematics: Thinking with the late, Spinozist Vygotsky*. Rotterdam: Sense Publishers.

Rotman, B. (2008). *Becoming beside ourselves: The alphabet, ghosts, and distributed human beings*. Durham: Duke University Press.

Rubel, L. H. (2002). *Probabilistic misconceptions: Middle and high school students' mechanisms for judgments under uncertainty*. Unpublished doctoral dissertation, Teachers College, Columbia University, New York, NY.

Rubel, L. H. (2006). Good things always come in threes: Three cards, three prisoners, and three doors. *Mathematics Teacher, 99*(6), 401–405.

Rubie-Davies, C. M. (2015). *Becoming a high expectation teacher: Raising the bar*. London, UK: Routledge.

Rubin, B. (2008). Detracking in context: How local constructions of ability complicate equity-geared reform. *Teachers College Record, 110*(3), 646–699.

Rubin, A., Bruce, B., & Tenney, Y. (1990). Learning about sampling: Trouble at the core of statistics. In D. Vere-Jones (Ed.), *Proceedings of the third International Conference on Teaching Statistics* (Vol. 1, pp. 314–319). Dunedin/Voorburg: International Statistical Institute Retrieved from http://iase-web.org/documents/papers/icots3/BOOK1/A9-4.pdf.

Russell, G. (2016). *Valued kinds of knowledge and ways of knowing in mathematics and the teaching and learning of mathematics: A worldview analysis.* Unpublished doctoral dissertation. University of Saskatchewan, Saskatoon, SK.

Russell, G. (2017). *Transreform radical humanism: A mathematics and teaching philosophy*. Rotterdam: Sense Publishers.

Russell, G. L., & Chernoff, E. J. (2013). The marginalisation of Indigenous students within school mathematics and the math wars: Seeking resolutions within ethical spaces. *Mathematics Education Research Journal, 25*, 109–127.

Rutter, M. (1979). Protective factors in children's responses to stress and disadvantage. In M. W. Kent & J. E. Rolf (Eds.), *Primary prevention of psychopathology: Social competence in children* (Vol. 3, pp. 49–74). Hanover: University Press of New England.

Saul, J. R. (2008). *A fair country: Telling truths about Canada*. Toronto: Viking Canada.

Savard, A. (2008). *Le développement d'une pensée critique envers les jeux de hasard et d'argent par l'enseignement des probabilités à l'école primaire: Vers une prise de décision*. Doctoral thesis. Université Laval: Laval, Québec. Retrieved from http://www.theses.ulaval.ca/2008/25308/25308.pdf

Savard, A. (2014). Developing probabilistic thinking: What about people's conceptions? In E. J. Chernoff & B. Sriraman (Eds.), *Probabilistic thinking: Presenting plural perspectives* (pp. 283–297). Dordrecht: Springer Science+Business.

Savard, A. (2015). Making decisions about gambling: The influence of risk on children's arguments. *The Mathematics Enthusiast, 12*(1,2 & 3), 226–245.

Savard, A., & DeBlois, L. (2005). Un cadre théorique pour éclairer l'apprentissage des probabilités à l'école primaire: Vers une prise de décision à l'égard des jeux de hasard et d'argent. In *Proceedings of the « Colloque annuel du Groupe de didactique des mathématiques du Québec»*, Montréal, Québec.

Savard, A., & Manuel, D. (2016). Teaching statistics in middle school mathematics classrooms: Creating an intersection for intra and interdisciplinarity. *Statistics Education Research Journal, 15*(2), 239–256.

Savard, A., & Polotskaia, E. (2017). Who's wrong? Tasks fostering understanding of mathematical relationships in word problems in elementary students. *ZDM Mathematics Education*. Retrieved from https://doi.org/10.1007/s11858-017-0865-5.

Scheaffer, R. L. (2006). Statistics and mathematics: On making a happy marriage. In G. F. Burrill & P. C. Elliott (Eds.), *Thinking and reasoning with data and chance* (pp. 309–321). Reston: National Council of Teachers of Mathematics.

Schleppegrell, M. J. (2007). The linguistic challenges of mathematics teaching and learning: A research review. *Reading & Writing Quarterly, 23*(2), 139–159.

Schmidt, W. H., Burroughs, N. A., Zoido, P., & Houang, R. T. (2015). The role of schooling in perpetuating educational equality: An international perspective. *Educational Researcher, 44*(7), 371–386.

Schoenfeld, A. (1985a). *Mathematical problem solving*. New York: Academic Press.

Schoenfeld, A. (1985b). *Mathematical problem solving*. Orlando: Academic Press.

Schoenfeld, A. H. (2014). What makes for powerful classrooms, and how can we support teachers in creating them? A story of research and practice, productively intertwined. *Educational Researcher, 43*(8), 404–412. https://doi.org/10.3102/0013189X14554450.

Scholastic. (2017). *The magic school bus: FAQs*. Retrieved from http://www.scholastic.com/magicschoolbus/parentteacher/faq.htm

Schön, D. (1983). *The reflective practitioner: How professionals think in action*. London: Temple Smith.

Schütz, A. (1932). *Der sinnhafte Aufbau der sozialen Welt* [The phenomenology of the social world]. Vienna: Julius Springer.

Seeley Brown, J., Collins, A., & Duguid, P. (1989). Situated cognition and the culture of learning. *Educational Researcher, 18*(1), 32–42.

Sellner, S. (2016, March). *85 break-ins today in ABQ*. Retrieved from http://www.securitysales.com/news/albuquerque_alarm_company_faces_heat_for_misrepresenting_burglary_data_on_b

Sensoy, O., & Diangelo, R. (2009). Developing social justice literacy: An open letter to our faculty colleagues. *The Phi Delta Kappan, 90*(5), 345–352.

Setati, M. (2005). Teaching mathematics in a primary multilingual classroom. *Journal for Research in Mathematics Education, 36*(5), 447–466.

Setati, M., & Adler, J. (2001). Between languages and discourses: Code switching practices in primary classrooms in South Africa. *Educational Studies in Mathematics, 43*, 243–269.

