Chapter 3
Measurable Functions

Measure and integration theory studies above all real-valued functions. To avoid unnecessary repetition, let us agree that, unless otherwise stipulated, the term "function" will be used for real-valued functions. Thus, when we say "function f on Ω", we mean that f is a function from Ω to $\mathbb{R}$. For functions whose range does not lie in $\mathbb{R}$ we will use the term "map" or "mapping".

The operations on functions will be understood pointwise. For example, $f_1 + f_2$ is the function on Ω given by the rule $(f_1 + f_2)(t) = f_1(t) + f_2(t)$, the function $\max\{f, g\}$ is defined as $\max\{f, g\}(t) = \max\{f(t), g(t)\}$, and so on. The limit of a sequence of functions will also be understood as the pointwise limit.

3.1 Measurable Functions and Operations on Them

In this section (Ω, Σ) will be a set endowed with a σ-algebra of its subsets. All functions, unless otherwise stipulated, will be assumed to be defined on Ω; the elements of the σ-algebra Σ will be referred to as measurable sets.

3.1.1 Measurability Criterion

Definition 1. Let (Ω_1, Σ_1) and (Ω_2, Σ_2) be sets endowed with σ-algebras of subsets. A map $f : \Omega_1 \to \Omega_2$ is said to be *measurable* if $f^{-1}(A) \in \Sigma_1$ for all $A \in \Sigma_2$.

As the definition indicates, measurable maps play in measure theory the same role that continuous maps do in the theory of topological spaces. Particular examples of measurable maps are the *measurable functions* introduced below.

V. Kadets, *A Course in Functional Analysis and Measure Theory*,
Universitext, https://doi.org/10.1007/978-3-319-92004-7_3

Definition 2. A function f on Ω is said to be *measurable* (more specifically, *measurable with respect to the σ-algebra Σ*, or *Σ-measurable*), if for any Borel subset $A \subset \mathbb{R}$ the set $f^{-1}(A)$ is measurable.

Theorem 1. *Let (Ω_1, Σ_1) and (Ω_2, Σ_2) be sets endowed with σ-algebras of their subsets, and let Λ be a family of subsets of Ω_2 that generates the σ-algebra Σ_2. In order for the map $f: \Omega_1 \to \Omega_2$ to be measurable it is necessary and sufficient that for any set $A \in \Lambda$ its preimage $f^{-1}(A)$ lies in the σ-algebra Σ_1.*

Proof. If f is measurable, then the preimage of any set $A \in \Sigma_2$ lies in Σ_1. In particular, Σ_1 contains the preimages of all sets $A \in \Lambda$.

Conversely, suppose that Σ_1 contains all sets of the form $f^{-1}(A)$ with $A \in \Lambda$. We need to show that the preimages of all elements of the family Σ_2 lie in Σ_1. To do this, we introduce the following family Λ_1 of subsets of the set Ω_2: a set A belongs to Λ_1 if $f^{-1}(A) \in \Sigma_1$. It is readily verified that Λ_1 is a σ-algebra and contains all elements of the family Λ. Since Σ_2 is the smallest σ-algebra containing Λ, it follows that $\Sigma_2 \subset \Lambda_1$, as we needed to show. □

Let $f: \Omega \to \mathbb{R}$ be a function and $a \in \mathbb{R}$. Denote $f^{-1}((a, +\infty))$ by $f_{>a}$, i.e., $f_{>a}$ is the set of all $t \in \Omega$ at which $f(t) > a$. Since (see Subsection 2.1.2, Proposition 2) the sets $(a, +\infty)$ with $a \in \mathbb{R}$ generate the σ-algebra $\mathfrak{B}$ of Borel set on $\mathbb{R}$, we obtain the following simple measurability criterion:

Corollary 1. *The function $f: \Omega \to \mathbb{R}$ is measurable if and only if all the sets $f_{>a}$ with $a \in \mathbb{R}$ are measurable.* □

Corollary 2. *Let (Ω, Σ), (Ω_1, Σ_1), and (Ω_2, Σ_2) be sets endowed with σ-algebras of subsets. Endow, as usual, the Cartesian product $\Omega_1 \times \Omega_2$ with the σ-algebra $\Sigma_1 \otimes \Sigma_2$ (see Subsection 2.1.3). Then for any measurable maps $f_1: \Omega \to \Omega_2$ and $f_2: \Omega \to \Omega_2$, the map $f: \Omega \to \Omega_1 \times \Omega_2$ given by the rule $f(t) = (f_1(t), f_2(t))$ is also measurable.*

Proof. By definition, the σ-algebra $\Sigma_1 \otimes \Sigma_2$ is generated by the sets $A_1 \times A_2$ with $A_1 \in \Sigma_1$ and $A_2 \in \Sigma_2$. We have $f^{-1}(A_1 \times A_2) = f_1^{-1}(A_1) \cap f_2^{-1}(A_2) \in \Sigma$. □

If we take for Ω a topological space and for Σ the σ-algebra $\mathfrak{B}$ of Borel sets on Ω, we obtain a particular case of measurability, Borel measurability:

Definition 3. A function f on the topological space Ω is said to be *Borel measurable* if the preimage $f^{-1}(A)$ of any Borel subset A of the real line is a Borel subset of Ω.

As an example of a Borel-measurable function one can take any continuous function. Indeed, for a continuous function f all the sets $f_{>a}$ are open, and hence belong to the σ-algebra $\mathfrak{B}$ of Borel sets, i.e., the above measurability criterion applies.

For an arbitrary set $A \in \Sigma$ we can consider the σ-algebra Σ_A of all measurable subsets of A. If the restriction of the function f to A is measurable with respect to the σ-algebra Σ_A, then f is said to be *measurable on the subset A*.

Exercises

1. If the function f is measurable, then for any $a \in \mathbb{R}$ the sets $f_{\neq a} = \{t \in \Omega : f(t) \neq a\}$, $f_{=a} = \{t \in \Omega : f(t) = a\}$, $f_{\leqslant a} = \{t \in \Omega : f(t) \leqslant a\}$, $f_{<a} = \{t \in \Omega : f(t) < a\}$, and $f_{\geqslant a} = \{t \in \Omega : f(t) \geqslant a\}$ are measurable.

2. Let f be a Borel-measurable function on the interval $[a, b]$. Then the set of maximum points of f is a Borel set.

3. The set of local maximum points of a Borel-measurable function on the real line is a Borel set.

4. Let (Ω_1, Σ_1) and (Ω_2, Σ_2) be sets endowed with σ-algebras of subsets, and let $\Omega_1 \times \Omega_2$ be endowed with the σ-algebra $\Sigma_1 \otimes \Sigma_2$. Prove that the projection maps P_1 and P_2, which send each element $(t_1, t_2) \in \Omega_1 \times \Omega_2$ into its coordinates t_1 and t_2, respectively, are measurable.

