On Higher Order Effective Boundary Conditions for a Coated Elastic Half-Space

Julius Kaplunov, Danila Prikazchikov and Leyla Sultanova

Abstract Higher order effective boundary conditions are derived for a coated half-space. Comparison with the long wavelength expansion of the exact solution of a plane time-harmonic problem for the coating demonstrates the validity of the proposed formulation. At the same time the corrections to the simplest leading order effective conditions, earlier obtained in the widely cited paper (Bövik (1996). J. Appl. Mech. 63(1), 162–167.) [1], are proven to be asymptotically inconsistent.

1 Introduction

Thin films and coatings find numerous applications, including in particular, engineering and biological sciences, see e.g. [2–5]. The effect of a thin coating is often modeled by imposing the so-called effective boundary conditions along the surface of a substrate. These conditions first were derived in [6] using *adhoc* assumptions originating from the classical theory of plate extensions. Later on, it was suggested in [1] that the results of [6] are not consistent, and refined boundary conditions were proposed starting from rather heuristic arguments. The asymptotic procedure exposed in [7] justifies at leading order the consistency of the effective boundary conditions in [6] and also reveals that the extra terms in [1] are in fact of a higher order. Moreover, as it was briefly mentioned in [7], the development in [1] is not asymptotically consistent at the next order as well.

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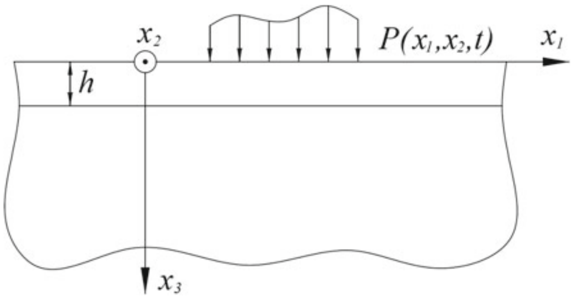

Fig. 1 A coated half-space

It is remarkable that the boundary conditions in [1] were exploited not only before but also after the publication of the critical comments in [7], e.g. see [8–10] along with [11–13]. This is partly an inspiration for revisiting the original problem for a coated elastic half-space aiming at establishing higher order effective conditions.

As in [7], we adapt the asymptotic methodology well established for the thin elastic structures, e.g. see [14, 15] and references therein. At leading order, we validate again the results in [6]. At next order, we arrive at refined effective conditions. They are tested by comparison with the exact solution of a plane strain time-harmonic problem. As it might be expected, the comparison demonstrates that the boundary conditions in [1] are not consistent at a higher order.

2 Statement of the Problem

We consider a linearly elastic isotropic layer of thickness h occupying the area $0 \le x_3 \le h$, lying on an elastic half-space $x_3 \ge h$. The prescribed vertical force $P = P(x_1, x_2, t)$ is acting on the free surface of the layer, see Fig. 1.

The 3D equations in linear elasticity can be written as

$$\begin{aligned} \frac{\partial \sigma_{ii}}{\partial x_i} + \frac{\partial \sigma_{ij}}{\partial x_j} + \frac{\partial \sigma_{i3}}{\partial x_3} &= \rho \frac{\partial^2 u_i}{\partial t^2}, \\ \frac{\partial \sigma_{i3}}{\partial x_i} + \frac{\partial \sigma_{j3}}{\partial x_j} + \frac{\partial \sigma_{33}}{\partial x_3} &= \rho \frac{\partial^2 u_3}{\partial t^2}. \end{aligned} \tag{1}$$

Here and below $i \neq j = 1, 2$ and $n = 1, 2, 3$, u_n are the displacements, σ_{in}, σ_{3n} are stresses, and ρ is the volume density. The constitutive relations are

$$\begin{aligned} \sigma_{ij} &= \mu \left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} \right), & \sigma_{ii} &= (\lambda + 2\mu) \frac{\partial u_i}{\partial x_i} + \lambda \left(\frac{\partial u_j}{\partial x_j} + \frac{\partial u_3}{\partial x_3} \right), \\ \sigma_{i3} = \sigma_{3i} &= \mu \left(\frac{\partial u_i}{\partial x_3} + \frac{\partial u_3}{\partial x_i} \right), & \sigma_{33} &= \lambda \left(\frac{\partial u_i}{\partial x_i} + \frac{\partial u_j}{\partial x_j} \right) + (\lambda + 2\mu) \frac{\partial u_3}{\partial x_3}, \end{aligned} \tag{2}$$

where λ and μ are the Lamé parameters. In addition, the wave speeds are given by

$$c_1 = \sqrt{\frac{\lambda + 2\mu}{\rho}}, \quad c_2 = \sqrt{\frac{\mu}{\rho}}. \tag{3}$$

In case of the coating, below we supply with suffix 0 the parameters in the Eqs. (1)–(3), using the notations ρ_0, λ_0, μ_0, c_{10} and c_{20}.

We impose the boundary conditions

$$\sigma_{33} = -P \quad \text{and} \quad \sigma_{i3} = 0 \tag{4}$$

at the surface of the coating $x_3 = 0$ and also assume continuity of the displacements u_n and stresses σ_{n3} along the interface $x_3 = h$.