Sfard, A. (2003). Balancing the unbalanceable: The NCTM standards in light of theories of learning mathematics. In J. Kilpatrick, W. G. Martin, & D. Schifter (Eds.), *A research companion to principles and standards for school mathematics* (pp. 353–392). Reston: National Council of Teachers of Mathematics.

Sfard, A., Nesher, P., Streefland, L., Cobb, P., & Mason, J. (1998). Learning mathematics through conversation: Is it as good as they say? *For the Learning of Mathematics, 18*(1), 41–51.

Shahbari, J. A., & Peled, I. (2015). Resolving cognitive conflict in a realistic situation with modeling characteristics: Coping with a changing reference in fraction. *International Journal of Science and Mathematics Education, 13*(4), 891–907.

Shaughnessy, J. M. (1992). Research in probability and statistics: Reflections and direction. In D. A. Grouws (Ed.), *Handbook of research on mathematics teaching and learning* (pp. 465–494). New York: National Council of Teachers of Mathematics.

Shem-Tov, N. (2011). Improvisational teaching as mode of knowing. *The Journal of Aesthetic Education, 45*(3), 103–113.

Shepard, L. A. (2001). The role of classroom assessment in teaching and learning. In V. Richardson (Ed.), *Handbook of research on teaching* (4th ed., pp. 1066–1101). Washington, DC: American Educational Research Association.

Sherin, M. G., Jacobs, V. R., & Philipp, R. A. (Eds.). (2011). *Mathematics teacher noticing: Seeing through teachers' eyes*. New York: Routledge.

Shulman, L. S. (1986). Those who understand: Knowledge growth in teaching. *Educational Researcher, 15*(2), 4–14.

Shulman, L. S., & Shulman, J. H. (2004). How and what teachers learn: A shifting perspective. *Journal of Curriculum Studies, 36*(2), 257–271.

Siegel, C. (2005). Implementing a research-based model of cooperative learning. *Journal of Educational Research, 98*(6), 339–349.
Silver, E. A. (1985). *Teaching and learning mathematical problem solving: Multiple research perspectives*. Mahwah: Erlbaum.
Silver, E. A., & Mesa, V. (2011). Coordinating characterizations of high quality mathematics teaching: Probing the intersection. In Y. Li & G. Kaiser (Eds.), *Expertise in mathematics instruction: An international perspective* (pp. 63–84). New York: Springer.
Silver, E. A., & Zawodjewsky, J. S. (1997). *Benchmarks of students' understanding (BOSUN) project* [technical guide]. Pittsburgh: LRDC.
Simon, M. (2006). Key developmental understandings in mathematics: A direction for investigating and establishing learning goals. *Mathematical Thinking and Learning, 8*(4), 359–371.
Sinclair, N. (2006). *Mathematics and beauty*. New York: Teacher's College Press.
Sinclair, N., & Watson, A. (2001). Wonder, the rainbow and the aesthetics of rare experiences. *For the Learning of Mathematics, 21*, 39–42.
Sinclair, N., Bartolini Bussi, M., de Villiers, M., Jones, K., Kortenkamp, U., Leung, A., & Owens, K. (2016). Recent research on geometry education: An ICME13 survey team report. *ZDM – The International Journal on Mathematics Education, 48*(5), 691–719.
Singapore Ministry of Education. (2012). *Mathematics syllabus: Primary one to five*. Retrieved from https://www.moe.gov.sg/docs/default-source/document/education/syllabuses/sciences/files/primary_mathematics_syllabus_pri1_to_pri5.pdf
Skemp, R. R. (1971). *The psychology of learning mathematics*. Bungay Suffolk: The Chaucer Press.
Skemp, R. (1976a). Relational understanding and instrumental understanding. *Mathematics Teaching, 3*, 20–26.
Skemp, R. R. (1976b). Relational understanding and instrumental understanding. *Mathematics Teaching in the Middle School, 12*(2), 88–95.
Skilling, K. (2014). Teacher practices: How they promote or hinder student engagement. In J. Anderson, M. Cavanagh, & A. Prescott (Eds.), *Curriculum in focus: Research guided practice*. Proceedings of the 37th annual conference of the Mathematics Education Research group of Australasia (pp. 589–596), MERGA.
Skovsmose, O. (1994). *Towards a philosophy of critical mathematics education*. Dordrecht: Kluwer.
Skovsmose, O. (2016). Mathematics: A critical rationality? In P. Ernest, B. Sriraman, & N. Ernest (Eds.), *Critical mathematics education: Theory, praxis, and reality* (pp. 1–22). Charlotte: Information Age Publishing Inc.
Slavin, R. E. (1990). Achievement effects of ability grouping in secondary schools: A best evidence synthesis. *Review of Educational Research, 60*(3), 471–499.
Slavin, R. E., & Lake, C. (2008). Effective programs in elementary mathematics: A best-evidence synthesis. *Review of Educational Research, 78*(3), 427–515.
Slavit, D., & Nelson, T. H. (2010). Collaborative teacher inquiry as a tool for building theory on the development and use of rich mathematical tasks. *Journal of Mathematics Teacher Education, 13*, 201–221.
Small, M. (2013). *Powerpoint presentation*. Toronto, ON.
Smith, M. S., Hughes, E. K., Engle, R. A., & Stein, M. K. (2009). Orchestrating discussions. *Mathematics Teaching in the Middle School, 14*(9), 548–556.
Soine, K. M., & Lumpe, A. (2014). Measuring characteristics of teacher professional development. *Teacher Development, 18*(3), 303–333.
Solomon, D., Battistich, V., Watson, M., Schaps, E., & Lewis, C. (2000). A six-district study of educational change: Direct and mediated effects of the Child Development Project. *Social Psychology of Education, 41*, 3–51.
Son, J.-W., & Sinclair, N. (2010). How preservice teachers interpret and respond to student geometric errors. *School Science and Mathematics, 110*(1), 31–46.
Sousa, D. A. (2006). *How the brain learns mathematics*. Thousand Oaks: Corwin Press.

