Chapter 16
Topological Vector Spaces

16.1 Supplementary Material from Topology

We have already encountered a very general type of convergence — convergence along a directed set. We now turn to yet another type, convergence along a filter, and apply this new technique to the study of compact topological spaces. Throughout this chapter we will have to frequently deal, within one and the same argument, with sets as well as some families of subsets. To make it easier to distinguish these objects, we will denote sets by upper case Roman italic letters A, B, X, Y, and so on, and use for families Gothic letters $\mathfrak{A}$, $\mathfrak{C}$, $\mathfrak{D}$, $\mathfrak{F}$. Of course, the difference here is rather conventional, since any family of sets is itself a set.

16.1.1 Filters and Filter Bases

Definition 1. A family $\mathfrak{F}$ of subsets of a set X is called a *filter on* X if it satisfies the following axioms:

(i) $\mathfrak{F}$ is not empty;

(ii) $\emptyset \notin \mathfrak{F}$;

(iii) if $A, B \in \mathfrak{F}$, then $A \cap B \in \mathfrak{F}$;

(iv) if $A \in \mathfrak{F}$ and $A \subset B \subset X$, then $B \in \mathfrak{F}$.
Let us note several consequences of the filter axioms:

(v) $X \in \mathfrak{F}$ (follows from (i) and (iv));

(vi) in view of (iii), the intersection of any finite number of elements of a filter is again an element of that filter; from (ii) we deduce that

(vii) the intersection of any finite number of elements of a filter is not empty.

V. Kadets, *A Course in Functional Analysis and Measure Theory*,
Universitext, https://doi.org/10.1007/978-3-319-92004-7_16

An example of a filter is provided by the family $\mathfrak{N}_x$ of all neighborhoods of a point x in a topological space X.

Definition 2. A non-empty family $\mathfrak{D}$ of subsets of a set X is called a *filter basis* (also *base* in the literature) if

(a) $\emptyset \notin \mathfrak{D}$, and

(b) for any sets $A, B \in \mathfrak{D}$ there exists a set $C \in \mathfrak{D}$ such that $C \subset A \cap B$.

Let $\mathfrak{D}$ be a filter basis. The *filter generated by the basis* $\mathfrak{D}$ is the family $\mathfrak{F}$ of all sets $A \subset X$ such that A contains as a subset at least one element of $\mathfrak{D}$.

We leave to the reader to verify that this definition is correct, i.e., that what we call the filter $\mathfrak{F}$ generated by the basis $\mathfrak{D}$ is indeed a filter.

If X is a topological space and $x_0 \in X$, and as a basis $\mathfrak{D}$ we take the family of all open sets that contain x_0, then the filter generated by the basis $\mathfrak{D}$ is precisely the filter $\mathfrak{N}_{x_0}$ of all neighborhoods of the point x_0.

Let us give one more example. Let $(x_n)_{n=1}^{\infty}$ be a sequence of elements of the set X. Then the family $\mathfrak{D}_{(x_n)}$ of "tails" of the sequence (x_n) (i.e., the family of sets of the form $\{x_n\}_{n=N}^{\infty}$, $N \in \mathbb{N}$) is a filter basis. The filter $\mathfrak{F}_{(x_n)}$ generated by the basis $\mathfrak{D}_{(x_n)}$ is called the *filter associated with the sequence* (x_n).

Theorem 1. *Let X, Y be sets, $f : X \to Y$ a mapping, and $\mathfrak{D}$ a filter basis in X. Then the family $f(\mathfrak{D})$ of all images $f(A)$ with $A \in \mathfrak{D}$ is a filter basis in Y.*

Proof. Axiom (a) in the definition of a filter basis is obvious. Further, let $f(A)$ and $f(B)$ be arbitrary elements of $f(\mathfrak{D})$, $A, B \in \mathfrak{D}$. By axiom (b), there exists a $C \in \mathfrak{D}$, such that $C \subset A \cap B$. Then $f(C) \subset f(A) \cap f(B)$, which proves (b) for $f(\mathfrak{D})$. □

In particular, if $\mathfrak{F}$ is a filter on X, then $f(\mathfrak{F})$ is a filter basis in Y.

Definition 3. The *image of the filter* $\mathfrak{F}$ under the mapping f is the filter $f[\mathfrak{F}]$ generated by the filter basis $f(\mathfrak{F})$. Equivalently, $A \in f[\mathfrak{F}]$ if and only if $f^{-1}(A) \in \mathfrak{F}$.

Recall (see Subsection 1.2.3) that a family of sets $\mathfrak{C}$ is said to be *centered* if the intersection of any finite collection of members of $\mathfrak{C}$ is not empty.

Theorem 2. *Let $\mathfrak{C} \subset 2^X$ be a non-empty family of sets. For the existence of a filter $\mathfrak{F}$ such that $\mathfrak{F} \supset \mathfrak{C}$ (i.e., such that all elements of $\mathfrak{C}$ are also elements of the filter $\mathfrak{F}$) it is necessary and sufficient that $\mathfrak{C}$ be a centered family.*

Proof. If $\mathfrak{F}$ is a filter and $\mathfrak{F} \supset \mathfrak{C}$, then any finite collection $A_1, \ldots, A_n$ of elements of the family $\mathfrak{C}$ will consist of elements of the filter $\mathfrak{F}$. Hence (property (vii) of filters), $\bigcap_{k=1}^{n} A_k \neq \emptyset$. Necessity is thus proved. Conversely, suppose $\mathfrak{C}$ is a centered family. Then the family $\mathfrak{D}$ of all sets of the form $\bigcap_{k=1}^{n} A_k$, where $n \in \mathbb{N}$ and $A_1, \ldots, A_n \in \mathfrak{C}$, is a filter basis. Now for $\mathfrak{F}$ one needs to take the filter generated by the basis $\mathfrak{D}$. □

Definition 4. Let $\mathfrak{F}$ be a filter on X. A family $\mathfrak{D}$ of subsets is said to be a *basis of the filter* $\mathfrak{F}$ if $\mathfrak{D}$ is a filter basis and the filter generated by $\mathfrak{D}$ coincides with $\mathfrak{F}$.

Theorem 3. *For the family $\mathfrak{D}$ to be a basis of the filter $\mathfrak{F}$, it is necessary and sufficient that the following two conditions be satisfied:*

— $\mathfrak{D} \subset \mathfrak{F}$;

— *for any $A \in \mathfrak{F}$ there exists a $B \in \mathfrak{D}$ such that $B \subset A$.*

Definition 5. Let $\mathfrak{F}$ be a filter on X and $A \subset X$. The *trace of the filter $\mathfrak{F}$ on A* is the family of subsets $\mathfrak{F}_A = \{A \cap B : B \in \mathfrak{F}\}$.

Theorem 4. *For the family $\mathfrak{F}_A$ to be a filter on A, it is necessary and sufficient that all intersections $A \cap B$ with $B \in \mathfrak{F}$ are non-empty. In particular, $\mathfrak{F}_A$ will be a filter whenever $A \in \mathfrak{F}$.*

Exercises

1. Prove Theorems 3 and 4.

Below we give examples of filters and filter bases. Many of these examples will be used in the sequel. The reader is invited to verify the corresponding axioms.

2. The *Fréchet filter* on $\mathbb{N}$: the elements of this filter are the complements of the finite sets of natural numbers. A basis of the Fréchet filter is provided by the sets $A_1 = \{1, 2, 3, \ldots\}$, $A_2 = \{2, 3, 4, \ldots\}$, $\ldots$, $A_n = \{n, n+1, n+2, \ldots\}$, $\ldots$.

3. The *neighborhood filter of infinity* in a normed space X: the set $A \subset X$ lies in this filter if the set $X \setminus A$ is bounded.

4. The filter $\mathfrak{N}_x^0$ of *deleted* (or *punctured*) *neighborhoods* of a given point x in a topolgical space X: a basis of this filter consists of the sets of the form $U \setminus \{x\}$, where U is a neighborhood of x. For this definition to be correct, it is necessary that the point x is not isolated.

5. The neighborhood filters of the point $+\infty$ in $\mathbb{R}$: a basis of the filter consists of the intervals $(a, +\infty)$ with $a \in \mathbb{R}$.

6. The filter of deleted neighborhoods of the "point" $a + 0$ in $\mathbb{R}$: a basis of this filter consists of the sets (a, b) with $b \in (a, +\infty)$.

7. The *statistical filter* $\mathfrak{F}_s$ on $\mathbb{N}$: $A \in \mathfrak{F}_s$ if $\lim_{n\to\infty} |A \cap \{1, 2, \ldots, n\}|/n = 1$. Here $|B|$ denotes the cardinality of the set B.

8. Let $(G, \succ)$ be a directed set. The *section filter* on G is the filter $\mathfrak{F}_{\succ}$, a basis of which consists of all sets of the form $\{x \in G : x \succ a\}$ with $a \in G$.

Prove that

9. The set of all filters on $\mathbb{N}$ is not countable. In fact, the cardinality of this set is bigger than the cardinality of the continuum.

10. The filters in Execises 3, 5, and 6 have countable bases.

11. The statistical filter (Exercise 7) does not have a countable basis.

12. Let $(x_n)_{n=1}^{\infty}$ be a sequence in X, and let the function $f : \mathbb{N} \to X$ act by the rule $f(n) = x_n$. Then the image of the Fréchet filter from Exercise 2 under f is the filter $\mathfrak{F}_{(x_n)}$ associated with the sequence (x_n).

16.1.2 Limits, Limit Points, and Comparison of Filters

Definition 1. Suppose given two filters $\mathfrak{F}_1$ and $\mathfrak{F}_2$ on the topological space X. We say that $\mathfrak{F}_1$ *majorizes* $\mathfrak{F}_2$ if $\mathfrak{F}_1 \supset \mathfrak{F}_2$; in other words, if every element of the filter $\mathfrak{F}_2$ is also an element of the filter $\mathfrak{F}_1$.

Example 1. Let $(x_n)_{n\in\mathbb{N}}$ be a sequence in X, and $(x_{n_k})_{k\in\mathbb{N}}$ be a subsequence of $(x_n)_{n\in\mathbb{N}}$. Then the filter $\mathfrak{F}_{(x_{n_k})}$ associated with the subsequence majorizes the filter $\mathfrak{F}_{(x_n)}$ associated with the sequence itself. Indeed, let $A \in \mathfrak{F}_{(x_n)}$. Then there exists an $N \in \mathbb{N}$ such that $\{x_n\}_{n=N}^{\infty} \subset A$. But then also $\{x_{n_k}\}_{k=N}^{\infty} \subset A$, that is, $A \in \mathfrak{F}_{(x_{n_k})}$.

Definition 2. Let X be a topological space, and $\mathfrak{F}$ a filter on X. The point $x \in X$ is called the *limit of the filter* $\mathfrak{F}$ (denoted $x = \lim \mathfrak{F}$) if $\mathfrak{F}$ majorizes the neighborhood filter of the point x. In other words, $x = \lim \mathfrak{F}$ if every neighborhood of the point x belongs to the filter $\mathfrak{F}$.

The point $x \in X$ is said to be a *limit point of the filter* $\mathfrak{F}$ if every neighborhood of x intersects all elements of the filter $\mathfrak{F}$. The set of all limit points of the filter $\mathfrak{F}$ is denoted by $\mathrm{LIM}(\mathfrak{F})$.

Example 2. Let $(x_n)_{n\in\mathbb{N}}$ be a sequence in the topological space X. Then $\lim \mathfrak{F}_{(x_n)} = \lim_{n\to\infty} x_n$, and $\mathrm{LIM}(\mathfrak{F}_{(x_n)})$ coincides with the set of limit points of the sequence $(x_n)_{n\in\mathbb{N}}$.

Theorem 1. *Let $\mathfrak{F}$ be a filter on the topological space X, and $\mathfrak{D}$ be a basis for the filter $\mathfrak{F}$. Then*

(a) $x = \lim \mathfrak{F}$ *if and only if for any neighborhood U of the point x there exists an element $A \in \mathfrak{D}$ such that $A \subset U$.*

(b) *If* $x = \lim \mathfrak{F}$*, then* x *is a limit point of the filter* $\mathfrak{F}$*. If, in addition,* X *is a Hausdorff space, then the filter* $\mathfrak{F}$ *has no other limit points. In particular, if a filter in a Hausdorff space has a limit, then this limit is unique.*

(c) *The set* $\mathrm{LIM}(\mathfrak{F})$ *coincides with the intersection of the closures of all elements of the filter* $\mathfrak{F}$*.*

Proof. (a) By definition, $x = \lim \mathfrak{F}$ if any neighborhood U of the point x belongs to the filter $\mathfrak{F}$. In its turn, $U \in \mathfrak{F}$ if and only if U contains some set $A \in \mathfrak{D}$.

(b) Let $x = \lim \mathfrak{F}$, and let U be a neighborhood of x. Then $U \in \mathfrak{F}$, hence any set $A \in \mathfrak{F}$ intersects U. That is, $x \in \mathrm{LIM}(\mathfrak{F})$.

Further, let $x = \lim \mathfrak{F}$, $y \in \mathrm{LIM}(\mathfrak{F})$, and let U and V be arbitrary neighborhoods of the points x and y, respectively. Then $U \in \mathfrak{F}$, and since any neighborhood of a limit point intersects all elements of the filter $\mathfrak{F}$, $U \cap V \neq \emptyset$. Since the space is Hausdorff, this is possible only if $x = y$.

(c) By definition, $x \in \mathrm{LIM}(\mathfrak{F})$ if and only if every element $A \in \mathfrak{F}$ intersects all neighborhoods of the point x. This is equivalent to x belonging to the closure of every element $A \in \mathfrak{F}$. □

Theorem 2. *Suppose* $\mathfrak{F}_1$, $\mathfrak{F}_2$ *are filters in the topological space* X*, and* $\mathfrak{F}_1 \subset \mathfrak{F}_2$*. Then:*

(i) *if* $x = \lim \mathfrak{F}_1$*, then* $x = \lim \mathfrak{F}_2$*;*

(ii) *if* $x \in \mathrm{LIM}(\mathfrak{F}_2)$*, then* $x \in \mathrm{LIM}(\mathfrak{F}_1)$*. In particular,*

(iii) *if* $x = \lim \mathfrak{F}_2$*, then* $x \in \mathrm{LIM}(\mathfrak{F}_1)$*.*

Proof. (i) $\mathfrak{F}_1$ majorizes the neighborhood filter $\mathfrak{N}_x$ of the point x and $\mathfrak{F}_1 \subset \mathfrak{F}_2$, therefore $\mathfrak{N}_x \subset \mathfrak{F}_2$.

(ii) Since as the collection of sets increases their intersection decreases, we have $\mathrm{LIM}(\mathfrak{F}_2) = \bigcap_{A \in \mathfrak{F}_2} \overline{A} \subset \bigcap_{A \in \mathfrak{F}_1} \overline{A} = \mathrm{LIM}(\mathfrak{F}_1)$. □

Definition 3. Let X be a set, Y a topological space, and $\mathfrak{F}$ a filter in X. The point $y \in Y$ is called the *limit of the mapping* $f : X \to Y$ *with respect to the filter* $\mathfrak{F}$ (denoted $y = \lim_{\mathfrak{F}} f$), if $x = \lim f[\mathfrak{F}]$. In other words, $y = \lim_{\mathfrak{F}} f$ if for any neighborhood U of the point y there exists an element $A \in \mathfrak{F}$ such that $f(A) \subset U$.