The leading order effective boundary conditions on the surface of the substrate, modelling the effect of the coating, can be written as, see (3.18) in [7],

$$\begin{aligned} \sigma_{33} &= \rho_0 h \frac{\partial^2 u_3}{\partial t^2} - P, \\ \sigma_{i3} &= \rho_0 h \left[\frac{\partial^2 u_i}{\partial t^2} - c_{20}^2 \left(\frac{\partial^2 u_i}{\partial x_j^2} + 4(1 - \kappa_0^{-2}) \frac{\partial^2 u_i}{\partial x_i^2} + (3 - 4\kappa_0^{-2}) \frac{\partial^2 u_j}{\partial x_i \partial x_j} \right) \right], \end{aligned} \tag{5}$$

where $\kappa_0 = c_{10}/c_{20}$. In absence of surface loading ($P = 0$) these conditions coincide with those in [6] derived starting from the 2D theory of plate extension. More recent developments in [1], see also [9] treating a similar anisotropic problem, claim that the effective conditions (5) ignore several essential h-terms. The formulae (35) and (36) in [1] rewritten in the notation specified in this section, similarly to [7], can be presented as

$$\begin{aligned} \sigma_{33} &= \rho_0 h \frac{\partial^2 u_3}{\partial t^2} \underline{- h \left(\frac{\partial \sigma_{i3}}{\partial x_i} + \frac{\partial \sigma_{j3}}{\partial x_j} \right)}, \\ \sigma_{i3} &= \rho_0 h \left[\frac{\partial^2 u_i}{\partial t^2} - c_{20}^2 \left(\frac{\partial^2 u_i}{\partial x_j^2} + 4(1 - \kappa_0^{-2}) \frac{\partial^2 u_i}{\partial x_i^2} \right.\right. \\ &\quad \left.\left. + (3 - 4\kappa_0^{-2}) \frac{\partial^2 u_j}{\partial x_i \partial x_j} \right) \right] \underline{- h(1 - 2\kappa_0^{-2}) \frac{\partial \sigma_{33}}{\partial x_i}}. \end{aligned} \tag{6}$$

The underlined terms in formulae (6) do not appear in the effective conditions (5). The former may be also transformed to

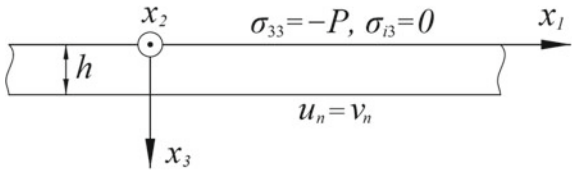

Fig. 2 Boundary value problem for a thin coating

$$\begin{aligned}
\sigma_{33} = {} & \rho_0 h \frac{\partial^2 u_3}{\partial t^2} - \rho_0 h^2 \left[\frac{\partial^3 u_i}{\partial t^2 \partial x_i} + \frac{\partial^3 u_j}{\partial t^2 \partial x_j} - c_{20}^2 \left(\frac{\partial^3 u_i}{\partial x_i \partial x_j^2} + \frac{\partial^3 u_j}{\partial x_i^2 \partial x_j} \right.\right. \\
& \left.\left. + 4(1-\kappa_0^{-2}) \left[\frac{\partial^3 u_i}{\partial x_i^3} + \frac{\partial^3 u_j}{\partial x_j^3} \right] + (3-4\kappa_0^{-2}) \left[\frac{\partial^3 u_j}{\partial x_i^2 \partial x_j} + \frac{\partial^3 u_i}{\partial x_i \partial x_j^2} \right] \right) \right] \\
& + h^2 (1-2\kappa_0^{-2}) \left(\frac{\partial^2 \sigma_{33}}{\partial x_i^2} + \frac{\partial^2 \sigma_{33}}{\partial x_j^2} \right), \\
\sigma_{i3} = {} & \rho_0 h \left[\frac{\partial^2 u_i}{\partial t^2} - c_{20}^2 \left(\frac{\partial^2 u_i}{\partial x_j^2} + 4(1-\kappa_0^{-2}) \frac{\partial^2 u_i}{\partial x_i^2} + (3-4\kappa_0^{-2}) \frac{\partial^2 u_j}{\partial x_i \partial x_j} \right) \right] \\
& - h^2 (1-2\kappa_0^{-2}) \left(\rho_0 \frac{\partial^3 u_3}{\partial t^2 \partial x_i} - \left[\frac{\partial^2 \sigma_{i3}}{\partial x_i^2} + \frac{\partial^2 \sigma_{j3}}{\partial x_i \partial x_j} \right] \right).
\end{aligned} \tag{7}$$

It is already pretty clear at this stage that all extra h^2-terms in (7) can be neglected at leading order. In what follows, this observation is asymptotically justified. We also show below that h^2-terms in (7) are not identical to a proper asymptotic correction to (5).

