Chapter 6
Normed Spaces

6.1 Normed Spaces, Subspaces, and Quotient Spaces

6.1.1 Norms. Examples

Let X be a linear space. A mapping $x \mapsto \|x\|$ that associates to each element of the space X a nonnegative number is called a *norm* if it obeys the following axioms:

(1) if $\|x\| = 0$, then $x = 0$ (non-degeneracy);

(2) $\|\lambda x\| = |\lambda| \|x\|$ for all $x \in X$ and all scalars λ;

(3) $\|x + y\| \leqslant \|x\| + \|y\|$ (triangle inequality).

Conditions (2) and (3) show that a norm is a particular case of a convex functional. In connection with this we suggest that the reader return to the exercises in Subsection 5.4.1 and examine which of the properties 1–5 of convex functionals hold in the case of a norm, and also which of the functionals p_i in Exercises 7–15 are norms.

Definition 1. A linear space X endowed with a norm is called a *normed space*.

Let us note that if a linear space X is endowed with some norm, then one has a normed space, but if the same linear space is endowed with another norm, then it already becomes a different normed space. Below we provide examples of normed spaces that will be repeatedly encountered in the sequel. The verification of the norm axioms in these examples is left to the reader.

Examples

1. Let K be a compact topological space. We let $C(K)$ denote the normed space of continuous scalar-valued functions on K with the norm $\|f\| = \max\{|f(t)| : t \in K\}$. An important particular case of the space $C(K)$ is the space $C[a, b]$ of continuous functions on the interval $[a, b]$.

V. Kadets, *A Course in Functional Analysis and Measure Theory*,
Universitext, https://doi.org/10.1007/978-3-319-92004-7_6

2. ℓ_1 is the space of numerical sequences $x = (x_1, x_2, \ldots, x_n, \ldots)$ that satisfy the condition $\sum_{n=1}^{\infty} |x_n| < \infty$, equipped with the norm $\|x\| = \sum_{n=1}^{\infty} |x_n|$. Since any sequence can be regarded as a function defined on the set $\mathbb{N}$ of natural numbers, the space ℓ_1 is a particular case of the space $L_1(\Omega, \Sigma, \mu)$ studied below in Subsection 6.1.3: $\Omega = \mathbb{N}$, Σ is the family of all subsets of $\mathbb{N}$, and μ is the counting measure.

3. ℓ_∞ denotes the space of all bounded numerical sequences with the norm $\|x\| = \sup_n |x_n|$.

4. c_0 is the space of all numerical sequences that tend to zero. The norm on c_0 is given in the same way as on ℓ_∞.

Definition 2. A mapping $x \mapsto p(x)$ that associates to each element of the linear space X a non-negative number is called a *seminorm* if it obeys the norm axioms (2) and (3).

Exercises

1. Give an example of a seminorm on $\mathbb{R}^2$ which is not a norm.

2. Give an example of a convex functional on $\mathbb{R}^2$ that is not a seminorm.

3. Let B be a convex absorbing set in the linear space X. Suppose, in addition, that B is a *balanced set*, i.e., $\lambda B \subset B$ for any scalar λ with $|\lambda| \leqslant 1$. Then the Minkowski functional of B (see Subsection 5.4.2) is a seminorm.

6.1.2 The Metric of a Normed Space and Convergence. Isometries

Let X be a normed space. The *distance between the elements* $x_1, x_2 \in X$ is defined by $\rho(x_1, x_2) = \|x_2 - x_1\|$. From the norm axioms is follows that ρ is indeed a metric on X. Hence, every normed space is simultaneously a metric space, and so all the notions defined in metric spaces — open and closed sets, compact sets, limit points, completeness, etc., — also make sense in normed spaces. In particular, a sequence (x_n) of elements of the normed space X converges to the element x if $\|x_n - x\| \to 0$ as $n \to \infty$. An essential difference in terminology between normed and metric spaces shows up in the definition of isometries: in a normed space one additionally requires that the map in question is linear.

A linear operator T acting from a normed space X to a normed space Y is called an *isometric embedding* if $\|Tx\| = \|x\|$ for all $x \in X$.

A bijective isometric embedding is called an *isometry*. The normed spaces X and Y are said to be *isometric* if there exists an isometry between them.

Exercises

1. Suppose the sequence (x_n) of elements of a normed space converges to the element x. Show that $\|x_n\| \to \|x\|$ as $n \to \infty$.

2. Consider in the space ℓ_1 the elements $x_n = (n^k/(n+1)^{k+1})_{k=1}^{\infty}$. Write in explicit form the coordinates of x_1 and x_2. What are the norms of x_1 and x_2? Calculate the norms $\|x_n\|$ for arbitrary n.

3. Show that convergence in $C(K)$ is the uniform convergence on K. In particular, convergence in $C[a, b]$ is uniform convergence on $[a, b]$, a type of convergence well known from calculus.

4. Show that for any $a < b$ the space $C[a, b]$ is isometric to the space $C[0, 1]$.

5. If the compact spaces K_1 and K_2 are homeomorphic, then the space $C(K_1)$ is isometric to $C(K_2)$. Conversely, if $C(K_1)$ is isometric to $C(K_2)$, then K_1 and K_2 are homeomorphic (this converse is far from trivial).

6. Show that in the space ℓ_1 the convergence of a sequence of vectors $x_n = (x_n^k)_{k=1}^{\infty}$ to a vector $x = (x^k)_{k=1}^{\infty}$ implies the coordinatewise convergence: $x_n^k \to x^k$ as $n \to \infty$, for all $k = 1, 2, \ldots$. On the other hand, coordinatewise convergence does not imply convergence in ℓ_1.

7. The sequence (x_n) in Exercise 2 above can be regarded as a sequence in ℓ_1, and also as one in c_0. What are the norms $\|x_n\|$ in c_0 equal to? Show that the sequence (x_n) converges coordinatewise to 0 and converges to 0 in c_0, but does not converge in ℓ_1.

8. Let X be some sequence space. The *positive cone* in X is defined to be the set of vectors of X all the coordinates of which are non-negative. Consider the three cases $X = c_0$, $X = \ell_1$, and $X = \ell_\infty$. In each of them prove that the positive cone is closed and convex, and describe its interior and boundary.

6.1.3 The Space L_1

Let (Ω, Σ, μ) be a (finite or infinite) measure space, E the linear space of all μ-integrable scalar-valued functions on Ω, and F the subspace of E consisting of all functions that vanish almost everywhere. By $L_1(\Omega, \Sigma, \mu)$ we denote the quotient space E/F. The analogous quotient space $L_0(\Omega, \Sigma, \mu)$ was mentioned in Subsection 5.2.2. To simplify the terminology, one usually says that the elements of the space $L_1(\Omega, \Sigma, \mu)$ are functions integrable on Ω, with the understanding that two functions that coincide almost everywhere are identified. The norm in $L_1(\Omega, \Sigma, \mu)$ is given by the formula $\|f\| = \int_\Omega |f(t)| d\mu$. An important particular case of the space

$L_1(\Omega, \Sigma, \mu)$ is the space $L_1[a, b]$ of Lebesgue-integrable functions on an interval $[a, b]$. In this case $\Omega = [a, b]$, Σ is the family of all Lebesgue-measurable subsets of the interval, and μ is the Lebesgue measure.