5. Prove the converse of Corollary 2: if the map $f : \Omega \to \Omega_1 \times \Omega_2$ given by $f(t) = (f_1(t), f_2(t))$ is measurable, then the maps f_1 and f_2 are also measurable.

6. Show that every monotone function on the real line is Borel measurable.

7. Let f be a Borel-measurable function on the interval $[a, b]$. Then the set of maximum points of f is a Borel set.

8. Let f be a measurable function on Ω. Prove that the functions $|f|$, sign f, f^+, and f^- are measurable.

9. If the function f is measurable, then λf is measurable for any $\lambda \in \mathbb{R}$.

10. Let the function f be measurable on Ω. Then f is measurable on any subset $A \in \Sigma$.

11. Suppose that Ω can be written as the union of two measurable subsets A and B, and the function f is measurable on both A and B. Then f is measurable on Ω.

12. Give an example of a bijective measurable map $f : \Omega_1 \to \Omega_2$ whose inverse is not measurable.

13. Let $g : \mathbb{R} \to \mathbb{R}$ be a continuous function and A be a Lebesgue-measurable set in $\mathbb{R}$.

(a) Is the set $g(A)$ necessarily Borel measurable?

(b) Lebesgue measurable?

(c) Can the set $g^{-1}(A)$ be not Lebesgue measurable?

14. Let $g : \mathbb{R} \to \mathbb{R}$ be a continuous function and A be an open subset of $\mathbb{R}$. Then $g(A)$ is a Borel set. Moreover, $g(A)$ is an F_σ-set.

15. Let $g : \mathbb{R} \to \mathbb{R}$ be a continuous function and A be a Borel set in $\mathbb{R}$. Can the set $g(A)$ be not Borel?

16. Let (Ω, Σ, μ) be a measure space. Two measurable functions f and g on Ω are said to be *equimeasurable*, if $\mu(f_{>a}) = \mu(g_{>a})$ for all $a \in \mathbb{R}$. Show that if f and g are equimeasurable then $\mu(f^{-1}(A)) = \mu(g^{-1}(A))$ for any Borel set A of real numbers.

3.1.2 Elementary Properties of Measurable Functions

Theorem 1. *Let* (Ω_1, Σ_1), (Ω_2, Σ_2), *and* (Ω_3, Σ_3) *be sets endowed with* σ*-algebras of subsets, and let* $f : \Omega_1 \to \Omega_2$ *and* $g : \Omega_2 \to \Omega_3$ *be measurable maps. Then the composition* $g \circ f : \Omega_1 \to \Omega_3$ *is also a measurable map.*

Proof. Let $A \in \Sigma_3$. Then $g^{-1}(A) \in \Sigma_2$, and so $(g \circ f)^{-1}(A) = f^{-1}(g^{-1}(A)) \in \Sigma_1$, as needed. □

Corollary 1.

1. *Suppose the function* $f : \Omega \to \mathbb{R}$ *is measurable and the function* $g : \mathbb{R} \to \mathbb{R}$ *is Borel measurable. Then the composition* $g \circ f$ *is also measurable.*
2. *In particular, if* $f : \Omega \to \mathbb{R}$ *is measurable and* $g : \mathbb{R} \to \mathbb{R}$ *is continuous, then* $g \circ f$ *is measurable.*
3. *Suppose the functions* $f_1, f_2 : \Omega \to \mathbb{R}$ *are measurable, and the function* $g : \mathbb{R}^2 \to \mathbb{R}$ *of two variables is continuous. Then the function* $f(t) = g(f_1(t), f_2(t))$ *is measurable.*

Proof. Only item 3 requires a proof. Consider the plane $\mathbb{R}^2 = \mathbb{R} \times \mathbb{R}$, endowed with the σ-algebra of Borel sets, or, which is the same, with the product of the σ-algebras of Borel sets on the line $\mathbb{R}$. By Corollary 2 in the preceding subsection, the function $F : \Omega \to \mathbb{R}^2$ defined by the rule $F(t) = (f_1(t), f_2(t))$ is measurable. It remains to note that $f = g \circ F$ and apply the preceding theorem. □

Theorem 2. *The class of measurable functions on* (Ω, Σ) *enjoys the following properties: if the functions* f *and* g *are measurable, then so are the functions* $f + g$, fg, $\max\{f, g\}$, *and* $\min\{f, g\}$. *Moreover, the functions* $|f|$, $\operatorname{sign} f$, $f^+ = \max\{f, 0\}$, $f^- = (-f)^+$, *and* λf *with any* $\lambda \in \mathbb{R}$ *are measurable. If* f *does not vanish at any point, then the function* $1/f$ *is measurable.*

Proof. The functions $g_1(x, y) = x + y$ and $g_2(x, y) = xy$ of two variables are continuous, and so are the functions $\max\{x, y\}$ and $\min\{x, y\}$. By item 3 of the last corollary, this implies that the functions $f + g$, fg, $\max\{f, g\}$, and $\min\{f, g\}$ are measurable. The continuity of the functions $|t|$, t^+, t^-, and λt, in conjunction with item 2 of the preceding corollary, guarantee the measurability of the functions $|f|$,

f^+, f^-, and λf. The measurability of the function sign f follows from item 1 of the same corollary and the Borel measurability of the function sign t. Finally, if f does not vanish at any point, then the function $1/f$ can be represented as the composition of the measurable function $f : \Omega \to \mathbb{R} \setminus \{0\}$ (where $\mathbb{R} \setminus \{0\}$ is endowed with the σ-algebra of Borel sets), and the continuous — and hence Borel-measurable — function $1/t : \mathbb{R}\setminus\{0\} \to \mathbb{R}$. □

Theorem 3. *Suppose the sequence (f_n) of measurable functions converges pointwise to a function f, i.e., for any $t \in \Omega$, $f_n(t) \to f(t)$ as $n \to \infty$. Then f is a measurable function.*

Proof. Fix a number $a \in \mathbb{R}$. The value of the function f at the point $t \in \Omega$ is larger than a if and only if there exist a rational number $r \in \mathbb{Q}$ and a number $n \in \mathbb{N}$ such that for any $m > n$ it holds that $f_m(t) > a + r$. Translating this statement into the language of measure theory, we conclude that $f_{>a} = \bigcup_{r\in\mathbb{Q}}(\bigcup_{n=1}^{\infty} \bigcap_{m=n+1}^{\infty} (f_m)_{>a+r} \in \Sigma$. □

Applying this theorem to the sequence of partial sums of a series we obtain the following statement.