The point $y \in Y$ is called a *limit point of the mapping* $f : X \to Y$ *with respect to the filter* $\mathfrak{F}$ if $y \in \mathrm{LIM}(f[F])$, i.e., if any neighborhood of the point y intersects the images of all elements of the filter $\mathfrak{F}$ under f.

Example 3. Let X be a topological space, $f : \mathbb{N} \to X$, and $\mathfrak{F}$ be the Fréchet filter on $\mathbb{N}$ (see Exercise 2 in Subsection 16.1.1). Then $\lim_{\mathfrak{F}} f = \lim_{n\to\infty} f(n)$.

Theorem 3. *Let* X, Y *be topological spaces,* $\mathfrak{F}$ *a filter in* X*,* $x = \lim \mathfrak{F}$*, and* $f : X \to Y$ *a continuous mapping. Then* $f(x) = \lim_{\mathfrak{F}} f$*.*

Proof. Let U be an arbitrary neighborhood of the point $f(x)$. Then there exists a neighborhood V of the point x such that $f(V) \subset U$. The condition $x = \lim \mathfrak{F}$ means that $V \in \mathfrak{F}$. That is, for any neighborhood U of the point $f(x)$ we have found the required element $V \in \mathfrak{F}$ for which $f(V) \subset U$. □

Exercises

To avoid complicating the formulations connected with the possible non-uniqueness of the limit, in the exercises below all the topological spaces are assumed to be Hausdorff.

1. Let $(G, \succ)$ be a directed set, X a topological space, $f: G \to X$ a mapping, and $\mathfrak{F}_{\succ}$ the section filter on G (see Exercise 8 in Subsection 16.1.1). Then $\lim\limits_{\mathfrak{F}_{\succ}} f = \lim\limits_{(G,\succ)} f$. Thus, the limit with respect to a directed set is a particular case of a limit with respect to a filter.

2. Suppose the subspace A of the topological space X intersects all the elements of the filter $\mathfrak{F}$. Let $\mathfrak{F}_A = \{A \cap B : B \in \mathfrak{F}\}$ be the trace of the filter $\mathfrak{F}$ on A. Then $\mathrm{LIM}(\mathfrak{F}_A) \subset \mathrm{LIM}(\mathfrak{F})$.

3. Let $A \in \mathfrak{F}$. Then the existence of $\lim \mathfrak{F}_A$ in the topology induced on A implies the existence of $\lim \mathfrak{F}$ and $\lim \mathfrak{F}_A = \lim \mathfrak{F}$.

4. Let $\lim \mathfrak{F} = a \in A$. Then $\lim \mathfrak{F}_A = a$.

5. Let X and Y be topological spaces, and $\mathfrak{N}_x$ be the neighborhood filter of the point $x \in X$. A mapping $f: X \to Y$ is continuous at the point x if and only if the limit $\lim\limits_{\mathfrak{N}_x} f$ exists. If this limit exists, then it is equal to $f(x)$.

6. Let X and Y be topological spaces, and $\mathfrak{N}_x^0$ be the filter of deleted neighborhoods of the point $x \in X$ (see Exercise 4 in Subsection 16.1.1), and suppose x is not an isolated point. Then the continuity of the mapping $f: X \to Y$ at the point x is equivalent to the condition $\lim\limits_{\mathfrak{N}_x^0} f = f(x)$.

7. In the topological space X, consider the filter $\mathfrak{F}$ consisting of all sets that contain a fixed set $A \subset X$. Then $\mathrm{LIM}(\mathfrak{F})$ coincides with the closure of the set A.

8. Based on Exercises 5 and 6 of Subsection 16.1.1, write for a function f of a real variable the expressions $\lim\limits_{x\to+\infty} f(x)$ and $\lim\limits_{x\to a+0} f(x)$ as the limits of the function with respect to appropriately chosen filters on $\mathbb{R}$.

9. For a function f of a real variable, write the expressions $\lim\limits_{x\to\infty} f(x)$, $\lim\limits_{x\to a} f(x)$, $\lim\limits_{x\to a-0} f(x)$, and $\lim\limits_{x\to-\infty} f(x)$ as limits with respect to a filter.

10. Let $\mathfrak{F}$ be a filter on the set X. A sequence $x_n \in X$ is said to be *cofinal* for the filter $\mathfrak{F}$ if $\mathfrak{F}_{\{x_n\}} \supset \mathfrak{F}$. If the filter $\mathfrak{F}$ has a countable basis, then there exists a cofinal sequence for $\mathfrak{F}$.

11. For the statistical filter $\mathfrak{F}_s$ (Exercise 7 in Subsection 16.1.1) there exists no cofinal sequence.

12. On the interval [0, 1] consider the filter consisting of all the sets with finite complement. This filter does not have a countable basis, yet it possesses a cofinal sequence. (More precisely, any sequence $x_n \in [0, 1]$ of pairwise distinct numbers is cofinal for this filter.)

13. Let X be a set, Y a topological space, $f : X \to Y$ a mapping, and $y = \lim\limits_{\mathfrak{F}} f$. If the sequence $x_n \in X$ is cofinal for the filter $\mathfrak{F}$, then $f(x_n) \to y$ as $n \to \infty$. In particular, if the filter $\mathfrak{F}$ on the set X has a countable basis, then there exists a sequence $x_n \in X$ such that $f(x_n) \to y$ as $n \to \infty$.

14. If the filter $\mathfrak{F}$ on the set X does not possess a cofinal sequence, then there exist a topological space Y and a mapping $f : X \to Y$, which has a limit y with respect to $\mathfrak{F}$, such that no sequence of the form $(f(x_n))$, with $x_n \in X$, converges to y.

16.1.3 Ultrafilters. Compactness Criteria

In the preceding subsection, we introduced the order relation $\supset$ on the family of filters given on a set X. The next lemma justifies the application of Zorn's lemma to the family of filters.

Lemma 1. *Let $\mathfrak{M}$ be a linearly ordered non-empty family of filters given on the set X, i.e., for any $\mathfrak{F}_1, \mathfrak{F}_2 \in \mathfrak{M}$, either $\mathfrak{F}_1 \supset \mathfrak{F}_2$, or $\mathfrak{F}_2 \supset \mathfrak{F}_1$. Then the union $\mathfrak{F}$ of all filters in the family $\mathfrak{M}$ is again a filter on X.*

Proof. We need to verify that the family of sets $\mathfrak{F}$ satisfies the filter axioms. The axioms (i) and (ii) are obvious here, so let us establish that the remaining two are satisfied.

(iii) Let $A, B \in \mathfrak{F}$. Then there exist filters $\mathfrak{F}_1, \mathfrak{F}_2 \in \mathfrak{M}$, such that $A \in \mathfrak{F}_1$ and $B \in \mathfrak{F}_2$. By hypothesis, one of the filters $\mathfrak{F}_1, \mathfrak{F}_2$ majorizes the other. Suppose, for instance, that $\mathfrak{F}_2 \supset \mathfrak{F}_1$. Then both sets A, B lie in $\mathfrak{F}_2$, and since $\mathfrak{F}_2$ is a filter, it follows that $A \cap B \in \mathfrak{F}_2 \subset \mathfrak{F}$.

(iv) Let $A \in \mathfrak{F}$ and $A \subset B \subset X$. Then there exists a filter $\mathfrak{F}_1 \in \mathfrak{M}$ such that $A \in \mathfrak{F}_1$. Since $\mathfrak{F}_1$ is a filter, also $B \in \mathfrak{F}_1 \subset \mathfrak{F}$. $\square$

Definition 1. An *ultrafilter* on X is a filter on X that is maximal with respect to inclusion. In detail, the filter $\mathfrak{A}$ on X is called an ultrafilter if any filter $\mathfrak{F}$ on X that majorizes $\mathfrak{A}$ necessarily coincides with $\mathfrak{A}$.

Zorn's lemma yields the following existence theorem.

Theorem 1. *For any filter $\mathfrak{F}$ on X there exists an ultrafilter that majorizes it.* □

Lemma 2. *Suppose $\mathfrak{A}$ is an ultrafilter, $A \subset X$, and all elements of $\mathfrak{A}$ intersect A. Then $A \in \mathfrak{A}$.*

Proof. It is readily seen that when one adds to the family of sets $\mathfrak{A}$ the set A as a new element one obtains a centered family of sets. By Theorem 2 in Subsection 16.1.1, there exists a filter $\mathfrak{F}$ which contains all elements of this centered family. We have that $\mathfrak{F} \supset \mathfrak{A}$, and $\mathfrak{A}$ is an ultrafilter, that is, $\mathfrak{F} = \mathfrak{A}$. On the other hand, by construction, $A \in \mathfrak{F}$. Hence, $A \in \mathfrak{A}$. □

Theorem 2 (ultrafilter criterion). *For the filter $\mathfrak{A}$ on X to be an ultrafilter it is necessary and sufficient that for any set $A \subset X$, either A itself or $X \setminus A$ belongs to $\mathfrak{A}$.*

Proof. Necessity. Suppose $\mathfrak{A}$ is an ultrafilter and $X \setminus A \notin \mathfrak{A}$. Then no set $B \in \mathfrak{A}$ is entirely contained in $X \setminus A$, i.e., every $B \in \mathfrak{A}$ intersects A. Hence, by Lemma 2, $A \in \mathfrak{A}$.

Sufficiency. Suppose that $\mathfrak{A}$ is not an ultrafilter. Then there exist a filter $\mathfrak{F} \supset \mathfrak{A}$ and a set $A \in \mathfrak{F} \setminus \mathfrak{A}$. By construction, $A \notin \mathfrak{A}$. On the other hand, $X \setminus A$ does not intersect A, $A \in \mathfrak{F}$, and consequently $X \setminus A$ cannot belong to the filter $\mathfrak{F}$, and the more so not to the filter $\mathfrak{A}$, which is smaller than $\mathfrak{F}$. □

Corollary 1. *The image of any ultrafilter is an ultrafilter.*

Proof. Let $f: X \to Y$ and let $\mathfrak{A}$ be an ultrafilter on X. Consider an arbitrary set $A \subset Y$. Then either $f^{-1}(A)$, or $f^{-1}(Y \setminus A) = X \setminus f^{-1}(A)$ belongs to $\mathfrak{A}$. It follows that either A or $Y \setminus A$ belongs to $f[\mathfrak{A}]$. □

Lemma 3. *Let $\mathfrak{A}$ be an ultrafilter on the Hausdorff topological space X and $x \in$ $\mathrm{LIM}(\mathfrak{A})$. Then $x = \lim \mathfrak{A}$. In particular, an ultrafilter can have at most one limit point.*

Proof. Let U be an arbitrary neighborhood of the point x. Then, by the definition of a limit point, U intersects all elements of $\mathfrak{A}$. By Lemma 2, $U \in \mathfrak{A}$. □

Theorem 3 (compactness criteria in terms of filters). *For a Hausdorff topological space X, the following conditions are equivalent:*

(1) *X is compact;*

(2) *every filter on X has a limit point;*

(3) *every ultrafilter on X has a limit.*

Proof. We will successively establish the equivalence of the listed conditions.

(1) $\Longrightarrow$ (2). The filter $\mathfrak{F}$ is a centered family of sets. All the more the family of closures of the elements of the filter is also centered. Consequently (Theorem 1 of Subsection 1.2.3), the intersection $\mathrm{LIM}(\mathfrak{F})$ of these closures is not empty.

(2) $\Longrightarrow$ (1). Let $\mathfrak{C}$ be an arbitrary centered family of closed subsets of the space X. By Theorem 2 of Subsection 16.1.1, there exists a filter $\mathfrak{F} \supset \mathfrak{C}$. Then $\bigcap_{A \in \mathfrak{C}} \overline{A} \supset \bigcap_{A \in \mathfrak{F}} \overline{A} = \mathrm{LIM}(\mathfrak{F}) \neq \emptyset$.

(2) $\Longrightarrow$ (3). By condition (2), every ultrafilter has a limit point, and by Lemma 3 this point is the limit of the ultrafilter.

(3) $\Longrightarrow$ (2). Consider an arbitrary filter $\mathfrak{F}$ on X and choose (Theorem 1) an ultrafilter $\mathfrak{A} \supset \mathfrak{F}$. By (3), the ultrafilter $\mathfrak{A}$ has a limit $x \in X$. By assertion (iii) of Theorem 2 in Subsection 16.1.2, x is a limit point of the filter $\mathfrak{F}$. □

Corollary 2. *Suppose $\mathfrak{A}$ is an ultrafilter on E, X a topological space, and the image of the mapping $f: E \to X$ lies in a compact subset $K \subset X$. Then there exists the limit* $\lim\limits_{\mathfrak{A}} f$.

Proof. Consider f as a mapping acting from E into K. Since (Corollary 1) $f[\mathfrak{A}]$ is an ultrafilter on the compact space K, there exists the limit $\lim f[\mathfrak{A}]$. But, by definition, $\lim\limits_{\mathfrak{A}} f = \lim f[\mathfrak{A}]$. □

Exercises

1. Let E be a set and e be an element of E. Verify that the family $\mathfrak{A}_e \subset 2^E$ of all sets containing e constitutes an ultrafilter on E. Ultrafilters of this form are called *trivial ultrafilters*.

2. Let E be a set, X a topological space, and $e \in E$. Then $f(e) = \lim\limits_{\mathfrak{A}_e} f$ for any mapping $f: E \to X$.

3. Prove that on any infinite set there exist non-trivial ultrafilters. It is interesting that to construct an explicit example of a non-trivial ultrafilter is in principle impossible: such a construction necessarily relies on the Axiom of Choice or on Zorn's lemma.

4. Let $\mathfrak{A}$ be an ultrafilter on E. Use induction on n to show that if an element $A \in \mathfrak{A}$ is covered a finite number of sets: $A \subset \bigcup_{k=1}^{n} A_k$, then at least one of the sets A_k belongs to $\mathfrak{A}$.

5. Every ultrafilter on a finite set E is trivial.

6. Let $\mathfrak{A}$ be an ultrafilter on $\mathbb{N}$. Then either $\mathfrak{A}$ is trivial, or $\mathfrak{A}$ majorizes the Fréchet filter.

7. Let $\mathfrak{A}$ be an ultrafilter on $\mathbb{N}$ which majorizes the Fréchet filter. Then $f \mapsto \lim_{\mathfrak{A}} f$ is a continuous linear functional on ℓ_∞ (recall that sequences $f = (f_1, f_2, \ldots)$, i.e., elements of the space ℓ_∞, can be regarded as bounded functions on $\mathbb{N}$ with values in the corresponding field of scalars $\mathbb{R}$ or $\mathbb{C}$). Based on this example, show that $(\ell_\infty)^* \neq \ell_1$.

8. Let $\mathfrak{A}_1, \mathfrak{A}_2$ be ultrafilters on $\mathbb{N}$, $\mathfrak{A}_1 \neq \mathfrak{A}_2$. Then there exists an $f \in \ell_\infty$ for which $\lim_{\mathfrak{A}_1} f \neq \lim_{\mathfrak{A}_2} f$.