3 Asymptotic Analysis

The aim of the paper is to determine an asymptotic correction to the leading order effective boundary conditions (5), in order to address consistency of (6), or equivalently, (7). Here we implement an asymptotic procedure similar to [7], modifying it slightly according to a more recent treatment in [16]. As usual, we study the boundary value problem for an elastic coating with the Dirichlet boundary conditions

$$u_n = v_n \tag{8}$$

at the interface $x_3 = h$, where $v_n = v_n(x_1, x_2, t)$ denote prescribed displacements, see Fig. 2.

We assume that the thickness of the coating h is small compared to typical wave length L, therefore, we introduce a geometric parameter given by

$$\varepsilon = \frac{h}{L} \ll 1. \tag{9}$$

We also specify dimensionless variables

$$\xi_i = \frac{x_i}{L}, \quad \eta = \frac{x_3}{h}, \quad \tau = \frac{tc_{20}}{L}. \tag{10}$$

According to the conventional asymptotic procedure, e.g. [7, 14], and ref. therein, we adopt the scaling

$$\begin{aligned} u_n &= Lu_n^*, \qquad v_n = Lv_n^*, \qquad P = \mu_0 \varepsilon p^* \\ \sigma_{ii} &= \mu_0 \sigma_{ii}^*, \quad \sigma_{ij} = \mu_0 \sigma_{ij}^*, \quad \sigma_{n3} = \mu_0 \varepsilon \sigma_{n3}^*, \end{aligned} \tag{11}$$

where all quantities with the asterisk are assumed to be of the same asymptotic order.

The Eq. (1) and the constitutive relations (2) rewritten in dimensionless form, become

$$\frac{\partial \sigma_{ii}^*}{\partial \xi_i} + \frac{\partial \sigma_{ij}^*}{\partial \xi_j} + \frac{\partial \sigma_{i3}^*}{\partial \eta} = \frac{\partial^2 u_i^*}{\partial \tau^2}, \tag{12}$$

$$\frac{\partial \sigma_{33}^*}{\partial \eta} + \varepsilon \left(\frac{\partial \sigma_{i3}^*}{\partial \xi_i} + \frac{\partial \sigma_{j3}^*}{\partial \xi_j} \right) = \frac{\partial^2 u_3^*}{\partial \tau^2}, \tag{13}$$

and

$$\sigma_{ij}^* = \frac{\partial u_i^*}{\partial \xi_j} + \frac{\partial u_j^*}{\partial \xi_i}, \tag{14}$$

$$\varepsilon \sigma_{ii}^* = (\kappa_0^2 - 2) \frac{\partial u_3^*}{\partial \eta} + \varepsilon \left(\kappa_0^2 \frac{\partial u_i^*}{\partial \xi_i} + (\kappa_0^2 - 2) \frac{\partial u_j^*}{\partial \xi_j} \right), \tag{15}$$

$$\varepsilon^2 \sigma_{i3}^* = \frac{\partial u_i^*}{\partial \eta} + \varepsilon \frac{\partial u_3^*}{\partial \xi_i}, \tag{16}$$

$$\varepsilon^2 \sigma_{33}^* = \kappa_0^2 \frac{\partial u_3^*}{\partial \eta} + \varepsilon (\kappa_0^2 - 2) \left(\frac{\partial u_i^*}{\partial \xi_i} + \frac{\partial u_j^*}{\partial \xi_j} \right), \tag{17}$$

with the transformed boundary conditions

$$\begin{aligned} &\sigma_{33}^* = -p^* \quad \text{and} \quad \sigma_{i3}^* = 0, \quad \eta = 0, \\ \text{and} \quad & \\ &u_n^* = v_n^*, \quad \eta = 1. \end{aligned} \tag{18}$$

First, expressing $\frac{\partial u_3^*}{\partial \eta}$ from (17) and substituting the result into (15), we obtain

$$\sigma_{ii}^* = 4(1-\kappa_0^{-2})\frac{\partial u_i^*}{\partial \xi_i} + 2(1-2\kappa_0^{-2})\frac{\partial u_j^*}{\partial \xi_j} + (1-2\kappa_0^{-2})\varepsilon\sigma_{33}^*. \tag{19}$$

Next, we expand the displacements and stresses as

$$\begin{pmatrix} u_n^* \\ \sigma_{ii}^* \\ \sigma_{ij}^* \\ \sigma_{3i}^* \\ \sigma_{33}^* \end{pmatrix} = \begin{pmatrix} u_n^{(0)} \\ \sigma_{ii}^{(0)} \\ \sigma_{ij}^{(0)} \\ \sigma_{3i}^{(0)} \\ \sigma_{33}^{(0)} \end{pmatrix} + \varepsilon \begin{pmatrix} u_n^{(1)} \\ \sigma_{ii}^{(1)} \\ \sigma_{ij}^{(1)} \\ \sigma_{3i}^{(1)} \\ \sigma_{33}^{(1)} \end{pmatrix} + \varepsilon^2 \begin{pmatrix} u_n^{(2)} \\ \sigma_{ii}^{(2)} \\ \sigma_{ij}^{(2)} \\ \sigma_{3i}^{(2)} \\ \sigma_{33}^{(2)} \end{pmatrix} + \cdots \tag{20}$$