Corollary 2. *If a series of measurable functions converges pointwise, then its sum is a measurable function.* □

Exercises

1. Prove directly that if the functions f and g are measurable, then for any $a \in \mathbb{R}$ the set $(f + g)_{>a}$ belongs to Σ. According to the criterion in the preceding section, this will provide another proof of the measurability of the sum of two measurable functions.

2. Express the sets $(\max\{f, g\})_{>a}$ and $(\min\{f, g\})_{>a}$ in terms of the analogous sets for the functions f and g.

3. If the functions f and g are measurable, then the sets of points $t \in \Omega$ in which $f = g$, $f \neq g$, $f > g$, and $f < g$, respectively, are measurable.

4. Let (f_n) be a pointwise bounded sequence of measurable functions. Then the functions $f = \sup_n f_n$ and $g = \overline{\lim}_{n\to\infty} f_n$ are also measurable.

5. Let A denote the set of all differentiability points of the function f on the line (see Exercise 13 in Subsection 2.1.2). Show that the function f' is Borel-measurable on A.

6. Identify in the standard way the field $\mathbb{C}$ of complex numbers with the plane $\mathbb{R}^2$, and endow $\mathbb{C}$ with the σ-algebra of Borel subsets of the plane. A measurable map $f : \Omega \to \mathbb{C}$ is called a *measurable complex-valued function*. Prove that $f : \Omega \to \mathbb{C}$ is measurable if and only if the real-valued functions Re f and Im f are measurable.

7. Prove the following properties of complex-valued functions:

(1) if the functions f and g are measurable, then so is their sum $f + g$;

(2) if the function f is measurable, then so is λf for any $\lambda \in \mathbb{C}$;

(3) if the functions f and g are measurable, then so is their product fg;

(4) if the function f is measurable, then $|f|$ is a measurable real-valued function.

3.1.3 The Characteristic Function of a Set

Let Ω be a set and A be a subset of Ω. The *characteristic function* of the set A is the function $\mathbb{1}_A$ on Ω equal to 1 on A and equal to zero on the complement $\Omega \setminus A$ of A. Alternative notations found in the literature are χ_A and I_A. We note that the last notation is most frequently encountered in probability theory, where the characteristic function of a set is called the *indicator* of that set, and the term "characteristic function" is used for a completely different object. Of course, it would be reasonable, in the notation for the characteristic function, to account not only for the set A, but also for the ambient set Ω. For instance, one and the same set A of real numbers can be regarded as a subset of an interval in one situation, and as a subset of the real line in another. In the first case the function $\mathbb{1}_A$ is defined on the interval, and in the second on the real line, and the same symbol is used in both situations. This slight ambiguity does not have unpleasant consequences: here, like in many other situations, a function defined on a subset is tacitly extended to the ambient set by zero.

The properties listed in Exercises 1–5 below will be used in the sequel, and for this reason the reader is advised to pay close attention to them.

Exercises

1. Let (Ω, Σ) be a set endowed with a σ-algebra of subsets, and $A \subset \Omega$. The function $\mathbb{1}_A$ is measurable if and only if the set A is measurable.

2. $\mathbb{1}_{A \cup B} = \max\{\mathbb{1}_A, \mathbb{1}_B\}$.

3. $\mathbb{1}_{A \cap B} = \min\{\mathbb{1}_A, \mathbb{1}_B\} = \mathbb{1}_A \cdot \mathbb{1}_B$.

4. If the sets A and B are disjoint, then $\mathbb{1}_{A \sqcup B} = \mathbb{1}_A + \mathbb{1}_B$.

5. Let $A = \bigsqcup_{n=1}^{\infty} A_n$. Then $\mathbb{1}_A = \sum_{n=1}^{\infty} \mathbb{1}_{A_n}$.

6. Let (A_n) be a sequence of sets. Then $\varlimsup_{n\to\infty} \mathbb{1}_{A_n}$ is the characteristic function of a set A, called the *upper limit* of the sequence (A_n). Express the set A in terms of the sets A_n by means of the usual operations of union and intersection.

7. Consider the set $2^{\mathbb{N}}$ of all subsets of the natural numbers with the topology described in Exercise 7 of Subsection 1.4.4. Verify that a sequence of sets converges in this topology if and only if the characteristic functions of the sets converge pointwise to the corresponding characteristic function.

3.1.4 Simple Functions. Lebesgue Approximation of Measurable Functions by Simple Ones. Measurability on the Completion of a Measure Space

Let (Ω, Σ) be a set endowed with a σ-algebra. A function f on Ω is called *simple* if it can be represented as $f = \sum_{n=1}^{\infty} a_n \mathbb{1}_{A_n}$, where $A_n \in \Sigma$ is a disjoint sequence of sets and a_n are numbers. Since the sets A_n are disjoint, the series $\sum_{n=1}^{\infty} a_n \mathbb{1}_{A_n}$ does not merely converge pointwise: for any point $t \in \Omega$ all the terms of the series, except possibly for one (with the index n for which $t \in A_n$), vanish at t. On each of the sets A_n the function f is equal to the constant a_n, and $f(t) = 0$ in the complement of the union of all A_n. Simple functions are also called *countably-valued functions* or, in more detail, *countably-valued measurable functions*. This terminology is justified by the following assertion.

Theorem 1. *The function f is simple if and only if it is measurable and the set of its values* (*i.e., its image, or range*) *is at most countable.*

Proof. The measurability of a simple function $f = \sum_{n=1}^{\infty} a_n \mathbb{1}_{A_n}$ can be verified directly (the preimage of any set under f is a finite or countable union of some of the sets A_n); alternatively, one can refer to the measurability of the sum of a series of measurable functions. Further, $f(\Omega) \subset \{a_n\}_{n=1}^{\infty} \cup \{0\}$, which shows that the set of all values of f is at most countab le. Conversely, suppose f is measurable and the set M of its values is at most countable. Then for any $t \in M$, the set $f^{-1}(t)$ is measurable and $f = \sum_{t \in M} t \mathbb{1}_{f^{-1}(t)}$. □

If the set of values of a simple function is finite, then the function is said to be *finitely-valued.*

Theorem 2. *The classes of finitely-valued and countably-valued functions are stable under taking sums and products, as well as the maximum and the minimum of two functions.*

Proof. We already know that the listed operations preserve measurability. Now let f and g be two functions on Ω, and let $f(\Omega)$ and $g(\Omega)$ be their images. If $f(\Omega)$ and $g(\Omega)$ are finite (countable), then the sets

$$f(\Omega) + g(\Omega) = \{t + r : t \in f(\Omega),\ r \in g(\Omega)\}$$

and

$$f(\Omega) \cdot g(\Omega) = \{t \cdot r : t \in f(\Omega),\ r \in g(\Omega)\}$$

are finite (respectively, countable). The assertion of the theorem follows from the fact that the images of the functions $f + g$, fg, $\max\{f, g\}$, and $\min\{f, g\}$ lie in $f(\Omega) + g(\Omega)$, $f(\Omega) \cdot g(\Omega)$, $f(\Omega) \cup g(\Omega)$, and $f(\Omega) \cup g(\Omega)$, respectively. □

Measurable functions can have a rather complicated structure. For this reason, to facilitate the study of their structure one uses approximations of measurable functions by simple functions.