9. For each set $A \subset \mathbb{N}$ we denote by $\mathcal{U}_A$ the family of all ultrafilters on $\mathbb{N}$ that have A as an element. We equip the set $\beta\mathbb{N}$ of all ultrafilters on $\mathbb{N}$ with the following topology: the neighborhoods of the ultrafilter $\mathfrak{A}$ are all sets $\mathcal{U}_A$ with $A \in \mathfrak{A}$, as well as all the larger sets. More formally: the topology on $\beta\mathbb{N}$ is specified by neighborhood bases (see Subsection 1.2.1); as a neighborhood basis of the element $\mathfrak{A} \in \beta\mathbb{N}$ one takes the family $\mathcal{U}_{\mathfrak{A}} = \{\mathcal{U}_A : A \in \mathfrak{A}\}$.[1] Verify the axioms given in Subsection 1.2.1, the satisfaction of which is necessary for the specification of a topology by means of neighborhoods.

10. Identify the trivial ultrafilter $\mathfrak{A}_n$, generated by the point $n \in \mathbb{N}$, with the point n itself. Under this identification, $\mathbb{N} \subset \beta\mathbb{N}$. Prove that $\mathbb{N}$ is a dense subset of the topological space $\beta\mathbb{N}$, i.e., $\beta\mathbb{N}$ is separable.

11. Let $\mathfrak{A}$ be an ultrafilter on $\mathbb{N}$ which majorizes the Fréchet filter. For each $x = (x_1, x_2, \ldots) \in \ell_\infty$ define $F(x)$ as the limit with respect to $\mathfrak{A}$ of the function f given by $f(n) = (x_1 + x_2 + \cdots + x_n)/n$. Verify that the functional F is invariant under translations. By this construction you will obtain a proof of the existence of the generalized Banach limit (see the exercises in Subsection 5.5.2) that does not resort to the Hahn–Banach theorem.

16.1.4 The Topology Generated by a Family of Mappings. The Tikhonov Product

Suppose that on the set X there is given a family of mappings $\mathcal{F}$, where the mappings $f \in \mathcal{F}$ act in respective (possibly different) topological spaces $f(X)$. For any point $x \in X$, any finite family of mappings $\{f_k\}_{k=1}^n \subset \mathcal{F}$, and any open neighborhoods V_k of the points $f_k(x)$ in the spaces $f_k(X)$, respectively, we introduce the sets

$$U_{n,\{f_k\}_{k=1}^n,\{V_k\}_{k=1}^n}(x) = \bigcap_{k=1}^n f_k^{-1}(V_k).$$

[1] What a splendid thing is the modern system of notations: $\mathcal{U}_{\mathfrak{A}}$ is a familiy of neighborhoods. Each neighborhood is a set of ultrafilters. Each ultrafilter is a family of sets of natural numbers. Thus, with one symbol $\mathcal{U}_{\mathfrak{A}}$ we managed to denote a set of sets of sets of sets of natural numbers!

Recall (Subsection 1.2.1) the following fact: Suppose that for each point $x \in X$ there is given a non-empty family $\mathcal{U}_x$ of subsets with the following properties:

— if $U \in \mathcal{U}_x$, then $x \in U$;

— if $U_1, U_2 \in \mathcal{U}_x$, then there exists a $U_3 \in \mathcal{U}_x$ such that $U_3 \subset U_1 \cap U_2$;

— if $U \in \mathcal{U}_x$ and $y \in U$, then there exists a set $V \in \mathcal{U}_y$ such that $V \subset U$.

Then there exists a topology τ on X for which the families $\mathcal{U}_x$ are neighborhood bases of the corresponding points.

Consequently, on X there exist a topology (possibly not separated) in which the sets $U_{n,\{f_k\}_{k=1}^n,\{V_k\}_{k=1}^n}(x)$ constitute a neighborhood basis of the point x, for any $x \in X$. We denote this topology by $\sigma(X, \mathcal{F})$. In particular, among the neighborhoods of the point $x \in X$ in the topology $\sigma(X, \mathcal{F})$ there are all the sets $f^{-1}(V)$, where $f \in \mathcal{F}$ and V is a neighborhood of the point $f(x)$ in the topological space $f(X)$. Therefore, all the mappings in the family $\mathcal{F}$ are continuous in $\sigma(X, \mathcal{F})$.

Theorem 1. *$\sigma(X, \mathcal{F})$ is the weakest topology on X in which all the mappings belonging to the family $\mathcal{F}$ are continuous.*

Proof. Let τ be some topology in which all the mappings in the family $\mathcal{F}$ are continuous. Let us show that any set of the form $U_{n,\{f_k\}_{k=1}^n,\{V_k\}_{k=1}^n}(x)$ will be a neighborhood of the point x in the topology τ. This will prove that $\tau \succ \sigma(X, \mathcal{F})$. By hypothesis, all mappings $f_k \colon X \to f_k(X)$ are continuous in the topology τ. Hence, the sets $f_k^{-1}(V_k)$ are open neighborhoods of the point x in τ. Therefore, the intersection $U_{n,\{f_k\}_{k=1}^n,\{V_k\}_{k=1}^n}(x)$ of such sets is also an open neighborhood of x. □

Definition 1. The topology $\sigma(X, \mathcal{F})$ is called the *topology generated by the family of mappings $\mathcal{F}$*. Another term (justified by the preceding theorem) is that of the *weakest topology in which all the mappings in the family $\mathcal{F}$ are continuous*.

Definition 2. A family of mappings $\mathcal{F}$ is said to *separate the points* of the set X if for any $x_1, x_2 \in X$, $x_1 \neq x_2$, there exists a mapping $f \in \mathcal{F}$ such that $f(x_1) \neq f(x_2)$.

Theorem 2. *Suppose all the spaces $f(X)$, $f \in \mathcal{F}$, are Hausdorff. For the topology $\sigma(X, \mathcal{F})$ to be Hausdorff it is necessary and sufficient that the family $\mathcal{F}$ separates the points of the set X.*

Proof. Sufficiency. Suppose $\mathcal{F}$ separates the points of the set X. Then for any $x_1, x_2 \in X$, $x_1 \neq x_2$, there exists an $f \in \mathcal{F}$ such that $f(x_1) \neq f(x_2)$. Since $f(X)$ is a Hausdorff space, there exist disjoint neighborhoods V_1 and V_2 of the points $f(x_1)$ and $f(x_2)$, respectively. The sets $f^{-1}(V_1)$ and $f^{-1}(V_2)$ are the required $\sigma(X, \mathcal{F})$-neighborhoods that separate the points x_1 and x_2.

Necessity. Suppose $\mathcal{F}$ does not separate the points of X. Then there exist points $x_1, x_2 \in X$, $x_1 \neq x_2$, such that $f(x_1) = f(x_2)$ for all $f \in \mathcal{F}$. Pick an arbitrary $\sigma(X, \mathcal{F})$-neighborhood $U_{n,\{f_k\}_{k=1}^n,\{V_k\}_{k=1}^n}(x_1)$ of the point x_1. Since $f_k(x_1) = f_k(x_2)$ for all $k = 1, 2, \ldots, n$, the point x_2 will also lie in $U_{n,\{f_k\}_{k=1}^n,\{V_k\}_{k=1}^n}(x_1)$. Thus, in the described situation $\sigma(X, \mathcal{F})$ not only is not Hausdorff, it even fails the first separation axiom. □

Theorem 3. *For the filter $\mathfrak{F}$ on X to converge in the topology $\sigma(X, \mathcal{F})$ to some $x \in X$, it is necessary and sufficient that $\lim_{\mathfrak{F}} f = f(x)$ for all $f \in \mathcal{F}$.*

Proof. In view of the continuity in the topology $\sigma(X, \mathcal{F})$ of all mappings $f \in \mathcal{F}$, the necessity follows from Theorem 3 of Subsection 16.1.2. Let us prove the sufficiency. Suppose $\lim_{\mathfrak{F}} f = f(x)$ for all $f \in F$. We need to show that every neighborhood of the form $U_{n,\{f_k\}_{k=1}^n,\{V_k\}_{k=1}^n}(x)$ will be an element of the filter $\mathfrak{F}$. By assumption, $\lim_{\mathfrak{F}} f_k = f_k(x)$, and so $f_k^{-1}(V_k) \in \mathfrak{F}$ for all $k = 1, 2, \dots, n$. Since a filter is stable under taking finite intersections of elements, $U_{n,\{f_k\}_{k=1}^n,\{V_k\}_{k=1}^n}(x) = \bigcap_{k=1}^n f_k^{-1}(V_k) \in \mathfrak{F}$. □

Let Γ be an index set (i.e., a set whose elements will henceforth referred to as indices). Suppose that to each index $\gamma \in \Gamma$ there is assigned a set X_γ. The *Cartesian product* of the sets X_γ with respect to $\gamma \in \Gamma$ is defined to be the set $\prod_{\gamma\in\Gamma} X_\gamma$ consisting of all mappings $x : \Gamma \to \bigcup_{\gamma\in\Gamma} X_\gamma$ with the property that $x(\gamma) \in X_\gamma$ for any $\gamma \in \Gamma$. In the particular case when all sets X_γ are equal to one and the same set X, the product consists of all functions $x : \Gamma \to X$; then the Cartesian product is called the *Cartesian power* and is denoted by X^Γ.

For the values of a function $x \in \prod_{\gamma\in\Gamma} X_\gamma$, instead of $x(\gamma)$ one uses the notation x_γ. In this notation the element $x \in \prod_{\gamma\in\Gamma} X_\gamma$ itself is usually written in the form $x = \{x_\gamma\}_{\gamma\in\Gamma}$ of an indexed set of values.

For any $\alpha \in \Gamma$, the mapping $P_\alpha : \prod_{\gamma\in\Gamma} X_\gamma \to X_\alpha$, acting by the rule $P_\alpha(x) = x_\alpha$, is called a *coordinate projection*.

Definition 3. Suppose all X_γ, $\gamma \in \Gamma$, are topological spaces. The *Tikhonov topology* on $\prod_{\gamma\in\Gamma} X_\gamma$ is the weakest topology in which all coordinate projections P_α, $\alpha \in \Gamma$, are continuous. The Cartesian product $\prod_{\gamma\in\Gamma} X_\gamma$, equipped with the Tikhonov topology, is called the *Tikhonov product*.

We note that, obviously, the coordinate projections separate the points of the product, and so, by Theorem 2, a Tikhonov product of Hausdorff spaces is again a Hausdorff space. Further, Theorem 3 yields the following assertion:

Convergence Criterion in a Tikhonov Product. *A filter $\mathfrak{F}$ on $\prod_{\gamma\in\Gamma} X_\gamma$ converges in the Tikhonov topology to an element $x = \{x_\gamma\}_{\gamma\in\Gamma}$ if and only if $x_\gamma = \lim_{\mathfrak{F}} P_\gamma$ for all $\gamma \in \Gamma$.*

Let us describe the Tikhonov topology explicitly, i.e., describe in more detail the form that the neighborhoods of the topology generated by a family of maps take in this particular case. Let $x \in \prod_{\gamma\in\Gamma} X_\gamma$; let $N \subset \Gamma$ be a finite set of indices, and $V_\gamma \subset X_\gamma$, $\gamma \in N$, be open neighborhoods of the corresponding points x_γ. Define

$$U_{N,\{V_\gamma\}_{\gamma\in N}}(x) = \left\{ y \in \prod_{\gamma\in\Gamma} X_\gamma : y_\alpha \in V_\alpha \text{ for all } \alpha \in N \right\}.$$

Theorem 4. *The sets of the form $U_{N,\{V_\gamma\}_{\gamma\in N}}(x)$ form a basis of neighborhoods of the point $x \in \prod_{\gamma\in\Gamma} X_\gamma$ in the Tikhonov topology.* □

Theorem 5 (Tikhonov's theorem on products of compact spaces). *Let X_γ, $\gamma \in \Gamma$ be compact topological spaces. Then the Tikhonov product $\prod_{\gamma\in\Gamma} X_\gamma$ is also compact.*

Proof. We use criterion (3) of Theorem 3 in Subsection 16.1.3. Let $\mathfrak{A}$ be an ultrafilter on $\prod_{\gamma\in\Gamma} X_\gamma$. Since all the spaces X_γ are compact, for each $\gamma \in \Gamma$ the coordinate projection P_γ has a limit. Denote it by $y_\gamma = \lim_{\mathfrak{A}} P_\gamma$. Then the element $y = \{y_\gamma\}_{\gamma\in\Gamma}$ is the limit of the ultrafilter $\mathfrak{A}$. □

Exercises

1. In the case where $\Gamma = \{1, 2\}$, the definition of the Tikhonov product $\prod_{\gamma\in\Gamma} X_\gamma$ coincides with the definition of the product $X_1 \times X_2$ of topological spaces introduced earlier in Subsection 1.2.2.

2. A particular case of the Tikhonov product — the Tikhonov power X^Γ of the topological space X — is the space of all functions $f : \Gamma \to X$. Write in explicit form the neighborhoods of a function f in the Tikhonov topology.

3. Prove that a sequence of functions $f_n \in X^\Gamma$ converges in the Tikhonov topology to a function f if and only if $f_n(x) \to f(x)$ for all $x \in X$. This justifies yet another name used for the Tikhonov topology — *topology of pointwise convergence*.

4. For a particular case of the Tikhonov power — the space $[0, 1]^{[0,1]}$ of all functions $f : [0, 1] \to [0, 1]$ — write explicitly the neighborhoods of a function f. Prove that the set of all polynomials with rational coefficients is dense in $[0, 1]^{[0,1]}$, i.e., $[0, 1]^{[0,1]}$ is a separable space.

A topological space X is said to be *sequentially compact* if from any sequence of elements in X one can extract a convergent subsequence.

5. The space $[0, 1]^{[0,1]}$, despite being compact, is not sequentially compact (see Exercise 10 in Subsection 3.2.2).

A subset A of a topological space X is said to be *sequentially dense* if for any $x \in X$ there exists a sequence $a_n \in A$ that converges to x. A topological space X is said to be *sequentially separable*, if X contains a countable sequentially dense set.

6. A sequentially separable Hausdorff topological space cannot have cardinality larger than the cardinality of the continuum.

7. The space $[0, 1]^{[0,1]}$, despite its separability, is not sequentially separable.

8. Let G_γ be a topological group. Equip the Tikhonov product $\prod_{\gamma\in\Gamma} G_\gamma$ with the operation $\{x_\gamma\}_{\gamma\in\Gamma} \cdot \{y_\gamma\}_{\gamma\in\Gamma} = \{x_\gamma \cdot y_\gamma\}_{\gamma\in\Gamma}$. Verify that $\prod_{\gamma\in\Gamma} G_\gamma$ is a topological group.

9. Equip the two-point set $\{0, 1\}$ with the discrete topology. Prove that the Tikhonov power $\{0, 1\}^{\mathbb{N}}$ is homeomorphic to the Cantor perfect set.

10. Let X be a fixed set. Identifying each subset $A \subset X$ with its characteristic function $\mathbb{1}_A$, we obtain a natural identification of the family 2^X of all subsets of the space X with the space $\{0, 1\}^X$ of all functions $f: X \to \{0, 1\}$. Since the two-point set is a (discrete) compact space, the space $\{0, 1\}^X = 2^X$ is compact in the Tikhonov topology. Describe explicitly the neighborhoods of the set $A \subset X$ in the Tikhonov topology on 2^X.