On substituting the latter into the Eqs. (12)–(17) and (19), we have at leading order

$$\begin{aligned}
&\frac{\partial \sigma_{ii}^{(0)}}{\partial \xi_i} + \frac{\partial \sigma_{ij}^{(0)}}{\partial \xi_j} + \frac{\partial \sigma_{i3}^{(0)}}{\partial \eta} = \frac{\partial^2 u_i^{(0)}}{\partial \tau^2}, \\
&\frac{\partial \sigma_{33}^{(0)}}{\partial \eta} = \frac{\partial^2 u_3^{(0)}}{\partial \tau^2}, \\
&\sigma_{ij}^{(0)} = \frac{\partial u_i^{(0)}}{\partial \xi_j} + \frac{\partial u_j^{(0)}}{\partial \xi_i}, \\
&\frac{\partial u_n^{(0)}}{\partial \eta} = 0, \\
&\sigma_{ii}^{(0)} = 4(1-\kappa_0^{-2})\frac{\partial u_i^{(0)}}{\partial \xi_i} + 2(1-2\kappa_0^{-2})\frac{\partial u_j^{(0)}}{\partial \xi_j},
\end{aligned} \tag{21}$$

with the boundary conditions

$$\begin{aligned}
&\sigma_{33}^{(0)} = -p^* \quad \text{and} \quad \sigma_{i3}^{(0)} = 0, \quad \eta = 0, \\
\text{and} \quad & \\
&u_n^{(0)} = v_n^*, \quad \eta = 1.
\end{aligned} \tag{22}$$

Integrating the leading order Eq. (21) together with the boundary conditions (22), gives

$$u_n^{(0)} = v_n^*, \tag{23}$$

$$\sigma_{33}^{(0)} = \eta\frac{\partial^2 v_3^*}{\partial \tau^2} - p^*, \tag{24}$$

$$\sigma_{ii}^{(0)} = 4(1-\kappa_0^{-2})\frac{\partial v_i^*}{\partial \xi_i} + 2(1-2\kappa_0^{-2})\frac{\partial v_j^*}{\partial \xi_j}, \tag{25}$$

$$\sigma_{i3}^{(0)} = \eta\left[\frac{\partial^2 v_i^*}{\partial \tau^2} - \frac{\partial^2 v_i^*}{\partial \xi_j^2} - 4(1-\kappa_0^{-2})\frac{\partial^2 v_i^*}{\partial \xi_i^2} - (3-4\kappa_0^{-2})\frac{\partial^2 v_j^*}{\partial \xi_i \xi_j}\right]. \tag{26}$$

At next asymptotic order, the governing equations take the form

$$\frac{\partial \sigma_{ii}^{(1)}}{\partial \xi_i} + \frac{\partial \sigma_{ij}^{(1)}}{\partial \xi_j} + \frac{\partial \sigma_{i3}^{(1)}}{\partial \eta} = \frac{\partial^2 u_i^{(1)}}{\partial \tau^2}, \tag{27}$$

$$\frac{\partial \sigma_{33}^{(1)}}{\partial \eta} + \frac{\partial \sigma_{i3}^{(0)}}{\partial \xi_i} + \frac{\partial \sigma_{j3}^{(0)}}{\partial \xi_j} = \frac{\partial^2 u_3^{(1)}}{\partial \tau^2}, \tag{28}$$

$$\sigma_{ij}^{(1)} = \frac{\partial u_i^{(1)}}{\partial \xi_j} + \frac{\partial u_j^{(1)}}{\partial \xi_i}, \tag{29}$$

$$\sigma_{ii}^{(0)} = (\kappa_0^2 - 2)\frac{\partial u_3^{(1)}}{\partial \eta} + \kappa_0^2 \frac{\partial u_i^{(0)}}{\partial \xi_i} + (\kappa_0^2 - 2)\frac{\partial u_j^{(0)}}{\partial \xi_j}, \tag{30}$$

$$\frac{\partial u_i^{(1)}}{\partial \eta} + \frac{\partial u_3^{(0)}}{\partial \xi_i} = 0, \tag{31}$$

$$\kappa_0^2 \frac{\partial u_3^{(1)}}{\partial \eta} + (\kappa_0^2 - 2)\left(\frac{\partial u_i^{(0)}}{\partial \xi_i} + \frac{\partial u_j^{(0)}}{\partial \xi_j}\right) = 0, \tag{32}$$

$$\sigma_{ii}^{(1)} = 4(1 - \kappa_0^{-2})\frac{\partial u_i^{(1)}}{\partial \xi_i} + (1 - 2\kappa_0^{-2})\left(2\frac{\partial u_j^{(1)}}{\partial \xi_j} + \sigma_{33}^{(0)}\right), \tag{33}$$

with the boundary conditions

$$\sigma_{n3}^{(1)} = 0, \quad \eta = 0, \tag{34}$$

and

$$u_n^{(1)} = 0, \quad \eta = 1. \tag{35}$$

First, we obtain from (31) and (32), respectively, satisfying (35)

$$u_i^{(1)} = (1 - \eta)\frac{\partial v_3^*}{\partial \xi_i},$$

and

$$u_3^{(1)} = (1 - 2\kappa_0^{-2})(1 - \eta)\left(\frac{\partial v_i^*}{\partial \xi_i} + \frac{\partial v_j^*}{\partial \xi_j}\right). \tag{36}$$