Theorem 3. *Let f be a measurable function on Ω. Then for any $\varepsilon > 0$ there exists a simple function $f_\varepsilon \leqslant f$ which at all points differs from f by at most ε. Moreover, if $f \geqslant 0$, then f_ε can also be chosen to be non-negative, and if f is bounded, then for f_ε one can take a finitely-valued function.*

Proof. For each integer n introduce the number $t_n = n\varepsilon$ and the intervals $\Delta_n = [t_n, t_{n+1})$. Denote the set $f^{-1}(\Delta_n)$ by A_n. Some of the sets A_n may be empty. In particular, if $f \geqslant 0$, then all the A_n with index $n < 0$ are empty. Further, if f is bounded in modulus by some constant C, then all the A_n with $|n| > (C/\epsilon) + 1$ are empty. The sets A_n are pairwise disjoint, their union is the whole Ω, and on A_n the values of the function f satisfy the inequalities $t_n \leqslant f(t) < t_{n+1}$. We define the function f_ε so that its value on A_n is equal to the corresponding t_n: $f_\varepsilon = \sum_{n=1}^{\infty} t_n \mathbb{1}_{A_n}$.

The function f_ε defined in this way enjoys all the properties stated in the theorem. Indeed, on each A_n we have $t_n = f_\varepsilon(t) \leqslant f(t) < t_{n+1}$, i.e., $f(t) - \varepsilon < f_\varepsilon(t) \leqslant f(t)$ at all points $t \in \Omega$. If $f \geqslant 0$, then f_ε cannot take negative values t_n: the sets A_n that correspond to negative t_n will be empty. If f is bounded, then all the A_n, except for a finite number of them, will be empty, and so f_ε will be finitely-valued. □

Corollary 1. *For any measurable function f there exists a non-decreasing sequence $f_1 \leqslant f_2 \leqslant \cdots$ of simple functions which converges uniformly to f. If, in addition, f is non-negative (bounded), then the functions f_n can be chosen to be non-negative (respectively, finitely-valued).*

Proof. We use the preceding theorem and chose a simple function f_1 such that $0 \leqslant f - f_1 \leqslant 1$. The function $f - f_1$ is measurable and non-negative, so by the preceding theorem there exists a simple non-negative function g_1 which satisfies the inequalities $0 \leqslant f - f_1 - g_1 \leqslant 1/2$. Put $f_2 = f_1 + g_1$. Then $f_1 \leqslant f_2$ and $0 \leqslant f - f_2 \leqslant 1/2$. The function $f - f_2$ is again measurable and non-negative, and so one can approximate it by a simple function g_2: $0 \leqslant f - f_2 - g_2 \leqslant 1/3$. Naturally, we define the function f_3 as $f_2 + g_2$. Continuing this process, we obtain an increasing sequence of simple functions satisfying the conditions $0 \leqslant f - f_n \leqslant 1/n$, which ensures that the sequence converges uniformly. Ensuring that the additional non-negativity or finite-valuedness requirements in the statement of the corollary are satisfied presents no difficulty. □

The proof of the next result is based on the fact that measurable functions can be approximated by simple ones.

Theorem 4. *Let* (Ω, Σ, μ) *be a measure space and* (Ω, Σ', μ) *be its completion. Then for any* Σ'*-measurable function* f *on* Ω*, there exists a* Σ*-measurable function* g *that coincides with* f *almost everywhere.*

Proof. First we will prove this assertion for simple functions. Let $f = \sum_{n=1}^{\infty} a_n \mathbb{1}_{A_n}$, where the sets A_n belong to the σ-algebra Σ' and are disjoint. In each of the sets A_n we choose a subset $B_n \in \Sigma$ for which $\mu(A_n \setminus B_n) = 0$ (see Exercise 3 in Subsection 2.1.5). Then $g = \sum_{n=1}^{\infty} a_n \mathbb{1}_{B_n}$ is the sought-for function. Now let f be an arbitrary Σ'-measurable function, let (f_n) be a sequence of simple Σ'-measurable functions that converges pointwise to f, and finally let g_n be Σ-measurable functions that coincide almost everywhere with the corresponding f_n. Denote by $A \subset \Omega$ the negligible set in the complement of which $f_n = g_n$ for all $n = 1, 2, \ldots$. By the definition of negligible sets, there exists a Σ-measurable set B of null measure such that $B \supset A$. Consider the full-measure set $C = \Omega \setminus A$. The functions $g_n \cdot \mathbb{1}_C$ are Σ-measurable, converge on C to f, and vanish in the complement of C. That is, the functions $g_n \cdot \mathbb{1}_C$ converge pointwise to $g = f \cdot \mathbb{1}_C$, and, by Theorem 3 of Subsection 3.1.2, this limit function is Σ-measurable. It remains to observe that $g = f$ almost everywhere, since the set B where this equality can fail is negligible. □

Exercises

1. The function f_ε figuring in the statement of Theorem 3 can be chosen so that $f_\varepsilon(\Omega) \subset f(\Omega)$.

2. Let X be a metric space endowed with the σ-algebra of Borel sets, and let $f : \Omega \to X$ be a measurable map. Then the following conditions are equivalent:

— for every $\varepsilon > 0$ there exists a countably-valued map $f_\varepsilon : \Omega \to X$ such that $\rho(f(t), f_\varepsilon(t)) \leqslant \varepsilon$ for all $t \in \Omega$;

— the set $f(\Omega)$ is separable.

3. In the setting of the preceding exercise, the following conditions are equivalent:

— for every $\varepsilon > 0$, there exist a finitely-valued measurable map $f_\varepsilon : \Omega \to X$ such that $\rho(f(t), f_\varepsilon(t)) \leqslant \varepsilon$ for all $t \in \Omega$;

— the set $f(\Omega)$ is precompact.

4. The map f_ε in the two preceding exercises can be chosen so that it will satisfy $f_\varepsilon(\Omega) \subset f(\Omega)$.

5. Show that for every Lebesgue-measurable function f on the interval one can find an equimeasurable decreasing function $\widetilde{f}$ (for the definition of equimeasurability, see Exercise 16 in Subsection 3.1.1). This function $\widetilde{f}$ is called a *decreasing rearrangement* of the function f.

3.2 Main Types of Convergence

In this section (Ω, Σ, μ) will be a fixed finite measure space, and the functions f, f_n, and all the others will be assumed, unless otherwise stipulated, to be defined on Ω, measurable, and real-valued.