11. The topological space $\beta\mathbb{N}$ defined in Exercise 9 of Subsection 16.1.3 is a closed subset of the compact space $2^{2^{\mathbb{N}}}$. Hence, $\beta\mathbb{N}$ is compact as well. The space $\beta\mathbb{N}$ is called the *Stone–Čech compactification* of the natural numbers.

12. Define the operator $T: C(\beta\mathbb{N}) \to \ell_\infty$ by the rule: $T(f)$ is the sequence with coordinates $x_n = f(\mathfrak{A}_n)$, where $\mathfrak{A}_n$ denotes the trivial ultrafilter generated by the point $n \in \mathbb{N}$. Prove that T is a linear bijective isometry. Therefore, the space ℓ_∞ is isometric to the space of continuous functions on a (admittedly rather exotic) compact space.

16.2 Background Material on Topological Vector Spaces

We have already encountered topologies and the corresponding types of convergence on linear spaces of functions with the feature that the convergence cannot be described as convergence with respect to a norm. These were, for instance, pointwise convergence and convergence in measure. Such types of convergence will, with rare exceptions, be the weak and weak* convergence in Banach spaces — the main objects of study in Chap. 17. An adequate language for describing such topologies and convergences is that of topological vector spaces.

16.2.1 Axiomatics and Terminology

Definition 1. A linear space X (real or complex) endowed with a topology τ is called a *topological vector space* if the topology τ is compatible with the linear structure, in the sense that the operations of addition of elements and multiplication of an element by a scalar are jointly continuous in their variables.

To avoid treating the real and complex cases separately each time, we will assume that all spaces are complex, leaving the simpler case of real spaces to the reader for independent study.

Let us explain Definition 1 in more detail. Let X be a topological vector space. Consider the mappings $F\colon X \times X \to X$ and $G\colon \mathbb{C} \times X \to X$, acting by the rules $F(x_1, x_2) = x_1 + x_2$ and $G(\lambda, x) = \lambda x$. The compatibility of the topology with the linear structure means that each of the mappings F and G is jointly continuous in its variables. We will use this continuity step by step to deduce geometric properties of neighborhoods in the topology compatible with the linear structure.

Theorem 1. *Let U be an open set in X. Then*

— *for any $x \in X$, the set $U + x$ is open;*

— *for any $\lambda \in \mathbb{C} \setminus \{0\}$, the set λU is open.*

Proof. Fix $x_2 = -x$ and use the continuity of the mapping $F(x_1, x_2) = x_1 + x_2$ in the first variable when the second variable is fixed. The mapping $f(x_1) = x_1 - x$ is continuous in x_1, and $U + x$ is the preimage of the open set U under f. Consequently, $U + x$ is open. The second property is deduced in exactly the same way, using the continuity of the mapping $g(x) = \frac{1}{\lambda}x$. □

It follows from Theorem 1 that the neighborhoods of an arbitrary element $x \in X$ are the sets $U + x$ with U a neighborhood of zero. Accordingly, the topology τ is uniquely determined by the family $\mathfrak{N}_0$ of neighborhoods of zero. For this reason, further properties of the topology τ will be formulated in the language of neighborhoods of zero. Below $\mathbb{C}_r$ will denote the disc of radius r in $\mathbb{C}$ centered at zero: $\mathbb{C}_r = \{\lambda \in \mathbb{C} : |\lambda| \leqslant r\}$.

Let us recall several definitions from Subsection 5.4.2. A subset A of a linear space X is said to be *absorbing* if for any $x \in X$ there exists an $n \in \mathbb{N}$ such that $x \in tA$ for all $t > n$. A subset $A \subset X$ is said to be *balanced* if for any scalar $\lambda \in \mathbb{C}_1$ it holds that $\lambda A \subset A$.

Theorem 2. *The family $\mathfrak{N}_0$ of neighborhoods of zero in the linear space X has the following properties:*

(i) *Any neighborhood of zero is an absorbing set.*

(ii) *Any neighborhood of zero contains a balanced neighborhood of zero.*

(iii) *For any neighborhood $U \in \mathfrak{N}_0$ there exists a balanced neighborhood $V \in \mathfrak{N}_0$ such that $V + V \subset U$.*

Proof. (i) Fix $x \in X$ and use the continuity of the mapping $f(\lambda) = \lambda x$. Since $f(0) = 0$, continuity at the point $\lambda = 0$ means that for any $U \in \mathfrak{N}_0$ there exists an $\varepsilon > 0$ such that $\lambda x \in U$ for all $\lambda \in \mathbb{C}_\varepsilon$. Defining $t = 1/\lambda$, we see that $x \in tU$ for all $t > 1/\varepsilon$.

(ii) Let $U \in \mathfrak{N}_0$. Thanks to the continuity of the mapping $G(\lambda, x) = \lambda x$ at the point $(0, 0)$, there exist an $\varepsilon > 0$ and a neighborhood $W \in \mathfrak{N}_0$ such that $\lambda x \in U$ for all $\lambda \in \mathbb{C}_\varepsilon$ and all $x \in W$. Set $V = \bigcup_{\lambda \in \mathbb{C}_\varepsilon} \lambda W$. Let us show that the set $V \subset U$ provides the requisite balanced neighborhood of zero. On one hand, $V \supset W$, whence $V \in \mathfrak{N}_0$. On the other hand, for any $\lambda_0 \in \mathbb{C}_1$ we have $\lambda_0 \mathbb{C}_\varepsilon \subset \mathbb{C}_\varepsilon$, and so

$$\lambda_0 V = \bigcup_{\lambda \in \mathbb{C}_\varepsilon} \lambda_0 \lambda W = \bigcup_{\mu \in \lambda_0 \mathbb{C}_\varepsilon} \mu W \subset \bigcup_{\mu \in \mathbb{C}_\varepsilon} \mu W = V;$$

this proves that the neighborhood V is balanced.

(iii) Thanks to the continuity of the mapping $F(x_1, x_2) = x_1 + x_2$ at $(0, 0)$, for any neighborhood $U \in \mathfrak{N}_0$ there exist neighborhoods $V_1, V_2 \in \mathfrak{N}_0$ such that $V_1 + V_2 \subset U$. Then based on item (ii) we choose the requisite balanced neighborhood V of zero so that V is contained in the neighborhood $V_1 \cap V_2$. □

We invite the reader to prove the converse result:

Theorem 3. *Suppose the system $\mathfrak{N}_0$ of neighborhoods of zero in a topology τ on the linear space X satisfies the conditions (i)–(iii) in Theorem 2, and for every point $x \in X$ the system $\mathfrak{N}_x$ of neighborhoods of x is obtained from $\mathfrak{N}_0$ by parallel translation by the vector x. Then the topology τ is compatible with the linear structure.* □

Remark 1. In view of the balancedness property, the condition $V + V \subset U$ of item (iii) of Theorem 2 can be rewritten as $V - V \subset U$.

Theorem 4. *For a topological vector space X to be Hausdorff it is necessary and sufficient that the system $\mathfrak{N}_0$ of neighborhoods of zero satisfies the following condition: for any $x \neq 0$ there exists a $U \in \mathfrak{N}_0$ such that $x \notin U$.*

Proof. Suppose $x \neq y$. Then $x - y \neq 0$ and there exists a neighborhood $U \in \mathfrak{N}_0$ which does not contain $x - y$. Pick a neighborhood $V \in \mathfrak{N}_0$ such that $V - V \subset U$. Then the neighborhoods $x + V$ and $y + V$ are disjoint: assuming the contrary, i.e., that there exists a point z which belongs to both $x + V$ and $y + V$, we have $z - x \in V$, $z - y \in V$, and so $x - y = (z - y) - (z - x) \in V - V \subset U$. □

Exercises

1. A balanced set in $\mathbb{C}$ is either the whole set $\mathbb{C}$, or a disc (open or closed) centered at zero, or, finally, consists only of zero.

2. Replacing $\lambda \in \mathbb{C}_1$ by $\lambda \in [-1, 1]$, formulate the analogue of being a balanced set for real spaces. Prove for real spaces the analogue of Theorem 2.

3. Describe the balanced sets in $\mathbb{R}$.

4. Suppose the topology τ on the linear space X is compatible with the linear structure and satisfies the first separation axiom: every point is a closed set. Then the topology τ is Hausdorff.

5. Every topological vector space is also a topological group with respect to addition.

6. Prove that the discrete topology (namely, all sets are open) on $\mathbb{C}$ is compatible with the additive group structure, but not with the linear structure.

Verify that the spaces listed below are topological vector spaces.

7. The space $L_0(\Omega, \Sigma, \mu)$ of measurable functions on a finite measure space, equipped with the topology of convergence in measure (Subsection 3.2.2). The standard neighborhood basis of the function f is provided by the sets of functions $\{g \in L_0(\Omega, \Sigma, \mu) : \mu\{t : |g(t) - f(t)| > \delta\} < \varepsilon\}$, $\delta, \varepsilon > 0$. In this space, as usual, functions that coincide almost everywhere are identified: without this convention, the space would not be separated.

8. Any normed space with the topology defined by its norm.

9. Any Tikhonov product $\prod_{\gamma \in \Gamma} X_\gamma$ of topological vector spaces X_γ, with the linear operations defined coordinatewise: $a\{x_\gamma\}_{\gamma\in\Gamma} + b\{y_\gamma\}_{\gamma\in\Gamma} = \{ax_\gamma + by_\gamma\}_{\gamma\in\Gamma}$.

10. Any linear subspace of a topological vector space, equipped with the induced topology.

Other natural examples will be given in Subsection 16.3.2. Prove that in a topological vector space:

11. The interior and closure of a convex set are convex.

12. The closure of any linear subspace is a linear subspace.

13. Any neighborhood of zero contains a balanced open neighborhood of zero.

14. Any neighborhood of zero contains a balanced closed neighborhood of zero.

Any metrizable topological vector space satisfies the first countability axiom: every point has a countable neighborhood basis. For Hausdorff topological vector spaces the converse is also true. The reader will obtain the proof by solving the following chain of exercises.

Suppose X is a Hausdorff topological vector space and the family of neighborhoods of zero of the space X has a countable basis. Then:

15. There exists a neighborhoods basis $\{V_n\}$ of zero consisting of balanced open sets that satisfy the condition $V_{n+1} + V_{n+1} \subset V_n$, $n = 1, 2, \ldots$.

16. Denote by D the set of dyadic rational numbers in the segment $(0, 1]$. For each $r \in D$, $r < 1$, write its dyadic fraction expansion: $r = \sum_{k=1}^{n(r)} c_k(r)2^{-k}$, where $c_k(r) \in \{0, 1\}$, and $n(r)$ can be arbitrarily large, and define $U(r) = \sum_{k=1}^{n(r)} c_k(r)V_k$. For $r \geqslant 1$, put $U(r) = X$. Then all the sets $U(r)$ are balanced, open, and satisfy $U(1/2^n) = V_n$, $n = 1, 2, \ldots$, and $U(r) + U(s) \subset U(r + s)$ for all $r, s \in D$.

17. For each $x \in X$, put $\theta(x) = \inf\{r \in D : x \in U(r)\}$. Then the quantity θ is symmetric: $\theta(-x) = \theta(x)$, and satisfies the triangle inequality $\theta(x + y) \leqslant \theta(x) + \theta(y)$ for all $x, y \in X$.

18. The function $\rho(x, y) = \theta(x - y)$ is a metric on X. The topology defined by the metric ρ coincides with the original topology of the space.

16.2.2 Completeness, Precompactness, Compactness

To work successfully with topological vector spaces, we need to define analogues of the basic notions that are used in the setting of normed spaces. Since in general a topological vector space is not metrizable, we need to renounce the language of sequences and use instead the language of neighborhoods and filters befitting our general situation.

Definition 1. A filter $\mathfrak{F}$ on X is called a *Cauchy filter* if for any neighborhood U of zero there exists an element $A \in \mathfrak{F}$ such that $A - A \subset U$. Such an element A is said to be *small of order U*.

Theorem 1. *If the filter $\mathfrak{F}$ has a limit, then $\mathfrak{F}$ is a Cauchy filter.*

Proof. Suppose $\lim \mathfrak{F} = x$ and $U \in \mathfrak{N}_0$. Pick a $V \in \mathfrak{N}_0$ such that $V - V \subset U$. By the definition of the limit, there exists an $A \in \mathfrak{F}$ such that $A \subset x + V$. Then $A - A \subset (x + V) - (x + V) \subset V - V \subset U$. □

Theorem 2. *Let $\mathfrak{F}$ be a Cauchy filter on a topological vector space X and x a limit point of $\mathfrak{F}$. Then* $\lim \mathfrak{F} = x$.

Proof. Let $x + U$ be an arbitrary neighborhood of the point x, with $U \in \mathfrak{N}_0$. Pick a neighborhood $V \in \mathfrak{N}_0$ with $V + V \subset U$ and a set $A \in \mathfrak{F}$, small of order V: $A - A \subset V$. By the definition of a limit point, the sets A and $x + V$ intersect, i.e., there exists a point $y \in A \cap (x + V)$. Then

$$x + U \supset x + V + V \supset y + V \supset y + A - A \supset y + A - y = A.$$

Hence, the neighborhood $x + U$ contains an element of $\mathfrak{F}$, and so $x + U \in \mathfrak{F}$. □

Definition 2. A set A in a topological vector space X is said to be *complete*[2] if any Cauchy filter on X that contains A as an element has a limit which belongs to A. In particular, a topological vector space X is said to be *complete* if every Cauchy filter on X has a limit.

[2]Here again the already mentioned terminological confusion is widespread. The current term is introduced to generalize the notion of complete metric space. Equally successfully one could have called complete a set whose linear span coincides with the space X (a term used in the theory of linear spaces) or, by analogy with the theory of normed spaces, call a set complete if its linear span is dense in X. We thus obtain identically named notions which however have nothing in common. The relevant meaning must be figured out from the context.

Theorem 3. *Let X be a subspace of a topological vector space E and $A \subset X$ a complete subset of X. Then A is also complete as a subset of the space E.*

Proof. Let $\mathfrak{F}$ be a Cauchy filter in E which contains A as an element. Then, in particular, $X \in \mathfrak{F}$, i.e., the trace $\mathfrak{F}_X$ on X of the filter $\mathfrak{F}$ is a filter. Next, $\mathfrak{F}_X$ is a Cauchy filter on X which contains A as an element. Hence, in view of the completeness of A in X, the filter $\mathfrak{F}_X$ has in X a limit $a \in A$. The same point a is the limit of the filter $\mathfrak{F}$ in E. □

Theorem 4. *Every complete subset A of a Hausdorff topological vector space X is closed. In particular, if a subspace of a Hausdorff topological vector space is complete in the induced topology, then this subspace is closed.*

Proof. Suppose the point $x \in X$ belongs to the closure of the set A. We need to show that $x \in A$. Consider the family $\mathfrak{D}$ of all intersections $(x + U) \cap A$, where $U \in \mathfrak{N}_0$. All such intersections are non-empty, and $\mathfrak{D}$ obeys all axioms of a filter basis. The filter $\mathfrak{F}$ generated by the basis $\mathfrak{D}$ majorizes the filter $\mathfrak{N}_x$ of all neighborhoods of the point x, and so $\lim \mathfrak{F} = x$. In particular, $\mathfrak{F}$ is a Cauchy filter. By construction, our complete set A is an element of the filter $\mathfrak{F}$. Hence, by Definition 2, $\mathfrak{F}$ must have a limit in A. Since the limit is unique, $x \in A$, as we needed to prove. □

Definition 3. A set A in a topological vector space X is called *precompact* if for any neighborhood U of zero there exists a finite set $B \subset X$ such that $A \subset B + U$. Such a set B is called, by analogy with an ε-net, a *U-net* of the set A.