Spillane, J. P. (2000). Cognition and policy implementation: District policymakers and the reform of mathematics education. *Cognition and Instruction, 18*(2), 141–179.
Sriraman, B., & Dickman, B. (2017). Mathematical pathologies as pathways into creativity. *ZDM Mathematics Education, 49*, 137–145.
Statistics Canada. (2011). *National household survey profile, Canada, 2011*. Retrieved from http://www12.statcan.gc.ca/nhs-enm/2011/dp-pd/prof/details/page.cfm?Lang=E&Geo1=PR&Code1=01&Data=Count&SearchText=canada&SearchType=Begins&SearchPR=01&A1=Education&B1=All&Custom=&TABID=1
Statistics Canada. (2013). *Immigration and ethnocultural diversity in Canada, DHHS Catalogue no. 99–010-X2011001*. Ottawa: Author.
Statistics South Africa. (2016). *Community survey 2016: Statistical release*. Retrieved from http://cs2016.statssa.gov.za/wp-content/uploads/2016/07/NT-30-06-2016-RELEASE-for-CS-2016-_Statistical-releas_1-July-2016.pdf
Stein, M. K., Smith, M. S., & Silver, E. (1999). The development of professional developers: Learning to assist teachers in new settings in new ways. *Harvard Educational Review, 69*(3), 237–270.
Stein, M. K., Smith, M. S., Henningsen, M. A., & Silver, E. A. (2000). *Implementing standards-based mathematics instruction: A casebook for professional development*. New York: Teachers College Press.
Steinbring, H. (1991). The theoretical nature of probability in the classroom. In R. Kapadia & M. Borovcnik (Eds.), *Chance encounters: Probability in education* (pp. 135–168). Amsterdam: Kluwer.
Steinbring, H. (1998). Elements of epistemological knowledge for mathematics teachers. *Journal of Mathematics Teacher Education, 1*(2), 157–189.
Sterenberg, G. (2013). Learning Indigenous and Western mathematics from place. *Mathematics Education Research Journal, 25*(1), 91–108.
Sterenberg, G., & Hogue, M. (2011). Reconsidering approaches to Aboriginal science and mathematics education. *Alberta Journal of Educational Research, 57*(1), 1–15.
Sternberg, R., & Zhang, L. F. (Eds.). (2001). *Perspectives on thinking, learning, and cognitive styles*. London: Lawrence Erlbaum.
Stiggins, R. J. (2007). Conquering the formative assessment frontier. In J. A. McMillan (Ed.), *Formative classroom assessment: Theory into practice* (pp. 8–28). New York: Teachers College Press.
Stocker, D. (2006). *Maththatmatters: A teacher resource linking math and social justice*. Ottawa: CCPA Education Project.
Stohl, H. (2005). Probability in teacher education and development. In G. A. Jones (Ed.), *Exploring probability in school: Challenges for teaching and learning* (pp. 345–366). New York: Springer.
Stoilescu, D., McDougall, D., & Egadawatte, G. (2015). Teachers' views of the challenges of teaching Grade 9 Applied mathematics in Toronto schools. *Educational Research for Policy and Practice, 15*(2), 83–97.
Struyven, K., Dochy, F., & Janssens, S. (2005). Students' perceptions about evaluation and assessment in higher education: A review. *Assessment & Evaluation in Higher Education, 30*(4), 325–341. https://doi.org/10.1080/02602930500099102.
Sullivan, P. (2011). *Australian education review teaching mathematics: Using research-informed strategies*. Camberwell: Australian Council for Educational Research.
Sullivan, P., & Lilburn, P. (2002). *Good questions for math teaching: Why ask them and what to ask? (Grades K-6)*. Sausalito: Math Solutions.
Sullivan, P., Tobias, S., & McDonough, A. (2006). Perhaps the decision of some students not to engage in learning mathematics in school is deliberate. *Educational Studies in Mathematics, 62*(1), 81–99.
Sullivan, P., Mornane, A., Prain, V., Campbell, C., Deed, C., Drane, S., Faulkner, M., McDonough, A., & Smith, C. (2009). Junior secondary students' perceptions of influences on their engagement with schools. *Australian Journal of Education, 53*(2), 176–191.