3.2.1 Almost Everywhere Convergence

The sequence of functions (f_n) is said to *converge almost everywhere* to the function f (written $f_n \xrightarrow{\text{a.e.}} f$) if the set of all points $t \in \Omega$ at which the numerical sequence $f_n(t)$ does not converge to $f(t)$ as $n \to \infty$ is negligible.

We note the following elementary properties of almost everywhere convergence, the verification of which is left to the reader.

A. If $f_n \xrightarrow{\text{a.e.}} f$ and $f_n \xrightarrow{\text{a.e.}} g$, then $f \overset{\text{a.e.}}{=} g$.

B. If $f_n \xrightarrow{\text{a.e.}} f$ and $f_n \overset{\text{a.e.}}{=} g_n$, then $g_n \xrightarrow{\text{a.e.}} f$.

C. If $f_n \xrightarrow{\text{a.e.}} f$, $g_n \xrightarrow{\text{a.e.}} g$, and $f_n \overset{\text{a.e.}}{\leqslant} g_n$, then $f \overset{\text{a.e.}}{\leqslant} g$.

D. If $G\colon \mathbb{R}^2 \to \mathbb{R}$ is a continuous function, $f_n \xrightarrow{\text{a.e.}} f$ and $g_n \xrightarrow{\text{a.e.}} g$, then $G(f_n, g_n) \xrightarrow{\text{a.e.}} G(f, g)$. This implies, in particular, the theorems on the limit of a sum and of a product.

Almost everywhere convergence plays an important role in the theory of the Lebesgue integral. Under relatively mild additional assumptions (see Subsection 4.4) the integral of the limit function can be calculated as the limit of the integrals of the terms of the sequence. Moreover, almost everywhere convergence is in many respects far more convenient to work with than the usual pointwise convergence. First of all, it is a more general type of convergence, so it is easier to verify. Next, here, as in general when one deals with properties that hold almost everywhere, we can ignore the behavior of functions on negligible sets. For example, for a piecewise-continuous or for a monotone function it is not at all necessary to define the values in discontinuity points, as they have no influence whatsoever on almost everywhere convergence! On the other hand, almost everywhere convergence has an essential drawback: this convergence is not generated by a metric or topology, so there is no natural way of defining a "rate of convergence" for it. Let us give an example of a problem where this drawback shows up.

Definition 1. Let X and Y be two families of measurable functions on Ω. We say that X is a.e. *dense in* Y (dense in the sense of almost everywhere convergence) if for any $f \in Y$ there exists a sequence (f_n) of elements of the family X such that $f_n \xrightarrow{\text{a.e.}} f$.

Theorem 1. *Suppose that X is* a.e. *dense in Y and Y is* a.e. *dense in Z. Then X is* a.e. *dense in Z.* □

This natural property is important not only from the point of view of the inner harmony of the theory of almost everywhere convergence, but also from the point of view of applications. For instance, it enables one to show that the family of continuous functions on an interval is a.e. dense in the family of all Lebesgue-measurable functions on that interval. Although these results can be established using only the definition of almost everywhere convergence, devising such proofs is far from simple (we invite the reader to have a try at it!). If, on the contrary, the convergence had been given by some topology, the problem would have been rather trivial (see Exercise 4 in Subsection 1.2.1). Fortunately, here the following subtle idea comes to the rescue. As it turns out, the space of measurable functions carries a topology for which the notion of denseness of a subset coincides precisely with a.e. denseness, though the convergence in it (the so-called convergence in measure) is not equivalent to almost everywhere convergence. The study of this topology and the corresponding type of convergence is addressed next.

3.2.2 Convergence in Measure. Examples

Let a and ε be strictly positive numbers, f a measurable function. We denote by $U_{a,\varepsilon}(f)$ the set of all measurable functions g for which $\mu\left(|g-f|_{>a}\right)<\varepsilon$. (Here, as earlier, the symbol $h_{>a}$ stands for the set of all points $t \in \Omega$ at which $h(t) > a$). The *topology of convergence in measure* on the space of all measurable functions on Ω is the topology in which a neighborhood basis of f is provided by the sets $U_{a,\varepsilon}(f)$ with $a, \varepsilon > 0$. Accordingly, a sequence of functions (f_n) is said to *converge in measure* to the function f (written $f_n \xrightarrow{\mu} f$) if for any $a > 0$,

$$\mu\left(|f_n - f|_{>a}\right) \to 0 \quad \text{as } n \to +\infty.$$

Theorem 1. *Convergence in measure enjoys the following properties:*

A. $f_n \xrightarrow{\mu} f$ *if and only if* $f_n - f \xrightarrow{\mu} 0$.

B. *If* $f_n \xrightarrow{\mu} f$ *and* $f_n \xrightarrow{\mu} g$, *then* $f \overset{\text{a.e.}}{=} g$.

C. *If* $f_n \xrightarrow{\mu} f$ *and* $f_n \overset{\text{a.e.}}{=} g_n$, *then* $g_n \xrightarrow{\mu} f$.

Proof. Properties A and C are obvious. We prove property B. Let A be the set of all points $t \in \Omega$ at which $f(t) \neq g(t)$, and A_n the set of all points $t \in \Omega$ at which $|f(t) - g(t)| > 1/n$. Since $A = \bigcup_{n\in\mathbb{N}} A_n$, it suffices to show that $\mu(A_n) = 0$ for all n. For any $k \in \mathbb{N}$, at each point $t \in A_n$ either $|f(t) - f_k(t)| > 1/(2n)$, or $|g(t) - f_k(t)| > 1/(2n)$. Hence, if we denote by $B_{n,k}$ the set of all points at which $|f(t) - f_k(t)| > 1/(2n)$, and by $C_{n,k}$ the set of all points where $|g(t) - f_k(t)| > 1/(2n)$, then $A_n \subset B_{n,k} \cup C_{n,k}$. By the definition of convergence in measure, for fixed n and $k \to \infty$, the measures of the sets $B_{n,k}$ and $C_{n,k}$ tend to 0. Hence, $\mu(A_n)$ can only be 0. □

Theorem 2. *Let X be a family of measurable functions on Ω. Then every point in the closure of X in the topology of convergence in measure is the limit of a sequence of elements of X that converges in measure.*

Proof. We use here the idea of Exercise 6 of Subsection 1.2.1. Let f be a point in the closure of the set X. Note that the neighborhood $U_{a,\varepsilon}(f)$ increases with the growth of a, as well as with the growth of ε. Consider the neighborhoods $U_n = U_{1/n,1/n}(f)$. Clearly, $U_1 \supset U_2 \supset \cdots$ and together the sets U_n constitute a neighborhood basis of f (if $U_{a,\varepsilon}(f)$ is an arbitrary neighborhood of f, then $U_{a,\varepsilon}(f) \supset U_n$ for $n > \max\{1/a, 1/\varepsilon\}$). By the definition of the closure, all sets $X \cap U_n$ are non-empty. Pick in each set $X \cap U_n$ an element f_n. Then (f_n) is the sought-for sequence of elements of the set X that converges in measure to f. □