Exercises

1. Show that $L_1(\Omega, \Sigma, \mu)$ is a normed space.

2. Show that for any $a < b$ the space $L_1[a, b]$ is isometric to the space $L_1[0, 1]$.

3. Show that the space $L_1[0, 1]$ is isometric to the space $L_1(-\infty, +\infty)$.

4. Show that the space $L_1[0, 1]$ is isometric to the space $L_1([0, 1] \times [0, 1])$.

5. The convergence of a sequence of functions in $L_1(\Omega, \Sigma, \mu)$ implies its convergence in measure, but if the measure is not purely atomic (a typical example is the space $L_1[a, b]$), then convergence in $L_1(\Omega, \Sigma, \mu)$ does not imply convergence almost everywhere.

6. If (Ω, Σ, μ) is a finite measure space and a sequence of integrable functions converges uniformly on Ω, then this sequence also converges in $L_1(\Omega, \Sigma, \mu)$.

7. Show that regardless of what norm the space $L_1[a, b]$ is endowed with, convergence in this norm cannot coincide with convergence in measure. (Compare with Exercise 6 in Subsection 4.3.3.)

8. Consider the positive cone in $L_1(\Omega, \Sigma, \mu)$, that is, the set G of all functions from $L_1(\Omega, \Sigma, \mu)$ that are almost everywhere greater than or equal to zero. Show that G is a closed convex set that has no interior points.

9. By analogy with the preceding exercise, consider the positive cone in $C(K)$. Show that this set is convex and closed, and describe its interior and boundary.

Let p be a seminorm on the space X. The *kernel* of the seminorm p is the set $\operatorname{Ker} p$ of all points $x \in X$ such that $p(x) = 0$.

10. $\operatorname{Ker} p$ is a linear subspace of X.

11. The expression $\rho(x_1, x_2) = p(x_2 - x_1)$ defines a pseudometric on X.

12. Show that for any $x \in X$ and any $y \in \operatorname{Ker} p$, we have $p(x + y) = p(x)$.

13. The expression $\|[x]\| = p(x)$ defines a norm on the quotient space $X/\operatorname{Ker} p$.

Since the expression $p(f) = \int_\Omega |f(t)| d\mu$ gives a seminorm on the linear space E of all scalar-valued μ-integrable functions on Ω, $F = \operatorname{Ker} p$ is the subspace of E consisting of all functions that vanish almost everywhere, the definition given above for the space $L_1(\Omega, \Sigma, \mu)$ is a particular case of the construction described in Exercises 10–13 of this subsection.

6.1.4 Subspaces and Quotient Spaces

A linear subspace Y of the normed space X, equipped with the norm of X, is called *subspace of the normed space* X. Hence, any subspace of a normed space is itself a normed space.

Let Y be a closed subspace of the normed space X, $x \in X$ an arbitrary element, and $[x] = x + Y$ the corresponding element of the quotient space X/Y. Define $\|[x]\| = \inf_{y \in Y} \|x + y\|$. In other words, $\|[x]\|$ is the distance in X from 0 to the set $x + Y$. Since Y is a subspace, and hence $Y = -Y$, the original definition is equivalent to the following one: $\|[x]\| = \inf_{y \in Y} \|x - y\|$. The geometric meaning of the latter is that $\|[x]\|$ is the distance in X from x to the subspace Y.

Proposition 1. *The expression* $\|[x]\|$ *introduced above gives a norm on the space* X/Y.

Proof. Let us verify the norm axioms.

1. Suppose $\|[x]\| = 0$. Then $\inf_{y \in Y} \|x - y\| = 0$, and so x is a limit point of the subset Y. Since Y is closed, $x \in Y$ and $[x] = Y = [0]$.

2. Since Y is a subspace, $\lambda Y = Y$ for any nonzero scalar λ. We have $\|[\lambda x]\| = \inf_{y \in Y} \|\lambda x + y\| = \inf_{y \in Y} \|\lambda x + \lambda y\| = |\lambda| \inf_{y \in Y} \|x + y\| = |\lambda| \cdot \|[x]\|$.

3. Let $x_1, x_2 \in X$ and $\varepsilon > 0$. By the definition of the infimum, there exist y_1, $y_2 \in Y$, such $\|x_1 + y_1\| < \|[x_1]\| + \varepsilon$ and $\|x_2 + y_2\| < \|[x_2]\| + \varepsilon$. It follows that

$$\begin{aligned}\|[x_1 + x_2]\| &= \inf_{y \in Y} \|x_1 + x_2 + y\| \leqslant \|x_1 + x_2 + y_1 + y_2\| \\ &\leqslant \|x_1 + y_1\| + \|x_2 + y_2\| \leqslant \|[x_1]\| + \|[x_2]\| + 2\varepsilon,\end{aligned}$$

which in view of the arbitrariness of ε means that the needed triangle inequality holds.

Henceforth we will always assume that the quotient space of a normed space is equipped with the norm described above.

Example Let (Ω, Σ, μ) be a measure space, X the space of all bounded measurable functions on Ω, endowed with the norm $\|f\| = \sup_{t \in \Omega} |f(t)|$, and Y the subspace of X consisting of the functions that vanish almost everywhere. The corresponding quotient space X/Y is denoted by $L_\infty(\Omega, \Sigma, \mu)$.

Exercises

1. Prove the following formula for the norm in $L_\infty(\Omega, \Sigma, \mu)$:

$$\|f\|_\infty = \inf_{A \in \Sigma,\, \mu(A) = 0} \left\{ \sup_{t \in \Omega \setminus A} |f(t)| \right\}.$$

2. Prove the inequality $|f| \overset{\text{a.e.}}{\leqslant} \|f\|_\infty$.

3. Show that $\|f\|_\infty$ is equal to the infimum of the set of all constants c such that $|f| \overset{\text{a.e.}}{\leqslant} c$.

4. In the space $C[a, b]$ consider the subspace Y consisting of the constants (i.e., constant functions). Show that the norm of the element $[f]$ of the quotient space $C[a, b]/Y$ can be calculated by the formula

$$\|[f]\| = \frac{1}{2}\Big(\max\{f(t) : t \in [a, b]\} - \min\{f(t) : t \in [a, b]\}\Big).$$

5. The space ℓ_1 can be regarded as a linear subspace of c_0, though it will not be a normed subspace of c_0: the norm given on ℓ_1 does not coincide with the norm on c_0. Show that ℓ_1 is not closed and is dense in c_0. Show that as a subset of c_0, ℓ_1 belongs to the class F_σ.

6. Show that the space c_0 of all sequences that converge to zero is closed in ℓ_∞.

7. Show that the norm of the element $[a]$ in the space ℓ_∞/c_0 is calculated by the formula $\|[a]\| = \varlimsup\limits_{n\to\infty} |a_n|$, where a_n are the coordinates of the element $a \in \ell_\infty$.[1]

6.2 Connection Between the Unit Ball and the Norm. L_p Spaces

6.2.1 Properties of Balls in a Normed Space

Let X be a normed space, $x_0 \in X$, and $r > 0$. As usual, we denote by $B_X(x_0, r)$ the open ball of radius r centered at x_0:

$$B_X(x_0, r) = \{x \in X : \|x - x_0\| < r\}.$$

The *unit ball* B_X in the space X is the open ball of unit radius centered at zero: $B_X = \{\, x \in X : \|x\| < 1 \,\}$. The *unit sphere* S_X and *closed unit ball* $\overline{B}_X$ are similarly defined as

$$S_X = \{\, x \in X : \|x\| = 1 \,\}, \quad \text{and} \quad \overline{B}_X = \{\, x \in X : \|x\| \leqslant 1 \,\}.$$

Let us list some of the simplest properties of the objects just introduced.

[1] In Soviet times, one of the Kharkiv newspapers published a paper on the fulfillment of the production plan by highly productive workers ("peredoviks"), entitled "The [production] norm is not a limit!". The last assertion above can be considered as a counterexample to this assertion.