Theorem 5. *For a set A of a Hausdorff topological vector space X to be compact it is necessary and sufficient that A be simultaneously precompact and a complete set in X.*

Proof. Necessity. Let A be a compact set and U be an arbitrary open neighborhood of zero in X. The neighborhoods of the form $x + U$ with $x \in A$ form an open cover of the compact set A, hence there exists a finite subcover $x_1 + U, x_2 + U, \ldots, x_n + U$, with $x_k \in A$. The set $B = \{x_1, x_2, \ldots, x_n\}$ is a U-net of the set A. This establishes the precompactness of the compact set A. Now let us prove the completeness. Suppose $\mathfrak{F}$ is a Cauchy filter in X which contains A as an element. Then the trace $\mathfrak{F}_A$ on A of the filter $\mathfrak{F}$ is a filter in the compact topological space A, so $\mathfrak{F}_A$ has a limit point $a \in A$. The same point is then a limit point for $\mathfrak{F}$. But a limit point of a Cauchy filter is the limit of that filter. Therefore, $\mathfrak{F}$ has a limit, and $\lim \mathfrak{F} = a \in A$.

Necessity. Let A be a complete precompact set in X. Let us prove that every ultrafilter $\mathfrak{A}$ on A has a limit. Consider the filter $\widetilde{\mathfrak{A}}$, given already not on A, but on the entire space X, for which $\mathfrak{A}$ is a filter basis: $B \in \widetilde{\mathfrak{A}}$ if and only if $B \cap A \in \mathfrak{A}$. Using the ultrafilter criterion (Theorem 2 in Subsection 16.1.3), it is readily verified that $\widetilde{\mathfrak{A}}$ is an ultrafilter. We claim that $\widetilde{\mathfrak{A}}$ is a Cauchy filter. Indeed, let $U \in \mathfrak{N}_0$. Pick a neighborhood $V \in \mathfrak{N}_0$ such that $V - V \subset U$. Let $B = \{x_1, x_2, ..., x_n\}$ be the corresponding V-net of the precompact set A. Since the sets $x_1 + V, x_2 + V, \ldots, x_n + V$ form a finite open cover of the element A of the ultrafilter $\widetilde{\mathfrak{A}}$, one of these sets, say, $x_j + V$, will be an element of $\widetilde{\mathfrak{A}}$ (Exercise 4 of Subsection 16.1.3). But $x_j + V$ is small of order U:

$$(x_j + V) - (x_j + V) = V - V \subset U.$$

Thus, $\widetilde{\mathfrak{A}}$ is a Cauchy filter, $A \in \widetilde{\mathfrak{A}}$, and A is a complete set in X. Therefore, there exists $\lim \widetilde{\mathfrak{A}} \in A$. The same element will also be the limit in A of the filter $\mathfrak{A}$, the trace of the filter $\widetilde{\mathfrak{A}}$ on A (Exercise 4 of Subsection 16.1.2). □

Definition 4. Let X be a topological vector space. We say that the neighborhood $U \in \mathfrak{N}_0$ of zero *absorbs* the set $A \subset X$ if there exists a number $t > 0$ such that $A \subset tU$. The set $A \subset X$ is said to be *bounded* if it is absorbed by every neighborhood of zero.

Theorem 6. *The family of bounded subsets of a topological vector space X enjoys the following properties:*

(a) *If $A \subset X$ is bounded, then for any neighborhood $U \in \mathfrak{N}_0$ there exists a number $N > 0$ such that $A \subset tU$ for all $t \geqslant N$.*

(b) *The union of any finite collection of bounded sets is a bounded set.*

(c) *Every finite set is bounded.*

(d) *Every precompact set in X is bounded.*

Proof. (a) Let $V \in \mathfrak{N}_0$ be a balanced neighborhood which is contained in U. Pick $N > 0$ such that $A \subset NV$. Then for every $t \geqslant N$ we have $A \subset NV = t((N/t)V) \subset tV \subset tU$.

(b) Let $A_1, A_2, \dots, A_n$ be bounded sets, and U be a neighborhood of zero. By (a), for each of the sets A_k there exists a number $N_k \in \mathbb{N}$ such that $A_k \subset tU$ for all $t > N_k$. Put $N = \max_{1\leqslant k\leqslant n} N_k$. Then for any $t \geqslant N$ all inclusions $A_k \subset tU$ hold simultaneously, that is, $\bigcup_{k=1}^{n} A_k \subset tU$.

(c) Any single-point set is bounded, since every neighborhood of zero is an absorbing set. It remains to use assertion (b).

(d) Let A be precompact in X and U be a neighborhood of zero. Pick a balanced neighborhood $V \in \mathfrak{N}_0$ such that $V + V \subset U$. By the definition of precompactness, there exists a finite set $B \subset X$ such that $A \subset B + V$. By (c), one can find a number $N > 1$ such that $B \subset NV$. Then $A \subset B + V \subset NV + V \subset N(V + V) \subset NU$. □

Exercises

1. Let $\mathfrak{F}$ be a Cauchy filter in a topological vector space X. Suppose the filter $\mathfrak{F}_1$ majorizes $\mathfrak{F}$ and $x = \lim \mathfrak{F}_1$. Show that $x = \lim \mathfrak{F}$.

A sequence (x_n) of elements of a topological vector space X is called a *Cauchy sequence* if the filter $\mathfrak{F}_{(x_n)}$ generated by the sequence (x_n) is a Cauchy filter. Prove that:

2. (x_n) is a Cauchy sequence if and only if for any $U \in \mathfrak{N}_0$ there exists a number $N \in \mathbb{N}$ such that $x_n - x_m \in U$ for all $n, m \geq N$.

3. (x_n) is a Cauchy sequence if and only if for every $U \in \mathfrak{N}_0$ there exists a number $N \in \mathbb{N}$, such that $x_n - x_N \in U$ for all $n \geq N$.

4. Suppose the topological vector space X has a countable basis of neighborhoods of zero, and every Cauchy sequence in X has a limit. Then X is a complete space.

5. Suppose the complete topological vector space X has a countable basis of neighborhoods of zero U_n, $n \in \mathbb{N}$, and the neighborhoods U_n are chosen so that $U_{n+1} + U_{n+1} \subset U_n$. Pick in each set U_n one element $x_n \in U_n$. Then show that the series $\sum_{n=1}^{\infty} x_n$ converges.

6. Extend Banach's theorem on the inverse operator (if $T: X \to Y$ is linear, bijective, and continuous, then T^{-1} is continuous) to the case where X and Y are complete metrizable topological vector spaces.

7. Prove the completeness of the space $L_0(\Omega, \Sigma, \mu)$ of all measurable functions on a finite measure space, equipped with the topology of convergence in measure.

A metric ρ on a linear space X is said to be *invariant* if $\rho(x, y) = \rho(x - y, 0)$ for any $x, y \in X$. Suppose the topology τ of the topological vector space X is given by an invariant metric ρ. Then:

8. The sequence $(x_n) \subset X$ is Cauchy in the topology τ if and only if it is Cauchy in the metric ρ.

9. The completeness of the topological vector space (X, τ) is equivalent with the completeness of the metric space (X, ρ).

10. The precompactness of a set A in (X, τ) is equivalent to the precompactness of A in the metric ρ.

11. Warning: the boundedness of a set A in (X, τ) **is not equivalent** to the boundedness of A in the metric ρ. More precisely, boundedness in (X, τ) implies ρ-boundedness, but the converse is not true. As an example consider $X = \mathbb{R}$ with the natural topology, and introduce an invariant metric by the formula $\rho(x, y) = \arctan|x - y|$. Then $A = \mathbb{R}$ is a ρ-bounded set, but obviously A is not a bounded subset of the topological vector space $\mathbb{R}$.

Let X_γ, $\gamma \in \Gamma$ be topological vector spaces. We equip the space $X = \prod_{\gamma \in \Gamma} X_\gamma$ with the Tikhonov topology and the coordinatewise-defined linear operations. As usual, denote by $P_\gamma: X \to X_\gamma$, $\gamma \in \Gamma$, the coordinate projectors. Prove that

12. The set $A \subset X$ is bounded if and only if all images $P_\gamma(A) \subset X_\gamma$ are bounded.

13. The set $A \subset X$ is precompact if and only if all the images $P_\gamma(A) \subset X_\gamma$ are precompact.

14. For the closedness and compactness of a set A in the Tikhonov product the analogous criteria are no longer valid. Give examples showing this in the space $X = \mathbb{R} \times \mathbb{R}$.

15. A filter $\mathfrak{F}$ in $X = \prod_{\gamma\in\Gamma} X_\gamma$ is a Cauchy filter if and only if $P_\gamma(F)$ are Cauchy filters in the corresponding spaces X_γ.

16. If all X_γ, $\gamma \in \Gamma$, are complete spaces, then the space $X = \prod_{\gamma\in\Gamma} X_\gamma$ is also complete.

Consider the space $\mathbb{R}^{\mathbb{N}}$, equipped with the Tikhonov product topology. $\mathbb{R}^{\mathbb{N}}$ can be regarded as the space of all infinite numerical sequences $x = (x_1, x_2, \ldots)$. A neighborhood basis of zero is provided by the sets $U_{n,\varepsilon} = \{x \in \mathbb{R}^{\mathbb{N}} : \max_{1\leqslant k\leqslant n} |x_k| < \varepsilon\}$. Prove that:

17. In $\mathbb{R}^{\mathbb{N}}$ there exists a countable neighborhood basis of zero, i.e., $\mathbb{R}^{\mathbb{N}}$ is metrizable.

18. A metrization of the space $\mathbb{R}^{\mathbb{N}}$ (under another commonly used name $\mathbb{R}^{\omega}$) was proposed in Exercise 11 of Subsection 1.3.1. Verify that the metric from that exercise generates the Tikhonov product topology on $\mathbb{R}^{\mathbb{N}}$.

19. Convergence in $\mathbb{R}^{\mathbb{N}}$ is equivalent to coordinatewise (or componentwise) convergence.

20. $\mathbb{R}^{\mathbb{N}}$ is a complete topological vector space.

21. A set $A \subset \mathbb{R}^{\mathbb{N}}$ is bounded if and only if there exists an element $b = (b_1, b_2, \ldots) \in (\mathbb{R}^+)^{\mathbb{N}}$ that majorizes all elements of A: for any $a = (a_1, a_2, \ldots) \in A$, the estimate $|a_n| \leqslant b_n$ holds for all $n \in \mathbb{N}$.

22. In $\mathbb{R}^{\mathbb{N}}$ the classes of bounded sets and precompact sets coincide.

23. Regard the sets $\overline{B}_{c_0}$ and $\overline{B}_{\ell_1}$, i.e., the closed unit balls in the spaces c_0 and ℓ_1, as subsets of the space $\mathbb{R}^{\mathbb{N}}$. Are these sets bounded in $\mathbb{R}^{\mathbb{N}}$? Closed in $\mathbb{R}^{\mathbb{N}}$? Precompact in $\mathbb{R}^{\mathbb{N}}$? Compact in $\mathbb{R}^{\mathbb{N}}$?

16.2.3 Linear Operators and Functionals

Throughout this subsection X and E will be topological vector spaces.

Theorem 1. *A linear operator $T : X \to E$ is continuous if and only if it is continuous at the point $x = 0$.*

Proof. A continuous operator is continuous at all points, in particular, at zero. Conversely, suppose the operator T is continuous at zero. Let us show that T is continuous at any point $x_0 \in X$. Let V be an arbitrary neighborhood of the point Tx_0 in E. Then $V - Tx_0$ is a neighborhood of zero in E. By assumption, $T^{-1}(V - Tx_0)$ is a neighborhood of zero in X. Thanks to the linearity of the operator, $T^{-1}(V) \supset T^{-1}(V - Tx_0) + x_0$, i.e., $T^{-1}(V)$ is a neighborhood of x_0. □

Definition 1. The linear operator $T: X \to E$ is said to be *bounded* if the image under T of any bounded subset of the space X is bounded in E.

Theorem 2. *Any continuous linear operator $T : X \to E$ is bounded.*

Proof. Let A be a bounded subset of X. We need to prove that the set $T(A)$ is bounded. Let V be an arbitrary neighborhood of zero in E and U a neighborhood of zero in X such that $T(U) \subset V$. Using the boundedness of A, pick an $N > 0$ such that $A \subset tU$ for all $t > N$. Then $T(A) \subset tT(U) \subset tV$ for all $t > N$. □

As we will show below, it is quite possible that two different topologies $\tau_1 \succ \tau_2$ on X (for instance, the strong and weak topologies of a normed space) generate one and the same system of bounded sets. In this case the identity operator, regarded as acting from (X, τ_2) into (X, τ_1), will be bounded, but discontinuous.

Theorem 3. *Suppose the operator $T: X \to E$ takes some neighborhood U of zero in the space X into a bounded set. Then T is continuous.*

Proof. Suppose $T(U)$ is a bounded set. For any neighborhood V of zero in E, there exists a $t > 0$ such that $T(U) \subset tV$. Then $\frac{1}{t}U \subset T^{-1}(V)$, i.e., $T^{-1}(V)$ is a neighborhood of zero in X. □

Next, we consider continuity conditions for linear functionals.

Theorem 4. *For a non-zero linear functional f on a topological vector space X, the following conditions are equivalent:*

(i) *f is continuous;*

(ii) *the kernel of the functional f is closed;*

(iii) *the kernel of the functional f is not dense in X;*

(iv) *there exists a neighborhood U of zero for which $f(U)$ is a bounded set.*

Proof.
(i) $\Longrightarrow$ (ii). The preimage of any closed set is closed; in particular, $\operatorname{Ker} f = f^{-1}(0)$ is a closed set.

(ii) $\Longrightarrow$ (iii). If the kernel is closed and dense in X, then $\operatorname{Ker} f = X$, i.e., $f \equiv 0$.

(iii) $\Longrightarrow$ (iv). Suppose $\operatorname{Ker} f$ is not dense. Then there exist a point $x \in X$ and a balanced neighborhood U of zero such that $(U + x) \cap \operatorname{Ker} f = \emptyset$. This means that the functional cannot take the value $-f(x)$ at any point $y \in U$. Therefore, $f(U)$ is a balanced set of complex numbers which does not coincide with the whole complex plane $((-f(x) \notin f(U))$. It follows that $f(U)$ is a disc centered at zero (in the real case it would be an interval in $\mathbb{R}$ symmetric with respect to zero).

(iv) $\Longrightarrow$ (i). This implication was already established in Theorem 3. □

As in the case of normed spaces, for a topological vector space X we denote by X^* the set of all continuous linear functionals on X.[3] The geometric form of the Hahn–Banach theorem admits a generalization to topological vector spaces.

Theorem 5 (Hahn–Banach separation theorem for topological vector spaces). *Let A and B be disjoint non-empty convex subsets of a real topological vector space X and let A be open. Then there exist a functional $f \in X^* \setminus \{0\}$ and a scalar $\theta \in \mathbb{R}$ such that $f(a) < \theta$ for all $a \in A$ and $f(b) \geqslant \theta$ for all $b \in B$.*

Using the connection between a linear functional and its real part (Subsection 9.1.1), one can obtain a version of the theorem for a complex space, replacing the conditions above by $\operatorname{Re} f(a) < \theta$ for all $a \in A$ and $\operatorname{Re} f(b) \geqslant \theta$ for all $b \in B$.