Suurtaam, C., & Vezina, N. (2010). Transforming pedagogical practice in mathematics: Moving from telling to listening. *International Journal for Mathematics Teaching and Learning*.
Suurtamm, C. (2012). Assessment can support reasoning and sense making. *Mathematics Teacher, 106*(1), 29–33.
Suurtamm, C., & Arden, A. (2017). Using assessment to enhance mathematics mathematics teaching and learning. In D. Spangler & J. Wanko (Eds.), *Enhancing classroom practice with research behind principles to actions* (pp. 141–152). Reston: NCTM.
Suurtamm, C., & Graves, B. (2007). *Curriculum implementation intermediate math: Research report executive summary*. Ottawa: University of Ottawa, Faculty of Education.
Suurtamm, C., Koch, M. J., & Arden, A. (2010). Teachers' emerging assessment practices in mathematics: Classrooms in the context of reform. *Assessment in Education: Principles, Policy, and Practice, 17*(4), 399–417.
Suurtamm, C., Thompson, D. R., Kim, R. Y., Moreno, L. D., Sayac, N., Schukajlow, S., Silver, E., Ufer, S., & Vos, P. (2016). Assessment in mathematics education: Large-scale assessment and classroom assessment. In G. Kaiser (Ed.), *ICME-13 topical surveys* (pp. 1–30). Hamburg: University of Hamburg.
Swain, M. (2000). The output hypothesis and beyond: Mediating acquisition through collaborative dialogue. In J. Lantolf (Ed.), *Sociocultural theory and second language learning* (pp. 97–114). Oxford: Oxford University Press.
Swain, M., & Lapkin, S. (2005). The evolving socio-political context of immersion education in Canada: Some implications for program development. *International Journal of Applied Linguistics, 15*(2), 169–186.
Swan, M., & Burkhardt, H. (2012). A designer speaks: Designing assessment of performance in mathematics. *Educational Designer – Journal of the International Society for Design and Development in Education*, *2*(5). Retrieved from http://www.educationaldesigner.org/ed/volume2/issue5/article19/
Sztajn, P., Confrey, J., Wilson, P. H., & Edgington, C. (2012). Learning trajectory based instruction: Toward a theory of teaching. *Educational Researcher, 41*(5), 147–156.
Talbert, J. E. (1995). Boundaries of teachers' professional communities in U. S. High Schools: Power and precariousness of the Subject Department. In L. Siskin & J. W. Little (Eds.), *The subjects in question: Departmental organization and the high school* (pp. 68–94). New York: Teachers College Press.
Taleb, N. N. (2010). *The black swan: The impact of the highly improbable* (2nd ed.). New York: Random House.
Tall, D. (2013). *How humans learn to think mathematically: Exploring the three worlds of mathematics*. New York: Cambridge University Press.
Tall, D., Yevdokimov, O., Koichu, B., Whiteley, W., Kondratieva, M., & Cheng, Y. H. (2012). Cognitive development of proof. In M. de Villiers & G. Hanna (Eds.), *ICMI Study 19, Proof and Proving in Mathematics Education*.
Taylor, P. (2017, October). The aesthetics of mathematical modelling for the classroom. Presentation at MathEd Forum, The Fields Institute for Research in Mathematical Sciences, Toronto.
Thames, M. H., & Ball, D. L. (2010). What math knowledge does teaching require? *Teaching Children Mathematics, 17*(4), 22–229.
The Western & Northern Canadian Protocol. (2006). *The common curriculum framework for K-9 mathematics*. Retrieved from https://www.wncp.ca/media/38765/ccfkto9.pdf
Theis, L., & Gagnon, N. (2013). *L'apprentissage à travers des situations-problèmes mathématiques : Bases théoriques et réalisation pratique*. Québec City: Presses de l'Université du Québec.
Thom, J. S., & Pirie, S. E. (2006). Looking at the complexity of two young children's understanding of number. *The Journal of Mathematical Behavior, 25*(3), 185–195.
Thomsen, K. (2002). *Building resilient students* (1st ed.). Thousand Oaks: Corwin Press.
Tirosh, D., & Graeber, A. O. (1990). Evoking cognitive conflict to explore preservice teachers' thinking about division. *Journal for Research in Mathematics Education, 21*(2), 98–108.

Törner, G., Schoenfeld, A. H., & Reiss, K. M. (2007). Problem solving around the world: Summing up the state of the art. *ZDM, 39*(5), 353–353.

Tranter, D., Carson, L., & Boland, T. (2018). *The third path: A relationship-based approach to student well-being and achievement*. Toronto: Nelson.

Treffers, A. (1987). *Three dimensions: A model of goal and theory description in mathematics education*. Dordrecht: Reidel.

Treffers, A., & Goffree, F. (1985). Rational analysis of realistic mathematics education – The Wiskobas program. In L. Streefland (Ed.), *Proceedings of PME 9* (Vol. 2, pp. 97–121). Utrecht: PME.

Truth and Reconciliation Commission (TRC). (2015). *Truth and Reconciliation Commission of Canada: Calls to action*. Winnipeg, MB: Author. Retrieved from http://www.trc.ca/websites/trcinstitution/File/2015/Findings/Calls_to_Action_English2.pdf

Truth and Reconciliation Commission (TRC). (2016). Home page. Retrieved from http://www.trc.ca/websites/trcinstitution/index.php?p=39

Truth and Reconciliation Commission of Canada. (2012). *Truth and Reconciliation Commission: Calls to action*. Retrieved from http://www.trc.ca/websites/trcinstitution/File/2015/Findings/Calls_to_Action_English2.pdf

Turnbull, M., Lapkin, S., & Hart, D. (2001). Grade 3 immersion students' performance in literacy and mathematics: Province-wide results from Ontario (1998–99). *The Canadian Modern Language Review/La revue canadienne des langues vivantes, 58*(1), 9–26.

Turner-Bisset, R. (2001). *Expert teaching: Knowledge and pedagogy to lead the profession*. London: David Fulton Publishers.

U.S. Congress House of Representatives Subcommittee on Early Childhood, Elementary and Secondary Education (CHRSECESE). (2008). *Challenges facing bureau of Indian education schools in improving student achievement*. Washington, DC: U.S. Government Printing Office.

Ungar, M. (2004). A constructionist discourse on resilience: Multiple contexts, multiple realities among at-risk children and youth. *Youth and Society, 35*(3), 341–365.

Ungar, M. (2015). *What is resilience?* Retrieved from http://resilienceresearch.org/

United Nations. (2008). *United Nations declaration on the rights of Indigenous peoples*. Retrieved from http://www.un.org/esa/socdev/unpfii/documents/DRIPS_en.pdf

University of Alaska Fairbanks. (n.d.). *Math in a cultural context: Lessons learned from Yup'ik Eskimo Elders project*. Fairbanks, AK: University of Alaska Geography Program, School of Natural Resources and Natural Sciences. Retrieved from http://www.uaf.edu/mcc/

Van de Walle, J. A., Folk, S., Karp, K. S., & Bay-Williams, J. M. (2011). *Elementary and middle school mathematics: Teaching developmentally* (3rd Canadian ed.). Toronto: Pearson Canada.

Van de Walle, J. A., Karp, K. S., Bay-Williams, J. M., McGarvey, L. M., & Folk, S. (2015). *Elementary and middle school mathematics: Teaching developmentally* (4th Canadian ed.). Toronto: Pearson Allyn & Bacon.

Van Houtte, M. (2004). Tracking effects on school achievement: A quantitative explanation in terms of the academic culture of school staff. *American Journal of Education, 110*(4), 354–388.

Van Maanen, J. (1991). L'Hôpital's weight problem. *For the Learning of Mathematics, 11*(2), 44–47.

Van Manen, M. (1986). *The tone of teaching*. Richmond Hill: Scholastic.

van Merriënboer, J. J., & Sweller, J. (2005). Cognitive load theory and complex learning: Recent developments and future directions. *Educational Psychology Review, 17*(2), 147.

Vangrieken, K., Meredith, C., Packer, T., & Kyndt, E. (2017). Teacher communities as a context for professional development: A systematic review. *Teaching and Teacher Education, 61*, 47–59.