— The unit ball is an open set, while the unit sphere and the closed unit ball are closed sets.

— $B_X(x_0, r) = x_0 + rB_X$.

— B_X is a convex absorbing set (see Exercise 2 in Subsection 5.4.2).

— B_X is a balanced set, i.e., for any scalar λ such that $|\lambda| \leqslant 1$, we have $\lambda B_X \subset B_X$.

— For any $x_0 \in X$ and $r > 0$, the linear span of the ball $B_X(x_0, r)$ coincides with the whole space X.

Exercises

1. Prove that the closure of the open ball $B_X(x_0, r)$ in a normed space is the closed ball $\overline{B}_X(x_0, r)$. Compare with Exercise 10 in Subsection 1.3.1.

2. The space of numerical rows $x = (x_1, x_2, \ldots, x_n)$ with the norm $\|x\| = \sum_{k=1}^{n} |x_n|$ is denoted by ℓ_1^n; the analogous space of rows with the norm $\|x\| = \sup_n |x_n|$ is denoted by ℓ_∞^n. The spaces ℓ_1^n and ℓ_∞^n are finite-dimensional analogues of the spaces ℓ_1 and ℓ_∞. Construct in the coordinate plane the unit balls of the spaces ℓ_1^2 and ℓ_∞^2. Exhibit an isometry between these two spaces.

3. Construct in the three-dimensional coordinate space the unit balls of the spaces ℓ_1^3 and ℓ_∞^3. Show that these normed spaces are not isometric.

4. Nested balls principle. Let X be a complete normed space, and $B_n = \overline{B}_X(x_n, r_n)$ be a decreasing (with respect to inclusion) sequence of closed balls. Show that the intersection $\bigcap_{n=1}^{\infty} B_n$ is not empty. (In contrast to the nested sets principle, here one does not assume that the diameters of the balls tend to zero, but neither does one assert that the intersection consists of a single point.)

5. Give an example of a complete metric space in which the assertion of the preceding exercise is not true.

6.2.2 Definition of the Norm by Means of a Ball. The Spaces L_p

Let B be a convex absorbing set in the linear space X. Recall (see Subsection 5.4.2) that the Minkowski functional of the set B is the function on X given by the formula $\varphi_B(x) = \inf\left\{t > 0 : \ t^{-1}x \in B\right\}$.

Theorem 1. *Let B be a convex, absorbing, balanced set in the space X which also has the following* algebraic boundedness *property: for each $x \in X \setminus \{0\}$ there exists an $a > 0$ such that $ax \notin B$. Then the Minkowski functional φ_B gives a norm on X.*

Proof. The fact that φ_B is a convex functional was already established in Subsection 5.4.2. Since the set B is balanced, $\varphi_B(\lambda x) = \varphi_B(|\lambda| x) = |\lambda|\varphi_B(x)$ for all $x \in X$ and all scalars λ, i.e., φ_B is a seminorm. Finally, if $x \in X \setminus \{0\}$, then thanks to the algebraic boundedness there exists an $a > 0$ such that $ax \notin B$. Hence, $\varphi_B(x) \geqslant \frac{1}{a} > 0$, which establishes the non-degeneracy of the Minkowski functional. □

Let (Ω, Σ, μ) be a (finite or not) measure space, and $p \in [1, \infty)$ a fixed number. We denote by $L_p(\Omega, \Sigma, \mu)$ the subset of the space $L_0(\Omega, \Sigma, \mu)$ of all measurable scalar-valued functions on Ω consisting of the functions for which the integral $\int_\Omega |f(t)|^p d\mu$ exists. Here, as in the case of the space $L_0(\Omega, \Sigma, \mu)$, functions in $L_p(\Omega, \Sigma, \mu)$ that are equal almost everywhere are regarded as one and the same element. For $f \in L_p(\Omega, \Sigma, \mu)$, we put $\|f\|_p = \left(\int_\Omega |f(t)|^p d\mu\right)^{1/p}$.

Theorem 2. *$L_p(\Omega, \Sigma, \mu)$ is a linear space, and $\|\cdot\|_p$ is a norm on $L_p(\Omega, \Sigma, \mu)$.*

Proof. Consider the set $B_p \subset L_p(\Omega, \Sigma, \mu)$ consisting of all functions for which $\int_\Omega |f(t)|^p d\mu < 1$. Let $f, g \in B_p$ and $\lambda \in [0, 1]$. Since the function $|x|^p$ is convex on $\mathbb{R}$, for any $t \in \Omega$ we have the numerical inequality

$$|\lambda f(t) + (1-\lambda)g(t)|^p \leqslant \lambda |f(t)|^p + (1-\lambda)|g(t)|^p.$$

Integrating this inequality we conclude that $\lambda f + (1-\lambda)g \in B_p$, and so B_p is a convex set. It is readily verified that the set B_p is balanced and algebraically bounded. From the fact that B_p is convex and balanced and the obvious equality $L_p(\Omega, \Sigma, \mu) = \bigcup_{n=1}^\infty nB_p$ it follows that $L_p(\Omega, \Sigma, \mu)$ is a linear space and B_p is an absorbing set in $L_p(\Omega, \Sigma, \mu)$ (Exercise 1 in Subsection 5.4.2). Consequently, the Minkowski functional of the set B_p is defined on $L_p(\Omega, \Sigma, \mu)$ and gives a norm on this linear space. It remains to observe that $\|\cdot\|_p$ coincides with φ_{B_p}. Indeed, for any $f \in L_p(\Omega, \Sigma, \mu)$, $\frac{1}{t} f \in B_p$ if and only if $t > \|f\|_p$, i.e., $\|f\|_p = \varphi_{B_p}(f)$. □

In what follows, $L_p(\Omega, \Sigma, \mu)$ will be regarded as a normed space equipped with the norm $\|\cdot\|_p$. Important particular cases are the spaces $L_p[a, b]$ (i.e., the case $\Omega = [a, b]$ with the Lebesgue measure) and the spaces ℓ_p, where the role of Ω is played by $\mathbb{N}$, $\Sigma = 2^{\mathbb{N}}$, and μ is the counting measure (the measure of a set is the number of its elements). Since every function defined on the set $\mathbb{N}$ of natural numbers can be regarded as a sequence, ℓ_p is usually defined as the space of numerical sequences $\mathrm{x} = (x_k)_{k\in\mathbb{N}}$ that satisfy the condition $\sum_{k=1}^\infty |x_k|^p < \infty$, equipped with the norm $\|\mathrm{x}\|_p = \left(\sum_{k=1}^\infty |x_k|^p\right)^{1/p}$.

Exercises

1. Suppose the linear space X is endowed with two norms, $\|\cdot\|_1$ and $\|\cdot\|_2$, and let B_1 and B_2 be the corresponding unit balls. Then $B_1 \subset B_2$ if and only if the inequality $\|\cdot\|_1 \geqslant \|\cdot\|_2$ holds in X.

2. Suppose the linear space X is endowed with three norms, $\|\cdot\|_1$, $\|\cdot\|_2$, $\|\cdot\|_3$, and let B_1, B_2, and and B_3 be the corresponding unit balls. Suppose $\|\cdot\|_3$ is expressed in terms of $\|\cdot\|_1$ and $\|\cdot\|_2$ as $\|x\|_3 = \max\{\|x\|_1, \|x\|_2\}$. Then $B_3 = B_1 \cap B_2$.

3. ℓ_p, regarded as a set, increases with the growth of p, while for fixed x the norm $\|x\|_p$ decreases with the growth of p.

4. The set ℓ_0 of terminating (finitely supported) sequences (i.e., sequences in which, starting with some index, all terms are equal to 0) is dense in ℓ_p for any $p \in [1, \infty)$.