Example 1 (sliding hump). In the interval $[0, 1]$ consider the subintervals $\mathrm{I}_{n,k} = [\frac{k-1}{2^n}, \frac{k}{2^n}]$, $n = 0, 1, 2, \ldots, k = 1, \ldots, 2^n$. For fixed n, the intervals $\mathrm{I}_{n,k}$, $k = 1, \ldots, 2^n$, cover the whole interval $[0, 1]$. Now consider the sequence of functions $f_1 = \mathbb{1}_{[0,1]}$, $f_2 = \mathbb{1}_{[0,1/2]}$, $f_3 = \mathbb{1}_{[1/2,1]}$, …, $f_{2^n+k} = \mathbb{1}_{\mathrm{I}_{n,k}}$, …. For each $a > 0$, the set of points $x \in [0, 1]$ where $|f_{2^n+k}(x)| > a$ is either empty (if $a \geqslant 1$), or coincides with $\mathrm{I}_{n,k}$. Since the lengths of the intervals $\mathrm{I}_{n,k}$ tend to zero when $n \to \infty$, the sequence (f_n) tends to zero in measure (with respect to the Lebesgue measure). At the same time, the sequence (f_n) does not tend to zero at *any point*, since every point of the interval $[0, 1]$ belongs to infinitely many intervals $\mathrm{I}_{n,k}$. This example allows one to get a feeling for the meaning of the convergence in measure, and at the same time shows that convergence in measure is not equivalent to almost everywhere convergence.

Exercises

1. In the preceding example, find a subsequence of the sequence (f_n) that tends to 0 at every point.

2. Why are the sets $|f_n - f|_{>a}$ in the definition of convergence in measure measurable?

3. Verify that our definition of convergence in measure is correct, i.e., that convergence in the topology of convergence in measure is indeed equivalent to the condition appearing in the definition.

4. If $f_n \xrightarrow{\mu} f$, $g_n \xrightarrow{\mu} g$, and $f_n \overset{\text{a.e.}}{\leqslant} g_n$, then $f \overset{\text{a.e.}}{\leqslant} g$.

5. On the segment [0, 1] consider the sequence of functions $g_n(x) = x^n$. Show that $g_n \xrightarrow{\mu} 0$ (in the sense of the Lebesgue measure). Does this sequence converge to zero pointwise? Almost everywhere?

6. Flesh out the proof of Theorem 2.

7. $\mu\left(|f-h|_{>a}\right) \leqslant \mu\left(|f-g|_{>\frac{a}{2}}\right) + \mu\left(|g-h|_{>\frac{a}{2}}\right)$ for any measurable functions f, g, h and any $a > 0$.

8. Let $f_n \xrightarrow{\mu} f$ and $g_n \xrightarrow{\mu} g$. Then $f_n + g_n \xrightarrow{\mu} f + g$.

9. By definition, (f_n) is a Cauchy sequence in the sense of convergence in measure if $\mu(|f_n - f_m|_{>a}) \to 0$ as $n, m \to \infty$. Prove that any sequence that converges in measure is a Cauchy sequence in the above sense.

10. The sequence of functions $\sin(\pi n x)$ on [0, 1] does not tend in measure to any function; moreover, it does not contain a subsequence that converges in measure.

11. Let f_n be an increasing sequence of functions and let $f_n \xrightarrow{\mu} f$. Then $f_n \xrightarrow{\text{a.e.}} f$.

12. The expression $\rho(f, g) = \inf_{a \in (0,+\infty)} \left\{a + \mu(|f-g|_{>a})\right\}$ is a pseudometric that generates the topology of convergence in measure.

13. Another example: the pseudometric $d(f, g) = \inf\left\{a > 0 : \mu(|f-g|_{>a}) \leqslant a\right\}$ also gives the topology of convergence in measure.

14. Let (Ω, Σ, μ) be a finite measure space and let the measure μ be purely atomic. Then for functions on Ω convergence in measure is equivalent to convergence almost everywhere. If μ is not purely atomic, then these two types of convergence are not equivalent.

3.2.3 Theorems Connecting Convergence in Measure to Convergence Almost Everywhere

Definition 1. The *upper limit of a sequence of sets* (A_n) is the set $\overline{\lim}\, A_n = \bigcap_{n=1}^{\infty} \bigcup_{k=n}^{\infty} A_k$.

Another commonly used name for the same object is the *limit superior*, with the corresponding notation $\limsup_{n\to\infty} A_n$. That our use of the terms "upper limit" or "limit superior" is natural will become clear once Exercise 6 in Subsection 3.1.3 is solved.

Lemma 1 (on the upper limit of a sequence of sets). *Let $A_n \in \Sigma$ and $A_\infty = \overline{\lim}\, A_n$. Then*

(i) $\mu(A_\infty) \geqslant \overline{\lim}\, \mu(A_n)$. *In particular, if $\mu(A_\infty) = 0$, then $\mu(A_n) \to 0$ as $n \to \infty$.*

(ii) *If $\sum_{n=1}^{\infty} \mu(A_n) < \infty$, then $\mu(A_\infty) = 0$.*

Proof. Consider the sets $B_n = \bigcup_{k=n}^{\infty} A_k$. Then $A_\infty = \bigcap_{n=1}^{\infty} B_n$. Since the sets B_n form a decreasing chain,

$$\lim_{n\to\infty} \mu(B_n) = \mu(A_\infty). \tag{3.1}$$

To prove assertion (i), it remains to note that $B_n \supset A_n$, and $\mu(B_n) \geqslant \mu(A_n)$. Further, if $\sum_{n=1}^{\infty} \mu(A_n) < \infty$, then $\mu(B_n) \leqslant \sum_{k=n}^{\infty} \mu(A_k) \to 0$ as $n \to \infty$, which in view of (3.1) yields assertion (ii). □

We note that in probability theory the assertion (ii) of the preceding lemma is known as the "Borel–Cantelli lemma".