Proof. As in the case of normed spaces (Subsection 9.3.2), the theorem reduces to the following particular case: Let $A \subset X$ be an open convex neighborhood of zero in X, and let $x_0 \in X \setminus A$. Then there exists a functional $f \in X^* \setminus \{0\}$ such that $f(a) \leqslant f(x_0)$ for all $a \in A$.

In this last case the Minkowski functional φ_A of the set A is a convex functional (Subsection 5.4.2). Consider the subspace $Y = \operatorname{Lin}\{x_0\}$ and a linear functional f on Y with the property that $f(x_0) = \varphi_A(x_0)$. Then on Y the linear functional f is majorized by the convex functional φ_A (see the proof of the lemma in Subsection 9.3.2).

Now using the analytic form of the Hahn–Banach theorem (Subsection 5.4.3) we extend f to the entire space X with preservation of the linearity and the majorization condition $f(x) \leqslant \varphi_A(x)$. By the definition of the Minkowski functional, $\varphi_A(a) \leqslant 1$ for all $a \in A$, whence $f(a) \leqslant \varphi_A(a) \leqslant 1$ on A. Since $x_0 \notin A$, $\varphi_A(x_0) \geqslant 1$. Therefore, $f(a) \leqslant 1 \leqslant \varphi_A(x_0) = f(x_0)$ for all $a \in A$.

Further, since $f(x_0) \geqslant 1$, f is not identically equal to zero. By Lemma 5 in Subsection 9.3.1, which generalizes with no difficulty to topological vector spaces, the strict inequality $f(a) < f(x_0)$ holds for all $a \in A$. This means that the kernel $\operatorname{Ker} f$ does not intersect the non-empty open set $A - x_0$. Hence, $\operatorname{Ker} f$ cannot be dense, and the functional f is continuous. □

Finally, let us generalize to finite-dimensional topological vector spaces properties of finite-dimensional normed spaces already known to us.

Theorem 6. *Let X be a Hausdorff topological vectors space, with* $\dim X = n$. *Then:*

(a) *Every linear functional on X is continuous.*

[3]Often, in textbooks on topological vector spaces, the symbol X^* is used to denote the set of **all** linear functionals on X, while the set of **continuous** linear functionals is denoted by X'. We will do exactly the opposite, in order to preserve the compatibility with the notations from the theory of normed spaces the reader is already familiar with.

(b) *For any topological vector space E, every linear operator $T: X \to E$ is continuous.*

(c) *X is isomorphic to the n-dimensional Hilbert space ℓ_2^n.*

(d) *X is complete.*

Proof. First note that for fixed n the implications (a) $\Longrightarrow$ (b) $\Longrightarrow$ (c) $\Longrightarrow$ (d) hold true. Indeed, (a) $\Longrightarrow$ (b), because if we choose in X a basis $\{x_k\}_{k=1}^n$ with the coordinate functionals $\{f_k\}_{k=1}^n$, the operator T can be represented in the form

$$T(x) = T\left(\sum_{k=1}^{n} f_k(x)x_k\right) = \sum_{k=1}^{n} f_k(x)Tx_k.$$

Thus, the calculation of $T(x)$ reduces to calculating the scalars $f_k(x)$ (this action is continuous due to assumption (a)), multiplying by them the constant vectors Tx_k, and summing the resulting products. But by the axioms of a topological vector space, multiplication by a scalar and taking the sum are continuous operations.

(b) $\Longrightarrow$ (c). Both spaces X and ℓ_2^n have the same dimension n, and so there exists a linear bijection $T: X \to \ell_2^n$. Both T and T^{-1} are continuous by condition (b).

Finally, (c) $\Longrightarrow$ (d) thanks to the completeness of the space ℓ_2^n.

The main assertion (a) is proved by induction on n. For $n = 0$ the space X reduces to $\{0\}$, and so the assertion is trivial. Let us perform the step $n \to n + 1$. Suppose $\dim X = n + 1$ and f is a non-zero linear functional on X. Then $\operatorname{Ker} f$ is an n-dimensional space. By the induction hypothesis, and using the already established implications (a) $\Longrightarrow$ (b) $\Longrightarrow$ (c) $\Longrightarrow$ (d), we conclude that $\operatorname{Ker} f$ is a complete space. Therefore, $\operatorname{Ker} f$ is closed in X, and, by Theorem 4, the functional f is continuous. □

Exercises

1. Suppose the space X has a countable neighborhood basis of zero. Then every bounded linear operator $T: X \to E$ is continuous.

2. Let X be a topological vector space, $Y \subset X$ a subspace, and $q: X \to X/Y$ the quotient mapping. Define a topology τ on X/Y as follows: the set $U \subset X/Y$ is declared to be open if $q^{-1}(U)$ is an open set. Verify that:

— the topology τ is compatible with the linear structure;

— τ is the strongest among all topologies on X/Y in which the quotient mapping q is continuous;

— if the subspace Y is closed, then the space X/Y is separated, even when the initial space X is not separated;

— in the case of normed spaces, the topology τ coincides with the topology given by the quotient norm.

3. Prove the following generalization of Theorem 5 from Subsection 11.2.1: if in a Hausdorff topological vector space X there exists a precompact neighborhood of zero, then X is finite-dimensional.

4. On the example of the identity operator in $\mathbb{R}^{\mathbb{N}}$, show that the sufficient condition for continuity proved in Theorem 3 is not necessary.

5. Where in the proof of Theorem 6 of Subsection 16.2.3 was the assumption that the space is separated used? Will the theorem remain valid if the separation assumption is discarded?

16.3 Locally Convex Spaces

16.3.1 Seminorms and Topology

Definition 1. A topological vector space X is said to be *locally convex* if for any neighborhood U of zero there exists a convex neighborhood V of zero such that $V \subset U$. In other words, the space X is locally convex if the neighborhood system $\mathfrak{N}_0$ of zero has a basis consisting of convex sets.

Theorem 1. *Every convex neighborhood U of zero contains a convex balanced open neighborhood of zero. In particular, in a locally convex space there exists a neighborhood basis of zero consisting of convex balanced open sets.*

Proof. Let $V \subset U$ be an open and balanced neighborhood of zero. Then $\operatorname{conv} V \subset U$. Let us show that $\operatorname{conv} V$ is a convex balanced and open neighborhood of zero. Convexity is obvious. Further, $\operatorname{conv} V \supset V$, and hence $\operatorname{conv} V$ is a neighborhood of zero. Let us verify that $\operatorname{conv} V$ is balanced. Take $\lambda \in \mathbb{C}_1$, i.e., $|\lambda| \leqslant 1$. Then $\lambda V \subset V$ (since V is balanced), and $\lambda \operatorname{conv} V = \operatorname{conv}(\lambda V) \subset \operatorname{conv} V$. Finally, let us verify that $\operatorname{conv} V$ is open. Since V is an open set and the operations of multiplication by non-zero scalars and taking the sum of sets leave the class of open sets invariant, all sets of the form $\sum_{k=1}^{n} \lambda_k V$ with $n \in \mathbb{N}$, $\lambda_k > 0$, and $\sum_{k=1}^{n} \lambda_k = 1$, are open. The conclusion follows from the fact that $\operatorname{conv} V$ is a union of sets of the form $\sum_{k=1}^{n} \lambda_k V$. □

Recall (Definition 2 in Subsection 6.1.1) that a function $p\colon X \to \mathbb{R}$ is called a seminorm if $p(x) \geqslant 0$, $p(\lambda x) = |\lambda| p(x)$ for any $x \in X$ and any scalar λ, and $p(x + y) \leqslant p(x) + p(y)$ for all $x, y \in X$. A seminorm differs from a norm by the fact that $p(x)$ may be equal to zero for some non-zero elements $x \in X$. See also Exercises 10–13 in Subsection 6.1.3.

As in the case of a norm, the unit ball of the seminorm p is the set $B_p = \{x \in X : p(x) < 1\}$. The set B_p is convex and balanced. The seminorm p can be recovered from its unit ball by means of the Minkowski functional: $p(x) = \varphi_{B_p}(x)$ (see Subsection 5.4.2).

Theorem 2. *A seminorm p on a topological vector space X is continuous if and only if B_p is a neighborhood of zero.*

Proof. $B_p = p^{-1}(-1, 1)$ is the preimage of an open set. If p is continuous, then this preimage is open. Conversely, suppose that B_p is a neighborhood of zero, and let us show that the seminorm p is continuous. Thus, given any $x \in X$ and any $\varepsilon > 0$, we need to find a neighborhood U of the point x such that $p(U) \subset (p(x) - \varepsilon, p(x) + \varepsilon)$. Such a neighborhood is provided by $U = x + \varepsilon B_p$. Indeed, any point $y \in U$ has the form $y = x + \varepsilon z$, where $p(z) < 1$. Hence, by the triangle inequality, $p(x) - \varepsilon < p(y) < p(x) + \varepsilon$. □

Definition 2. Let G be a family of seminorms on a linear space X. Denote by $\mathfrak{D}_G$ the collection of all finite intersections of sets rB_p, where $p \in G$ and $r > 0$. The *locally convex topology generated by the family of seminorms G* is the topology τ_G on X, in which a neighborhood basis of zero is $\mathfrak{D}_G$, and a neighborhood basis of a point $x \in X$ is, correspondingly, the collection of sets $x + U$ with $U \in \mathfrak{D}_G$.

A family G of seminorms is said to be *non-degenerate* if for any $x \in X \setminus \{0\}$ there exists a $p \in G$ such that $p(x) \neq 0$.

Theorem 3. I. *Let G be a family of seminorms on a linear space X. Then the topology τ_G generated by the family G is compatible with the linear structure and is locally convex.*
II. *The topology τ_G is separated if and only if the family of seminorms G is non-degenerate.*
III. *The topological vector space X is locally convex if and only if its topology is generated by a family of seminorms.*

Proof. I. Since a ball of a seminorm is a convex balanced and absorbing set and these properties are inherited by finite intersections, a neighborhood basis of zero $\mathfrak{D}_G$ consists of convex balanced absorbing sets. Further, for any $U \in \mathfrak{D}_G$ we have $V = (1/2)U \in \mathfrak{D}_G$, and thanks to convexity, $V + V \subset U$. Thus, we have verified conditions (i)–(iii) of Theorem 3 in Subsection 16.2.1. The proof of the fact that the conditions ensuring the existence of the topology given by families of open neighborhoods are satisfied is left to the reader. The compatibility of the topology with the linear structure follows from Theorem 4 of Subsection 16.2.1.

II. The characterization of the separation property follows from Theorem 4 of Subsection 16.2.1.

III. Let X be a locally convex space. By Theorem 1, X has a neighborhood basis $\mathfrak{D}$ of zero that consists of convex balanced open sets. Then as the elements of the sought-for family of seminorms one takes the seminorms whose unit balls are precisely the elements of the basis $\mathfrak{D}$. □

Theorem 4. *Let X be a topological vector space and f be a linear functional on X. Then for f to be continuous it is necessary and sufficient that there exists a continuous seminorm p on X such that $|f(x)| \leqslant p(x)$ for all $x \in X$.*

Proof. Suppose f is continuous. Then $p(x) = |f(x)|$ is the required seminorm. Conversely, suppose that $|f(x)| \leqslant p(x)$, and p is a continuous seminorm. Then f is bounded on the neighborhood B_p of zero. □

Theorem 5 (Hahn–Banach extension theorem in locally convex spaces). *Let f be a continuous linear functional given on a subspace Y of a locally convex space X. Then f can be extended to the entire space X with preservation of linearity and continuity.*

Proof. By assumption, the set $U = \{y \in Y : |f(y)| < 1\}$ is an open neighborhood of zero in Y. By the definition of the topology induced on a subspace, there exists a neighborhood V of zero in X such that $U \supset V \cap Y$. Since the space X is locally convex, one can take for the neighborhood V the unit ball of some continuous seminorm p given on X. By construction, for any $y \in Y$, if $p(y) < 1$, then $y \in U$ and $|f(y)| < 1$. That is, $|f(y)| \leqslant p(y)$ everywhere on Y.

Now, like for normed spaces, one needs to argue separately for the real and complex cases. If f is a real functional, then by the analytic form of the Hahn–Banach theorem, f can be extended to the entire space X with preservation of the inequality $f(x) \leqslant p(x)$. Replacing x by $-x$, we also obtain the inequality $-f(x) \leqslant p(x)$. Therefore, $|f(x)| \leqslant p(x)$, and the extended functional f is continuous. In the complex case, the extension can be performed so that the condition $\operatorname{Re} f(x) \leqslant p(x)$ is preserved on the entire space X. Taking for x the element $e^{-i \arg f(x)}x$, we arrive again at the inequality $|f(x)| \leqslant p(x)$, which establishes the continuity on X of the functional f. □

Remark 1. A set of linear functionals $E \subset X'$ will separate the points if and only if for every $x \neq 0$ there exists an $f \in E$ such that $f(x) \neq 0$. Indeed, if the set E separates the points, then, in particular, E separates x from 0. Conversely, if $x \neq y$ are arbitrary points, then $x - y \neq 0$. A functional $f \in E$ for which $f(x - y) \neq 0$ will separate the points x and y.

Corollary 1. *The set X^* of all continuous linear functionals on a Hausdorff locally convex space X separates the points of X.*

Proof. For any $x \neq 0$ there exists a linear functional f on $\operatorname{Lin}\{x\}$ such that $f(x) \neq 0$. It remains to extend f to X by means of the Hahn–Banach theorem. □

Exercises

1. On the example of the family of seminorms G consisting of a single norm, verify that the locally convex topology τ_G generated by the family of seminorms G

(Definition 2) does not coincide with the topology $\sigma(X, G)$ generated by the family of mappings G (Subsection 16.1.4). Moreover, $\sigma(X, G)$ is not compatible with the linear structure.

2. Let G be a family of seminorms, and F be the family of all functions of the form $f_x(y) = p(x + y)$, with $p \in G$ and $x \in X$. Then $\tau_G = \sigma(X, F)$.

3. A sequence $x_n \in X$ converges to $x \in X$ in the topology τ_G if and only if $p(x_n - x) \to 0$ as $n \to \infty$ for all $p \in G$.

4. Verify that the spaces listed below are indeed separated locally convex spaces, and describe the convergence of sequences in them. Prove the completeness and metrizability of the spaces in the first three examples. Is the fourth space metrizable? Complete?

— The space $\mathcal{H}(D)$ of holomorphic functions in a domain (i.e., connected open subset) $D \subset \mathbb{C}$, equipped with the locally convex topology generated by the family of all seminorms of $p_K(f) = \max_{z \in K} |f(z)|$, where K is a compact subset of D.

— The space $C^\infty[0, 1]$ of all infinitely differentiable functions on $[0, 1]$, equipped with the locally convex topology generated by the family of seminorms $p_n(f) = \max_{t \in [0,1]} |f^{(n)}(t)|$, $n = 0, 1, 2, \ldots$.

— The space $C^\infty(0, +\infty)$ of all infinitely differentiable functions on $(0, +\infty)$, equipped with the topology generated by the family of seminorms $p_n(f) = \max_{t \in (n^{-1}, n)} |f^{(n-1)}(t)|$, $n \in \mathbb{N}$.