Then, using (29), we have

$$\sigma_{ij}^{(1)} = 2(1 - \eta)\frac{\partial^2 v_3^*}{\partial \xi_i \partial \xi_j}. \tag{37}$$

Next, we deduce from (28) and (34)

$$\sigma_{33}^{(1)} = \frac{\eta}{\kappa_0^2}\left((\eta - 2 + \kappa_0^2 - \eta\kappa_0^2)\left[\frac{\partial^3 v_i^*}{\partial\xi_i\partial\tau^2} + \frac{\partial^3 v_j^*}{\partial\xi_j\partial\tau^2}\right]\right.$$
$$\left. +2\eta(\kappa_0^2 - 1)\left[\frac{\partial^3 v_i^*}{\partial\xi_i\partial\xi_j^2} + \frac{\partial^3 v_j^*}{\partial\xi_i^2\partial\xi_j} + \frac{\partial^3 v_i^*}{\partial\xi_i^3} + \frac{\partial^3 v_j^*}{\partial\xi_j^3}\right]\right). \tag{38}$$

As a result, (33) becomes

$$\sigma_{ii}^{(1)} = 2(\eta - 1)\left[2(\kappa_0^{-2} - 1)\frac{\partial^2 v_3^*}{\partial\xi_i^2} - (1 - 2\kappa_0^{-2})\frac{\partial^2 v_3^*}{\partial\xi_j^2}\right]$$
$$+(1 - 2\kappa_0^{-2})\left[\eta\frac{\partial^2 v_3^*}{\partial\tau^2} - p^*\right]. \tag{39}$$

Therefore, (27) implies

$$\sigma_{i3}^{(1)} = -\eta\left[(\eta - 1 - \eta\kappa_0^{-2})\frac{\partial^3 v_3^*}{\partial\xi_i\partial\tau^2} + 2(\kappa_0^{-2} - 1)(\eta - 2)\right.$$
$$\left.\left(\frac{\partial^3 v_3^*}{\partial\xi_i\partial\xi_j^2} + \frac{\partial^3 v_3^*}{\partial\xi_i^3}\right) - (1 - 2\kappa_0^{-2})\frac{\partial p^*}{\partial\xi_i}\right]. \tag{40}$$

Finally, substituting the leading order formulae (24) and (26) and $O(\varepsilon)$ corrections (38) and (40) into the expansions (20), we arrive at

$$\sigma_{33}^* = \eta\frac{\partial^2 v_3^*}{\partial\tau^2} - p^* + \varepsilon\frac{\eta}{\kappa_0^2}\left[(\eta - 2 + \kappa_0^2 - \eta\kappa_0^2)\left(\frac{\partial^3 v_i^*}{\partial\xi_i\partial\tau^2} + \frac{\partial^3 v_j^*}{\partial\xi_j\partial\tau^2}\right)\right.$$
$$\left. +2\eta(\kappa_0^2 - 1)\left(\frac{\partial^3 v_i^*}{\partial\xi_i\partial\xi_j^2} + \frac{\partial^3 v_j^*}{\partial\xi_i^2\partial\xi_j} + \frac{\partial^3 v_i^*}{\partial\xi_i^3} + \frac{\partial^3 v_j^*}{\partial\xi_j^3}\right)\right] + ...,$$
$$\sigma_{i3}^* = \eta\left[\frac{\partial^2 v_i^*}{\partial\tau^2} - \frac{\partial^2 v_i^*}{\partial\xi_j^2} - 4(1 - \kappa_0^{-2})\frac{\partial^2 v_i^*}{\partial\xi_i^2} - (3 - 4\kappa_0^{-2})\frac{\partial^2 v_j^*}{\partial\xi_i\xi_j}\right] \tag{41}$$
$$-\varepsilon\eta\left[(\eta - 1 - \eta\kappa_0^{-2})\frac{\partial^3 v_3^*}{\partial\xi_i\partial\tau^2} + 2(\kappa_0^{-2} - 1)(\eta - 2)\right.$$
$$\left.\left(\frac{\partial^3 v_3^*}{\partial\xi_i\partial\xi_j^2} + \frac{\partial^3 v_3^*}{\partial\xi_i^3}\right) - (1 - 2\kappa_0^{-2})\frac{\partial p^*}{\partial\xi_i}\right] + ...$$