Vescio, V., Ross, D., & Adams, A. (2008). A review of research on the impact on professional learning communities on teaching practice and student learning. *Teaching and Teacher Education: An International Journal of Research and Studies, 24*(1), 80–91.

Vézina, H., Jomphe, M., Lavoie, E., Moreau, C., & Labuda, D. (2012). L'apport des données génétiques à la mesure généalogique des origines amérindiennes des Canadiens français. *Cahiers Québécois de Démographie, 41*, 87–105.

Vinner, S. (1991). The role of definitions in the teaching and learning of mathematics. In D. Tall (Ed.), *Advanced mathematical thinking* (pp. 65–81). Dordrecht: Kluwer.

Von Glasersfeld, E. (1988). The reluctance to change a way of thinking. *Irish Journal of Psychology, 9*(1), 83–90.

Vygotsky, L. S. (1978). In M. Cole, V. John-Steiner, S. Scribner, & E. Souberman (Eds.), *Mind in society: The development of higher psychological processes*. Cambridge, MA: Harvard University Press.

Vygotsky, L. S. (1987). *The collected works of L. S. Vygotsky, Vol. 1: Problems of general psychology*. New York: Springer.

Vygotsky, L. S. (1989). Concrete human psychology. *Soviet Psychology, 27*(2), 53–77.

Vygotsky, L. S. (2010). Two fragments of personal notes by L. S. Vygotsky from the Vygotsky family archive. *Journal of Russian and East European Psychology, 48*(1), 91–96.

Wagner, D., & Lunney Borden, L. (2012). Aiming for equity in mathematics education research in a marginalized community: A perspective from Canada. In J. Choppin, B. Herbel-Eisenmann, D. Pimm, & D. Wagner (Eds.), *Equity in discourse for mathematics education: Theories, practices and policies* (pp. 69–87). New York: Springer.

Wakeling, E. (1995). *Rediscovered Lewis Carroll puzzles*. New York: Dover Publications, Inc.

Walsh, F. (2002). A family resilience framework: Innovative practice applications. *Family Relations, 51*(2), 130–137. https://doi.org/10.1111/j.1741-3729.2002.00130.x.

Walshaw, M. (1999). *Paradox, partiality and promise: A politics for girls in school mathematics*. Unpublished doctoral dissertation. Massey University, Palmerston North, New Zealand.

Walter, E. V. (1988). *Placeways: A theory of the human environment*. Chapel Hill: University of North Carolina Press.

Warren-Little, J. (1999). Organizing schools for teacher learning. In L. Darling-Hammond & G. Sykes (Eds.), *Teaching as the learning profession: Handbook of policy and practice* (pp. 51–60). San Francisco: Jossey-Bass.

Wasserman, N. (2016). Nonlocal mathematical knowledge for teaching. In C. Csíkos, A. Rausch, & J. Szitányi (Eds.), *Proceedings of the 40th conference of the international group for the psychology of mathematics education* (Vol. 4, pp. 379–386). Szeged: PME.

Watson, A. (2006a). *Raising achievement in secondary mathematics*. UK: McGraw-Hill Education.

Watson, A. (2006b). Some difficulties in informal assessment in mathematics. *Assessment in Education, 13*(3), 289–303.

Watson, J. (2007). The role of cognitive conflict in developing students' understanding of average. *Educational Studies in Mathematics, 65*(1), 21–47.

Watson, A. (2008). School mathematics as a special kind of mathematics. *For the Learning of Mathematics, 28*(3), 3–7.

Watson, J. (2010). BODMAS, BOMDAS and DAMNUS. *Cross Section, 20*(4), 6–9.

Watson, H., & Chambers, D. W. (1989). *Singing the land, signing the land*. Geelong: Deakin University Press.

Watson, A., & Winbourne, P. (2008). *New directions for situated cognition in mathematics education*. New York: Springer.

Waxter, M., & Morton, J. B. (2012). Cognitive conflict and learning. In N. M. Seel (Ed.), *Encyclopedia of the sciences of learning* (pp. 585–587). Boston: Springer.

Webb, R., Vulliamy, G., Sarja, A., Hämäläinen, S., & Poikonen, P. (2009). Professional learning communities and teacher well-being? A comparative analysis of primary schools in England and Finland. *Oxford Review of Education, 35*(3), 405–422.

Wenger, E. (1998). *Communities of practice: Learning, meaning and identity*. New York: Cambridge University Press.

Werner, E. E., & Smith, R. S. (1982). *Vulnerable but invincible: A longitudinal study of resilient children and youth*. New York: McGraw-Hill.

Western and Northern Canadian Protocol. (2006). *The common curriculum framework for K-9 mathematics*. Edmonton: Alberta Education Retrieved from http://www.wncp.ca/media/38765/ccfkto9.pdf.

Western and Northern Canadian Protocol (WNCP). (2008a). *The common curriculum framework for grades 10–12 mathematics*. Edmonton: Alberta Education. Retrieved from https://www.bced.gov.bc.ca/irp/pdfs/mathematics/WNCPmath1012/2008math1012wncp_ccf.pdf

Western and Northern Canadian Protocol. (2008b). *The common curriculum framework for grades 10–12 mathematics*. Edmonton: Alberta Education. Retrieved from http://www.wncp.ca/media/38771/math10to12.pdf

What is a health promoting school?. (n.d.). *World Health Organization*. Retrieved from http://www.who.int/school_youth_health/gshi/hps/en/

Wheels. (2008, July 8). *Solo to James Bay and back again*. Retrieved from https://www.wheels.ca/news/solo-to-james-bay-and-back-again/

Whitehead, A. N. (1929). *Aims of education*. New York: The Free Press.