— An infinite-dimensional linear space X, equipped with the *strongest locally convex topology*, i.e., the topology generated by the family of **all** seminorms on X.

5. Any Tikhonov product of locally convex spaces is locally convex.

6. Any subspace of a locally convex space is locally convex.

7. Any quotient space of a locally convex space (see Exercise 2 in Subsection 16.2.3) is locally convex.

8. Show that if U is a balanced set and f is a linear functional such that $\operatorname{Re} f(x) \leqslant a$ for all $x \in U$, then also $|f(x)| \leqslant a$ on U.

9. Applying the geometric form of the Hahn–Banach theorem to a set U and an open neighborhood V of the point x_0, prove the following corollary: Let U be a closed, balanced, and convex subset of a Hausdorff locally convex space X and let $x_0 \in X \setminus U$. Then there exists a continuous linear functional f such that $|f(y)| \leqslant 1$ for all $y \in U$ and $|f(x_0)| > 1$.

10. A series $\sum_{k=1}^\infty x_k$ in a locally convex space X is said to be *absolutely convergent* if $\sum_{k=1}^\infty p(x_k) < \infty$ for any continuous seminorm p on X. Prove that in a complete locally convex space every absolutely convergent series converges.

11. The space $L_0[0, 1]$ with the topology of convergence in measure is not a locally convex space. Moreover, any convex closed neighborhood of zero in $L_0[0, 1]$ coincides with the entire space. In particular, the only continuous linear functional on $L_0[0, 1]$ is the functional identically equal to zero.

16.3.2 Weak Topologies

Definition 1. Let X be a linear space, X' its algebraic dual (i.e., the space of all linear functionals on X), and $E \subset X'$ a subset. The *weak topology on X generated by the set of functionals E* is the weakest topology in which all functionals from E are continuous. This topology is a particular case of the topology defined by a family of mappings (Subsection 16.1.4). Accordingly, we denote it by the same symbol $\sigma(X, E)$.

Let us explain this definition in more detail. For any finite collection of functionals $G = \{g_1, g_2, \ldots, g_n\}$ and any $\varepsilon > 0$, define

$$U_{G,\varepsilon} = \bigcap_{g \in G} \{x \in X : |g(x)| < \varepsilon\} = \left\{x \in X : \max_{g \in G} |g(x)| < \varepsilon\right\}.$$

The family of sets $U_{G,\varepsilon}$ with $G = \{g_1, g_2, \ldots, g_n\} \subset E$ and $\varepsilon > 0$ constitutes a neighborhood basis of zero in the topology $\sigma(X, E)$. For an arbitrary element $x_0 \in X$, a neighborhood basis is provided by the sets of the form

$$\bigcap_{g \in G} \{x \in X : |g(x - x_0)| < \varepsilon\} = x_0 + U_{G,\varepsilon}.$$

This shows that $\sigma(X, E)$ is the locally convex topology generated by the family of seminorms $p_G(x) = \max_{g \in G} |g(x)|$, where G runs over all finite subsets of the set E. For this topology to be separated it is necessary and sufficient that the set of functionals E separates the points of the space X.

As we already remarked in Subsection 16.1.4, a filter $\mathfrak{F}$ on X converges in the topology $\sigma(X, E)$ to the element x if and only if $\lim_{\mathfrak{F}} f = f(x)$ for all $f \in E$. In particular, this convergence criterion is also valid for sequences: $x_n \to x$ in the topology $\sigma(X, E)$ if $f(x_n) \to f(x)$ for all $f \in E$.

We begin our more detailed study of weak topologies with a lemma that was proposed earlier as an exercise on the subject "functionals and codimension" (Subsection 5.3.3, Exercise 16). Here, for the reader's convenience, we provide a direct proof.

Lemma 1. *Let f and $\{f_k\}_{k=1}^n$ be linear functionals on X such that* $\operatorname{Ker} f \supset \bigcap_{k=1}^n \operatorname{Ker} f_k$. *Then* $f \in \operatorname{Lin}\{f_k\}_{k=1}^n$.

Proof. We use induction on n. The induction base is $n = 1$. If $f_1 = 0$, then Ker $f \supset$ Ker $f_1 = X$, i.e., $f = 0$. Now let f_1 be a non-zero functional. Then $Y =$ Ker f_1 is a subspace in X. Therefore, there exists a vector $e \in X \setminus Y$, such that Lin$\{e, Y\} = X$. Let $a = f(e)$ and $b = f_1(e)$. The functional $f - (a/b) f_1$ vanishes on Y as well as at the point e. Hence, $f - (a/b) f_1$ vanishes on the whole space $X =$ Lin$\{e, Y\}$, i.e., $f \in$ Lin$\{f_1\}$.

Step $n \to n+1$. We introduce the subspace $Y = \bigcap_{k=1}^{n}$ Ker f_k. The condition Ker $f \supset \bigcap_{k=1}^{n+1}$ Ker f_k may be interpreted as saying that the kernel of the restriction of the functional f to Y contains the kernel of the restriction of the functional f_{n+1} to Y. Therefore (by the case $n = 1$), there exists a scalar α such that $f - \alpha f_{n+1}$ vanishes on the whole space $Y = \bigcap_{k=1}^{n}$ Ker f_k. That is, Ker$(f - \alpha f_{n+1}) \supset \bigcap_{k=1}^{n}$ Ker f_k. By the induction hypothesis, $f - \alpha f_{n+1} \in$ Lin $\{f_k\}_{k=1}^{n}$, i.e., $f \in$ Lin $\{f_k\}_{k=1}^{n+1}$. □

Lemma 2. *Let Y be a subspace of the linear space X, let $f \in X'$, and suppose there exists an $a > 0$ such that $|f(y)| \leqslant a$ for all Y. Then $f(y) = 0$ for all $y \in Y$.*

Proof. Suppose that there exists an $y_0 \in Y$ such that $f(y_0) \neq 0$. Then for the element $y = (2a/f(y_0))y_0 \in Y$ one has $|f(y)| = 2a > a$, a contradiction. □

We are now ready to describe the functionals that are continuous in a weak topology.

Theorem 1. *A functional $f \in X'$ is continuous in the topology $\sigma(X, E)$ if and only if $f \in$ Lin E. In particular, if $E \subset X'$ is a linear subspace, then the set $(X, \sigma(X, E))^*$ of all $\sigma(X, E)$-continuous functionals on X coincides with E.*

Proof. By the definition of the topology $\sigma(X, E)$, all elements of the set E are $\sigma(X, E)$-continuous functionals. Hence, linear combinations of such functionals are also continuous. Conversely, suppose the functional $f \in X'$ is continuous in the topology $\sigma(X, E)$. Then there exist a finite set of functionals $G = \{g_1, g_2, \ldots, g_n\} \subset E$ and an $\varepsilon > 0$ such that on the neighborhood $U_{G,\varepsilon} = \{x \in X : \max_{g \in G} |g(x)| < \varepsilon\}$ all values of the functional f are bounded in modulus by some number $a > 0$. The same number will also bound the values of f on the subspace $Y = \bigcap_{k=1}^{n}$ Ker $g_k \subset U_{G,\varepsilon}$. By Lemma 2, the functional f vanishes on Y, which in turn means (Lemma 1) that $f \in$ Lin$\{g_k\}_{k=1}^{n} \subset$ Lin E. □

Exercises

1. Prove the equality of topologies $\sigma(X, \text{Lin } E) = \sigma(X, E)$.

2. The Tikhonov topology (topology of coordinatewise convergence) on $\mathbb{R}^{\mathbb{N}}$ coincides with the weak topology generated by the family $E = \{e_n^*\}_{n \in \mathbb{N}}$ of coordinate functionals. What is $\left(\mathbb{R}^{\mathbb{N}}\right)^*$ equal to?

3. Let $E \subset X'$ be a subspace. Then the topology $\sigma(X, E)$ has a countable neighborhood basis of zero if and only if the linear space E has an at most countable Hamel basis.

4. Suppose that on X there exists a norm that is continuous in the topology $\sigma(X, E)$. Then the space X is finite-dimensional.

5. Every set that is bounded in the topology $\sigma(X, E)$ is precompact in this topology.

6. Kolmogorov's theorem: *If in the topological vector space X there exists a bounded neighborhood U of zero, then the system of neighborhoods of zero has the countable basis $\{(1/n)U\}_{n\in\mathbb{N}}$. In particular, if this bounded neighborhood U is convex, then the topology of the space can be given by a single seminorm* (*a single norm, if the space is assumed to be separated*).

7. Let X be an infinite-dimensional linear space and let the family of functionals $E \subset X'$ separate points. Then none of the $\sigma(X, E)$-neighborhoods of zero is a $\sigma(X, E)$-bounded set.

8. The space $X = c_0$ is not complete in the topology $\sigma(X, X^*)$.

9. General result: no infinite-dimensional Banach space X is complete in the topology $\sigma(X, X^*)$.

16.3.3 Eidelheit's Interpolation Theorem

Lemma 1. *Let X be a topological vector space and $Y \subset X$ a closed subspace of finite codimension. If the functional $f \in X'$ is discontinuous, then the restriction of f to Y is also discontinuous.*

Proof. We use the properties of quotient spaces of topological vector spaces given in Exercise 2 of Subsection 16.2.3. Suppose, by contradiction, that the restriction of the functional f to Y is continuous. Then $\widetilde{Y} = Y \cap \operatorname{Ker} f$ is a closed subspace of finite codimension. By the definition of the codimension, the quotient space $X/\widetilde{Y}$ is finite-dimensional. Define the functional $\widetilde{f}$ on $X/\widetilde{Y}$ by the rule $\widetilde{f}(q(x)) = f(x)$, where $q: X \to X/\widetilde{Y}$ is the quotient mapping. Since the space $X/\widetilde{Y}$ is finite-dimensional, $\widetilde{f}$ is continuous on $X/\widetilde{Y}$. Therefore, f is continuous, being the composition of $\widetilde{f}$ and the quotient mapping q. □

Lemma 2. *Let X be a topological vector space, and $f \in X'$ a discontinuous functional. Then for any scalar a the hyperplane $f_{=a} = \{x \in X : f(x) = a\}$ is dense in X.*

Proof. The fact that the kernel of f is dense is guaranteed by Theorem 4 of Subsection 16.2.3. The hyperplane $f_{=a}$ is obtained from $\operatorname{Ker} f$ by parallel translation by any fixed vector $y \in f_{=a}$. □

Theorem 1. (M. Eidelheit [52]). *Let X be a complete locally convex subspace, the topology of which is given by a sequence of seminorms $p_1 \leqslant p_2 \leqslant p_3 \leqslant \cdots$. Suppose that the sequence of linear functionals $f_n \in X^*$ has the following property: for any $n \in \mathbb{N}$, the functional f_n is discontinuous with respect to the seminorm p_n (i.e., discontinuous in the topology generated by the single seminorm p_n), but is continuous with respect to p_{n+1}, and so also continuous with respect to all seminorms p_k with $k > n$. Then for any sequence of scalars a_n there exists an element $x \in X$ such that $f_n(x) = a_n$, $n = 1, 2, \ldots$.*

Proof. We construct the required element $x \in X$ as the sum of a series $\sum_{k=1}^{\infty} x_k$, the elements of which satisfy the following conditions:

(a) $p_n(x_n) \leqslant \dfrac{1}{2^n}$;

(b) $f_n\left(\sum_{k=1}^{n} x_k\right) = a_n$;

(c) $f_n(x_k) = 0$ for $k > n$.

Condition (a) guarantees the absolute convergence of the series $\sum_{k=1}^{\infty} x_k$ (see Exercise 10 in Subsection 16.3.1). Indeed, if p is a continuous seminorm, then its unit ball contains one of the balls of the seminorms p_n. Hence, starting with some m, the estimate $p \leqslant Cp_n$ holds for all $n \geqslant m$. Consequently, $\sum_{k=m}^{\infty} p(x_k) < C \sum_{k=m}^{\infty} p_k(x_k) < \infty$. This shows that the element $x = \sum_{k=1}^{\infty} x_k$ exists. The conditions (b) and (c) ensure that $f_n(x) = a_n$.

Hence, all we need (if not in general in life, at least in the setting of this proof) is to construct a sequence (x_n) with the properties (a)–(c). The construction will be carried out recursively.

The functional f_1 is discontinuous with respect to the seminorm p_1; hence, the hyperplane $X_1 = \{y \in X \colon f_1(y) = a_1\}$ is p_1-dense in X. In particular, X_1 intersects the ball $B_1 = \{y \in X \colon p_1(y) < 1/2\}$. Now as x_1 we take any element of the set $X_1 \cap B_1$.

Next, suppose the vectors $x_1, \ldots, x_{n-1}$ are already constructed; let us construct x_n. Consider the finite-codimensional subspace $Y = \bigcap_{k=1}^{n-1} \operatorname{Ker} f_k$. Since the functionals f_k are p_n-continuous for $k < n$, Y is a p_n-closed subspace. By Lemma 1, the restriction of the functional f_n to Y is discontinuous with respect to the seminorm p_n. Therefore, the hyperplane $X_n = \{y \in Y \colon f_n(y) = a_n - \sum_{k=1}^{n-1} f_n(x_k)\}$ is dense in Y with respect to the seminorm p_{n+1}. It follows that the ball $B_n = \{y \in Y \colon p_n(y) < 2^{-n}\}$ intersects the hyperplane X_n. For x_n we take an arbitrary element of $X_n \cap B_n$. The fact that the vector x_n belongs to B_n, X_n, and Y guarantees the fulfillment of condition (a), (b), and (c), respectively. □

Let us give a couple of examples which demonstrate how the interpolation theorem just proved applies in problems of mathematical analysis.

Theorem 2. *For any sequence of scalars a_n, $n = 0, 1, 2, \ldots$, there exists an infinitely differentiable function $x(t)$ on the interval* $[0, 1]$ *such that* $x(0) = a_0$, $x'(0) = a_1, \ldots$, $x^{(n)}(0) = a_n, \ldots$.

Proof. Observe that the natural approach to give the solution in the form of a Taylor series $x(t) = \sum_{n=0}^{\infty} \frac{a_n}{n!} t^n$ fails: if a_n tends rapidly to infinity, then the radius of convergence of the Taylor series will be equal to zero. The interpolation theorem, however, provides a very economical solution to the problem.

In the space $C^\infty[0, 1]$ of infinitely differentiable functions on [0, 1], consider the sequence of seminorms $p_0 \leqslant p_1 \leqslant p_2 \leqslant \cdots$ given by

$$p_0 = 0, \quad p_1(x) = \max_{t \in [0,1]} |x(t)|, \quad p_2(x) = \max\{p_1(x), p_1(x')\}, \ldots,$$
$$p_n(x) = \max\{p_1(x), p_1(x'), \ldots, p_1(x^{(n-1)})\}, \ldots,$$

and the sequence of functionals

$$f_0(x) = x(0), \ f_1(x) = x'(0), \ \ldots, \ f_n(x) = x^{(n)}(0), \ldots.$$

The chosen sequence of seminorms gives on $C^\infty[0, 1]$ the topology of uniform convergence of functions and all their derivatives. In this topology the space $C^\infty[0, 1]$ is complete. Further, $|f_n(x)| \leqslant p_{n+1}(x)$, i.e., the functional f_n is continuous with respect to p_{n+1}. However, there is no constant C such that the inequality $|f_n| \leqslant Cp_n$ is satisfied, as one can readily verify by substituting into the inequality, say, the sequence of functions $x_m(t) = \sin(\pi m t)$. All the conditions of Theorem 1 are satisfied, so it remains only to apply it. □

Theorem 3. *For any sequence of scalars* (a_n), $n = 1, 2, \ldots$, *there exists a function* $x(z)$ *such that* $x(n) = a_n$ *for all* n.