The continuity of the displacements, see (8), and stresses at the interface $x_3 = h$ readily results in refined effective boundary conditions for the substrate $x_3 \geq h$. In the original variables they take the form

$$\begin{aligned}\sigma_{33} &= \rho_0 h \frac{\partial^2 u_3}{\partial t^2} - P + \frac{\rho_0 h^2}{\kappa_0^2}\left[2c_2^2(\kappa_0^2-1)\left(\frac{\partial^3 u_i}{\partial x_i \partial x_j^2} + \frac{\partial^3 u_j}{\partial x_i^2 \partial x_j}\right.\right.\\ &\quad \left.\left. + \frac{\partial^3 u_i}{\partial x_i^3} + \frac{\partial^3 u_j}{\partial x_j^3}\right) - \left(\frac{\partial^3 u_i}{\partial x_i \partial t^2} + \frac{\partial^3 u_j}{\partial x_j \partial t^2}\right)\right],\\ \sigma_{i3} &= \rho_0 h\left[\frac{\partial^2 u_i}{\partial t^2} - c_{20}^2\left(\frac{\partial^2 u_i}{\partial x_j^2} + 4(1-\kappa_0^{-2})\frac{\partial^2 u_i}{\partial x_i^2} + (3-4\kappa_0^{-2})\frac{\partial^2 u_j}{\partial x_i \partial x_j}\right)\right]\\ &\quad + \frac{\rho_0 h^2}{\kappa_0^2}\left[\frac{\partial^3 u_3}{\partial x_i \partial t^2} + 2c_2^2(1-\kappa_0^2)\left(\frac{\partial^3 u_3}{\partial x_i \partial x_j^2} + \frac{\partial^3 u_3}{\partial x_i^3}\right)\right] + h\frac{\kappa_0^2-2}{\kappa_0^2}\frac{\partial P}{\partial x_i}.\end{aligned} \tag{42}$$

Comparing these formulae at $P = 0$ with (7) we may expect that higher order h^2-terms will not coincide.

4 Comparison with the Exact Solution of a Plane Strain Problem

In order to validate the asymptotic results obtained in the previous section, let us consider a time-harmonic plane strain problem for the coating over the plane Ox_1x_3. In this case the displacements can be taken as

$$u_1 = \frac{\partial \varphi}{\partial x_1} + \frac{\partial \psi}{\partial x_3}, \quad u_3 = \frac{\partial \varphi}{\partial x_3} - \frac{\partial \psi}{\partial x_1}, \tag{43}$$

where φ and ψ are Lamé elastic potentials. The wave equations of motion become

$$\Delta\varphi - \frac{1}{c_{10}^2}\frac{\partial^2 \varphi}{\partial t^2} = 0, \quad \Delta\psi - \frac{1}{c_{20}^2}\frac{\partial^2 \psi}{\partial t^2} = 0, \tag{44}$$

where $\Delta = \frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_3^2}$. The solutions of (44) are sought for in the form

$$\varphi = f(x_3)e^{ik(x_1-ct)}, \quad \psi = g(x_3)e^{ik(x_1-ct)}. \tag{45}$$

Substituting the latter into (44), we deduce

$$f(x_3) = A_1 e^{kx_3\alpha} + A_2 e^{-kx_3\alpha} \quad \text{and} \quad g(x_3) = A_3 e^{kx_3\beta} + A_4 e^{-kx_3\beta}, \tag{46}$$

where $A_m,\ m = 1, 2, 3, 4,$ are arbitrary constants, and $\alpha = \sqrt{1 - \frac{c^2}{c_{10}^2}}$ and $\beta = \sqrt{1 - \frac{c^2}{c_{20}^2}}$.

We consider a traction free upper face ($P = 0$), i.e. at $x_3 = 0$

$$\sigma_{k3} = 0, \quad k = 1, 3, \tag{47}$$

imposing the boundary conditions (8) at the lower face $x_3 = h$ with

$$v_k = h B_k \mathrm{e}^{ik(x_1 - ct)}, \tag{48}$$

where B_k are certain prescribed values.

On satisfying the boundary conditions, we have

$$\begin{pmatrix} i\alpha & -i\alpha & \gamma^2 & \gamma^2 \\ \gamma^2 & \gamma^2 & -i\beta & i\beta \\ ik\mathrm{e}^{kh\alpha} & ik\mathrm{e}^{-kh\alpha} & \beta k\mathrm{e}^{kh\beta} & -\beta k\mathrm{e}^{-kh\beta} \\ \alpha k\mathrm{e}^{kh\alpha} & -\alpha k\mathrm{e}^{-kh\alpha} & -ik\mathrm{e}^{kh\beta} & -ik\mathrm{e}^{-kh\beta} \end{pmatrix} \begin{pmatrix} A_1 \\ A_2 \\ A_3 \\ A_4 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ hB_1 \\ hB_3 \end{pmatrix} \tag{49}$$

where $\gamma = \sqrt{1 - \frac{1}{2}\frac{c^2}{c_{20}^2}}$, and coefficients A_m expressed through the given constants B_k are presented in Appendix.