Whiteley, W., & Davis, B. (2003a). A mathematics curriculum manifesto. In E. Simmt and B. Davis, (Eds.), *Proceedings of the 2003 annual meeting of the Canadian Mathematics Education Study Group/Groupe Canadien d'Etude en Didactique des Mathématiques* (pp. 79–83). Edmonton: CMESG/GCEDM. Retrieved from http://www.cmesg.org/wp-content/uploads/2015/01/CMESG2003.pdf

Whiteley, W., & Davis, B. (2003b). A mathematics curriculum manifesto. In E. Simmt & B. Davis (Eds.), *Proceedings of the 2003 annual meeting of the Canadian Mathematics Education Study Group* (pp. 79–83). Edmonton, AB: CMESG/GCEDM. Retrieved from http://www.cmesg.org/wp-content/uploads/2015/01/CMESG2003.pdf

Whiteley, W., & Davis, B. (2016). A mathematics curriculum manifesto. In P. Liljedahl, D. Allan, O. Chapman, F. Gourdeau, C. Lajoie, S. Oesterle, E. Simmt, & P. Taylor (Eds.), *Canadian Mathematics Education Study Group: 40 years of CMESG (Les 40 ans du GCEDM)* (pp. 193– 197). Burnaby, BC: CMESG/GCEDM. (Reprinted from *Proceedings of the 2003 annual meeting of the Canadian Mathematics Education Study Group/Groupe Canadien d'Etude en Didactique des Mathématiques*, pp. 79–83, by E. Simmt & B. Davis, Eds., 2003, Edmonton, AB: CMESG/GCEDM)

Wiggins, G. P. (1993). *Assessing student performance: Exploring the purpose and limits of testing*. San Francisco: Jossey-Bass.

Wild, C. J., & Pfannkuch, M. (1999). Statistical thinking in empirical inquiry. *International Statistical Review, 67*, 223–265.

Wilder, R. L. (1981). *Mathematics as a cultural system*. Oxford: Pergamon Press.

Wiliam, D. (2007). Keeping learning on track: Classroom assessment and the regulation of learning. In F. K. Lester Jr. (Ed.), *Second handbook of research on mathematics teaching and learning* (pp. 1053–1098). Charlotte: Information Age Publishing.

Wiliam, D. (2015). Assessment: A powerful focus of the improvement of mathematics education. In C. Suurtamm & A. Roth McDuffie (Eds.), *Assessment to enhance teaching and learning* (pp. 247–254). Reston: NCTM.

Wiliam, D., & Leahy, S. (2007). A theoretical foundation for formative assessment. In J. A. McMillan (Ed.), *Formative classroom assessment: Theory into practice* (pp. 29–42). New York: Teachers College Press.

William, D. (2003). Constructing difference: Assessment in mathematics education. In L. Burton (Ed.), *Which way social justice in mathematics education?* (pp. 189–207). Westport: Praeger.

Williams, H. (1989). *Tuning-in to young children: An exploration of contexts for learning mathematics*. Unpublished M. Phil. thesis, Open University, Milton Keynes, England.

Willingham, D. T. (2002, Summer). Allocating student study time: "Massed" versus "distributed" practice. *American Educator, 26*, 37–39.

Willoughby, S. S. (1967). *Contemporary teaching of secondary school mathematics*. New York: Wiley.

Wilson, S. M., Floden, R. E., & Ferrini-Mundy, J. (2001). Teacher preparation research: Current knowledge, gaps, and recommendations. A research report prepared for the U.S. Department of Education. Center for the Study of Teaching and Policy, Seattle, WA.

Wiseman, D., Glanfield, F., & Lunney Borden, L. (2017). How we are coming to know: Ways in which Indigenous and non-indigenous ways of knowing, being, and doing might circulate together in mathematics and science teaching and learning. Report submitted to the Social Sciences and Humanities Research Council of Canada. Retrieved from http://showmeyour-math.ca/comingtoknow/report/

Witmer, T. (Trans.). (1968). *Ars magna or the rules of algebra, Girolamo Cardano*. New York: Dove.

Woolford, A., Benvenuto, J., & Hinton, A. L. (Eds.). (2014). *Colonial genocide in Indigenous North America*. London: Duke University Press.

World Bank. (n.d.). *GINI index (World Bank estimate)*. Retrieved from https://data.worldbank.org/indicator/SI.POV.GINI

Yackel, E., & Cobb, P. (1996). Sociomathematical norms, argumentation, and autonomy in mathematics. *Journal for Research in Mathematics Education, 27*(2), 458–477. https://doi.org/10.2307/749877.

Yackel, E., & Rasmussen, C. (2002). Beliefs and norms in the mathematics classroom. In G. Leder, E. Pehkonen, & G. Törner (Eds.), *Beliefs: A hidden variable in mathematics education?* (pp. 313–330). London: Kluwer Academic Publishing.

Yorke, M. (2011). Summative assessment: Dealing with the 'measurement fallacy. *Studies in Higher Education, 36*(3), 251–273.

Youcubed. (2016). *Painting youcubed*. Retrieved from https://www.youcubed.org/task/painting-youcubed/

Yusof, Y., & Tall, D. (1999). Changing attitudes to university mathematics through problem- solving. *Educational Studies in Mathematics, 37*, 67–82.

Zaslavsky, O., & Peled, I. (1996). Inhibiting factors in generating examples by mathematics teachers and student-teachers: The case of binary operation. *Journal for Research in Mathematics Education, 27*(1), 67–78.

Zazkis, R. (2011). *Relearning mathematics: A challenge for prospective elementary school teachers*. Charlotte: Information Age Publishing.

Zazkis, R., & Chernoff, E. (2008). What makes a counterexample exemplary. *Educational Studies in Mathematics, 68*(3), 195–208.

Zazkis, R., & Kontorovich, I. (2016). A curious case of superscript (−1): Prospective secondary mathematics teachers explain. *Journal of Mathematical Behavior, 43*, 98–110.

Zazkis, R., & Mamolo, A. (2011). Reconceptualising knowledge at the mathematical horizon. *For the Learning of Mathematics, 31*(2), 8–13.

Zazkis, R., & Rouleau, A. (2018). Order of operations: On convention and met-before acronyms. *Educational Studies in Mathematics, 97*(2), 143–162.

Zazkis, R., & Zazkis, D. (2014). Script writing in the mathematics classroom: Imaginary conversations on the structure of numbers. *Research in Mathematics Education, 16*(1), 54–70.

Zazkis, R., Liljedahl, P., & Sinclair, N. (2009). Lesson plays: Planning teaching vs. teaching planning. *For the Learning of Mathematics, 29*(1), 40–47.