Proof. In the space $\mathcal{H}(\mathbb{C})$ of entire functions consider the sequence of seminorms $p_1 \leq p_2 \leq \cdots$, $p_n(x) = \max_{|z| \leqslant n-1} |x(z)|$, and the functionals $f_n(x) = x(n)$. This sequence of seminorms gives the topology of uniform convergence on compact sets, in which the space $\mathcal{H}(\mathbb{C})$ is complete. Again, as in the preceding theorem, $|f_n(x)| \leqslant p_{n+1}(x)$, whereas there is no C such that $|f_n| \leqslant Cp_n$ (substitute the functions $x_m(z) = z^m$). And again, the sought-for function $x(z)$ is obtained by applying Theorem 1. □

A slightly more general variant of the interpolation theorem above and its application to the moment problem can be found in B. M. Makarov's work [70].

Exercises

1. Verify the correctness of the definition of the functional $\widetilde{f}$ in the proof of Lemma 1, namely, that if $q(x) = q(y)$, then $f(x) = f(y)$. That is, that $\widetilde{f}(qx) = f(x)$ depends on qx, but not on x.

2. Let $t_1, \ldots, t_j$ be a finite collection of distinct points of the interval [0, 1], and $\{a_{n,k}\}_{n \in \mathbb{N} \cup \{0\}, k \in \{1,\ldots,j\}}$ be numbers. Show that there exists a function $x \in C^\infty[0, 1]$ such that $x^{(n)}(t_k) = a_{n,k}$ for all $n \in \mathbb{N} \cup \{0\}$ and $k \in \{1, \ldots, j\}$.

3. Let $z_n \in \mathbb{C}$ be an arbitrary collection of points ("interpolation nodes"). The following conditions are equivalent:

— for any sequence of scalars a_n, $n = 1, 2, \ldots$, there exists an entire function $x(z)$ such that $x(z_n) = a_n$ for all n;

— $z_n \neq z_m$ for $n \neq m$, and $|z_n| \to \infty$ as $n \to \infty$.

4. For any sequence of scalars a_n, $n = 1, 2, \ldots$, and any sequence of indices $m_1 < m_2 < m_3 < \ldots$, there exists a "lacunary" entire function $x(z)$ of the form $x(z) = \sum_{k=1}^{\infty} b_k z^{m_k}$ such that $x(n) = a_n$ for all $n \in \mathbb{N}$.

A sequence x_n of elements of a topological vector space X is said to be *ω-linearly independent* if for any sequence of scalars b_n, the equality $\sum_{k=1}^{\infty} b_k x_k = 0$ implies that all b_n are equal to zero. The Erdős–Straus theorem (P. Erdős, E.G. Straus, 1953)[4] asserts that in a normed space from any linearly independent sequence one can extract an ω-linearly independent subsequence.

5. Suppose that the topological vector space X carries a continuous norm. Then from any linearly independent sequence in X one can extract an ω-linearly independent subsequence.

6. Consider the vectors $x_1 = (1, 2, 3, \ldots, n, \ldots)$, $x_2 = (1^2, 2^2, 3^2, \ldots, n^2, \ldots)$, $x_3 = (1^3, 2^3, 3^3, \ldots, n^3, \ldots), \ldots$ in the space $\mathbb{R}^{\mathbb{N}}$. This sequence is linearly independent, but it contains no ω-linearly independent subsequence. (Use Exercise 4 above.)

7. The Bessaga–Pełczyński theorem (C. Bessaga, A. Pełczyński, 1957). If a complete metrizable locally convex space X admits no continuous norm, then X contains a real subspace isomorphic to $\mathbb{R}^{\mathbb{N}}$.

8. Deduce from the three preceding exercises that for a complete metrizable locally convex space X the following conditions are equivalent:

— from any linearly independent sequence in X one can extract an ω-linearly independent subsequence;

— there exists a continuous norm on X.

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[4]During the preparation for publication of the second volume of his monograph [40], I. Singer discovered a gap in the original proof of Erdős and Straus. He distributed a letter to other specialists in the theory of bases, asking for an alternative proof of the result. Such proofs were obtained by P. Terenzi and at about the same time by V.I. Gurariĭ, who back then, in 1980, was an active participant in our Kharkiv seminar on the theory of Banach spaces. I have nostalgic memories about those times: in the spring of 1980 I was a third-year student, and this was the first "mature" problem to which I devoted serious thought. The example in Exercise 6 — the fruit of this pondering — was later mentioned by Singer in his monograph. One can imagine how proud I was for discovering this example …It is amusing that I published this observation only after 10 years and a bit [56].

16.3.4 Precompactness and Boundedness

Definition 1. A topological vector space space X is said to belong to the *Montel class* (or to be a *Montel space*) if any closed bounded set in X is compact.

By Riesz's theorem, a normed space is Montel only if it is finite-dimensional. At the same time, many of the topological vector spaces arising naturally in problems of analysis are Montel, despite being infinite-dimensional. In this subsection we shall give examples of Montel space. The name "Montel class" comes from Montel's theorem, which establishes a compactness criterion in the space $\mathcal{H}(D)$ of holomorphic functions. In modern complex analysis courses this theorem serves as the basis for the proof of Riemann's theorem on the existence of conformal maps (the Riemann mapping theorem).

Definition 2. Let A, B be subsets of the linear space X. We say that the set A is *B-precompact* (and write $A \prec_c B$) if for any $\varepsilon > 0$ there exists a finite set Q such that $A \subset \varepsilon B + Q$.

If X is a normed space, then a subset $A \subset X$ is precompact if and only if $A \prec_c B_X$. A subset A of the topological vector space X is precompact if and only if $A \prec_c U$ for all neighborhoods U of zero in the space X.

Theorem 1. *The relation $\prec_c$ between subsets of a linear space X has the following properties:*

(a) *if $A \prec_c B$ and $A_1 \subset A$, then $A_1 \prec_c B$;*

(b) *if $A \prec_c B$ and $B \subset B_1$, then $A \prec_c B_1$;*

(c) *if $A \prec_c B$ and $t > 0$, then $A \prec_c tB$;*

(d) *if $A_1 \prec_c B$ and $A_2 \prec_c B$, then $A_1 \cup A_2 \prec_c B$;*

(e) *if $A \prec_c B$, Y is a linear subspace, and $T : X \to Y$ is a linear operator, then $T(A) \prec_c T(B)$;*

(f) *if $A_1 \prec_c B$, $A_2 \prec_c B$, and B is a convex set, then $A_1 + A_2 \prec_c B$;*

(g) *if $A \prec_c B$ and $B - B \subset U$, then $A \prec_c U$; moreover, for any $\varepsilon > 0$ there exists a finite set Q such that $Q \subset A$ and $A \subset \varepsilon U + Q$;*

(h) *let $A \prec_c B$, B a convex balanced set, Y a linear space, $T : Y \to X$ a linear operator, and $A \subset T(Y)$. Then $T^{-1}(A) \prec_c T^{-1}(B)$.*

Proof. Properties (a)–(e) are obvious. Let us prove the remaining properties.

(f) Fix $\varepsilon > 0$. Let $Q_1, Q_2 \subset X$ be finite sets for which $A_1 \subset (\varepsilon/2)B + Q_1$, and $A_2 \subset (\varepsilon/2)B + Q_2$. Then $A_1 + A_2 \subset (\varepsilon/2)B + (\varepsilon/2)B + Q_1 + Q_2$. Thanks to convexity, $(1/2)B + (1/2)B \subset B$, and hence $A_1 + A_2 \subset \varepsilon B + Q_1 + Q_2$. It remains to note that the set $Q_1 + Q_2$ is finite.

(g) By hypothesis, there exists a finite set $Q_1 \subset X$ such that $A \subset \varepsilon B + Q_1$. Let us introduce a mapping $f : Q_1 \to A$ with the property that for any $q \in Q_1$, if $q + \varepsilon B$

intersects A, then $f(q) \in (q + \varepsilon B) \cap A$. We claim that $A \subset \varepsilon U + f(Q_1)$, i.e., that $f(Q_1)$ can be taken as the required set Q. Indeed, for every $a_0 \in A$ there exists a $q \in Q_1$ such that $a_0 \in q + \varepsilon B$. For this vector q the sets $q + \varepsilon B$ and A intersect (a_0 is one of the intersection points), and so $f(q) \in (q + \varepsilon B) \cap A$. We have

$$a_0 \in q + \varepsilon B = f(q) + \varepsilon B + (q - f(q)) \subset f(q) + \varepsilon B - \varepsilon B \subset f(q) + \varepsilon U.$$

(h) Since, by property (c), $A \prec_c (1/2)B$ and $(1/2)B - (1/2)B \subset B$, the property (g) proved above says that for any $\varepsilon > 0$ there exists a finite set $Q \subset A$ such that $A \subset \varepsilon B + Q$. Then $Q \subset T(Y)$, and we can construct a mapping $f : Q \to Y$ such that $T(f(q)) = q$ for all $q \in Q$. Let us show that $T^{-1}(A) \subset \varepsilon T^{-1}(B) + f(Q)$. Let $y \in T^{-1}(A)$. Then $T(y) \in A$, and there exist a $b \in B$ and a $q \in Q$ such that $T(y) = q + \varepsilon b$. Since $T(f(q)) = q$, we have $T(y - f(q)) = \varepsilon b$, i.e., $(y - f(q))/\varepsilon \in T^{-1}(B)$. Consequently, $y = \varepsilon(y - f(q))/\varepsilon + f(q) \in \varepsilon T^{-1}(B) + f(Q)$. □

Theorem 2. *Suppose X is a complete topological vector space and for every neighborhood U of zero there exists a neighborhood V of zero such that $V \prec_c U$. Then X belongs to the Montel class.*

Proof. In view of the completeness of the space X, it suffices to prove that every bounded subset $A \subset X$ is precompact (Theorem 5 in Subsection 16.2.2). So, let A be bounded and U be an arbitrary neighborhood of zero. By hypothesis, there exists a neighborhood of zero V with $V \prec_c U$. By the definition of boundedness, $A \subset nV$ for n large enough. By items (a) and (c) in the preceding theorems, $A \subset nV \prec_c nU$, that is, $A \prec_c U$. □

Example 1. The space $\mathcal{H}(D)$ of holomorphic functions on a domain $D \subset \mathbb{C}$ belongs to the Montel class.

To verify this, we use Theorem 2. Let U be an arbitrary neighborhood of zero in $\mathcal{H}(D)$. Recalling the definition of the topology on $\mathcal{H}(D)$ (Exercise 4 in Subsection 16.3.1), we may assume that U is the unit ball of the seminorm $p_K(f) = \max_{z \in K} |f(z)|$, where K is a compact subset of D, and without loss of generality we may assume that K is a finite union of closed disks. Consider a rectifiable contour $\Gamma \subset D$ which includes K in its interior; denote by K_1 the compact set which includes K and has Γ as its boundary, and by V the unit ball of the seminorm p_{K_1}. We show that $V \prec_c U$. Let $\delta = \min\{|z - \zeta| : z \in K, \zeta \in \Gamma\}$, and l be the length of the contour Γ. By the Cauchy integral formula for the derivative, for any function $f \in V$ and any $z \in K$ we have

$$|f'(z)| \leqslant \frac{1}{2\pi} \left| \int_\Gamma \frac{f(\zeta) d\zeta}{(z - \zeta)^2} \right| \leqslant \frac{1}{2\pi} \frac{p_{K_1}(f)}{\delta^2} l \leqslant \frac{l}{2\pi\delta^2}.$$

Thus, the first derivatives of the functions in the family V are uniformly bounded on K. Further, the family V itself is bounded in modulus on K (and even on the larger compact set K_1) by 1. By Arzelà's theorem, V is precompact if regarded as a subset of $C(K)$.

Now consider the operator $T\colon \mathcal{H}(D) \to C(K)$ which maps each function into the restriction of the function to K. The fact that we just proved can be formulated as follows: the set $T(V)$ is precompact in $C(K)$. In other words, $T(V) \prec_c B_{C(K)}$. Since $T^{-1}(B_{C(K)}) = U$, item (h) of Theorem 1 shows that $V \prec_c U$. □

Example 2. The space $C^\infty[0, 1]$ belongs to the Montel class.

Recall that the topology of the space $C^\infty[0, 1]$ is given by the family of seminorms $p_n(f) = \max_{t\in[0,1]} |f^{(n)}(t)|$, $n = 0, 1, 2, \ldots$. Denote the unit ball of the seminorm p_n by B_n. A neighborhood basis of zero is provided by the sets rU_n, where $r > 0$, and $U_n = \bigcap_{k=0}^n B_k = \{f \in C^\infty[0, 1] : \max_{k=0,1,\ldots,n} \max_{t\in[0,1]}|f^{(k)}(t)| < 1\}$. By Theorem 2, to justify our example it suffices to show that $U_{n+1} \prec_c U_n$ for all $n = 0, 1, 2, \ldots$. We proceed by induction on n.

$n = 0$. Consider the identity embedding operator $T\colon C^\infty[0, 1] \to C[0, 1]$. The set $T(U_1)$ (which coincides with U_1) consists of infinitely differentiable functions that obey the conditions $|f(t)| < 1$ and $|f'(t)| < 1$ for all $t \in [0, 1]$. By Arzelà's theorem, $T(U_1)$ is precompact in $C[0, 1]$, i.e., $T(U_1) \prec_c B_{C[0,1]}$. According to item (h) of Theorem 1, $U_1 \prec_c T^{-1}(B_{C[0,1]}) = U_0$.

$n \to n+1$. Suppose $U_{n+1} \prec_c U_n$. Consider the integration operator $G\colon C^\infty[0, 1] \to C^\infty[0, 1]$, $(Gf)(t) = \int_0^t f(\tau)d\tau$. By item (e) of Theorem 1, $G(U_{n+1}) \prec_c G(U_n)$. Since $G(U_n) \subset U_{n+1}$, we deduce that

$$G(U_{n+1}) \prec_c U_{n+1}. \tag{1}$$

On the other hand, since every function $f \in U_{n+2}$ can be represented as $f(t) = f(0) + \int_0^t f'(\tau)d\tau$, where $f' \in U_{n+1}$, and $|f(0)| < 1$, we have $U_{n+2} \subset A + G(U_{n+1})$, where A consists of constants smaller than 1 in modulus. Condition (1) combined with the obvious condition $A \prec_c U_{n+1}$ (A is a one-dimensional bounded set) allows us to apply assertion (f) of Theorem 2: $U_{n+2} \subset A + G(U_{n+1}) \prec_c U_{n+1}$, as we needed to prove.

Exercises

1. Show that $C^\infty(0, +\infty)$ is a Montel space.

2. Every linear space X equipped with the strongest locally convex topology is a Montel space. Moreover, in such a space every bounded set is finite-dimensional.

3. Any Tikhonov product of Montel spaces is a Montel space.

4. Any closed subspace of a Montel space is itself a Montel space.