5. If $p_1 < p$, then the set ℓ_{p_1} is dense in the space ℓ_p.

6. Let B be a convex, absorbing, balanced, and algebraically bounded set in the normed space X. Endow X with the norm defined by the Minkowski functional of B. In order for the unit ball of this norm to coincide with B it is necessary and sufficient that B have the following property: for every $x \in B$, there exists an $\varepsilon > 0$, such that $(1 + \varepsilon)x \in B$.

7. For $1 \leqslant p < \infty$, the set of bounded functions is dense in $L_p[a, b]$.

8. For $1 \leqslant p < \infty$, the set of continuous functions is dense in $L_p[a, b]$.

9. For $1 \leqslant p < \infty$, the set of all polynomials is dense in $L_p[a, b]$.

10. For $1 \leqslant p < \infty$, the set of continuous functions satisfying the condition $f(0) = 0$ is dense in $L_p[0, b]$.

11. The set of continuous functions is not dense $L_\infty[a, b]$.

6.3 Banach Spaces and Absolutely Convergent Series

A *Banach space* is a complete normed space, i.e., a normed space in which every Cauchy sequence converges. Banach spaces constitute the most important class of normed spaces: they are the spaces most often encountered in applications, and many of the most important results of functional analysis revolve around the notion of Banach space.[2]

6.3.1 Series. A Completeness Criterion in Terms of Absolute Convergence

Let (x_n) be a sequence of elements of the normed space X. The *partial sums* of the series $\sum_{n=1}^{\infty} x_n$ are the vectors $s_n = \sum_{k=1}^{n} x_k$. If the partial sums of the series

[2] At least in the opinion of the author of these lines, who specializes in the theory of Banach spaces.

$\sum_{n=1}^{\infty} x_n$ converge to an element x, the series is said to *converge* and the element x is called the *sum* of the series. The equality $\sum_{n=1}^{\infty} x_n = x$ is the generally adopted short way of writing that "the series $\sum_{n=1}^{\infty} x_n$ converges and its sum is equal to x". The series $\sum_{n=1}^{\infty} x_n$ is called *absolutely convergent* if $\sum_{n=1}^{\infty} \|x_n\| < \infty$.

Proposition 1 (Cauchy convergence criterion for series). *For the series $\sum_{n=1}^{\infty} x_n$ of elements of a Banach space X to converge it is necessary and sufficient that $\left\|\sum_{k=n}^{m} x_k\right\| \to 0$ as $n, m \to \infty$.*

Proof. Convergence of a series is equivalent to convergence of the sequence of its partial sums s_n. In turn, in a complete space convergence of a sequence is equivalent to the sequence being Cauchy. It remains to note that $s_m - s_n = \sum_{k=n+1}^{m} x_k$. $\square$

Proposition 2. *Suppose the series $\sum_{n=1}^{\infty} x_n$ of elements of the Banach space X converges absolutely. Then the series $\sum_{n=1}^{\infty} x_n$ converges.*

Proof. Since the numerical series $\sum_{n=1}^{\infty} \|x_n\|$ converges, $\sum_{k=n}^{m} \|x_k\| \to 0$ as $n, m \to \infty$. Consequently, $\|\sum_{k=n}^{m} x_k\| \leqslant \sum_{k=n}^{m} \|x_k\| \to 0$ as $n, m \to \infty$. To complete the proof, it remains to apply Proposition 1. $\square$

Proposition 3. *Let X be normed space that is not complete. Then in X there exists an absolutely convergent, but not convergent series.*

Proof. Since X is not complete, there exists a Cauchy sequence $v_n \in X$ which does not have a limit. By the definition of a Cauchy sequence, $\|v_n - v_m\| \to 0$ as $n, m \to \infty$. It follows that there exists an $n_1 \in \mathbb{N}$ such that $\|v_n - v_m\| < \frac{1}{2}$ for all $n, m \geqslant n_1$. Analogously, pick an $n_2 \geqslant n_1$ such that $\|v_n - v_m\| < \frac{1}{4}$ for all $n, m \geqslant n_2$. Continuing this argument, we obtain an increasing sequence of indices n_j such that $\|v_n - v_m\| < \frac{1}{2^j}$ for all $n, m \geqslant n_j$. Then for the sequence v_{n_j} it holds that

$$\|v_{n_2} - v_{n_1}\| < \frac{1}{2}, \ \|v_{n_3} - v_{n_2}\| < \frac{1}{4}, \ \ldots, \ \|v_{n_{j+1}} - v_{n_j}\| < \frac{1}{2^j}, \ \ldots .$$

Now we define the sought-for series $\sum_{j=1}^{\infty} x_j$ by $x_1 = v_{n_1}$, $x_2 = v_{n_2} - v_{n_1}$, ..., $x_j = v_{n_j} - v_{n_{j-1}}$, and so on. The constructed series is absolutely convergent: $\sum_{j=2}^{\infty} \|x_j\| < \frac{1}{2} + \frac{1}{4} + \cdots = 1$. At the same time, its partial sums are equal to v_{n_j}, and so they form (see Exercise 1 in Subsection 1.3.3) a divergent sequence. $\square$

Propositions 2 and 3 provide the following characterization of complete normed spaces.

Theorem 1. *For the normed space X to be complete it is necessary and sufficient that every absolutely convergent series in X be convergent.* $\square$

6.3.2 *Completeness of the Space* L_1

Sobriety is a life norm …
True, but is life complete with this norm?[3]

We begin by proving a reformulation of Levi's theorem, essentially stated above in Exercise 3 of Subsection 4.4.3.

Lemma 1. *Suppose the series* $\sum_{n=1}^{\infty} f_n$ *of functions from* $L_1 = L_1(\Omega, \Sigma, \mu)$ *converges absolutely in the norm of this space. Then the series* $\sum_{n=1}^{\infty} f_n$ *converges almost everywhere to an integrable function* f *and* $\|f\| \leqslant \sum_{n=1}^{\infty} \|f_n\|$.

Proof. By the definition of the norm in L_1, the absolute convergence means that $\sum_{n=1}^{\infty} \int_\Omega |f_n| d\mu < \infty$. By Levi's theorem, the series $\sum_{n=1}^{\infty} |f_n|$ converges almost everywhere to an integrable function g, and $\int_\Omega g\, d\mu = \sum_{n=1}^{\infty} \int_\Omega |f_n| d\mu$. Denote the set of measure 0 in the complement of which $\sum_{n=1}^{\infty} |f_n|$ converges by A. For each point $t \in \Omega \setminus A$, the numerical series $\sum_{n=1}^{\infty} f_n(t)$ converges absolutely to some number $f(t)$. Thus, we defined on $\Omega \setminus A$ (i.e., almost everywhere on Ω) a function f, and the series $\sum_{n=1}^{\infty} f_n$ converges to f at all points of $\Omega \setminus A$. Extend f to the set A by 0. The function f is measurable on $\Omega \setminus A$, being the pointwise limit of a sequence of measurable functions; moreover, f has an integrable majorant, namely, the function g. Hence, f is integrable and

$$\int_\Omega |f| d\mu \leqslant \int_\Omega g\, d\mu = \sum_{n=1}^{\infty} \int_\Omega |f_n| d\mu. \qquad \square$$

Theorem 1. *L_1 is a Banach space.*

Proof. We use the theorem of the preceding subsection, i.e., the completeness criterion in terms of absolute convergence. Suppose the series $\sum_{n=1}^{\infty} f_n$ of L_1-functions converges absolutely. By the preceding lemma, $\sum_{n=1}^{\infty} f_n$ converges almost everywhere to an integrable function f. We claim that the series $\sum_{n=1}^{\infty} f_n$ converges to f in the norm of the space L_1. Indeed, $\|f - \sum_{n=1}^{k} f_n\| = \|\sum_{n=k}^{\infty} f_n\| \leqslant \sum_{n=k}^{\infty} \|f_n\| \to 0$ as $k \to \infty$. $\square$

Exercise

Prove the completeness of the space L_p.