Zazkis, R., Sinclair, N., & Liljedahl, P. (2013). *Lesson play in mathematics education: A tool for research and professional development*. Dordrecht: Springer.

Zimmer, B. J. (2003). *Reflections for tending the sacred garden: Embracing the art of slowing down*. Lincoln: IUniverse.

Additional Suggestions for Further Reading

Again, I recommend the Mathematics Education and Society website, which aims to promote discussion about the social, ethical, and political dimensions of mathematics education. The website provides access to the past conference proceedings on the latest research in mathematics education: http://www.mescommunity.info

Aikenhead, G. S. (2017b). Enhancing school mathematics culturally: A path of reconcilia- tion. *Canadian Journal of Science Mathematics and Technology Education, 17*(2.: Special Monograph Issue), 73–140.

Alaska Native Knowledge Network. (n.d.). *Curriculum resources*. Retrieved from http://www.ankn.uaf.edu/

Alberta Education. (2016). *Walking together*. Retrieved from http://www.learnalberta.ca/content/aswt/index.html#/home

Avital, S., & Barbeau, E. (1992). Commutativity of polynomials. *College Mathematics Journal, 23*(5), 386–395.

Baird, J. A., Hopfenbeck, T. N., Newton, P., Stobart, G., & Steen-Utheim, A. T. S. (2014). *State of the field review: Assessment and learning (OUCEA Report 13/4697)*. Retrieved from University of Oxford, Oxford University Centre for Educational Assessment website http:// bit.ly/1qZ52Mq

Boaler, J., & Humphreys, C. (2005). *Connecting mathematical ideas: Middle school video cases to support teaching and learning*. Portsmouth: Heinemann.

Carnellor, Y. (2004). *Encouraging mathematical success for children with learning difficulties*. Victoria: Social Science Press.

Clark, M., & Lovric, M. (2008). Suggestion for a theoretical model for the secondary-tertiary transition in mathematics. *Mathematics Education Research Journal, 20*(2), 25–37.

Clark, M., & Lovric, M. (2009). Understanding secondary-tertiary transition in mathematics. *International Journal of Mathematical Education in Science and Technology, 40*(6), 755–776.

Dan Meyer's blog (including additional Three-act tasks): http://blog.mrmeyer.com/

De Guzman, M., Hodgson B., Robert, A., & Villani, V. (1998). Difficulties in passage from secondary to tertiary education. In *Proceedings of the International Congress of Mathematicians* (pp. 747–762). Berlin 1998 (Vol. III: Invited Lectures.

Ehrenpreis, W., & Iannone, M. (1972). Commutativity of linear functions. *Mathematics Teacher, 65*(3), 217–220.

Fenstermacher, G. (1986). Philosophy of research on teaching: Three aspects. In M. C. Whittrock (Ed.), *Handbook of research on teaching* (3rd ed., pp. 37–49). New York: Macmillan.

Fenstermacher, G. (1994, revised 1997). *On the distinction between being a student and being a learner*. Paper presented at the annual meeting of the American Educational Research Association, New Orleans, LA.

Gal, I. (2002). Adults' statistical literacy: Meanings, components, responsibilities. *International Statistics Review, 70*(1), 1–51.

Groth, R. E. (2017). Classroom data analysis with the five strands of mathematical proficiency. *The Clearing House: A. Journal of Educational Strategies, Issues, and Ideas, 90*(3), 103–109.

Hewitt, D. (1999). Arbitrary and necessary part I: A way of viewing the mathematics curriculum. *For the Learning of Mathematics, 19*(3), 2–9.

Hoshino, R. (2015). *The math Olympian*. Victoria: Friesen Press.

Hoshino, R., Polotskaia, E., & Reid, D. (2016). Problem solving: Definition, role, and pedagogy. In S. Oesterle, D. Allan, & J. Holm (Eds.), *Proceedings of the 2016 annual meeting of the Canadian Mathematics Education Study Group /Groupe Canadien d'Étude en Didactique des Mathématiques* (pp. 151–162). Kingston: CMESG/GCEDM.

I recommend you to take a look on the Organization for Economic Cooperation and Development (OECD) website, section Program for International Student Assessment (PISA). This international organization provides information about teaching mathematics in international environments: http://www.oecd.org/pisa/

Jackson, D. B. (2014). Algebra homework: A sandwich! *The Mathematics Teacher, 107*(7), 528–533.

Kajander, A. (2013–2016). MB4T: Math by and for teachers. Monthly column in the *Ontario Mathematics Gazette*.

Kajander, A., & Boland, T. (2014). *Mathematical models for teaching; reasoning without memorization*. Toronto: Canadian Scholars Press.

Kajander, A., & Lovric, M. (2005). Transition from secondary to tertiary mathematics: McMaster University experience. *International Journal of Mathematics Education in Science and Technology, 36*(2–3), 149–160.

Kajander, A., & Lovric, M. (2009). Mathematics textbooks and their potential role in supporting misconceptions. *International Journal of Mathematics Education in Science and Technology, 40*(2), 173–181.

Kajander, A., & Zuke, C. (2008). Listening for their voices: A study of intermediate students at risk in mathematics. *delta K, 45*(2), 21–24.

Learning goals and success criteria video library: http://www.edugains.ca/newsite/aer/aervideo/learninggoals.html

Liljedahl, P. (2015). Building thinking classrooms: Conditions for problem solving. In S. Oesterle & D. Allan (Eds.), *Proceedings of the 2015 annual meeting of the Canadian Mathematics Education Study Group/Groupe Canadien d'Étude en Didactique des Mathématiques* (pp. 131–138). Moncton: CMESG/GCEDM.

Liljedahl, P., & Allan, D. (2013). Studenting: The case of homework. In M. Martinez & A. Castro Superfine (Eds.), *Proceedings of the 35th annual meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education* (pp. 489–492). Chicago: University of Illinois at Chicago.

Lockhart, P. (2009). *A mathematician's lament*. New York: Bellevue Literary Press.