Theorem 1 (Lebesgue). *Convergence almost everywhere implies convergence in measure. Precisely, if f, f_n are measurable functions on Ω and $f_n \to f$ almost everywhere, then $f_n \xrightarrow{\mu} f$.*

Proof. By hypothesis, the set D of all points at which f_n does not converge to f is negligible (of measure zero). Fix $a > 0$. Consider the sets $A_n = |f_n - f|_{>a}$ and $A_\infty = \overline{\lim}\, A_n$. By the definition of the upper limit, $A_\infty = \bigcap_{n=1}^{\infty} \bigcup_{k=n}^{\infty} A_k$, i.e., A_∞ is the set of all points $t \in \Omega$ with the property that for any $n \in \mathbb{N}$ there exists a $k > n$ such that $|f_n(t) - f(t)| > a$. Hence, $A_\infty \subset D$ and $\mu(A_\infty) = 0$. By the preceding lemma, $\mu(A_n) \to 0$ as $n \to \infty$, i.e., $\mu\left(|f_n - f|_{>a}\right) \to 0$ as $n \to \infty$. □

Lemma 2. *Let f_n be measurable functions, and a_n and ε_n be positive numbers such that $a_n \to 0$ as $n \to \infty$ and $\sum_{n=1}^{\infty} \varepsilon_n < \infty$. Moreover, suppose that f_n satisfy the condition $\mu\left(|f_n|_{>a_n}\right) < \varepsilon_n$. Then $f_n \xrightarrow{\text{a.e.}} 0$.*

Proof. Denote by D the set of all points where f_n does not tend 0, and set $A_n = |f_n|_{>a_n}$, $B_n = \bigcup_{k=n}^{\infty} A_k$, $A_\infty = \overline{\lim}\, A_n = \bigcap_{n=1}^{\infty} B_n$. Let $t \in \Omega$ be an arbitrary point such that $f_n(t)$ does not tend to zero. For any $n \in \mathbb{N}$, there exists $k \geqslant n$ such $f_k(t) > a_k$, that is, $t \in B_n$. Hence, $D \subset B_n$ for all n, and $D \subset A_\infty$. At the same time, $\sum_{n=1}^{\infty} \mu(A_n) < \sum_{n=1}^{\infty} \varepsilon_n < \infty$ by hypothesis. Applying assertion (ii) of the lemma on the upper limit of sequence of sets, we conclude that $\mu(D) \leqslant \mu(A_\infty) = 0$. □

Theorem 2 (F. Riesz). *Any sequence of measurable functions that converges in measure contains a subsequence that converges almost everywhere.*

Proof. Suppose that $f_n \xrightarrow{\mu} f$. Fix $a_n, \varepsilon_n > 0$, such that the conditions of the preceding lemma are satisfied, and choose an increasing sequence of indices m_n such that $\mu\left(|f_{m_n} - f|_{>a_n}\right) < \varepsilon_n$. By Lemma 2, $f_{m_n} - f \xrightarrow{\text{a.e.}} 0$, hence $f_{m_n} \xrightarrow{\text{a.e.}} f$. □

Theorem 3 (convergence in measure criterion). *The sequence of measurable functions (f_n) converges in measure to the function f if and only if any subsequence of the sequence (f_n), in its turn, contains a subsequence that converges to f almost everywhere.*

Proof. Suppose $f_n \xrightarrow{\mu} f$. Then each subsequence of the sequence (f_n) also converges in measure, so by the preceding theorem, it contains a subsequence that converges to f almost everywhere. Conversely, suppose that f_n does not converge in measure to f. Then there exist a, $\varepsilon > 0$ and a subsequence (g_n) of (f_n) such that none of the functions g_n lies in the neighborhood $U_{a,\varepsilon}(f)$. It follows that the subsequence (g_n) does not contain subsequences that converge in measure to f, and hence, by Theorem 1, neither does it contain subsequences that converge almost everywhere to f. □

Corollary 1. *If $G: \mathbb{R}^2 \to \mathbb{R}$ is a continuous function, $f_n \xrightarrow{\mu} f$ and $g_n \xrightarrow{\mu} g$, then $G(f_n, g_n) \xrightarrow{\mu} G(f, g)$. In particular, it follows that $f_n + g_n \xrightarrow{\mu} f + g$ and $f_n g_n \xrightarrow{\mu} fg$.*

Proof. Use the preceding criterion and the corresponding properties of convergence almost everywhere. □

Corollary 2 *(**Theorem 1 in Subsection 3.2.1**). Let X, Y and Z be sets of measurable functions on Ω. If X is* a.e. *dense in Y and Y is* a.e. *dense in Z, then X is* a.e. *dense in Z.*

Proof. By Theorem 1, in the topology of convergence in measure X is dense in Y and Y is dense in Z. Therefore (Exercise 4 of Subsection 1.2.1), X is dense in Z in the topology of convergence in measure. Hence, by Theorem 2 of Subsection 3.2.2, the set X is *sequentially dense* in Z in the sense of convergence in measure, i.e., for any $f \in Z$ there exists a sequence (f_n) of elements of the set X such that $f_n \xrightarrow{\mu} f$. It remains to apply Theorem 2. □

Exercises

1. Solve Exercise 4 in Subsection 3.2.2 based on the results obtained in the current subsection.

2. Let (f_n) be a Cauchy sequence in the sense of convergence in measure (see Exercise 9 in Subsection 3.2.2). Then (f_n) contains a subsequence that converges almost everywhere.

3. If a sequence of measurable functions is Cauchy in the sense of convergence in measure, then it has a limit in the same sense.

4. Suppose that in some space X of measurable functions on a finite measure space almost everywhere convergence coincides with convergence in some topology τ on X. Then in X almost everywhere convergence coincides with convergence in measure.

5. Almost everywhere convergence in the space of all measurable functions on an interval cannot be given by a topology.

6. The subset of all continuous functions is a.e. dense in the space of all measurable functions on an interval.

7. Let (A_n) be a decreasing chain of sets. Then $\overline{\lim}\, A_n = \bigcap_{n=1}^{\infty} A_n$ and $\mu(\overline{\lim} A_n) = \lim_{n\to\infty} \mu(A_n)$.

8. For any increasing chain of sets A_n it also holds that $\mu(\overline{\lim}\, A_n) = \lim_{n\to\infty} \mu(A_n)$, because in this case $\overline{\lim}\, A_n = \bigcup_{n=1}^{\infty} A_n$.

9. Give an example in which $\mu(\overline{\lim}\, A_n) \neq \overline{\lim\limits_{n\to\infty}}\, \mu(A_n)$.

10. A point $t \in \Omega$ belongs to $\overline{\lim}\, A_n$ if and only if t belongs to infinitely many of the sets A_n.

11. Consider the functions $f_n = \mathbb{1}_{(n,\infty)}$ on $\mathbb{R}$. Verify that f_n converge almost everywhere on $\mathbb{R}$ to 0 but does not converge in measure. This example shows that Theorem 1 does not extend to σ-finite measure spaces.

12. Let (Ω, Σ, μ) be a σ-finite measure space, then any sequence (f_n) of measurable functions on Ω that converges in measure to a measurable function f contains a subsequence that converges to f almost everywhere. In other words, Theorem 2 remains valid in σ-finite measure spaces.