The completeness of the space L_p will be established later, in Chap. 14, by an indirect argument. Nevertheless, the reader will profit from finding a direct proof of this fact.

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[3] A joke from the times of the Gorbachev anti-alcoholism campaign in Soviet Union, 1985–1990. Quoted from a toast given by Ya.G. Prytula at the banquet for the International Conference on Functional Analysis and its Applications dedicated to the 110th anniversary of Stefan Banach, May 28–31, 2002, Lviv, Ukraine.

6.3.3 Subspaces and Quotient Spaces of Banach Spaces

Let X be a Banach space. A linear subspace $Y \subset X$, equipped with the norm of X, is called a *subspace of the Banach space* X if Y is closed in X. Hence, a subspace of a Banach space is itself a Banach space. As the reader had undoubtedly noticed, the meaning of the term "subspace" depends on where this subspace is considered. Since a Banach space is simultaneously a metric as well as a linear and normed space, the term "subspace" is somewhat overloaded. For this reason we emphasize once again that in Banach spaces subspaces will be tacitly understood to be closed linear subspaces. If for some reason we need to consider a non-closed linear subspace, we will state explicitly that the subspace is not closed.

Theorem 1. *Let X be a Banach space and Y be a subspace of X. Then the quotient space X/Y is also a Banach space.*

Proof. Let $x_n \in X$ be such that the corresponding equivalence classes form an absolutely convergent series: $\sum_n \|[x_n]\| < \infty$. By the completeness criterion, we need to prove that the series $\sum_n [x_n]$ converges to some element of the quotient space. To this end we pick in each class $[x_n]$ a representative y_n such that $\|y_n\| \leqslant \|[x_n]\| + \frac{1}{2^n}$. Then $\sum_n y_n$ is an absolutely convergent series in X, which in view of the completeness of X means that the series $\sum_n y_n$ converges in X to some element x. We claim that $\sum_n [y_n] = [x]$. Indeed,

$$\left\| [x] - \sum_{k=1}^{n} [x_k] \right\| = \left\| [x] - \sum_{k=1}^{n} [y_k] \right\| = \left\| [x - \sum_{k=1}^{n} y_k] \right\| \leqslant \left\| x - \sum_{k=1}^{n} y_k \right\| \to 0$$

as $n \to \infty$. □

Exercises

12. Let X be a Banach space, and let $x_n \in X$ be a fixed sequence of nonzero vectors. We introduce the space E of all numerical sequences $a = (a_n)_1^\infty$ for which the series $\sum_{n=1}^\infty a_n x_n$ converges. We endow the space E with the norm $\|a\| = \sup\{\|\sum_{n=1}^N a_n x_n\| : N = 1, 2, \ldots\}$. Verify that E is a Banach space.

13. Let X be a Banach space, Y a nontrivial subspace of X (i.e., Y is closed and $Y \neq X$). Prove that Y is nowhere dense in X.

14. Show that a Banach space cannot be represented as a countable union of nontrivial subspaces.

15. Show that a Hamel basis of an infinite-dimensional Banach space is not countable.

16. Let $\mathcal{P}$ be the space of all polynomials (of arbitrarily large degree) with real coefficients, equipped with the norm $\|a_0 + a_1 t + \cdots + a_n t^n\| = |a_0| + |a_1| \cdots + |a_n|$. Is $\mathcal{P}$ complete?

17. Denote by $\{e_n\}_1^\infty$ the *standard basis* of the space ℓ_1: $e_1 = (1, 0, 0, \ldots)$, $e_2 = (0, 1, 0, \ldots)$,.... Show that for every $a = (a_n)_1^\infty \in \ell_1$ the series $\sum_{n=1}^\infty a_n e_n$ converges to a. Is the convergence absolute?

18. Consider in ℓ_∞ the sequence $\{e_n\}_1^\infty$ from the previous exercise. What are the partial sums of the series $\sum_{n=1}^\infty e_n$ equal to? Does this series converge to the element $\mathrm{x} = (1, 1, \ldots) \in \ell_\infty$? Describe the elements $a = (a_n)_1^\infty \in \ell_\infty$ for which the series $\sum_{n=1}^\infty a_n e_n$ converges to a. For which a will the convergence be absolute?

19. Prove that the space $L_\infty(\Omega, \Sigma, \mu)$ is complete.

20. Prove that in each of the spaces $L_p(\Omega, \Sigma, \mu)$ with $1 \leqslant p \leqslant \infty$, the subspace of finite-valued measurable functions is dense.

21. The space ℓ_p with $1 \leqslant p < \infty$ is separable, whereas ℓ_∞ is not separable.

6.4 Spaces of Continuous Linear Operators

6.4.1 A Continuity Criterion for Linear Operators

Definition 1. Let X and Y be normed spaces. A linear operator $T : X \to Y$ is said to be *bounded* if it maps bounded sequences into bounded sequences. In other words, if $x_n \in X$ and $\sup_n \|x_n\| < \infty$ imply $\sup_n \|T x_n\| < \infty$.

The main purpose of this subsection is to prove that for a linear operator continuity and boundedness are equivalent.

Theorem 1. *Let X and Y be normed spaces. For a linear operator $T : X \to Y$ the following conditions are equivalent:*

(1) *T is continuous;*

(2) *T maps sequences that converge to zero into sequences that converge to zero;*

(3) *T maps sequences that converge to zero into bounded sequences;*

(4) *T is bounded.*

Proof. The implications (1) $\Longrightarrow$ (2) $\Longrightarrow$ (3) $\Longleftarrow$ (4) are obvious: indeed, condition (2), i.e., the continuity of the operator at zero, is a particular case of condition (1); condition (3) follows from (2) as well as from (4), because sequences that converge to zero are bounded. Now let us prove the converse implications.

(2) $\Longrightarrow$ (1). Suppose the sequence of vectors $x_n \in X$ converges to the vector $x \in X$. Then $x_n - x \to 0$ as $n \to \infty$, so by condition (2), $Tx_n - Tx = T(x_n - x) \to 0$ as $n \to \infty$. That is, convergence of x_n to x implies convergence of Tx_n to Tx.

(3) $\Longrightarrow$ (2). We proceed by reductio ad absurdum. Suppose condition (2) is not satisfied: there exists a sequence (x_n) in X which converges to zero, but such that Tx_n does not converge to zero. Then one can extract from (x_n) a subsequence, denoted (v_n), for which $\inf_n \|Tv_n\| = \varepsilon > 0$. Consider the vectors $w_n = \frac{1}{\sqrt{\|v_n\|}} v_n$. The sequence w_n still converges to 0, but $\|Tw_n\| \geqslant \frac{\varepsilon}{\sqrt{\|v_n\|}} \to \infty$, which contradicts condition (3).

(3) $\Longrightarrow$ (4). Suppose condition (4) is not satisfied: there exists a bounded sequence (x_n) in X such that $\sup_n \|Tx_n\| = \infty$. Then one can extract from (x_n) a subsequence, denoted (v_n), for which $\|Tv_n\| \to \infty$. Consider the vectors $w_n = \frac{1}{\sqrt{\|Tv_n\|}} v_n$. The sequence (w_n) already converges to 0, but $\|Tw_n\| = \sqrt{\|Tv_n\|} \to \infty$, which contradicts condition (3). □

Exercises

1. Let X and Y be normed spaces, $T\colon X \to Y$ a continuous linear operator. Then $\operatorname{Ker} T = T^{-1}(0)$ is a closed linear subspace in X. (**N.B.!**) This is a simple yet important fact, and in the sequel will be used without further clarifications.