Macaulay, A. (2015). *Effective practices in Grade 9 Applied mathematics*. Unpublished doctoral dissertation, Ontario Institute for Studies in Education. Toronto: University of Toronto. Retrieved from https://tspace.library.utoronto.ca/handle/1807/71461

Manninen, J. L. (2014). Teaching perseverance through differentiated homework. *Mathematics Teaching in the Middle School, 19*(8), 502–504.

Martin, V., & Mai Huy, K. (2015). Une réflexion didactique sur des activités pour penser l'enseignement-apprentissage des probabilités et des statistiques à l'école primaire. *Bulletin de l'AMQ, 55*(3), 50–67.

Martin, V., & Theis, L. (2016). A study of the teaching of probability to students judged or not with learning difficulties in mathematics in regular elementary classes in Québec. In *Proceedings of the thirteenth International Congress on Mathematical Education*. Germany: Hamburg.

Marzano, R. J., & Pickering, D. J. (2007). The case for and against homework. *Educational Leadership, 64*(6), 74–79.

Mason, J. (2002). *Researching your own practice: The discipline of noticing*. New York: Routledge.

Mason, J., Burton, L., & Stacey, K. (1982). *Thinking mathematically*. London: Addison-Wesley Publishing Company.

McFeetors, P. J. (2015). Opportunities for learning-based conversations in high school mathematics. *Alberta Journal of Educational Research, 61*(2), 147–165.

Miller, J. P. (2010). *Whole child education*. Toronto: University of Toronto Press.

Nasir, N. S., Cabana, C., Shreve, B., Woodbury, E., & Louie, N. (2014). *Mathematics for equity: A framework for successful practice*. New York: Teachers' College Press.

Ontario Ministry of Education. (2016). *Edugains: Math home page*. Retrieved from http://www.edugains.ca/newsite/math/index.html

Otten, S., Cirillo, M., & Herbel-Eisenmann, B. A. (2015). Making the most of going over homework. *Mathematics Teaching in the Middle School, 21*(2), 98–105.

Overwijk, A. *Slam dunk mathematics* [blog]. Retrieved from http://slamdunkmath.blogspot.ca

Pólya, G. (1988). *How to solve it* (2nd ed.). Princeton: Princeton University Press.

Radical Math. http://radicalmath.org/main.php?id=SocialJusticeMath

Resources from the Creating Balance Conference: http://www.creatingbalanceconference.org/resources

Rossman, A. J., & Short, T. H. (1995). Conditional probability and education reform: Are they compatible? *Journal of Statistics Education, 3*(2). Retrieved from https://ww2.amstat.org/publications/JSE/v3n2/rossman.html

Roth, W.-M., & Radford, L. (2011). *A cultural-historical perspective on mathematics teaching and learning*. Rotterdam: Sense Publishers.

Schoenfeld, A. H. (2004). The math wars. *Educational Policy, 18*(1), 253–286.

Seguin, C. P. (1965). Commuting linear functions and fixed points. *Mathematics Teacher, 58*(8).

Shaugnessy, J. M., Chance, B., & Kranendonk, H. (2009). *Focus in high school mathematics: Reasoning and sense making in statistics and probability*. Reston: National Council of Teachers of Mathematics.
Small, M. Gr. 4-8 (2009). Gr K-3 (2010) and Gr 9–12 (2011). *Big ideas from Dr. Small: Creating a comfort zone for teaching mathematics*. Toronto: Nelson Education.
Smith, C. M. (1999). Meta-learning in mathematics: How can teachers help students learn how to learn? In F. Hitt & M. Santos (Eds.), *Proceedings of the twenty-first annual meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education* (pp. 792–797). Columbus: ERIC Clearinghouse for Science, Mathematics, and Environmental Education.
Sterenberg, G., Barrett, L., Blood, N., Glanfield, F., Lunney Borden, L., McDonnell, T., Nicol, C., & Weston, H. (2010). To become wise to the world around us: Multiple perspectives of relating Indigenous knowledges and mathematics education. *delta-K, 47*(2), 21–29.
Stocker, D. (2008). *Maththatmatters: A teacher resource linking math and social justice* (2nd ed.). Ottawa: CCPA Education Project.
Stocker, D. (2017). *Maththatmatters 2: A teacher resource linking math and social justice*. Toronto: Between the Lines.
Suh, J. M. (2007). Tying it all together: Classroom practices that promote mathematical proficiency for all students. *Teaching Children Mathematics, 14*(3), 163–169.
Tall, D. O. (1995). Mathematical growth in elementary and advanced mathematical thinking. In L. Meira & D. Carraher (Eds.), Proceedings of PME-19 (Vol. 1, pp. 61–75). Recife, Brazil.
Tall, D. (1997). From school to university: The effects of learning styles in the transition from elementary to advanced mathematical thinking. In M. O. J. Thomas (Ed.), *Proceedings of the seventh annual Australasian Bridging Network mathematics conference* (pp. 9–26). Auckland: University of Auckland.
Third path website: http://www.thirdpath.ca/
This book chapter focuses on the implication of modelize or mathematize a situation or a problem. In order to support your students to create models before computing, I would recommend this website on virtual manipulatives. Even older students need support when mathematize: http://nlvm.usu.edu
TinkerPlots 2 software website: http://www.tinkerplots.com/
TinkerPlots data sets and classroom activities: http://www.tinkerplots.com/activities
Ungar, M. (2004). *Nurturing hidden resilience in troubled youth*. Toronto: Buffalo.
Ungar, M. (2014). *I still love you: Nine things troubled kids need from their parents* (1st ed.). Toronto: Dundurn.
Vakil, R. (1996). *A mathematical mosaic: Patterns & problem solving*. Burlington: Brendan Kelly Publishing Inc.
Ward, C., Bochner, S., & Furnham, A. (2001). *The psychology of culture shock*. Hove: Routhledge.
Yasnitsky, A., & van der Veer, R. (2016). *Revisionist revolution in Vygotsky studies: The state of the art*. East Sussex: Routledge.
Zager, T. J. (2017). *Becoming the math teacher you wish you'd had: Ideas and strategies from vibrant classrooms*. Portland: Stenhouse Publishers.

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