3.2.4 Egorov's Theorem

The functions $g_n(x) = x^n$ on the interval $[0, 1]$ provide a typical example of a sequence that converges at each point, but does not converge uniformly. At the same time, the convergence can be improved if one removes an arbitrarily small neighborhood of the point 1: on the remaining interval $[0, 1-\varepsilon]$ the convergence will already be uniform. A similar situation arises in the theory of power series: a series converges to its sum uniformly not in the entire disc of convergence, but in any disc of a slightly smaller radius. These facts are particular cases of the following very general result.

Theorem 1 (Egorov's theorem). *Suppose that* $f_n \xrightarrow{\text{a.e.}} f$ *on* Ω. *Then for every* $\varepsilon > 0$ *there exists a set* $A = A_\varepsilon \in \Sigma$ *with* $\mu(A) < \varepsilon$, *on the complement of which* (f_n) *converges uniformly to* f.

Proof. Fix $a_n, \varepsilon_n > 0$ such that $a_n \to 0$ as $n \to \infty$ and $\sum_{n=1}^{\infty} \varepsilon_n < \varepsilon$. Consider the sets $A_{m,n} = |f_m - f|_{>a_n}$ and $B_{m,n} = \bigcup_{k=m}^{\infty} A_{k,n}$. For fixed n, the sets $B_{m,n}$ form a chain decreasing with m, and $\mu\left(\bigcap_{m=1}^{\infty} B_{m,n}\right) = 0$ (since $\bigcap_{m=1}^{\infty} B_{m,n}$ is included in the negligible set D consisting of all points at which f_n does not tend to f). Consequently, $\mu(B_{m,n}) \to 0$ as $m \to \infty$. Now for each n pick an index m_n such that $\mu(B_{m_n,n}) < \varepsilon_n$. Let us prove that $A = \bigcup_{n=1}^{\infty} B_{m_n,n}$ is the required set. First, $\mu(A) \leqslant \sum_{n=1}^{\infty} \varepsilon_n < \varepsilon$. Further, $\Omega \setminus A \subset \Omega \setminus B_{m_n,n}$, that is, for every $k > m_n$ the set $A_{k,n} = |f_k - f|_{>a_n}$ does not contain points of $\Omega \setminus A$. It follows that $\sup_{t \in \Omega \setminus A} |f_k(t) - f(t)| \leqslant a_n$ for $k > m_n$, which establishes the uniform convergence on $\Omega \setminus A$. □

Exercises

1. Use Exercise 6 in Subsection 3.2.3 and Egorov's theorem to obtain the following result: **Luzin's theorem.** *For any Lebesgue-measurable function f on the interval [a,b] and any* $\varepsilon > 0$ *there exists a measurable set A with* $\mu(A) < \varepsilon$, *such that the restriction of f to* $[a, b]\setminus A$ *is continuous.*

2. Show that in the statement of Luzin's theorem the set A can be chosen to be open.

3. In the statement of Egorov's theorem, can the condition $\mu(A) < \varepsilon$ be replaced by $\mu(A) = 0$? What about the analogous question for Luzin's theorem?

4. In the statement of Egorov's theorem, can the sequence f_n, which converges almost everywhere, be replaced by a sequence which converges in measure?

5. Where in Egorov's theorem did the measurability of the involved functions play a role?

Comments on the Exercises

Subsection 3.1.1

Exercise 2. Denote the supremum of the values of the function f on $[a, b]$ by m. Then the set of maximum points of f coincides with $f_{=m}$.

Exercise 3. Write all intervals with rational endpoints as a sequence (a_n, b_n), $n \in \mathbb{N}$, and denote the set of points of "true" maximum of the function f on (a_n, b_n) by M_n. The sought-for set of local maxima of f coincides with $\bigcup_{n=1}^{\infty} M_n$.

Exercise 4. Take as (Ω_1, Σ_1) the interval $[0, 1]$ endowed with the σ-algebra of Lebesgue-measurable sets, and take for (Ω_2, Σ_2) the same interval with the σ-algebra of Borel sets, and for f the identity map.

Exercise 13. (a) No (even for the function $g(x) = x$).

(b) No. Let g be the Cantor staircase (Subsection 2.3.6), extended to $(-\infty, 0)$ by 0, and to $(1, +\infty)$ by 1. Let $B \subset [0, 1]$ be a set that is not Lebesgue measurable. With no loss of generality, we may assume that B consists only of irrational points (otherwise one can replace B by $B \setminus \mathbb{Q}$). As the required A take $g^{-1}(B)$. Then A is a subset of the Cantor set, whence $\lambda(A) = 0$, so A is Lebesgue measurable. However, $f(A) = B$ is not measurable.

(c) It can. To produce an example, one needs to come up with a continuous strictly monotone function which maps some set of positive measure into a set of measure 0.

Exercise 14. One needs to represent A as the union of a sequence of compact sets and recall that the image of a compact set under a continuous map is again compact.

Exercise 15. It can. The author is not aware of a simple example. A set that is the image of a Borel set under a continuous map is called an *analytic set*, or a *projective set of class* 1. The existence of an analytic set that is not Borel is a particular case of Theorem VI in §38 of the monograph [25, vol. 1].

Subsection 3.2.3

Exercise 6. Continuous functions can be used to approximate characteristic functions of intervals; linear combinations of characteristic functions of intervals can in turn be used to approximate characteristic functions of open sets; then characteristic functions of open sets to approximate characteristic functions of arbitrary Lebesgue-measurable sets; then linear combinations of characteristic functions of measurable sets (i.e., finitely-valued functions) to approximate simple functions; and finally, simple functions to approximate arbitrary measurable functions. An analogous statement will be proved in a considerably more general situation in Subsection 8.3.3.

Exercise 12. Write Ω as a disjoint union of sets Ω_m, $m = 1, 2, \ldots$, of finite measure. Successively applying on each set Ω_m the theorem asserting that from any sequence that converges in measure one can extract an almost-everywhere convergent subsequence, we construct an infinite sequence of sets of indices $\mathbb{N} \supset N_1 \supset N_2 \supset N_3 \supset \cdots$ such that on each of the sets Ω_m the sequence $\{f_n\}_{n \in N_m}$ converges almost everywhere. Picking a diagonal subsequence n_m (i.e., one for which $n_1 \in N_1$, $n_2 \in N_2$ and $n_2 > n_1$, $n_3 \in N_3$ and $n_3 > n_2$, and so on), we obtain a subsequence f_{n_m} which converges almost everywhere on each set Ω_j, i.e., converges almost everywhere on $\Omega = \bigsqcup_{j=1}^{\infty} \Omega_j$.

Subsection 3.2.4

Exercise 1. In a more general situation Luzin's theorem will be proved in Subsection 8.3.3.