2. The image (range) of a continuous operator is not necessarily closed. Examine this in the case of the integration operator $T\colon C[0, 1] \to C[0, 1]$, $(Tf)(t) = \int_0^t f(\tau)d\tau$.

6.4.2 The Norm of an Operator

The *norm* of the linear operator T, acting from the normed space X into the normed space Y, is defined as

$$\|T\| = \sup_{x \in S_X} \|Tx\|.$$

Proposition 1. *Let $\|T\| < \infty$. Then $\|Tx\| \leqslant \|T\| \cdot \|x\|$ for any $x \in X$.*

Proof. For $x = 0$ the inequality holds trivially. Now let $x \neq 0$. Since $x/\|x\| \in S_X$, we have $\|T(x/\|x\|)\| \leqslant \|T\|$. Therefore, $\|Tx\| = \|x\| \cdot \|T(x/\|x\|)\| \leqslant \|T\| \cdot \|x\|$, as claimed. □

Proposition 2. *Let X and Y be normed spaces. For a linear operator $T\colon X \to Y$ the following conditions are equivalent:*

(1) *T is bounded*;

(2) $\|T\| < \infty$;

(3) *there exists a constant $C > 0$ such that $\|Tx\| \leqslant C\|x\|$ for all $x \in X$.*

Proof. (1) $\Longrightarrow$ (2). Suppose $\|T\| = \infty$. Then for any positive integer n there exists a vector $x_n \in S_X$ such that $\|Tx_n\| > n$. The sequence (x_n) is bounded, and the images of its terms tend in norm to infinity. This contradicts condition (1). The implication (2) $\Longrightarrow$ (3) was proved in Proposition 1 (with $C = \|T\|$). It remains to show that (3) $\Longrightarrow$ (1). Let $x_n \in X$ be a bounded sequence: $\|x_n\| \leqslant K$ for some constant K. Then, by condition (3), $\|Tx_n\| \leqslant CK$ for all n. Hence, the operator T maps bounded sequences into bounded sequences, as we needed to prove. □

Remark 1. If condition (3) of the preceding theorem is satisfied, then

$$\|T\| = \sup_{x \in S_X} \|Tx\| \leqslant \sup_{x \in S_X} C\|x\| = C.$$

That is, if $\|Tx\| \leqslant C\|x\|$ for all $x \in X$, then $\|T\| \leqslant C$. This observation is often used in the estimation of norms of operators.

Remark 2. In the literature one encounters quite a few equivalent definitions of the norm of an operator:

— $\|T\| = \sup_{x \in B_X} \|Tx\|$;

— $\|T\| = \sup_{x \in \overline{B}_X} \|Tx\|$;

— $\|T\| = \sup_{x \in X \setminus \{0\}} \dfrac{\|Tx\|}{\|x\|}$;

— $\|T\|$ is the infimum of all constants $C \geqslant 0$ such that the inequality $\|Tx\| \leqslant C\|x\|$ is satisfied for all $x \in X$.

The verification of the equivalence of these definitions is left to the reader.

We let $L(X, Y)$ denote the space of all continuous linear operators acting from the normed space X into the normed space Y. $L(X, Y)$ is naturally endowed with linear operations: if $T_1, T_2 \in L(X, Y)$ are operators and λ_1, λ_2 are scalars, then the operator $\lambda_1 T_1 + \lambda_2 T_2 \in L(X, Y)$ acts according to the rule $(\lambda_1 T_1 + \lambda_2 T_2)\, x = \lambda_1 T_1 x + \lambda_2 T_2 x$. We described above how to introduce a norm on $L(X, Y)$ — the norm of the operator, but it remains to verify that the norm axioms are indeed satisfied.

Proposition 3. *The space $L(X, Y)$ of continuous operators is a normed space.*

Proof. Let us verify the norm axioms (Subsection 6.1.1).

1. Suppose $\|T\| = 0$. Then the operator T is equal to 0 on all elements of the unit sphere of the space X, which in view of its linearity means that T is equal to zero on the entire space X.

2. $\|\lambda T\| = \sup\limits_{x \in S_X} \|\lambda T x\| = |\lambda| \sup\limits_{x \in S_X} \|T x\| = |\lambda| \, \|T\|$.

3. Let $T_1, T_2 \in L(X, Y)$ and $x \in X$. By Proposition 1,

$$\|(T_1 + T_2)x\| \leqslant \|T_1 x\| + \|T_2 x\| \leqslant \|T_1\| \cdot \|x\| + \|T_2\| \cdot \|x\| = (\|T_1\| + \|T_2\|) \cdot \|x\|.$$

By Remark 1, this yields the needed triangle inequality: $\|T_1 + T_2\| \leqslant \|T_1\| + \|T_2\|$.
□

The norm of an operator is an important concept that will be frequently used in our text. For this reason the reader who has no experience working with norms is strongly advised to seriously pay attention to the exercises given below.

Exercises

1. Let $T \in L(X, Y)$ and $x_1, x_2 \in X$. Then $\|T x_1 - T x_2\| \leqslant \|T\| \cdot \|x_1 - x_2\|$.

2. Let $T_1, T_2 \in L(X, Y)$ and $x \in X$. Then $\|T_1 x - T_2 x\| \leqslant \|T_1 - T_2\| \cdot \|x\|$.

3. Let X, Y, Z be normed spaces, $T_1 \in L(X, Y)$, $T_2 \in L(Y, Z)$. Prove the *multiplicative triangle inequality* for the composition of operators: $\|T_2 \circ T_1\| \leqslant \|T_2\| \cdot \|T_1\|$.

4. Let X be a Banach space, $(x_n)_1^\infty$ a bounded sequence in X, and $\{e_n\}_1^\infty$ the standard basis in the space ℓ_1 (see Exercise 6 in Subsection 6.3.3). Define the operator $T : \ell_1 \to X$ by the formula $Ta = \sum_{n=1}^\infty a_n x_n$, for any element $a = (a_n)_1^\infty$ of the space ℓ_1. Show that T is a continuous linear operator, $T e_n = x_n$, and $\|T\| = \sup_n \|x_n\|$. Show that any continuous linear operator $T : \ell_1 \to X$ can be described as indicated above.

5. Let X be a normed space and X_1 a closed subspace of X. Show that the quotient map q of the space X onto the space X/X_1 (see Subsection 5.2.2) is a continuous linear operator. Calculate the norm $\|q\|$. Show that $q(B_X) = B_{X/X_1}$.

6. Let X and Y be normed spaces, and $T : X \to Y$ a linear operator. Prove that the injectivization $\widetilde{T}$ of the operator T (see Subsection 5.2.3) is a continuous linear operator and $\|\widetilde{T}\| = \|T\|$.

7. In the setting of the preceding exercise, suppose that $T(B_X) = B_Y$. Show that in this case $\widetilde{T}$ is a bijective isometry of the spaces $X/\mathrm{Ker}\, T$ and Y.

8. Let $\mathcal{P}$ be the space of all polynomials, as in Exercise 5 in Subsection 6.3.3, and let $D_m : \mathcal{P} \to \mathcal{P}$ be the m-th derivative operator. Verify that D_m is a linear operator and calculate its norm. Is D_m a continuous operator?