Then, substituting (45) and (46) into (43), we get

$$\begin{aligned} u_1 &= k\left[\beta(A_3\mathrm{e}^{2kx_3\beta} - A_4)\mathrm{e}^{-kx_3\beta} + i(A_1\mathrm{e}^{2kx_3\alpha} + A_2)\mathrm{e}^{-kx_3\alpha}\right], \\ u_3 &= k\left[\alpha(A_1\mathrm{e}^{2kx_3\alpha} - A_2)\mathrm{e}^{-kx_3\alpha} - i(A_3\mathrm{e}^{2kx_3\beta} + A_4)\mathrm{e}^{-kx_3\beta}\right]. \end{aligned} \tag{50}$$

Here and below the factor $\mathrm{e}^{ik(x_1-ct)}$ is omitted. Next, using the expressions above and the constitutive relations (2), we have for the stresses at $x_3 = h$

$$\begin{aligned} \sigma_{33} &= 2\mu_0 k^2\left[\gamma^2(A_1\mathrm{e}^{2kh\alpha} + A_2)\mathrm{e}^{-kh\alpha} - i\beta(A_3\mathrm{e}^{2kh\beta} - A_4)\mathrm{e}^{-kh\beta}\right], \\ \sigma_{13} &= 2\mu_0 k^2\left[\gamma^2(A_3\mathrm{e}^{2kh\beta} + A_4)\mathrm{e}^{-kh\beta} + i\alpha(A_1\mathrm{e}^{2kh\alpha} - A_2)\mathrm{e}^{-kh\alpha}\right]. \end{aligned} \tag{51}$$

The last expressions can be expanded into asymptotic series in the small parameter $\varepsilon = kh \ll 1$ $\left(L = k^{-1} \text{ in } (9)\right)$ to get

$$\begin{aligned} \frac{\sigma_{33}}{\varepsilon^2\mu_0} &= -B_3\zeta^2 - iB_1\left[2 - \kappa_0^{-2}(2 + \zeta^2)\right]\varepsilon + \cdots, \\ \frac{\sigma_{13}}{\varepsilon^2\mu_0} &= B_1\left[4(1 - \kappa_0^{-2}) - \zeta^2\right] + iB_3\left[2 - \kappa_0^{-2}(2 + \zeta^2)\right]\varepsilon \\ &\quad - \frac{B_1}{3}\left[20 + \zeta^2(\zeta^2 - 8) + \kappa_0^{-2}(6\zeta^2 - 44) + 4\kappa_0^{-4}(\zeta^2 + 6)\right]\varepsilon^2 + \cdots, \end{aligned} \tag{52}$$

where the dimensionless velocity is

$$\zeta = \frac{c}{c_{20}}. \tag{53}$$

The asymptotic effective conditions (42) for the same displacements (48) prescribed at the lower face, become

$$\begin{aligned}
\sigma_{33} &= k^2h^2\rho_0\left[-B_3c^2 - iB_1kh\left[2c_{20}^2 - \kappa_0^{-2}(2c_{20}^2 + c^2)\right]\right],\\
\sigma_{13} &= k^2h^2\rho_0\left[B_1\left[4c_{20}^2(1-\kappa_0^{-2}) - c^2\right] + iB_3kh\left[2c_{20}^2 - \kappa_0^{-2}(2c_{20}^2 + c^2)\right]\right],
\end{aligned} \tag{54}$$

or, rewritten in terms of ε and ζ,

$$\begin{aligned}
\frac{\sigma_{33}}{\varepsilon^2\mu_0} &= -B_3\zeta^2 - iB_1\left[2 - \kappa_0^{-2}(2+\zeta^2)\right]\varepsilon,\\
\frac{\sigma_{13}}{\varepsilon^2\mu_0} &= B_1\left[4(1-\kappa_0^{-2}) - \zeta^2\right] + iB_3\left[2 - \kappa_0^{-2}(2+\zeta^2)\right]\varepsilon.
\end{aligned} \tag{55}$$

These formulae coincide with the two-term expansion of the exact solution (52). Thus, the validity of the asymptotic results in Sect. 3 is confirmed.

Let us now test the conditions in [1] in a similar manner. In case of the displacements (48) the relation (6) takes the form

$$\begin{aligned}
\sigma_{33} &= -\frac{h^2\rho_0\left[iB_1k^3h(4c_{20}^2 - c^2) + B_3c^2k^2\right]}{1 + k^2h^2(1-2\kappa_0^{-2})},\\
\sigma_{13} &= \frac{h^2\rho_0\left[B_1k^2(4c_{20}^2(1-\kappa_0^2) - c^2) + iB_3kh(1-2\kappa_0^{-2})\right]}{1 + k^2h^2(1-2\kappa_0^{-2})},
\end{aligned} \tag{56}$$

or, expanding the latter in ε,

$$\begin{aligned}
\frac{\sigma_{33}}{\varepsilon^2\mu_0} &= -B_3\zeta^2 - iB_1\left[4(1-\kappa_0^{-2}) - \zeta^2\right]\varepsilon + B_3\zeta^2(1-2\kappa_0^{-2})\varepsilon^2 + \cdots,\\
\frac{\sigma_{13}}{\varepsilon^2\mu_0} &= B_1\left[4(1-\kappa_0^{-2}) - \zeta^2\right] + iB_3\zeta^2(1-2\kappa_0^{-2})\varepsilon\\
&\quad + B_1(1-2\kappa_0^{-2})\left[4(\kappa_0^{-2} - 1) + \zeta^2\right]\varepsilon^2 + \cdots.
\end{aligned} \tag{57}$$