9. Equip the linear space $\mathcal{P}$ of polynomials with the norm $\|a_0 + a_1 t + \cdots + a_n t^n\|_1 = \sum_{k=0}^n k! \, |a_k|$. Denote the resulting normed space by $\mathcal{P}_1$. Is the m-th derivative operator $D_m : \mathcal{P}_1 \to \mathcal{P}_1$ continuous? What is its norm?

10. Let X and Y be normed spaces and $T: X \to Y$ be a bijective linear operator. Show that the operator T is an isometry if and only if $\|T\| = \|T^{-1}\| = 1$.

6.4.3 Pointwise Convergence

Theorem 1. *Suppose X and Y are normed spaces, $T_n: X \to Y$ is a linear operator, and the limit $\lim_{n\to\infty} T_n x$ exists for all $x \in X$. Then the map $T: X \to Y$ given by the recipe $T(x) = \lim_{n\to\infty} T_n x$ is a linear operator.*

Proof. Indeed,

$$T(ax_1 + bx_2) = \lim_{n\to\infty} T_n(ax_1 + bx_2)$$

$$= a \lim_{n\to\infty} T_n(x_1) + b \lim_{n\to\infty} T_n(x_2) = aT(x_1) + bT(x_2). \qquad \square$$

Definition 1. A sequence of operators $T_n: X \to Y$ is said to *converge pointwise* to the operator $T: X \to Y$ if $Tx = \lim_{n\to\infty} T_n x$ for all $x \in X$.

Theorem 2. *Suppose the sequence of operators $T_n \in L(X, Y)$ converges pointwise to the operator $T: X \to Y$ and $\sup_n \|T_n\| = C < \infty$. Then $T \in L(X, Y)$ and $\|T\| \leqslant C$.*

Proof. The estimate $\|Tx\| = \lim_{n\to\infty} \|T_n x\| \leqslant C\|x\|$ holds for all $x \in X$. $\square$

Theorem 3. *If the sequence of operators $T_n \in L(X, Y)$ converges to the operator T in the norm of the space $L(X, Y)$, then it also converges pointwise to T.*

Proof. Indeed,

$$\|T_n x - Tx\| = \|(T_n - T)x\| \leqslant \|T_n - T\| \cdot \|x\| \to 0 \quad \text{as } n \to \infty. \qquad \square$$

Exercises

1. Let $X = C[0, 1]$, $Y = \mathbb{R}$, and let the operators $T_n \in L(X, Y)$ act as $T_n(f) = f(0) - f(1/n)$. Calculate the norms of T_n.

2. Pointwise convergence does not imply convergence in norm. Example: the sequence of operators from the preceding exercise tends to 0 pointwise, but not in norm.

3. The following general fact is known (Josefson and Nissenzweig, [55, 72], see also [47]): *On any infinite-dimensional normed space there exists a sequence of linear functionals that converges to 0 pointwise, but not in norm.* Give corresponding examples in all infinite-dimensional normed spaces you know.

4. Under the assumptions of Theorem 2, show that $\|T\| \leqslant \underline{\lim}_{n\to\infty}\|T_n\|$. In other words, the norm on $L(X, Y)$ is lower semicontinuous with respect to pointwise convergence.

5. Introduce on $L(X, Y)$ a topology in which convergence coincides with pointwise convergence.

6.4.4 Completeness of the Space of Operators. Dual Space

Theorem 1. *Let X be a normed space and Y a Banach space. Then $L(X, Y)$ is a Banach space.*

Proof. We use the definition. Suppose the operators $T_n \in L(X, Y)$ form a Cauchy sequence: $\|T_n - T_m\| \to 0$ as $n, m \to \infty$. Then for any point $x \in X$ the values $T_n x$ form a Cauchy sequence in the complete space Y, because $\|T_n x - T_m x\| \leqslant \|T_n - T_m\| \cdot \|x\| \to 0$ as $n, m \to \infty$. Hence, for any $x \in X$ the sequence $(T_n x)$ has a limit. Define the operator $T : X \to Y$ by the rule $Tx = \lim_{n\to\infty} T_n x$. By Theorem 1 of the preceding subsection, the operator T is linear. Since every Cauchy sequence is bounded, Theorem 2 of the preceding subsection shows that $T \in L(X, Y)$. It remains to verify that $T = \lim_{n\to\infty} T_n$ in the norm of the space $L(X, Y)$. Since the sequence T_n is Cauchy, for any $\varepsilon > 0$ there exists a number $N(\varepsilon)$ such that $\|T_N - T_M\| < \varepsilon$ for all $M > N > N(\varepsilon)$. Then for any point x of the unit sphere S_X of X it also holds that $\|T_N x - T_M x\| < \varepsilon$ for $M > N > N(\varepsilon)$. Letting here $M \to \infty$ in the last inequality, we obtain $\|T_N x - Tx\| < \varepsilon$. Now if in the left-hand side of this inequality we take the supremum over $x \in S_X$, we get $\|T_N - T\| \leqslant \varepsilon$ for $N > N(\varepsilon)$, i.e., $T = \lim_{n\to\infty} T_n$, as needed. □

The *dual* (or *conjugate*) *space* of the normed space X is the space X^* of all continuous linear functionals on X, equipped with the norm $\|f\| = \sup_{x\in S_X} |f(x)|$. In other words, if X is a real space, then $X^* = L(X, \mathbb{R})$, while if X is a complex space, $X^* = L(X, \mathbb{C})$. Since $\mathbb{R}$ and $\mathbb{C}$ are complete spaces, the theorem above shows that the space X^* is complete, regardless of whether the space X itself is complete or not. The space X^* will also be referred to simply as the *dual* of X.

As was the case with the norm of an operator (see Remark 2 in Subsection 6.4.2), there are other standard definitions for the norm of a functional. We provide one of them that is specific for functionals rather than for general operators.

Remark 1. Let X be a real normed space, and let $f \in X^*$. Then $\|f\| = \sup_{x\in S_X} f(x)$.

Proof. We use the symmetry of the sphere: $x \in S_X$ if and only if $-x \in S_X$. Hence, $\sup_{x\in S_X} f(x) = \sup_{x\in S_X} f(-x)$. Consequently,

$$\|f\| = \sup_{x\in S_X} |f(x)| = \sup_{x\in S_X} \max\{f(x), -f(x)\}$$

$$= \max\left\{\sup_{x\in S_X} f(x), \sup_{x\in S_X} f(-x)\right\} = \sup_{x\in S_X} f(x). \qquad \square$$

Exercises

1. Let X be a real normed space, and $f \in X^*$. Then $\|f\| = \sup_{x \in \overline{B}_X} f(x)$.

2. Let X be a complex normed space, and $f \in X^*$. Then $\|f\| = \sup_{x \in S_X} \operatorname{Re} f(x)$.

3. On the space ℓ_∞ of all bounded numerical sequences $x = (x_1, x_2, \dots)$, equipped with the norm $\|x\| = \sup_n |x_n|$, define the functional f by the formula $f(x) = \sum_{n=1}^{\infty} a_n x_n$, where $a = (a_1, a_2, \dots)$ is a fixed element of the space ℓ_1. Show that $\|f\| = \sum_{n=1}^{\infty} |a_n|$.

4. On the space $C[0,1]$ consider the linear functional F defined by the rule $F(x) = \int_0^{1/2} x(t)\,dt - \int_{1/2}^{1} x(t)\,dt$. Show that $\|F\| = 1$ and that $|F(x)| < 1$ for all $x \in S_{C[0,1]}$. This example shows that the supremum in the definition of the norm of a functional (or operator) is not necessarily attained.

6.5 Extension of Operators

In this section we consider several simple yet useful conditions under which a continuous operator can be extended from a subspace of a normed space to the entire ambient space.

6.5.1 Extension by Continuity

Theorem 1. *Let X_1 be a dense subspace of the normed space X, Y a Banach space, and $T_1 \in L(X_1, Y)$. Then the operator T_1 admits a unique extension $T \in L(X, Y)$.*

Proof. Since the subspace X_1 is dense, for any $x \in X$ there exists a sequence of vectors $x_n \in X_1$ which converges to x. Then $(T_1 x_n)$ is a Cauchy sequence in Y:

$$\|T_1 x_n - T_1 x_m\| \leqslant \|T_1\| \cdot \|x_n - x_m\| \to \infty \quad \text{as } n, m \to \infty.$$

Denote the limit of this sequence by Tx. Then

$$\|Tx\| = \lim_{n \to \infty} \|T_1 x_n\| \leqslant \|T_1\| \lim_{n \to \infty} \|x_n\| = \|T_1\| \cdot \|x\|.$$

Note that Tx does indeed depend only on x, and not on the choice of the sequence x_n: if $x_n' \in X_1$ is some other sequence that converges to x, then $\|T_1 x_n - T_1 x_n'\| \leqslant \|T_1\| \cdot \|x_n - x_n'\| \to 0$ as $n \to \infty$, and so the sequences $(T_1 x_n)$ and $(T_1 x_n')$ have the same limit. Hence, for every $x \in X$ we defined a map $T: X \to Y$ by the rule $Tx = \lim_{n \to \infty} T_1 x_n$, where $x_n \in X_1$ form a sequence that converges to x. It remains to show that T is the sought-for operator. Let us verify that the operator T is linear. Let $x_1, x_2 \in X$, $x_1^n, x_2^n \in X_1$, $x_2^n \to x_2$, $x_1^n \to x_1$ as $n \to \infty$. Then

$$\begin{aligned} T(a_1x_1 + a_2x_2) &= \lim_{n\to\infty} T_1(a_1x_1^n + a_2x_2^n) \\ &= a_1 \lim_{n\to\infty} T_1x_1^n + a_2 \lim_{n\to\infty} T_1x_2^n = a_1Tx_1 + a_2Tx_2 \end{aligned}$$

for all scalars a_1, a_2. Thanks to the already established inequality $\|Tx\| \leqslant \|T_1\| \cdot \|x\|$, the operator T is continuous, i.e., $T \in L(X, Y)$. This proves the existence of the extension. Its uniqueness follows from the fact that two continuous functions which coincide on a dense set coincide everywhere. □

Exercises

1. In the argument above we skipped the verification of the fact that the operator T is an extension of the operator T_1. Complete this step.

2. Show that under the conditions of the preceding theorem $\|T\| \leqslant \|T_1\|$.

3. Let X and Y be normed spaces, $X_1 \subset X$ be an arbitrary subspace, and $T \in L(X, Y)$ be an extension of the operator $T_1 \in L(X_1, Y)$. Show that $\|T\| \geqslant \|T_1\|$.

4. Combining Exercises 2 and 3 above, show that under the assumptions of Theorem 1, $\|T\| = \|T_1\|$.

5. Give an example of a continuous function which is defined on a dense subset of the interval [0, 1], but which cannot be extended to a continuous function on the whole interval.

6. Show that every continuous linear operator is a uniformly continuous mapping. Deduce the main theorem of the present subsection from the theorem, given in Subsection 1.3.4, on the extension of uniformly continuous mappings. Moreover, the linearity of the extended operator can be deduced from the uniqueness of the extension.

6.5.2 *Projectors; Extension from a Closed Subspace*

Let X_1 be a subspace of the normed space X. The operator $P \in L(X, X)$ is called a *projector* onto X_1 if $P(X) \subset X_1$ and $Px = x$ for all $x \in X_1$.

Theorem 1. *For a subspace X_1 of the normed space X, the following conditions are equivalent*:

(1) *in X there exists a projector onto X_1;*

(2) *for any normed space Y, any operator $T_1 \in L(X_1, Y)$ extends to an operator $T \in L(X, Y)$.*

Proof. (1) $\Longrightarrow$ (2). Define $T \in L(X, Y)$ by the rule $Tx = T_1(Px)$.

(2) $\Longrightarrow$ (1). Take $Y = X_1$ and define $T_1 \in L(X_1, Y)$ by the rule $T_1 x = x$. Let $T \in L(X, Y)$ be an extension of the operator T_1. Since in our case $Y \subset X$, we can regard T as an operator from X to X. We have $T(X) \subset Y = X_1$, and $Tx = T_1 x = x$ for all $x \in X_1$. Hence, T is the required projector onto X_1. □

Exercises

1. Provide the details of the proof of the implication (1) $\Longrightarrow$ (2) in the preceding theorem.

2. Let X_1 be a subspace of the normed space X and $P \in L(X, X)$ be a projector onto X_1. Then $P(X) = X_1 = \operatorname{Ker}(I - P)$ and the subspace X_1 is closed in X.

3. Suppose that under the conditions of the preceding exercise $X_1 \neq \{0\}$. Then $\|P\| \geqslant 1$.

4. For a subspace X_1 of the normed space X the following conditions are equivalent:

— in X there exists a projector P onto X_1 with $\|P\| = 1$;

— for any normed space Y, any operator $T_1 \in L(X_1, Y)$ extends to an operator $T \in L(X, Y)$ with $\|T\| = \|T_1\|$.

5. Let $X = \ell_1^3$ (see Exercise 2 in Subsection 6.2.1 for the definition), and let X_1 be the subspace consisting of all elements for which the sum of their coordinates is equal to zero. Show that in X there is no projector P onto X_1 with $\|P\| = 1$.

Comments on the Exercises

Subsection 6.1.2

Exercise 1. Since $\|x\| = \rho(0, x)$, the result follows from the continuity of the distance (Subsection 1.3.2).

Exercise 5. See Subsection 18.2.1.

Subsection 6.2.2

Exercise 3. See Theorem 2 in Subsection 14.1.2.

Exercise 7. Let $g \in L_p[a, b]$. Consider the sequence of truncations

$$g_n = \min\{n, \max\{g, -n\}\}.$$

The sequence of functions $|g_n - g|^p$ converges almost everywhere to zero and admits the integrable majorant $|g|^p$. Hence, by the Lebesgue dominated convergence theorem, $\|g_n - g\|_p \to 0$ as $n \to \infty$.

Exercise 8. By the preceding exercise, it suffices to show that any bounded function $f \in L_p[a, b]$ can be approximated in the metric of L_p by continuous functions. By Exercise 6 in Subsection 3.2.3, there exists a sequence of continuous functions (f_n) that converges to f a.e. With no loss of generality we can assume that all f_n are bounded in modulus by the same constant C as f (otherwise we replace f_n by the truncations $\widetilde{f_n} = \min\{C, \max\{f_n, -C\}\}$). The convergence of $\|f_n - f\|_p$ to 0 follows from the Lebesgue dominated convergence theorem.

Subsection 6.4.2

Exercise 5. $[x] \in q(B_X) \iff \exists y \in B_X : [y] = [x] \iff \|[x]\| < 1 \iff [x] \in B_{X/X_1}$.

Exercise 6.

$$\begin{aligned}\|\widetilde{T}\| &= \sup_{[x]\in B_{X/X_1}} \|\widetilde{T}[x]\| = \sup_{[x]\in q(B_X)} \|\widetilde{T}[x]\| \\ &= \sup_{x\in B_X} \|\widetilde{T}[x]\| = \sup_{x\in B_X} \|Tx\| = \|T\|.\end{aligned}$$