These conditions coincide with the asymptotic expansion of the exact solution (52) only at leading order. This means that the effect of the underlined terms in (6) appears only at next order; in doing so, it is different from $O(\varepsilon)$ correction in the asymptotic expansion (52). As an illustration, in Fig. 3 for $\nu = 0.3$ we plot the normalized coefficients χ_{k3}^E and χ_{k3}^B, $k = 1, 3$, at ε-terms in (52) and (57). They are

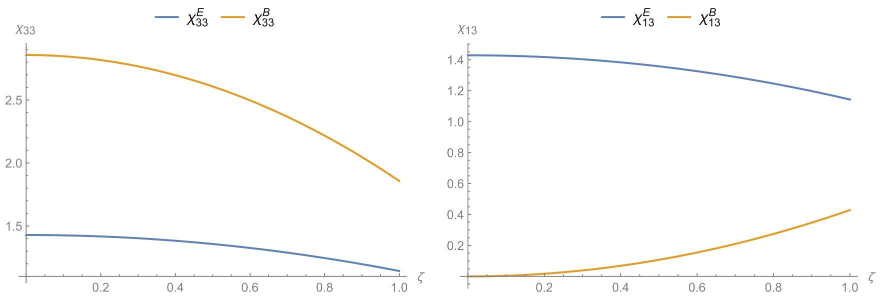

Fig. 3 Comparison of coefficients at ε-terms

$$\begin{aligned} \chi^E_{33} &= 2 - \kappa_0^{-2}(2+\zeta^2), \quad \chi^B_{33} = 4(1-\kappa_0^{-2}) - \zeta^2, \\ \chi^E_{13} &= 2 - \kappa_0^{-2}(2+\zeta^2), \quad \chi^B_{13} = \zeta^2(1-2\kappa_0^{-2}). \end{aligned} \tag{58}$$

5 Conclusion

In this paper, we derive an asymptotic correction to the leading order effective boundary conditions for a coated elastic half-space. The derived conditions are tested by comparison with the exact solution of a plane time-harmonic problem. As a result, the formulation in [6] is validated at leading order, whereas its corrections proposed in [1] appears to be asymptotically inconsistent. The obtained conditions are of general interest for elastodynamics, e.g. for developing refined asymptotic models for surface waves, see [17, 18]. The latter provide a useful framework for modelling coated solids subject to high-speed moving loads, see [19, 20].

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Appendix

The constants in (49) are

$$A_1 = h\frac{N_1}{D}, \quad A_2 = \mathrm{e}^{kh\alpha}h\frac{N_2}{D}, \quad A_3 = -h\frac{N_3}{D}, \quad A_4 = -\mathrm{e}^{kh\beta}h\frac{N_4}{D}, \tag{59}$$

where

$$
\begin{aligned}
N_1 &= i B_1 \left(\mathrm{e}^{kh\alpha}(D_1\alpha\beta + D_2\gamma^4) - 2\mathrm{e}^{kh\beta}\alpha\beta\gamma^2\right)\\
&\qquad - B_3\beta\left(\mathrm{e}^{kh\alpha}(D_2\alpha\beta + D_1\gamma^4) - 2\mathrm{e}^{kh\beta}\gamma^2\right),\\
N_2 &= i B_1 \left(D_1\alpha\beta - 2\mathrm{e}^{kh(\alpha+\beta)}\alpha\beta\gamma^2 - \gamma^4 D_2\right)\\
&\qquad + B_3\beta\left(D_1\gamma^4 - D_2\alpha\beta - 2\mathrm{e}^{kh(\alpha+\beta)}\gamma^2\right),\\
N_3 &= i B_3 \left(\mathrm{e}^{kh\beta}(D_3\alpha\beta + D_4\gamma^4) - 2\mathrm{e}^{kh\alpha}\alpha\beta\gamma^2\right)\\
&\qquad + B_1\alpha\left(\mathrm{e}^{kh\beta}(D_4\alpha\beta + D_3\gamma^4) - 2\mathrm{e}^{kh\alpha}\gamma^2\right),\\
N_4 &= i B_3 \left(D_3\alpha\beta - 2\mathrm{e}^{kh(\alpha+\beta)}\alpha\beta\gamma^2 - \gamma^4 D_4\right)\\
&\qquad - B_1\alpha\left(D_3\gamma^4 - D_4\alpha\beta - 2\mathrm{e}^{kh(\alpha+\beta)}\gamma^2\right),
\end{aligned}
\tag{60}
$$

and

$$
D = k\left[8\mathrm{e}^{kh(\alpha+\beta)}\alpha\beta\gamma^2 + D_2 D_4(\alpha^2\beta^2 + \gamma^4) - D_1 D_3\alpha\beta(1+\gamma^4)\right],
$$

with

$$
D_1 = 1 + \mathrm{e}^{2kh\beta}, \quad D_2 = 1 - \mathrm{e}^{2kh\beta}, \quad D_3 = 1 + \mathrm{e}^{2kh\alpha}, \quad D_4 = 1 - \mathrm{e}^{2kh\alpha}. \tag{61}
$$

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